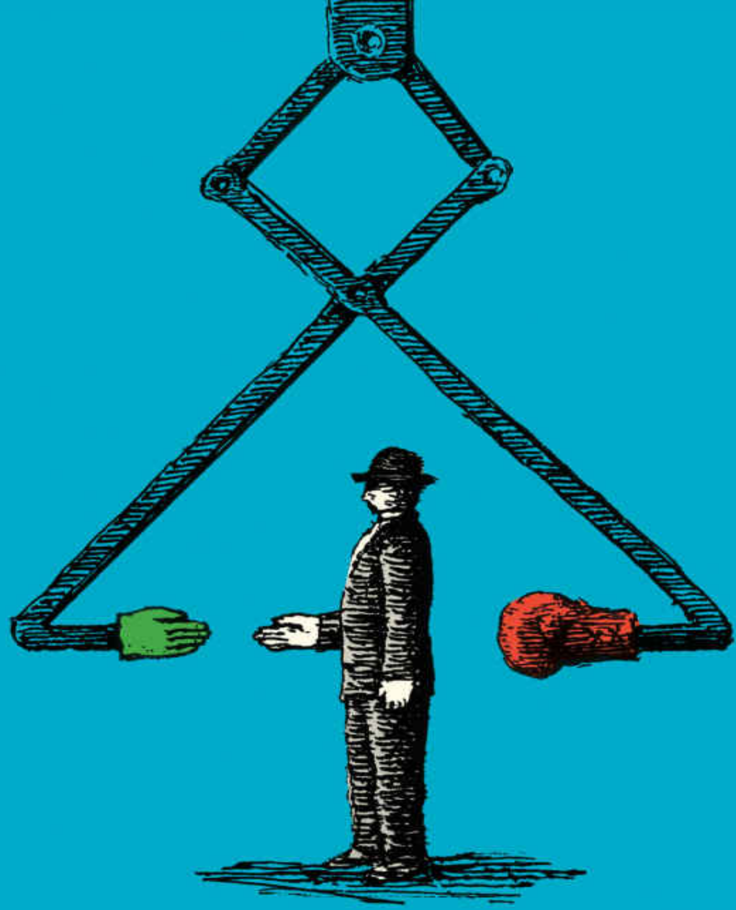

Games of Strategy

FIFTH EDITION

Avinash Dixit • Susan Skeath
David McAdams

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FIFTH EDITION

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Dedication

■

To the memory of my father,

Kamalakar Ramachandra Dixit

—Avinash Dixit

To the memory of my father,

James Edward Skeath

—Susan Skeath

To my mother,

Elizabeth Emery McAdams

—David McAdams

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■

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Preface

■

We wrote this textbook to make possible the teaching of game theory to first- or second-year college students at an introductory or "principles" level without requiring any prior knowledge of the fields where game theory is used—economics, political science, evolutionary biology, and so forth—and requiring only minimal high school mathematics. Our aim has succeeded beyond our expectations. Many such courses now exist where none did twenty-five years ago; indeed, some of these courses have been inspired by our textbook. An even better sign of success is that competitors and imitators are appearing on the market.

However, success does not justify complacency. We have continued to improve the material in each new edition in response to feedback from teachers and students in these courses and from our own experiences of using the book.

The main news item for this, the fifth, edition is a switch of Davids. David Reiley, who was a coauthor on the Third and Fourth Editions, got too busy with his day job in tech and left the team. Dixit and Skeath were fortunate to recruit David McAdams, who brought new ideas and skills to the team, leading to many tripartite discussions that have yielded perhaps the most substantial revision and improvement to any of our editions. We highlight several changes in this preface; they, and many more, will be evident throughout the book.

The new material and improvements in the Fifth Edition include the following, in rough order of the level of substantive change involved: (1) The chapter on auctions, now Chapter 15, was completely rewritten and brought up to date with recent developments and examples from Internet auctions. (2) The case study of the Cuban missile crisis (Chapter 13) was revised in the light of recent disclosures of participants and analyses of historians. It was also enriched by numerical calculations based on a dynamic version of the chicken game with asymmetric information, which itself is a significant addition to Chapter 9. (3) The terminology and theory of strategic moves in Chapter 8 was substantially clarified and improved. This material relates to a similar improvement in Chapter 6, clarifying what it means for a game to have simultaneous versus sequential moves and the advantages of being first mover versus second mover. (4) The mathematics of situations involving risk aversion in Chapters 9 and 14, previously explained using expected utility theory, was simplified by using the related and empirically well-supported concept of loss aversion instead. (5) We introduce in Chapter 4, and use later, the concept of superdominance as well as some special features of games where only the players' ranking of the outcomes matters (ordinal payoffs). (6) We broadened our discussion in Chapter 2 of what games are and why game theory is useful, highlighting a distinction between small-numbers and massively multiplayer games. (7) We updated the discussion of very complex games like Chess and Go in the light of recent advances in artificial intelligence and machine learning, and updated empirical evidence on mixed strategies, citing new research on tennis serves at Wimbledon. (8) New and updated examples and stories appear throughout the book. Two are especially noteworthy: For the coordination game in Chapter 4 and later, we used the recently popular Holmes and Watson movies and TV series. And we used an episode from the movie *Ransom* to illustrate the idea of variable-threat bargaining in Chapter 17.

Our main innovation in the Fourth Edition was a simpler treatment of mixed strategies. That seems to have worked well, and we have retained it here. Chapter 7 develops the solution and interpretation of mixed-strategy equilibria in two-by-two games and provides a simple example of a game with more than two pure strategies in which some strategies may go unused in equilibrium. Some of the more complex material on the theory of mixed strategy equilibria is available as online appendices for those readers who want to know more about the advanced topics.

Many of the games in Chapters 8–17, Parts Three and Four of the text, are simple enough to be grasped without drawing game trees or showing payoff tables. However, some readers wanted these tools incorporated into the later chapters to show the applications of the methods developed in earlier chapters more explicitly. We started doing so in the Fourth Edition and have continued and expanded that practice here.

Our effort to expand and improve the stock of exercises throughout the book and its associated Web sites continues in this edition. We have updated exercise examples where necessary and added new exercises to almost every chapter, with the chapters in the latter half of the book (which had fewer exercises in previous editions) getting particular attention in terms of added exercises. As before, the exercises in each chapter are split into two sets—solved and unsolved. These usually run in parallel: for each solved exercise, there is a corresponding unsolved one with some variation to give students further practice. Solutions to the solved sets are available to all readers on the text's digital landing page at digital.wwnorton.com/gamesofstrategy5. Solutions to the unsolved sets are reserved for instructors who have adopted the textbook. Instructors should contact the publisher about getting access to those resources. In each of the solved and unsolved sets, there are two kinds of exercises. Some provide repetition and drill in the techniques developed in the chapter. In others—and, in our view, those with the most educational value—we take the student step-by-step through the process of constructing a game-theoretic model to analyze an issue or problem. Such experience, gained in some solved exercises and repeated in corresponding unsolved ones, will best develop the students’ skills in strategic thinking.

Finally, some suggestions for teachers: The first seven chapters comprise the basic concepts and tools needed in the chapters that follow. Chapters 8–17 apply these to several separate topics and fields, including voting, auctions, incentive design, and so on. Teachers who want to focus on some of these special topics can take their pick and leave out the others without loss of continuity. Also, while the book is aimed at the introductory-level learner, we want to give students a glimpse of the rich applications of game theory that exist at higher levels. Therefore, each of the chapters in the second half of the book contains one or two sections with such material: Chapter 9 has examples illustrating dynamic chicken and bluffing (semi-separating equilibria), Chapter 12 gives a detailed treatment of the evolutionary Hawk-Dove game, and so on. Teachers of courses that do not cover these topics or levels can choose to omit them without compromising student learning.

The substance and writing in the book have been improved by the perceptive and constructive pieces of advice offered by faculty who have used the text in their courses and numerous others who have read all or parts of the book in other contexts. We thanked many of these readers for their comments in the prefaces to previous editions. For the Fifth Edition, we have had the added benefit of extensive comments from Fiona Carmichael (University of Birmingham, UK), Jihui Chen (Illinois State University), Martin Dufwenberg (University of Arizona), Henry Duncanson (University of Bristol, UK), Jens Grosser (Florida State University), Katerina Raoukka (University of Bristol, UK), and Hanzhe Zhang (Michigan State University). Thank you all.

Finally, it is a pleasure to thank Eric Svendsen, our editor at Norton, and his team, Layne Broadwater, Victoria Reuter, Sean Mintus, and Elizabeth Bowles, for their efficient shepherding of the book through the production process. We want to make a special mention of Norma Sims Roche, our copyeditor. She improved our writing in many places, caught some ambiguities, inconsistencies, and lack of precision or clarity in our use of terminology, and more generally contributed to making this edition much better.

Avinash Dixit

Susan Skeath

David McAdams

PART ONE

Introduction and General Principles

1 ■ Basic Ideas and Examples

ALL INTRODUCTORY TEXTBOOKS begin by attempting to convince their student readers that the subject is of great importance in the world and therefore merits their attention. The physical sciences and engineering claim to be the basis of modern technology and therefore of modern life; the social sciences discuss big issues of governance, such as democracy and taxation; the humanities claim to revive your soul after it has been deadened by exposure to the physical and social sciences and to engineering. Where does the subject of games of strategy, often called game theory, fit into this picture, and why should you study it?

We offer a practical motivation that is much more individual and probably closer to your personal concerns than most other subjects. You play games of strategy all the time: with your parents, siblings, friends, and enemies, and even with your professors. You have probably acquired a lot of instinctive expertise in playing such games, and we hope you will be able to connect what you have already learned to the discussion that follows. We will build on your experience, systematize it, and develop it to the point where you will be able to improve your strategic skills and use them more methodically. Opportunities for such uses will appear throughout your life; you will go on playing such games with your employers, employees, spouses, children, and even strangers.

Not that the subject lacks wider importance. Similar games are played in business, politics, diplomacy, and war—in fact, whenever people interact to strike mutually agreeable deals or to resolve conflicts. Being able to recognize such games will enrich your understanding of the world around you and will make you a better participant in all its affairs. Understanding games of strategy will also have a more immediate payoff in your study of many other subjects.

Economics and business courses already use a great deal of game-theoretic thinking. Political science, psychology, and philosophy are also using game theory to study interactions, as is biology, which has been importantly influenced by the concepts of evolutionary games and has in turn exported these ideas to economics. Psychology and philosophy also interact with the study of games of strategy. Game theory provides concepts and techniques of analysis for many disciplines—one might even say all disciplines except those dealing with completely inanimate objects.

1 WHAT IS A GAME OF STRATEGY?

The word *game* may convey an impression that our subject is frivolous or unimportant in the larger scheme of things—that it deals with trivial pursuits such as gambling and sports when the world is full of weightier matters such as war and business and your education, career, and relationships. Actually, games of strategy are not "just a game"; all of these weightier matters are instances of games, and game theory helps us understand them all. But it will not hurt to start with game theory as applied to gambling or sports.

Most games include chance, skill, and strategy in varying proportions. Playing double or nothing on the toss of a coin is a game of pure chance, unless you have exceptional skill in doctoring or tossing coins. A hundred-yard dash is a game of pure skill, although some chance elements can creep in; for example, a runner may simply have a slightly off day for no clear reason.

Strategy is a skill of a different kind. In the context of sports, it is a part of the mental skill needed to play well; it is the calculation of how best to use your physical skill. For example, in tennis, you develop physical skill by practicing your serves (first serves hard and flat, second serves with spin or kick) and passing shots (hard, low, and accurate). The strategic skill is knowing where to put your serve (wide, or on the T) or passing shot (crosscourt, or down the line). In football, you develop physical skills such as blocking and tackling, running and catching, and throwing. Then the coach, knowing the physical skills of his own team and those of the opposing team, calls the plays that best exploit his team's skills and the other team's weaknesses. The coach's calculation constitutes the strategy. The physical game of football is played on the gridiron by jocks;

the strategic game is played in the offices and on the sidelines by coaches and by nerdy assistants.

A hundred-yard dash is a matter of exercising your physical skill as best you can; it offers no opportunities to observe and react to what other runners in the race are doing and therefore no scope for strategy. Longer races do entail strategy—whether you should lead to set the pace, how soon before the finish you should try to break away, and so on.

Strategic thinking is essentially about your interactions with others, as they do similar thinking at the same time and about the same situation. Your competitors in a marathon may try to frustrate or facilitate your attempts to lead, given what they think best suits their interests. Your opponent in tennis tries to guess where you will put your serve or passing shot. The opposing coach in football calls the play that will best counter what he thinks your coach will call. Of course, just as you must take into account what the other player is thinking, he is taking into account what you are thinking. Game theory is the analysis, or science, if you like, of such interactive decision making.

When you think carefully before you act—when you are aware of your objectives or preferences and of any limitations or constraints on your actions, and choose your actions in a calculated way to do the best according to your own criteria—you are said to be behaving rationally. Game theory adds another dimension to rational behavior—namely, interaction with other equally rational decision makers. In other words, game theory is the science of rational behavior in interactive situations.

However, rationality does have limits. Recent research in psychology and behavioral economics has shown that many decisions are made instinctively and are based on rules or heuristics. Good strategic thinking will recognize the possibility that other players may not be rational and will also anticipate and prepare for one's own instinctive departures from rationality. We will include these aspects of thinking as we pursue our analyses of strategy.

We do not claim that game theory will teach you the secrets of perfect play or ensure that you will never lose. For one thing, your opponent can read the same book, and both of you cannot win all the time. More importantly, many games are complex and subtle, and most actual situations include enough idiosyncratic or chance elements that game theory cannot hope to offer surefire recipes for action. What it can do is provide some general principles for thinking about strategic interactions. You have to supplement these principles with details specific to your situation before you can devise a successful strategy for it. Good strategists mix the science of game theory with their own experience; one might say that game playing is as much art as science. We will develop the general ideas of the science but will also point out its limitations and tell you when the art is more important.

You may think that you have already acquired the art of game playing from your experience or instinct, but you will find the study of the science useful nonetheless. The science systematizes general principles that are common to many seemingly different contexts or applications. Without general principles, you would have to figure out from scratch each new situation that requires strategic thinking. That would be especially difficult to do in new areas of application—for example, if you learned your art by playing games against parents and siblings and must now practice strategy against business competitors. The general principles of game theory provide you with a ready reference point. With this foundation in place, you can proceed much more quickly and confidently to acquire and add the situation-specific features or elements of the art to your thinking and action.

2 SOME EXAMPLES AND STORIES OF STRATEGIC GAMES

With the aims announced in Section 1, we will begin by offering you some simple examples, many of them taken from situations that you have probably encountered in your own life, where strategy is of the essence. In each case, we will point out the crucial strategic principle. Each of these principles will be discussed more fully in a later chapter, and after each example we will tell you where the details can be found. But don’t jump to those details right away; just read all the examples here to get a preliminary idea of the whole scope of strategy and of strategic games.

A. Which Passing Shot?

Tennis at its best consists of memorable duels between top players: Roger Federer versus Rafael Nadal, Serena Williams versus Venus Williams, Pete Sampras versus Andre Agassi, and Martina Navratilova versus Chris Evert. Picture the 1983 U.S. Open final between Evert and Navratilova.[1] Navratilova at the net has just volleyed to Evert on the baseline. Evert is about to hit a passing shot. Should she go down the line or crosscourt? And should Navratilova expect a down-the-line shot and lean slightly that way or expect a crosscourt shot and lean the other way?

Conventional wisdom favors the down-the-line shot. The ball has a shorter distance to travel to the net, so the other player has less time to react. But this does not mean that Evert should use that shot all the time. If she did, Navratilova would confidently come to expect it and prepare for it, and the shot would not be so successful. To improve the success of the down-the-line passing shot, Evert has to use the crosscourt shot often enough to keep Navratilova guessing on any single instance. Similarly, in football, with a yard to go on third down, a run up the middle is the percentage play—that is, the one used most often—but the offense must throw a pass occasionally in such situations "to keep the defense honest."

Thus, the most important general principle of such situations is not what Evert *should* do but what she *should not* do: She should not do the same thing all the time or systematically. If she did, then Navratilova would learn to cover that shot, and Evert's chances of success would fall.

Not doing any one thing systematically means more than not playing the same shot in every situation of this kind. Evert should not even mechanically switch back and forth between the two shots. Navratilova would spot and exploit this pattern or, indeed, any other detectable system. Evert must make the choice on each particular occasion *at random* to prevent this exploitation.

This general idea of “mixing one’s plays” is well known, even to sports commentators on TV. But there is more to the idea, and these further aspects require analysis in greater depth. Why is down-the-line the percentage shot? Should one play it 80% of the time, or 90%, or 99%? Does it make any difference if the occasion is particularly big; for example, does one throw that pass on third down in the regular season but not in the Super Bowl? In actual practice, just how does one mix one’s plays? What happens when a third possibility (the lob) is introduced? We will examine and answer such questions in Chapter 7.

The movie *The Princess Bride* (1987) illustrates the same idea in the “battle of wits” between the hero (Westley) and a villain (Vizzini). Westley is to poison one of two wineglasses out of Vizzini’s sight, and Vizzini is to decide who will drink from which glass. Vizzini goes through a number of convoluted arguments as to why Westley should poison one glass. But all of the arguments are innately contradictory, because Westley can anticipate Vizzini’s logic and choose to put the poison in the other glass. Conversely, if Westley uses any specific logic or system to choose one glass, Vizzini can anticipate that and drink from the other glass, leaving Westley to drink from the poisoned one. Thus, Westley’s strategy has to be random or unsystematic.

The scene illustrates something else as well. In the film, Vizzini loses the game and with it his life. But it turns out that Westley had poisoned both glasses; over the last several years, he had built up immunity to the poison. So Vizzini was actually playing the game under a fatal information disadvantage. Players can sometimes cope with such asymmetries of information; Chapters 9 and 14 examine when and how they can do so.

B. The GPA Rat Race

Imagine that you are enrolled in a course that is graded on a curve. No matter how well you do in absolute terms, only 40% of the students will get As, and only 40% will get Bs. Therefore, you must work hard, not just in absolute terms, but relative to how hard your classmates (actually, "class enemies" seems a more fitting term in this context) work. All of you recognize this, and after the first lecture you hold an impromptu meeting in which all students agree not to work too hard. As weeks pass by, the temptation to get an edge on the rest of the class by working just that little bit harder becomes overwhelming. After all, the others are not able to observe your work in any detail, nor do they have any real hold over you. And the benefits of an improvement in your grade point average are substantial. So you hit the library more often and stay up a little longer.

The trouble is, everyone else in the class is doing the same. Therefore, your grade is no better than it would have been if you and everyone else had abided by the agreement. The only difference is that all of you have spent more time working than you would have liked.

This scenario is an example of the prisoners' dilemma.[2] In the original story, two suspects in a crime are being separately interrogated and invited to confess. One of them, say A, is told, "If the other suspect, B, does not confess, then you can cut a very good deal for yourself by confessing. But if B does confess, then you would do well to confess, too; otherwise the court will be especially tough on you. So you should confess no matter what B does." B is urged to confess with the use of similar reasoning. Faced with these choices, both A and B confess. But it would have been better for both if neither had confessed, because the police had no really compelling evidence against them.

Your situation in the class is similar. If the others slack off, then you can get a much better grade by working hard; if the others work hard, then you had better do the same, or else you will get a poor grade. You may even think that the label "prisoner" is fitting for a group of students trapped in a required course.

Professors and schools have their own prisoners' dilemmas. Each professor can make her course look good or attractive by grading it slightly more liberally, and each school can place its students in better jobs or attract better applicants by grading all of its courses a little more liberally. Of course, when all do this, none has any advantage over the others; the only result is rampant grade inflation, which compresses the spectrum of grades and therefore makes it difficult to distinguish abilities.

People often think that in every game there must be a winner and a loser. The prisoners' dilemma is different—all players can come out losers. People play, and lose, such games every day, and their losses can range from minor inconveniences to potential disasters. Spectators at a sports event stand up to get a better view, but when all stand, no one has a better view than when they were all sitting. Superpowers acquire more weapons to get an edge over their rivals, but when both do so, the balance of power is unchanged; all that has happened is that both have spent economic resources that they could have used for better purposes, and the risk of accidental war has escalated. The magnitude of the potential cost of such games to all players makes it important to understand the ways in which mutually beneficial cooperation can be achieved and sustained. All of Chapter 10 deals with the study of this game.

The prisoners’ dilemma is potentially a lose-lose game, but there are win-win games, too. International trade is an example: When each country produces more of what it can produce relatively well, all share in the fruits of this international division of labor. But successful bargaining about the division of the pie is needed if the full potential of trade is to be realized. The same applies to many other bargaining situations. We will study these situations in Chapter 17.

C. “We Can’t Take the Exam Because We Had a Flat Tire”

Here is a story, probably apocryphal, that circulates on the undergraduate e-mail networks and which each of us has independently received from our students:

> There were two friends taking chemistry at Duke. Both had done pretty well on all of the quizzes, the labs, and the midterm, so that going into the final they each had a solid A. They were so confident the weekend before the final that they decided to go to a party at the University of Virginia. The party was so good that they slept all day Sunday and got back to Duke too late to study for the chemistry final that was scheduled for Monday morning. Rather than take the final unprepared, they went to the professor with a sob story. They said they had gone up to UVA and had planned to come back in good time to study for the final, but had a flat tire on the way back. Because they didn’t have a spare, they had spent most of Sunday night looking for help. Now they were really too tired, so could they please have a makeup final the next day? The professor thought it over and agreed.
>
> The two studied all of Monday evening and came well prepared on Tuesday morning. The professor placed them in separate rooms and handed the test to each. The first question on the first page, worth 10 points, was very easy. Each of them wrote a good answer and, greatly relieved, turned the page. It had just one question, worth 90 points. It was: “Which tire?”

The story has two important strategic lessons for future partygoers. The first is to recognize that the professor may be an intelligent game player. He may suspect some trickery on the part of the students and may use some device to catch them. Given their excuse, the test question was the likeliest such device. They should have foreseen it and prepared their answer in advance. This idea that one should look ahead to future moves in the game and then reason backward to calculate one’s best current action is a very general principle of strategy, which we will elaborate on in Chapter 3. We will also use it, most notably, in Chapter 8.

But it may not be possible to foresee all such professorial countermoves; after all, professors have much more experience seeing through students’ excuses than students have making up such excuses. If the two students in the story are unprepared, can they independently produce a mutually consistent lie? If each picks a tire at random, the chances are only 25% that the two will pick the same one. (Why?) Can they do better?

You may think that the front tire on the passenger side is the one most likely to suffer a flat because a nail or a shard of glass is more likely to lie closer to that side of the road than to the middle, so the front tire on that side will encounter the nail or glass first. You may think this is good logic, but that is not enough to make your choice a good one. What matters is not the logic of the choice, but making the same choice as your friend does. Therefore, you have to think about whether your friend would use the same logic and would consider that choice equally obvious. But even that is not the end of the chain of reasoning. Would your friend think that the choice would be equally obvious to you? And so on. The point is not whether a choice is obvious or logical, but whether it is obvious to the other that it is obvious to you that it is obvious to the other . . . In other words, what is needed is a convergence of expectations about what should be chosen in such circumstances. Such a commonly expected strategy on which game players can successfully coordinate is called a focal point.

There is nothing general or intrinsic to the structure of these games that creates such convergence. In some games, a focal point may exist because of chance circumstances surrounding the labeling of strategies or because of some experience or knowledge shared by the players. For example, if the front passenger side of a car were for some reason called the Duke side, then two Duke students would be very likely to choose that tire without any need for explicit prior understanding. Or, if the front driver’s side of all cars were painted orange (for safety, to be easily visible to oncoming cars), then two Princeton students would be very likely to choose that tire, because orange is the Princeton color. But without some such clue, tacit coordination might not be possible at all.

We will study focal points in more detail in Chapter 4. Here, in closing, we merely point out that when asked to make this choice in classrooms, more than 50% of students choose the front driver’s side. They are generally unable to explain why, except to say that it seems the obvious choice.

D. Why Are Professors So Mean?

Many professors have inflexible rules against giving makeup exams and accepting late submission of problem sets or term papers. Students think these professors must be really hardhearted to behave in this way. The true strategic reason is often exactly the opposite. Most professors are kindhearted and would like to give their students every reasonable break and accept any reasonable excuse. The trouble lies in judging what is reasonable. It is hard to distinguish between similar excuses and almost impossible to verify their truth. The professor knows that on each occasion she will end up giving the student the benefit of the doubt. But the professor also knows that this is a slippery slope. As students come to know that the professor is a soft touch, they will procrastinate more and produce ever-flimsier excuses. Deadlines will cease to mean anything, and examinations will become a chaotic mix of postponements and makeup tests.

Often the only way to avoid this slippery slope is to refuse to take even the first step down it. Refusal to accept any excuses is the only realistic alternative to accepting them all. By making an advance commitment to the “no excuses” strategy, the professor avoids the temptation to give in to all.

But how can a softhearted professor maintain such a hardhearted commitment? She must find some way to make a refusal firm and credible. The simplest way is to hide behind an administrative procedure or university-wide policy. “I wish I could accept your excuse, but the university won’t let me” not only puts the professor in a nicer light, but also removes the temptation by genuinely leaving her no choice in the matter. Of course, the rules may be made by the same collectivity of professors that hides behind them, but once they are made, no individual professor can unmake the rules in any particular instance.

If the university does not provide such a general shield, then the professor can try to make up commitment devices of her own. For example, she can make a clear and firm announcement of the policy at the beginning of the course. Any time an individual student asks for an exception, the professor can then invoke a fairness principle, saying, “If I do this for you, I would have to do it for everyone.” Or the professor can acquire a reputation for toughness by acting tough a few times. This may be unpleasant and run against her true inclination, but it helps in the long run over her whole career. If a professor is believed to be tough, few students will try excuses on her, so she will actually suffer less pain in denying them.

We will study commitments, and related strategies such as threats and promises, in considerable detail in Chapter 8.

E. Roommates and Families on the Brink

You are sharing an apartment with one or more other students. You notice that the apartment is nearly out of dishwasher detergent, paper towels, cereal, beer, and other items. You have an agreement to share the actual expenses, but the trip to the store takes time. Do you spend your own time going to the store or do you hope that someone else will spend his, leaving you more time to study or relax? Do you go and buy the soap or stay in and watch TV to catch up on the soap operas?[3]

In many situations of this kind, the waiting game goes on for quite a while before someone who is really impatient for one of the items (usually beer) gives in and spends the time for the shopping trip. Things may deteriorate to the point of serious quarrels or even breakups among the roommates.

This game of strategy can be viewed from two perspectives. In one, each of the roommates is regarded as having a simple binary choice—to do the shopping or not. The best outcome for you is that someone else does the shopping and you stay at home; the worst is that you do the shopping while the others get to use their time better. If more than one roommate does the shopping (unknown to one another, on the way home from school or work), there is unnecessary duplication and perhaps some waste of perishables; if no one does the shopping, there can be serious inconvenience or even disaster if the toilet paper runs out at a crucial time.

This binary choice game is analogous to the game of chicken that used to be played by American teenagers. Two of them drove their cars toward each other. The first to swerve to avoid a collision was the loser (chicken); the one who kept driving straight was the winner. We will analyze the game of chicken further in Chapter 4 and in Chapters 7, 11, and 12.

A more interesting dynamic perspective on the same situation regards it as a war of attrition, where each roommate tries to wait out the others, hoping that someone else's patience will run out first. In the meantime, the risk escalates that the apartment will run out of something critical, leading to serious inconvenience or a blowup. Each player lets the risk escalate to the point of his own tolerance; the one revealed to have the least tolerance loses. Each sees how close to the brink of disaster the others will let the situation go. Hence the name *brinkmanship* for this strategy and this game. It is a dynamic version of chicken, offering richer and more interesting possibilities.

One of us (Dixit) was privileged to observe a brilliant example of brinkmanship at a dinner party one Saturday evening. Before dinner, the company was sitting in the living room when the host's 15-year-old daughter appeared at the door and said, "Bye, Dad." The father asked, "Where are you going?" and the daughter replied, "Out." After a pause that was only a couple of seconds but seemed much longer, the host said, "All right, bye."

Your strategic observer of this scene was left thinking how it might have gone differently. The host might have asked, "With whom?" and the daughter might have replied, "Friends." The father could have refused permission unless the daughter told him exactly where and with whom she would be. One or the other might have capitulated at some such later stage of this exchange, or it could have led to a blowup.

This was a risky game for both the father and the daughter to play. The daughter might have been punished or humiliated in front of strangers; an argument could have ruined the father's evening with his friends. Each had to judge how far to push the process, without being fully sure whether and when the other might give in or whether there would be an unpleasant scene. The risk of an explosion would increase as the father tried harder to force the daughter to answer and as she defied each successive demand.

In this respect, the game played by the father and the daughter was just like that between a union and a company’s management who are negotiating a labor contract, or between two superpowers that are encroaching on each other’s sphere of influence in the world. Neither side can be fully sure of the other’s intentions, so each side explores them through a succession of small incremental steps, each of which escalates the risk of mutual disaster. The daughter in our story was exploring previously untested limits on her freedom; the father was exploring previously untested—and perhaps unclear even to himself—limits on his authority.

This exchange was an example of brinkmanship—a game of escalating mutual risk—par excellence. Such games can end in one of two ways. In the first, one of the players reaches the limit of his own tolerance for risk and concedes. (The father in our story conceded quickly, at the very first step. Other fathers might be more successful disciplinarians, and their daughters might not even initiate a game like this.) In the second, before either player has conceded, the risk that they both fear comes about, and the blowup (the strike or the war) occurs. The feud in our host’s family ended “happily”; although the father conceded and the daughter won, a blowup would have been much worse for both.

We will analyze the strategy of brinkmanship and the dynamic version of chicken more fully in Chapter 9; in Chapter 13, we will examine a particularly important instance of it—namely, the Cuban missile crisis of 1962.

F. The Dating Game

When you go on a date, you want to show off the best attributes of your personality to your date and to conceal the worst ones. Of course, you cannot hope to conceal them forever if the relationship progresses, but you are resolved to improve or hope that by that stage the other person will accept the bad things about you with the good ones. And you know that the relationship will not progress at all unless you make a good first impression; you won’ t get a second chance to do so.

Of course, you want to find out everything, good and bad, about the other person. But you know that if the other is as good at the dating game as you are, he or she will similarly try to show the best side and hide the worst. You will think through the situation more carefully and try to figure out which signs of good qualities are real and which ones can easily be put on for the sake of making a good impression. Even the worst slob can easily appear well groomed for a big date; ingrained habits of courtesy and manners that are revealed in a hundred minor details may be harder to simulate for a whole evening. Flowers are relatively cheap; more expensive gifts may have value not for intrinsic reasons, but as credible evidence of how much the other person is willing to sacrifice for you. And the currency in which the gift is given may have differing significance depending on the context; from a millionaire, a diamond may be worth less in this regard than the act of giving up valuable time for your company or for time spent on some activity at your request.

You should also recognize that your date will similarly scrutinize your actions for their information content. Therefore, you should take actions that are credible signals of your true good qualities, not actions that anyone can imitate. This is important not only on a first date; revealing, concealing, and eliciting information about each partner’s deepest intentions remain important throughout a relationship. Here is a story to illustrate this principle:

> Once upon a time in New York City there lived a man and a woman who had separate rent-controlled apartments, but their relationship had reached the point at which they were using only one of them. The woman suggested to the man that they give up the other apartment. The man, an economist, explained to her a fundamental principle: It is always better to have more choice available. The probability of their splitting up might be small, but given even a small risk, it would be useful to retain the second low-rent apartment. The woman took this very badly and promptly ended the relationship!

Someone who hears this story might say that it just confirms the principle that greater choice is better. But strategic thinking offers a very different and more compelling explanation. The woman was not sure of the man’s commitment to the relationship, and her suggestion was a brilliant strategic device to elicit the truth. Words are cheap; anyone can say, “I love you.” If the man had put his property where his mouth was and had given up his rent-controlled apartment, this would have been concrete evidence of his love. The fact that he refused to do so constituted hard evidence of the opposite, and the woman did right to end the relationship.

This story, designed to appeal to your immediate experience, is an example of a very important class of games—namely, those where the real strategic issue is manipulation of information. Strategies that convey good information about yourself are called signals; strategies that induce others to act in ways that will credibly reveal private information about themselves, good or bad, are called screening devices.

Thus, the woman’s suggestion of giving up one of the apartments was a screening device, which put the man in the situation of either offering to give up his apartment or revealing his lack of commitment. We will study games of information, as well as signaling and screening, in Chapters 9 and 14.

Endnotes

- Evert and Navratilova’s rivalry throughout the 1970s and 1980s is arguably the greatest in women’s tennis history and, indeed, one of the greatest in sports history. We introduce an example based on their rivalry in Chapter 4 and return to it in several later chapters, most notably in Chapter 7. Return to reference 1
- There is some disagreement regarding the appropriate grammatical placement of the apostrophe in the term *prisoners’ dilemma.* Our placement acknowledges the facts that there must be at least two prisoners in order for there to be any dilemma at all and that the (at least two) prisoners therefore jointly possess the dilemma. Return to reference 2
- This example comes from Michael Grunwald’s “At Home” column, “A Game of Chicken,” in the *Boston Globe Magazine*, April 28, 1996. Return to reference 3

3 OUR STRATEGY FOR STUDYING GAMES OF STRATEGY

In Section 2, we presented several examples that relate to your experiences as an amateur strategist in real life to illustrate some basic concepts of strategic thinking and strategic games. We could continue, building a stock of dozens of similar stories. The hope would be that when you faced an actual strategic situation, you might recognize a parallel with one of these stories, which would help you decide the appropriate strategy for your own situation. This is the *case study* approach taken by most business schools. It offers a concrete and memorable vehicle for presenting the underlying concepts. However, each new strategic situation typically consists of a unique combination of so many variables that an intolerably large stock of cases would be needed to cover all of them.

An alternative approach focuses on the general principles behind the examples and so constructs a *theory* of strategic action—namely, formal game theory. The hope here would be that when you faced an actual strategic situation, you might recognize which principle or principles apply to it. This is the route taken by the more academic disciplines, such as economics and political science. A drawback to this approach is that the theory may be presented in a very abstract and mathematical manner, without enough cases or examples. This might make it difficult for most beginners to understand or remember the theory and to connect the theory with reality afterward.

But knowing some general theory has an overwhelming compensating advantage: It gives you a deeper understanding of games and of *why* they have the outcomes they do. This understanding will help you play better than you would if you merely read some cases and knew the recipes for *how* to play some specific games. With the knowledge of why, you can think through new and unexpected situations where a mechanical follower of a “how” recipe would be lost. A world champion of checkers, Tom Wiswell, has expressed this beautifully: “The player who knows how will usually draw; the player who knows why will usually win.”[4] This statement is not to be taken literally for all games; some games may be hopeless situations for one of the players no matter how knowledgeable he may be. But the statement contains the germ of an important general truth: Knowing why gives you an advantage beyond what you can get if you merely know how. For example, knowing the why of a game can help you foresee a hopeless situation and avoid getting into such a game in the first place.

In this book, we take an intermediate route that combines some of the advantages of both approaches—case studies (how) and theory (why). Chapters 3–7 (Part Two) are organized around the general principles of game theory, but we develop these principles through illustrative cases rather than abstractly. That way, the context and scope of each idea will be clearer and more evident. In other words, we will focus on theory, but build it up through cases. Starting with Chapter 8, we will then apply the theory to several types of strategic situations.

Of course, such an approach requires some compromises of its own. Most importantly, you should remember that each of our examples serves the purpose of conveying some general idea or principle of game theory. Therefore, we will leave out many details of each case that are incidental to the principle at stake. If some examples seem somewhat artificial, please bear with us; we have generally considered the omitted details and left them out for good reasons.

A word of reassurance: Although the examples that motivate the development of our conceptual or theoretical frameworks are deliberately selected for that purpose (even at the cost of leaving out some other features of reality), once the theory has been constructed, we will pay a lot of attention to its connection with reality. Throughout the book, we will examine factual and experimental evidence concerning how well the theory explains reality. The frequent answer—very well in some respects and less well in others—should give you cautious confidence in using the theory and should be a spur to the formulation of better theories. In appropriate places, we will examine in great detail how institutions evolve in practice to solve problems highlighted by the theory. For example, Chapter 10 explores how prisoners' dilemmas arise and are solved in reality, Chapter 11 provides a similar discussion of more general collective-action problems, Chapter 13 examines the use of brinkmanship in the Cuban missile crisis, and Chapter 15 discusses how to design an auction and avoid the winner's curse.

To pursue our approach, in which examples lead to general theories that are then tested against reality and used to interpret reality, we must first identify the general principles that serve to organize the discussion. We will do so in Chapter 2 by classifying games along several key dimensions, such as whether players have aligned or conflicting interests, whether one player acts first (sequential games) or all players act at once (simultaneous games), whether all players have the same information about the game, and so on. Once this general framework has been constructed in Chapter 2, the chapters that follow will build on it, systematically developing ideas and principles that you can deploy when analyzing each player's strategic choice and the interaction of all players' strategies in games.

Endnotes

- Quoted in Victor Niederhoffer, *The Education of a Speculator* (New York: Wiley, 1997), p. 169. We thank Austin Jaffe of Pennsylvania State University for bringing this aphorism to our attention. Return to reference 4

2 ■ How to Think about Strategic Games

CHAPTER 1 GAVE SOME simple examples of strategic games and strategic thinking. In this chapter, we begin a more systematic and analytical approach to the subject. We choose some crucial conceptual categories or dimensions of strategic games, each of which has a dichotomy of types of strategic interactions. For example, one such dimension concerns the timing of the players' actions, and the two pure types are games where the players act in strict turns (sequential moves) and where they act at the same time (simultaneous moves). We consider some matters that arise in thinking about each pure type in this dichotomy as well as in similar dichotomies, such as whether the game is played only once or repeatedly and what the players know about one another.

In Chapters 3–7, we will examine each of these categories or dimensions in more detail; in Chapters 8–17, we will show how game-theoretic analysis can be used in several contexts. Of course, most actual situations to which these analyses are applied are not purely of just one type, but are rather a mixture. Moreover, in each application, two or more of the categories have some relevance. The lessons learned from the study of the pure types must therefore be combined in appropriate ways. We will show how to do this by using the context of our applications.

In this chapter, we present some basic concepts and terms—such as strategies, payoffs, and equilibrium—that are used in game-theoretic analysis, and we briefly describe methods of solving a game. We also provide a brief discussion of the uses of game theory and an overview of the structure of the remainder of the book.

1 STRATEGIC GAMES

The very word *game* conjures up many meanings or connotations. The two that are most common, but misleading in the present context, are that a game involves two players or teams, and that one of them wins and the other loses. Our study of games of strategy does include two-player, win-lose games, but covers much more. Most of you have probably played “massively multiplayer online games,” where hundreds or thousands of players interact simultaneously on the same server. Or you have done battle against your future self when avoiding procrastination or overeating. And you constantly hear or read pundits and analysts who discuss whether international trade can be a win-win game or assert that nuclear war is only a lose-lose game. These other types of games are also within the purview of game theory, as are a broad range of interactions you have on a regular basis.

Our broad concept of a game of strategy (we will usually just say game, because we are not concerned with games of pure skill or pure chance) applies to any situation in which strategies of the participants (players) interact, so that the outcome for each participant depends on the choices of all. In this very broad sense, it is difficult to think of a decision that gets made by any person or team that is not part of a game. You may think that your decision to brush your teeth in the morning is purely personal. But if you are a waiter, it will influence your customers’ satisfaction with the service, and that will affect your tips. Even in your private life, it may have a bearing on your close personal relationships![1] Indeed, strategic interactions are everywhere, but their nature depends on how many players are involved and how those players interact.

Consider a market for bread with thousands of customers and bakers. Each customer decides whether to buy bread, and how much to buy, at the going price; each baker decides how much bread to make. Their demand and supply decisions collectively determine the market price of bread. Thus, there is a feedback loop from all of the customers' and bakers' choices (strategies) to the price, and from the price back to their choices. But each individual has only a tiny effect on the price. Therefore, no one needs to be concerned with whether or how much any other specific individual is buying or selling. Their strategies respond to, and collectively feed back on, only a summary or aggregate measure—namely, the market price.[2] In most multiplayer games, there will be similar summary measures that channel strategic interactions. For example, each driver choosing from alternative routes to work responds to the expected travel time on each route, and their collective choices determine those travel times.

Contrast this situation with a market that has only a few sellers, such as the one for mobile phones. Each firm has to be concerned with what choices each of the other few sellers is making and plan its own strategy in response. Samsung looks at the specific choices of features, prices, and advertising being made by Apple, Nokia, and the others when making its own choices. It does not merely respond to broad market-wide statistics on phone attributes.

Game theory encompasses multiplayer games as well as small-numbers games, but the applicable concepts, the nature of strategies, and the ranges of outcomes can differ across these types of games—as does the usefulness of game theory itself. In games with few players, where each must think about the others' thinking, there is much more scope for manipulating information and actions in the game. In such situations, game theory often yields useful advice on how to play or change a game to your advantage; see, for instance, our analyses of the timing of moves in Chapter 6, strategic moves in Chapter 8, signaling and screening in Chapter 9, repeated games in Chapter 10, and incentive design in Chapter 14.

In multiplayer games, where each player’s decision has a negligible effect on others and the optimal strategy for each player merely involves responding to an overall variable such as a price, there is much less scope for advice or advantage. Even so, game theory can be useful in such games for motivating and coordinating collective action (as we will show in Chapter 11) as well as for generating insights that can help outside observers understand the interactions and guide policy efforts aimed at achieving better outcomes for society as a whole. As such, game theory can be especially useful to leaders and policymakers looking to change a multiplayer game. For example, game theory helps us identify the nature of the game (prisoners’ dilemma) that leads to excessive congestion on roads, pollution of air and water, and depletion of fisheries, and thereby helps society design institutions that can ameliorate these problems.

Sometimes multiplayer and small-numbers games are interwoven. For instance, the game of finding a romantic partner often proceeds in two stages, initially with many players all looking to meet others and then later in a relationship with just one other person. When formulating one’s strategy in the initial multiplayer phase, it pays to think ahead about the two-player game that will unfold with one’s chosen partner. A similar set of considerations arises when building a house. During the earliest planning stage, a customer can choose any of several dozen local contractors, and each contractor can similarly choose from multiple potential customers. Once a customer and contractor choose to work together, however, the customer pays an initial installment and the builder buys materials that can be used only on that customer’s house. The two become tied to each other, separately from the broader housing market—creating a small-numbers game between *one* customer and *one* contractor.

Endnotes

- Alan S. Blinder, "The Economics of Brushing Teeth," *Journal of Political Economy*, vol. 82, no. 4 (July–August 1974), pp. 887–91. Return to reference 1
- Some multiplayer games may have players of very unequal sizes. For example, in the game among nations to deal with climate change, emissions from China and the United States have a substantial direct effect on smaller nations. When formulating their climate-change strategies, China and the United States should therefore bear in mind how smaller nations will react (collectively) to their individual moves. Return to reference 2

Glossary

game (game of strategy)
: An action situation where there are two or more mutually aware players, and the outcome for each depends on the actions of all.

game (game of strategy)
: An action situation where there are two or more mutually aware players, and the outcome for each depends on the actions of all.

2 CLASSIFYING GAMES

Games of strategy arise in many different contexts and cannot all be analyzed in the same way. That said, some games that seem quite different on the surface share underlying features that allow them to be studied and understood very similarly from a game-theoretic point of view. When faced with a strategic situation, how can you tell what type of game it is and how it should be analyzed? In this section, we show how games can be (roughly but usefully) divided into a relatively small number of categories on the basis of the answers to a few driving questions discussed in the subsections that follow. As Stephen Jay Gould once wrote,[3] "Dichotomies are useful or misleading, not true or false. They are simplifying models for organizing thought, not ways of the world. . . . They do not blend, but dwell together in tension and fruitful interaction." In the same spirit, we present the categorization of games here as a useful framework for organizing one's thinking about games and for sparking creative connections.

A. Are Players' Interests in Total Alignment, Total Conflict, or a Mix of Both?

Perhaps the most fundamental question to ask about a game is whether the players have aligned or conflicting individual interests. Players on the same sports team share a broad goal, to win, but also play strategic games with one another during the contest. For example, during a football passing play, the quarterback decides where to throw the ball and the receiver decides where to run. This interaction constitutes a game because each player's choice affects the success of the play, but it is a game in which both players want the same thing, to complete the pass. Players with perfectly aligned interests in such games can get their best outcome relatively easily by communicating and coordinating their moves—for instance, by the quarterback telling the receiver where to run during the huddle before the play.

Competing sports teams can also be viewed as players in games, albeit now with totally conflicting interests, as each team wants to win while the other loses. Similarly, in gambling games, one player's winnings are the others' losses, with the sum of their earnings equal to 0. This is why such situations are called zero-sum games. More generally, players' interests will be in complete conflict whenever they are dividing up any fixed amount of total benefit, whether it be measured in yards, dollars, acres, or scoops of ice cream. Because the available gain need not always be exactly zero, the term constant-sum game is often substituted for *zero-sum game*; we will use the two terms interchangeably.

Most economic and social games are not zero-sum.[4] Trade, or economic activity more generally, offers scope for deals that benefit everyone. Joint ventures can combine the participants' different skills and generate synergy to produce more than the sum of what they could have produced separately. However, the partners' interests are typically not completely aligned either. They can cooperate to create a larger total pie, but they will tend to clash when it comes to deciding how to split this pie among themselves.

Even wars and strikes are not zero-sum games. A nuclear war is the most striking example of a situation where there can only be losers, but the concept is far older. Pyrrhus, the king of Epirus, defeated the Romans at Heraclea in 280 B.C., but at such great cost to his own army that he exclaimed, "Another such victory and we are lost!" Hence the phrase "Pyrrhic victory." In the 1980s, at the height of the frenzy of business takeovers, the battles among rival bidders led to such costly escalation that the successful bidder's victory was often similarly Pyrrhic.

Most games in reality have this tension between conflict and cooperation, and many of the most interesting analyses in game theory come from the need to handle it. The players' attempts to resolve their conflict—for example, over the distribution of territory or profit—are influenced by the knowledge that if they fail to agree, the outcome will be bad for all of them. One side's threat of a war or a strike is its attempt to frighten the other side into conceding to its demands.

B. Are the Moves in the Game Sequential or Simultaneous?

Moves in chess are sequential: White moves first, then Black, then White again, and so on. In contrast, participants in an auction for an oil-drilling lease or a part of the airwave spectrum make their bids simultaneously, without knowing competitors' bids. Most actual games combine aspects of both types of moves. In a race to research and develop a new product, the firms act simultaneously, but each competitor has partial information about the others' progress and can respond. During one play in football, the opposing offensive and defensive coaches simultaneously send out teams with the expectation of carrying out certain plays, but after seeing how the defense has set up, the quarterback can change the play at the line of scrimmage or call a time-out so that the coach can change the play.

The distinction between sequential moves and simultaneous moves is important because sequential- and simultaneous-move games require different sorts of strategic thinking. In a sequential-move game, each player must think: "If I do this, how will my opponent react?" Your current move is governed by your calculation of its *future* consequences. With simultaneous moves, you have the trickier task of trying to figure out what your opponent is going to do *right now*. But you must recognize that, in making his own calculation, your opponent is also trying to figure out your current move, while at the same time recognizing that you are doing the same with him . . . Both of you have to think your way out of this logical circle.

In Chapter 3, we will examine sequential-move games, where you must look ahead to act now. In Chapters 4 and 5, we will move on to simultaneous-move games, where you must square the circle of “He thinks that I think that he thinks . . .” In each case, we will devise some simple tools for thinking through each type of game—trees for sequential-move games and payoff tables for simultaneous-move games—and derive some simple rules to guide our actions.

Studying sequential-move games will also shed light on when it is advantageous to move first or to move second in a game, a topic we will consider in Chapter 6. Roughly speaking, which order a player prefers depends on the relative importance of commitment and flexibility in the game in question. For example, a game of economic competition among rival firms in a market has a first-mover advantage if one firm, by making a commitment to compete aggressively, can get its rivals to back off. On the other hand, in a political competition, a candidate who has taken a firm stand on an issue may give her rivals a clear focus for their attack ads, so that game has a second-mover advantage. Understanding these considerations can help you devise ways to manipulate the order of moves to your own advantage. That in turn leads to the study of strategic moves such as threats and promises, which we will take up in Chapter 8.

C. Are the Rules of the Game Fixed or Manipulable?

The rules of chess, card games, or sports are given, and every player must follow them, no matter how arbitrary or strange they seem. But in games of business, politics, and ordinary life, the players can make their own rules to a greater or lesser extent. For example, in the home, parents constantly try to make the rules, and children constantly look for ways to manipulate or circumvent those rules. In legislatures, rules for the progress of a bill (including the order in which amendments and main motions are voted on) are fixed, but the game that sets the agenda—which amendments are brought to a vote first—can be manipulated. This is where political skill and power have the most scope. We will address these matters in detail in Chapter 16.

In such situations, the real game is the "pregame" where rules are made, and your strategic skill must be deployed at that point. The actual playing out of the subsequent game can be more mechanical; you could even delegate it to someone else. However, if you snooze through the pregame, you might find that you have lost the game before it ever began. For example, American firms ignored the rise of foreign competition in just this way for many years and ultimately paid the price. But some entrepreneurs, such as oil magnate John D. Rockefeller Sr., adopted the strategy of limiting their play to games in which they could also participate in making the rules.[5]

A recurring theme throughout this book is the distinction between how best to *play* a given game and how best to *change the rules* of the game being played. In Chapter 8, we will consider strategic moves (such as promises and threats) whereby one player seeks to shape the outcome of the game by committing to how he will play the game and/or by changing the timing of moves. Similarly, in Chapters 14 and 15, we will consider how one player (“the principal” in Chapter 14 or “the auctioneer” in Chapter 15) can shape the options and/or incentives of other players (“the agent” or “the bidders”) to get a better outcome for himself.

In each case, the rules of the game are determined in a pregame. But what determines the rules of the pregame? In some cases, only one player has the opportunity to set the rules. For instance, an auctioneer selling an item gets to decide what sort of auction to conduct, while in an employment relationship, the boss gets to decide what sort of incentive contract to use to motivate an employee. In other cases, the nature of the pregame may depend on players’ innate abilities. In business competition, one firm can take preemptive actions that alter subsequent games between it and its rivals—for instance, by expanding a factory or by advertising in a way that affects subsequent price competition. Which firm can do this best or most easily depends on which has the managerial or organizational resources to make the necessary moves, and on which has sufficient understanding of game theory to recognize and seize the opportunity to do so.

Players may also be unsure of their rivals’ abilities. This important nuance often makes the pregame one of asymmetric information (see Section 2.D), requiring more subtle strategies and occasionally resulting in big surprises. We will comment on these matters in Chapter 9 and elsewhere in the chapters that follow.

D. Do the Players Have Full or Equal Information?

In chess, each player knows exactly what the current situation is and all the moves that led to it, and each knows that the other aims to win. This situation is exceptional; in most other games, the players face some limitations on information. Such limitations come in two kinds. First, a player may not know all the information that is pertinent to the choices that he has to make at every point in the game. This type of information problem arises because of the player’s uncertainty about relevant variables, both internal and external to the game. For example, he may be uncertain about external circumstances, such as the weekend weather or the quality of a product he wishes to purchase; we call this situation one of external uncertainty. Or he may be uncertain about exactly what moves his opponent has made in the past or is making at the same time he makes his own move; we call this situation one of strategic uncertainty. If a game has neither external nor strategic uncertainty, we call it a game with perfect information; otherwise, the game has imperfect information. We will give a more precise technical definition of perfect information in Chapter 6, Section 3.A, after we have introduced the concept of an information set. We will develop the theory of games with imperfect information (uncertainty) in three future chapters. In Chapter 4, we will discuss games with contemporaneous (simultaneous) actions, which entail strategic uncertainty, and we will analyze methods for making choices under external uncertainty in Chapter 9 and its appendix.

Trickier strategic situations arise when one player knows more than another does, what is referred to as asymmetric information. In such situations, the players’ attempts to infer, conceal, or sometimes convey their private information become an important part of the game and the strategies. In bridge or poker, each player has only partial knowledge of the cards held by the others. The actions of the players (bidding and play in bridge, the number of cards taken and the betting behavior in poker) give information to their opponents. Each player tries to manipulate her actions to mislead the opponents (and, in bridge, to inform her partner truthfully), but in doing so, she must be aware that the opponents know this and that they will use strategic thinking to interpret her actions.

You may think that if you have superior information, you should always conceal it from other players. But that is not true. For example, suppose you are the CEO of a pharmaceutical firm that is engaged in an R&D competition to develop a new drug. If your scientists make a discovery that is a big step forward, you may want to let your competitors know that, in the hope that they will give up their own searches and you won’t face any future competition. In war, each side wants to keep its tactics and troop deployments secret, but in diplomacy, if your intentions are peaceful, then you desperately want other countries to know and believe that fact.

The general principle here is that you want to release your information selectively. You want to reveal the good information (the kind that will draw responses from the other players that work to your advantage) and conceal the bad (the kind that may work to your disadvantage).

This principle raises a problem. Your opponents in a strategic game are purposive and rational, and they know that you are, too. They will recognize your incentive to exaggerate or even to lie. Therefore, they are not going to accept your unsupported declarations about your progress or capabilities. They can be convinced only by objective evidence or by actions that are credible proof of your information. Such actions on the part of the more informed player are called signals, and strategies that use them are called signaling. Conversely, the less informed party can create situations in which the more informed player will have to take some action that credibly reveals his information; such strategies are called screening, and the methods they use are called screening devices. The word *screening* is used here in the sense of testing in order to sift or separate, not in the sense of concealing.

Sometimes the same action may be used as a signal by the informed player or as a screening device by the uninformed player. Recall that in the dating game in Section 2.F of Chapter 1, the woman was screening the man to test his commitment to their relationship, and her suggestion that the pair give up one of their two rent-controlled apartments was the screening device. If the man had been committed to the relationship, he might have acted first and volunteered to give up his apartment; this action would have been a signal of his commitment.

Now we see how, when different players have different information, the manipulation of information itself becomes a game, and often a vitally important one. Such information games are ubiquitous, and playing them well is essential for success in life. We will study more games of this kind in greater detail in Chapter 9 (games with signaling and/or screening), Chapter 14 (incentive design), and Chapter 15 (auctions).

E. Is the Game Played Once or Repeatedly, and with the Same or Changing Opponents?

A game played just once is in some respects simpler and in others more complicated than one that includes many interactions. You can think about a one-shot game without worrying about its repercussions on other games you might play in the future against the same person or against others who might hear of your actions in this one. Therefore, actions in one-shot games are more likely to be unscrupulous or ruthless. For example, an automobile repair shop is much more likely to overcharge a passing motorist than a regular customer.

In one-shot encounters, no player knows much about the others—for example, what their capabilities and priorities are, whether they are good at calculating their best strategies, or whether they have any weaknesses that can be exploited. Therefore, in one-shot games, secrecy or surprise is likely to be an important component of good strategy.

Games with ongoing relationships require the opposite considerations. You have an opportunity to build a reputation (for example, for toughness, fairness, honesty, or reliability, depending on the circumstances) and to find out more about your opponent. The players together can better exploit mutually beneficial prospects by arranging to divide the spoils over time (taking turns to "win") or to punish a cheater in future plays (an eye for an eye, or tit-for-tat). We will consider these possibilities in Chapter 10 on the prisoners' dilemma.

More generally, a game may be zero-sum in the short run but have scope for mutual benefit in the long run. For example, each football team likes to win, but they all recognize that close competition generates more spectator interest, which benefits all teams in the long run. That is why they agree to a drafting scheme where teams get to pick players in reverse order of their current standing, thereby reducing the inequality of talent. In long-distance races, runners or cyclists often develop a lot of cooperation; two or more of them can help one another by taking turns following in one another's slipstream. But near the end of the race, the cooperation collapses as all of them dash for the finish line.

Here is a useful rule of thumb for your own strategic actions in life: In a game that has some conflict and some scope for cooperation, you will often think up a great strategy for winning big and grinding a rival into the dust but have a nagging worry at the back of your mind that you are behaving badly. In such a situation, the chances are that the game has a repeated or ongoing aspect that you have overlooked. Your aggressive strategy may gain you a short-run advantage, but the cost of its long-run side effects outweigh your gains. Therefore, you should dig deeper, discover the cooperative element, and alter your strategy accordingly. You will be surprised how often niceness, integrity, and the golden rule of doing to others as you would have them do to you turn out to be not just old nostrums, but good strategies as well when you consider the whole complex of games that you will be playing in the course of your life.[6]

F. Are Agreements to Cooperate Enforceable?

We have argued that most strategic interactions consist of a mixture of conflict and common interest. Under these circumstances, there is a case to be made that all participants should get together and reach an agreement about what everyone should do, balancing their mutual interest in maximizing the total benefit with their conflicting interests in the division of gains. Such negotiations can take several rounds in which agreements are made on a tentative basis, better alternatives are explored, and the deal is finalized only when no group of players can find anything better. However, even after the completion of such a process, additional difficulties often arise in putting the final agreement into practice. For instance, all the players must perform, in the end, the actions that were stipulated for them in the agreement. When all others do what they are supposed to do, any one participant can typically get a better outcome for herself by doing something different. And, if any participant suspects that the others may cheat in this way, she would be foolish to adhere to her stipulated cooperative action.

Agreements to cooperate can succeed if all players act immediately and in the presence of the whole group, but agreements with such immediate implementation are quite rare. More often the participants disperse after the agreement has been reached and then take their actions in private. Still, if these actions are observable to the others, and a third party—for example, a court of law—can enforce compliance, then the agreement of joint action can prevail.

However, in many other instances, individual actions are neither directly observable nor enforceable by external forces. Without enforceability, agreements will stand only if it is in all participants' individual interests to abide by them. Games among sovereign countries are of this kind, as are many games with private information or games where the players' actions cannot be proven to the standard of evidence required by courts of law. In fact, games where agreements for joint action are not enforceable constitute a vast majority of strategic interactions.

Game theory uses special terminology to capture the distinction between situations in which agreements are enforceable and those in which they are not. Games in which joint-action agreements are enforceable are called cooperative games; those in which such enforcement is not possible, and individual participants must be allowed to act in their own interests, are called noncooperative games. This terminology has become standard, but it is somewhat unfortunate because it gives the impression that the former will produce cooperative outcomes and the latter will not. In fact, individual actions can be compatible with the achievement of a lot of mutual gain, especially in repeated interactions. The important distinction is that in so-called noncooperative games, cooperation will emerge only if it is in the participants' separate and individual interests to continue to take the prescribed actions. This emergence of cooperative outcomes from noncooperative behavior is one of the most interesting findings of game theory, which we will develop in Chapters 10, 11, and 12.

We will adhere to the standard terminology in this book, but emphasize that the terms *cooperative* and *noncooperative* refer to the way in which actions are implemented or enforced—collectively in the former mode and individually in the latter—and not to the nature of the outcomes. In practice, as we said earlier, most games do not have adequate mechanisms for external enforcement of joint-action agreements. Bearing that in mind, most of our analytical discussion will deal with noncooperative games. The one exception comes in our discussion of bargaining in Chapter 17.

Endnotes

- Stephen Jay Gould, *Time's Arrow, Time's Cycle: Myth and Metaphor in the Discovery of Geologic Time* (Cambridge, Mass.: Harvard University Press, 1987), pp. 199–200. Return to reference 3
- Even when a game is constant-sum for all players, if the game has three (or more) players, two of them may cooperate at the expense of the third. Such situations lead to the study of alliances and coalitions, topics we explore in Chapter 17, on bargaining. Return to reference 4
- For more on Rockefeller's rise to power, see Ron Chernow, *Titan* (New York: Random House, 1998). Return to reference 5
- For more on the strategic advantages of being honest and kind, see David McAdams, "Game Theory and Cooperation: How Putting Others First Can Help Everyone," *Frontiers for Young Minds: Mathematics*, vol. 5, no. 66 (December, 2017), available at https://kids.frontiersin.org/article/10.3389/frym.2017.00066. Return to reference 6

Glossary

zero-sum game
: A game where the sum of the payoffs of all players equals zero for every configuration of their strategy choices. (This is a special case of a *constant-sum game*, but in practice no different because adding a constant to all the payoff numbers of any one player makes no difference to his choices.)

constant-sum game
: A game in which the sum of all players' payoffs is a constant, the same for all their strategy combinations. Thus, there is a strict conflict of interests among the players—a higher payoff to one must mean a lower payoff to the collectivity of all the other players. If the payoff scales can be adjusted to make this constant equal to zero, then we have a *zero-sum game*.

sequential moves
: The moves in a game are sequential if the rules of the game specify a strict order such that at each action node only one player takes an action, with knowledge of the actions taken (by others or himself) at previous nodes.

simultaneous moves
: The moves in a game are simultaneous if each player must take his action without knowledge of the choices of others.

external uncertainty
: A player's uncertainty about external circumstances such as the weather or product quality.

strategic uncertainty
: A player's uncertainty about an opponent's moves made in the past or made at the same time as her own.

perfect information
: A game is said to have perfect information if players face neither strategic nor external uncertainty.

imperfect information
: A game is said to have perfect information if each player, at each point where it is his turn to act, knows the full history of the game up to that point, including the results of any random actions taken by nature or previous actions of other players in the game, including pure actions as well as the actual outcomes of any mixed strategies they may play. Otherwise, the game is said to have imperfect information.

asymmetric information
: Information is said to be asymmetric in a game if some aspects of it—rules about what actions are permitted and the order of moves if any, payoffs as functions of the players strategies, outcomes of random choices by "nature," and of previous actions by the actual players in the game—are known to some of the players but are not common knowledge among all players.

signals
: Devices used for signaling.

signaling
: Strategy of a more-informed player to convey his "good" information credibly to a less-informed player.

screening
: Strategy of a less-informed player to elicit information credibly from a more-informed player.

screening devices
: Methods used for screening.

cooperative game
: A game in which the players decide and implement their strategy choices jointly, or where joint-action agreements are directly and collectively enforced.

noncooperative game
: A game where each player chooses and implements his action individually, without any joint-action agreements directly enforced by other players.

3 SOME TERMINOLOGY AND BACKGROUND ASSUMPTIONS

When one thinks about a strategic game, the logical place to begin is by specifying its structure. A game’s structure includes all the strategies available to all the players, their information, and their objectives. The first two aspects will differ from one game to another along the dimensions discussed in the preceding section, and one must locate one’s particular game within that framework. The objectives raise some new and interesting considerations. Here we consider all of these aspects of game structure.

A. Strategies

Strategies are simply the choices available to the players, but even this basic notion requires some further study and elaboration. If a game has purely simultaneous moves made only once, then each player's strategy is simply the action taken on that single occasion. But if a game has sequential moves, then a player who moves later in the game can respond to what other players have done (or what he himself has done) at earlier points. Therefore, each such player must make a complete plan of action—for example, "If the other does A, then I will do X, but if the other does B, then I will do Y." This complete plan of action constitutes the strategy in such a game.

A very simple test determines whether your strategy is complete: Does it specify such full detail about how you would play the game—describing your action in every contingency—that if you were to write it all down, hand it to someone else, and go on vacation, this other person, acting as your representative, could play the game just as you would have played it? Would that person know what to do on each occasion that could conceivably arise in the course of play without ever needing to disturb your vacation for instructions on how to deal with some situation that you had not foreseen? The use of this test will become clearer in Chapter 3, when we will develop and apply it in some specific contexts. For now, you should simply remember that a strategy is a complete plan of action.

This notion is similar to the common usage of the word *strategy* to denote a longer-term or larger-scale plan of action as distinct from tactics, which pertain to a shorter term or a smaller scale. For example, generals in the military make strategic plans for a war or a large-scale battle, while lower-level officers devise tactics for a smaller skirmish or a particular theater of battle based on local conditions. But game theory does not use the term *tactics* at all. The term *strategy* covers all situations, meaning a complete plan of action when necessary and meaning a single move if that is all that is needed in the particular game being studied.

The word *strategy* is also commonly used to describe a person’s decisions over a fairly long time span and sequence of choices, even when there is no game in our sense of purposive interaction with other people. Thus, you have probably already chosen a career strategy. When you start earning an income, you will make saving and investment strategies and eventually plan a retirement strategy. This usage of the term *strategy* has the same sense as ours: a plan for a succession of actions in response to evolving circumstances. The only difference is that we are reserving it for a situation—namely, a game—where the circumstances evolve because of actions taken by other purposive players.

B. Payoffs

When asked what a player's objective in a game is, most newcomers to strategic thinking respond that it is "to win," but matters are not always so simple. Sometimes the margin of victory matters; for example, in R&D competition, if your product is only slightly better than your nearest rival's, your patent may be more open to challenge. Sometimes there may be smaller prizes for several participants, so winning isn't everything. Most importantly, very few games of strategy are purely zero-sum or win-lose; most games combine some common interests and some conflict among the players. Thinking about such mixed-motive games requires more refined calculations than the simple dichotomy of winning and losing—for example, comparisons of the gains from cooperating versus cheating.

To enable such comparisons, we will assume that each player is able to assign a *number* to each logically conceivable outcome of the game, corresponding to each logically possible combination of choices made by all the players. The number that a player assigns to a given outcome is that player's payoff for that outcome. Higher payoff numbers are given to outcomes that better achieve that player's objectives.

Sometimes a player's payoffs will be represented as a simple ranking of the outcomes, the worst labeled 1, the next worst 2, and so on, all the way to the best. In other games, there may be other more natural numerical scales—for example, money income or profit for firms, viewer-share ratings for TV networks, and so on. In still other situations, the payoff numbers that we use are only educated guesses. In such cases, we need to make sure that the results of our analysis do not change significantly if we vary these guesses within some reasonable margin of error.

Two important points about payoffs need to be understood clearly. First, a player's payoffs in a game capture everything that the player cares about vis-à-vis that game. In particular, players need not be selfish; their concern about others can be included in their payoffs. Second, we will assume that, if the player faces a random prospect of outcomes, then the number associated with this prospect is the average of the payoffs associated with each potential outcome, each weighted by its probability. Thus, if in one player's ranking, outcome A has payoff 0 and outcome B has payoff 100, then the prospect of a 75% probability of A and a 25% probability of B should have the payoff $0.75 \times 0 + 0.25 \times 100 = 25$. This number is often called the expected payoff from the random prospect. The word *expected* has a special connotation in the jargon of probability theory. It does not mean what you think you will get or expect to get; it is the mathematical or probabilistic or statistical expectation, meaning an average of all possible outcomes, where each is given a weight proportional to its probability.

The second point creates a potential difficulty. Consider a game where players get or lose money and where payoffs are measured simply in money amounts. In reference to the preceding example, if a player has a 75% chance of getting nothing and a 25% chance of getting $100, then the expected payoff, as calculated in that example, is $25. That is also the payoff that the player would get from a simple nonrandom outcome of $25. In other words, in this way of calculating payoffs, a person should be indifferent to whether she receives $25 for sure or faces a risky prospect of which the average amount is $25. But one would think that most people would be averse to risk, preferring a sure $25 to a gamble that yields only $25 on the average.

Aversion to risk can be incorporated into the theory in several ways. One is to measure payoffs not in money terms, but with a nonlinear rescaling of dollar amounts. This approach, called expected utility theory, is widely used, although it is not without its own difficulties; it is also mathematically complex. Therefore, in the text, we adopt a simpler approach that has some support from recent behavioral research: We capture aversion to risk by attaching a higher weight to losses than to gains, both measured from some average or status quo level. We will develop this idea in Chapter 9 and use it later in Chapter 14.

C. Rationality

Each player's aim in a game is to achieve as high a payoff for himself as possible. But how good is each player at pursuing this aim? This question is not about whether and how other players pursuing their own interests will impede him; that is in the very nature of a game of strategic interaction. Rather, achieving a high payoff depends on how good each player is at calculating the strategy that is in his own best interests and at following this strategy in the actual course of play.

Traditional game theory assumes that players are perfect calculators and flawless followers of their best strategies. In other words, it assumes that players will exhibit rational behavior. Observe the precise sense in which the term *rational* is being used here. It means that each player has a consistent payoff ranking of all the logically possible outcomes of the game and calculates the strategy that best serves his interests. Thus, rationality has two essential ingredients: complete knowledge of one's own interests, and flawless calculation of what actions will best serve those interests.

It is equally important to understand what is *not* included in this concept of rational behavior. It does not mean that players are selfish; a player may rank the well-being of some other player(s) highly and incorporate this high ranking into his payoffs. It does not mean that players are short-term thinkers; in fact, calculation of future consequences is an important part of strategic thinking, and actions that seem irrational from an immediate perspective may have valuable long-term strategic roles. Most importantly, being rational does not mean having the same value system that other players, or sensible people, or ethical or moral people would use; it means merely pursuing one’s own value system consistently. Therefore, when one player carries out an analysis of how other players will respond (in a game with sequential moves) or of successive rounds of thinking about thinking (in a game with simultaneous moves), he must recognize that the other players calculate the consequences of their choices by using their own value or ranking system. You must not impute your own value system or standards of rationality to others and assume that they would act as you would in the same situation. Thus, many “experts” commenting on the Persian Gulf conflict in late 1990 and again in 2002–2003 predicted that Saddam Hussein would back down “because he is rational”; they failed to recognize that Saddam’s value system was different from the one held by most Western governments and Western experts.

In most games, no player really knows the other players’ value systems; this is part of the reason that in reality, many games have asymmetric information. In such games, trying to find out the values of others and trying to conceal or convey one’s own values become important components of strategy.

Although the assumption of rational behavior remains the basis for much of game theory and for the majority of our exposition, significant departures from it have been observed and built into modern theories. Research in psychology and behavioral economics has found that people often take actions based on instincts, fixed rules, or heuristics. We will incorporate these considerations throughout this book.

Even for players who have clear preferences and want to pursue optimal strategies, calculation of the optimal strategy is often far from easy. Most games in real life are very complex, and most real players are limited in their thinking and computational abilities. In games such as chess, it is known that the calculation of the best strategy can be performed in a finite number of steps, but that number is mind-bogglingly large. Only in the last few years has the advent of artificial intelligence and neural networks in computing enabled such calculations. We discuss advances in games like chess further in Chapter 3.

The assumption of rationality may be closer to reality when the players are regulars who play the game often. In this case, they benefit from having experienced the different possible outcomes. They understand how the strategic choices of various players lead to the outcomes and how well or badly they themselves fare. Thus we can hope that their choices, even if not made with full and conscious computations, closely approximate the results of such computations. We can think of the players as implicitly choosing the optimal strategy or behaving as if they were perfect calculators. We will offer some experimental evidence in Chapter 5 that the experience of playing the game generates more rational behavior.

The advantage of making a complete calculation of your best strategy, taking into account the corresponding calculations of a similar strategically calculating rival, is that you are not making mistakes that the rival can exploit. In many actual situations, you may have specific knowledge of the ways in which the other players fall short of this standard of rationality, and you can exploit this information in devising your own strategy. We will say something about such calculations, but very often they are a part of the "art" of game playing, not easily codifiable in rules to be followed. You must always beware of the danger that the other players are merely pretending to have poor skills or strategy, losing small sums through bad play and hoping that you will then raise the stakes, at which time they can raise the level of their play and exploit your gullibility. When this risk is real, the safest advice to a player may be to assume that the rivals are perfect and rational calculators and to choose your own best response to their actions. In other words, you should play to your opponents’ capabilities instead of their limitations.

D. Common Knowledge of Rules

We assume that, at some level, the players have a common understanding of the rules of the game. In a *Peanuts* cartoon, Lucy thought that body checking was allowed in golf and decked Charlie Brown just as he was about to take his swing. We do not allow this.

The qualification “at some level” is important. We saw how the rules of the immediate game could be manipulated. But this merely admits that there is another game being played at a deeper level—namely, the pregame where the players choose the rules of the superficial game. Then the question is whether the rules of this deeper game are fixed. For example, in the legislative context, what are the rules of the agenda-setting game? They may be that the committee chairs have the power. Then how are the committees and their chairs elected? And so on. At some basic level, the rules are fixed by a nation’s constitution, by the technology of campaigning, or by general social norms of behavior. We ask that all players recognize the given rules of this basic game, and that is the focus of the analysis. Of course, that is an ideal; in practice, we may not be able to proceed to a deep enough level of analysis.

Strictly speaking, the rules of a game consist of (1) the list of players, (2) the strategies available to each player, (3) the payoffs to each player for all possible combinations of strategies pursued by all the players, and (4) the assumption that each player is a rational maximizer.

Game theory cannot properly analyze a situation where one player does not know whether another player is participating in the game, what the entire sets of actions available to the other players are from which they can choose, what their value systems are, or whether they are conscious maximizers of their own payoffs. But in actual strategic interactions, some of the biggest gains are to be made by taking advantage of the element of surprise and doing something that your rivals never thought you capable of. Several vivid examples can be found in historic military conflicts. For example, in 1967, Israel launched a preemptive attack that destroyed the Egyptian air force on the ground; in 1973, it was Egypt’s turn to spring a surprise on Israel by launching a tank attack across the Suez Canal.

It would seem, then, that the strict definition of game theory leaves out a very important aspect of strategic behavior, but in fact the problem is not that serious. The theory can be reformulated so that each player attaches some small probability to the situation in which such dramatically different strategies are available to the other players. Of course, each player knows her own set of available strategies. Therefore, the game becomes one of asymmetric information and can be handled by using the methods developed in Chapter 9.

The concept of common knowledge itself requires some explanation. For some fact or situation X to be common knowledge between two people, A and B, it is not enough for each of them separately to know X. Each should also know that the other knows X; otherwise, for example, A might think that B does not know X and might act under this misapprehension in the midst of a game. But then A should also know that B knows that A knows X, and the other way around, otherwise A might mistakenly try to exploit B’s supposed ignorance of A’s knowledge. Of course, it doesn’t even stop there. A should know that B knows that A knows that B knows, and so on ad infinitum. Philosophers have a lot of fun exploring the fine points of this infinite regress and the intellectual paradoxes that it can generate. For us, the general notion that the players have a common understanding of the rules of their game will suffice.

E. Equilibrium

Finally, what happens when rational players' strategies interact? Our answers will generally apply the framework of equilibrium. This term simply means that each player is using the strategy that is the best response to the strategies of the other players. We will develop game-theoretic concepts of equilibrium in Chapters 3 - 7 and then use them in subsequent chapters.

Equilibrium does not mean that things don't change; in sequential-move games, the players' strategies are complete plans of action and reaction, and any move that one player makes affects the circumstances in which later moves are made and responded to. Nor does equilibrium mean that everything is for the best; the interaction of rational strategic choices by all players can lead to bad outcomes for all, as in the prisoners' dilemma. But we will generally find that the idea of equilibrium is a useful descriptive tool and organizing principle for our analysis. We will consider this idea in greater detail later in this book, in connection with specific equilibrium concepts. We will also see how the concept of equilibrium can be augmented or modified to remove some of its flaws and to incorporate behavior that falls short of full calculating rationality.

Just as the rational behavior of individual players can be the result of experience in playing the game, the fitting of their choices into an overall equilibrium can come about after some plays that involve trial and error and nonequilibrium outcomes. We will look at this matter in Chapter 5.

Defining an equilibrium is not hard, but finding an equilibrium in a particular game—that is, solving the game—

can be a lot harder. Throughout this book, we will solve many simple games in which there are two or three players, each of them having two or three strategies or one move each in turn. Many people believe this to be the limit of the reach of game theory and therefore believe that the theory is useless for the more complex games that take place in reality. That is not true.

Humans are severely limited in their speed of calculation and in their patience for performing long calculations. Therefore, humans can easily solve only simple games with two or three players and strategies. But computers are very good at speedy and lengthy calculations. Many games that are far beyond the power of human calculators are easy for computers. The complexity of many games in business and politics is already within the power of computers. Even for games such as chess that are far too complex to solve completely, computers have reached a level of ability comparable to that of the best human players; we will consider chess in more detail in Chapter 3.

A number of computer programs for solving complex games exist, and more are appearing rapidly. Mathematica and similar program packages contain routines for finding equilibria in simultaneous-move games, even when such equilibria entail strategies in which players incorporate randomness into their moves. Gambit, a National Science Foundation project led by Professors Richard D. McKelvey of the California Institute of Technology and Andrew McLennan of the University of Minnesota, is producing a comprehensive set of routines for finding equilibria in games with sequential and simultaneous moves, with pure and mixed strategies, and with varying degrees of uncertainty and asymmetric information. We will refer to this project again in several places in the next several chapters. The biggest advantage of the project is that its programs are open source and can easily be obtained from its Web site, www.gambit-project.org.

Why, then, do we set up and solve several simple games in detail in this book? Understanding the concepts of game theory is an important prerequisite for making good use of the mechanical solutions that computers can deliver, and understanding comes from doing simple cases yourself. This is exactly how you learned, and now use, arithmetic. You came to understand the ideas of addition, subtraction, multiplication, and division by doing many simple problems mentally or using paper and pencil. With this grasp of basic concepts, you can now use calculators and computers to do far more complicated sums than you would ever have the time or patience to do manually. But if you did not understand the concepts, you would make errors in using calculators; for example, you might solve $3 + 4 \times 5$ by grouping additions and multiplications incorrectly as $(3 + 4) \times 5 = 35$ instead of correctly as $3 + (4 \times 5) = 23$.

Thus, the first step of understanding the concepts and tools is essential. Without it, you would never learn to set up correctly the games that you ask the computer to solve. You would not be able to inspect the solution with any feeling for whether it was reasonable, and if it was not, you would not be able to go back to your original specifications, improve them, and solve the game again until the specifications and the calculations correctly captured the strategic situation that you wanted to study. Therefore, please pay serious attention to the simple examples that we will solve and the exercises that we will ask you to solve, especially in Chapters 3–7.

F. Dynamics and Evolutionary Games

The theory of games described thus far, based on assumptions of rationality and equilibrium, has proved very useful, but it would be a mistake to rely on it totally. When games are played by novices who do not have the necessary experience to perform the calculations to choose their best strategies, explicitly or implicitly, their choices, and therefore the outcome of the games, can differ significantly from the predictions of analyses based on the concept of equilibrium.

However, we should not abandon all notions of good choice; we should recognize the fact that even poor calculators are motivated to do better for their own sakes and will learn from experience and by observing others. We should allow for a dynamic process in which strategies that proved to be better in previous plays of the game are more likely to be chosen in later plays.

The concept of evolutionary_games, derived from the idea of evolution in biology, does just this. Any individual animal’s genes strongly influence its behavior. Some behaviors succeed better in the prevailing environment, in the sense that the animals exhibiting those behaviors are more likely to reproduce successfully and pass their genes to their progeny. An evolutionarily stable state, relative to a given environment, is the ultimate outcome of this process over several generations.

The analogy in games would be to assume that strategies are not chosen by conscious rational maximizers, but instead that each player comes to the game with a particular strategy “hardwired” or “programmed” in. The players then confront other players who may be programmed to apply the same or different strategies. The payoffs to all the players in such games are then obtained. The strategies that fare better—in the sense that the players programmed to play them get higher payoffs in the games—multiply faster, whereas the strategies that fare worse decline. In biology, the mechanism of this growth or decay is purely genetic transmission through reproduction. In the context of strategic games in business and society, the mechanism is much more likely to be social or cultural—observation and imitation, teaching and learning, greater availability of capital for the more successful ventures, and so on.

The object of study is the dynamics of this process. Does it converge to an evolutionarily stable state? Does just one strategy prevail over all others in the end, or can a few strategies coexist? Interestingly, in many games, the evolutionarily stable state is the same as the equilibrium that would result if the players were consciously rational calculators. Therefore, the evolutionary approach gives us a backdoor justification for equilibrium analysis.

The concept of evolutionary games has thus imported biological ideas into game theory, but there has been an influence in the opposite direction, too. Biologists have recognized that significant parts of animal behavior consist of strategic interactions with other animals. Members of a given species compete with one another for space or mates; members of different species relate to one another as predators and prey along a food chain. The payoff in such games in turn contributes to reproductive success and therefore to biological evolution. Just as game theory has benefited by importing ideas from biological evolution for its analysis of choice and dynamics, biology has benefited by importing game-theoretic ideas of strategies and payoffs for its characterization of basic interactions among animals. We have here a true instance of synergy or symbiosis. We will provide an introduction to the study of evolutionary games in Chapter 12.

G. Observation and Experiment

All of Section 3 to this point has concerned how to think about games or how to analyze strategic interactions—in other words, it has been about theory. This book will cover an extremely simple level of theory, developed through cases and illustrations instead of formal mathematics or theorems, but it will be theory just the same. All theory should relate to reality in two ways: Reality should help structure the theory, and reality should provide a check on the results of the theory.

We can find out the reality of strategic interactions in two ways: (1) by observing them as they occur naturally and (2) by conducting special experiments that help us pin down the effects of particular conditions. Both methods have been used, and we will mention several examples of each in the proper contexts.

Many people have studied strategic interactions—the participants' behavior and the outcomes—under experimental conditions, in classrooms among "captive" players, or in special laboratories with volunteers. Auctions, bargaining, the prisoners' dilemma, and several other games have been studied in this way. The results are mixed. Some conclusions of the theoretical analysis are borne out; for example, in games of buying and selling, the participants generally settle quickly on the economic equilibrium. In other contexts, the outcomes differ significantly from the theoretical predictions; for example, prisoners' dilemmas and bargaining games show more cooperation than theory based on the assumption of selfish, maximizing behavior would lead us to expect, whereas auctions show some gross overbidding.

At several points in the chapters that follow, we will review the knowledge that has been gained by observation and experiments, discuss how it relates to the theory, and consider what reinterpretations, extensions, and modifications of the theory have been made or should be made in light of this knowledge.

Glossary

strategy

A complete plan of action for a player in a game, specifying the action he would take at all nodes where it is his turn to act according to the rules of the game (whether these nodes are on or off the equilibrium path of play). If two or more nodes are grouped into one information set, then the specified action must be the same at all these nodes.

payoff

The objective, usually numerical, that a player in a game aims to maximize.

expected_payoff

The probability-weighted average (statistical mean or expectation) of the payoffs of one player in a game, corresponding to all possible realizations of a random choice of nature or mixed strategies of the players.

rational_behavior

Perfectly calculating pursuit of a complete and internally consistent objective (payoff) function.

equilibrium

A configuration of strategies where each player's strategy is his best response to the strategies of all the other players.

evolutionary_game

A situation where the strategy of each player in a population is fixed genetically, and strategies that yield higher payoffs in random matches with others from the same population reproduce faster than those with lower payoffs.

4 THE USES OF GAME THEORY

We began Chapter 1 by saying that games of strategy are everywhere—in your personal and working life; in the functioning of the economy, society, and polity around you; in sports and other serious pursuits; in war and in peace. This should be motivation enough to study such games systematically, and that is what game theory is about. But your study can be better directed if you have a clearer idea of just how you can put game theory to use. We suggest a threefold perspective.

The first use is in *explanation*. Many events and outcomes prompt us to ask, Why did that happen? When a situation requires the interaction of decision makers with different aims, game theory often supplies the key to understanding that situation. For example, cutthroat competition in business is the result of the rivals being trapped in a prisoners' dilemma. At several points in this book, we will mention actual cases where game theory has helped us to understand how and why the events unfolded as they did. These cases include Chapter 13's detailed study of the Cuban missile crisis from the perspective of game theory.

The other two uses evolve naturally from the first. The second use is in *prediction*. When looking ahead to situations where multiple decision makers will interact strategically, we can use game theory to foresee what actions they will take and what outcomes will result. Of course, prediction in a particular context depends on its details, but we will prepare you to use prediction by analyzing several broad classes of games that arise in many applications

The third use is in *advice* or *prescription*. We can act in the service of one participant in a future interaction and tell him which strategies are likely to yield good results and which ones are likely to lead to disaster. Once again, such work is context specific, but we can equip you with several general principles and techniques and show you how to apply them to some general types of contexts. For example, in Chapter 7, we will show how to mix moves; in Chapter 8, we will examine how to make commitments, threats, and promises credible; and in Chapter 10, we will examine alternative ways of overcoming prisoners' dilemmas.

The theory is far from perfect in performing any of these three functions. To explain an outcome, one must first have a correct understanding of the motives and behavior of the participants. As we saw earlier, most of game theory takes a specific approach to these matters—namely, by applying a framework in which individual players make rational choices and those choices constitute an equilibrium of the game. Actual players and interactions in a game might not conform to this framework. But the proof of the pudding is in the eating. Game-theoretic analysis has greatly improved our understanding of many phenomena, as reading this book should convince you. The theory continues to evolve and improve as the result of ongoing research. This book will equip you with the basics so that you can more easily learn and profit from the new advances as they appear.

When explaining a past event, we can often use historical records to get a good idea of the motives and the behavior of the players in the game. When attempting prediction or advice, we have the additional problem of determining what motives will drive the players' actions, what informational or other limitations they will face, and sometimes even who the players will be. Most importantly, if game-theoretic analysis assumes that a player is a rational maximizer of his own objectives when in fact he is unable to do the required calculations or is a clueless person acting at random, the advice based on that analysis may prove wrong. This risk is reduced as more and more players recognize the importance of strategic interaction and think through their strategic choices or get expert advice on the matter, but some risk remains. Even then, the systematic thinking made possible by the framework of game theory helps keep the errors down to this irreducible minimum by eliminating the errors that arise from faulty logical thinking about the strategic interaction. Furthermore, game theory can take into account many kinds of uncertainty and asymmetric information, including uncertainties about the strategic possibilities and the rationality of the players. We will consider a few examples in the chapters to come.

5 THE STRUCTURE OF THE CHAPTERS TO FOLLOW

In this chapter, we introduced several considerations that arise in almost every game in reality. We also introduced some basic concepts that prove useful for game-theoretic analysis, to understand or predict the outcomes of games. However, trying to cope with all of these concepts at once merely leads to confusion and a failure to grasp any of them. Therefore, we will build up the theory one concept at a time. We will develop the appropriate technique for applying each concept and illustrate it.

In Part Two of this book, in Chapters 3–7, we will construct and illustrate the most important of these concepts and techniques. We will examine purely sequential-move games in Chapter 3 and introduce the techniques—game trees and rollback reasoning—that are used to analyze and solve such games. In Chapters 4 and 5, we will turn to games with simultaneous moves and develop for them another set of concepts—payoff tables, dominance, and Nash equilibrium. Both chapters will focus on games where players use pure strategies; in Chapter 4, we will restrict players to a finite set of pure strategies, and in Chapter 5, we will allow strategies that are continuous variables. Chapter 5 will also examine some mixed empirical evidence, conceptual criticisms, and counterarguments on Nash equilibrium, as well as a prominent alternative to Nash equilibrium—namely, rationalizability. In Chapter 6, we will show how games that have some sequential moves and some simultaneous moves can be studied by combining the techniques developed in Chapters 3–5. In Chapter 7, we will turn to simultaneous-move games that require the use of randomization or mixed strategies. We will start by introducing the basic ideas about mixing in two-by-two games, develop the simplest techniques for finding mixed-strategy Nash equilibria, and then consider more complex examples along with the empirical evidence on mixing.

The ideas and techniques developed in Chapters 3 – 7 are the most basic ones: (1) correct forward-looking reasoning for sequential-move games, and (2) equilibrium strategies—pure and mixed—for simultaneous-move games. Equipped with these concepts and tools, we can apply them to study some broad classes of games and strategies in Part Three (Chapters 8 – 12).

In Chapter 8, we will examine strategies that players use to manipulate the rules of a game, such as seizing a first-mover advantage and/or making a strategic move. Such moves are of three kinds: commitments, threats, and promises. In each case, credibility is essential to the success of the move, and we will outline some ways of making such moves credible. In Chapter 9, we will study what happens in games when players are subject to uncertainty or when they have asymmetric information. We will examine strategies for coping with risk and even for using risk strategically. We will also study the important strategies of signaling and screening that are used for manipulating and eliciting information. We will develop the appropriate generalization of Nash equilibrium in the context of uncertainty—namely, Bayesian Nash equilibrium—and show the different kinds of equilibria that can arise. We will also consider situations in which the asymmetry of information is two-way and the implications of such two-way private information for dynamic games (games of timing).

In Chapter 10, we will move on to study the best-known game of them all: the prisoners' dilemma. We will study whether and how cooperation can be sustained, most importantly in a repeated or ongoing relationship. Then, in Chapter 11, we will turn to situations where large populations, rather than pairs or small groups of players, interact strategically—games that concern problems of collective action. Each person’s actions have an effect—in some instances beneficial, in others, harmful—on the others. The outcomes are generally not the best from the aggregate perspective of the society as a whole. We will clarify the nature of these outcomes and describe some simple policies that can lead to better outcomes.

All these theories and applications are based on the supposition that the players in a game fully understand the nature of the game and deploy calculated strategies that best serve their objectives in the game. Such rationally optimal behavior is sometimes too demanding of information and calculating power to be believable as a good description of how people really act. Therefore, Chapter 12 will look at games from a very different perspective. Here, the players are not calculating and do not pursue optimal strategies. Instead, each player is tied, as if genetically programmed, to a particular strategy. The population is diverse, and different players have different predetermined strategies. When players from such a population meet and act out their strategies, which strategies perform better? And if the more successful strategies proliferate better in the population, whether through inheritance or imitation, then what will the eventual structure of the population look like? It turns out that such evolutionary dynamics often favor exactly those strategies that would be used by rational optimizing players. Thus, our study of evolutionary games lends useful indirect support to the theories of optimal strategic choice and equilibrium that we will have studied in the previous chapters.

In Part Four, Chapters 13–17, we will take up specific applications of game theory to situations of strategic interactions. Here, we will use as needed the ideas and methods from all the earlier chapters. Chapter 13 will apply ideas from Chapters 8 and 9 to examine a particularly interesting dynamic version of a threat, known as the strategy of brinkmanship. We will elucidate its nature and apply the model developed in Chapter 9 to study the Cuban missile crisis of 1962. Chapter 14 will analyze strategies that people and firms use to deal with others who have some private information. We will illustrate the screening mechanisms that are used for eliciting information—for example, the multiple fares with different restrictions that airlines use for separating the business travelers who are willing to pay more from the tourists who are more price sensitive. We will also develop the methods for designing incentive payments to elicit effort from workers when direct monitoring is difficult or too costly. Chapter 15 will examine auctions as a means to allocate valuable economic resources, emphasizing the roles of information and attitudes toward risk (developed in Chapter 9) in the formulation of optimal strategies for both bidders and sellers.

Chapter 16 will consider voting in committees and elections. We will look at the variety of voting rules available and some paradoxical results that can arise. In addition, we will address the potential for strategic behavior not only by voters but also by candidates in a variety of election types. Finally, Chapter 17 will present bargaining in both cooperative and noncooperative settings.

All of these chapters together provide a lot of material; how might readers or teachers with more specialized interests choose from it? Chapters 3–7 constitute the core theoretical ideas that are needed throughout the rest of the book. Chapters 8 and 10 are likewise important for the general classes of games and strategies considered therein. Beyond that, there is a lot from which to pick and choose. Section 3 of Chapter 5, Section 6 of Chapter 7, and Section 5 of Chapter 11, for example, all consider more advanced topics. These sections will appeal to those with more scientific and quantitative backgrounds and interests, but those who come from the social sciences or humanities and have less quantitative background can omit them without loss of continuity.

Chapters 8 and 10 are key to understanding many phenomena in the real world, and most teachers will want to include them in their courses. Chapter 9 deals with an important topic in that most games, in practice, have asymmetric information, and the players' attempts to manipulate information is a critical aspect of many strategic interactions. However, the concepts and techniques for analyzing information games are inherently complex. Therefore, some readers and teachers may choose to study just the examples that convey the basic ideas of signaling and screening and leave out the rest. We have placed this chapter early in Part Three, however, in view of the importance of the subject.

Chapters 11 and 12 both look at games with large numbers of players. In Chapter 11, the focus is on social interactions; in Chapter 12, the focus is on evolutionary biology. The topics in Chapter 12 will be of greatest interest to those with interests in biology, but similar themes are emerging in the social sciences, and students from that background should aim to get the gist of the ideas even if they skip the details. Chapter 14 is most important for students of business and organization theories, while Chapters 13 and 16 present topics from political science (international diplomacy and elections, respectively). Chapters 15 and 17 cover topics from economics (auctions and bargaining). Those teaching courses with more specialized audiences may choose a subset from Chapters 11–17 and, indeed, expand on the ideas considered therein.

Whether you come from mathematics, biology, economics, politics, or other sciences, or from history or sociology, the theory and examples of strategic games will stimulate and challenge your intellect. We urge you to enjoy the subject even as you are studying or teaching it.

SUMMARY

A game of strategy is any situation with multiple decision makers, referred to as players, whose choices affect one another. Games are everywhere: at school and at work, on social media or out on a date. There is no escaping them! Being able to recognize what types of games you are playing, and knowing how to play them, will give you an advantage in life. Of course, games differ in many important ways and can be classified along several dimensions, such as whether there are many players or a small number, the timing of play, the extent to which players have common or conflicting interests, the number of times an interaction occurs, what players know about one another, whether the rules can be changed, and whether coordinated action is feasible.

Learning the terminology for a game's structure is crucial for game-theoretic analysis. Players have *strategies* that lead to different *outcomes* with different associated *payoffs*. Payoffs incorporate everything that is important to a player about a game and are calculated by using probabilistic averages or *expectations* if outcomes are random or include some risk. *Rationality*, or consistent behavior, is assumed of all players, who must also be aware of all of the relevant rules of conduct. *Equilibrium* arises when all players use strategies that are best responses to others' strategies; some classes of games allow learning from experience and the study of dynamic movements toward equilibrium. The study of behavior in actual game situations provides additional information about the performance of game theory.

Game theory may be used for explanation, prediction, or prescription in various circumstances. Although not perfect in any of these roles, the theory continues to evolve; the importance of strategic interaction and strategic thinking has also become more widely understood and accepted.

KEY TERMS[7]

asymmetric information (24)

constant-sum game (20)

cooperative game (27)

equilibrium (33)

evolutionary game (35)

expected payoff (29)

external uncertainty (23)

game (game of strategy) (18)

imperfect information (24)

noncooperative game (27)

payoff (29)

perfect information (24)

rational behavior (30)

screening (24)

screening device (24)

sequential moves (21)

signal (24)

signaling (24)

simultaneous moves (21)

strategic uncertainty (24)

strategy (27)

zero-sum game (20)

Endnotes

- The number in parentheses after each key term is the page on which that term is defined or discussed. Return to reference 7

Glossary

asymmetric information
: Information is said to be asymmetric in a game if some aspects of it—rules about what actions are permitted and the order of moves if any, payoffs as functions of the players strategies, outcomes of random choices by "nature," and of previous actions by the actual players in the game—are known to some of the players but are not common knowledge among all players.

constant-sum game
: A game in which the sum of all players' payoffs is a constant, the same for all their strategy combinations. Thus, there is a strict conflict of interests among the players—a higher payoff to one must mean a lower payoff to the collectivity of all the other players. If the payoff scales can be adjusted to make this constant equal to zero, then we have a *zero-sum game*.

cooperative game
: A game in which the players decide and implement their strategy choices jointly, or where joint-action agreements are directly and collectively enforced.

equilibrium
: A configuration of strategies where each player's strategy is his best response to the strategies of all the other players.

evolutionary game
: A situation where the strategy of each player in a population is fixed genetically, and strategies that yield higher payoffs in random matches with others from the same population reproduce faster than those with lower payoffs.

expected payoff
: The probability-weighted average (statistical mean or expectation) of the payoffs of one player in a game,

corresponding to all possible realizations of a random choice of nature or mixed strategies of the players.

external uncertainty

A player's uncertainty about external circumstances such as the weather or product quality.

game (game of strategy)

An action situation where there are two or more mutually aware players, and the outcome for each depends on the actions of all.

imperfect information

A game is said to have perfect information if each player, at each point where it is his turn to act, knows the full history of the game up to that point, including the results of any random actions taken by nature or previous actions of other players in the game, including pure actions as well as the actual outcomes of any mixed strategies they may play. Otherwise, the game is said to have imperfect information.

noncooperative game

A game where each player chooses and implements his action individually, without any joint-action agreements directly enforced by other players.

payoff

The objective, usually numerical, that a player in a game aims to maximize.

perfect information

A game is said to have perfect information if players face neither strategic nor external uncertainty.

rational behavior

Perfectly calculating pursuit of a complete and internally consistent objective (payoff) function.

screening

Strategy of a less-informed player to elicit information credibly from a more-informed player.

screening devices

Methods used for screening.

sequential moves

The moves in a game are sequential if the rules of the game specify a strict order such that at each action node only one player takes an action, with knowledge of the actions taken (by others or himself) at previous nodes.

signals

Devices used for signaling.

signaling

Strategy of a more-informed player to convey his “good” information credibly to a less-informed player.

simultaneous moves

The moves in a game are simultaneous if each player must take his action without knowledge of the choices of others.

strategic uncertainty

A player’s uncertainty about an opponent’s moves made in the past or made at the same time as her own.

strategy

A complete plan of action for a player in a game, specifying the action he would take at all nodes where it is his turn to act according to the rules of the game (whether these nodes are on or off the equilibrium path of play). If two or more nodes are grouped into one information set, then the specified action must be the same at all these nodes.

zero-sum game

A game where the sum of the payoffs of all players equals zero for every configuration of their strategy choices. (This is a special case of a *constant-sum game*, but in practice no different because adding a constant to all the payoff numbers of any one player makes no difference to his choices.)

SOLVED EXERCISES[8]

1. Determine which of the following scenarios describe decisions, which of them describe games with a small number of players, and which of them describe games with many players. In each case, indicate what specific features of the scenario caused you to classify it as you did.
   1. A group of grocery shoppers in the dairy section, with each shopper choosing a flavor of yogurt to purchase
   2. A pair of teenage girls choosing dresses for their prom
   3. A college student considering what type of postgraduate education to pursue
   4. The *New York Times* and the *Wall Street Journal* choosing the prices for their online subscriptions this year
   5. College students in a class, deciding how hard to study for the final exam
   6. A presidential candidate picking a running mate
2. Consider the strategic games described below. In each case, state how you would classify the game according to the six dimensions outlined in Sections 2.A–2.F of this chapter: (i) Are players' interests totally aligned, totally in conflict, or a mix of both? (ii) Are moves sequential or simultaneous? (iii) Are the rules fixed or not? (iv) Is it a game with imperfect information, and if so, is it one with asymmetric information? (v) Is the game repeated? (vi) Are cooperative agreements possible or not? If you do not have enough information to classify a game in a particular dimension, explain why not.
   1. *Rock-Paper-Scissors:* On the count of three, each player makes the shape of one of the three items with

her hand. Rock beats Scissors, Scissors beats Paper, and Paper beats Rock.
   2. *Roll-call voting:* Voters cast their votes orally as their names are called. The choice with the most votes wins.
   3. *Sealed-bid auction:* Bidders on a bottle of wine seal their bids in envelopes. The highest bidder wins the item and pays the amount of his bid.
3. "A game player would never prefer an outcome in which every player gets a little profit to an outcome in which he gets all the available profit." Is this statement true or false? Explain why in one or two sentences.
4. You and a rival are engaged in a game in which there are three possible outcomes: you win, your rival wins (you lose), or the two of you tie. You get a payoff of 50 if you win, a payoff of 20 if you tie, and a payoff of 0 if you lose. What is your expected payoff in each of the following situations?
   1. There is a 50% chance that the game ends in a tie, but only a 10% chance that you win. (There is thus a 40% chance that you lose.)
   2. There is a 50–50 chance that you win or lose. There are no ties.
   3. There is an 80% chance that you lose, a 10% chance that you win, and a 10% chance that you tie.
5. Explain the difference between game theory's use as a predictive tool and its use as a prescriptive tool. In what types of real-world settings might these two uses be most important?

UNSOLVED EXERCISES

1. Determine which of the following scenarios describe decisions, which of them describe games with a small number of players, and which of them describe games with many players. In each case, indicate what specific features of the scenario caused you to classify it as you did.
   1. A party nominee for president of the United States must choose whether to use private financing or public financing for her campaign.
   2. Frugal Fred receives a $20 gift card for downloadable music and must choose whether to purchase individual songs or whole albums.
   3. Beautiful Belle receives 100 replies to her online dating profile and must choose whether to reply to each of them.
   4. After news that gold has been discovered, Americans living on the East Coast decide whether to join the Gold Rush to California.
   5. NBC chooses how to distribute its TV shows online this season. The executives consider Amazon.com, iTunes, and/or NBC.com. The fee they might pay to Amazon or to iTunes is open to negotiation.
   6. China chooses a level of tariffs to apply to American imports.
2. Consider the strategic games described below. In each case, state how you would classify the game according to the six dimensions outlined in the text. (i) Are players' interests totally aligned, totally in conflict, or a mix of both? (ii) Are moves sequential or simultaneous? (iii) Are the rules fixed or not? (iv) Is there imperfect information, and if so, is there asymmetric information? (v) Is the game repeated? (vi) Are cooperative agreements possible or not? If you do not

have enough information to classify a game in a particular dimension, explain why not.

1. Garry and Ross are sales representatives for the same company. Their manager informs them that of the two of them, whoever sells more this year wins a Cadillac.
2. On the game show *The Price Is Right*, four contestants are asked to guess the price of a TV set. Play starts with the leftmost player, and each player's guess must be different from the guesses of the previous players. The person who comes closest to the real price, without going over it, wins the TV set.
3. Six thousand players each pay $10,000 to enter the World Series of Poker. Each starts the tournament with $10,000 in chips, and they play No-Limit Texas Hold 'Em (a type of poker) until someone wins all the chips. The top 600 players each receive prize money according to their order of finish, with the winner receiving more than $8,000,000.
4. Passengers on Desert Airlines are not assigned seats; passengers choose seats once they board. The airline assigns the order of boarding according to the time the passenger checks in, either on the Web site up to 24 hours before takeoff or in person at the airport.

3. "Any gain by the winner must harm the loser." Is this statement true or false? Explain your reasoning in one or two sentences.
4. Alice, Bob, and Confucius are bored during recess, so they decide to play a new game. Each of them puts a dollar in the pot, and each tosses a quarter. Alice wins if the coins land all heads or all tails. Bob wins if two heads and one tail land, and Confucius wins if one head and two tails land. The quarters are fair. The winner receives a net payment of $2 ($3 − $1 = $2), and the losers lose their $1.
   1. What is the probability that Alice will win and the probability that she will lose?

2. What is Alice's expected payoff?
3. What is the probability that Confucius will win and the probability that he will lose?
4. What is Confucius's expected payoff?
5. Is this a constant-sum game? Please explain your answer.

5. "When one player surprises another, this indicates that the players did not have common knowledge of the rules." Give an example that illustrates this statement and give a counterexample that shows that the statement is not always true.

Endnotes

- **Note to Students:** The solutions to the **Solved Exercises** are found on the Web site **digital.wwnorton.com/gamesofstrategy5**. Instructions on how to download this resource are available at that location. Return to reference 8

PART TWO

Fundamental Concepts and Techniques

3 ■ Games with Sequential Moves

SEQUENTIAL-MOVE GAMES entail strategic situations in which there is a strict order of play. Players take turns making their moves, and they know what the players who have gone before them have done. To play well in such a game, participants must use a particular type of interactive thinking. Each player must consider how her opponent will respond if she makes a particular move. Whenever actions are taken, players need to think about how their current actions will influence future actions, both for their rivals and for themselves. Players thus decide their current moves on the basis of calculations of future consequences.

Most actual games combine aspects of sequential- and simultaneous-move situations. But the concepts and methods of analysis are more easily understood if they are first developed separately for the two pure cases. Therefore, in this chapter, we study purely sequential-move games. Chapters 4 and 5 deal with purely simultaneous-move games, and Chapter 6 and parts of Chapter 7 show how to combine the two types of analysis in more realistic mixed situations. The analysis presented here can be used whenever a game includes sequential decision making. Analysis of sequential-move games also provides information about when it is to a player's advantage to move first and when it is better to move second. Players can then devise ways, called *strategic moves*, to manipulate the order of play to their advantage. The analysis of such moves is the focus of Chapter 8.

1 GAME TREES

We begin by developing a graphical technique for displaying and analyzing sequential-move games, called a game tree. This tree is referred to as the extensive form of a game. It shows all the component parts of the game that we introduced in Chapter 2: players, actions, and payoffs.

You have probably come across decision trees in other contexts. Such trees show all the successive decision points, or nodes, for a single decision maker in a neutral environment. Decision trees also include branches corresponding to the available choices emerging from each node. Game trees are just joint decision trees for all the players in a game. The trees illustrate all the possible actions that can be taken by all the players and indicate all the possible outcomes of the game.

A. Nodes, Branches, and Paths of Play

Figure 3.1 shows the tree for a particular sequential-move game. We do not supply a story for this game, because we want to omit circumstantial details and help you focus on general concepts. Our game has four players: Ann, Bob, Chris, and Deb. The rules of the game give the first move to Ann; this is shown at the leftmost node, which is called the initial node or root of the game tree. At this node, which may also be called an action node or decision node, Ann has two choices available to her. Ann's possible choices, labeled Stop and Go (remember that these labels are abstract and have no necessary significance), are shown as branches emerging from the initial node.

If Ann chooses Stop, then it will be Bob's turn to move. At his action node, he has three available choices, labeled 1, 2, and 3. If Ann chooses Go, then Chris gets the next move, with choices Risky and Safe. Other nodes and branches follow successively; rather than listing them all in words, we draw your attention to a few prominent features.

If Ann chooses Stop and then Bob chooses 1, Ann gets another turn, with new choices, Up and Down. It is quite common in actual sequential-move games for a player to get to move several times and to have her available moves differ at different turns. In chess, for example, two players make alternate moves; each move changes the board, and therefore the available moves are changed at subsequent turns.

B. Uncertainty and “Nature’s Moves”

If Ann chooses Go and then Chris chooses Risky, something happens at random: A fair coin is tossed, and the outcome of the game is determined by whether that coin comes up heads or tails. This aspect of the game is an example of external uncertainty and is handled in the tree by introducing an outside player called “Nature.” Control over the random event is ceded to the player known as Nature, who chooses, as it were, one of two branches, each with 50% probability. Although Nature makes its choices with the specified probabilities, it is otherwise a passive, neutral player that receives no payoffs of its own. In this example, the probabilities are fixed by the type of random event—a coin toss—but could vary in other circumstances. For example, with the throw of a die, Nature could specify six possible outcomes, each with 16⅔% probability. Use of the player Nature allows us to introduce external uncertainty in a game and gives us a mechanism to allow things to happen that are outside the control of any of the actual players.

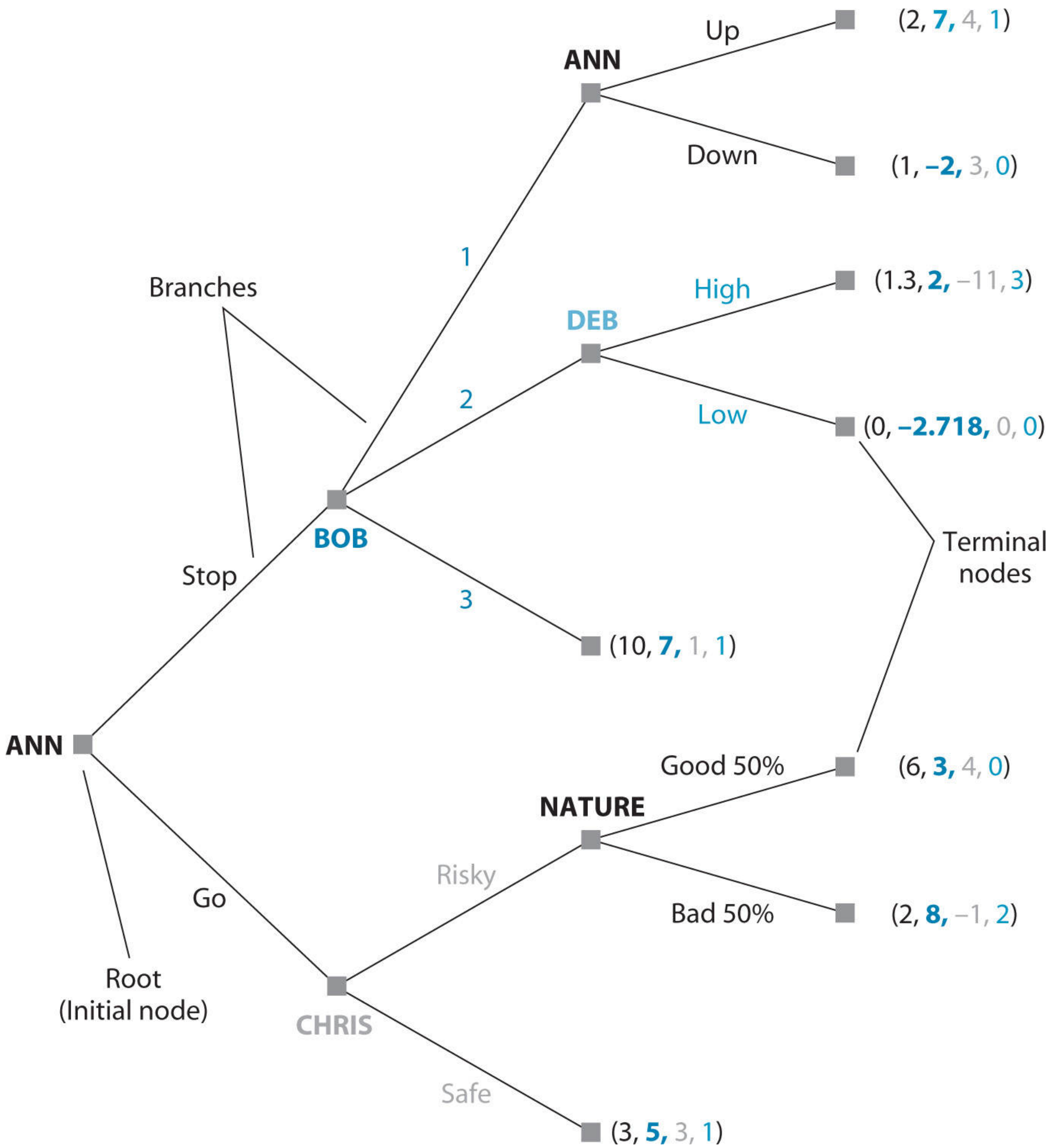


Figure 3.1 An Illustrative Game Tree

You can trace a number of different paths through the game tree by following successive branches. In Figure 3.1, each path leads you to an end point of the game after a finite number of moves. An end point is not a necessary feature of all games; some may, in principle, go on forever. But most applications that we will consider are finite games.

C. Outcomes and Payoffs

At the last node along each path, called a terminal node, no player has another move. (Note that terminal nodes are thus distinguished from *action* nodes.) Instead, we show the outcome of that particular sequence of actions, as measured by the payoffs for the players. For our four players, we list the payoffs in order (Ann, Bob, Chris, Deb). It is important to specify which payoff belongs to which player. The usual convention is to list payoffs in the order in which the players make their moves. But this method may sometimes be ambiguous; in our example, it is not clear whether Bob or Chris should be said to have the second move. Thus, we have used alphabetical order. Further, we have color-coded everything so that Ann's name, choices, and payoffs are all in black; Bob's in dark blue; Chris's in gray; and Deb's in light blue. When drawing trees for any games that you analyze, you can choose any specific convention you like, but you should state and explain it clearly for the reader.

The payoffs are numerical, and generally, for each player, a higher number means a better outcome. Thus, for Ann, the outcome of the bottommost path (payoff 3) is better than that of the topmost path (payoff 2) in Figure 3.1. But there is no necessary comparability across players. Thus, there is no necessary sense in which, at the end of the topmost path, Bob (payoff 7) does better than Ann (payoff 2). Sometimes—if payoffs are dollar amounts, for example—such interpersonal comparisons may be meaningful.

Players use information about payoffs when deciding among the various actions available to them. The inclusion of a random event (a choice made by Nature) means that players need to determine what they get on average when Nature moves. For example, if Ann chooses Go at the game's first move, Chris may then choose Risky, giving rise to the coin toss and Nature's "choice" of Good or Bad. In this situation, Ann could anticipate a payoff of 6 half the time and a payoff of 2 half the time, or a statistical average, or *expected payoff,* of 4 = (0.5 × 6) + (0.5 × 2).

D. Strategies

Finally, we use the tree in Figure 3.1 to explain the concept of a strategy. A single action taken by a player at a node is called a move. But players can, do, and should make plans for the succession of moves that they expect to make in all the various eventualities that might arise in the course of a game. Such a complete plan of action is called a *strategy*. (Although the term *strategy* always refers to a *complete* plan of action, as we described in Chapter 2, we will sometimes use the term *complete strategy* to remind you of the need to include actions for all possible eventualities in your description of a player’s strategy.)

In this tree, Bob, Chris, and Deb each get to move at most once; Chris, for example, gets a move only if Ann chooses Go on her first move. For them, there is no distinction between a move and a strategy. We can qualify their moves by specifying the contingencies in which they get made; thus, a (complete) strategy for Bob might be, “Choose 1 if Ann has chosen Stop.” But Ann has two opportunities to move, so her strategy needs a fuller specification. One (complete) strategy for her is, “Choose Stop, and then if Bob chooses 1, choose Down.”

In more complex games such as chess, where there are long sequences of moves with many choices available at each, descriptions of strategies get very complicated; we will consider this aspect in more detail later in this chapter. But the general principle for constructing strategies is simple, except for one peculiarity. If Ann chooses Go on her first move, she never gets to make a second move. Should a strategy in which she chooses Go also specify what she would do in the hypothetical case in which she somehow found herself at the action node of her second move? Your first instinct may be to say *no*, but formal game theory says *yes*, for two reasons.

First, Ann’s choice of Go at her first move may be influenced by her consideration of what she would have to do at her second move if she were to choose Stop originally instead. For example, if she chooses Stop, Bob may then choose 1; then Ann gets a second move, and her best choice would be Up, giving her a payoff of 2. If she chooses Go at her first move, Chris may then choose Safe (because his payoff of 3 from Safe is better than his expected payoff of 1.5 from Risky), and that outcome would yield Ann a payoff of 3. To make this thought process clearer, we state Ann's strategy as, "Choose Go at the first move, and choose Up if the next move arises."

The second reason for this seemingly pedantic specification of strategies has to do with the stability of equilibrium. When considering this stability, we ask what would happen if players' choices were subjected to small disturbances. One such disturbance is that players make small mistakes. If choices are made by pressing a key, for example, Ann may intend to press the Go key, but there is a small probability that her hand will tremble and she will press the Stop key instead. In such a setting, it is important to specify how Ann will follow up when she discovers her error because Bob chooses 1 and it is Ann's turn to move again. More advanced levels of game theory require such stability analyses, and we want to prepare you for that by insisting on your specifying strategies as *complete* plans of action right from the beginning.

E. Tree Construction

Let’ s sum up the general concepts illustrated by the tree in Figure 3.1. Game trees consist of nodes and branches. The nodes, which are connected to one another by the branches, come in two types. The first type is called a decision node. Each decision node is associated with the player who chooses an action at that node. Every tree has one decision node that is the game’ s initial node, the starting point of the game. The second type of node is called a terminal node. Each terminal node has associated with it a set of outcomes for the players taking part in the game; these outcomes are the payoffs received by each player if the game has followed the branches that lead to this particular terminal node.

The branches of a game tree represent the possible actions that can be taken at any decision node. Each branch leads from a decision node either to another decision node, generally for a different player, or to a terminal node. The tree must account for all the possible choices that could be made by a player at each node, so some game trees include branches associated with the choice Do Nothing. There must be at least one branch leading from each decision node, but there is no maximum number. No decision node can have more than one branch leading to it, however.

Game trees are often drawn from left to right across a page. However, game trees can be drawn in any orientation that suits the game at hand: bottom up, sideways, top down, or even radiating outward from a center. The tree is a metaphor, and its important feature is the idea of successive branching as decisions are made at the tree nodes.

Glossary

game tree
: Representation of a game in the form of nodes, branches, and terminal nodes and their associated payoffs.

extensive form
: Representation of a game by a game tree.

decision tree
: Representation of a sequential decision problem facing one person, shown using nodes, branches, terminal nodes, and their associated payoffs.

node
: This is a point from which branches emerge, or where a branch terminates, in a decision or game tree.

initial node
: The starting point of a sequential-move game. (Also called the root of the tree.)

root
: Same as initial node.

action node
: A node at which one player chooses an action from two or more that are available.

decision node
: A decision node in a decision or game tree represents a point in a game where an action is taken.

terminal node
: This represents an end point in a game tree, where the rules of the game allow no further moves, and payoffs for each player are realized.

move
: An action at one node of a game tree.

branch
: Each branch emerging from a node in a game tree represents one action that can be taken at that node.

2 SOLVING GAMES BY USING TREES

We illustrate the use of trees in finding equilibrium outcomes of sequential-move games in a very simple context that many of you have probably confronted—whether to smoke. As we mentioned in Chapter 2, players in a game need not be physically distinct persons. The question of whether to smoke, like many other similar one-player strategic situations, can be described as a game if we recognize that future choices are made by the player's future self, who will be subject to different influences and have different views about the ideal outcome of the game.

Take, for example, a teenager named Carmen who is deciding whether to smoke. First, her current self, who we call "Today's Carmen," has to decide whether to try smoking at all. If she does try it, she creates a future version of herself—a "Future Carmen"—who will have a different ranking of the alternatives available in the future. Future Carmen is the one who makes the choice of whether to continue smoking. When Today's Carmen makes her choice, she has to look ahead, consider Future Carmen's probable addiction to nicotine, and factor that into her current decision, which she should make on the basis of her current preferences. In other words, when Today's Carmen makes her decision, she has to play a game against Future Carmen.

To illustrate this example, we can use the simple game tree shown in Figure 3.2. At the initial node, Today's Carmen decides whether to try smoking. If her decision is to try, then the addicted Future Carmen comes into being and chooses whether to continue. We show the healthy, nonsmoking Today's Carmen, her actions, and her payoffs in blue and the addicted Future Carmen, her actions, and her payoffs in black, the color that her lungs have become.

In order to solve this game—that is, to determine the strategies that players choose in equilibrium—we need to use the payoffs shown at the terminal nodes of the tree. As shown in Figure 3.2, we have chosen the outcome of never smoking at all as the standard of reference, and we call its payoff 0. There is no special significance to the number 0 in this context; all that matters for comparing outcomes, and thus for making choices, is whether this payoff is bigger or smaller than the others. Suppose Today's Carmen likes best the outcome in which she tries smoking for a while but Future Carmen chooses not to continue. The reason may be that she just likes to have experienced many things firsthand, or so that she can more convincingly be able to say, "I have been there and know it to be a bad situation," when she tries in the future to dissuade her children from smoking. We give this outcome the payoff +1 for Today's Carmen. The outcome in which Today's Carmen tries smoking and then Future Carmen continues is the worst for Today's Carmen. In addition to the long-term health hazards, she foresees other problems —her hair and clothes will smell bad, and her friends will avoid her. We give this outcome the payoff −1 for Today's Carmen. Future Carmen has become addicted to smoking and has different preferences from Today's Carmen. She will enjoy continuing to smoke and will suffer terrible withdrawal symptoms if she does not continue. In the tree, we show Future Carmen's payoff from "Continue" as +1 and that from "Not" as −1.

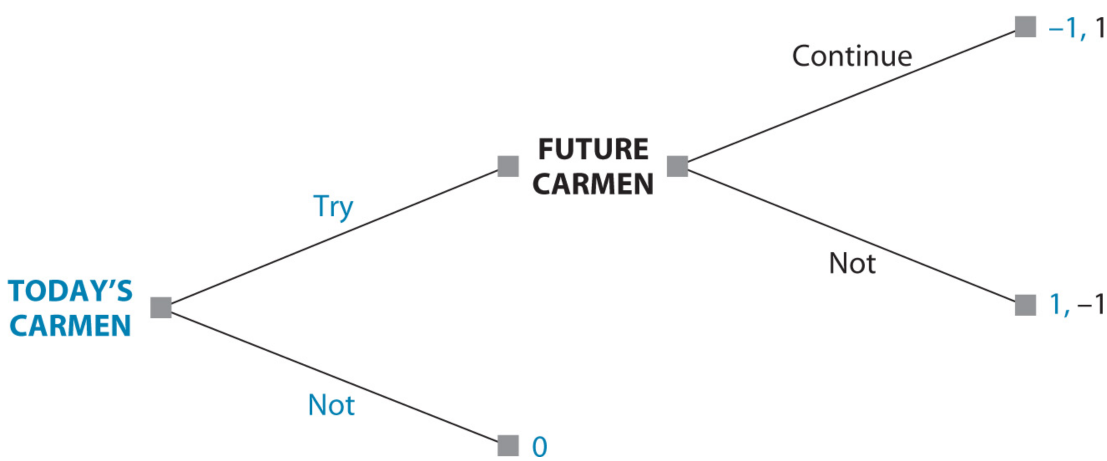


Figure 3.2 The Smoking Game

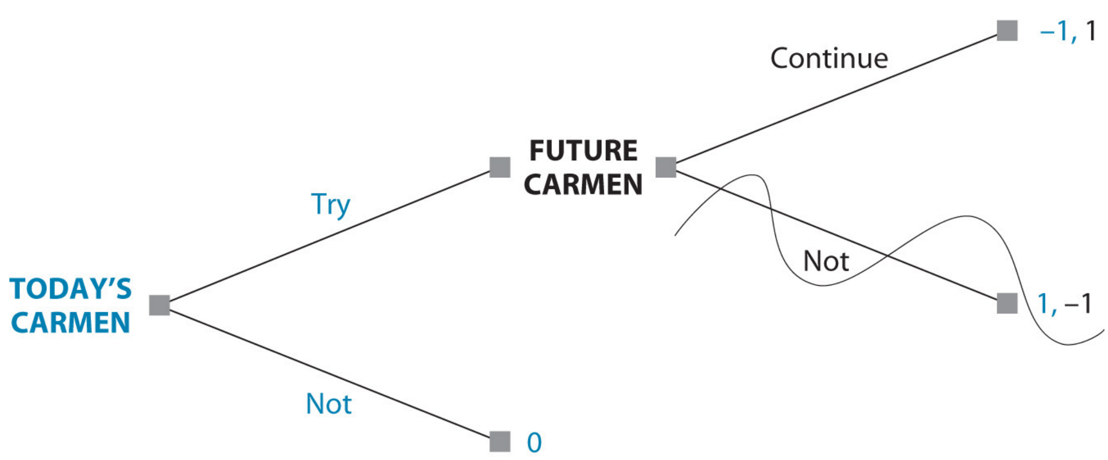
Given the preferences of the addicted Future Carmen, she will choose Continue at her decision node. Today's Carmen should look ahead to that prospect and fold it into her current decision, recognizing that the choice to try smoking will inevitably lead to continuing to smoke. Even though Today's Carmen does not want to continue to smoke in the future, given her preferences today, she will not be able to implement her currently preferred choice in the future because Future Carmen, who has different preferences, will make that choice. So Today's Carmen should foresee that the choice Try will lead to

Continue and get her the payoff −1, as judged by her today, whereas the choice Not will get her the payoff 0. So she should choose Not.

This argument is shown more formally, and with greater visual effect, in Figure 3.3. In Figure 3.3a, we cut off, or prune, the branch Not emerging from the second node. This pruning corresponds to the fact that Future Carmen, who makes the choice at that node, will not choose the action associated with that branch, given her preferences, as shown in black.

The tree that remains has two branches emerging from the first node, where Today's Carmen makes her choice; each of these branches now leads directly to a terminal node. The pruning allows Today's Carmen to forecast completely the eventual consequence of each of her choices. Try will be followed by Continue and will yield a payoff of −1, as judged by the preferences of Today's Carmen, while Not will yield 0. Carmen's choice today should then be Not rather than Try. Therefore, we can prune the Try branch emerging from the first node (along with its foreseeable continuation). This pruning is done in Figure 3.3b. The tree shown there is now fully pruned, leaving only one branch emerging from the initial node and leading to a terminal node. Following the only remaining path through the tree shows what will happen in the game when all players make their best choices with correct forecasting of all future consequences.

(a) Pruning at second node:



(b) Full pruning:

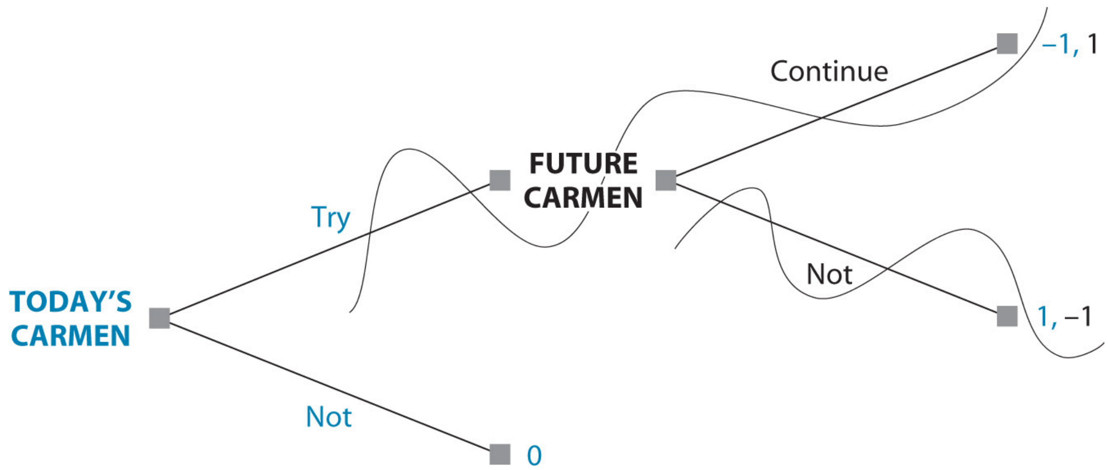


Figure 3.3Pruning the Tree of the Smoking Game

In pruning the tree in Figure 3.3, we crossed out the branches not chosen. An equivalent but alternative way of showing player choices is to highlight the branches that *are* chosen. To do so, we can place check marks or arrowheads on those branches, or show them as thicker lines. Any method will do; Figure 3.4 shows them all. You can choose whether to prune or to highlight, but the latter method, especially in its arrowhead form, has some advantages. First, it produces a cleaner picture. Second, the mess of the pruning picture sometimes does not clearly show the order in which various branches were cut. For example, in Figure 3.3b, a reader may get confused and incorrectly think that the Continue branch at the second node was cut first and that the Try branch at the initial node followed by the Not branch at the second node were cut next. Finally, and most importantly, the arrowheads show the outcome of the sequence of optimal choices most visibly, as a continuous sequence of arrowheads from the initial node to a terminal node. Therefore, in subsequent diagrams of this type, we will generally use highlighting with arrowheads instead of pruning. When you draw game trees, you should practice both methods for a while; when you are comfortable with trees, you can choose either one to suit your taste.

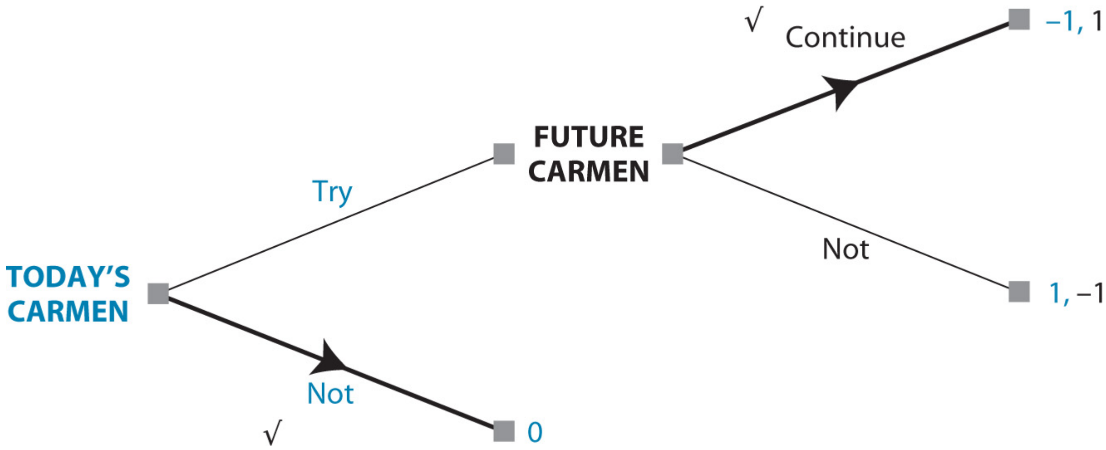


Figure 3.4 Showing Branch Selection on the Tree of the Smoking Game

No matter how you display your thinking in a game tree, the logic of the analysis is the same. You must start your analysis by considering those action nodes that lead directly to terminal nodes. The optimal choices for a player making a move at such a node can be found immediately by comparing her payoffs at the relevant terminal nodes. By using these end-of-game choices to forecast consequences of earlier actions, we can determine the optimal choices at nodes just preceding the final decision nodes. Then the same can be done for the nodes before them, and so on. By working backward along the tree in this way, we can solve the whole game.

This method of looking ahead and reasoning back to determine optimal behavior in sequential-move games is known as rollback. As the name suggests, using rollback requires starting to think about what will happen at each terminal node and literally "rolling back" through the tree to the initial node as you do your analysis. Because this reasoning requires working backward one step at a time, the method is also called backward induction. We use the term *rollback* because it is simpler and is becoming more widely used, but other sources on game theory will use the older term *backward induction*. Just remember that the two are equivalent.

When all players use rollback to choose their optimal strategies, we call this set of strategies the rollback equilibrium of the game; the outcome that arises from playing these strategies is the *rollback equilibrium outcome*. More advanced game-theory texts refer to this concept as *subgame-perfect equilibrium*, and your instructor may prefer to use that term. We will provide more formal explanation and analysis of subgame-perfect equilibrium in Chapter 6, but we generally prefer the simpler and more intuitive term *rollback equilibrium*. Game theory predicts this outcome as the equilibrium of a sequential-move game in which all players are rational calculators in pursuit of their respective best payoffs. Later in this chapter, we will address how well this prediction is borne out in practice. For now, you should know that all finite sequential-move games presented in this book have at least one rollback equilibrium. In fact, most have exactly one. Only in those exceptional cases where a player gets equal payoffs from two or more different sets of moves, and is therefore indifferent among them, will games have more than one rollback equilibrium.

In the smoking game, the rollback equilibrium is the list of strategies where Today’s Carmen chooses the strategy Not and Future Carmen chooses the strategy Continue. When Today’s Carmen takes her optimal action, the addicted Future Carmen does not come into being at all, and therefore gets no actual opportunity to make a move. But Future Carmen’s shadowy presence, and the strategy that she would choose if Today’s Carmen chose Try and gave her an opportunity to move, are important parts of the game. In fact, they are instrumental in determining the optimal move for Today’s Carmen.

We have introduced the ideas of the game tree and rollback in a very simple example, where the solution was obvious from verbal argument. Now we will proceed to use these ideas in successively more complex situations, where verbal analysis becomes harder to conduct and visual analysis with the use of the tree becomes more important.

Glossary

pruning
: Using rollback analysis to identify and eliminate from a game tree those branches that will not be chosen when the game is rationally played.

rollback
: Analyzing the choices that rational players will make at all nodes of a game, starting at the terminal nodes and working backward to the initial node. Also called backward induction.

backward induction
: Same as rollback.

rollback equilibrium
: The strategies (complete plans of action) for each player that remain after rollback analysis has been used to prune all the branches that can be pruned.

3 ADDING MORE PLAYERS

The techniques developed in Section 2 in the simplest setting of two players and two moves can be readily extended. The game trees get more complex, with more branches and nodes, but the basic concepts and the rollback method remain unchanged. In this section, we consider a game with three players, each of whom has two choices; this game, with slight variations, reappears in many subsequent chapters.

The three players, Emily, Nina, and Talia, all live on the same small street. Each has been asked to contribute toward the creation of a flower garden where their small street intersects with the main highway. The ultimate size and splendor of the garden depends on how many of them contribute. Furthermore, although each player is happy to have the garden—and happier as its size and splendor increase—each is reluctant to contribute because of the cost that she must incur to do so.

Suppose that, if two or all three contribute, there will be sufficient resources for the initial planting and subsequent maintenance of the garden; it will then be quite attractive and pleasant. However, if one or none contribute, it will be too sparse and poorly maintained to be pleasant. From each player's perspective, there are thus four distinguishable outcomes:

- She does not contribute, but both of the others do (resulting in a pleasant garden and saving the cost of her own contribution).
- She contributes, and one or both of the others do as well (resulting in a pleasant garden, but incurring the cost of her own contribution).
- She does not contribute, and only one or neither of the others does (resulting in a sparse garden, but saving the cost of her own contribution).
- She contributes, but neither of the others does (resulting in a sparse garden and incurring the cost of her own contribution).

Of these outcomes, the one listed first is clearly the best, and the one listed last is clearly the worst. We want higher payoff numbers to indicate outcomes that are more highly regarded, so we give the first outcome the payoff 4 and the last one the payoff 1. (Sometimes payoffs are associated with an outcome's rank order, so with four outcomes, 1 would be best and 4 worst, and smaller numbers would denote more preferred outcomes. When reading, you should carefully note which convention the author is using; when writing, you should carefully state which convention you are using.)

There is some ambiguity about the two middle outcomes. Let us suppose that each player regards a pleasant garden more highly than her own contribution. Then the outcome listed second gets payoff 3, and the outcome listed third gets payoff 2.

Suppose the players move sequentially. Emily has the first move, and chooses whether to contribute. Then, after observing what Emily has chosen, Nina chooses between contributing and not contributing. Finally, having observed what Emily and Nina have chosen, Talia makes a similar choice.[1]

Figure 3.5 shows the tree for this game. We have labeled the action nodes for easy reference. Emily moves at the initial node, *a*, and the branches corresponding to her two choices, Contribute and Don't, lead to nodes *b* and *c*, respectively. At each of these nodes, Nina gets to move and to choose between Contribute and Don't. Her choices lead to nodes *d*, *e*, *f*, and *g*, at each of which Talia gets to move. Her choices lead to eight terminal nodes, where we list the payoffs for the players in order (Emily, Nina, Talia).[2] For example, if Emily contributes, then Nina does not, and finally Talia does, then the garden is pleasant, and the two contributors each get a payoff of 3, while the noncontributor gets her top outcome, with a payoff of 4; in this case, the payoff list is (3, 4, 3).

To apply rollback to this game, we begin with the action nodes that come immediately before the terminal nodes—namely, *d*, *e*, *f*, and *g*. Talia moves at each of these nodes. At *d*, she faces the situation where both Emily and Nina have contributed. The garden is already assured to be pleasant, so, if Talia chooses Don’t, she gets her best payoff, 4, whereas, if she chooses Contribute, she gets the next best, 3. Her preferred choice at this node is Don’t. We show this preference both by thickening the branch for Don’t and by adding an arrowhead; either one would suffice to illustrate Talia’s choice. At node *e*, Emily has contributed and Nina has not; so Talia’s contribution is crucial for a pleasant garden. Talia gets the payoff 3 if she chooses Contribute and 2 if she chooses Don’t. Her preferred choice at *e* is Contribute. You can check Talia’s choices at the other two nodes similarly.

Now we roll back the analysis to the preceding stage—namely, nodes *b* and *c*, where it is Nina’s turn to choose. At *b*, Emily has contributed. Nina’s reasoning now goes as follows: “If I choose Contribute, that will take the game to node *d*, where I know that Talia will choose Don’t, and my payoff will be 3. (The garden will be pleasant, but I will have incurred the cost of my contribution.) If I choose Don’t, the game will go to node *e*, where I know that Talia will choose Contribute, and I will get a payoff of 4. (The garden will be pleasant, and I will have saved the cost of my contribution.) Therefore, I should choose Don’t.” Similar reasoning shows that at *c*, Nina will choose Contribute.

Finally, consider Emily’s choice at the initial node, *a*. She can foresee the subsequent choices of both Nina and Talia. Emily knows that, if she chooses Contribute, these later choices will be Don’t for Nina and Contribute for Talia. With two contributors, the garden will be pleasant, but Emily will have incurred a cost; so her payoff will be 3. If Emily chooses Don’t, then the subsequent choices will both be Contribute, and, with a pleasant garden and no cost incurred, Emily’s payoff will be 4. So her preferred choice at *a* is Don’t.

The result of rollback analysis for this street-garden game is now easily summarized. Emily will choose Don’t, then Nina will choose Contribute, and finally Talia will choose Contribute. These choices trace a particular path of play through the tree—along the lower branch from the initial node, *a*, and then along the upper branches at each of the two subsequent nodes reached, *c* and *f.* In Figure 3.5, the path of play is easily seen as the continuous sequence of arrowheads joined tail to tip from the initial node to the terminal node fifth from the top of the tree. The payoffs that accrue to the players are shown at this terminal node.

Rollback is a simple and appealing method of analysis. Here, we emphasize some features that emerge from it. First, notice that the equilibrium path of play of a sequential-move game (the one that results in the rollback equilibrium outcome) misses most of the branches and nodes. Calculating the best actions that could be taken if these other nodes were reached, however, is an important part of determining the ultimate equilibrium. Choices made early in the game are affected by players' expectations of what would happen if they chose to do something other than their best actions and by what would happen if any opposing player chose to do something other than what was best for her. These expectations, based on predicted actions at out-of-equilibrium nodes (nodes associated with branches pruned in the process of rollback), keep players choosing optimal actions at each node. For instance, Emily's optimal choice of Don't at the first move is governed by the knowledge that, if she chooses Contribute, then Nina will choose Don't, followed by Talia choosing Contribute; this sequence will give Emily the payoff 3, instead of the 4 that she can get by choosing Don't at the first move.

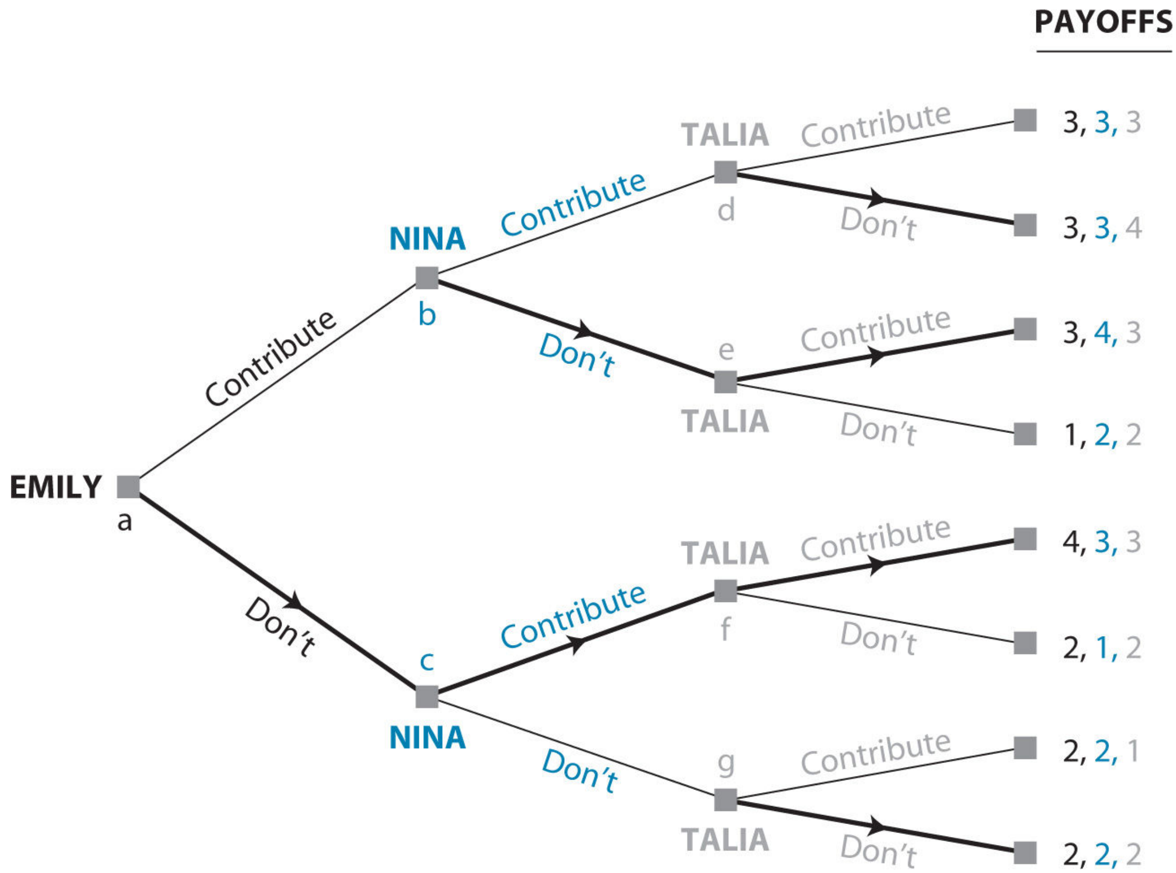


Figure 3.5 The Street-Garden Game

The rollback equilibrium gives a complete statement of all this analysis by specifying the optimal *strategy* for each player. Recall that a strategy is a complete plan of action. Emily moves first and has just two choices, so her strategy is quite simple and is effectively the same thing as her move. Emily has available to her two (complete) strategies, of which her optimal strategy is Don't, as shown in Figure 3.5. But Nina, moving second, acts at one of two nodes: at one if Emily has chosen Contribute and at the other if Emily has chosen Don't. Nina's complete plan of action has to specify what she will do in either case. One such plan, or strategy, might be "Choose Contribute if Emily has chosen Contribute, choose Don't if Emily has chosen Don't." We know from our rollback analysis that Nina will not choose this strategy, but our interest at this point is in describing all the available strategies from which Nina can choose within the rules of the game. We can abbreviate and write C for Contribute and D for Don't; thus this strategy can be written as “C if Emily chooses C so that the game is at node *b*, D if Emily chooses D so that the game is at node *c*,” or, more simply, “C at *b*, D at *c*,” or even “CD” if the circumstances in which each of the stated actions is taken are evident or previously explained. Now it is easy to see that, because Nina has two choices available at each of the two nodes where she might be acting, she has available to her four plans, or (complete) strategies: “C at *b*, C at *c*,” “C at *b*, D at *c*,” “D at *b*, C at *c*,” and “D at *b*, D at *c*”; or “CC,” “CD,” “DC,” and “DD.” Of these strategies, the rollback analysis and the arrowheads at nodes *b* and *c* of Figure 3.5 show that her optimal strategy is DC.

Matters are even more complicated for Talia. When her turn comes, the history of play can, according to the rules of the game, be any one of four possibilities. Talia's turn to act comes at one of four nodes in the tree: the first after Emily has chosen C and Nina has chosen C (node *d*), the second after Emily's C and Nina's D (node *e*), the third after Emily's D and Nina's C (node *f*), and the fourth after both Emily and Nina choose D (node *g*). Each of Talia's strategies, or complete plans of action, must specify one of her two possible actions for each of these four scenarios—that is, at each of her four possible action nodes. With four nodes at which to specify an action and with two actions from which to choose at each node, there are 2 × 2 × 2 × 2, or 16 possible combinations of actions. So Talia has available to her 16 possible (complete) strategies. One of them could be written as

C at *d*, D at *e*, D at *f*, C at *g*,

or CDDC for short, where we have fixed the order of the four scenarios (the histories of moves by Emily and Nina) in the order of nodes *d*, *e*, *f*, and *g*. Then, with the use of the same abbreviations, the full list of 16 strategies available to Talia is

CCCC, CCCD, CCDC, CCDD, CDCC, CDCD, CDDC, CDDD,

DCCC, DCCD, DCDC, DCDD, DDCC, DDCD, DDDC, DDDD.

Of these strategies, the rollback analysis in Figure 3.5 and the arrowheads that follow nodes *d*, *e*, *f*, and *g* show that Talia's optimal strategy is DCCD.

Now we can express the findings of our rollback analysis by stating the strategy choices of each player: Emily chooses D from the 2 strategies available to her, Nina chooses DC from the 4 strategies available to her, and Talia chooses DCCD from the 16 strategies available to her. When each player looks ahead in the tree to forecast the eventual outcomes of her current choices, she is calculating the optimal strategies of the other players. This configuration of strategies—D for Emily, DC for Nina, and DCCD for Talia—then constitutes the rollback equilibrium of the game.

We can put together the optimal strategies of the players to find the equilibrium path of play. Emily will begin by choosing D. Nina, following her strategy DC, chooses the action C in response to Emily's D. (Remember that Nina's DC means "Choose D if Emily has played C, and choose C if Emily has played D.") According to the convention that we have adopted, Talia's actual action after Emily's D and then Nina's C—at node *f*—is the third letter in the four-letter specification of her strategies. Because Talia's optimal strategy is DCCD, her action along the equilibrium path of play is C. Thus the equilibrium path of play consists of Emily playing D, followed successively by Nina and Talia playing C.

To sum up, we have three distinct concepts:

1. The list of available strategies, or complete plans of action, for each player. The list, especially for later players, may be very long because their actions in situations corresponding to all conceivable preceding moves by other players must be specified.
2. The optimal strategy for each player. This strategy must specify the player's *best* choice at each node where the rules of the game specify that she moves, even though many of these nodes will never be reached in the actual path of play. This specification is, in effect, the preceding movers'

forecasting of what would happen if they took different actions and is therefore an important part of their calculations of their own best actions at the earlier nodes. The optimal strategies of all players together yield the rollback equilibrium.

3. The equilibrium path of play, found by putting together the optimal strategies for all the players.

Endnotes

- In later chapters, we vary the rules of this game—the order of moves and payoffs—and examine how such variation changes the outcomes. Return to reference 1
- Recall from the discussion of game trees in Section 1 that the usual convention for sequential-move games is to list payoffs in the order in which the players move; however, in case of ambiguity, or simply for clarity, it is good practice to specify the order explicitly. Return to reference 2

Glossary

path of play
: A route through the game tree (linking a succession of nodes and branches) that results from a configuration of strategies for the players that are within the rules of the game. (See also *equilibrium path of play*.)

equilibrium path of play
: The *path of play* actually followed when players choose their rollback equilibrium strategies in a sequential game.

4 ORDER ADVANTAGES

In the rollback equilibrium of the street-garden game, Emily gets her best outcome (payoff 4) because she can take advantage of the opportunity to make the first move. When she chooses not to contribute, she puts the onus on the other two players—each of whom can get her next best outcome if and only if both of them choose to contribute. Most casual thinkers about strategic games have the preconception that such a first-mover advantage should exist in all games. However, that is not the case. It is easy to think of games in which an opportunity to move second is an advantage. Consider the strategic interaction between two firms that sell similar merchandise from catalogs—say, Lands' End and L.L.Bean. If one firm had to release its catalog first, and the second firm could see what prices the first had set before printing its own catalog, then the second mover could undercut its rival on all items and gain a tremendous competitive edge.

First-mover advantage comes from the ability to commit oneself to an advantageous position and to force the other players to adapt to it; second-mover advantage comes from the flexibility to adapt oneself to the others' choices. Whether commitment or flexibility is more important in a specific game depends on the particular details of the players' strategies and payoffs, and we will come across examples of both kinds of advantages throughout this book. The general point that there need not be a first-mover advantage—a point that runs against much common perception—is so important that we felt it necessary to emphasize at the outset.

When a game has a first- or second-mover advantage, each player may try to manipulate the order of play so as to secure for herself the advantageous position. We will discuss a player’s ability to change the order of moves in greater detail in Chapter 6, and we will take up the manipulation of a game’s order of play by way of strategic moves in Chapter 8.

Glossary

first-mover advantage
: This exists in a game if, considering a hypothetical choice between moving first and moving second, a player would choose the former.

second-mover advantage
: A game has this if, considering a hypothetical choice between moving first and moving second, a player would choose the latter.

5 ADDING MORE MOVES

We saw in Section 3 that adding more players increases the complexity of the analysis of sequential-move games. In this section, we consider another type of complexity, which arises from adding more moves to the game. We can do so most simply in a two-person game by allowing players to alternate moves more than once. In this case, the tree is enlarged in the same fashion that a multiple-player game tree would be, but later moves in the tree are made by the same players who have made decisions earlier in the game.

Many common board games, such as tic-tac-toe, checkers, chess, and Go, are two-person strategic games with such alternating sequential moves. The use of game trees and rollback should allow us to solve such games—that is, to determine the rollback equilibrium outcome and the optimal strategies leading to that outcome. Unfortunately, as the complexity of the game grows, and as strategies become more and more intricate, the search for optimal strategies and equilibria becomes more and more difficult as well. Fortunately, or perhaps unfortunately for the humans who used to be experts and champions at these games, computer science has evolved to a point where artificial intelligence (AI) methods can go where no human has been before. In this section, we give a brief overview of the search for solutions to such games.

A. Tic-Tac-Toe

Let' s start with the simplest of the four examples mentioned in the preceding paragraph, tic-tac-toe, and consider an easier-than-usual version in which each of two players (X and O) tries to be the first to get two of their symbols to fill any row, column, or diagonal of a two-by-two game board. The first player has four possible actions, or positions in which to put her X. The second player then has three possible actions at each of four decision nodes. When the first player gets to her second turn, she has two possible actions at each of 12 (4 × 3) decision nodes. As Figure 3.6 shows, even this mini-game of tic-tac-toe has quite a complex game tree. This tree is actually not too complex, because the game is guaranteed to end after the first player moves a second time, but there are still 24 terminal nodes to consider.

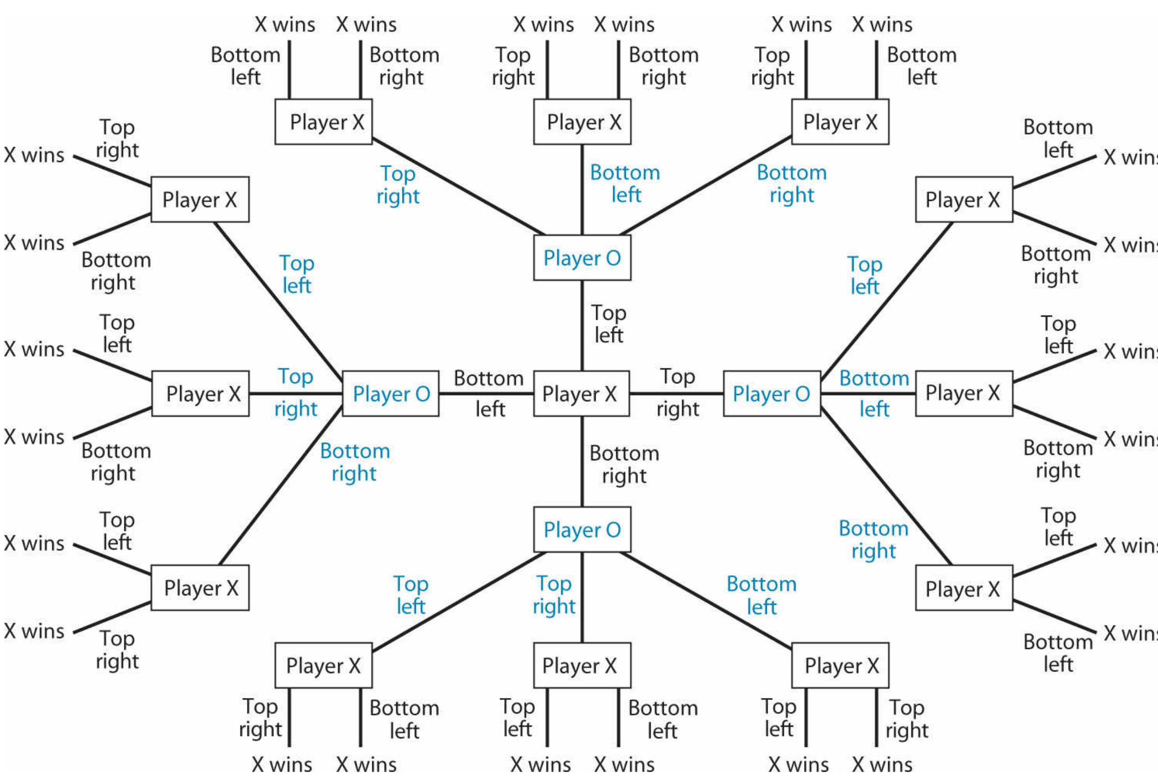


Figure 3.6 The Complex Tree for Simple Two-by-Two Tic-Tac-Toe

We show this tree merely as an illustration of how complex game trees can become even in simple (or simplified) games. As it turns out, using rollback on this mini-game of tic-tac-toe leads us quickly to an equilibrium. Rollback shows that all the choices for the first player at her second move lead to the same outcome. There is no optimal action; any move is as good as any other move. Thus, when the second player makes her first move, she also sees that each possible move yields the same outcome, and she, too, is indifferent among her three choices at each of her four decision nodes. Finally, the same is true for the first player on her first move; any choice is as good as any other, so she is guaranteed to win the game.

Although this version of tic-tac-toe has an interesting tree, its solution is not as interesting. The first mover always wins, so no choices made by either player can affect the ultimate outcome. Most of us are more familiar with the three-by-three version of tic-tac-toe. To illustrate that version with a game tree, we would have to show that the first player has nine possible actions at the initial node; that the second player has eight possible actions at each of nine decision nodes; that the first player, on her second turn, has seven possible actions at each of $8 \times 9 = 72$ nodes; and that the second player, on her second turn, has six possible actions at each of $7 \times 8 \times 9 = 504$ nodes. This pattern continues until eventually the tree stops branching so rapidly because certain combinations of moves lead to a win for one player, and the game ends. But no win is possible until at least the fifth move. Drawing the complete tree for this game requires a very large piece of paper or very tiny handwriting.

Most of you know, however, how to achieve, at worst, a tie when you play three-by-three tic-tac-toe. So there is a simple solution to this game that can be found by rollback, and a learned strategic thinker can reduce the complexity of the game considerably in the quest for such a solution. It turns out that, as in the two-by-two version, many of the possible paths through the game tree are strategically identical. Of the nine possible initial moves, there are only three types: You put your X in a corner position (of which there are four possibilities), a side position (of which there are also four possibilities), or the (one) middle position. Using this method to simplify the tree can help reduce the complexity of the problem and lead you to a rollback equilibrium. Specifically, we can show that the player who moves second can always guarantee at least a tie by choosing an appropriate first move and then by continually blocking the first player's attempts to get three symbols in a row.[3]

B. Checkers

Although relatively small games, such as tic-tac-toe, can be solved using rollback, we saw in Figure 3.6 how rapidly the complexity of game trees can increase, even in two-player games, as the number of possible moves increases. Thus, when we consider more complex games, finding a complete solution becomes much more difficult. Consider checkers, a two-player game played on an eight-by-eight board in which each player has 12 round game pieces of different colors, as shown in Figure 3.7. Players take turns moving their pieces diagonally on the board, jumping (and capturing) the opponent's pieces when possible. The game ends, and Player A wins, when Player B is either out of pieces or unable to move; the game can also end in a draw if both players agree that neither can win. There are about 5×10^{20} possible arrangements of pieces on the board, so drawing a game tree is out of the question.

Conventional wisdom and evidence from world checkers championships over many years suggested that good play should lead to a draw, but there was no proof. That is, not until 2007, when a computer scientist in Canada proved that a computer program named Chinook could play to a guaranteed tie. Checkers was solved.[4]

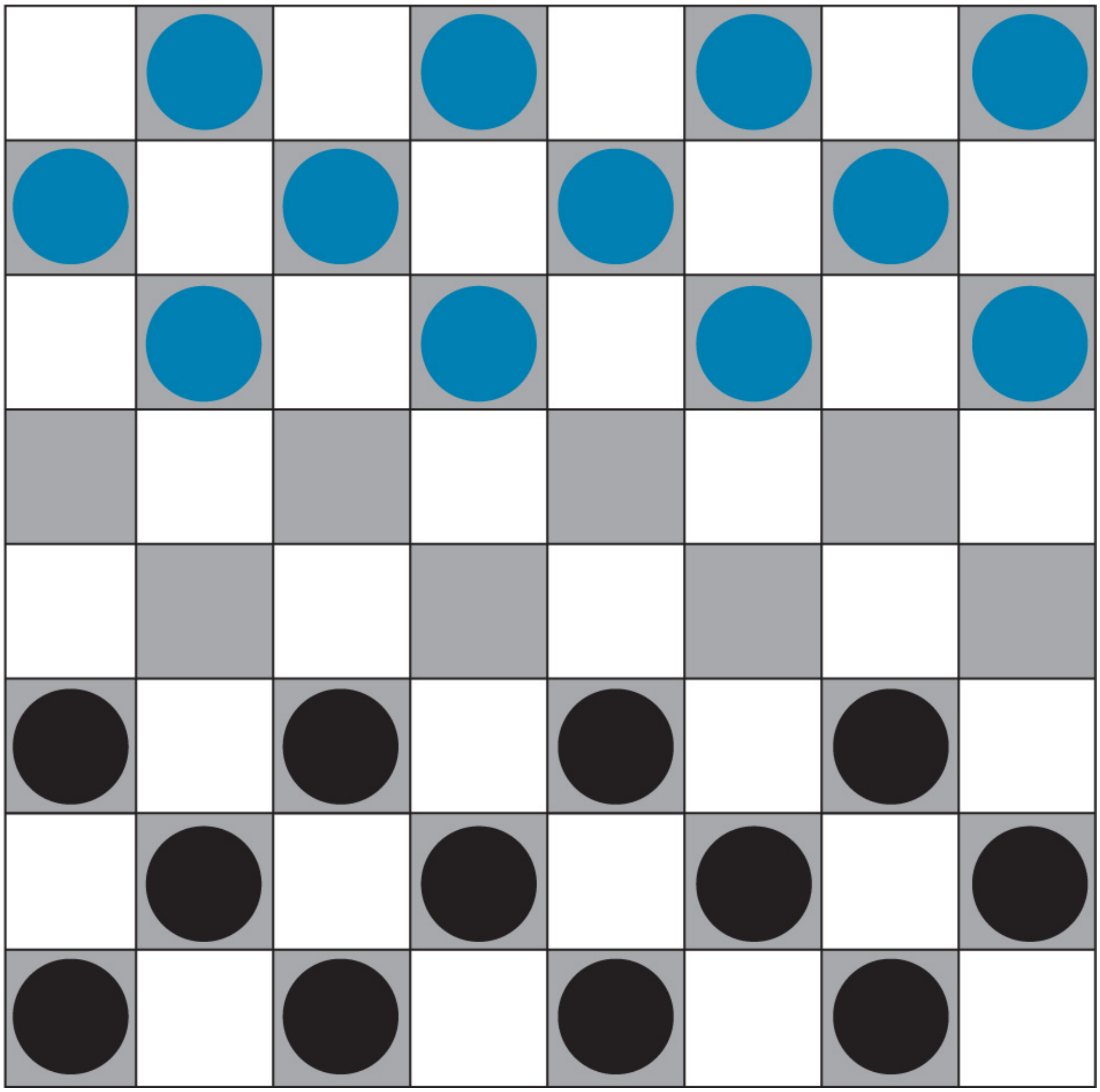


Figure 3.7 Checkerboard

Chinook, first created in 1989, played the world champion, Marion Tinsley, in 1992 (losing 4 to 2, with 33 draws) and again in 1994 (when Tinsley’s health failed during a series of draws). It was put on hold between 1997 and 2001 while its creators waited for computer technology to improve. And it finally exhibited a loss-proof algorithm in the spring of 2007. That algorithm uses a combination of endgame rollback analysis and starting position forward analysis, along with calculations designed to determine the value of intermediate positions, to trace out the best moves within a database including all possible positions on the board.

The creators of Chinook describe the full game of checkers as "weakly solved"; they know that they can generate a tie, and they have a strategy for reaching that tie from the start of the game. For all 39×10^{12} possible positions that include 10 or fewer pieces on the board, they describe checkers as "strongly solved"; not only do they know they can play to a tie, but they can reach that tie from any of the possible positions that can arise once only 10 pieces remain. Their algorithm first solved these 10-piece endgames, then went back to the start to search out paths of play in which both players make optimal choices. The search mechanism, involving a complex system of evaluating the value of each intermediate position, invariably led to those 10-piece positions that generate a draw.

Thus, our hope for the future of rollback analysis may not be misplaced. We know that for very simple games, we can find the rollback equilibrium by verbal reasoning without having to draw the game tree explicitly. For games having an intermediate range of complexity, verbal reasoning is too hard, but a complete tree can be drawn and used for rollback. Sometimes we can enlist the aid of a computer to draw and analyze a moderately complicated game tree. For the most complex games, such as checkers (and chess and Go, as we will see), we can draw only a small part of the game tree, and we must use a combination of two methods: (1) calculation based on the logic of rollback, and (2) rules of thumb for valuing intermediate positions on the basis of experience. The computational power of current algorithms has shown that even some games in this category are amenable to solution, provided one has the time and resources to devote to the problem.

C. Chess

In chess, each of the players, White and Black, has a collection of 16 pieces in six distinct shapes, each of which is bound by specified rules of movement on the eight-by-eight game board shown in Figure 3.8.[5] White opens with a move, Black responds with one, and so on, in turns. All the moves are visible to the other player, and nothing is left to chance, as it would be in card games that include shuffling and dealing. Moreover, a chess game must end in a finite number of moves. The rules declare that a game is drawn if a given position on the board is repeated three times in the course of play. Because there are a finite number of ways to place the 32 (or fewer after captures) pieces on 64 squares, a game could not go on infinitely without running up against this rule. Therefore, in principle, chess is amenable to full rollback analysis.[6]

That rollback analysis has not been carried out, however. Chess has not been solved, as tic-tac-toe and checkers have been. And the reason is that, for all its simplicity of rules, chess is a bewilderingly complex game. From the initial set position of the pieces, illustrated in Figure 3.8, White can open with any one of 20 moves,[7] and Black can respond with any of 20. Therefore, 20 branches emerge from the first node of the game tree, each leading to a second node, from each of which 20 more branches emerge. After only two moves, there are already 400 branches, each leading to a node from which many more branches emerge. And the total number of possible moves in chess has been estimated to be 10^{120}, or a one with 120 zeroes after it. A supercomputer a thousand times as fast as your PC, making a trillion calculations a second, would need more than 10^{100} years to check out all these moves.[8] Astronomers offer us less than 10^{10} years before the sun turns into a red giant and swallows the earth.

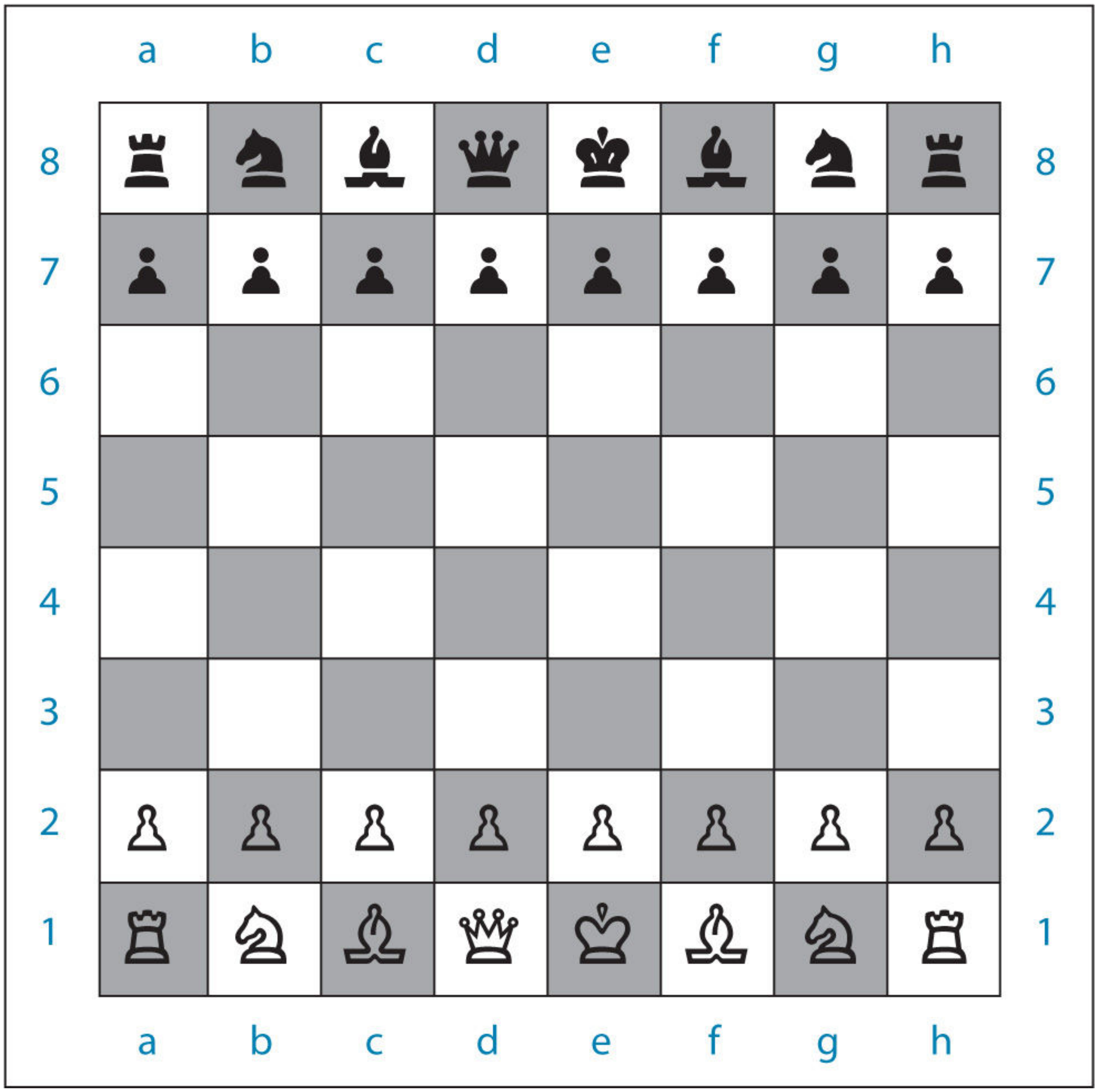


Figure 3.8 Chessboard

The general point is that, although a game may be amenable in principle to solution by rollback, its tree may be too complex to permit such a solution in practice. Faced with such a situation, what is a player to do? We can learn a lot about this problem by reviewing the history of attempts to program computers to play chess.

When computers first started to prove their usefulness for complex calculations in science and business, many mathematicians and computer scientists thought that a chess-playing computer program would soon beat the world champion. It took a lot longer, because even though computer technology improved rapidly, human thought progressed much more slowly. Finally, in December 1992, a German chess program called Fritz2 beat world champion Gary Kasparov in some blitz (high-speed, time-limited) games. Under regular rules, where each player gets two and a half hours to make 40 moves, humans retained their superiority longer. A team sponsored by IBM put a lot of effort and resources into the development of a specialized chess-playing computer and its associated software. In February 1996, this package, called Deep Blue, was pitted against Kasparov in a best-of-six series. Deep Blue caused a sensation by winning the first game, but Kasparov quickly figured out its weaknesses, improved his counterstrategies, and won the series handily. In the next 15 months, the IBM team improved Deep Blue's hardware and software, and the resulting Deeper Blue beat Kasparov in another best-of-six series in May 1997.

To sum up, chess-playing computers progressed in a combination of slow patches and some rapid spurts, while human players held some superiority, but were not able to improve sufficiently fast to keep ahead. Closer examination reveals that the two used quite different approaches to think through the very complex game tree of chess.

When contemplating a move in chess, looking ahead to the end of the whole game is too hard (for humans and computers both). How about looking part of the way—say, 5 or 10 moves ahead—and working back from there? The game need not end within this limited horizon; that is, the nodes that you reach after 5 or 10 moves will not generally be terminal nodes. Only terminal nodes have payoffs specified by the rules of the game. Therefore, you need some indirect way of assigning plausible payoffs to nonterminal nodes if you are not able to explicitly roll back from a full look-ahead. A rule that assigns such payoffs is called an intermediate valuation function.

In chess, humans and computer programs have both used such partial look-ahead in conjunction with an intermediate valuation function. The typical method assigns a numerical value to each piece and to positional and combinational advantages that can arise during play. Values for different positions are generally quantified on the basis of the whole chess-playing community’s experience of play in past games starting from such positions or patterns; this experience is called *knowledge*. The sum of all the numerical values attached to pieces and their combinations in a position is the intermediate value of that position. A move is judged by the value of the position to which it is expected to lead after an explicit forward-looking calculation for a certain number—say, 5 or 6—of moves.

The valuation of intermediate positions has progressed furthest with respect to chess openings—that is, the first dozen or so moves of a game. Each opening can lead to any one of a vast multitude of further moves and positions, but experience enables players to sum up certain openings as being more or less likely to favor one player or the other. This knowledge has been written down in massive books of openings, and all top players and computer programs remember and use this information.

At the end stages of a game, when only a few pieces are left on the board, rollback is often simple enough to yield a complete solution from that point forward in the game. The midgame, when positions have evolved to a level of complexity that will not simplify within a few moves, is the hardest to analyze. To find a good move from a midgame position, a well-built intermediate valuation function is likely to be more valuable than the ability to calculate another few moves further ahead. Gradually, computer scientists improved their chess-playing programs by building on this knowledge. When modifying Deep Blue in 1996 and 1997, IBM enlisted the help of human experts to improve the intermediate valuation function in its software. These consultants played repeatedly against the machine, noted its weaknesses, and suggested how the valuation function should be modified to correct the flaws. Deep Blue benefited from the contributions of the experts and their subtle kind of thinking, which results from long experience and an awareness of complex interconnections among the pieces on the board.

The art of the midgame in chess is also an exercise in recognizing and evaluating patterns, something humans have always excelled at, but computer scientists used to find it difficult to program into exact algorithms. This is where Kasparov had his greatest advantage over Fritz2 or Deep Blue. It also explains why computer programs do better against humans at blitz or limited-time games: The humans do not have the time to marshal their art of the midgame.

This state of affairs has changed dramatically in the last decade with the advent of artificial intelligence, including machine learning and deep learning. Instead of having programs that perform one very specific calculation, computers are programmed to develop knowledge and to learn to perform broader sets of tasks. Machine learning "trains" the computer by feeding it huge quantities of relevant data, allowing its algorithms to draw and test inferences and, gradually, to improve its ability. Perhaps the best example is in facial recognition, where deep learning improves the process by constructing the machine as an artificial neural network, mimicking the neural connection structure of the brain.

Applying these techniques to chess means having a machine play against itself numerous times in order to build its own "knowledge" and "art" of the game and to improve its strategies.[9] So successful is this new method that in December 2017, AlphaZero, a game-playing AI program created by Deep Mind (another subsidiary of Google's parent company, Alphabet), achieved an amazing feat. Programmers gave it only the formal rules of chess and fed in no prior knowledge at all. Then, after only four hours of playing against itself, AlphaZero beat Stockfish 8 (the then-world-champion specialized chess program) in a 100-game match without losing a single game, winning 28 and drawing 72![10] Similar AI techniques have been applied to other games, including Go, which we consider next.

D. Go

The complexity of chess may be mind-boggling, but it pales in comparison with that of Go. In this two-player sequential-move game, played on a 19-by-19 board (although some simpler variants use smaller boards), each player has a large supply of tokens (“stones”) of one color, usually black for one and white for the other. The players take turns placing one stone at a time at an empty intersection of the grid lines that divide the board. There are additional rules about capturing and removing the opponent’s stones, and about ending and scoring the game. Figure 3.9 shows a board and a game in progress.

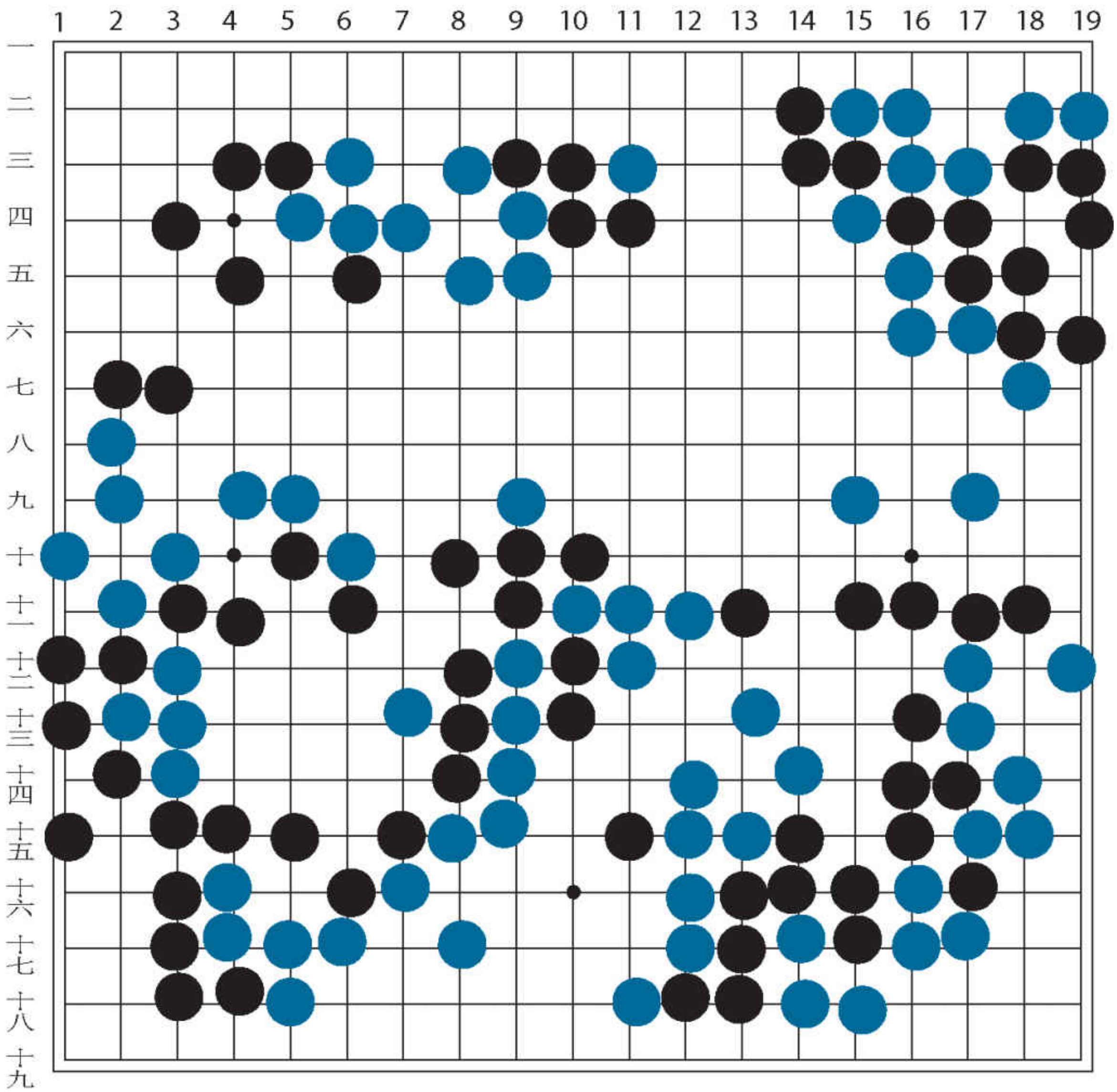


Figure 3.9 Go Game Board, © Natalya Erofeeva/Shutterstock

It has been estimated that the number of board positions in Go is about 10^{172}, and that a game tree would have about 10^{360} nodes. By contrast, chess, with its 10^{120} possible moves, may seem almost trivial! For a long time, it was believed that human skills at pattern recognition and knowledge development would always give people an advantage in this game against the brute-force calculations of a computer. But in the last two or three years, artificial intelligence has progressed to a level where it can successfully challenge top players. Much of this progress has been achieved by AlphaGo, a special deep-learning neural network created by Deep Mind for playing Go. In October 2015, AlphaGo beat Europe’s Go champion, Fan Hui, 5 to 0. In March 2016, it beat Lee Sedol, one of the world’s top players, 4 to 1. And in 2017, it won a three-game match against Ke Jie, who at the time had been the world’s top-ranked player for two successive years. AlphaGo benefited from “reinforcement learning” by being fed data from human play and then improving by playing against itself. Then, later in 2017, an improved version called AlphaGoZero (similar to the chess-playing AlphaZero), using skills attained from playing against itself for a mere eight hours, beat AlphaGo by 60 games to 40 in a 100-game match.[11] It seems that human superiority at Go is Go-ne.

Thankfully, most of the strategic games that we encounter in economics, politics, sports, business, and daily life are far less complex than Go, chess, or even checkers. These games may have a number of players who move a number of times; they may even have a large number of players or a large number of moves. But we have a chance at being able to draw a reasonable-looking tree for those games that are sequential in nature. The logic of rollback remains valid, and once you understand the idea of rollback, you can often carry out the necessary logical thinking and solve the game without explicitly drawing a tree. Moreover, it is precisely at this intermediate level of difficulty, between the simple examples that we solved explicitly in this chapter and the insoluble cases such as chess, that computer software such as Gambit is most likely to be useful; this is indeed fortunate for the prospect of applying the theory to solve many games in practice.

Endnotes

- If the first player puts her first symbol in the middle position, the second player must put her first symbol in a corner position. Then the second player can guarantee a tie by taking the third position in any row, column, or diagonal that the first player tries to fill. If the first player goes to a corner or a side position first, the second player can guarantee a tie by going to the middle first and then following the same blocking technique. Note that if the first player picks a corner, the second player picks the middle, and the first player then picks the corner opposite from her original play, then the second player must not pick one of the remaining corners if she is to ensure at least a tie. For a beautifully detailed picture of the complete contingent strategy in tic-tac-toe, see the online comic strip at http://xkcd.com/832/. Return to reference 3
- Our account is based on two reports in the journal *Science*. See Adrian Cho, "Program Proves That Checkers, Perfectly Played, Is a No-Win Situation," *Science*, vol. 317 (July 20, 2007), pp. 308–9, and Jonathan Schaeffer, Neil Burch, Yngvi Björnsson, Akihiro Kishimoto, Martin Müller, Robert Lake, Paul Lu, and Steve Sutphen, "Checkers Is Solved," *Science*, vol. 317 (September 14, 2007), pp. 1518–22. Return to reference 4
- An easily accessible statement of the rules of chess and much more is at Wikipedia, at http://en.wikipedia.org/wiki/Chess. Return to reference 5
- In fact, John Von Neumann, the polymath who was one of the pioneers of game theory, called chess merely "a well-defined form of computation," and wanted to reserve the name "game theory" for the analysis of "bluffing, . . . little tactics of deception, . . . asking yourself what is the other man going to think I mean to do." See William Poundstone, *Prisoner's Dilemma* (New York: Anchor Books, 1992), p. 6. Return to reference 6
- He can move one of the eight pawns forward either one square or two, or he can move one of the two knights in one of two

ways (to square a3, c3, f3, or h3). Return to reference 7

- This would have to be done only once, because after the game has been solved, anyone will be able to use the solution and no one will actually need to play. Everyone will know whether White has a win or whether Black can force a draw. Players will toss a coin to decide who gets which color. They will then know the outcome, shake hands, and go home. Return to reference 8
- This process was perhaps foreshadowed in the 1983 movie *War Games,* where a military computer, by playing multiple times against itself, learns that in the game of Global Thermonuclear War, both sides always lose. Or, as the computer famously put it, "The only winning move is not to play." Return to reference 9
- "AlphaZero AI Beats Champion Chess Program after Teaching Itself in Four Hours," *The Guardian,* December 7, 2017. Return to reference 10
- For more information on these AI accomplishments, see https://en.wikipedia.org/wiki/DeepMind, and the article referenced in note 10. Return to reference 11

Glossary

intermediate valuation function

A rule assigning payoffs to nonterminal nodes in a game. In many complex games, this must be based on knowledge or experience of playing similar games, instead of explicit rollback analysis.

6 EVIDENCE CONCERNING ROLLBACK

As we have seen, rollback calculations may be infeasible in some complex multistage games. But do actual participants, even in simpler sequential-move games, act as rollback reasoning predicts? Classroom and research experiments with some games have yielded outcomes that appear to counter the predictions of game theory. Some of these experiments and their outcomes have interesting implications for the strategic analysis of sequential-move games.

Many experimenters have had subjects play a single-round bargaining game in which two players, designated A and B, are chosen from a class or a group of volunteers. The experimenter provides a dollar (or some known total amount of money), which can be divided between the players according to the following procedure: Player A proposes a split—for example, "75% to me, 25% to B." If Player B accepts this proposal, the dollar is divided as proposed by A. If B rejects the proposal, neither player gets anything. Because A is, in effect, offering B a stark choice of "this or nothing," the game is known as the *ultimatum game*.

Rollback in this case predicts that B should accept any sum, no matter how small, because the alternative payoff is even worse—namely, $0—and that A, foreseeing this, should propose "99% to me, 1% to B." This particular split almost never happens. Most players assigned the A role propose a much more equal split. In fact, 50:50 is the single most common proposal. Furthermore, most players assigned the B role turn down proposals that leave them with 25% or less of the total and walk away with nothing; some reject proposals that would give them 40% of the pie.[12]

Many game theorists remain unpersuaded that these findings undermine the theory. They counter with some variant of the following argument: "The sums are so small as to make the whole thing trivial in the players' minds. The B players lose 25 or 40 cents, which is almost nothing, and perhaps gain some private satisfaction that they walked away from a humiliatingly small award. If the total were a thousand dollars, so that 25% of it amounted to real money, the B players would accept." But this argument does not seem to be valid. Experiments with much larger stakes show similar results. The findings from experiments conducted in Indonesia with sums that were small in dollars, but amounted to as much as three months' earnings for the participants, showed no clear tendency on the part of the A players to make less equal offers, although the B players tended to accept somewhat smaller shares as the sums increased; similar experiments conducted in the Slovak Republic found the behavior of inexperienced players unaffected by large changes in payoffs.[13]

The participants in these experiments typically have no prior knowledge of game theory and no special computational abilities. But the game is extremely simple; surely even the most naive player can see through the reasoning, and players' answers to direct questions after the experiment generally show that most of them do. The results show not so much the failure of rollback as the theorist's error in supposing that each player cares only about her own money earnings. People have an innate sense of fairness; most societies instill in their members such a sense, which causes the B players to reject anything that is grossly unfair. Anticipating this, the A players offer relatively equal splits.

Supporting evidence comes from the new field of neuroeconomics. Alan Sanfey and his colleagues took MRI readings of players' brains as they made their choices in the ultimatum game. They found stimulation of "activity in a region well known for its involvement in negative emotion" in the brains of responders (B players) when they rejected "unfair" (less than 50:50) offers. Thus, deep instincts or emotions of anger and disgust seem to be implicated in these rejections. They also found that "unfair" (less than 50:50) offers were rejected less often when responders knew that the offerer was a computer than when they knew that the offerer was human.[14]

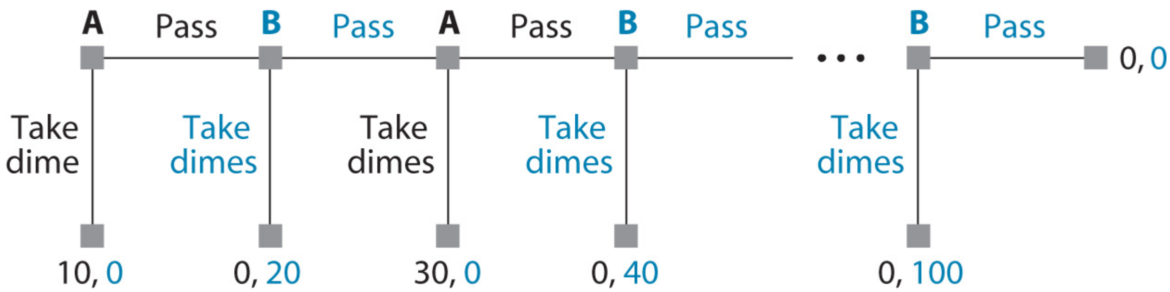


Figure 3.10 The Centipede Game

But this sense of fairness also has a social or cultural component, because different societies exhibit it to different degrees. A group of researchers carried out identical experiments, including the ultimatum game, in 15 societies. In none was the rollback equilibrium with purely selfish behavior observed, but the extent of departure from it varied widely. Most interestingly, "the higher the degree of market integration and the higher the payoffs to cooperation in everyday life, the greater the level of prosociality expressed in experimental games."[15]

Notably, A players have some tendency to be generous even without the threat of retaliation. In a drastic variant of the ultimatum game, called the *dictator game*, where the A player decides on the split and the B player has no choice at all, many As still give significant shares to the Bs, suggesting that the players have some intrinsic preference for relatively equal splits.[16] However, the offers by the A players are noticeably less generous in the dictator game than in the ultimatum game, suggesting that a credible fear of retaliation is also a strong motivator. Other people's perceptions of ourselves also appear to matter. When the experimental design is changed so that not even the experimenter can identify who proposed (or accepted) the split, the extent of sharing drops noticeably.

Another experimental bargaining game with similarly paradoxical outcomes goes as follows: Two players are chosen and designated A and B. The experimenter puts a dime on the table. Player A can take it or pass. If A takes the dime, the game is over, with A getting the 10 cents and B getting nothing. If A passes, the experimenter adds a dime, and now B has the choice of taking the 20 cents or passing. The turns alternate, and the pile of money grows until reaching some limit—say, a dollar—that is known in advance by both players.

We show the tree for this game in Figure 3.10. Because of the appearance of the tree, this type of game is often called the *centipede game*. You may not even need the tree to use rollback on this game. Player B is sure to take the dollar at the last stage, so A should take the 90 cents at the penultimate stage, and so on. Thus, A should take the very first dime and end the game.

In experiments, however, the centipede game typically goes on for at least a few rounds. Remarkably, by behaving "irrationally," the players as a group make more money than they would if they followed the logic of backward reasoning. Sometimes A does better and sometimes B does, but sometimes they even solve this conflict or bargaining problem. In a classroom experiment that one of us (Dixit) conducted, the game went all the way to the end. Player B collected the dollar, and quite voluntarily gave 50 cents to Player A. Dixit asked A, "Did you two conspire? Is B a friend of yours?" and A replied, "No, we didn't even know each other before. But he is a friend now." We will come across some similar evidence of cooperation that seems to contradict backward reasoning when we look at finitely repeated prisoners' dilemma games in Chapter 10.

The centipede game points out a possible problem with the logic of rollback in non-zero-sum games, even for players who care only about their monetary payoffs. Note that if Player A passes in the first round, he has already shown himself not to be playing rollback. So what should Player B expect him to do in round 3? Having passed once, he might pass again, which would make it rational for Player B to pass in round 2. Eventually, someone will take the pile of money, but an initial deviation from rollback equilibrium makes it difficult to predict exactly when this will happen. And because the size of the pie keeps growing, if I see you deviate from rollback equilibrium, I might want to deviate as well, at least for a little while. A player might deliberately pass in an early round in order to signal a willingness to pass in future rounds. This problem does not arise in zero-sum games, where there is no incentive to cooperate by passing.

Steven Levitt, John List, and Sally Sadoff, who conducted experiments with world-class chess players, found more rollback behavior in zero-sum sequential-move games than in the non-zero-sum centipede game. Their centipede game involved 6 nodes, with total payoffs increasing quite steeply across rounds.[17] While there are considerable gains to players who can manage to pass back and forth to each other, the rollback equilibrium specifies playing Take at each node. In stark contrast to the predictions of game theory, only 4% of players played Take at node 1, providing little support for rollback equilibrium even in this simple six-move game. (The fraction of players who played Take increased over the course of the game.[18])

By contrast, in a zero-sum sequential-move game whose rollback equilibrium involves 20 moves (you are invited to solve such a game in Exercise S7), the chess players played the exact rollback equilibrium 10 times as often as in the six-move centipede game.[19] Levitt and his coauthors also experimented with a similar but more difficult zero-sum game (a version of which you are invited to solve in Exercise U5). There, the chess players played the complete rollback equilibrium only 10% of the time (20% for the highest-ranked grandmasters), although by the last few moves, the agreement with rollback equilibrium was nearly 100%. Given that world-class chess players spend tens of thousands of hours trying to win chess games using rollback, these results indicate that even highly experienced players usually cannot immediately carry their experience over to a new game; they need a little experience with the new game before they can figure out the optimal strategy. An advantage of learning game theory is that you can more easily spot underlying similarities between seemingly different situations and thus devise good strategies more quickly in any new games you may face.

Some other instances of non-optimal, short-sighted, or even irrational behavior are observed, not in games involving two or more people, but in one person’s decisions over time, which amount to games between one's current self and future self. For example, when issuers of credit cards offer favorable initial interest rates or no fees for the first year, many people fall for these offers without realizing that they may have to pay much more later. And when people start new jobs, they may be offered a choice among several savings plans, and fully intend to sign up for one, but keep postponing the action. Psychologists and behavioral economists have offered various explanations for these observations: a tendency to discount the immediate future much more heavily than the longer-term future (so-called *hyperbolic discounting*), or framing of choices that implicitly favors inaction or the status quo. Some of these tendencies can be countered using different framing or "nudges" toward the choice that one's future self would prefer. For example, the default for new employees could be a modest savings plan, and an employee who actually did not want to save would have to take action to drop out of that plan.[20] Thus, game-theoretic analysis of rollback and rollback equilibria can serve an advisory or prescriptive role as much as it does a descriptive role. People equipped with an understanding of rollback are in a position to make better strategic decisions and to get higher payoffs, no matter what they include in their payoff calculations. And game theorists can use their expertise to give valuable advice to those who are placed in complex strategic situations but lack the skill to determine their own best strategies.

To sum up, we have seen several reasons and explanations for observed departures from rollback behavior. Section 5 argued that rollback equilibrium may be too complex to compute in games like chess and Go, and other look-ahead methods may have to be devised. In this section, we learned that people's preferences (which determine their payoffs in games) may include additional factors, such as a concern about fairness or about other people's welfare. People may also procrastinate or choose inaction to favor the status quo. Or they may simply be inexperienced and miscalculate, in which case their actions may come into better conformity with rollback logic as their experience of using it in a game increases.

Thus, the theory has three types of uses: (1) normative, telling us how a player with certain stipulated objectives should behave to achieve those objectives as fully as possible—in other words, what their “rational behavior” is; (2) descriptive, telling us how experienced players with known objectives will behave; and (3) prescriptive, telling players how to behave in light of their own cognitive and other limitations, and yielding suggestions for policies like nudges.

Endnotes

- For a detailed account of this game and related ones, read Richard H. Thaler, "Anomalies: The Ultimate Game," *Journal of Economic Perspectives*, vol. 2, no. 4 (Fall 1988), pp. 195–206; and Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton, NJ: Princeton University Press, 1993), pp. 263–69. Return to reference 12
- The results of the Indonesian experiment are reported in Lisa Cameron, "Raising the Stakes in the Ultimatum Game: Experimental Evidence from Indonesia," *Economic Inquiry*, vol. 37, no. 1 (January 1999), pp. 47–59. Robert Slonim and Alvin Roth report results similar to Cameron's, but they also found that offers (in all rounds of play) were rejected less often as the payoffs were raised. See Robert Slonim and Alvin Roth, "Learning in High Stakes Ultimatum Games: An Experiment in the Slovak Republic," *Econometrica*, vol. 66, no. 3 (May 1998), pp. 569–96. Return to reference 13
- See Alan Sanfey, James Rilling, Jessica Aronson, Leigh Nystrom, and Jonathan Cohen, "The Neural Basis of Economic Decision-Making in the Ultimatum Game," *Science*, vol. 300 (June 13, 2003), pp. 1755–58. Return to reference 14
- Joseph Henrich et al., " 'Economic Man' in Cross-cultural Perspective: Behavioral Experiments in 15 Small-Scale Societies," *Behavioral and Brain Sciences,* vol. 28, no. 6 (December 2005), pp. 795–815. Return to reference 15
- One could argue that this social norm of fairness may actually have value in the ongoing evolutionary game being played by the whole society. Players with a sense of fairness reduce transaction costs and the costs of fights; that can be beneficial to society in the long run. This idea is supported by the findings of the cross-cultural experiments cited in note 15). The correlation between individuals' prosociality and society's state of development may be cause and effect. These matters will be discussed in more detail in Chapters 10 and 11. Return to reference 16
- See Steven D. Levitt, John A. List, and Sally E. Sadoff, "Checkmate: Exploring Backward Induction among Chess

Players," *American Economic Review*, vol. 101, no. 2 (April 2011), pp. 975–90. The details of the game tree are as follows. If A plays Take at node 1, then A receives $4 while B receives $1. If A passes and B plays Take at node 2, then A receives $2 while B receives $8. This pattern of doubling continues until node 6, where if B plays Take, the payoffs are $32 for A and $128 for B, but if B plays Pass, the payoffs are $256 for A and $64 for B. Return to reference 17

- Different results were found in an earlier paper by Ignacio Palacios-Huerta and Oscar Volij, "Field Centipedes," *American Economic Review*, vol. 99, no. 4 (September 2009), pp. 1619–35. Of the chess players they studied, 69% played Take at the first node, with the more highly rated chess players being more likely to play Take at the first opportunity. These results indicated a surprisingly high ability of players to carry experience with them to a new game context, but these results were not reproduced in the later paper discussed in note 17. Return to reference 18
- As you will see in the exercises, another key distinction of this zero-sum game is that there is a way for one player to guarantee victory, regardless of what the other player does. By contrast, a player's best move in the centipede game depends on what she expects the other player to do. Return to reference 19
- See Richard Thaler and Cass Sunstein, *Nudge: Improving Decisions about Health, Wealth, and Happiness* (New Haven, CT: Yale University Press, 2008). Return to reference 20

SUMMARY

Sequential-move games require players to consider the future consequences of their current moves before choosing their actions. Analysis of pure sequential-move games generally requires the creation of a *game tree*. The tree is made up of *nodes* and *branches* that show all the possible actions available to each player at each of her opportunities to move, as well as the payoffs associated with all possible outcomes of the game. Strategies for each player are complete plans that describe actions at each of the player's decision nodes contingent on all possible combinations of actions made by players who acted at earlier nodes. The equilibrium concept employed in sequential-move games is that of *rollback equilibrium*, in which players' optimal strategies are found by looking ahead to subsequent nodes and the actions that would be taken there and by using these forecasts to calculate one's current best action. This process is known as *rollback*, or *backward induction*.

Different types of games entail advantages for different players, such as *first-mover advantages or second-mover advantages.* The inclusion of many players or many moves enlarges the game tree of a sequential-move game but does not change the solution process. In some cases, drawing the full tree for a particular game requires more space or time than is feasible. Such games can often be solved by identifying strategic similarities between actions that reduce the size of the tree, or by simple logical thinking.

When solving games with more moves, verbal reasoning can lead to the rollback equilibrium if the game is simple enough, or a complete tree may be drawn and analyzed. If the game is so complex that verbal reasoning is too difficult and a complete tree is too large to draw, we can enlist the help of a computer program. Checkers has been solved with the use of such a program, and artificial intelligence techniques are rapidly improving computer skills in the play of games like chess and Go.

Tests of the theory of sequential-move games seem to suggest that actual play shows the irrationality of the players or the failure of the theory to predict behavior adequately. The counterargument points out the complexity of actual preferences for different possible outcomes and the usefulness of game theory for identifying optimal actions when actual preferences are known.

KEY TERMS

action node (48)

backward induction (55)

branch (48)

decision node (48)

decision tree (48)

equilibrium path of play (58)

extensive form (48)

first-mover advantage (61)

game tree (48)

initial node (48)

intermediate valuation function (68)

move (50)

node (48)

path of play (58)

pruning (53)

rollback (55)

rollback equilibrium (55)

root (48)

second-mover advantage (61)

terminal node (50)

Glossary

action node
: A node at which one player chooses an action from two or more that are available.

backward induction
: Same as rollback.

branch
: Each branch emerging from a node in a game tree represents one action that can be taken at that node.

decision node
: A decision node in a decision or game tree represents a point in a game where an action is taken.

decision tree
: Representation of a sequential decision problem facing one person, shown using nodes, branches, terminal nodes, and their associated payoffs.

equilibrium path of play
: The *path of play* actually followed when players choose their rollback equilibrium strategies in a sequential game.

extensive form
: Representation of a game by a game tree.

first-mover advantage
: This exists in a game if, considering a hypothetical choice between moving first and moving second, a player would choose the former.

game tree
: Representation of a game in the form of nodes, branches, and terminal nodes and their associated payoffs.

initial node
: The starting point of a sequential-move game. (Also called the root of the tree.)

intermediate valuation function

A rule assigning payoffs to nonterminal nodes in a game. In many complex games, this must be based on knowledge or experience of playing similar games, instead of explicit rollback analysis.

move

An action at one node of a game tree.

node

This is a point from which branches emerge, or where a branch terminates, in a decision or game tree.

path of play

A route through the game tree (linking a succession of nodes and branches) that results from a configuration of strategies for the players that are within the rules of the game. (See also *equilibrium path of play*.)

pruning

Using rollback analysis to identify and eliminate from a game tree those branches that will not be chosen when the game is rationally played.

rollback

Analyzing the choices that rational players will make at all nodes of a game, starting at the terminal nodes and working backward to the initial node. Also called backward induction.

rollback equilibrium

The strategies (complete plans of action) for each player that remain after rollback analysis has been used to prune all the branches that can be pruned.

root

Same as initial node.

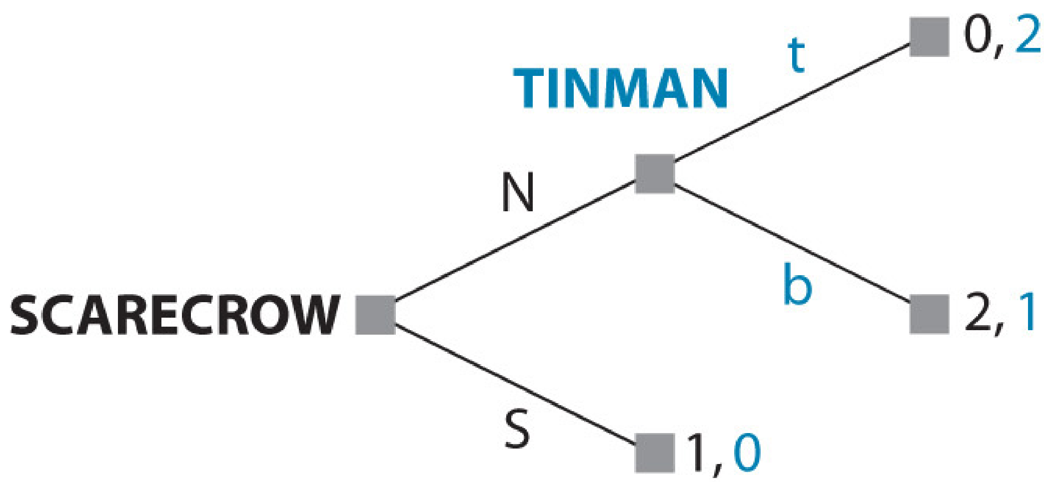
second-mover advantage

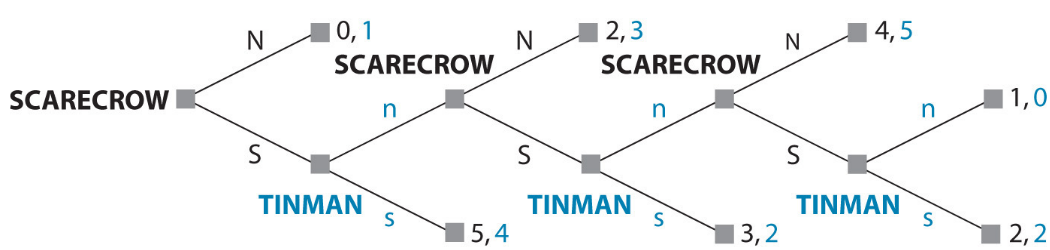
A game has this if, considering a hypothetical choice between moving first and moving second, a player would choose the latter.

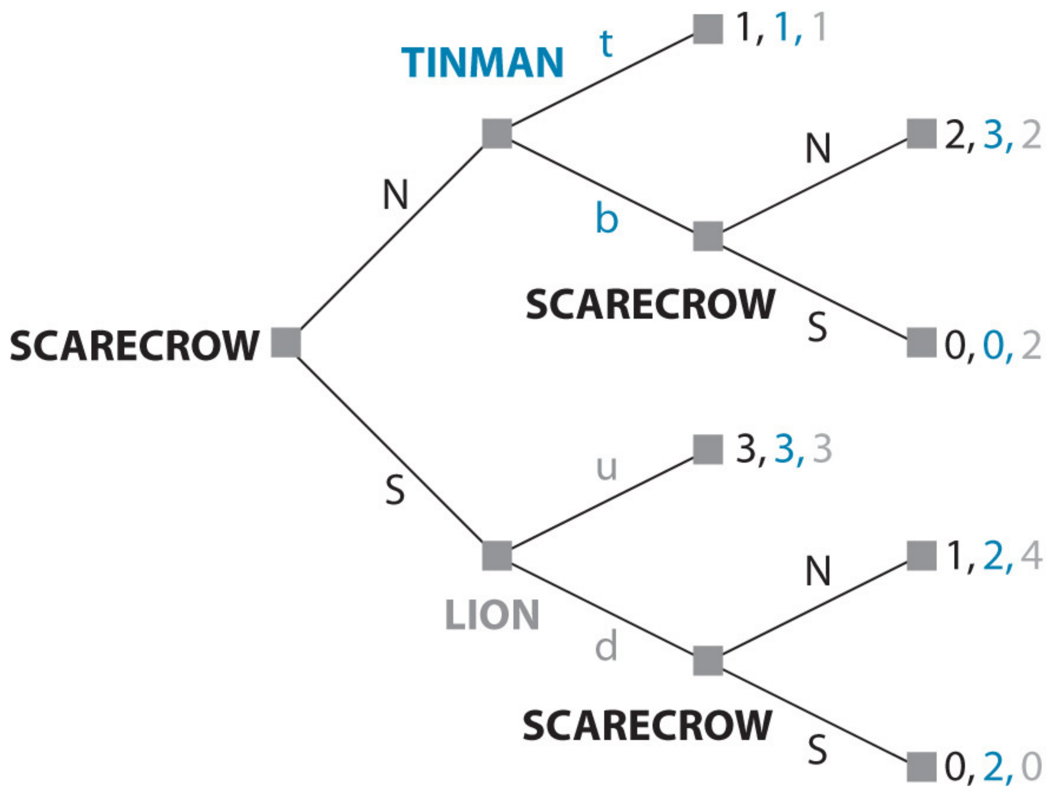
terminal node

This represents an end point in a game tree, where the rules of the game allow no further moves, and payoffs for each player are realized.

SOLVED EXERCISES

1. Suppose two players, Hansel and Gretel, take part in a sequential-move game. Hansel moves first, Gretel moves second, and each player moves only once.
   1. Draw a game tree for a game in which Hansel has two possible actions (Up or Down) at each node and Gretel has three possible actions (Top, Middle, or Bottom) at each node. How many of each node type—decision and terminal—are there?
   2. Draw a game tree for a game in which Hansel and Gretel each have three possible actions (Sit, Stand, or Jump) at each node. How many of each node type are there?
   3. Draw a game tree for a game in which Hansel has four possible actions (North, South, East, or West) at each node and Gretel has two possible actions (Stay or Go) at each node. How many of each node type are there?
2. In each of the following games, how many strategies (complete plans of action) are available to each player? List all the possible strategies for each player.
   1. 

   2. 



3.

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3. For each of the games illustrated in Exercise S2, identify the rollback equilibrium outcome and the optimal (complete) strategy for each player.
4. Consider the rivalry between Airbus and Boeing to develop a new commercial jet aircraft. Suppose Boeing is ahead in the development process, and Airbus is considering whether to enter the competition. If Airbus stays out, it earns a profit of $0, whereas Boeing enjoys a monopoly and earns a profit of $1 billion. If Airbus decides to enter and develop a rival airplane, then Boeing has to decide whether to accommodate Airbus peaceably or to wage a price war. In the event of peaceful competition, each firm will make a profit of $300 million. If there is a price war, each will lose $100 million because the prices of airplanes will fall so low that neither firm will be able to recoup its development costs.

   Draw the tree for this game. Find the rollback equilibrium and describe each firm's optimal strategy.

5. Consider a game in which two players, Fred and Barney, take turns removing matchsticks from a pile. They start with 21 matchsticks, and Fred goes first. On each turn, each player may remove either 1, 2, 3, or 4 matchsticks. The player to remove the last matchstick wins the game.

   1. Suppose there are only 6 matchsticks left, and it is Barney’s turn. What move should Barney make to guarantee himself victory? Explain your reasoning.
   2. Suppose there are 12 matchsticks left, and it is Barney’s turn. What move should Barney make to guarantee himself victory? (Hint: Use your answer to part (a) and roll back.)
   3. Now start from the beginning of the game. If both players play optimally, who will win?
   4. What are the optimal (complete) strategies for each player?
6. Consider the game in the previous exercise. Suppose the players have reached a point where it is Fred’s move and there are just 5 matchsticks left.
   1. Draw the game tree for the game starting with 5 matchsticks.
   2. Find the rollback equilibria for this game starting with 5 matchsticks.
   3. Would you say this 5-matchstick game has a first-mover advantage or a second-mover advantage?
   4. Explain why you found more than one rollback equilibrium. How is your answer related to the optimal strategies you found in part (c) of the previous exercise?
7. Elroy and Judy play a game that Elroy calls “the race to 100.” Elroy goes first, and the two players take turns choosing numbers between 1 and 9. On each turn, they add the new number to a running total. The player who brings the total exactly to 100 wins the game.
   1. If both players play optimally, who will win the game? Does this game have a first-mover advantage? Explain your reasoning.
   2. What are the optimal (complete) strategies for each player?
8. A slave has just been thrown to the lions in the Roman Colosseum. Three lions are chained down in a line, with Lion 1 closest to the slave. Each lion’s chain is short enough that he can only reach the two players immediately adjacent to him.

   The game proceeds as follows. First, Lion 1 decides whether or not to eat the slave.

   If Lion 1 has eaten the slave, then Lion 2 decides whether or not to eat Lion 1 (who is then too heavy to defend himself). If Lion 1 has not eaten the slave, then Lion 2 has no choice: He cannot try to eat Lion 1, because a fight would kill both lions.

   Similarly, if Lion 2 has eaten Lion 1, then Lion 3 decides whether or not to eat Lion 2.

   Each lion’s preferences are fairly natural: best (4) is to eat and stay alive, next best (3) is to stay alive but go hungry, next (2) is to eat and be eaten, and worst (1) is to go hungry and be eaten.

1. Draw the game tree, with payoffs, for this three-player game.
2. What is the rollback equilibrium of this game? Make sure to describe the players' optimal strategies, not just the payoffs.
3. Is there a first-mover advantage to this game? Explain why or why not.
4. How many (complete) strategies does each lion have? List them.

9. Consider three major department stores—Big Giant, Titan, and Frieda's—that are contemplating opening a branch in one of two new Boston-area shopping malls. Urban Mall is located close to the large and rich population center of the area; it is relatively small and can accommodate at most two department stores as anchors for the mall. Rural Mall is farther out in a rural and relatively poor area; it can accommodate as many as three anchor stores. None of the three stores wants to have branches in both malls, because there is sufficient overlap of customers between the malls that locating in both would just mean competing with itself. Each store would rather be in a mall with one or more other department stores than be alone in the same mall, because a mall with multiple department stores will attract enough additional customers that each store's profit will be higher. Further, each store prefers Urban Mall to Rural Mall because of the richer customer base. Each store must choose between trying to get a space in Urban Mall (knowing that if the attempt fails, it will try for a space in Rural Mall) and trying to get a space in Rural Mall right away (without even attempting to get into Urban Mall).

   In this case, the stores rank the five possible outcomes as follows: 5 (best), in Urban Mall with one other department store; 4, in Rural Mall with one or two other department stores; 3, alone in Urban Mall; 2, alone in Rural Mall; and 1 (worst), alone in Rural Mall after having attempted to get into Urban Mall and failed, by which time other non-department stores have signed up the best anchor locations in Rural Mall.

   The three stores are sufficiently different in their managerial structures that they experience different lag times in doing the paperwork required to request a space in a new mall. Frieda's moves quickly, followed by Big Giant, and finally by Titan, which is the least efficient in readying a location plan. When all three have made their requests, the malls decide which stores to let in. Because of the name recognition that both Big Giant and Titan have with potential customers, a mall would take either (or both) of those stores before it took Frieda's. Thus, Frieda's does not get one of the two spaces in Urban Mall if all three stores request those spaces; this is true even though Frieda' s moves first.

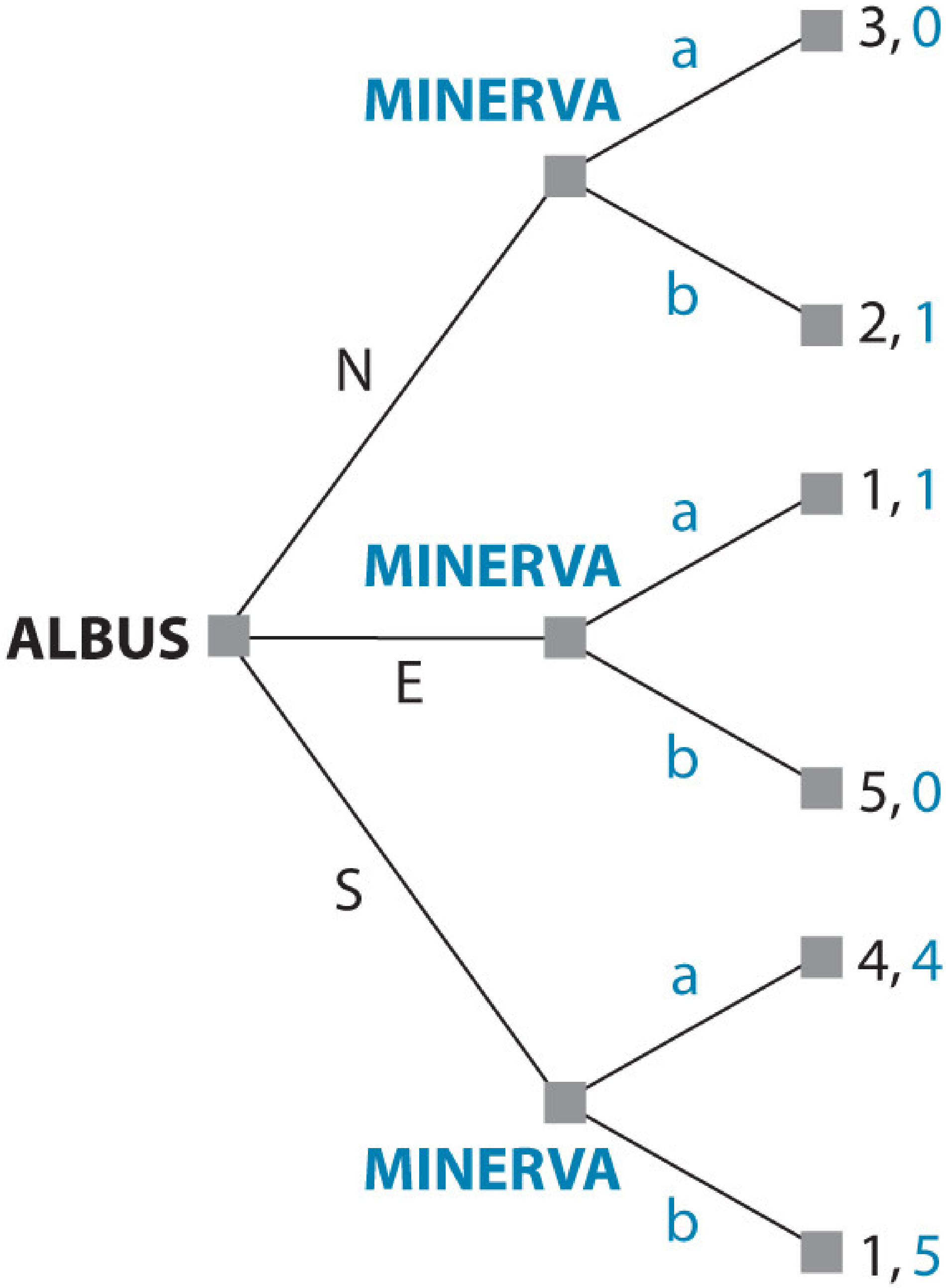
1. Draw the game tree for this mall location game.
2. Illustrate the pruning process on your game tree and use the fully pruned tree to find the rollback equilibrium. Describe the equilibrium by specifying the optimal strategies employed by each department store. What are the payoffs to each store at the rollback equilibrium outcome?

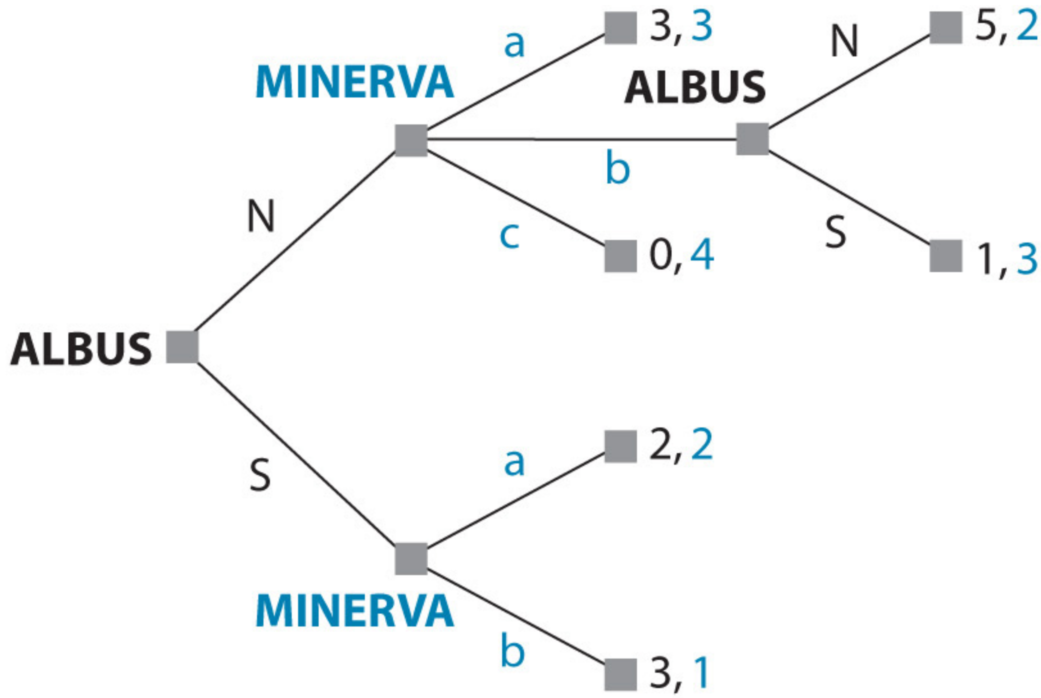
10. (**Optional**) Consider the following ultimatum game, which has been studied in laboratory experiments. The Proposer moves first, and proposes a split of $10 between himself and the Responder. Any whole-dollar split may be proposed. For example, the Proposer may offer to keep the whole $10 for himself, he may propose to keep $9 for himself and give $1 to the Responder, $8 to himself and $2 to the Responder, and so on. (Note that the Proposer therefore has 11 possible choices.) After seeing the split, the Responder can choose to accept the split or reject it. If the Responder accepts, both players get the proposed amounts. If she rejects, both players get $0.
    1. Draw the game tree for this game.
    2. How many complete strategies does each player have?
    3. What is the rollback equilibrium of this game, assuming the players care only about their cash payoffs?
    4. Suppose Rachel the Responder would accept any offer of $3 or more, and reject any offer of $2 or less. Suppose Pete the Proposer knows Rachel' s strategy, and he wants to maximize his cash payoff. What strategy should he use?
    5. Rachel' s true payoff might not be the same as her cash payoff. What other aspects of the game might she care about? Given your answer, propose a set of payoffs for Rachel that would make her strategy optimal.
    6. In laboratory experiments, players typically do not play the rollback equilibrium. Proposers typically offer an amount between $2 and $5 to Responders. Responders often reject offers of $3, $2, and especially $1. Explain why you think this might occur.

UNSOLVED EXERCISES

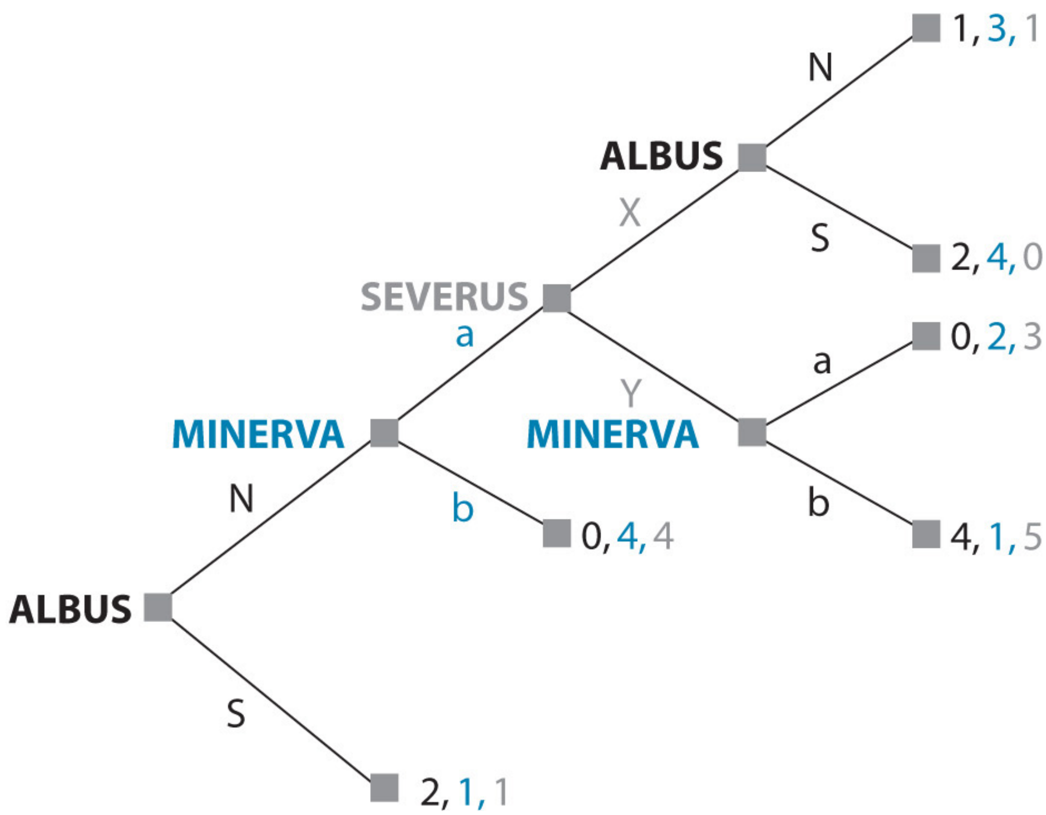
1. “In a sequential-move game, the player who moves first is sure to win.” Is this statement true or false? State the reason for your answer in a few brief sentences, and give an example of a game that illustrates your answer.
2. In each of the following games, how many strategies (complete plans of action) are available to each player? List all the possible strategies for each player.

1.





2.



3.

3. For each of the games illustrated in Exercise U2, identify the rollback equilibrium outcome and the optimal strategy for each

player.

4. Two distinct proposals, A and B, are being debated in Washington. Congress likes Proposal A, and the president likes Proposal B. The proposals are not mutually exclusive; either or both or neither may become law. Thus, there are four possible outcomes, and they are ranked by the two sides are as follows, where a larger number represents a more favored outcome:

Outcome	Congress	President
A becomes law	4	1
B becomes law	1	4
Both A and B become law	3	3
B Neither (status quo prevails)	2	2

The moves in the game are as follows: First, Congress decides whether to pass a bill and whether the bill is to contain A or B or both. Then the president decides whether to sign or veto the bill. Congress does not have enough votes to override a veto.

1. Draw a tree for this game and find the rollback equilibrium.
2. Now suppose the rules of the game are changed in only one respect: The president is given the extra power of a line-item veto. Thus, if Congress passes a bill containing both A and B, the president may choose not only to sign or veto the bill as a whole, but also to veto just one of the two items. Show the new tree and find the rollback equilibrium.
3. Explain intuitively why the difference between the two equilibria arises.

5. Two players, Amy and Beth, play the following game with a jar containing 100 pennies. The players take turns; Amy goes first. Each time it is a player’ s turn, she takes between 1 and 10 pennies out of the jar. The player whose move empties the jar wins.
   1. If both players play optimally, who will win the game? Does this game have a first-mover advantage? Explain your reasoning.

1. What are the optimal strategies for each player?

6. Consider a slight variant to the game in Exercise U5. Now the player whose move empties the jar loses.
   1. Does this game have a first-mover advantage?
   2. What are the optimal strategies for each player?
7. Kermit and Fozzie play a game with two jars, each containing 100 pennies. The players take turns; Kermit goes first. Each time it is a player’s turn, he chooses one of the jars and removes anywhere from 1 to 10 pennies from it. The player whose move leaves both jars empty wins. (Note that when a player empties the second jar, the first jar must already have been emptied in some previous move by one of the players.)
   1. Does this game have a first-mover advantage or a second-mover advantage? Explain which player can guarantee victory and how he can do it. (Hint: Simplify the game by starting with a smaller number of pennies in each jar, and see if you can generalize your finding to the actual game.)
   2. What are the optimal strategies for each player? (Hint: First think of a starting situation in which both jars have equal numbers of pennies. Then consider starting positions in which the two jars differ by 1 to 10 pennies. Finally, consider starting positions in which the jars differ by more than 10 pennies.)
8. Modify Exercise S8 so that there are now four lions.
   1. Draw the game tree, with payoffs, for this four-player game.
   2. What is the rollback equilibrium of this game? Make sure to describe the players’ (complete) optimal strategies, not just the payoffs.
   3. Is the additional lion good or bad for the slave? Explain.
9. To give Mom a day of rest, Dad plans to take his two children, Bart and Cassie, on an outing on Sunday. Bart prefers to go to the amusement park (A), whereas Cassie prefers to go to the science museum (S). Each child gets 3 units of value from his/her more preferred activity and only 2 units of value from his/her less preferred activity. Dad gets 2 units of value for either of the two activities.

   To choose their activity, Dad plans first to ask Bart for his preference, then to ask Cassie after she hears Bart’s choice. Each child can choose either the amusement park (A) or the science museum (S). If both children choose the same activity, then that is what they will all do. If the children choose different activities, Dad will make a tie-breaking decision. As the parent, Dad has an additional option: He can choose the amusement park, the science museum, or his personal favorite, the mountain hike (M). Bart and

Cassie each get 1 unit of value from the mountain hike, and Dad gets 3 units of value from the mountain hike.

Because Dad wants his children to cooperate with each other, he gets 2 extra units of value if the children choose the same activity (no matter which one of the two it is).

1. Draw the game tree for this three-person game, with payoffs based on the units of value specified for each outcome.
2. What is the rollback equilibrium of this game? Make sure to describe the players’ optimal strategies, not just the payoffs.
3. How many different (complete) strategies does Bart have? Explain.
4. How many (complete) strategies does Cassie have? Explain.

4 ■ Simultaneous-Move Games: Discrete Strategies

RECALL FROM CHAPTER 2 that a game is said to have *simultaneous moves* if each player must move without knowing what other players have chosen to do. This is obviously true if players choose their actions at exactly the same time (have *literally* simultaneous moves), but it is also true in any situation in which players make choices at different times but do not have any information about others' moves when deciding what to do. (For this reason, simultaneous-move games have *imperfect information* in the sense defined in Chapter 2, Section 2.D.)

Many familiar strategic situations can be described as simultaneous-move games. Firms that make TV sets, stereos, or automobiles make decisions about product design and features without knowing what rival firms are doing with their own products. Voters in U.S. elections cast their votes without knowing how others are voting, and hence make simultaneous moves when deciding each election. The interaction between a soccer goalie and an opposing striker during a penalty kick requires both players to make their decisions simultaneously—the kicker does not know which way the goalie will be jumping, but the goalie also cannot afford to wait until the ball has actually been kicked to decide which way to go, because the ball travels too fast to watch and then respond to.

In Chapters 2 and 3, we emphasized that a strategy is a complete plan of action. But in a simultaneous-move game, each player has exactly one opportunity to act. Therefore, there is no real distinction between *strategy* and *action* in simultaneous-move games, and the terms are often used as synonyms in this context. There is one more complication: In simultaneous-move games, a strategy can be a probabilistic choice from the basic actions initially specified. For example, in football, a coach may deliberately randomize whether to run or pass the ball in order to keep the other side guessing. Such probabilistic strategies, called mixed strategies, are the focus of Chapter 7. In this chapter, we confine our attention to the initially specified actions available to the players, called pure strategies.

In many games, each player has available to her a finite number of discrete pure strategies—for example, Dribble, Pass, or Shoot in basketball. In other games, each player’s pure strategy can be any number from a continuous range—for example, the price charged for a product by a firm.[1] This distinction makes no difference to the general concept of equilibrium in simultaneous-move games, but the key ideas are more easily conveyed with discrete strategies. Therefore, in this chapter, we restrict our analysis to the simpler case of discrete pure strategies, then take up continuously variable strategies in Chapter 5.

When a player in a simultaneous-move game chooses her action, she obviously does so without any knowledge of the choices made by other players. She also cannot look ahead to how they will react to her choice, because they do not know what she is choosing. Rather, each player must figure out what others are choosing to do while the others are figuring out what she is choosing to do. This situation may seem complex, but the analysis is not difficult once you understand the relevant concepts and methods. We begin by showing how to depict simultaneous-move games in a way that makes them especially easy to analyze.

Endnotes

- In fact, prices must be denominated in the minimum unit of coinage—for example, whole cents—and can therefore take on only a finite number of discrete values. But this unit is usually so small that it makes more sense to think of price as a continuous variable. Return to reference 1

Glossary

mixed strategy
: A mixed strategy for a player consists of a random choice, to be made with specified probabilities, from his originally specified pure strategies.

pure strategy
: A rule or plan of action for a player that specifies without any ambiguity or randomness the action to take in each contingency or at each node where it is that player’s turn to act.

1 DEPICTING SIMULTANEOUS-MOVE GAMES WITH DISCRETE STRATEGIES

A simultaneous-move game with discrete strategies is most often depicted using a game table (also called a payoff table or payoff matrix). This table is referred to as the normal form or strategic form of the game. Games with any number of players can be illustrated by using a game table, but its dimensions must equal the number of players. For a two-player game, the table is two-dimensional and appears similar to a spreadsheet. The row and column headings of the table correspond to the strategies available to the first and second players, respectively. The size of the table, then, is determined by the number of strategies available to each player.[2] Each cell within the table lists the payoffs to all players that arise under the configuration of strategies that places players in that cell. Games with three players require three-dimensional tables; we will consider them later in this chapter.

		COLUMN		
		Left	Middle	Right
ROW	Top	3, 1	2, 3	10, 2
	High	4, 5	3, 0	6, 4
	Low	2, 2	5, 4	12, 3
	Bottom	5, 6	4, 5	9, 7

You may need to scroll left and right to see the full figure.

FIGURE 4.1 Representing a Simultaneous-Move Game in a Table

We illustrate the concept of a game table for a simple game in Figure 4.1. The game here has no special interpretation, so we can develop the concepts without the distraction of a "story." The players are named Row and Column. Row has four choices (strategies or actions), labeled Top, High, Low, and Bottom; Column has three choices, labeled Left, Middle, and Right. Each selection by Row and Column generates a potential outcome of the game. The payoffs associated with each outcome are shown in the cell corresponding to that row and that column. By convention, the first number in each cell is Row's payoff in that outcome, while the second number is Column's payoff. For example, if Row chooses High and Column chooses Right, payoffs are 6 to Row and 4 to Column. For additional convenience, we show everything pertaining to Row—player name, strategies, and payoffs—in black, and everything pertaining to Column in blue.

Next, we consider a second example of a game table with more of a story attached. Figure 4.2 represents a very simplified version of a single play in American football. Offense attempts to move the ball forward to improve its chances of kicking a field goal. It has four possible strategies: a run and passes of three different lengths (short, medium, and long). Defense can adopt one of three strategies to try to keep Offense at bay: a run defense, a pass defense, or a blitz of the quarterback. Offense tries to gain yardage while Defense tries to prevent it from doing so. Suppose we have enough information about the underlying strengths of the two teams to work out the probabilities of their completing different plays and to determine the average gain in yardage that could be expected under each combination of strategies. For example, when Offense chooses Medium Pass and Defense counters with its Pass defense, we estimate Offense's payoff to be a gain of 4.5 yards, or +4.5.[3] Defense's "payoff" is a loss of 4.5 yards, or −4.5. The other cells similarly show our estimates of each team's gain or loss of yardage.

		DEFENSE		
		Run	Pass	Blitz
OFFENSE	Run	2, −2	5, −5	13, −13
	Short Pass	6, −6	5.6, −5.6	10.5, −10.5
	Medium Pass	6, −6	4.5, −4.5	1, −1
	Long Pass	10, −10	3, −3	−2, 2
You may need to scroll left and right to see the full figure.				

Figure 4.2 A Single Play in American Football

Sometimes, for obviously zero-sum games like this one, only the row player's payoffs are shown in the game table, and the column player's payoffs are understood to be the negatives of those numbers. We will not adopt that approach here, however. Always showing both payoffs helps reduce the possibility of any confusion.

Endnotes

- In a game where each firm can set its price at any number of cents in a range that extends over a dollar, each firm has 100 distinct strategies, and the size of the table becomes 100 × 100. That is surely too unwieldy to analyze. Algebraic formulas with prices as continuous variables provide a simpler approach—not a more complicated one, as some readers might fear. We will develop this “algebra is our friend” method in Chapter 5. Return to reference 2
- Here is how the payoffs for this case were constructed. When Offense chooses Medium Pass and Defense counters with its Pass defense, our estimate is a probability of 50% that the pass will be completed for a gain of 15 yards, a probability of 40% that the pass will fall incomplete (0 yards), and a probability of 10% that the pass will be intercepted for a loss of 30 yards; this makes an average of $(0.5 \times 15) + (0.4 \times 0) + (0.1 \times (-30)) = 4.5$ yards. The numbers in the table were constructed by a small panel of expert neighbors and friends convened by Dixit on one fall Sunday afternoon. They received a liquid consultancy fee. Return to reference 3

Glossary

game table
: A spreadsheetlike table whose dimension equals the number of players in the game; the strategies available to each player are arrayed along one of the dimensions (row, column, page, . . .); and each cell shows the payoffs of all the players in a specified order, corresponding to the configuration of strategies that yield that cell. Also called payoff table.

payoff table
: Same as game table.

payoff matrix
: Same as payoff table and game table.

normal form
: Representation of a game in a game matrix, showing the strategies (which may be numerous and complicated if the game has several moves) available to each player along a separate dimension (row, column, etc.) of the matrix and the outcomes and payoffs in the multidimensional cells. Also called strategic form.

strategic form
: Same as normal form.

2 NASH EQUILIBRIUM

To analyze simultaneous-move games, we need to consider how players choose their actions. Return to the game table in Figure 4.1. Focus on one specific outcome—namely, the one where Row chooses Low and Column chooses Middle; the payoffs there are 5 to Row and 4 to Column. Each player wants to pick an action that yields her the highest payoff, and in this outcome each indeed makes such a choice, given what her opponent chooses. Given that Row is choosing Low, can Column do any better by choosing something other than Middle? No, because Left would give her the payoff 2, and Right would give her 3, neither of which is better than the 4 she gets from Middle. Thus, Middle is Column's best response to Row's choice of Low. Conversely, given that Column is choosing Middle, can Row do better by choosing something other than Low? Again, no, because the payoffs from switching to Top (2), High (3), or Bottom (4) would all be worse than what Row gets with Low (5). Thus, Low is Row's best response to Column's choice of Middle.

These two choices, Low for Row and Middle for Column, have the property that each is the chooser's best response to the other's action. If the players were making these choices, neither would want to switch to a different response *on her own*. By the definition of a noncooperative game, the players are making their choices independently; therefore, such unilateral changes are all that each player can contemplate. Because neither wants to make such a change, it is natural to call this state of affairs an equilibrium. This is exactly the concept of a Nash equilibrium.

To state it a little more formally, a Nash equilibrium[4] in a game is a list of strategies, one for each player, such that no player can get a better payoff by switching to some other strategy that is available to her while all the other players adhere to the strategies specified for them in that list.

A. Some Further Explanation of the Concept of Nash Equilibrium

To understand the concept of Nash equilibrium better, we take another look at the game table in Figure 4.1. Consider now a cell other than (Low, Middle)—say, the one where Row chooses High and Column chooses Left. Can these strategies be a Nash equilibrium? No, because if Column is choosing Left, Row does better to choose Bottom and get the payoff 5 rather than to choose High, which gives her only 4. Similarly, (Bottom, Left) is not a Nash equilibrium, because Column can do better by switching to Right, thereby improving her payoff from 6 to 7.

The definition of Nash equilibrium does not require equilibrium choices to be strictly better than other available choices. Figure 4.3 is the same as Figure 4.1 except that Row’s payoff from (Bottom, Middle) is changed to 5, the same as that from (Low, Middle). It is still true that, given Column’s choice of Middle, Row *could not do any better* than she does when choosing Low. So neither player has a reason to change her action, and that qualifies (Low, Middle) as a Nash equilibrium.[5]

More importantly, a Nash equilibrium does not have to be jointly best for the players. In Figure 4.1, the strategy pair (Bottom, Right) gives payoffs (9, 7), which are better for both players than the (5, 4) of the Nash equilibrium. However, playing independently, they cannot sustain (Bottom, Right). Given that Column plays Right, Row would want to deviate from Bottom to Low and get 12 instead of 9. Getting the jointly better payoffs of (9, 7) would require cooperative action that made such “cheating” impossible. We examine how to support cooperative behavior later in this chapter and in more detail in Chapter 10. For now, we merely point out the fact that a Nash equilibrium may not be in the joint interests of the players.

		COLUMN		
		Left	Middle	Right
ROW	Top	3, 1	2, 3	10, 2
	High	4, 5	3, 0	6, 4
	Low	2, 2	5, 4	12, 3
	Bottom	5, 6	4, 5	9, 7
You may need to scroll left and right to see the full figure.				

FIGURE 4.3 Variation on Game in Figure 4.1 with a Tie in Payoffs

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To reinforce the concept of Nash equilibrium, look at the football game table in Figure 4.2. If Defense is choosing the Pass defense, then the best choice for Offense is Short Pass (payoff of 5.6 versus 5, 4.5, or 3). Conversely, if Offense is choosing the Short Pass, then Defense's best choice is the Pass defense—it holds Offense down to 5.6 yards, whereas the Run defense and the Blitz would concede 6 and 10.5 yards, respectively. (Remember that the entries in each cell of a zero-sum game are the row player's payoffs; therefore, the best choice for the column player is the one that yields the smallest number, not the largest.) In this game, the strategy combination (Short Pass, Pass) is a Nash equilibrium, and the resulting payoff to Offense is 5.6 yards.

How does one find Nash equilibria in games? One can always check every cell to see if the strategies that generate it satisfy the definition of a Nash equilibrium. Such a systematic analysis is foolproof, but tedious and unmanageable except in simple games or with the help of a good computer program. In the next few sections, we develop other methods (based on the concepts of *dominance* and *best response*) that will not only help us find Nash equilibria much more quickly, but also shed light on how players form beliefs and make choices in games.

B. Nash Equilibrium as a System of Beliefs and Choices

Before we proceed with further study and use of the Nash equilibrium concept, we should try to clarify something that may have bothered some of you. We said that, in a Nash equilibrium, each player chooses her "best response" to the other's choice. But the two choices are made simultaneously. How can one *respond* to something that has not yet happened, at least when one does not *know* what has happened?

People play simultaneous-move games all the time in real life, and they do make such choices. To do so, they must find a substitute for actual knowledge or observation of others' actions. Players can make blind guesses and hope that they turn out to be inspired ones, but luckily there are more systematic ways to try to figure out what the other players are doing. One method is experience and observation: if the players play this game, or similar games, with similar players all the time, they may develop a pretty good idea of what the other players do. Then choices that are not the best will be unlikely to persist for long. Another method is the logical process of thinking through the others' thinking. You put yourself in the position of the other players and think what they are thinking, which of course includes their putting themselves in your position and thinking what you are thinking. The logic seems circular, but there are several ways of breaking into the circle, as we demonstrate through specific examples in the sections that follow. Nash equilibrium can be thought of as a culmination of this process of thinking about thinking where each player has correctly figured out the others' choices.

Whether by observation, logical deduction, or some other method, you, the game player, acquire some notion of what the other players are choosing in simultaneous-move games. It is not easy to find a word to describe this process or its outcome. It is not anticipation, nor is it forecasting, because the others' actions do not lie in the future—they occur simultaneously with your own. The word most frequently used by game theorists is belief. This word is not perfect either, because it seems to connote more confidence or certainty than is intended; in fact, in Chapter 7, we will allow for the possibility that beliefs are held with some uncertainty. But for lack of a better word, it will have to suffice.

This concept of belief also relates to our discussion of uncertainty in Chapter 2, Section 2.D. There, we introduced the concept of strategic uncertainty. Even when a player knows all the rules of a game—the strategies available to all players and the payoffs for each as functions of those strategies—they may be uncertain about what actions the others are taking at the same time. Similarly, if past actions are not observable, each player may be uncertain about what actions the others took in the past. How can players choose in the face of this strategic uncertainty? They must form some subjective views or estimates of the others' actions. That is exactly what the notion of belief captures.

Now think of Nash equilibrium in this light. We defined it as a configuration of strategies such that each player's strategy is her best response to the others' strategies. If she does not know the others' actual choices but has beliefs about them, in Nash equilibrium those beliefs are assumed to be correct—the others' actual actions are just what she believes them to be. Thus, we can define Nash equilibrium in an alternative and equivalent way: It is a set of beliefs and strategies, one for each player, such that (1) each player has correct beliefs about the others' strategies and (2) each player’s strategy is best for herself, given her beliefs about the others’ strategies.[6]

This way of thinking about Nash equilibrium has two advantages. First, the concept of “best response” is no longer logically flawed. Each player is choosing her best response not to the as yet unobserved actions of the others, but only to her own already formed beliefs about their actions. Second, in Chapter 7, where we will allow mixed strategies, the randomness in one player’s strategy may be better interpreted as uncertainty in the other players’ beliefs about that player’s action. For now, we proceed by using both definitions of Nash equilibrium in parallel.

You might think that forming correct beliefs and calculating best responses is too daunting a task for mere humans. We will discuss some criticisms of this kind, as well as empirical and experimental evidence concerning Nash equilibrium, in Chapter 5 for pure strategies and in Chapter 7 for mixed strategies. For now, we simply say that the proof of the pudding is in the eating. We will develop and illustrate the Nash equilibrium concept by applying it. We hope that seeing it in use will prove a better way to understand its strengths and drawbacks than would an abstract discussion at this point.

Endnotes

- This concept is named for the mathematician and economist John Nash, who developed it in his doctoral dissertation at Princeton in 1949. Nash also proposed a solution to cooperative games, which we consider in Chapter 17. He shared the 1994 Nobel Prize in economics with two other game theorists, Reinhard Selten and John Harsanyi; we will treat some aspects of their work in Chapters 8, 9, and 14. Sylvia Nasar's biography of Nash, *A Beautiful Mind* (New York: Simon & Schuster, 1998), was the (loose) basis for a movie starring Russell Crowe. Unfortunately, the movie failed in its attempt to explain the concept of Nash equilibrium. We explain this failure in Exercise S15 of this chapter and in Exercise S14 of Chapter 7. Return to reference 4
- But note that (Bottom, Middle) with the payoffs of (5, 5) is not itself a Nash equilibrium. If Row was choosing Bottom, Column's own best choice would not be Middle; she could do better by choosing Right. In fact, you can check all the other cells in in Figure 4.3 to verify that none of them can be a Nash equilibrium. Return to reference 5
- In this chapter, we consider only Nash equilibria in pure strategies—namely, those strategies initially listed in the rules of the game, not mixtures of two or more of them. Therefore, in such an equilibrium, each player has certainty about the actions of the others; strategic uncertainty is removed. When we consider mixed-strategy equilibria Chapter 7, the strategic uncertainty for each player will consist of the probabilities with which the various strategies are played in the other players' equilibrium mixtures. Return to reference 6

Glossary

best response
: The strategy that is optimal for one player, given the strategies actually played by the other players, or the belief of this player about the other players’ strategy choices.

Nash equilibrium
: A configuration of strategies (one for each player) such that each player’s strategy is best for him, given those of the other players. (Can be in pure or mixed strategies.)

belief
: The notion held by one player about the strategy choices of the other players and used when choosing his own optimal strategy.

3 DOMINANCE

In some games, one or more players have a strategy that is uniformly better than all their other strategies, no matter what strategies other players adopt. When that is the case for some player, that player's uniformly best strategy is referred to as her dominant strategy, or more precisely, her *strictly dominant strategy*.[7] Other times, when a player doesn't have a dominant strategy, there may still be some strategy that is uniformly worse for that player than some other strategy no matter what other players do. The uniformly worse strategy is then referred to as a dominated strategy, and we say that it is *dominated* by the uniformly better strategy. (When a player has a dominant strategy, all other strategies are dominated by the dominant strategy.) The presence of dominant strategies simplifies the search for and interpretation of Nash equilibrium because any player with a dominant strategy has an incentive to play that dominant strategy no matter what her belief about others' strategies.

A. Both Players Have Dominant Strategies

The game known as the prisoners' dilemma nicely illustrates the concept of a dominant strategy. Consider a story line of the type that appears regularly in the TV program *Law and Order*. A husband and wife have been arrested under the suspicion that they conspired in the murder of a young woman. Detectives Green and Lupo place the suspects in separate detention rooms and interrogate them one at a time, thereby ensuring that the game between the two suspects, Husband and Wife, has simultaneous moves. There is little concrete evidence linking the pair to the murder, although there is some evidence that they were involved in kidnapping the victim. The detectives explain to each suspect that they are both looking at jail time for the kidnapping charge, probably 3 years, even if there is no confession from either of them. In addition, Husband and Wife are told individually that the detectives "know" what happened and "know" how one was coerced by the other to participate in the crime; it is implied that jail time for a solitary confessor will be significantly reduced if the whole story is committed to paper. (In a scene common to many similar programs, a yellow legal pad and a pencil are produced and placed on the table at this point.) Finally, the suspects are told that if both confess, jail terms could be negotiated down for both, but not as much as they would be if there were one confession and one denial.

	WIFE	
	Confess (Defect)	Deny (Cooperate)

You may need to scroll left and right to see the full figure.

		WIFE	
		Confess (Defect)	Deny (Cooperate)
HUSBAND	Confess (Defect)	10 yr, 10 yr	1 yr, 25 yr
	Deny (Cooperate)	25 yr, 1 yr	3 yr, 3 yr
You may need to scroll left and right to see the full figure.			

FIGURE 4.4 Prisoners' Dilemma

In this situation, both Husband and Wife are players in a two-person, simultaneous-move game in which each has to choose between confessing and not confessing to the crime of murder. They both know that no confession leaves them each with a 3-year jail sentence for involvement with the kidnapping. They also know that if one of them confesses, he or she will get a short sentence of 1 year for cooperating with the police, while the other will go to jail for a minimum of 25 years. If both confess, they figure that they can negotiate for jail terms of 10 years each.

The choices and outcomes for this game are summarized by the game table in Figure 4.4. The strategies Confess and Deny can also be called Defect and Cooperate to capture their roles in the relationship between the *two players*; thus, Defect means to defect from any tacit arrangement with the spouse, and Cooperate means to take the action that helps the spouse (not cooperate with the cops).

The payoffs in this game are the lengths of the jail sentences associated with each outcome, so lower numbers are better for each player. In that sense, this example differs from most of the games that we analyze, in which larger payoffs are better and smaller payoffs are worse. We take this opportunity to remind you that "large is good" is not always true. Also, as mentioned in Section 1, in the game table for a zero-sum game that shows only one player's bigger-is-better payoffs, smaller numbers are better for the other player. In the prisoners' dilemma here, smaller numbers are better for both players. Thus, if you ever draw a payoff table where larger numbers are worse, you should alert the reader by pointing it out clearly. And when reading someone else's work, be aware of that possibility.

Now consider the prisoners' dilemma in Figure 4.4 from Husband's perspective. He has to think about what Wife will choose. Suppose he believes that she will confess. Then his best choice is to confess; he gets a sentence of only 10 years, whereas denial would have meant 25 years. What if he believes that Wife will deny? Again, his own best choice is to confess; he gets only 1 year instead of the 3 that his own denial would bring in this case. Thus, in this game, Confess is better than Deny for Husband *regardless of his belief about Wife's choice.* We say that, for Husband, the strategy Confess is a *dominant strategy* or that the strategy Deny is a *dominated strategy*. Equivalently, we could say that the strategy Confess *dominates* the strategy Deny or that the strategy Deny is *dominated* by the strategy Confess.

If an action is clearly best for a player no matter what other players might be doing, then there is compelling reason to think that a rational player would choose it. And if an action is clearly bad for a player no matter what others might be doing, then there is equally compelling reason to think that a rational player would avoid it. When it exists, dominance therefore provides a shortcut in our search for solutions to simultaneous-move games.

Games like the prisoners' dilemma are simple to analyze because the logic that led us to determine that Confess is dominant for Husband applies equally to Wife's choice. Her own strategy Confess dominates her own strategy Deny, so both players should choose Confess. Therefore, (Confess, Confess) is the predicted outcome and the unique Nash equilibrium of this game.

In this game, the best choice for each player is independent of whether his or her beliefs about the other are correct. Indeed, this is the meaning of dominance. But if each player attributes to the other the same rationality that he or she practices, then both of them should be able to form correct beliefs. And the actual action of each is the best response to the actual action of the other. Note that the fact that Confess dominates Deny for both players is completely independent of whether they are actually guilty, as in many episodes of *Law and Order*, or are being framed, as happened in the movie *L.A. Confidential.* It depends only on the pattern of payoffs dictated by the various jail terms.

We will examine the prisoners' dilemma in great detail throughout the book, including later in this chapter, in all of Chapter 10, and in several sections in other chapters as well. Why is this game so important to the study of game theory that it appears multiple times in this book? There are two main reasons. The first reason is that many important, though seemingly quite different, economic, social, political, and even biological strategic situations can be interpreted as examples of the prisoners' dilemma. Understanding the prisoners' dilemma will give you better insight into such games.

The second reason why the prisoners' dilemma is integral to any discussion of games of strategy is the somewhat curious nature of the equilibrium outcome achieved in such games. Both players choose their dominant strategies, but the resulting equilibrium outcome yields them payoffs that are lower than they could have achieved if they had each chosen their dominated strategies. Thus, the equilibrium outcome in the prisoners' dilemma is actually a bad outcome for the players. There is another outcome that they both prefer to the equilibrium outcome; the problem is how to guarantee that someone will not cheat. This particular feature of the prisoners' dilemma has received considerable attention from game theorists who have asked an obvious question: What can players in a prisoners' dilemma do to achieve the better outcome? We leave this question for now to continue our discussion of simultaneous-move games, but will return to it in later chapters, where we will develop some of the tools that can be used to "change the game" to improve outcomes.[8]

B. One Player Has a Dominant Strategy

In a simultaneous-move game in which only one player has a dominant strategy, we predict that the player who has a dominant strategy will choose it and that the other player will form a correct belief about the first player’s strategy and choose her own best response to it.

As an illustration of such a situation, consider a game frequently played between the U.S. Congress, which is responsible for fiscal policy (taxes and government expenditures), and the Federal Reserve (Fed), which is in charge of monetary policy (primarily interest rates).[9] To simplify the game to its essential features, suppose that Congress’s fiscal policy can result in either a balanced budget or a deficit, and that the Fed can set interest rates either high or low. In reality, the timing of moves in this game is not entirely clear (and may differ from one country to another). For now, we focus on the simultaneous-move version of the game, then return to it in Chapter 6 to consider how its outcomes may differ if Congress or the Fed moves first.

Almost everyone wants lower taxes. But there is no shortage of good claims on government funds for defense, education, health care, and so on. There are also various politically powerful special interest groups—including farmers and industries hurt by foreign competition—who want government subsidies. Therefore, Congress is under constant pressure both to lower taxes and to increase spending. But such behavior runs the budget into deficit, which can lead to inflation. The Fed’s primary task is to prevent inflation. However, it also faces political pressure for lower interest rates from many important groups, especially homeowners, who benefit from lower mortgage rates. Lower interest rates lead to higher demand for automobiles, housing, and capital investment by firms, and that can cause inflation. The Fed is generally happy to lower interest rates, but only so long as inflation is not a threat. And there is less threat of inflation when the government’s budget is in balance. With all these factors in mind, we construct the payoff matrix for this game in Figure 4.5.

		FEDERAL RESERVE	
		Low interest rates	**High interest rates**
CONGRESS	Budget balance	3, 4	1, 3
	Budget deficit	4, 1	2, 2

FIGURE 4.5 Game of Fiscal and Monetary Policies

Congress’s best outcome (payoff 4) is a budget deficit and low interest rates, as this pleases all of its immediate political constituents. This outcome may entail trouble for the future, but political time horizons are short. For the same reason, Congress’s worst outcome (payoff 1) is a balanced budget and high interest rates. Of the other two outcomes, Congress likes balanced budget and low interest rates (payoff 3) better than a budget deficit and high interest rates (payoff 2), as low interest rates please the important home-owning middle classes. Moreover, with low interest rates, less money is needed to service the government debt, leaving room even in a balanced budget for many other expenditures or tax cuts.

The Fed’s worst outcome (payoff 1) is a budget deficit and low interest rates, because this combination is the most inflationary. The Fed’s best outcome (payoff 4) is a balanced budget and low interest rates, because this combination can sustain a high level of economic activity without much risk of inflation. Comparing the other two outcomes with high interest rates, the Fed prefers the one with a balanced budget because it reduces the risk of inflation.

We look now for dominant strategies in this game. The Fed does better by choosing low interest rates if it believes that Congress is opting for a balanced budget (the Fed’s payoff is 4 rather than 3), but it does better by choosing high interest rates if it believes that Congress is choosing to run a budget deficit (the Fed’s payoff is 2 rather than 1). Therefore, the Fed does not have a dominant strategy. What about Congress? If Congress believes that the Fed is choosing low interest rates, it does better for itself by choosing a budget deficit rather than a balanced budget (Congress’s payoff is 4 instead of 3). If Congress believes that the Fed is choosing high interest rates, again, it does better for itself by choosing a budget deficit rather than a balanced budget (Congress’s payoff is 2 instead of 1). So, running a budget deficit is Congress’s dominant strategy.

The choice for Congress is now clear. No matter what Congress believes the Fed is doing, Congress will choose to run a budget deficit. The Fed can now take this choice into account when making its own decision. The Fed should believe that Congress will choose its dominant strategy (budget deficit) and therefore choose the best strategy for itself, given this belief. That means that the Fed should choose high interest rates. Indeed, the unique Nash equilibrium of this game is a budget deficit and high interest rates.

In the Nash equilibrium, each side gets payoff 2. But an inspection of Figure 4.5 shows that, just as in the prisoners’ dilemma, there is another outcome (namely, a balanced budget and low interest rates) that can give both players higher payoffs (namely, 3 for Congress and 4 for the

Fed). Why is that outcome not achievable as an equilibrium? The problem is that Congress would be tempted to deviate from its stated strategy and sneakily run a budget deficit. The Fed, knowing this temptation, and knowing that it would then get its worst outcome (payoff 1), also deviates from its stated strategy by choosing high interest rates. In Chapters 6 and 8, we consider how the two players can get around this difficulty to achieve their mutually preferred outcome. But we should note that, in many countries, these two policy authorities are indeed often stuck in the worse outcome. In these countries, the fiscal policy is too loose and the monetary policy has to be tightened to keep inflation down.

C. Successive Elimination of Dominated Strategies

The games considered so far have had only two pure strategies available to each player. In such games, if one strategy is dominant, the other is dominated, so choosing the dominant strategy is equivalent to eliminating the dominated one. In games where players have a large number of pure strategies available to them, some of a player’s strategies may be dominated even though no single strategy dominates all the others. If players find themselves in a game of this type, they may be able to reach an equilibrium by removing dominated strategies from consideration as possible choices. Removing dominated strategies reduces the size of the game, and the “new” game that results may have other dominated strategies that can also be removed. Or the “new” game may even have a dominant strategy for one of the players. Successive or iterated elimination of dominated strategies is the process of removing dominated strategies and reducing the size of a game until no further reductions can be made. If this process ends in a single outcome, then the game is said to be dominance solvable, and the strategies that yield that outcome constitute the (unique) Nash equilibrium of the game.

We can use the game in Figure 4.1 to provide an example of this process. Consider first Row’s strategies. If any one of Row’s strategies always provides worse payoffs for Row than another of her strategies, then that strategy is dominated and can be eliminated from consideration as Row’s choice. Here, the only dominated strategy for Row is High, which is dominated by Bottom: If Column plays Left, Row gets 5 from Bottom and only 4 from High; if Column plays Middle, Row gets 4 from Bottom and only 3 from High; and if Column plays Right, Row gets 9 from Bottom and only 6 from High. So we can eliminate High. We now turn to Column’s choices to see if any of them can be eliminated. We find that Column’s Left is now dominated by Right (with similar reasoning, as $1 < 2$, $2 < 3$, and $6 < 7$). Note that we could not say this before Row’s High was eliminated because, against Row’s High, Column would get 5 from Left but only 4 from Right. Thus, the first step of eliminating Row’s High makes possible the second step of eliminating Column’s Left. Then, within the remaining set of strategies (Top, Low, and Bottom for Row, and Middle and Right for Column), Row’s Top and Bottom are both dominated by her Low. When Row is left with only Low, Column chooses her best response—namely, Middle.

The game is thus dominance solvable, and the Nash equilibrium is (Low, Middle), with payoffs (5, 4). We identified this list of strategies as a Nash equilibrium when we first illustrated that concept using this game. Now we can see in better detail the thought processes of the players that lead to the formation of correct beliefs. A rational Row will not choose High. A rational Column will recognize this and, thinking about how her various strategies perform for her against Row’s remaining strategies, will not choose Left. In turn, Row will recognize this, and therefore will not choose either Top or Bottom. Finally, Column will see through all this, and choose Middle.

Other games may not be dominance solvable, or successive elimination of dominated strategies may not yield a unique Nash equilibrium. Even in such cases, some elimination may reduce the size of the game and make it easier to solve by using one or more of the methods described in the following sections. Thus, eliminating dominated strategies can be a useful step toward solving large simultaneous-move games, even when this method does not completely solve the game.

Endnotes

- A strategy is *strictly dominant* for a player when it is strictly better than any other strategy, no matter what strategies other players adopt. By contrast, a strategy is *weakly dominant* (as discussed below in Section 4) when it is no worse than any other strategy, no matter what others do. When we use the term *dominant* without any additional qualifier, we always mean strictly dominant, and similarly with *dominated*. Return to reference 7
- In his book *Game-Changer: Game Theory and the Art of Transforming Strategic Situations* (New York: W.W. Norton, 2014), McAdams identifies five sorts of "escape routes" from the prisoners' dilemma: "cartelization" (discussed briefly in Chapter 5), "retaliation," "relationships," and "trust," (all discussed at least indirectly in Chapter 10), and "regulation" (which is beyond the scope of our book). Return to reference 8
- Similar games are played in many other countries that have central banks with operational independence in their choice of monetary policy. Fiscal policies may be chosen by different political entities—the executive or the legislature—in different countries. Return to reference 9

Glossary

dominant strategy
: A strategy X is dominant for a player if the outcome when playing X is always better than the outcome when playing any other strategy, no matter what strategies other players adopt.

dominated strategy
: A strategy X is dominated by another strategy Y for a player if the outcome when playing X is always worse than the outcome when playing Y, no matter what strategies other players adopt.

prisoners' dilemma
: A game where each player has two strategies, say Cooperate and Defect, such that [1] for each player, Defect dominates Cooperate, and [2] the outcome (Defect, Defect) is worse for both than the outcome (Cooperate, Cooperate).

successive elimination of dominated strategies
: Same as iterated elimination of dominated strategies.

successive elimination of dominated strategies
: Same as iterated elimination of dominated strategies.

dominance solvable
: A game where iterated elimination of dominated strategies leaves a unique outcome, or just one strategy for each player.

4 STRONGER AND WEAKER FORMS OF DOMINANCE

Recall that a strategy is dominant (or, more precisely, strictly dominant) for a player if it gives that player a higher payoff than any other strategy, no matter what strategies other players employ. This definition seems simple at first glance, but it masks some important subtleties. In this section, we examine dominance in more depth by defining stronger and weaker versions of the concept and exploring their implications.

A. Superdominance

In a simultaneous-move game, a strategy is superdominant for a player if the worst possible outcome when playing that strategy is better than the best possible outcome when playing any other strategy. To see what that means, consider a generic game with Row's payoffs as shown in Figure 4.6. (We have omitted Column's payoffs for clarity.) For Up to be superdominant for Row, both outcomes that could conceivably occur if she were to play Up must be better for her than both possible outcomes if she were to play Down. In other words, if Row were to rank the four outcomes—with 4 being best and 1 being worst—the outcomes ranked 4 and 3 must be in the Up row and the outcomes ranked 2 and 1 must be in the Down row.

		COLUMN	
		Left	Right
ROW	Up	3 or 4	3 or 4
	Down	1 or 2	1 or 2

FIGURE 4.6 Possible Ranking of Outcomes for Row When Up Is Superdominant

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		COLIN	
		Ask her out	Don't ask
ROWAN	Ask her out	50%, 50%	100%, 0%
	Don't ask	0%, 100%	0%, 0%

FIGURE 4.7 The Dating Game

Any superdominant strategy must also be dominant, but a dominant strategy need not be superdominant. For instance, in the prisoners' dilemma shown in Figure 4.4, each player's dominant strategy Confess is *not* superdominant because the worst possible outcome when playing Confess (10 years in jail) is worse than the best possible outcome when playing Deny (3 years in jail).

For a simple example of superdominance, consider a game played between two high-school boys, Rowan and Colin, each of whom has a crush on the same girl (Winnie) and would like to take her to the big dance.[10] Each boy decides whether to ask Winnie out, knowing that Winnie likes them equally; so, if both boys ask her out, Winnie will flip a coin to decide whose offer to accept. Each boy wants to maximize his own chance of taking Winnie to the dance. The payoffs to each boy, which correspond to the probability that Winnie will go with him to the dance, are shown in Figure 4.7. Asking Winnie out is superdominant for each of them, because the worst possible outcome after asking her out (50% chance) is better than the best possible outcome after not asking (0% chance).

The distinction between dominance and superdominance can be especially important in games with sequential moves. A player with a dominant strategy in a simultaneous-move game might prefer not to play that strategy if the game were changed to have sequential moves. By contrast, a player with a superdominant strategy will always prefer to play it, no matter what the timing of moves. We will return to discuss this issue in depth later in the book, especially in Chapters 6 and 8.

B. Weak Dominance

In a simultaneous-move game, a strategy is weakly dominant for a player if the outcome when playing that strategy is *never worse* than the outcome when playing any other strategy, no matter what strategies other players adopt. Recall that a strategy is dominant for a player if it is *always better* than any other strategy, no matter what strategies other players adopt. The difference is that weak dominance allows for the possibility that a player may be indifferent between her weakly dominant strategy and some other strategy; that is, that for some subset of choices made by other players, she gets the same payoff from her weakly dominant strategy that she gets from some other strategy available to her, but for the remainder of the other players' choices, the weakly dominant strategy gives her a higher payoff than any other strategy. Dominance (or, more precisely, strict dominance) requires that she always get the highest payoff from her dominant strategy. This distinction may seem insignificant, but it has important implications for how one should analyze games.

Recall from Section 3 that when a rational player in a simultaneous-move game has a dominated strategy, we expect that she will not play that strategy because its outcome is always worse than that of any other strategy. This allows us to eliminate the dominated strategy from further consideration and lets us focus on the resulting simpler game. But what if a player has only a *weakly* dominated strategy? Eliminating such a strategy from consideration entails an extra assumption: that the player with a weakly dominant strategy will always "break ties" in favor of playing it. But we cannot be sure that she would do that.

In fact, in some games, there are Nash equilibria in which one or more players use weakly dominated strategies. Consider the game illustrated in Figure 4.8, in which, for Rowena, Up is weakly dominated by Down. (If Colin plays Left, then Rowena gets a better payoff by playing Down than by playing Up, and, if Colin plays Right, then Rowena gets the same payoffs from her two strategies.) Similarly, for Colin, Right weakly dominates Left. So the game is dominance solvable, and solving it tells us that (Down, Right) is a Nash equilibrium. However, while (Down, Right) might be the predicted outcome in many cases, there are other possible equilibrium outcomes of this game. (Down, Left) and (Up, Right) are also Nash equilibria. When Rowena is playing Down, Colin cannot improve his payoff by switching to Right, and when Colin is playing Left, Rowena's best response is clearly to play Down; similar reasoning verifies that (Up, Right) is also a Nash equilibrium.

		COLIN	
		Left	Right
ROWENA	Up	0, 0	1, 1
	Down	1, 1	1, 1

FIGURE 4.8 Elimination of Weakly Dominated Strategies

Therefore, if you eliminate weakly dominated strategies, it is a good idea to use other methods (such as the one described in the next section) to see if you have missed any other equilibria. The iterated dominance solution seems to be a reasonable outcome to predict as the Nash equilibrium of this simultaneous-move game, but it is also important to consider the significance of the multiple equilibria as well as that of the other equilibria themselves. We will address these issues in later chapters, taking up a discussion of multiple equilibria in Chapter 5 and the interconnections between sequential- and simultaneous-move games in Chapter 6.

Endnotes

- We use these names for players in some of our examples and end-of-chapter exercises, along with Rowena and Collette, in the hope that they will aid you in remembering which player chooses the row and which chooses the column. We acknowledge Robert Aumann, who shared the Nobel Prize in Economics with Thomas Schelling in 2005 (and whose ideas will be prominent in Chapter 8), for inventing this clever naming idea. Return to reference 10

Glossary

superdominant
: A strategy is superdominant for a player if the worst possible outcome when playing that strategy is better than the best possible outcome when playing any other strategy.

weakly dominant
: A strategy is weakly dominant for a player if the outcome when playing that strategy is never worse than the outcome when playing any other strategy, no matter what strategies other players adopt.

5 BEST-RESPONSE ANALYSIS

Many simultaneous-move games have no dominant strategies and no dominated strategies. Others have one or several dominated strategies, but iterated elimination of dominated strategies does not yield a unique outcome. In such cases, we need to take another step in the process of solving the game. We are still looking for a Nash equilibrium in which every player does the best she can given the actions of the other players, but we must now rely on subtler strategic thinking than the simple elimination of dominated strategies requires.

A. Identifying Best Responses

In this section, we develop another systematic method for finding Nash equilibria that will prove very useful in later analyses. We begin without imposing a requirement of correctness of beliefs about other players' actions. We take each player's perspective in turn and ask the following question: For each of the choices that the other players might be making, what is the best choice for this player? Thus, we find the best responses of each player to all available strategies of the others. In mathematical terms, we find each player's best-response strategy depending on, or as a function of, the other players' available strategies.

		COLUMN		
		Left	Middle	Right
ROW	Top	3, 1	2, 3	10, 2
	High	4, 5	3, 0	6, 4
	Low	2, 2	5, 4	12, 3
	Bottom	5, 6	4, 5	9, 7

You may need to scroll left and right to see the full figure.

FIGURE 4.9 Best-Response Analysis

Let's return to the game played by Row and Column in Figure 4.1 and reproduce it as Figure 4.9. We first consider Row's responses. If Column chooses Left, Row's best response is Bottom, yielding 5. We show this best response by circling that payoff in the game table. If Column chooses Middle, Row's best response is Low (also yielding 5). And if Column chooses Right, Row's best choice is again Low (now yielding 12). Again, we show Row's best choices by circling the appropriate payoffs. Similarly, Column's best responses are shown by circling her payoffs: 3 (Middle as best response to Row's Top), 5 (Left to Row's High), 4 (Middle to Row's Low), and 7 (Right to Row's Bottom).[11] We see that one cell —namely, (Low, Middle)—has both its payoffs circled. Therefore, the strategies Low for Row and Middle for Column are simultaneously best responses to each other. We have found the Nash equilibrium of this game. (Again.)

Best-response analysis is a comprehensive way of locating *all* possible Nash equilibria of a game. You should improve your understanding of it by trying it out on the other games that have been used as examples in this chapter. The game in Figure 4.8, where both players have a weakly dominant strategy, is one of interest. You will find that there are ties for best responses in that game. Rowena's Up and Down both yield her a payoff of 1 in response to Colin's choice of Right. And Colin's Left and Right both yield him a payoff of 1 in response to Rowena's choice of Down. In such cases, you should circle both (or all) of the payoffs that tie for best response. Your final analysis will show three cells with both payoffs circled, a confirmation that there are indeed three Nash equilibria in that game.

Comparing best-response analysis with the dominance method is also enlightening. If Row has a dominant strategy, that same strategy is her best response to all of Column's strategies; therefore, her best responses are all lined up horizontally in the same row. Similarly, if Column has a dominant strategy, her best responses are all lined up vertically in the same column. You should see for yourself how the Nash equilibria in the Husband–Wife prisoners' dilemma shown in Figure 4.4 and the Congress–Federal Reserve game depicted in Figure 4.5 emerge from such an analysis.

There will be some games for which best-response analysis does not find a Nash equilibrium, just as not all games are dominance solvable. But in this case, we can say something more specific than can be said when the iterated dominance fails. When best-response analysis of a discrete-strategy game does not find a Nash equilibrium, then the game has no equilibrium in pure strategies. We will discuss games of this type in Section 8 of this chapter. In Chapter 5, we will extend best-response analysis to games where the players' strategies are continuous variables—for example, prices or advertising expenditures. Moreover, we will construct best-response *curves* to help us find Nash equilibria, and we will see that such games are less likely—by virtue of the continuity of strategy choices—to have no equilibrium in pure strategies.

B. Ordinal Payoffs

Best-response analysis depends (only) on how players rank different outcomes. Each player's ranking of the possible outcomes is referred to as her ordinal payoff. Even if two game tables show different payoff numbers (*cardinal payoffs*) in each cell, the players' rankings over the cells may be the same in each table. For instance, compare the new version of the prisoners' dilemma shown in Figure 4.10a with that shown earlier in Figure 4.4. The numbers of years that Husband and Wife would spend in jail in each possible outcome have been changed in Figure 4.10a. However, the way in which the players rank the four possible outcomes is the same in both games. These ordinal payoffs for the prisoners' dilemma, shown in Figure 4.10b, are identical for *any* prisoners' dilemma regardless of the cardinal payoff numbers involved. In every prisoners' dilemma, "only I confess" is best (ordinal payoff of 4), "both deny" is second-best (ordinal payoff of 3), "both confess" is third-best (ordinal payoff of 2), and "only I deny" is worst (ordinal payoff of 1).

A prisoners' dilemma has three essential defining features. First, each player has two strategies: Cooperate (with the other player—deny any involvement with the crime, in our example) or Defect (from cooperation with the other player—here, confess to the crime). Second, each player has a dominant strategy: Defect. Third, both players are worse off when they both play their dominant strategy, Defect, than when they both play Cooperate. These defining features of the prisoners' dilemma imply that the players' ordinal payoffs must be as shown in Figure 4.10b. To see why, consider the generic row player, Rowan. (The same logic applies to the generic column player, Colette.) Because Defect is Rowan's dominant strategy, (Defect, Cooperate) must be better than

(Cooperate, Cooperate) and (Defect, Defect) must be better than (Cooperate, Defect). Since both players are worse off when they both confess than when they both deny, Rowan's ranking of the possible outcomes must therefore be (Defect, Cooperate) > (Cooperate, Cooperate) > (Defect, Defect) > (Cooperate, Defect), as shown in Figure 4.10b.

(a) Cardinal payoffs for new prisoners' dilemma

		WIFE	
		Confess (Defect)	Deny (Cooperate)
HUSBAND	Confess (Defect)	5 yr, 5 yr	0 yr, 50 yr
	Deny (Cooperate)	50 yr, 0 yr	1 yr, 1 yr

You may need to scroll left and right to see the full figure.

(b) Ordinal payoffs for all prisoners' dilemmas

		COLLETTE	
		Confess (Defect)	Deny (Cooperate)
ROWAN	Confess (Defect)	2, 2	4, 1

You may need to scroll left and right to see the full figure.

		COLLETTE	
		Confess (Defect)	Deny (Cooperate)
	Deny (Cooperate)	1, 4	3, 3

You may need to scroll left and right to see the full figure.

FIGURE 4.10 Cardinal and Ordinal Payoffs in the Prisoners' Dilemma

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More generally, the best responses among the pure strategies available in a game depend only on the way players rank the outcomes associated with those strategies. Therefore, Nash equilibria in pure strategies also depend only on ordinal payoffs, not on the actual payoff numbers. The same will not be true in games with Nash equilibria in mixed strategies. When we consider mixed strategies in Chapter 7, we will need to take averages of payoff numbers weighted by the probabilities of choosing those strategies, so the actual numbers will matter.

Endnotes

- Alternatively and equivalently, one could mark in some way the choices that are *not* made. For example, in Figure 4.3, Row will not choose Top, High, or Bottom as responses to Column’s Right; one could show this by drawing slashes through Row’s payoffs in these cases: 10, 6, and 9, respectively. When this is done for all strategies of both players, (Low, Middle) has both of its payoffs unslashed; it is thus the Nash equilibrium of the game. The alternatives of circling choices that are made and slashing choices that are not made stand in a relation to each other that is conceptually similar to that between the alternatives of showing chosen branches with arrows and pruning unchosen branches for sequential-move games. We prefer the first alternative in each case because the resulting picture looks cleaner and tells the story better. Return to reference 11

Glossary

best-response analysis
: Finding the Nash equilibria of a game by calculating the best-response functions or curves of each player and solving them simultaneously for the strategies of all the players.

ordinal payoffs
: Each player’s ranking of the possible outcomes in a game.

6 THREE PLAYERS

So far, we have analyzed only games between two players. All the methods of analysis that have been discussed, however, can be used to find pure-strategy Nash equilibria in any simultaneous-move game among any number of players. When a game is played by more than two players, each of whom has a relatively small number of pure strategies available, we can use a game table for our analysis, as we did in the first five sections of this chapter.

In Chapter 3, we described a game among three players, each of whom had two pure strategies. The three players, Emily, Nina, and Talia, had to choose whether to contribute toward the creation of a flower garden for their small street. We assumed that the garden would be no better when all three contributed than when only two contributed, and that a garden with just one contributor would be so sparse that it was as bad as no garden at all. Now let us suppose instead that the three players make their choices simultaneously and that there is a somewhat richer variety of possible outcomes and payoffs. In particular, the size and splendor of the garden will now differ according to the exact number of contributors: Three contributors will produce the largest and best garden, two contributors will produce a medium-sized garden, and one contributor will produce a small garden.

Suppose Emily is contemplating the possible outcomes of the street-garden game. There are six possible choices for her to consider. Emily can choose either to contribute or not to contribute when both Nina and Talia contribute, or when neither of them contributes, or when just one of them contributes. From her perspective, the best possible outcome, with a rank of 6, would be to take advantage of her good-hearted neighbors and to have both Nina and Talia contribute while she does not. Emily could then enjoy a medium-sized garden without putting up her own hard-earned cash. If both of the others contribute and Emily also contributes, she gets to enjoy a large, splendid garden, but at the cost of her own contribution; she ranks this outcome second best, or 5.

At the other end of the spectrum are the outcomes that arise when neither Nina nor Talia contributes to the garden. If that is the case, Emily would again prefer not to contribute, because she would foot the entire bill for a small public garden that everyone could enjoy; she would rather have the flowers in her own yard. Thus, when neither of the other players is contributing, Emily ranks the outcome in which she contributes as a 1 and the outcome in which she does not as a 2.

In between these cases are the situations in which either Nina or Talia contributes to the flower garden, but not both of them. When one of them contributes, Emily knows that she can enjoy a small garden without contributing; she also feels that the cost of her contribution outweighs the increase in benefit that she would get by increasing the size of the garden. Thus, she ranks the outcome in which she does not contribute but still enjoys the small garden as a 4 and the outcome in which she does contribute, thereby providing a medium-sized garden, as a 3. Because Nina and Talia have the same views as Emily on the costs and benefits of contributions and garden size, each of them ranks the different outcomes in the same way—the worst outcome being the one in which each contributes and the other two do not, and so on.

If all three women decide whether to contribute to the garden without knowing what their neighbors will do, we have a three-person simultaneous-move game. To find the Nash equilibrium of this game, we then need a game table. For a three-player game, the table must be three-dimensional, and the third player’s strategies must correspond to the new dimension. The easiest way to add a third dimension to a two-dimensional game table is to add pages. The first page of the table shows payoffs for the third player’ s first strategy, the second page shows payoffs for the third player’ s second strategy, and so on.

TALIA chooses:

Contribute

		NINA	
		Contribute	**Don’ t**
EMILY	Contribute	5, 5, 5	3, 6, 3
	Don’ t	6, 3, 3	4, 4, 1

Don’ t Contribute

		NINA	
		Contribute	**Don’ t**
EMILY	Contribute	3, 3, 6	1, 4, 4
	Don’ t	4, 1, 4	2, 2, 2

You may need to scroll left and right to see the full figure.

Figure 4.11 Street-Garden Game

We show the three-dimensional table for the street-garden game in Figure 4.11. It has two rows for Emily’ s two strategies, two columns for Nina’ s two strategies, and two pages for Talia’ s two strategies. We show the pages side by side here so that you can see everything at the same time. In each cell, payoffs are listed for the row player first, the column player second, and the page player third; in this case, the order is Emily, Nina, Talia.

Our first test should be to determine whether there are dominant strategies for any of the players. In one-page game tables, we found this test to be simple: we just compared the outcomes associated with one of a player’s strategies with the outcomes associated with another of her strategies. In practice, this comparison required, for the row player, a simple check within the columns of the single page of the table, and vice versa for the column player. Here, we must check both pages of the table to determine whether any player has a dominant strategy.

For Emily, we compare the two rows of both pages of the table and note that when Talia contributes, Emily has a dominant strategy, Don’t Contribute, and when Talia does not contribute, Emily also has a dominant strategy, Don’t Contribute. Thus, the best thing for Emily to do, regardless of what either of the other players does, is not to contribute. Similarly, we see that Nina’s dominant strategy—in both pages of the table—is Don’t Contribute. When we check for a dominant strategy for Talia, we have to be a bit more careful. We must compare outcomes that keep Emily’s and Nina’s behavior constant, checking Talia’s payoffs from choosing Contribute versus Don’t Contribute. That is, we must compare cells across pages of the table—the top-left cell in the first page (on the left) with the top-left cell in the second page (on the right), and so on. As for the first two players, this process indicates that Talia also has a dominant strategy, Don’t Contribute.

Each player in this game has a dominant strategy, which must therefore be her equilibrium strategy. The pure-strategy Nash equilibrium of the street-garden game entails all three players choosing not to contribute to the street garden and getting their second-worst payoffs: The garden is not planted, but no one has to contribute to it, either.

TALIA chooses:

Contribute

	NINA	
	Contribute	Don’t

		NINA	
		Contribute	Don't
EMILY	Contribute	5, 5, 5	3, 6, 3
	Don't	6, 3, 3	4, 4, 1

Don't Contribute

		NINA	
		Contribute	Don't
EMILY	Contribute	3, 3, 6	1, 4, 4
	Don't	4, 1, 4	2, 2, 2
You may need to scroll left and right to see the full figure.			

Figure 4.12 Best-Response Analysis in the Street-Garden Game

Notice that this game is yet another example of a prisoners' dilemma. It has a unique Nash equilibrium in which all players receive a payoff of 2. Yet there is another outcome in the game—in which all three neighbors contribute to the garden—that yields higher payoffs of 5 for all three players. Even though it would be beneficial to each of them for all of them to pitch in to build the garden, none of them has the individual incentive to do so. As a result, gardens of this type are often not planted at all, or they are paid for with tax dollars because the town government can require its citizens to pay such taxes. In Chapter 11, we will encounter more such dilemmas of collective action and study some methods for resolving them.

The Nash equilibrium of this game can also be found using best-response analysis, as shown in Figure 4.12. Because each player has Don't Contribute as her dominant strategy, all of

Emily’s best responses are on her Don’t Contribute row, all of Nina’s best responses are on her Don’t Contribute column, and all of Talia’s best responses are on her Don’t Contribute page. The bottom-right cell on the right page contains all three best responses; therefore, it gives us the Nash equilibrium.

7 MULTIPLE EQUILIBRIA IN PURE STRATEGIES

Most of the games considered in the preceding sections have had a unique pure-strategy Nash equilibrium. In general, however, games need not have unique Nash equilibria. We illustrate this possibility using a class of games that has many applications. As a group, they may be labeled coordination games. The players in such games have some (but not always completely) common interests, but because they act independently (by virtue of the nature of noncooperative games), the coordination of actions needed to achieve a jointly preferred outcome is problematic.

A. Pure Coordination: Will Holmes Meet Watson?

To illustrate the idea behind coordination games, we can imagine a situation involving Sherlock Holmes and Dr. Watson early in 1881, the first year of their partnership. They are hurriedly leaving their Baker Street apartment, each with a specific task to accomplish regarding a new case. As they rush off in different directions, Holmes shouts back to Watson that they should rendezvous at the end of the day to compare notes, at “four o’clock at our meeting place.” Later, while collecting evidence for the case, each realizes that in their rush that morning they failed to be precise about the meaning of “our meeting place.” Had Holmes meant to imply “the place where we first met each other” in January—St. Bartholomew’s Hospital? Or had he meant to imply “the place where we met recently for a meal”—Simpson’s in the Strand? Unfortunately, these two locations are sufficiently far apart across the crowded city of London that it will be impossible to try both. Having no other way to contact the other, each is left with a quandary: Which meeting place should he choose?

The game table in Figure 4.13 illustrates this situation. Each player has two choices: St. Bart’s and Simpson’s. The payoffs for each are 1 if they meet and 0 if they do not. Best-response analysis quickly reveals that the game has two Nash equilibria, one where both choose St. Bart’s and the other where both choose Simpson’s. It is important for both that they achieve one of the equilibria, but which one is immaterial because the two yield equal payoffs. All that matters is that they coordinate on the same action; it does not matter which action. That is why the game is said to be one of pure coordination.

But will they coordinate successfully? Or will they end up in different locations, each thinking that the other has left him or succumbed to some disastrous fate? Alas, that risk exists. Holmes might think that Watson will go to St. Bart's because he said something about following up a lead near the hospital that morning. But Watson might have the opposite belief about what Holmes will do. When there are multiple Nash equilibria, if the players are to select one successfully, they need some way to coordinate their beliefs or expectations about each other's actions.

Their situation is similar to that of the heroes of the "Which tire?" game in Chapter 1, where we labeled the coordination device a focal point. In the present context, one of the two locations might be closer to where each of the partners was heading that day. But it is not enough that Holmes knows this to be the case. He must know that Watson knows, and that he knows that Holmes knows, and so on. In other words, their expectations must *converge* on the focal point. Otherwise, Holmes may be doubtful about where Watson will go because he does not know what Watson is thinking about where Holmes will go, and similar doubts may arise at the third or fourth or higher levels of thinking about thinking.[12]

		WATSON	
		St. Bart's	Simpson's
HOLMES	St. Bart's	1, 1	0, 0
	Simpson's	0, 0	1, 1

FIGURE 4.13 Pure Coordination

Over time, the two players here might come to understand "our meeting place" to mean exactly one of the possible interpretations. (As, in fact, they did. Simpson's became "our Strand restaurant" for Holmes and Watson, but not until much later in their relationship.) But without the benefit of a lengthy partnership, they have no obvious choice.

In general, whether players in coordination games can find a focal point depends on whether they have some commonly known point of contact, whether historical, cultural, or linguistic.

B. Assurance: Will Holmes Meet Watson? And Where?

Now let’s change the game’s payoffs a little. It may be that our pair is not quite indifferent about which location they choose. Meeting at the hospital might be safer, as they will be protected from the prying eyes of other Londoners at dinner. Or they may both want to choose the location at which they can acquire a spot of tea in nice surroundings. Suppose they both prefer to meet at Simpson’s; then the payoff to each is 2 when they meet there versus 1 when they meet at St. Bart’s. The new payoff matrix is shown in Figure 4.14.

Again, there are two Nash equilibria. But in this version of the game, each prefers the equilibrium where both choose Simpson’s. Unfortunately, their mere liking of that outcome is not guaranteed to bring it about. First of all (and as always in our analysis), the payoffs have to be common knowledge—both have to know the entire payoff matrix, both have to know that both know, and so on. Such detailed knowledge about the game could arise if the two had discussed and agreed on the relative merits of the locations, but Holmes simply forgot to clarify that they should meet at Simpson’s. Even then, Holmes might think that Watson has some other reason for choosing St. Bart’s, or he might think that Watson thinks that he does, and so on. Without genuine convergence of expectations about actions, the players may choose the worse equilibrium, or worse still, they may fail to coordinate actions and get payoffs of 0 each.

		WATSON	
		St. Bart’s	Simpson’s
HOLMES	St. Bart’s	1, 1	0, 0

		WATSON	
		St. Bart’s	Simpson’s
	Simpson’s	0, 0	2, 2

FIGURE 4.14 Assurance

Thus, players in the game illustrated in Figure 4.14 can get the preferred equilibrium outcome only if each has enough certainty or assurance that the other is choosing the appropriate action. For this reason, such games are called assurance games.[13]

In many real-life situations of this kind, such assurance is easily obtained, given even a small amount of communication between the players. Their interests are perfectly aligned, so if one of them says to the other, “I will go to Simpson’s,” the other has no reason to doubt the truth of this statement and will go to Simpson’s to get the mutually preferred outcome. That is why we had to construct the story with the two friends dashing off in different directions with no later means of communication. If the players’ interests conflict, truthful communication becomes more problematic. We examine this problem further when we consider strategic manipulation of information in games in Chapter 9.

In larger groups, communication can be achieved by scheduling meetings or by making announcements. These devices work only if everyone knows that everyone else is paying attention to them, because successful coordination requires the desired equilibrium to be a focal point. The players’ expectations must converge on it; everyone should know that everyone knows that . . . everyone is choosing it. Many social institutions and arrangements play this role. Meetings where the participants sit in a circle facing inward ensure that everyone sees everyone else paying attention. Advertisements during the Super Bowl, especially when they are proclaimed in advance as major attractions, assure each viewer that many others are viewing them also. That makes such ads especially attractive to companies making products that are more desirable for any one buyer when many others are buying them, too; such products include those produced by the computer, telecommunication, and Internet industries.[14]

		WATSON	
		St. Bart's	Simpson's
HOLMES	St. Bart's	2, 1	0, 0
	Simpson's	0, 0	1, 2

FIGURE 4.15 Battle of the Sexes

C. Battle of the Sexes: Will Holmes Meet Watson? And Where?

Now let's introduce another complication to the coordination game: Both players want to meet, but prefer different locations. So Holmes might get a payoff of 2 and Watson a payoff of 1 from meeting at St. Bart's, and the other way around from meeting at Simpson's. This payoff matrix is shown in Figure 4.15.

This game is called the battle of the sexes. The name derives from the story concocted for this payoff structure by game theorists in the sexist 1950s. A husband and wife were supposed to choose between going to a boxing match and going to a ballet, and (presumably for evolutionary genetic reasons) the husband was supposed to prefer the boxing match and the wife the ballet. The name has stuck, and we will keep it, but our example should make it clear that it does not necessarily have sexist connotations.

What will happen in this game? There are still two Nash equilibria. If Holmes believes that Watson will choose St. Bart's, it is best for him to do likewise. For similar reasons, Simpson's is also a Nash equilibrium. To achieve either of these equilibria and avoid the outcomes where the two go to different locations, the players need a focal point, or convergence of expectations, as in the pure-coordination and assurance games. But the risk of coordination failure is greater in the battle of the sexes. The players are initially in quite symmetric situations, but each of the two Nash equilibria gives them asymmetric payoffs, and their preferences between the two outcomes are in conflict. Holmes prefers the outcome where they meet at St. Bart's, and Watson prefers to meet at Simpson's. They must find some way of breaking the symmetry.

In an attempt to achieve his preferred equilibrium, each player may try to "act tough" and follow the strategy leading to that equilibrium. In Chapter 8, we will consider in detail such advance devices, called *strategic moves*, that players in such games can adopt to try to achieve their preferred outcomes. Or each may try to be nice, leading to the unfortunate situation where Holmes goes to Simpson's because he wants to please Watson, only to find that Watson has chosen to please him and has gone to St. Bart's, like the couple choosing Christmas presents for each other in O. Henry's short story titled "The Gift of the Magi." Or if the game is repeated, successful coordination may be negotiated and maintained as an equilibrium. For example, Holmes and Watson may arrange to alternate between meeting locations on various days or during different cases. In Chapter 10, we will examine such tacit cooperation in repeated games in the context of a prisoners' dilemma.

D. Chicken: Will James Meet Dean?

Our final example in this section is a slightly different kind of coordination game. In this game, the players want to avoid, not choose, the same strategies. Further, the consequences of coordination failure in this kind of game are far more drastic than in the other kinds.

The story comes from a game that was supposedly played by American teenagers in the 1950s. Two teenagers take their cars to opposite ends of Main Street, Middle-of-Nowhere, USA, at midnight and start to drive toward each other. The one who swerves to prevent a collision is the "chicken," and the one who keeps going straight is the winner. If both maintain a straight course, there is a collision in which both cars are damaged and both players injured.[15]

The payoffs for games of chicken depend on how negatively one rates the "bad" outcome—being hurt and damaging your car in this case—against being labeled chicken. As long as words hurt less than crunching metal, a reasonable payoff table for the 1950s version of chicken is the one found in Figure 4.16. Each player's highest-ranked outcome is to win, having the other be chicken, and each player's lowest-ranked outcome is the crash of the two cars. In between these two extremes, it is better to have your rival be chicken (to save face) than to be chicken yourself.

This story has four essential features that define any game of chicken. First, each player has one strategy that is the "tough" strategy and one that is the "weak" strategy. Second, there are two pure-strategy Nash equilibria. These are the outcomes in which exactly one of the players is chicken, or weak. Third, each player strictly prefers that equilibrium in which the other player chooses chicken.

Fourth, the payoffs when both players are tough are very bad for both players. In games such as this one, the real game becomes a test of how to achieve one's preferred equilibrium.

		DEAN	
		Swerve (Chicken)	Straight (Tough)
JAMES	Swerve (Chicken)	0, 0	−1, 1
	Straight (Tough)	1, −1	−2, −2

FIGURE 4.16 Chicken

We are now back in a situation similar to the battle of the sexes. One expects most real-life games of chicken to be even worse as battles than most battles of the sexes—the benefit of winning is larger, as is the cost of a crash, so all the problems of conflict of interest and asymmetry between the players are aggravated. Each player will want to influence the outcome. One player may try to create an aura of toughness that everyone recognizes so as to intimidate all rivals.[16] Another possibility is to come up with some other way to convince your rival that you will not be chicken by making a visible and irreversible commitment to going straight. (In Chapter 8, we consider just how to make such commitment moves.) In addition, both players also want to try to prevent the worst outcome (a crash) if at all possible.

As with the battle of the sexes, if the game is repeated, tacit coordination is a better route to a solution. That is, if two teenagers played chicken every Saturday night at midnight, they would have the benefit of knowing that the game had both a history and a future when deciding on their strategies. In such a situation, they might logically choose to alternate between the two equilibria, taking turns being the winner every other week. (But if the others found out about this deal, both players would lose face.)

There is one final point arising from coordination games that must be addressed: The concept of Nash equilibrium requires each player to have the correct belief about the other’s choice of strategy. When we look for Nash equilibria in pure strategies, the concept requires each player to be confident about the other’s choice. But our analysis of coordination games shows that thinking about the other’s choice in such games is fraught with strategic uncertainty. How can we incorporate such uncertainty in our analysis? In Chapter 7, we introduce the concept of a mixed strategy, where actual choices are made randomly among the available actions. This approach generalizes the concept of Nash equilibrium to situations where the players may be unsure about each other’s actions.

Endnotes

- Thomas Schelling presented the classic treatment of coordination games and developed the concept of a focal point in his book *The Strategy of Conflict* (Cambridge, Mass.: Harvard University Press, 1960); see pp. 54–58, 89–118. His explanation of focal points included the results garnered when he posed several questions to his students and colleagues. The best remembered of these is, "Suppose you have arranged to meet someone in New York City on a particular day, but have failed to arrange a specific place or time, and have no way of communicating with the other person. Where will you go and at what time?" Fifty years ago, when the question was first posed, the clock at Grand Central Station was the usual focal place; now it might be the stairs at TKTS in Times Square. The focal time remains 12:00 noon. Return to reference 12
- The classic example of an assurance game usually offered is the stag hunt described by the eighteenth-century French philosopher Jean-Jacques Rousseau. Several people can successfully hunt a stag, thereby getting a large quantity of meat, if they collaborate. If any one of them is sure that all the others will collaborate, he also stands to benefit by joining the group. But if he is unsure whether the group will be large enough, he will do better to hunt for a smaller animal, a hare, on his own. However, it can be argued that Rousseau believed that each hunter would prefer to go after a hare regardless of what the others were doing, which would make the stag hunt a multiplayer prisoners' dilemma, not an assurance game. We discuss this example in the context of collective action Chapter 11. Return to reference 13
- Michael Chwe develops this theme in *Rational Ritual: Culture, Coordination, and Common Knowledge* (Princeton,

N.J.: Princeton University Press, 2001). Return to reference 14

- A slight variant was made famous by the 1955 James Dean movie *Rebel without a Cause*. There, two players drove their cars in parallel, very fast, toward a cliff. The first to jump out of his car before it went over the cliff was the chicken. The other, if he left too late, risked going over the cliff in his car to his death. The characters in the film referred to this as a "chicky game." In the mid-1960s, the British philosopher Bertrand Russell and other peace activists used the game of chicken as an analogy for the nuclear arms race between the United States and the USSR, and the game theorist Anatole Rapoport gave a formal game-theoretic statement to that effect. Other game theorists have chosen to interpret the arms race as a prisoners' dilemma or as an assurance game. For a review and interesting discussion, see Barry O'Neill, "Game Theory Models of Peace and War," in *The Handbook of Game Theory*, vol. 2, ed. Robert J. Aumann and Sergiu Hart (Amsterdam: North Holland, 1994), pp. 995–1053. Return to reference 15
- Why would a potential rival play chicken against someone with a reputation for never giving in? The problem is that participation in chicken, as in lawsuits, is not really voluntary. Put another way, choosing whether to play chicken is itself a game of chicken. As Thomas Schelling says, "If you are publicly invited to play chicken and say you would rather not, then you have just played [and lost]." *Arms and Influence* (New Haven, Conn.: Yale University Press, 1965), p. 118. Return to reference 16

Glossary

coordination game
: A game with multiple Nash equilibria, where the players are unanimous about the relative merits of the equilibria, and prefer any equilibrium to any of the nonequilibrium possibilities. Their actions must somehow be coordinated to achieve the preferred equilibrium as the outcome.

pure coordination game
: A coordination game where the payoffs of each player are the same in all the Nash equilibria. Thus, all players are indifferent among all the Nash equilibria, and coordination is needed only to ensure avoidance of a nonequilibrium outcome.

focal point
: A configuration of strategies for the players in a game, which emerges as the outcome because of the convergence of the players' expectations on it.

convergence of expectations
: A situation where the players in a noncooperative game can develop a common understanding of the strategies they expect will be chosen.

assurance game
: A game where each player has two strategies, say, Cooperate and Not, such that the best response of each is to Cooperate if the other cooperates, Not if not, and the outcome from (Cooperate, Cooperate) is better for both than the outcome of (Not, Not).

battle of the sexes
: A game where each player has two strategies, say, Hard and Soft, such that [1] (Hard, Soft) and (Soft, Hard) are both Nash equilibria, [2] of the two Nash equilibria, each player prefers the outcome where he is Hard and the other is Soft, and [3] both prefer the Nash equilibria to

the other two possibilities, (Hard, Hard) and (Soft, Soft).

chicken

A game where each player has two strategies, say Tough and Weak, such that [1] both (Tough, Weak) and (Weak, Tough) are Nash equilibria, [2] of the two, each prefers the outcome where she plays Tough and the other plays Weak, and [3] the outcome (Tough, Tough) is worst for both.

8 NO EQUILIBRIUM IN PURE STRATEGIES

Each of the games considered so far has had at least one Nash equilibrium in pure strategies. Some of these games, such as the coordination games in Section 7, had more than one Nash equilibrium, whereas games in earlier sections had exactly one. Unfortunately, not all games that we come across in the study of strategy and game theory will have easily definable outcomes in which players always choose one particular action as an equilibrium strategy. In this section, we look at games in which there is not even one pure-strategy Nash equilibrium—games in which none of the players would consistently choose one strategy as his equilibrium action.

A simple example of a game with no equilibrium in pure strategies is a single point in a tennis match. Recall the tennis match we first introduced in Chapter 1, Section 2.A, between the two all-time best women players—Martina Navratilova and Chris Evert.[17] Navratilova, at the net, has just volleyed a ball to Evert on the baseline, and Evert is about to attempt a passing shot. She can try to send the ball either down the line (DL; a hard, straight shot) or crosscourt (CC; a softer, diagonal shot). Navratilova must likewise prepare to cover one side or the other. Each player is aware that she must not give any indication of her planned action to her opponent, knowing that such information will be used against her: Navratilova will move to cover the side to which Evert is planning to hit, or Evert will hit to the side that Navratilova is not planning to cover. Both must act in a fraction of a second, and both are equally good at concealing their intentions until the last possible moment; therefore, their actions are effectively simultaneous, and we can analyze the point as a two-player simultaneous-move game.

Payoffs in this tennis-point game are given by the fraction of times a player wins the point in any particular combination of passing shot and covering play. Given that a down-the-line passing shot is stronger than a crosscourt shot and that Evert is more likely to win the point when Navratilova moves to cover the wrong side of the court, we can work out a reasonable set of payoffs. Suppose Evert is successful with a down-the-line passing shot 80% of the time if Navratilova covers crosscourt, but only 50% of the time if Navratilova covers down the line. Similarly, Evert is successful with her crosscourt passing shot 90% of the time if Navratilova covers down the line. This success rate is higher than when Navratilova covers crosscourt, in which case Evert wins only 20% of the time.

		NAVRATILOVA	
		DL	CC
EVERT	DL	50, 50	80, 20
	CC	90, 10	20, 80

FIGURE 4.17 No Equilibrium in Pure Strategies

Clearly, the percentage of times that Navratilova wins this tennis point is simply the difference between 100% and the percentage of times that Evert wins. Thus, the game is zero-sum (even though the two payoffs technically sum to 100), and we can represent all the necessary information in the payoff table with just the payoff to Evert in each cell. Figure 4.17 shows that payoff table and the percentage of times that Evert wins the point against Navratilova in each of the four possible combinations of their strategy choices.

The rules for solving simultaneous-move games tell us to look first for dominant or dominated strategies and then to use best-response analysis to find a Nash equilibrium. It is a useful exercise to verify that no dominant strategies exist here. Going on to best-response analysis, we find that Evert’ s best response to DL is CC, and that her best response to CC is DL. By contrast, Navratilova’ s best response to DL is DL, and her best response to CC is CC. None of the cells in the game table is a Nash equilibrium, because someone always prefers to change her strategy. For example, if we start in the upper-left cell of the table, we find that Evert prefers to deviate from DL to CC, increasing her own payoff from 50% to 90%. But in the lower-left cell of the table, we find that Navratilova, too, prefers to switch from DL to CC, raising her payoff from 10% to 80%. As you can verify, Evert similarly prefers to deviate from the lower-right cell, and Navratilova prefers to deviate from the upper-right cell. In every cell, one player always wants to change her play, and we cycle through the table endlessly without finding an equilibrium.

An important message is contained in the absence of a Nash equilibrium in this game and similar ones. What is important in games of this type is not what players should do, but what players should *not* do. In our example, each tennis player should neither always nor systematically pick the same shot when faced with the same situation. If either player engages in any determinate behavior of that type, the other can take advantage of it. (So if Evert consistently went crosscourt with her passing shot, Navratilova would learn to cover crosscourt every time and would thereby reduce Evert’ s chances of success with her crosscourt shot.) The most reasonable thing for players to do here is to act somewhat unsystematically, hoping for the element of surprise. An unsystematic approach entails choosing each strategy part of the time. (Evert should be using her weaker shot with enough frequency to guarantee that Navratilova cannot predict which shot will come her way. She should not, however, use the two shots in any set pattern, because that, too, would cause her to lose the element of surprise.) This approach, in which players randomize their actions, is known as mixing strategies, and it is the focus of Chapter 7. The game illustrated in Figure 4.17 may not have an equilibrium in pure strategies, but it can still be solved by looking for an equilibrium in mixed strategies, as we will do in Chapter 7, Section 2.

Endnotes

- For those among you who remember only the latest phenom who shines for a couple of years and then burns out, here are some amazing facts about these two women, who were at the top levels of the game for almost two decades and ran a memorable rivalry all that time. Navratilova was a left-handed serve-and-volley player. In grand-slam tournaments, she won 18 singles titles, 31 doubles, and 7 mixed doubles. In all tournaments, she won 167, a record. Evert, a right-handed baseliner, had a record win-loss percentage (90% wins) in her career and 150 titles, of which 18 were for singles in grand-slam tournaments. She probably invented (and certainly popularized) the two-handed backhand that is now so common. From 1973 to 1988, the two played each other 80 times, and Navratilova ended up with a slight edge, 43 – 37. Return to reference 17

SUMMARY

In simultaneous-move games, players make their strategy choices without knowledge of the choices being made by other players. Such games are illustrated by *game tables*, in which cells show the payoff of each choice to each player, and the dimensionality of the table equals the number of players. Two-person *zero-sum games* may be illustrated in shorthand with only one player’s payoff in each cell of the game table.

Nash equilibrium is the concept used to solve simultaneous-move games, It consists of a list of strategies, one for each player, such that each player has chosen her best response to the other’s choice. Nash equilibrium can also be defined as a list of strategies in which each player has correct *beliefs* about the others’ strategies and chooses the best strategy for herself given those beliefs. Nash equilibria can be found by searching for *dominant strategies*, by *successive elimination of dominated strategies*, or with *best-response analysis.*

There are many classes of simultaneous-move games. *Prisoners’ dilemma* games appear in many contexts. Coordination games, such as *assurance*, *chicken*, and *battle of the sexes*, have multiple equilibria, and the solution of such games requires players to achieve coordination by some means. If a game has no equilibrium in *pure strategies*, we must look for an equilibrium in *mixed strategies*, the analysis of which is presented in Chapter 7.

KEY TERMS

assurance game (110)

battle of the sexes (111)

belief (91)

best response (88)

best-response analysis (102)

chicken (112)

convergence of expectations (110)

coordination game (107)

dominance solvable (97)

dominant strategy (92)

dominated strategy (92)

focal point (108)

game table (86)

iterated elimination of dominated strategies (97)

mixed strategy (86)

Nash equilibrium (89)

normal form (86)

ordinal payoff (103)

payoff matrix (86)

payoff table (86)

prisoners' dilemma (92)

pure coordination game (108)

pure strategy (86)

strategic form (86)

successive elimination of dominated strategies (97)

superdominant (98)

weakly dominant (100)

Glossary

assurance game
: A game where each player has two strategies, say, Cooperate and Not, such that the best response of each is to Cooperate if the other cooperates, Not if not, and the outcome from (Cooperate, Cooperate) is better for both than the outcome of (Not, Not).

battle of the sexes
: A game where each player has two strategies, say, Hard and Soft, such that [1] (Hard, Soft) and (Soft, Hard) are both Nash equilibria, [2] of the two Nash equilibria, each player prefers the outcome where he is Hard and the other is Soft, and [3] both prefer the Nash equilibria to the other two possibilities, (Hard, Hard) and (Soft, Soft).

belief
: The notion held by one player about the strategy choices of the other players and used when choosing his own optimal strategy.

best response
: The strategy that is optimal for one player, given the strategies actually played by the other players, or the belief of this player about the other players' strategy choices.

best-response analysis
: Finding the Nash equilibria of a game by calculating the best-response functions or curves of each player and solving them simultaneously for the strategies of all the players.

chicken
: A game where each player has two strategies, say Tough and Weak, such that [1] both (Tough, Weak) and (Weak, Tough) are Nash equilibria, [2] of the two, each prefers the outcome where she plays Tough and the other plays

Weak, and [3] the outcome (Tough, Tough) is worst for both.

convergence of expectations

A situation where the players in a noncooperative game can develop a common understanding of the strategies they expect will be chosen.

coordination game

A game with multiple Nash equilibria, where the players are unanimous about the relative merits of the equilibria, and prefer any equilibrium to any of the nonequilibrium possibilities. Their actions must somehow be coordinated to achieve the preferred equilibrium as the outcome.

dominance solvable

A game where iterated elimination of dominated strategies leaves a unique outcome, or just one strategy for each player.

dominant strategy

A strategy X is dominant for a player if the outcome when playing X is always better than the outcome when playing any other strategy, no matter what strategies other players adopt.

dominated strategy

A strategy X is dominated by another strategy Y for a player if the outcome when playing X is always worse than the outcome when playing Y, no matter what strategies other players adopt.

focal point

A configuration of strategies for the players in a game, which emerges as the outcome because of the convergence of the players' expectations on it.

game table

A spreadsheetlike table whose dimension equals the number of players in the game; the strategies available to each player are arrayed along one of the dimensions (row, column, page, . . .); and each cell shows the payoffs of all the players in a specified order, corresponding to the configuration of strategies that yield that cell. Also called payoff table.

iterated elimination of dominated strategies

Considering the players in turns and repeating the process in rotation, eliminating all strategies that are dominated for one at a time, and continuing doing so until no such further elimination is possible. Also called successive elimination of dominated strategies.

mixed strategy

A mixed strategy for a player consists of a random choice, to be made with specified probabilities, from his originally specified pure strategies.

Nash equilibrium

A configuration of strategies (one for each player) such that each player's strategy is best for him, given those of the other players. (Can be in pure or mixed strategies.)

normal form

Representation of a game in a game matrix, showing the strategies (which may be numerous and complicated if the game has several moves) available to each player along a separate dimension (row, column, etc.) of the matrix and the outcomes and payoffs in the multidimensional cells. Also called strategic form.

ordinal payoffs

Each player's ranking of the possible outcomes in a game.

payoff matrix

Same as payoff table and game table.

payoff table

Same as game table.

prisoners' dilemma

A game where each player has two strategies, say Cooperate and Defect, such that [1] for each player, Defect dominates Cooperate, and [2] the outcome (Defect, Defect) is worse for both than the outcome (Cooperate, Cooperate).

pure coordination game

A coordination game where the payoffs of each player are the same in all the Nash equilibria. Thus, all players are indifferent among all the Nash equilibria, and coordination is needed only to ensure avoidance of a nonequilibrium outcome.

pure strategy

A rule or plan of action for a player that specifies without any ambiguity or randomness the action to take in each contingency or at each node where it is that player’s turn to act.

strategic form

Same as normal form.

successive elimination of dominated strategies

Same as iterated elimination of dominated strategies.

superdominant

A strategy is superdominant for a player if the worst possible outcome when playing that strategy is better than the best possible outcome when playing any other strategy.

weakly dominant

A strategy is weakly dominant for a player if the outcome when playing that strategy is never worse than the outcome when playing any other strategy, no matter what strategies other players adopt.

SOLVED EXERCISES

1. Find all Nash equilibria in pure strategies for the following games. First, check for dominant strategies. If there are none, solve the games using iterated elimination of dominated strategies. Explain your reasoning.

1.

		COLIN	
		Left	Right
ROWENA	Up	4, 0	3, 1
	Down	2, 2	1, 3

You may need to scroll left and right to see the full figure.

2.

		COLIN	
		Left	Right
ROWENA	Up	2, 4	1, 0
	Down	6, 5	4, 2

You may need to scroll left and right to see the full figure.

3.

		COLIN		
		Left	Middle	Right
ROWENA	Up	1, 5	2, 4	5, 1
	Straight	2, 4	4, 2	3, 3

You may need to scroll left and right to see the full figure.

	COLIN		
	Left	Middle	Right
Down	1, 5	3, 3	3, 3

You may need to scroll left and right to see the full figure.

4.

		COLIN		
		Left	Middle	Right
ROWENA	Up	5, 2	1, 6	3, 4
	Straight	6, 1	1, 6	2, 5
	Down	1, 6	0, 7	0, 7

You may need to scroll left and right to see the full figure.

---

2. For each of the four games in Exercise S1, identify whether the game is zero-sum or non-zero-sum. Explain your reasoning.
3. For each of the four games in Exercise S1, identify which players (if any) have a superdominant strategy. Explain your reasoning.
4. Another method for solving zero-sum games, important because it was developed long before Nash developed his concept of equilibrium for non-zero-sum games, is the *minimax* method. To use this method, assume that no matter which strategy a player chooses, her rival will choose to give her the worst possible payoff from that strategy. For each zero-sum game identified in Exercise S2, use the minimax method to find the game's equilibrium strategies by doing the following:
   1. For each of Rowena's strategies, write down the minimum possible payoff to her (the worst that Colin can do to her in each case). For each of Colin's strategies, write down the minimum possible payoff to

him (the worst that Rowena can do to him in each case).

2. Determine the strategy (or strategies) that gives each player the best of these worst payoffs. This strategy is called a *minimax strategy*.

(Because we are considering zero-sum games, players' best responses do indeed involve minimizing each other's payoffs, so the minimax strategies are the same as the Nash equilibrium strategies. John von Neumann proved the existence of a minimax equilibrium in zero-sum games in 1928, more than 20 years before Nash generalized the theory to include zero-sum games.)

5. Find all Nash equilibria in pure strategies in the following non-zero-sum games. Describe the steps that you used in finding the equilibria.

1.

		COLIN	
		Left	Right
ROWENA	Up	3, 2	2, 3
	Down	4, 1	1, 4

You may need to scroll left and right to see the full figure.

2.

		COLIN	
		Left	Right
ROWENA	Up	1, 1	0, 1
	Down	1, 0	1, 1

You may need to scroll left and right to see the full figure.

3.

		COLIN		
		Left	Middle	Right
ROWENA	Up	0, 1	9, 0	2, 3
	Straight	5, 9	7, 3	1, 7
	Down	7, 5	10, 10	3, 5

You may need to scroll left and right to see the full figure.

4.

		COLIN		
		West	Center	East
ROWENA	North	2, 3	8, 2	7, 4
	Up	3, 0	4, 5	6, 4
	Down	10, 4	6, 1	3, 9
	South	4, 5	2, 3	5, 2

You may need to scroll left and right to see the full figure.

---

6. Consider the following game table:

		COLIN			
		North	South	East	West
ROWENA	Earth	1, 3	3, 1	0, 2	1, 1
	Water	1, 2	1, 2	2, 3	1, 1
	Wind	3, 2	2, 1	1, 3	0, 3
	Fire	2, 0	3, 0	1, 1	2, 2

You may need to scroll left and right to see the full figure.

---

1. Does either Rowena or Colin have a dominant strategy? Explain why or why not.
2. Use iterated elimination of dominated strategies to reduce the game as much as possible. Give the order in which the eliminations occur and give the reduced form of the game.
3. Is this game dominance solvable? Explain why or why not.
4. State the Nash equilibrium (or equilibria) of this game.

7. "If a player has a dominant strategy in a simultaneous-move game, then she is sure to get her best possible outcome." True or false? Explain and give an example of a game that illustrates your answer.
8. An old lady is looking for help crossing the street. Only one person is needed to help her; if more people help her, this is no better. You and I are the two people in the vicinity who can help; we have to choose simultaneously whether to do so. Each of us will get pleasure worth a payoff of 3 from her success (no matter who helps her). But each one of us who helps will bear a cost of 1, this being the value of our time taken up in helping her. If neither player helps, the payoff for each player is 0. Set up this situation as a game. Draw the game table and find all pure-strategy Nash equilibria.
9. A university is contemplating whether to build a new laboratory or a new theater on campus. The science faculty would rather see a new lab built, and the humanities faculty would prefer a new theater. However, the funding for the project (whichever it may turn out to be) is contingent on unanimous support from the faculty. If there is disagreement, neither project will go forward, leaving each group with no new building and their worst payoff. Meetings of the two separate faculty groups to discuss which proposal to support occur simultaneously. Payoffs are given in the following table.

		HUMANITIES FACULTY	
		Lab	Theater
SCIENCE FACULTY	Lab	4, 2	0, 0
	Theater	0, 0	1, 5

---

1. What are the pure-strategy Nash equilibria of this game?
2. Which game described in this chapter is most similar to this game? Explain your reasoning.

10. Suppose two game-show contestants, Alex and Bob, each separately select one of three doors, numbered 1, 2, and 3. Both players get dollar prizes if their choices match, as indicated in the following payoff table:

		BOB		
		1	2	3
ALEX	1	10, 10	0, 0	0, 0
	2	0, 0	15, 15	0, 0
	3	0, 0	0, 0	15, 15

You may need to scroll left and right to see the full figure.

---

1. What are the Nash equilibria of this game? Which, if any, is likely to emerge as the focal point? Explain.
2. Consider a slightly changed game in which the choices are again doors 1, 2, and 3, but the payoffs in the two cells with (15, 15) in the table become (25, 25). What is the expected (average) payoff to each player if each flips a coin to decide whether to play 2 or 3? Is this outcome better than the outcome of both of them choosing 1 as a focal point? How should you account for the risk that Alex might do one thing while Bob does the other?

11. At the very end of the British game show Golden Balls, two players (Rowena and Colin) simultaneously decide whether to split or steal a large cash prize. If both choose Split, they each walk away with half of the prize. If one player chooses Split and the other chooses Steal, the stealer gets all the money. Finally, if both choose Steal, both get nothing.
    1. Draw the game table for this game, assuming for concreteness that the cash prize is worth $10,000 and that the players care only about how much money they wind up winning.
    2. In this game, both players view Steal as a _______ strategy. Fill in the blank with one of the following answers: "superdominant," "strictly dominant (but not superdominant)," "weakly dominant (but not strictly dominant)," or "not dominant." Explain your answer.
    3. Find all pure-strategy Nash equilibria of this game.
    4. Suppose that, in addition to wanting to win money, each player wants to avoid looking foolish. In particular, suppose that each player views the outcome in which they Split and the other player Steals as the very worst possibility. How does this extra consideration change your answer to part (b), and how does it change the set of pure-strategy Nash equilibria relative to part (c)? Explain your answers.
    5. Now, suppose that Rowena has preferences as described in part (d), but Colin is a kinder-hearted soul who would rather Rowena get all the money than for both of them to get nothing.[18] How does this change your answer to part (b), and how does it change the set of pure-strategy Nash equilibria relative to part (d)? Explain your answers.
12. Marta has three sons: Arturo, Bernardo, and Carlos. She discovers a broken lamp in her living room and knows that one of her sons must have broken it at play. Carlos was

actually the culprit, but Marta doesn't know this. She cares more about finding out the truth than she does about punishing the child who broke the lamp, so Marta announces that her sons are to play the following game.

Each child will write down his name on a piece of paper and write down either "Yes, I broke the lamp," or "No, I didn't break the lamp." If at least one child claims to have broken the lamp, she will give the normal allowance of \$2 to each child who claims to have broken the lamp, and \$5 to each child who claims not to have broken the lamp. If all three children claim not to have broken the lamp, none of them receives any allowance (each receives \$0).

1. Draw the game table. Make Arturo the row player, Bernardo the column player, and Carlos the page player.
2. Find all Nash equilibria of this game.
3. This game has multiple Nash equilibria. Which one would you consider to be a focal point?

13. Consider a game in which there is a prize worth \$30. There are three contestants, Larry, Curly, and Moe. Each can buy a ticket worth \$15 or \$30 or not buy a ticket at all. They make these choices simultaneously and independently. Then, knowing the ticket-purchase decisions, the game organizer awards the prize. If no one has bought a ticket, the prize is not awarded. Otherwise, the prize is awarded to the buyer of the highest-cost ticket if there is only one such player or is split equally between two or three if there are ties among the highest-cost ticket buyers. Show this game in strategic form, using Larry as the row player, Curly as the column player, and Moe as the page player. Find all pure-strategy Nash equilibria.
14. Anne and Bruce would like to rent a movie, but they can't decide what kind of movie to choose. Anne wants to

rent a comedy, and Bruce wants to rent a drama. They decide to choose randomly by playing "Evens or Odds." On the count of three, each of them shows one or two fingers. If the sum of all the fingers is even, Anne wins, and they rent the comedy; if the sum is odd, Bruce wins, and they rent the drama. Each of them earns a payoff of 1 for winning and 0 for losing Evens or Odds.

1. Draw the game table for Evens or Odds.
2. Demonstrate that this game has no Nash equilibrium in pure strategies.

15. In the film *A Beautiful Mind*, John Nash and three of his graduate-school colleagues find themselves faced with a dilemma while at a bar. There are five young women at the bar, four brunettes and one blonde. Each young man wants to win the attention of one young woman, most of all the blonde, but can approach only one of them. The catch is that if two or more of the young men approach the blonde, she will reject all of them and then the brunettes will also reject the men because they don't like being approached second. Each young man gets a payoff of 10 if he gains the blonde's attention, a payoff of 5 if he gains a brunette's attention, and a payoff of 0 if he is rejected by all of the women. (Thus, the only way that one of the men gets a payoff of 10 is if he is the sole person to approach the blonde.)
    1. First, consider a simpler situation in which there are only two young men instead of four. (And there are only two brunettes and one blonde, but these women merely respond in the manner just described and are not active players in the game.) Show the payoff table for the game and find all pure-strategy Nash equilibria of the game.
    2. Now show the (three-dimensional) payoff table for the case in which there are three young men (and three brunettes and one blonde who are not active players). Again, find all Nash equilibria of the game.

3. Without the use of a table, give all Nash equilibria for the case in which there are four young men (as well as four brunettes and a blonde).
4. (**Optional**) Use your results in parts (a), (b), and (c) to generalize your analysis to the case in which there are n young men. Do not attempt to draw an n-dimensional payoff table; merely find the payoff to one player when k of the others approach Blonde and $(n - k - 1)$ approach Brunette, for $k = 0, 1, \ldots (n - 1)$. Can the outcome specified in the movie as the Nash equilibrium of the game—that all the young men approach brunettes—ever really be a Nash equilibrium of the game?

UNSOLVED EXERCISES

1. Find all Nash equilibria in pure strategies for the following games. First, check for dominated strategies. If there are none, solve the games using successive elimination of dominated strategies.

    1.

		COLIN	
		Left	Right
ROWENA	Up	3, 1	4, 2
	Down	5, 2	2, 3

You may need to scroll left and right to see the full figure.

    2.

		COLIN		
		Left	Middle	Right
ROWENA	Up	2, 9	5, 5	6, 2
	Straight	6, 4	9, 2	5, 3
	Down	4, 3	2, 7	7, 1

You may need to scroll left and right to see the full figure.

    3.

		COLIN		
		Left	Middle	Right
ROWENA	Up	5, 3	3, 5	2, 6
	Straight	6, 2	4, 4	3, 5

You may need to scroll left and right to see the full figure.

	COLIN		
	Left	Middle	Right
Down	1, 7	6, 2	2, 6

You may need to scroll left and right to see the full figure.

4.

		COLIN			
		North	South	East	West
ROWENA	Up	6, 4	7, 3	5, 5	6, 4
	High	7, 3	3, 7	4, 6	5, 5
	Low	8, 2	6, 4	3, 7	2, 8
	Down	3, 7	5, 5	4, 6	5, 5

You may need to scroll left and right to see the full figure.

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2. For each of the four games in Exercise U1, identify whether the game is zero-sum or non-zero-sum. Explain your reasoning.
3. For each of the four games in Exercise U1, identify which players (if any) have a superdominant strategy. Explain your reasoning.
4. As in Exercise S4 above, use the minimax method to find the Nash equilibria for the zero-sum games identified in Exercise U2.
5. Find all Nash equilibria in pure strategies for the following games. Describe the steps that you used in finding the equilibria.

1.

	COLIN	
	Left	Right

You may need to scroll left and right to see the full figure.

		COLIN	
		Left	Right
ROWENA	Up	1, -1	4, -4
	Down	2, -2	3, -3

You may need to scroll left and right to see the full figure.

2.

		COLIN	
		Left	Right
ROWENA	Up	0, 0	0, 0
	Down	0, 0	1, 1

You may need to scroll left and right to see the full figure.

3.

		COLIN	
		Left	Right
ROWENA	Up	1, 3	2, 2
	Down	4, 0	3, 1

You may need to scroll left and right to see the full figure.

4.

		COLIN		
		Left	Middle	Right
ROWENA	Up	5, 3	7, 2	2, 1
	Straight	1, 2	6, 3	1, 4

You may need to scroll left and right to see the full figure.

	COLIN			
		Left	Middle	Right
	Down	4, 2	6, 4	3, 5
You may need to scroll left and right to see the full figure.				

6. Use successive elimination of dominated strategies to solve the following game. Explain the steps you followed. Show that your solution is a Nash equilibrium.

		COLIN		
		Left	Middle	Right
ROWENA	Up	4, 3	2, 7	0, 4
	Down	5, 0	5, −1	−4, −2
You may need to scroll left and right to see the full figure.				

7. Find all pure-strategy Nash equilibria for the following game. Describe the process that you used to find the equilibria. Use this game to explain why it is important to describe an equilibrium by using the strategies employed by the players, not merely the payoffs received in equilibrium.

		COLIN		
		Left	Center	Right
ROWENA	Up	1, 2	2, 1	1, 0
	Level	0, 5	1, 2	7, 4
	Down	−1, 1	3, 0	5, 2
You may need to scroll left and right to see the full figure.				

8. Consider the following game table:

		COLIN		
		Left	Center	Right
ROWENA	Top	4, __	__, 2	3, 1
	Middle	3, 5	2, __	2, 3
	Down	__, 3	3, 4	4, 2

You may need to scroll left and right to see the full figure.

1. Complete the payoffs in the game table so that Colin has a dominant strategy. State which strategy is dominant and explain why. (Note: There are many equally correct answers.)
2. Complete the payoffs in the game table so that neither player has a dominant strategy, but also so that each player does have a dominated strategy. State which strategies are dominated and explain why. (Again, there are many equally correct answers.)

9. The *Battle of the Bismarck Sea* (named for that part of the southwestern Pacific Ocean separating the Bismarck Archipelago from Papua New Guinea) was a naval engagement between the United States and Japan during World War II. In 1943, a Japanese admiral was ordered to move a convoy of ships to New Guinea; he had to choose between a rainy northern route and a sunnier southern route, both of which required three days' sailing time. The Americans knew that the convoy would sail and wanted to send bombers after it, but they did not know which route it would take. The Americans had to send reconnaissance planes to scout for the convoy, but they had only enough reconnaissance planes to explore one route at a time. Both the Japanese and the Americans had to make their

decisions with no knowledge of the plans being made by the other side.

If the convoy was on the route that the Americans explored first, they could send bombers right away; if not, they lost a day of bombing. Poor weather on the northern route would also hamper bombing. If the Americans explored the northern route and found the Japanese right away, they could expect only two (of three) good bombing days; if they explored the northern route and found that the Japanese had gone south, they could also expect two days of bombing. If the Americans chose to explore the southern route first, they could expect three full days of bombing if they found the Japanese right away, but only one day of bombing if they found that the Japanese had gone north.

1. Illustrate this game in a game table.
2. Identify any dominant strategies in the game and solve for the Nash equilibrium.

10. Two players, Jack and Jill, are put in separate rooms, and each is told the rules of the game. Each is to pick one of six letters: G, K, L, Q, R, or W. If the two happen to choose the same letter, both get prizes as follows:

Letter	G	K	L	Q	R	W
Jack's Prize	3	2	6	3	4	5
Jill's Prize	6	5	4	3	2	1

If they choose different letters, each gets 0. This whole schedule is revealed to both players, and both are told that both know the schedules, and so on.

1. Draw the game table for this game. What are the Nash equilibria in pure strategies?
2. Can one of the equilibria be a focal point? Which one? Why?

11. The widget market is currently monopolized by Widgets R Us (or simply Widgets), but another firm (Wadgets) is deciding whether to enter that market. If Wadgets stays out of the market, Widgets will earn $100 million profit. However, if Wadgets enters, Widgets can *either* share the market, in which case the two companies enjoy a total $20 million in profit, *or* wage a ruinous price war, in which case both companies lose big and go bankrupt (call this $0 profit for concreteness). The only sane choice for Widgets is to share the market, but before Wadgets chooses whether to enter, the Widgets Board of Directors has the opportunity to hire a new CEO—and this new CEO might just be crazy enough to wage a price war!
    1. Draw the game table for this game, where the relevant players are Wadgets, which decides whether to enter the market, and the Widgets Board, which decides whether to hire a crazy CEO who will wage a price war if Wadgets enters or hire a sane CEO who will share the market if Wadgets enters. (Assume that Wadgets has no way of knowing if the newly hired Widgets CEO is crazy or sane, making this a simultaneous-move game.)
    2. In this game, the Widgets Board views Hire a Sane CEO as a _______ strategy. Fill in the blank with one of the following answers: "superdominant," "strictly dominant (but not superdominant)," "weakly dominant (but not strictly dominant)," or "not dominant." Explain your answer.
    3. Find all pure-strategy Nash equilibria of this game.

4. Suppose that, in addition to wanting to maximize profits and avoid bankruptcy, the Widgets Board would prefer not to have a crazy CEO. How does this extra consideration change your answer to part (b), and how does it change the set of pure-strategy Nash equilibria relative to part (c)? Explain your answers.

12. Three friends (Julie, Kristin, and Larissa) independently go shopping for dresses for their high-school prom. On reaching the store, each girl sees only three dresses worth considering: one black, one lavender, and one yellow. Each girl, furthermore, knows that her two friends would consider the same set of three dresses because all three have somewhat similar tastes.

    Each girl would prefer to have a unique dress, so each girl's payoff is 0 if she ends up purchasing the same dress as at least one of her friends. All three know that Julie strongly prefers black to both lavender and yellow, so she would get a payoff of 3 if she were the only one wearing the black dress, and a payoff of 1 if she were either the only one wearing the lavender dress or the only one wearing the yellow dress. Similarly, all know that Kristin prefers lavender and secondarily prefers yellow, so her payoff would be 3 for uniquely wearing lavender, 2 for uniquely wearing yellow, and 1 for uniquely wearing black. Finally, all know that Larissa prefers yellow and secondarily prefers black, so she would get 3 for uniquely wearing yellow, 2 for uniquely wearing black, and 1 for uniquely wearing lavender.

    1. Provide the game table for this three-player game. Make Julie the row player, Kristin the column player, and Larissa the page player.
    2. Identify any dominated strategies in this game, or explain why there are none.
    3. What are the pure-strategy Nash equilibria in this game?

13. Bruce, Colleen, and David are all getting together at Bruce's house on Friday evening to play their favorite game, Monopoly. They all love to eat sushi while they play. They all know from previous experience that two orders of sushi are just the right amount to satisfy their hunger. If they wind up with fewer than two orders, they all end up going hungry and don't enjoy the evening. More than two orders would be a waste, however, because they can't manage to eat a third order, and the extra sushi just goes bad. Their favorite restaurant, Fishes in the Raw, packages its sushi in such large containers that each individual person can feasibly purchase at most one order of sushi. Fishes in the Raw offers takeout, but unfortunately doesn't deliver.

    Suppose that each player enjoys \$20 worth of utility from having enough sushi to eat on Friday evening, and \$0 from not having enough to eat. The cost to each player of picking up an order of sushi is \$10.

    Unfortunately, the players have forgotten to communicate about who should be buying sushi this Friday, and none of the players has a cell phone, so they must each make an independent decision about whether to buy (B) or not buy (N) an order of sushi.

    1. Write down this game in strategic form.
    2. Find all the Nash equilibria in pure strategies.
    3. Which equilibrium would you consider to be a focal point? Explain your reasoning.

14. Roxanne, Sara, and Ted all love to eat cookies, but there's only one left in the package. No one wants to split the cookie, so Sara proposes the following extension of "Evens or Odds" (see Exercise S14) to determine who gets to eat it. On the count of three, each of them will show one or two fingers, they'll add them up, and then divide the sum by 3. If the remainder is 0,

Roxanne gets the cookie, if the remainder is 1, Sara gets it, and if it is 2, Ted gets it. Each of them receives a payoff of 1 for winning (and eating the cookie) and 0 otherwise.

1. Represent this three-player game in normal form, with Roxanne as the row player, Sara as the column player, and Ted as the page player.
2. Find all the pure-strategy Nash equilibria of this game. Is this game a fair mechanism for allocating the cookie? Explain why or why not.

15. (**Optional**) Construct the payoff matrix for your own two-player game that satisfies the following requirements: First, each player should have three strategies. Second, the game should not have any dominant strategies. Third, the game should not be solvable using the minimax method. Fourth, the game should have exactly two pure-strategy Nash equilibria. Provide your payoff matrix, then demonstrate that all of the above conditions are true.

Endnotes

- In an episode of *Golden Balls* that WNYC Studio's *Radiolab* called "one of the strangest moments in game show history," one player (Nick Corrigan) made a brilliant move to ensure that the other player (Ibrahim Hussein) strictly preferred to Split even while believing that Nick was certain to Steal. Learn more and watch the episode at https://www.wnycstudios.org/story/golden-rule. Return to reference 18

5 ■ Simultaneous-Move Games: Continuous Strategies, Discussion, and Evidence

THE DISCUSSION OF SIMULTANEOUS-MOVE GAMES in Chapter 4 focused on games in which each player had a discrete set of actions from which to choose. Discrete-strategy games of this type include sporting contests in which a small number of well-defined plays can be used in a given situation—such as soccer penalty kicks, in which the kicker can choose to go high, low, to a corner, or to the center. Other examples include coordination and prisoners' dilemma games in which players have only two or three available strategies. Such games are amenable to analysis with the use of a game table, at least for situations with a reasonable number of players and available actions.

Many simultaneous-move games differ from those considered so far in that they entail players choosing strategies from a wide range of possibilities. Games in which manufacturers choose prices for their products, philanthropists choose charitable contribution amounts, or contractors choose project bid levels are examples in which players have a virtually infinite set of choices. Technically, prices and other amounts of money do have a minimum unit, such as a cent, so there is actually a finite set of discrete price strategies available. But in practice, the minimum unit is very small, and allowing discreteness in our analyses would require us to give each player too many distinct strategies and make the game table too large; therefore, it is simpler and better to regard such choices as continuously variable real numbers. When players have such a large range of actions available, game tables become virtually useless as analytical tools; they become too unwieldy to be of practical use. For these games, we need a different solution technique. We present the analytical tools for handling games with such continuous strategies in the first part of this chapter.

This chapter also takes up some broader matters relevant to player behavior in simultaneous-move games and to the concept of Nash equilibrium. We review the empirical evidence on Nash equilibrium play that has been collected both from the laboratory and from real-life situations. We also present some theoretical criticisms of the Nash equilibrium concept and rebuttals of these criticisms. You will see that game-theoretic predictions are often a reasonable starting point for understanding actual behavior, with some caveats.

Glossary

continuous strategy
: A choice over a continuous range of real numbers available to a player.

1 PURE STRATEGIES THAT ARE CONTINUOUS VARIABLES

In Chapter 4, we developed the method of best-response analysis for finding all pure-strategy Nash equilibria of simultaneous-move games. Now we extend that method to games in which each player—for example, a firm setting the price of a product—has available a continuous range of choices. To calculate best responses in this type of game, we find, for each possible value of one firm's price, the value of the other firm's price that is best for it (maximizes its payoff). The continuity of the sets of strategies allows us to use algebraic formulas to show how strategies generate payoffs and to show the best responses as curves in a graph, with each player's price (or any other continuous strategy) on one of the axes. In such an illustration, the Nash equilibrium of the game occurs where the two curves meet. We develop this idea and technique by using two stories.

A. Price Competition

Our first story is set in a small town, Eten, which has two restaurants, Xavier's Tapas Bar and Yvonne's Bistro. To keep the story simple, we assume that each place has a set menu, and that Xavier and Yvonne must set the prices of the meals on their respective menus. Prices are their strategic choices in the game of competing with each other; each restaurant's goal is to set its prices to maximize profit, the payoff in this game. We assume that they get their menus printed separately without knowing each other's prices, so the game has simultaneous moves.[1] Because prices can take any value within an (almost) infinite range, we start with general or algebraic symbols for them. We then find best-response rules that we can use to solve the game and to determine equilibrium prices. Let us call Xavier's price P_x and Yvonne's price P_y.

In setting its price, each restaurant has to calculate the consequences for its profit. To keep things relatively simple, we put the two restaurants in a very symmetric relationship, but readers with a little more mathematical skill can do a similar analysis by using much more general numbers or even algebraic symbols. Suppose the cost of serving each customer is $8 for each restaurateur. Suppose further that experience or market surveys have shown that when Xavier's price is P_x and Yvonne's price is P_y, the numbers of customers they serve, Q_x and Q_y, respectively (measured in hundreds per month), are given by the demand equations[2]

$$Q_x = 44 - 2P_x + P_y,$$

$$Q_y = 44 - 2P_y + P_x.$$

The key idea in these equations is that, if one restaurant raises its price by $1 (say, if Yvonne increases P_y by $1), its sales will go down by 200 per month (Q_y changes by -2) and those of the other restaurant will go up by 100 per month (Q_x changes by 1).

(Presumably, 100 of Yvonne's customers switch to Xavier's and another 100 stay at home.)

Xavier's profit per week (in hundreds of dollars per week), which we call Π_x—the Greek letter Π (pi) is the traditional economic symbol for profit—is given by the product of the net revenue per customer (price less cost, or $P_x - 8$) and the number of customers served:

$$\Pi_x = (P_x - 8)\, Q_x = (P_x - 8)\,(44 - 2P_x + P_y).$$

By multiplying out and rearranging the terms on the right-hand side of this expression, we can write profit as a function of increasing powers of P_x:

$$\Pi_x = -8(44 + P_y) + (16 + 44 + P_y)P_x - 2(P_x)^2$$

$$= -8(44 + P_y) + (60 + P_y)P_x - 2(P_x)^2.$$

Xavier sets his price P_x to maximize this payoff. Doing so for each possible level of Yvonne's price P_y gives us Xavier's best-response rule, and we can then graph it.

Many simple illustrative examples where one real number (such as a price) is chosen to maximize another real number that depends on it (such as a profit or payoff) have a similar form. (In mathematical jargon, we would describe the second number as a function of the first.) In the appendix to this chapter, we develop a simple general technique for performing such maximization; you will find many occasions to use it. Here we simply state the formula.

The function we want to maximize takes the general form

$$Y = A + BX - CX^2,$$

where we have used the descriptor Y for the number we want to maximize and X for the number we want to choose so as to maximize that Y. In our specific example, profit, Π_x, would be represented by Y, and the price, P_x, by X. Similarly, although in any specific problem the terms A, B, and C in the equation above would be known numbers, we have denoted them with general algebraic symbols here so that our formula can be applied across a wide variety of similar problems. (The technical term for the terms A, B, and C is *parameters*, or *algebraic constants*.) Because most of our applications involve nonnegative X entities, such as prices, and the maximization of the Y entity, we require that $B > 0$ and $C > 0$. Then the formula giving the choice of X to maximize Y in terms of the known parameters A, B, and C is simply $X = B/(2C)$. Observe that A does not appear in the formula, although it will of course affect the value of Y that results.

Comparing the general function in the preceeding equation and the specific example of the profit function in the restaurant pricing game, we have[3]

$B = 60 + P_y$ and $C = 2$.

Therefore, Xavier's choice of price to maximize his profit will satisfy the formula $B/(2C)$ and will be

$P_x = 15 + 0.25P_y$.

This equation determines the value of P_x that maximizes Xavier's profit, given a particular value of Yvonne's price, P_y. In other words, it is exactly what we want: the rule for Xavier's best response.

Yvonne's best-response rule can be found similarly. Because the costs and sales of the two restaurants are entirely symmetric, that equation is obviously going to be

$P_y = 15 + 0.25P_x$.

Both rules are used in the same way to develop best-response graphs. If Xavier sets a price of 16, for example, then Yvonne plugs this value into her best-response rule to find $P_y = 15 + 0.25(16) = 19$; similarly, Xavier's best response to Yvonne's P_y

= 16 is P_x = 19, and each restaurant's best response to the other's price of 4 is 16, that to 8 is 17, and so on.

Figure 5.1 shows the graphs of these two rules, called the best-response curves. Owing to the special features of our example—namely, the linear relationship between quantity sold and prices charged, and the constant cost of producing each meal—each of the two best-response curves is a straight line. For other specifications of demands and costs, the curves can be other than straight, but the method of obtaining them is the same—namely, first holding one restaurant's price (say, P_y) fixed and finding the value of the other's price (say, P_x) that maximizes that other restaurant's profit, and then the other way around.

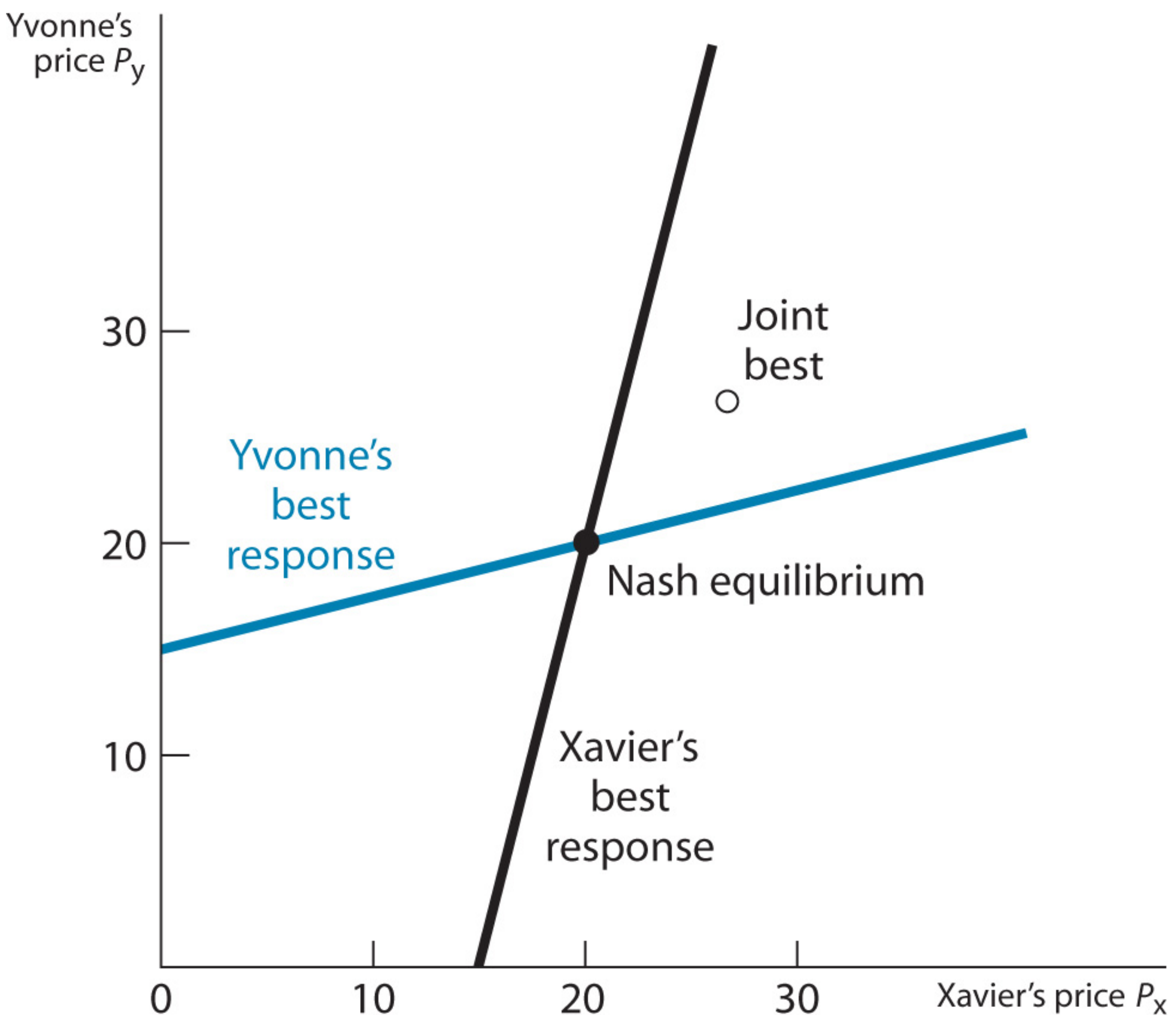

Figure 5.1 Best-Response Curves and Nash Equilibrium in the Restaurant Pricing Game

The point of intersection of the two best-response curves is the Nash equilibrium of the pricing game between the two restaurants. That point represents the pair of prices, one for each restaurant, that are best responses to each other. The specific values for each restaurant's pricing strategy in equilibrium can be found algebraically by solving the two best-response rules jointly for P_x and P_y. We deliberately chose our example to make the equations linear, and the solution is easy. In this case, we simply substitute the expression for P_x into the expression for P_y to find

$$P_y = 15 + 0.25P_x = 15 + 0.25(15 + 0.25P_y) = 18.75 + 0.0625\ P_y.$$

This last equation simplifies to $P_y = 20$. Given the symmetry of the problem, it is simple to determine that $P_x = 20$ also.[4] Thus, in equilibrium, each restaurant charges $20 for its menu and makes a profit of $12 on each of the 2,400 customers [2,400 = (44−(2 × 20) + 20) hundred] that it serves each month, for a total profit of $28,800 per month.

B. Some Economics of Oligopoly

Our main purpose in presenting the restaurant pricing example was to illustrate how the Nash equilibrium can be found in a game where the strategies are continuous variables, such as prices. But it is interesting to take a further look into some of the economics behind pricing strategies and profits when a small number of firms (here just two) compete. In the jargon of economics, such competition is referred to as *oligopoly*, from the Greek words for “a small number of sellers.”

Begin by observing that each firm’s best-response curve slopes upward. Specifically, when one restaurant raises its price by $1, the other’s best response is to raise its own price by 0.25, or 25 cents. When one restaurant raises its price, some of its customers switch to the rival restaurant, which can then profit from these new customers by raising its price part of the way. Thus, a restaurant that raises its price is also helping to increase its rival’s profit. In Nash equilibrium, where each restaurant chooses its price independently and out of concern for its own profit, it does not take into account this benefit that it conveys to the other. Could the two restaurants get together and cooperatively agree to raise their prices, thereby raising profits for both? Yes. Suppose the two restaurants charged $24 each. Then each would make a profit of $16 on each of the 2,000 customers [2,000 = (44 − (2 × 24) + 24) hundred] that it would serve each month, for a total profit of $32,000 per month.

This pricing game is exactly like the prisoners’ dilemma game presented in Chapter 4, but now the strategies are continuous variables. In the story in Chapter 4, Husband and Wife were each tempted to cheat the other and confess to the police, but when they both did so, both ended up with longer prison sentences (worse outcomes). In the same way, the more profitable price of $24 is not a Nash equilibrium. The separate calculations of the two restaurants will lead them to undercut such a price. Suppose that Yvonne somehow starts charging $24. Using the best-response formula, we see that Xavier will then charge 15 + 0.25 × 24 =

21. Then Yvonne will come back with her best response to that price: $15 + 0.25 \times 21 = 20.25$. Continuing this process, the prices of both will converge toward the Nash equilibrium price of $20.

But what price is jointly best for the two restaurants? Given the symmetry, suppose both charge the same price P. Then the profit of each will be

$$\Pi_x = \Pi_y = (P - 8)(44 - 2P + P) = (P - 8)(44 - P) = -352 + 52P - P^2.$$

The two can choose P to maximize this expression. Using the formula provided in Section 1.A, we see that the solution is $P = 52/2 = 26$. The resulting profit for each restaurant is $32,400 per month.

In the jargon of economics, a group of firms that collude to raise prices to the jointly optimal level is called a *cartel*. The high prices hurt consumers, and regulatory agencies of the U.S. government often try to prevent the formation of cartels and to make firms compete with one another. Explicit collusion over price is illegal, but it may be possible to maintain tacit collusion in a repeated prisoners' dilemma; we examine such repeated games in Chapter 10.[5]

But collusion need not always lead to higher prices. In the preceding example, if one restaurant lowers its price, its sales increase, in part because it draws some customers away from its rival because the products (meals) of the two restaurants are *substitutes* for each other. In other contexts, however, two firms may be selling products that are *complements* to each other—for example, hardware and software. In that case, if one firm lowers its price, the sales of both firms increase. In a Nash equilibrium, where the firms act independently, they do not take into account the benefit that would accrue to each of them if they both lowered their prices. Therefore, they keep prices higher than they would if they were able to coordinate their actions. Allowing them to cooperate would lead to lower prices and thus be beneficial to the consumers as well. We examine such collective-action problems in more detail in Chapter 11.

Competition need not always involve the use of prices as the strategic variables. For example, fishing fleets may compete to bring a larger catch to market; this is quantity competition as opposed to the price competition considered in this section. We will consider quantity competition later in this chapter and in several of the end-of-chapter exercises.

C. Political Campaign Advertising

Our second example is one drawn from politics. It requires just a little more mathematics than we normally use, but we explain the intuition behind the calculations in words and with a graph.

Consider an election contested by two parties or candidates. Each is trying to win votes away from the other by advertising—either with positive ads that highlight the good things about themselves or with negative ads that emphasize the bad things about their opponent. To keep matters simple, suppose the voters start out entirely ignorant and unconcerned and form opinions solely as a result of the ads. (Many people would claim that this is an accurate description of U.S. politics, but more advanced analyses in political science do recognize that there are informed and strategic voters. We address the behavior of such voters in detail in Chapter 16.) Even more simply, suppose the vote share of a party equals its share of the total campaign advertising that is done. Call the parties or candidates L and R; when L spends $\$x$ million on advertising and R spends $\$y$ million, L will get a share $x/(x+y)$ of the votes and R will get $y/(x+y)$. (Readers who get interested in this application can find more general treatments in specialized political science writings.)

Raising money to pay for these ads includes a cost: money to send letters and make phone calls; the time and effort of the candidates, party leaders, and activists; and future political payoffs to large contributors, along with possible future political costs if these payoffs are exposed and lead to scandals. For simplicity of analysis, let us suppose that all these costs are proportional to the direct campaign expenditures for advertising, x and y. Specifically, let us suppose that party L's payoff is measured by its vote percentage minus its advertising expenditure, $100x/(x+y) - x$. Similarly, party R's payoff is $100y/(x+y) - y$.

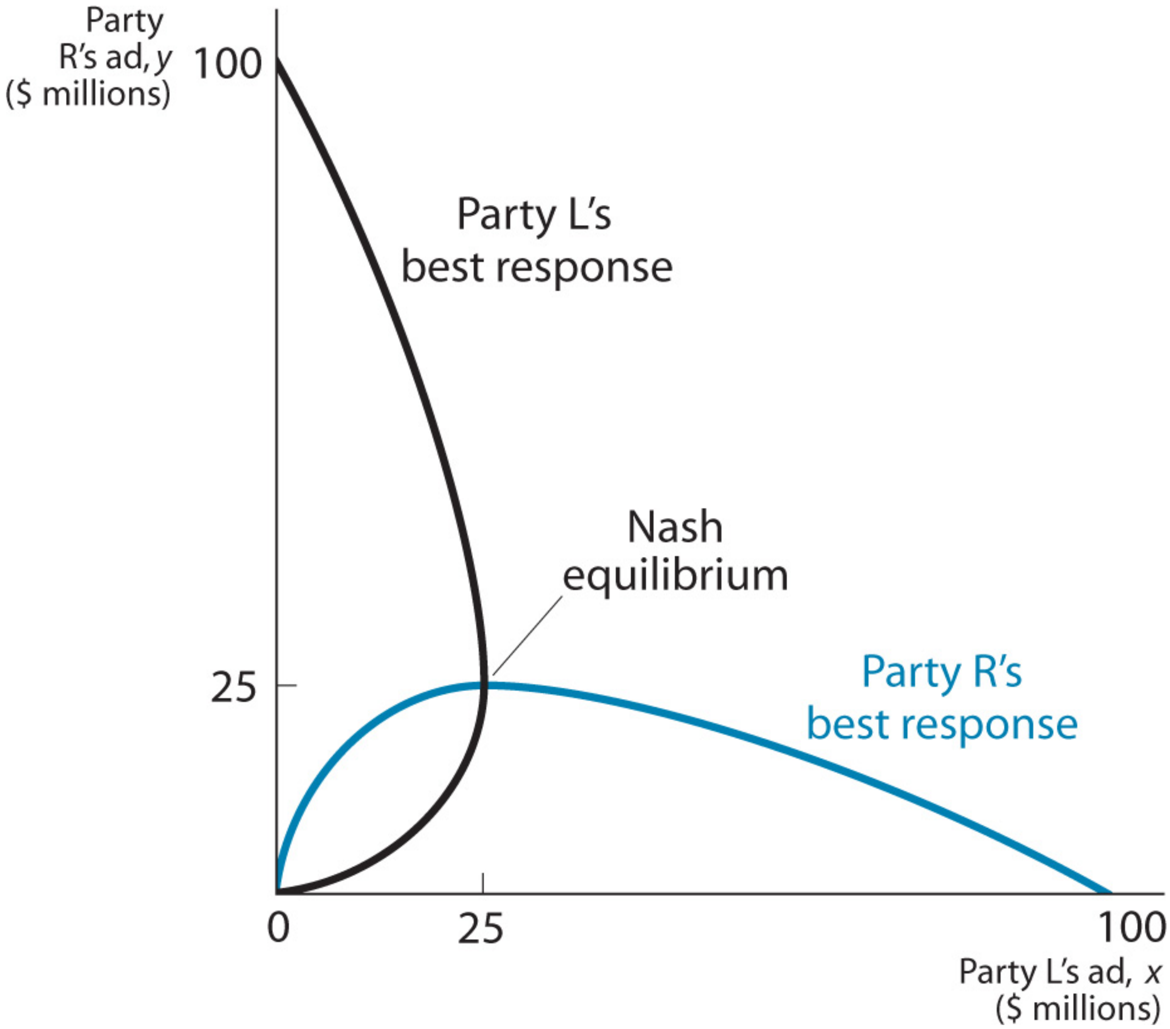

Figure 5.2 Best Responses and Nash Equilibrium in the Campaign Advertising Game

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Now we can find the best responses. Because we cannot do so without calculus, we derive the formula mathematically and then explain in words its general meaning intuitively. For a given strategy x of party L, party R chooses y to maximize its payoff. The calculus first-order condition is found by holding x fixed and setting the derivative of $100y/(x+y) - y$ with respect to y equal to 0. It is $100x/(x+y)^2 - 1 = 0$, or

$$y = 10\sqrt{x} - x.$$

Figure 5.2 shows its graph and that of

the analogous best-response function of party L—namely,

$$x = 10\sqrt{y} - y.$$

Look at the best-response curve of party R. As the value of party L's x increases, party R's y increases for a while and then decreases. If the other party is advertising very little, then one's own ads have a high reward in the form of votes, and it pays to respond to a small increase in the other party's expenditures by spending more oneself to compete harder. But if the other party already spends a great deal on ads, then one's own ads get only a small return in relation to their cost, so it is better to respond to the other party's increase in spending by scaling back.

As it happens, the two parties' best-response curves intersect at their peak points. Again, some algebraic manipulation of the equations for the two curves yields us the equilibrium values of x and y. You should verify that here x and y are each equal to 25, or $25 million. (This is presumably a congressional election; Senate and presidential elections cost much more these days.)

As in the pricing game, we have a prisoners' dilemma. If both parties were to cut back on their ads in equal proportions, their vote shares would be entirely unaffected, but both would save on their expenditures, and so both would have a larger payoff. Unlike a producers' cartel for substitute products (which keeps prices high and hurts consumers), a politicians' cartel to advertise less would probably benefit voters and society, just as a producers' cartel for complementary products would lead to lower prices and benefit consumers. We could all benefit from finding ways to resolve this particular prisoners' dilemma. The United States currently does not limit spending on political advertising by candidates, but some European countries do impose such limits, and the United Kingdom prohibits paid advertising of any kind on the part of political candidates.

What if the parties are not symmetrically situated? Two kinds of asymmetries can arise. One party (say, R) may be able to advertise at a lower cost because it has favored access to the media. Or R’s advertising dollars may be more effective than L’s—for example, L’s vote share may be $x/(x + 2y)$, while R’s vote share is $2y/(x + 2y)$.

In the first of these cases, R exploits its cheaper access to advertising by choosing a higher level of expenditures y for any given x for party L—that is, R’s best-response curve in Figure 5.2 shifts upward. The Nash equilibrium shifts to the northwest along L’s unchanged best-response curve. Thus, R ends up advertising more and L ends up advertising less than before. It is as if the advantaged party uses its muscle and the disadvantaged party gives up to some extent in the face of this adversity.

In the second case, both parties’ best-response curves shift in more complex ways. The outcome is that both spend equal amounts, but less than the $25 million that they spent in the symmetric case. In our example where R’s dollars are twice as effective as L’s, it turns out that their common expenditure level is $200/9 = 22.2 < 25$. (Thus the symmetric case is the one of most intense competition.) When R’s spending is more effective, it is also true that the best-response curves are asymmetric in such a way that the new Nash equilibrium, rather than being at the peak points of the two best-response curves, is on the downward part of L’s best-response curve and on the upward part of R’s best-response curve. That is to say, although both parties spend the same dollar amount, the favored party, R, spends more than the amount that would bring forth the maximum response from party L, and the underdog party, L, spends less than the amount that would bring forth the maximum response from party R. We include an optional exercise (Exercise U12) in this chapter that lets mathematically advanced students derive these results.

D. General Method for Finding Nash Equilibria

Although the strategies (prices or campaign expenditures) and payoffs (profits or vote shares) in the two previous examples are specific to the context of competition between firms or political parties, the method for finding the Nash equilibrium of a game with continuous strategies is perfectly general. Here we state its steps so that you can use it as a recipe for solving other games of this kind.

Suppose the players are numbered 1, 2, 3, ... Label their strategies x, y, z, . . . in the same order, and their payoffs with the corresponding uppercase letters X, Y, Z, ... The payoff of each player is, in general, a function of the choices of all; label the respective functions F, G, H, ... Construct payoffs from the information about the game, and write them as

$X = F(x, y, z, \ldots)$, $Y = G(x, y, z, \ldots)$, $Z = H(x, y, z, \ldots)$.

Using this general format to describe our example of price competition between two players (firms) makes the strategies x and y become the prices P_x and P_y. The payoffs X and Y are the profits Π_x and Π_y. The functions F and G are the quadratic formulas

$\Pi_x = -8(44 + P_y) + (16 + 44 + P_y)P_x - 2(P_x)$,

and similarly for Π_y.

In the general approach, player 1 regards the strategies of players 2, 3, . . . as outside his control, and chooses his own strategy to maximize his own payoff. Therefore, for each given set of values of y, z, . . . , player 1's choice of x maximizes $X = F(x, y, z, \ldots)$. If you use calculus, the condition for this maximization is that the derivative of X with respect to x

holding y, z, . . . constant (the partial derivative) equals 0. For special functions, simple formulas are available, such as the one we stated and used above for the quadratic. And even if an algebra or calculus formulation is too difficult, computer programs can tabulate or graph best-response functions for you. Whatever method you use, you can find an equation for the choice of x for given y, z, . . . that is player 1's best-response function. Similarly, you can find the best-response functions for each of the other players.

The best-response functions are equal in number to the number of strategies in the game and can be solved simultaneously while regarding the strategy variables as the unknowns. The solution is the Nash equilibrium we seek. Some games may have multiple solutions, yielding multiple Nash equilibria. Other games may have no solution, requiring further analysis, such as inclusion of mixed strategies.

Endnotes

- In reality, the competition extends over time, so each can observe the other's past choices. This repetition of the game introduces new considerations, which we cover in Chapter 10. Return to reference 1
- Readers who know some economics will recognize that the equations linking quantities to prices are demand functions for the two products X and Y. The quantity demanded of each product is decreasing with its own price (demands are downward sloping) and increasing with the price of the other product (the two products are substitutes). Return to reference 2
- Although P_y, chosen by Yvonne, is a variable in the full game, here we are considering only a part of the game—namely, Xavier's best response, where he regards Yvonne's choice as outside his control and therefore like a constant. Return to reference 3
- Without this symmetry, the two best-response equations will be different, but given our other specifications, still linear. So it is not much harder to solve the asymmetric case. You will have a chance to do so in Exercise S2 at the end of this chapter. Return to reference 4
- Firms do try to achieve explicit collusion when they think they can get away with it. An entertaining and instructive story of one such episode is in *The Informant*, by Kurt Eichenwald (New York: Broadway Books, 2000). Return to reference 5

Glossary

best-response rule
: A function expressing the strategy that is optimal for one player, for each of the strategy combinations actually played by the other players, or the belief of this player about the other players' strategy choices.

best-response curve
: A graph showing the best strategy of one player as a function of the strategies of the other player(s) over the entire range of those strategies.

2 CRITICAL DISCUSSION OF THE NASH EQUILIBRIUM CONCEPT

Although Nash equilibrium is the primary solution concept for simultaneous-move games, it has been subject to several theoretical criticisms. In this section, we briefly review some of these criticisms and some rebuttals, in each case by using an example.[6] Some of the criticisms are mutually contradictory, and some can be countered by thinking of the games themselves in a better way. Others tell us that the Nash equilibrium concept by itself is not enough and suggest some augmentations or relaxations of it that have better properties. We develop one such alternative here and point to some others that will appear in later chapters. We believe our presentation will leave you with renewed but cautious confidence in using the Nash equilibrium concept. But some serious doubts remain unresolved, indicating that game theory is not yet a settled science. Even this should give encouragement to budding game theorists because it shows that there is a lot of room for new thinking and new research in the subject. A totally settled science would be a dead science.

We begin by considering the basic appeal of the Nash equilibrium concept. Most of the games in this book are noncooperative, in the sense that every player takes her action independently. Therefore, it seems natural to suppose that if her action is not the best according to her own value system (payoff scale) given what everyone else does, she will change it. In other words, it is appealing to suppose that every player’s action will be the best response to the actions of all the others. Nash equilibrium has just this property of “simultaneous best responses”; indeed, that is its very definition. In any purported final outcome that is not a Nash equilibrium, at least one player could have done better by switching to a different action.

This consideration led Nobel laureate Roger Myerson to rebut those criticisms of the Nash equilibrium that were based on the intuitive appeal of playing a different strategy. His rebuttal simply shifted the burden of proof onto the critic. “When asked why players in a game should behave as in some Nash equilibrium,” he said, “my favorite response is to ask ‘Why not?’ and to let the challenger specify what he thinks the players should do. If this specification is not a Nash equilibrium, then . . . we can show that it would destroy its own validity if the players believed it to be an accurate description of each other’s behavior.”[7]

A. The Treatment of Risk in Nash Equilibrium

Some critics argue that the Nash equilibrium concept does not pay due attention to risk. In some games, people might find strategies different from their Nash equilibrium strategies to be safer and might therefore choose those strategies. We offer two examples of this kind. The first comes from John Morgan, an economics professor at the University of California, Berkeley; Figure 5.3 shows the game table for this example.

Best-response analysis quickly reveals that this game has a unique Nash equilibrium—namely, (A, A), yielding the payoffs (2, 2). But you may think, as did several participants in an experiment conducted by Morgan, that playing C has a lot of appeal, for the following reasons: First, it *guarantees* you the same payoff as you would get in the Nash equilibrium—namely, 2—whereas if you play your Nash equilibrium strategy A, you will get a 2 only if the other player also plays A. Why take that chance? What is more, if you think the other player might use this rationale for playing C, then you would be making a serious mistake by playing A; you would get only a 0 when you could have gotten a 2 by playing C.

Myerson would respond to Morgan, “Not so fast. If you really believe that the other player would think this way and play C, then you should play B to get the payoff 3. And if you think the other person would think this way and play B, then your best response to B should be A. And if you think the other person would figure this out, too, you should be playing your best response to A—namely, A. Back to the Nash equilibrium!” As you can see, criticizing Nash equilibrium and rebutting the criticisms is itself something of an intellectual game, and quite a fascinating one.

		COLUMN		
		A	B	C
ROW	A	2, 2	3, 1	0, 2
	B	1, 3	2, 2	3, 2
	C	2, 0	2, 3	2, 2

You may need to scroll left and right to see the full figure.

FIGURE 5.3 A Game with a Questionable Nash Equilibrium

		B	
		Left	Right
A	Up	9, 10	8, 9.9
	Down	10, 10	−1000, 9.9

FIGURE 5.4 Disastrous Nash Equilibrium?

The second example, which comes from David Kreps, an economist at Stanford Business School, is even more dramatic. The payoff matrix is shown in Figure 5.4. Before doing any theoretical analysis of this game, pretend that you are actually playing the game and that you are player A. Which of the two actions would you choose?

Keep in mind your answer to the preceding question as we proceed to analyze the game. If we start by looking for dominant strategies, we see that player A does not have a dominant strategy, but player B does. Playing Left guarantees B a payoff of 10, no matter what A does, versus the payoff of 9.9 earned from playing Right (also no matter what A does). Thus, player B should play Left. Given that player B is going to go Left, player A does better to go Down. The unique pure-strategy Nash equilibrium of this game is therefore (Down, Left); each player achieves a payoff of 10 with this outcome.

The problem that arises here is that many people assigned to be Player A would not choose to play Down. (What did you choose?) This is true for those who have been students of game theory for years as well as for those who have never heard of the subject. If A has *any* doubts about *either* B’s payoffs *or* B’s rationality, then it is a lot safer for A to play Up than to play her Nash equilibrium strategy of Down. What if A thought the payoffs were as illustrated in Figure 5.4, but in reality B’s payoffs were the reverse—the 9.9 payoff went with Left and the 10 payoff went with Right? What if the 9.9 payoff were only an approximation and the exact payoff was actually 10.1? What if B was a player with a value system substantially different from A’s, or was not a truly rational player and might choose the “wrong” action just for fun? Obviously, our assumptions of perfect information and rationality can be crucial to the analysis that we use in the study of strategy. Doubts about players can alter equilibria from those that we would normally predict and can call the reasonableness of the Nash equilibrium concept into question.

However, the real problem with many such examples is not that the Nash equilibrium concept is inappropriate, but that the examples illustrate it in an inappropriately simplistic way. In this example, if there are any doubts about B’s payoffs, then this fact should be made an integral part of the analysis. If A does not know B’s payoffs, the game is one of asymmetric information (which we won’t have the tools to discuss fully until Chapter 9). But this particular example is a relatively simple game of that kind, and we can figure out its equilibrium very easily.

Suppose A thinks there is a probability p that B’s payoffs from Left and Right are the reverse of those shown in Figure 5.4; $(1 - p)$ is then the probability that B’s payoffs are as stated in that figure. Because A must take her action without knowing what B’s actual payoffs are, she must choose her strategy to be "best on average." In this game, the calculation is simple because in each case B has a dominant strategy; the only problem for A is that in the two different cases, different strategies are dominant for B. With probability $(1 - p)$, B's dominant strategy is Left (the case shown in the figure), and with probability p, it is Right (the opposite case). Therefore, if A chooses Up, then with probability $(1 - p)$, he will meet B playing Left and so get a payoff of 9; with probability p, he will meet B playing Right and so get a payoff of 8. Thus, A's statistical, or probability-weighted, average payoff from playing Up is $9(1 - p) + 8p$. Similarly, A's statistical average payoff from playing Down is $10(1 - p) - 1{,}000p$. Therefore, it is better for A to choose Up if

$$9(1 - p) + 8p > 10(1 - p) - 1{,}000p, \text{ or } p > 1/1{,}009.$$

Thus, even if there is only a very slight chance that B's payoffs are the opposite of those in Figure 5.4, it is optimal for A to play Up. In this case, analysis based on rational behavior, when done correctly, contradicts neither the intuitive suspicion nor the experimental evidence after all.

B. Multiplicity of Nash Equilibria

Another criticism of the Nash equilibrium concept is based on the observation that many games have multiple Nash equilibria. Thus, the argument goes, the concept fails to pin down outcomes of games sufficiently precisely to give unique predictions. This argument does not automatically require us to abandon the Nash equilibrium concept. Rather, it suggests that if we want a unique prediction from our theory, we must add some criterion for deciding which one of the multiple Nash equilibria we want to select.

In Chapter 4, we studied many games of coordination with multiple equilibria. From among these equilibria, the players may be able to select one as a focal point if they have some common social, cultural, or historical knowledge. Consider the following coordination game played by students at Stanford University. One player was assigned the city of Boston and the other was assigned San Francisco. Each was then given a list of nine other U.S. cities—Atlanta, Chicago, Dallas, Denver, Houston, Los Angeles, New York, Philadelphia, and Seattle—and asked to choose a subset of those cities. The two chose simultaneously and independently. If and only if their choices divided up the nine cities completely and without any overlap between them, both got a prize. Despite the existence of 512 different Nash equilibria, when both players were Americans or long-time U.S. residents, more than 80% of the time they chose a unique equilibrium based on geography. The student assigned Boston chose all the cities east of the Mississippi, and the student assigned San Francisco chose all the cities west of the Mississippi. Such coordination was much less likely when one or both students were non-U.S. residents. In such pairs, the choices were sometimes made alphabetically, but with much less likelihood of achieving a non-overlapping split of the full list.[8]

The features of the game itself, combined with shared cultural background, can help players' expectations to converge. As another example of multiplicity of equilibria, consider a game where two players write down, simultaneously and independently, the share that each wants from a total prize of $100. If the amounts that they write down add up to $100 or less, each player receives the amount she wrote down. If the two add up to more than $100, neither gets anything. For any x, one player writing x and the other writing $(100 - x)$ is a Nash equilibrium. Thus, the game has an (almost) infinite range of Nash equilibria. But, in practice, 50:50 emerges as a focal point. This social norm of equality or fairness seems so deeply ingrained as to be almost an instinct; players who choose 50 say that it is the obvious answer. To be a true focal point, not only should that answer be obvious to each, but everyone should know that it is obvious to each, and everyone should know that everyone knows, and so on; in other words, its obviousness should be common knowledge. That is not always the case, as we see when we consider a situation in which one player is a woman from an enlightened and egalitarian society who believes that 50:50 is the obvious choice and the other is a man from a patriarchal society who believes it is obvious that, in any matter of division, a man should get three times as much as a woman. Then each will do what is obvious to her or him, and they will end up with nothing, because neither's obvious solution is obvious as common knowledge to both.

The existence of focal points is often a matter of coincidence, and creating them where none exist is basically an art that requires a lot of attention to the historical and cultural context of a game and not merely its mathematical description. This bothers many game theorists, who would prefer that the outcome depend only on an abstract specification of a game—that players and their strategies be identified by numbers without any external associations. We disagree. We think that historical and cultural contexts are just as important to a game as is its purely mathematical description, and if such context helps us select a unique outcome from multiple Nash equilibria, that is all to the good.

In Chapter 6, we will see that sequential-move games can have multiple Nash equilibria. There, we will introduce the requirement of *credibility* that enables us to select a particular equilibrium; it turns out that this equilibrium is in fact the rollback equilibrium of Chapter 3. For more complex games with information asymmetries or additional complications, other restrictions, called refinements, have been developed to identify and rule out Nash equilibria that are unreasonable in some way. In Chapter 9, we will consider one such refinement process that selects an outcome called a *perfect Bayesian equilibrium*. The motivation for a refinement is often specific to a particular type of game. A refinement stipulates how players update their information when they observe what moves other players made or failed to make. Each such stipulation is often perfectly reasonable in its context, and in many games it is not difficult to eliminate most of the Nash equilibria and therefore to reduce the ambiguity in predictions.

The opposite of the criticism that some games may have too many Nash equilibria is that some games may have none at all. In Section 8 of Chapter 4, where we presented an example, we argued that by extending the concept of strategy to random mixtures, Nash equilibrium could be restored. In Chapter 7, we will explain and consider Nash equilibria in mixed strategies. In higher reaches of game theory, there are esoteric examples of games that have no Nash equilibrium in mixed strategies either. However, this added complication is not relevant for the types of analyses and applications that we deal with in this book, so we do not attempt to address it here.

C. Requirements of Rationality for Nash Equilibrium

Remember that Nash equilibrium can be regarded as a system of the strategy choices of each player and the belief that each player holds about the other players' choices. In equilibrium, (1) the choice of each should give her the best payoff given her belief about the others' choices, and (2) the belief of each player should be correct—that is, the other players' actual choices should be the same as what this player believes them to be. These features seem to be natural expressions of the requirements of the mutual consistency of individual rationality. If all players have common knowledge that they are all rational, how can any one of them rationally believe something about others' choices that would be inconsistent with a rational response to her own actions?

To begin to address this question, we consider the three-by-three game in Figure 5.5. Best-response analysis quickly reveals that it has only one Nash equilibrium—namely, (R2, C2), leading to payoffs (3, 3). In this equilibrium, Row plays R2 because she believes that Column is playing C2. Why does she believe this? Because she knows Column to be rational, Row must simultaneously believe that Column believes that Row is choosing R2, because C2 would not be Column's best choice if she believed Row would be playing either R1 or R3. Thus, the claim goes, in any rational process of formation of beliefs and responses, beliefs would have to be correct.

		COLUMN	
	C1	C2	C3

You may need to scroll left and right to see the full figure.

		COLUMN		
		C1	C2	C3
ROW	R1	0, 7	2, 5	7, 0
	R2	5, 2	3, 3	5, 2
	R3	7, 0	2, 5	0, 7

You may need to scroll left and right to see the full figure.

FIGURE 5.5 Justifying Choices by Chains of Beliefs and Responses

The trouble with this argument is that it stops after one round of thinking about beliefs. If we allow it to go far enough, we can justify other choice combinations. We can, for example, rationally justify Row's choice of R1. To do so, we note that R1 is Row's best choice if she believes Column is choosing C3. Why does she believe this? Because she believes that Column believes that Row is playing R3. Row justifies this belief by thinking that Column believes that Row believes that Column is playing C1, believing that Row is playing R1, believing in turn . . . . This is a chain of beliefs, each link of which is perfectly rational.

Thus, rationality alone does not justify Nash equilibrium. There are more sophisticated arguments of this kind that do justify a special form of Nash equilibrium in which players can condition their strategies on a publicly observable randomization device. But we leave that to more advanced treatments. In the next section, we develop a simpler concept that captures what is logically implied by the players' common knowledge of their rationality alone.

Endnotes

- David M. Kreps, *Game Theory and Economic Modelling* (Oxford: Clarendon Press, 1990), gives an excellent in-depth discussion. Return to reference 6
- Roger Myerson, *Game Theory* (Cambridge, Mass.: Harvard University Press, 1991), p. 106. Return to reference 7
- See David Kreps, *A Course in Microeconomic Theory* (Princeton, N.J.: Princeton University Press, 1990), pp. 392–93, 414–15. Return to reference 8

Glossary

refinement
: A restriction that narrows down possible outcomes when multiple Nash equilibria exist.

3 RATIONALIZABILITY

What strategy choices in games can be justified on the basis of rationality alone? For the two-player game shown in Figure 5.5, we can justify any pair of strategies, one for each player, by using the same type of logic that we used in Section 2.C. In other words, we can justify any one of the nine logically conceivable combinations. Thus, rationality alone does not give us any power to narrow down or predict outcomes of this game. Is this a general feature of all games? No. For example, if a strategy is dominated, rationality alone can rule it out of consideration. And when players recognize that other players, being rational, will not play dominated strategies, iterated elimination of dominated strategies can be performed on the basis of common knowledge of rationality. Is this the best that can be done? No. Some more ruling out of strategies can be done by using a property slightly stronger than dominance in pure strategies. This property identifies strategies that are never a best response. The set of strategies that survive elimination on this ground are called rationalizable, and the concept itself is known as rationalizability.

Why introduce this additional concept, and what does it do for us? As for why, it is useful to know how far we can narrow down the possible outcomes of a game on the basis of the players' rationality alone, without invoking correctness of beliefs about the other player's actual choice. It is sometimes possible to figure out that the other player *will not* choose some available action or actions, even when it is not possible to pin down the single action that she *will* choose. As for what it achieves, that depends on the context. In some cases, rationalizability may not narrow down the outcomes at all. This was so in the three-by-three example game shown in Figure 5.5. In other cases, it narrows the possibilities to some extent, but not all the way down to the Nash equilibrium, if the game has a unique one, or to the set of Nash equilibria, if there are several. We will consider an example of such a situation in Section 3.A. In still other cases, the narrowing goes all the way down to the Nash equilibrium; in these cases, we have a more powerful justification for the Nash equilibrium that relies on rationality alone, without assuming correctness of beliefs. In Section 3.B below, we will present an example of quantity competition in which the rationalizability argument takes us all the way to the game’s unique Nash equilibrium.

A. Applying the Concept of Rationalizability

Consider the game in Figure 5.6, which is the same as Figure 5.5 but with an additional strategy for each player.[9] We indicated in Section 2.C that the original nine strategy combinations in Figure 5.5 can all be justified by a chain of the players' beliefs about each other's beliefs. That remains true in this enlarged matrix. But can R4 and C4 be justified in this way?

Could Row ever believe that Column would play C4? Such a belief would have to be justified by Column's beliefs about Row's choice. What might Column believe about Row's choice that would make C4 Column's best response? Nothing. If Column believes that Row will play R1, then Column's best choice is C1. If Column believes that Row will play R2, then Column's best choice is C2. If Column believes that Row will play R3, then C3 is Column's best choice. And, if Column believes that Row will play R4, then C1 and C3 are tied for her best choice. Thus, C4 is never a best response for Column.[10] This means that Row, knowing Column to be rational, can never attribute to Column any belief about Row's choice that would justify Column's choice of C4. Therefore, Row should never believe that Column would choose C4.

		COLUMN			
		C1	C2	C3	C4
ROW	R1	0, 7	2, 5	7, 0	0, 1
	R2	5, 2	3, 3	5, 2	0, 1
	R3	7, 0	2, 5	0, 7	0, 1
	R4	0, 0	0, −2	0, 0	10, −1

You may need to scroll left and right to see the full figure.

FIGURE 5.6 Rationalizable Strategies

Note that, although C4 is never a best response, it is not dominated by any of Column’s other strategies, C1, C2, or C3. For Column, C4 does better than C1 against Row’s R3, better than C2 against Row’s R4, and better than C3 against Row’s R1. If a strategy *is* dominated, it also can never be a best response. Thus, “never a best response” is a more general concept than “dominated.” Eliminating strategies that are never a best response may be possible even when eliminating dominated strategies is not. So eliminating strategies that are never a best response can narrow down the set of possible outcomes more than can elimination of dominated strategies.[11]

The elimination of never-a-best-response strategies can also be carried out iteratively. Because a rational Row can never believe that a rational Column will play C4, a rational Column should foresee this. Because R4 is Row’s best response only against C4, Column should never believe that Row will play R4. Thus, R4 and C4 can never figure in the set of rationalizable strategies. The concept of rationalizability allows us to narrow down the set of possible outcomes of this game to this extent.

If a game has a Nash equilibrium, it is rationalizable and, in fact, can be sustained by a simple one-round system of beliefs, as we saw in Section 2.C above. But, more generally, even if a game does not have a Nash equilibrium, it may have rationalizable outcomes. Consider the two-by-two game we can obtain from Figure 5.5 or Figure 5.6 by retaining just the strategies R1 and R3 for Row and C1 and C3 for Column. It is easy to see that this game has no Nash equilibrium in pure strategies. But all four outcomes are rationalizable with the use of exactly the chain of beliefs, constructed earlier, that went around and around these strategies.

Thus, the concept of rationalizability provides a possible way of solving games that do not have a Nash equilibrium. And more importantly, it tells us how far we can narrow down the possibilities in a game on the basis of rationality alone.

B. Rationalizability Can Take Us All the Way to Nash Equilibrium

In some games, iterated elimination of never-a-best-response strategies can narrow things down all the way to Nash equilibrium. Note that we said *can*, not *must*. But if it does, that is useful, because in these games we can strengthen the case for Nash equilibrium by arguing that it follows purely from the players' rational thinking about each other's thinking. Interestingly, one class of games that can be solved in this way is very important in economics. This class consists of games of competition between firms that choose the quantities that they produce, knowing that the total quantity that is put on the market will determine the price.

We illustrate a game of this type in the context of a small coastal town. It has two fishing boats that go out every evening and return the following morning to put their night's catch on the market. The game is played in an era before modern refrigeration, so all the fish has to be sold and eaten the same day it is caught. Fish are plentiful in the ocean near the town, so the owner of each boat can decide how much to catch each night. But each knows that if the total brought to the market is too large, the glut of fish will mean a low price and low profits.

Specifically, we suppose that, if one boat brings R barrels and the other brings S barrels of fish to the market, the price P (measured in ducats per barrel) will be $P = 60 - (R + S)$. We also suppose that the two boats and their crews are somewhat different in their fishing efficiency. Fishing costs the first boat 30 ducats per barrel and the second boat 36 ducats per barrel.

Now we can write down the profits U and V of the two boat owners in terms of their strategies R and S:

$U = [(60 - R - S) - 30]R = (30 - S)R - R^2,$

$V = [(60 - R - S) - 36]S = (24 - R)S - S^2.$

With these payoff expressions, we can construct best-response curves and find the Nash equilibrium. As in our price competition example from Section 1, each player’s payoff is a quadratic function of his own strategy, holding the strategy of the other player constant. Therefore, the same mathematical methods we develop there and in the appendix to this chapter can be applied.

The first boat’s best response R should maximize U for each given value of the other boat’s S. With the use of calculus, this means that we should differentiate U with respect to R, holding S fixed, and set the derivative equal to 0, which gives

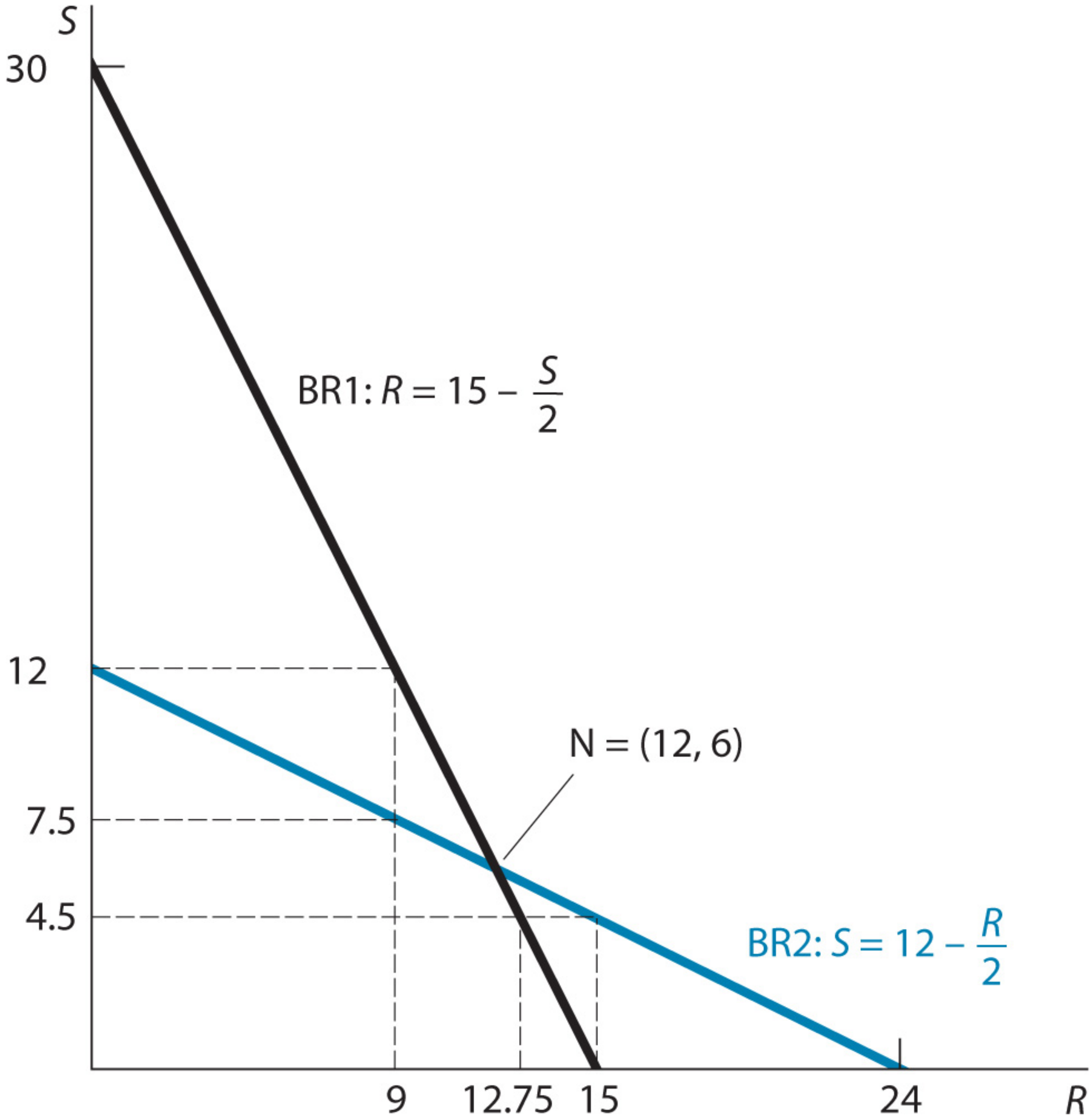

Figure 5.7 Nash Equilibrium through Rationalizability

$$(30 - R) - 2R = 0; \text{ so } R = 15 - \frac{S}{2}.$$

The noncalculus approach uses the result that the U-maximizing value of $R = B/(2C)$, where in this case $B = 30 - S$ and $C = 1$.

This gives $R = (30 - S)/2$, or $R = 15 - S/2$.

Similarly, the best-response equation for the second boat is found by choosing S to maximize V for each fixed R, yielding

$$S = \frac{24 - R}{2}; \text{ so } S = 12 - \frac{R}{2}.$$

The Nash equilibrium is found by solving the two best-response equations jointly for R and S, which is easy to do.[12] So we state only the results here: quantities are $R = 12$ and $S = 6$; price is $P = 42$; and profits are $U = 144$ and $V = 36$.

Figure 5.7 shows the two fishermen's best-response curves (labeled BR1 and BR2 with the equations displayed) and the Nash equilibrium (labeled N with its coordinates displayed) at the intersection of the two curves. Figure 5.7 also shows how the players' beliefs about each other's choices can be narrowed down by iteratively eliminating strategies that are never best responses.

What values of S can the first boat owner rationally believe the second owner will choose? That depends on what the second owner thinks the first owner will produce. But no matter what that quantity might be, the whole range of the second owner's best responses is between 0 and 12. So the first owner cannot rationally believe that the second owner will choose anything else; all negative choices of S (obviously) and all choices of S greater than 12 (less obviously) are eliminated. Similarly, the second owner cannot rationally think that the first owner will produce anything less than 0 or greater than 15.

Now take this process to the second round. When the first owner has restricted the second owner's choices of S to the range between 0 and 12, her own choices of R are restricted to the range of best responses to S's range. The best response to $S = 0$ is $R = 15$, and the best response to $S = 12$ is $R = 15 - 12/2 = 9$.

Because BR1 has a negative slope throughout, the whole range of R allowed at this round of thinking is between 9 and 15. Similarly, the second owner's choice of S is restricted to the range of best responses to R between 0 and 15—namely, values between S = 12 and S = 12 − 15/2 = 4.5. Figure 5.7 shows these restricted ranges on the axes.

The third round of thinking narrows the ranges still further. Because R must be at least 9 and BR2 has a negative slope, S can be at most the best response to 9—namely, S = 12 − 9/2 = 7.5. In the second round, S was already shown to be at least 4.5. Thus, S is now restricted to the range between 4.5 and 7.5. Similarly, because S must be at least 4.5, R can be at most 15 − 4.5/2 = 12.75. In the second round, R was shown to be at least 9, so now it is restricted to the range between 9 and 12.75.

This succession of rounds can be carried on as far as you like, but it is already evident that the successive narrowing of the two ranges is converging on the Nash equilibrium, R = 12 and S = 6. Thus, the Nash equilibrium is the only outcome that survives the iterated elimination of strategies that are never best responses.[13] We know that in general, rationalizability need not narrow down the outcomes of a game to its Nash equilibria, so that is a special feature of this example. Actually, the iterative process works for an entire class of games: those games that have a unique Nash equilibrium at the intersection of downward-sloping best-response curves.[14]

This argument should be carefully distinguished from an older one based on a succession of best responses. The old reasoning proceeded as follows: Start at any strategy for one of the players—say, R = 18. Then, the best response of the other is S = 12 − 18/2 = 3. The best response of R to S = 3 is then R = 15 − 3/2 = 13.5. In turn, the best response of S to R = 13.5 is 12 − 13.5/2 = 5.25. Then, in its turn, the best R against this S is R = 15 − 5.25/2 = 12.375. And so on.

The chain of best responses in the old argument also converges to the Nash equilibrium. But the argument is flawed. The game is played once with simultaneous moves. Therefore, it is not possible for one player to respond to what the other player has chosen, then have the first player respond in turn, and so on. If such dynamics of actual play were allowed, would each player not foresee how the other was going to respond and so do something different in the first place?

The rationalizability argument is different. It clearly incorporates the fact that the game is played only once and with simultaneous moves. All the thinking regarding the chain of best responses is done in advance, and all the successive rounds of thinking and responding are purely conceptual. Players are not responding to actual choices, but are merely calculating those choices that will never be made. The dynamics are purely in the minds of the players.

Endnotes

- This example comes from Douglas Bernheim, "Rationalizable Strategic Behavior," *Econometrica*, vol. 52, no. 4 (July 1984), pp. 1007–28, an article that originally developed the concept of rationalizability. See also Andreu Mas-Colell, Michael Whinston, and Jerry Green, *Microeconomic Theory* (New York: Oxford University Press, 1995), pp. 242–45. Return to reference 9
- Note that in each case, the best choice is strictly better than C4 for Column. Thus, C4 is never even tied for best response. We can distinguish between weak and strong senses of never being a best response just as we distinguished between weak and strict dominance. Here, we have the strong sense. Return to reference 10
- When one allows for mixed strategies, as we will do in Chapter 7, there arises the possibility of a pure strategy being dominated by a mixture of other pure strategies. With such an expanded definition of a dominated strategy, iterated elimination of strictly dominated strategies turns out to be equivalent to rationalizability. The details are best left for a more advanced course in game theory. Return to reference 11
- Although they are incidental to our purpose, some interesting properties of the solution are worth pointing out. The quantities differ because the costs differ; the more efficient (lower-cost) boat gets to sell more. The cost and quantity differences together imply even bigger differences in the resulting profits. The cost advantage of the first boat over the second is only 20%, but it makes four times as much profit as the second boat. Return to reference 12
- This example can also be solved by iteratively eliminating dominated strategies, but proving dominance is harder and needs more calculus, whereas the never-a-best-response property is obvious from Figure 5.7, so we use the simpler argument. Return to reference 13
- A similar argument works with upward-sloping best-response curves, such as those in the pricing game of Figure 5.1, for

narrowing the range of best responses starting at low prices. Narrowing from the higher end is possible only if there is some obvious starting point. This starting point might be a very high price that can never be exceeded for some externally enforced reason—if, for example, people simply do not have the money to pay prices beyond a certain level.
Return to reference 14

Glossary

never a best response
: A strategy is never a best response for a player if, for each list of strategies that the other players choose (or for each list of strategies that this player believes the others are choosing), some other strategy is this player's best response. (The other strategy can be different for different lists of strategies of the other players.)

rationalizable
: A strategy is called rationalizable for a player if it is his optimal choice given some belief about what (pure or mixed strategy) the other player(s) would choose, provided this belief is formed recognizing that the other players are making similar calculations and forming beliefs in the same way. (This concept is more general than that of the Nash equilibrium and yields outcomes that can be justified on the basis only of the players' common knowledge of rationality.)

rationalizability
: A solution concept for a game. A list of strategies, one for each player, is a rationalizable outcome of the game if each strategy in the list is rationalizable for the player choosing it.

4 EMPIRICAL EVIDENCE CONCERNING NASH EQUILIBRIUM

In Chapter 3, when we considered empirical evidence on sequential-move games and rollback, we presented empirical evidence from observations on games played in the real world as well as games deliberately constructed for testing the theory in the laboratory. There, we pointed out the different merits and drawbacks of the two methods for assessing the validity of rollback equilibrium predictions. Similar issues arise in securing and interpreting evidence on Nash equilibrium in simultaneous-move games.

Real-world games are played for substantial stakes, often by experienced players who have the knowledge and the incentives to employ good strategies. But these situations include many factors beyond those considered in the theory. In particular, in real-life situations, it is difficult to observe the quantitative payoffs that players would have earned for all possible combinations of strategies. Therefore, if their behavior does not bear out the predictions of the theory, we cannot tell whether the theory is wrong or whether some other factors overwhelm the strategic considerations.

Laboratory experiments attempt to control for other factors to provide cleaner tests of the theory. But they bring in inexperienced players and provide them with little time and relatively weak incentives to learn the game and play it well. Confronted with a new game, most of us would initially flounder and try things out at random. Thus, the first several plays of the game in an experimental setting may represent this learning phase and not the equilibrium strategies that experienced players would learn to play. Experiments often control for inexperience and learning by discarding several initial plays from their data, but the learning phase may last longer than the one morning or one afternoon that is the typical limit of laboratory sessions.

A. Laboratory Experiments

Researchers have conducted numerous laboratory experiments in the past three decades to test discover how people act when placed in certain interactive strategic situations. In particular, such research asks, Do participants play their Nash equilibrium strategies? Reviewing this work, Douglas Davis and Charles Holt conclude that “in the laboratory, the Nash equilibrium appears to explain behavior fairly well when it is unique (but [with] some important exceptions).” [15] But the theory’s success is more mixed in more complex situations, such as when multiple Nash equilibria exist, when emotional factors modify payoffs beyond the stated cash amounts, when the calculations for finding a Nash equilibrium are more complex, or when the game is repeated with the same partners. We briefly consider the predictive performance of Nash equilibrium in several of these circumstances.

I. CHOOSING AMONG MULTIPLE EQUILIBRIA In Section 2.B above, we presented examples demonstrating that focal points sometimes emerge to help players choose among multiple Nash equilibria. Players may not manage to coordinate 100% of the time, but circumstances often enable players to achieve much more coordination than would result from random choices across possible equilibrium strategies. Here we present a coordination game designed with an interesting trade-off: The equilibrium with the highest payoff to all players also happens to be the riskiest one to play, in the sense of Section 2.A above.

John Van Huyck, Raymond Battalio, and Richard Beil describe a 16-player game in which each player simultaneously chooses an “effort” level between 1 and 7. Individual payoffs depend on group “output,” a function of the minimum effort level chosen by any player in the group, minus the cost of one’s individual effort. The game has exactly seven Nash equilibria in pure strategies; any outcome in which all players choose the same effort level is an equilibrium. The highest possible payoff ($1.30 per player) occurs when all players choose an effort level of 7, while the lowest equilibrium payoff ($0.70 per player) occurs when all players choose an effort level of 1. The highest-payoff equilibrium is a natural candidate for a focal point, but in this case there is a risk to choosing the highest effort level: If just one other player chooses a lower effort level than you, then your extra effort is wasted. For example, if you play 7 and at least one other player chooses 1, you get a payoff of just $0.10, far worse than the worst equilibrium payoff of $0.70. This makes players nervous about whether others will choose the maximum effort level, and as a result, large groups typically fail to coordinate on the best equilibrium. A few players inevitably choose lower than the maximum effort, and in repeated rounds, play converges toward the lowest-effort equilibrium.[16]

II. EMOTIONS AND SOCIAL NORMS In Chapter 3, we saw several examples in sequential-move games where players were more generous to each other than Nash equilibrium would predict. Similar observations occur in simultaneous-move games such as the prisoners' dilemma. One reason may be that the players' payoffs are different from those assumed by the experimenter: In addition to cash, their payoffs may include the experience of emotions such as empathy, anger, or guilt. In other words, the players' value systems may have internalized some social norms of niceness and fairness that have proved useful in the larger social context and that therefore carry over to their behavior in the experimental game.[17] Seen through this lens, these observations do not show any deficiency of the Nash equilibrium concept itself, but they do warn us against using the concept under naive or mistaken assumptions about people's payoffs. It might be a mistake, for example, to assume that players are always driven by the selfish pursuit of money.

III. COGNITIVE ERRORS As we saw in the experimental evidence on rollback equilibrium in Chapter 3, players do not always fully think through the entire game before playing, nor do they always expect other players to do so. Behavior in a game known as the travelers' dilemma illustrates a similar limitation of Nash equilibrium in simultaneous-move games. In this game, two travelers purchase collections of souvenirs of identical value while on vacation, and the airline loses both of their bags on the return trip. The airline announces to the two players that it intends to reimburse them for their losses, but it does not know the exact amount to reimburse. It knows that the correct amount is between $80 and $200 per traveler, so it designs a game as follows. Each player may submit a claim between $80 and $200. The airline will reimburse both players at an amount equal to the lower of the two claims submitted. In addition, if the two claims differ, the airline will pay a reward of $5 to the person making the smaller claim and deduct a penalty of $5 from the reimbursement of the person making the larger claim.

With these rules, irrespective of the actual value of the lost luggage, each player has an incentive to undercut the other's claim. In fact, it turns out that the only Nash equilibrium—and indeed, the only rationalizable outcome—is for both players to report the minimum value of $80. However, in the laboratory, players rarely claim $80; instead, they claim amounts much closer to $200. (Real payoff amounts in the laboratory are typically in cents rather than in dollars.) Interestingly, if the penalty/reward parameter is increased by a factor of 10, from $5 to $50, player behavior conforms much more closely to the Nash equilibrium, with reported amounts generally near $80. Thus, behavior in this experiment varies tremendously with a parameter that does not affect the Nash equilibrium at all; the unique Nash equilibrium is $80, regardless of the penalty/reward amount.

To explain these results from their laboratory, Monica Capra and her coauthors employed a theoretical model called quantal-response equilibrium (QRE), originally proposed by Richard McKelvey and Thomas Palfrey. This model's mathematics are beyond the scope of this text, but its main contribution is allowing for the possibility that players make errors, with the probability of a given error being much smaller for costly mistakes than for mistakes that reduce one's payoff by very little. Furthermore, the model assumes that players expect each other to make errors in this way. It turns out that QRE can explain the data quite well. Reporting a high claim is not very costly when the penalty is only $5, so players are more willing to report values near $200—especially knowing that their rivals are likely to behave similarly, so that the payoff for reporting a high number could be large. However, with a penalty/reward of $50 instead of $5, reporting a high claim becomes quite costly, so players are less likely to expect each other to make such a mistake. This expectation pushes behavior toward the Nash equilibrium claim of $80. Building on this success, QRE has become a very active area of game-theoretic research.[18]

IV. COMMON KNOWLEDGE OF RATIONALITY We have just seen that to better explain experimental results, QRE allows for the possibility that players do not believe that others are perfectly rational. Another way to explain these results is to allow for the possibility that different players engage in different levels of reasoning. A strategic guessing game that is often used in classrooms or laboratories asks each participant to choose a number between 0 and 100. Typically, the players are handed cards on which to write their name and their choice, so this game is a simultaneous-move game. When the cards are collected, the average of the numbers is calculated. The person whose choice is closest to a specified fraction—say, two-thirds—of the average is the winner. The rules of the game (this whole procedure) are announced in advance.

The Nash equilibrium of this game is for everyone to choose 0. In fact, the game is dominance solvable. Even if everyone chooses 100, two-thirds of the average can never exceed 67, so for each player, any choice above 67 is dominated by 67.[19] But all players should rationally figure this out, so the average can never exceed 67, and two-thirds of it can never exceed 44, and so any choice above 44 is dominated by 44. The iterated elimination of dominated strategies goes on in this way until only 0 is left.

However, when a group actually plays this game for the first time, the winner is not a person who plays 0. Typically, the winning number is somewhere around 15 or 20. The most commonly observed choices are 33 and 22, suggesting that a large number of players perform one or two rounds of iterated elimination of dominated strategies without going further. That is, “level-1” players imagine that all other players will choose numbers randomly, with an average of 50, so they best-respond with a choice of two-thirds of this amount, or 33. Similarly, “level-2” players imagine that everyone else will be a “level-1” player, so they best-respond by playing two-thirds of 33, or 22. Note that all of these choices are far from the Nash equilibrium of 0. It appears that many players follow a limited number of steps of iterated elimination of dominated strategies, in some cases because they expect others to be limited in their number of rounds of thinking as well.[20]

V. LEARNING AND MOVING TOWARD EQUILIBRIUMWhat happens when the strategic guessing game is repeated with the same group of players? In classroom experiments, we find that the winning number can easily drop 50% in each subsequent round, as the students expect all their classmates to play numbers as low as the previous round’ s winning number or lower. By the third round of play, winning numbers tend to be as low as 5 or less.

How should one interpret this result? Critics would say that unless the exact Nash equilibrium is reached, the theory is refuted. Indeed, they would argue, if you have good reason to believe that other players will not play their Nash equilibrium strategies, then your best response is not your Nash equilibrium strategy either. If you can figure out how others will deviate from their Nash equilibrium strategies, then you should play your best response to what you believe they are choosing. Others would argue that theories in social science can never hope for the kind of precise prediction that we expect in sciences such as physics and chemistry. If observed outcomes are close to the Nash equilibrium, that is a vindication of the theory. In this case, the experiment not only produces such a vindication, but illustrates the process by which people gather experience and learn to play strategies close to Nash equilibrium. We tend to agree with this latter viewpoint.

Interestingly, we have found that people learn a game somewhat faster by observing others playing it than by playing it themselves. This may be because, as observers, they are free to focus on the game as a whole and think about it analytically. Players' brains are occupied with the task of making their own choices, and they are less able to take the broader perspective.

We should clarify the concept of gaining experience by playing a game. The quotation from Davis and Holt at the start of this section spoke of "repetitions with different partners." In other words, experience should be gained by playing the game frequently, but with different opponents each time. However, for any learning process to generate outcomes increasingly closer to the Nash equilibrium, the whole *population* of learners needs to be stable. If novices keep appearing on the scene and trying new experimental strategies, then the original group members may unlearn what they had learned by playing against one another.

If a game is played repeatedly between two players, or even among the same small group of known players, then any pair is likely to play each other repeatedly. In such a situation, the whole set of repetitions becomes a game in its own right. And that game can have Nash equilibria very different from those that simply repeat the Nash equilibrium of a single play. For example, tacit cooperation may emerge in repeated prisoners' dilemmas, owing to the expectation that any temporary gain from cheating will be more than offset by the subsequent loss of trust. If games are repeated in this way, then learning about them must come from playing whole sets of repetitions frequently, against different partners each time.

B. Real-World Evidence

While the field environment does not allow for as much direct observation as the laboratory does, observations outside the laboratory can also provide valuable evidence about the relevance of Nash equilibrium. Conversely, Nash equilibrium often provides a valuable starting point for social scientists seeking to make sense of the real world.

I. APPLICATIONS OF NASH EQUILIBRIUMOne of the earliest applications of the Nash equilibrium concept to real-world behavior was in the area of international relations. Thomas Schelling pioneered the use of game theory to explain phenomena such as the escalation of arms races, even between countries that have no intention of attacking each other, and to evaluate the credibility of deterrent threats. Subsequent applications in this area have included the questions of when and how a country can credibly signal its resolve in diplomatic negotiations or in the face of a potential war. Game theory began to be used systematically in economics and business in the mid-1970s, and such applications continue to proliferate.[21]

As we saw earlier in this chapter, price competition is one important application of Nash equilibrium in business. Other strategic choices by firms include product quality, investment, and R&D. The theory has also helped us to understand when and how the established firms in an industry can make credible commitments to deter new competition—for example, to wage a destructive price war against any new entrant to the market. Game-theoretic models based on the Nash equilibrium concept and its dynamic generalizations fit the data for many major industries, such as automobile manufacturers, reasonably well. They also give us a better understanding of the determinants of competition than older models, which assumed perfect competition and estimated supply and demand curves.[22]

Pankaj Ghemawat, a professor at the Stern School of Business, New York University, has developed a number of case studies of individual firms or industries, supported by statistical analysis of the data. His game-theoretic models are remarkably successful in improving our understanding of several initially puzzling observed business decisions on pricing, capacity, innovation, and so on. For example, DuPont constructed an enormous amount of manufacturing capacity for titanium dioxide in the 1970s. It added capacity in excess of the projected growth in worldwide demand over the next decade. At first glance, this choice looked like a terrible strategy because the excess capacity could lead to lower market prices for this commodity. However, DuPont successfully foresaw that by having excess capacity in reserve, it could punish competitors that cut prices by increasing its production and driving prices even lower. This ability made it a price leader in the industry, and it enjoyed high profit margins. The strategy worked quite well, and DuPont continued to be a worldwide leader in titanium dioxide 50 years later.[23]

More recently, game theory has become the tool of choice for the study of political systems and institutions. As we will see in Chapter 16, game theory has shown how agenda setting and voting in committees and elections can be strategically manipulated in pursuit of one's ultimate objectives. Part Four of this book will develop other applications of Nash equilibrium to auctions, voting, and bargaining. We will also develop our own case study of the Cuban missile crisis in Chapter 13.

Some critics remain unpersuaded of the value of Nash equilibrium, claiming that the same understanding of these phenomena can be obtained using previously known general principles of economics, political science, and so on. In one sense they are right: A few of these analyses existed before Nash equilibrium came along. For example, the equilibrium of the interaction between two price-setting firms, which we developed in Section 1 of this chapter, has been known in economics for more than 100 years. One can think of Nash equilibrium as just a general formulation of that equilibrium concept for all games. Some theories of strategic voting date to the eighteenth century, and some notions of credibility can be found in history as far back as Thucydides's *Peloponnesian War*. Nash equilibrium, however, unifies all these applications and thereby facilitates the development of new ones.

Furthermore, the development of game theory has also led directly to a wealth of new ideas and applications and thus to new discoveries—for example, how the existence of a second-strike capability reduces the fear of surprise attack, how different auction rules affect bidding behavior and seller revenues, how governments can successfully manipulate fiscal and monetary policies to achieve reelection even when voters are sophisticated and aware of such attempts, and so on. If these examples had all been amenable to previously known approaches, they would have been discovered long ago.

II. REAL-WORLD EXAMPLES OF LEARNINGWe conclude by offering a beautiful example of equilibrium and the learning process in the real-world game of major-league baseball, discovered by Stephen Jay Gould.[24] In this game, the stakes are high, and players play more than 100 games per year, so players have strong motivation and good opportunities to learn. The best batting average recorded in a baseball season declined over most of the twentieth century. In particular, instances of a player averaging .400 or better used to be much more frequent than they are now. Devotees of baseball history often explain this decline by invoking nostalgia: "There were giants in those days." A moment's thought should make one wonder why there were no corresponding pitching giants who would have kept batting averages low. But Gould demolishes such arguments in a more systematic way: He points out that we should look at all batting averages, not just the top ones. The worst batting averages are not as bad as they used to be; there are also many fewer .150 hitters in the major leagues than there used to be. Gould argues that this overall decrease in *variation* is a standardization or stabilization effect:

> When baseball was very young, styles of play had not become sufficiently regular to foil the antics of the very best. Wee Willie Keeler could "hit 'em where they ain't" (and compile an average of .432 in 1897) because fielders didn't yet know where they should be. Slowly, players moved toward *optimal* methods of positioning, fielding, pitching, and batting—and variation inevitably declined. The best [players] now met an opposition too finely honed to its own perfection to permit the extremes of achievement that characterized a more casual age [emphasis added].

In other words, through a succession of adjustments of strategies to counter one another, the system settled down into its (Nash) equilibrium.

Gould marshals decades of hitting statistics to demonstrate that such a decrease in variation actually occurred, except for occasional "blips." And indeed, the blips confirm his thesis, because they occur soon after an equilibrium is disturbed by an externally imposed change. Whenever the rules of baseball are altered (the strike zone is enlarged or reduced, the pitching mound is lowered, or new teams and many new players enter when an expansion takes place) or its technology changes (a livelier ball is used or, perhaps in the future, aluminum bats are allowed), the preceding system of mutual best responses is thrown out of equilibrium.

Variation increases for a while as players experiment, and some strategies succeed while others fail. Finally, a new equilibrium is attained, and variation goes down again. That is exactly what we should expect in the framework of learning and adjustment leading to a Nash equilibrium.

Michael Lewis's 2003 book *Moneyball* (later made into a movie starring Brad Pitt) describes a related example of movement toward equilibrium in baseball. Instead of focusing on the strategies of individual players, it focuses on the teams' back-office strategies for choosing players. The book documents Oakland A's general manager Billy Beane's decision to use "sabermetrics" in hiring decisions—that is, paying close attention to baseball statistics with the aim of maximizing runs scored and minimizing runs given up to opponents. His hiring decisions involved paying more attention to attributes undervalued by the market, such as a player's documented ability to earn walks. Such decisions arguably led to the A's becoming a very strong team, going to the playoffs in five out of seven seasons despite having less than half the payroll of larger-market teams such as the New York Yankees. Beane's innovative hiring strategies have subsequently been adopted by other teams, such as the Boston Red Sox. Using Beane's methods, Boston, under general manager Theo Epstein, managed to break the "curse of the Bambino" in 2004 to win its first World Series in 86 years, and it has remained a winning or strongly contending team in the 15 years since then. In that same era, over the course of a decade, nearly a dozen teams decided to hire full-time sabermetricians, with Beane noting in September 2011 that he was once again "fighting uphill" against larger-market teams that had learned to best-respond to his strategies. Real-world games often involve innovation followed by gradual convergence to equilibrium; the two examples from baseball both give evidence of movement toward equilibrium, although full convergence may sometimes take years or even decades to complete.[25]

We take up additional evidence about other game-theoretic predictions at appropriate points in later chapters. For now, the experimental and empirical evidence that we have presented should make you cautiously optimistic about using Nash equilibrium, especially as a first approach. On the whole, we believe you should have considerable confidence in using the Nash equilibrium concept when the game you are interested in is played frequently by players from a reasonably stable population and under relatively unchanging rules and conditions. When the game is new, or is played just once, and the players are inexperienced, you should use the equilibrium concept more cautiously and should not be surprised if the outcome that you observe is not the equilibrium that you calculate. But even then, your first step in the analysis should be to look for a Nash equilibrium; then you can judge whether it seems a plausible outcome and, if not, proceed to the further step of asking why not.[26] Often the reason will be your misunderstanding of the players' objectives, not the players' failure to play the game correctly giving their true objectives.

Endnotes

- Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton, N.J.: Princeton University Press, 1993), pp. 101–102. Return to reference 15
- See John B. Van Huyck, Raymond C. Battalio, and Richard O. Beil, "Tacit Coordination Games, Strategic Uncertainty, and Coordination Failure," *American Economic Review*, vol. 80, no. 1 (March 1990), pp. 234–48. Subsequent research has suggested methods that can promote coordination on the best equilibrium. Subhasish Dugar, "Non-monetary Sanction and Behavior in an Experimental Coordination Game," *Journal of Economic Behavior & Organization*, vol. 73, no. 3 (March 2010), pp. 377–86, shows that players gradually manage to coordinate on the highest-payoff outcome merely by allowing players, between rounds, to express the numeric strength of their disapproval for each other player's decision. Roberto A. Weber, "Managing Growth to Achieve Efficient Coordination in Large Groups," *American Economic Review*, vol. 96, no. 1 (March 2006), pp. 114–26, shows that starting with a small group and slowly adding additional players can sustain the highest-payoff equilibrium, suggesting that a firm may do well to expand slowly and make sure that employees understand the corporate culture of cooperation. Return to reference 16
- The distinguished game theorist Jörgen Weibull argues this position in detail in "Testing Game Theory," in *Advances in Understanding Strategic Behaviour: Game Theory, Experiments and Bounded Rationality: Essays in Honour of Werner Güth*, ed. Steffen Huck (Basingstoke, UK: Palgrave MacMillan, 2004), pp. 85–104. Return to reference 17
- See Kaushik Basu, "The Traveler's Dilemma," *Scientific American*, vol. 296, no. 6 (June 2007), pp. 90–95. The

experiments and modeling can be found in C. Monica Capra, Jacob K. Goeree, Rosario Gomez, and Charles A. Holt, “Anomalous Behavior in a Traveler’s Dilemma?” *American Economic Review*, vol. 89, no. 3 (June 1999), pp. 678–90. Quantal-response equilibrium (QRE) was first proposed by Richard D. McKelvey and Thomas R. Palfrey, “Quantal Response Equilibria for Normal Form Games,” *Games and Economic Behavior*, vol. 10, no. 1 (July 1995), pp. 6–38. Return to reference 18

- If you factor in your own choice, the calculation is strengthened. Suppose there are N players. In the “worst-case scenario,” where all the other $(N - 1)$ players choose 100 and you choose x, the average is $[x + (N - 1)100]/N$. Then your best choice is two-thirds of this, so $x = (2/3)[x + (N - 1)100]/N$, or $x = 100(2N - 2)/(3N - 2)$. If $N = 10$, then $x = (18/28)/100 = 64$ (approximately). So any choice above 64 is dominated by 64. The same reasoning applies to the successive rounds. Return to reference 19
- You will analyze similar games in Exercises S12 and U11. For a summary of results from large-scale experiments run in European newspapers with thousands of players, see Rosemarie Nagel, Antoni Bosch-Domènech, Albert Satorra, and Juan Garcia-Montalvo, “One, Two, (Three), Infinity: Newspaper and Lab Beauty-Contest Experiments,” *American Economic Review*, vol. 92, no. 5 (December 2002), pp. 1687–1701. Return to reference 20
- For those who would like to see more applications, here are some suggested sources. Thomas Schelling’s *The Strategy of Conflict* (Cambridge, Mass.: Harvard University Press, 1960) and *Arms and Influence* (New Haven, Conn.: Yale University Press, 1966) are still required reading for all students of game theory. The classic textbook on game-theoretic treatment of industries is Jean Tirole, *The Theory of Industrial Organization* (Cambridge, Mass.: MIT Press, 1988). In political science, an early classic is William H. Riker,

Liberalism against Populism (San Francisco: W. H. Freeman, 1982). For advanced-level surveys of research, see several articles in *The Handbook of Game Theory with Economic Applications*, ed. Robert J. Aumann and Sergiu Hart (Amsterdam: North-Holland/Elsevier, 1992, 1994, 2002), particularly Barry O'Neill, "Game Theory Models of Peace and War," in volume 2, and Kyle Bagwell and Asher Wolinsky, "Game Theory and Industrial Organization," and Jeffrey Banks, "Strategic Aspects of Political Systems," both of which are in volume 3. Return to reference 21

- For simultaneous-move models of price competition, see Timothy F. Bresnahan, "Empirical Studies of Industries with Market Power," in *Handbook of Industrial Organization*, vol. 2, ed. Richard L. Schmalensee and Robert D. Willig (Amsterdam: North-Holland/Elsevier, 1989), pp. 1011–57. For models of entry, see Steven Berry and Peter Reiss, "Empirical Models of Entry and Market Structure," in *Handbook of Industrial Organization*, vol. 3, ed. Mark Armstrong and Robert Porter (Amsterdam: North-Holland/Elsevier, 2007), pp. 1845–86. Return to reference 22
- Pankaj Ghemawat, "Capacity Expansion in the Titanium Dioxide Industry," *Journal of Industrial Economics*, vol. 33, no. 2 (December 1984), pp. 145–63. For more examples, see Pankaj Ghemawat, *Games Businesses Play: Cases and Models* (Cambridge, Mass.: MIT Press, 1997). Return to reference 23
- Stephen Jay Gould, "Losing the Edge," in *The Flamingo's Smile: Reflections in Natural History* (New York: W. W. Norton, 1985), pp. 215–29. Return to reference 24
- Susan Slusser, "Michael Lewis on A's 'Moneyball' Legacy," *San Francisco Chronicle*, September 18, 2011, p. B-1. The original book is Michael Lewis, *Moneyball: The Art of Winning an Unfair Game* (New York: W. W. Norton, 2003). Return to reference 25

- In an article probing the weaknesses of Nash equilibrium in experimental data and proposing QRE-style alternative models for dealing with them, two prominent researchers write, “We will be the first to admit that we begin the analysis of a new strategic problem by considering the equilibria derived from standard game theory before considering” other possibilities. Jacob K. Goeree and Charles A. Holt, “Ten Little Treasures of Game Theory and Ten Intuitive Contradictions,” *American Economic Review*, vol. 91, no. 5 (December 2001), pp. 1402–22.
  Return to reference 26

Glossary

quantal-response equilibrium (QRE)

Solution concept that allows for the possibility that players make errors, with the probability of a given error smaller for more costly mistakes.

SUMMARY

When players in a simultaneous-move game have a continuous range of actions to choose from, best-response analysis yields mathematical *best-response rules* that can be solved simultaneously to obtain Nash equilibrium. The best-response rules can be shown as *best-response curves* on a graph, where the intersection of the two curves represents the Nash equilibrium. Games played among firms choosing price or quantity from a large range of possible values and among political parties choosing campaign advertising expenditure levels are examples of games with *continuous strategies*.

Theoretical criticisms of the Nash equilibrium concept have argued that the concept does not adequately account for risk, that it is of limited use because many games have multiple equilibria, and that it cannot be justified on the basis of rationality alone. In many cases, a better description of the game and its payoff structure or a *refinement* of the Nash equilibrium concept can lead to better predictions or fewer potential equilibria. *Quantal response equilibrium* is one such alternative. Another is *rationalizability*, which relies on the iterated elimination of strategies that are *never a best response* to obtain a set of *rationalizable* outcomes. When a game has a Nash equilibrium, that outcome will be rationalizable, but rationalizability also allows one to predict equilibrium outcomes in games that have no Nash equilibria.

The results of laboratory tests of the Nash equilibrium concept show that a common cultural background is essential for coordination in games with multiple equilibria. Repeated play of some games shows that players can learn from experience and begin to choose strategies that approach their Nash equilibrium choices. Further, predictions of equilibria are accurate only when the experimenters’ assumptions match the true preferences of players. Real-world applications of game theory have helped economists and political scientists, in particular, to understand important consumer, firm, voter, legislative, and government behaviors.

KEY TERMS

best-response curve (133)

best-response rule (132)

continuous strategy (131)

never a best response (146)

quantal-response equilibrium (QRE) (154)

rationalizability (146)

rationalizable (146)

refinement (144)

Glossary

best-response curve
: A graph showing the best strategy of one player as a function of the strategies of the other player(s) over the entire range of those strategies.

best-response rule
: A function expressing the strategy that is optimal for one player, for each of the strategy combinations actually played by the other players, or the belief of this player about the other players' strategy choices.

continuous strategy
: A choice over a continuous range of real numbers available to a player.

never a best response
: A strategy is never a best response for a player if, for each list of strategies that the other players choose (or for each list of strategies that this player believes the others are choosing), some other strategy is this player's best response. (The other strategy can be different for different lists of strategies of the other players.)

quantal-response equilibrium (QRE)
: Solution concept that allows for the possibility that players make errors, with the probability of a given error smaller for more costly mistakes.

rationalizability
: A solution concept for a game. A list of strategies, one for each player, is a rationalizable outcome of the game if each strategy in the list is rationalizable for the player choosing it.

rationalizable
: A strategy is called rationalizable for a player if it is his optimal choice given some belief about what (pure or mixed strategy) the other player(s) would choose,

provided this belief is formed recognizing that the other players are making similar calculations and forming beliefs in the same way. (This concept is more general than that of the Nash equilibrium and yields outcomes that can be justified on the basis only of the players' common knowledge of rationality.)

refinement
: A restriction that narrows down possible outcomes when multiple Nash equilibria exist.

SOLVED EXERCISES

1. In the political campaign advertising game in Section 1.C, party L chooses an advertising budget x (millions of dollars), and party R similarly chooses an advertising budget y (millions of dollars). We showed there that the best-response rules in that game are $y = 10\sqrt{x} - x$ for party R and $x = 10\sqrt{y} - y$ for party L.
   1. What is party R’ s best response if party L spends \$16 million?
   2. Use the specified best-response rules to verify that the Nash equilibrium advertising budgets are $x = y = 25$, or \$25 million.
2. The restaurant pricing game illustrated in Figure 5.1 defines customer demand equations for meals at Xavier’ s (Q_x) and Yvonne’ s (Q_y) as $Q_x = 44 - 2P_x + P_y$, and $Q_y = 44 - 2P_y + P_x$. Profits for each firm also depend on the costs of serving each customer. Suppose that Yvonne’ s is able to reduce its costs to a mere \$2 per customer by completely eliminating the waitstaff (customers pick up their orders at a counter, and a few remaining employees bus the tables). Xavier’ s continues to incur a cost of \$8 per customer.
   1. Recalculate the best-response rules and the Nash equilibrium prices for the two restaurants given this change in one restaurant’ s costs.
   2. Graph the two best-response curves and describe the differences between your graph and Figure 5.1. In particular, which curve has moved, and by how much? Explain why these changes occurred in the graph.
3. The small town of Eten has two food stores, La Boulangerie, which sells bread, and La Fromagerie, which sells cheese. It costs \$1 to make a loaf of bread and \$2 to make a pound of cheese. If La Boulangerie’ s price is P_1 dollars per loaf of bread and La Fromagerie’ s price is P_2 dollars per pound of cheese, then their

respective weekly sales, Q_1 thousand loaves of bread and Q_2 thousand pounds of cheese, are given by the following equations:

$Q_1 = 14 - P_1 - 0.5\ P_2$

$Q_2 = 19 - 0.5P_1 - P_2.$

1. For each of the stores, write the profit as a function of P_1 and P_2 (in the exercises that follow, we will call this "the profit function" for brevity). Then find their respective best-response rules. Graph the best-response curves, and find the Nash equilibrium prices in this game.
2. Suppose that the two stores collude and set prices jointly to maximize the sum of their profits. Find the joint profit-maximizing prices for the stores.
3. Provide a short intuitive explanation for the differences between the Nash equilibrium prices and those that maximize joint profit. Why is joint profit maximization not a Nash equilibrium?
4. In this problem, bread and cheese are *complements* to each other. They are often consumed together, so a drop in the price of one increases the sales of the other. The products in our restaurant example in Section 1.A are *substitutes* for each other. How does this difference explain the differences between your findings for the best-response rules, the Nash equilibrium prices, and the joint profit-maximizing prices in this question and the corresponding entities in the restaurant example in the text?

4. The game illustrated in Figure 5.3 has a unique Nash equilibrium in pure strategies. However, all nine outcomes in that game are rationalizable. Confirm this assertion, explaining your reasoning for each outcome.
5. For the game presented in Exercise S6 in Chapter 4, what are the rationalizable strategies for each player? Explain your reasoning.
6. Section 3.B of this chapter describes a fishing-boat game played in a small coastal town. When the best-response rules for the two boats have been derived, rationalizability can be used to justify the Nash equilibrium in the game. In the description in the text, we take the process of narrowing down strategies that can never be best responses through three rounds. By the third round, we know that R (the number of barrels of fish brought home by boat

1) must be at least 9, and that S (the number of barrels of fish brought home by boat 2) must be at least 4.5. The iterative elimination process in that round restricted R to the range between 9 and 12.75 while restricting S to the range between 4.5 and 7.5. Take this process of narrowing through one additional (fourth) round and show the reduced ranges of R and S that are obtained at the end of that round.

7. Two carts selling coconut milk (from the coconut) are located at points 0 and 1, 1 mile apart on the beach in Rio de Janeiro. (They are the only two coconut-milk carts on the beach.) The carts—Cart 0 and Cart 1—charge prices p_0 and p_1, respectively, for each coconut. One thousand beachgoers buy coconut milk, and these customers are uniformly distributed along the beach between carts 0 and 1. Each beachgoer will purchase one coconut milk in the course of her day at the beach, and in addition to the price, each will incur a transport cost of $0.5 \times d^2$, where d is the distance (in miles) from her beach blanket to the coconut-milk cart. In this system, Cart 0 sells to all the beachgoers located between 0 and x, and Cart 1 sells to all the beachgoers located between x and 1, where x is the location of the beachgoer who pays the same total price whether she goes to Cart 0 or Cart 1. Location x is then defined by the expression

   $$p_0 + 0.5\ x^2 = p_1 + 0.5(1 - x)^2.$$

   Each cart will set its prices to maximize its profit, Π, which is determined by revenue (the cart's price times its number of customers) and cost (each cart incurs a cost of \$0.25 per coconut times the number of coconuts sold).

   1. For each cart, determine the expression for the number of customers served as a function of p_0 and p_1. [Recall that Cart 0 gets the customers between 0 and x, or just x, while Cart 1 gets the customers between x and 1, or $1 - x$. That is, Cart 0 sells to x customers, where x is measured in thousands, and Cart 1 sells to $(1 - x)$.]
   2. Write the profit functions for the two carts. Find the best-response rule for each cart as a function of its rival's price.
   3. Graph the best-response rules, and then calculate (and show on your graph) the Nash equilibrium price for coconut milk on the beach.

8. Crude oil is transported across the globe in enormous tanker ships called Very Large Crude Carriers (VLCCs). The vast majority of all new VLCCs are built in South Korea and Japan. Assume that the price of new VLCCs (in millions of dollars) is determined by the function $P = 180 - Q$, where $Q = q_{\text{Korea}} + q_{\text{Japan}}$, the sum of the quantities produced in each of South Korea and Japan (assuming that only these two countries produce VLCCs, so they are a duopoly.) Assume that the cost of building each ship is \$30 million in both Korea and Japan. That is, $c_{\text{Korea}} = c_{\text{Japan}} = 30$, where the cost per ship is measured in millions of dollars.
   1. Write the profit function for each country in terms of q_{Korea} and q_{Japan} and in terms of either c_{Korea} or c_{Japan}. Find each country's best-response function.
   2. Using the best-response functions found in part (a), solve for the Nash equilibrium quantity of VLCCs produced by each country per year. What is the price of a VLCC? How much profit is made in each country?
   3. Labor costs in Korean shipyards are actually much lower than in their Japanese counterparts. Assume now that the cost per ship in Japan is \$40 million and that in Korea it is only \$20 million. Given $c_{\text{Korea}} = 20$ and $c_{\text{Japan}} = 40$, what is the market share of each country (that is, the percentage of ships that each country sells relative to the total number of ships sold)? What are the profits for each country?
9. Extending the previous problem, suppose China decides to enter the VLCC construction market. The duopoly now becomes a triopoly, so that although price is still $P = 180 - Q$, quantity is now given by $Q = q_{\text{Korea}} + q_{\text{Japan}} + q_{\text{China}}$. Assume that all three countries have a per-ship cost of \$30 million: $c_{\text{Korea}} = c_{\text{Japan}} = c_{\text{China}} = 30$.
   1. Write the profit function for each of the three countries in terms of q_{Korea}, q_{Japan}, and q_{China}, and in terms of c_{Korea}, c_{Japan}, or c_{China}. Find each country's best-response rule.
   2. Using your answer to part (a), find the quantity produced, the market share captured [see Exercise S8, part (c)], and the profits earned by each country. This will require the solution of three equations in three unknowns.
   3. What happens to the price of a VLCC in the new triopoly relative to that in the duopoly situation in Exercise S8, part (b)? Why?

10. Monica and Nancy have formed a business partnership to provide consulting services in the golf industry. They each have to decide how much effort to put into the business. Let m be the amount of effort put into the business by Monica, and n be the amount of effort put in by Nancy.

    The joint profits of the partnership are given by $4m + 4n + mn$, in tens of thousands of dollars, and the two partners split these profits equally. However, each partner's payoff also includes the cost of her own effort; the cost to Monica of her effort is m^2, while the cost to Nancy of her effort is n^2 (both measured in tens of thousands of dollars). Each partner must make her effort decision without knowing what effort decision the other player has made.

    1. If Monica and Nancy each put in effort of $m = n = 1$, then what are their payoffs?
    2. If Monica puts in effort of $m = 1$, then what is Nancy's best response?
    3. What is the Nash equilibrium of this game?

11. Nash equilibrium can be achieved through rationalizability in games with upward-sloping best-response curves if the rounds of eliminating strategies that are never a best response begin with the smallest possible values. Consider the pricing game between Xavier's Tapas Bar and Yvonne's Bistro that is illustrated in Figure 5.1. Use Figure 5.1 and the best-response rules from which it is derived to begin rationalizing the Nash equilibrium in that game. Start with the lowest possible prices for the two restaurants and describe (at least) two rounds of narrowing the set of rationalizable prices toward the Nash equilibrium.
12. A professor presents the following game to Elsa and her 49 classmates. Each of the students simultaneously and privately writes down a number between 0 and 100 on a piece of paper, and they all hand in their numbers. The professor then computes the mean of these numbers and defines X to be the mean of the students' numbers. The student who submits the number closest to one-half of X wins \$50. If multiple students tie, they split the prize equally.
    1. Show that choosing the number 80 is a dominated strategy.
    2. What would the set of best responses be for Elsa if she knew that all of her classmates would submit the number 40? That

is, what is the range of numbers for which each number in the range is closer to the winning number than 40?

3. What would the set of best responses be for Elsa if she knew that all of her classmates would submit the number 10?
4. Find a symmetric Nash equilibrium for this game. That is, what number is a best response to everyone else submitting that same number?
5. Which strategies are rationalizable in this game?

UNSOLVED EXERCISES

1. Diamond Trading Company (DTC), a subsidiary of De Beers, is the dominant supplier of high-quality diamonds for the wholesale market. For simplicity, assume that DTC has a monopoly on wholesale diamonds. The quantity that DTC chooses to sell thus has a direct impact on the wholesale price of diamonds. Let the wholesale price of diamonds (in hundreds of dollars) be given by the following (inverse) demand equation: $P = 120 - Q_{\text{DTC}}$. Assume that DTC has a cost of 12 (hundred dollars) per high-quality diamond.
   1. Write DTC’ s profit function in terms of Q_{DTC}, and solve for DTC’ s profit-maximizing quantity. What will be the wholesale price of diamonds at that quantity? What will DTC’ s profit be?

   Frustrated with DTC’ s monopoly, several diamond mining interests and large retailers collectively set up a joint venture called Adamantia to act as a competitor to DTC in the wholesale market for diamonds. The wholesale price is now given by $P = 120 - Q_{\text{DTC}} - Q_{\text{ADA}}$. Assume that Adamantia has a cost of 12 (hundred dollars) per high-quality diamond.

   1. (b) Write the best-response functions for both DTC and Adamantia. What quantity does each wholesaler supply to the market in equilibrium? What wholesale price do these quantities imply? What will the profit of each supplier be in this duopoly situation?
   2. (c) Describe the differences between the market for wholesale diamonds under the duopoly of DTC and Adamantia and under the monopoly of DTC. What happens to the quantity supplied in the market and the market price when Adamantia enters? What happens to the collective profit of DTC and Adamantia?
2. There are two movie theaters in the town of Harkinsville: Modern Multiplex, which shows first-run movies, and Sticky Shoe, which shows movies that have been out for a while at a cheaper price. The demand for movies at Modern Multiplex is given by $Q_{\text{MM}} = 14 - P_{\text{MM}} + P_{\text{SS}}$, while the demand for movies at Sticky Shoe is $Q_{\text{SS}} = 8 - 2P_{\text{SS}} + P_{\text{MM}}$, where prices are in dollars and quantities are measured in hundreds of moviegoers. Modern Multiplex has a per-

customer cost of \$4, while Sticky Shoe has a per-customer cost of only \$2.

1. In the demand equations alone, what indicates whether Modern Multiplex and Sticky Shoe offer services that are substitutes or complements?
2. Write the profit function for each theater in terms of P_{SS} and P_{MM}. Find each theater's best-response rule.
3. Find the Nash equilibrium price, quantity, and profit for each theater.
4. What would each theater's price, quantity, and profit be if the two decided to collude to maximize their joint profits in this market? Why isn't the outcome of this collusion a Nash equilibrium?

3. Fast forward a decade beyond the situation in Exercise S3. Eten's demand for bread and cheese has decreased, and the town's two food stores, La Boulangerie and La Fromagerie, have been bought out by a third company: L'Épicerie. It still costs \$1 to make a loaf of bread and \$2 to make a pound of cheese, but the quantities of bread and cheese sold (Q_1 and Q_2 respectively, measured in thousands) are now given by the equations

   $$Q_1 = 8 - P_1 - 0.5P_2, \quad Q_2 = 16 - 0.5P_1 - P_2.$$

   Again, P_1 is the price in dollars of a loaf of bread, and P_2 is the price in dollars of a pound of cheese.

   1. Initially, L'Épicerie runs La Boulangerie and La Fromagerie as if they were separate firms, with independent managers who each try to maximize their own profit. What are the Nash equilibrium quantities, prices, and profits for the two divisions of L'Épicerie, given the new quantity equations?
   2. The owners of L'Épicerie think that they can make more total profit by coordinating the pricing strategies of the two Eten divisions of their company. What are the joint profit-maximizing prices for bread and cheese under this arrangement? What quantities of each good do La Boulangerie and La Fromagerie sell, and what is the profit that each division earns separately?
   3. In general, why might companies sell some of their goods at prices below cost? That is, explain the rationale of such *loss leaders*, using your answer from part (b) as an illustration.

4. The coconut-milk carts from Exercise S7 set up again the next day. Nearly everything is exactly the same as in Exercise S7: The carts are in the same locations, the number and distribution of beachgoers is identical, and the demand of the beachgoers for exactly one coconut milk each is unchanged. The only difference is that it is a particularly hot day, so that now each beachgoer incurs a higher transport cost of $0.6d^2$. Again, Cart 0 sells to all the beachgoers located between 0 and x, and Cart 1 sells to all the beachgoers located between x and 1, where x is the location of the beachgoer who pays the same total price whether she goes to Cart 0 or Cart 1. However, location x is now defined by the expression

   $$p_0 + 0.6x^2 = p_1 + 0.6(1 - x)^2.$$

   Again, each cart has a cost of \$0.25 per coconut sold.

   1. For each cart, determine the expression for the number of customers served as a function of p_0 and p_1. [Recall that Cart 0 gets the customers between 0 and x, or just x, while Cart 1 gets the customers between x and 1, or $1 - x$. That is, Cart 0 sells to x customers, where x is measured in thousands, and Cart 1 sells to $(1 - x)$.]
   2. Write out profit functions for the two carts and find the two best-response rules.
   3. Calculate the Nash equilibrium price for coconuts on the beach. How does this price compare with the price found in Exercise S7? Why?

5. The game illustrated in Figure 5.4 has a unique Nash equilibrium in pure strategies. Find that Nash equilibrium, and then show that it is also the unique rationalizable outcome in that game.
6. What are the rationalizable strategies of the game of "Evens or Odds" from Exercise S14 in Chapter 4?
7. In the fishing-boat game of Section 3.B, we showed that it is possible for a game to have a uniquely rationalizable outcome in continuous strategies that is also a Nash equilibrium. However, this is not always the case; there may be many rationalizable strategies, and not all of them will necessarily be part of a Nash equilibrium.

Returning to the political advertising game of Exercise S1, find the set of rationalizable strategies for party L. (Due to their symmetric payoffs, the set of rationalizable strategies will be the same for party R.) Explain your reasoning.

8. Intel and AMD, the primary producers of computer central processing units (CPUs), compete with each other in the mid-range chip category (among other categories). Assume that global demand for mid-range chips depends on the quantity that the two firms make, so that the price (in dollars) for mid-range chips is given by $P = 210 - Q$, where $Q = q_{\text{Intel}} + q_{\text{AMD}}$ and where the quantities are measured in millions. Each mid-range chip costs Intel \$60 to produce. AMD' s production process is more streamlined; each chip costs it only \$48 to produce.
   1. Write the profit function for each firm in terms of q_{Intel} and q_{AMD}. Find each firm' s best-response rule.
   2. Find the Nash equilibrium price, quantity, and profit for each firm.
   3. (**Optional**) Suppose Intel acquires AMD, so that it now has two separate divisions with two different production costs. The merged firm wishes to maximize total profits from the two divisions. How many chips should each division produce? (Hint: You may need to think carefully about this problem, rather than blindly applying mathematical techniques.) What is the market price and the total profit for the firm?
9. Return to the VLCC triopoly game of Exercise S9. In reality, the three countries do not have identical production costs. China has been gradually entering the VLCC construction market for several years, and its production costs started out rather high due to its lack of experience.
   1. Solve for the triopoly quantities, market shares, prices, and profits for the case where the per-ship costs are \$20 million for Korea, \$40 million for Japan, and \$60 million for China ($c_{\text{Korea}} = 20$, $c_{\text{Japan}} = 40$, and $c_{\text{China}} = 60$).

   After it gains experience and adds production capacity, China' s per-ship cost will decrease dramatically. Because labor is even cheaper in China than in Korea, eventually the per-ship cost will be even lower in China than it is in Korea.

   1. (b) Repeat part (a) with the adjustment that China' s per-ship cost is \$16 million ($c_{\text{Korea}} = 20$, $c_{\text{Japan}} = 40$, and $c_{\text{China}} =$

16).

10. Return to the story of Monica and Nancy from Exercise S10. After some additional professional training, Monica is more productive on the job, so that the joint profits of their company are now given by $5m + 4n + mn$, in tens of thousands of dollars. Again, m is the amount of effort put into the business by Monica, n is the amount of effort put in by Nancy, and the costs of their efforts are m^2 and n^2 to Monica and Nancy respectively (in tens of thousands of dollars).

    The terms of their partnership still require that the joint profits be split equally, despite the fact that Monica is more productive. Assume that their effort decisions are made simultaneously.

    1. What is Monica's best response if she expects Nancy to put in an effort of $n = 4/3$?
    2. What is the Nash equilibrium of this game?
    3. Compared with the old Nash equilibrium found in Exercise S10, part (c), does Monica now put in more, less, or the same amount of effort? What about Nancy?
    4. What are the final payoffs to Monica and Nancy in the new Nash equilibrium (after splitting the joint profits and accounting for the costs of their efforts)? How do these payoffs compare to the payoffs to each of them under the old Nash equilibrium? In the end, who receives more benefit from Monica's additional training?
11. A professor presents a new game to Elsa and her 49 classmates (similar to the situation in Exercise S12). As before, each of the students simultaneously and privately writes down a number between 0 and 100 on a piece of paper, and the professor computes the mean of these numbers and calls it X. This time, the student who submits the number closest to $2/3 \times (X + 9)$ wins \$50. Again, if multiple students tie, they split the prize equally.
    1. Find a symmetric Nash equilibrium for this game. That is, what number is a best response to everyone else submitting the same number?
    2. Show that choosing the number 5 is a dominated strategy. (Hint: What would class average X have to be for the target number to be 5?)
    3. Show that choosing the number 90 is a dominated strategy.
    4. What are all of the dominated strategies?

5. Suppose Elsa believes that none of her classmates will play the dominated strategies you found in part (d). Given her beliefs, what strategies are never a best response for Elsa?
6. Which strategies do you think are rationalizable in this game? Explain your reasoning.

12. (**Optional**—requires calculus) Recall the political campaign advertising example from Section 1.C concerning parties L and R. In that example, when L spends $\$x$ million on advertising and R spends $\$y$ million, L gets a share $x/(x + y)$ of the votes and R gets a share $y/(x + y)$. We also mentioned that two types of asymmetries can arise between the parties in that model: One party—say, R—may be able to advertise at a lower cost, or R's advertising dollars may be more effective in generating votes than L's. To allow for both possibilities, we can write the payoff functions of the two parties as

$$V_L = \frac{x}{x+ky} - x \text{ and } V_n = \frac{ky}{x+ky} - cy, \text{ where } k>0 \text{ and } c>0.$$

These payoff functions show that R has an advantage in the relative effectiveness of its ads when k is high and that R has an advantage in the cost of its ads when c is low.

1. Use the payoff functions to derive the best-response functions for R (which chooses y) and L (which chooses x).
2. Use your calculator or your computer to graph these best-response functions when $k = 1$ and $c = 1$. Compare the graph with the one for the case in which $k = 1$ and $c = 0.8$. What is the effect of having an advantage in the cost of advertising?
3. Compare the graph from part (b), when $k = 1$ and $c = 1$, with the one for the case in which $k = 2$ and $c = 1$. What is the effect of having an advantage in the effectiveness of advertising dollars?
4. Solve the best-response functions that you found in part (a), jointly for x and y, to show that the campaign advertising expenditures in Nash equilibrium are

$$x = \frac{ck}{(c+k)^2} - x \text{ and } y = \frac{k}{(c+k)^2}.$$

---

5. Let $k = 1$ in the equilibrium equations in part (d) and show how the two equilibrium spending levels vary with changes in c (that is, interpret the signs of dx/dc and dy/dc). Then let $c = 1$ and show how the two equilibrium spending levels vary with changes in k (that is, interpret the signs of dx/dk and dy/dk). Do your answers support the effects that you observed in parts (b) and (c) of this exercise?

■ Appendix: Finding a Value to Maximize a Function

Here we develop in a simple way the method for choosing a variable X to obtain the maximum value of a variable that is a function of it, say $Y = F(X)$. Our applications will mostly be to cases where the function is quadratic, such as $Y = A + BX - CX^2$. For such functions we derive the formula $X = B/(2C)$ that was stated and used in the chapter text. We develop the general idea using calculus, and then offer an alternative approach that does not use calculus but applies only to the quadratic function.[27]

The calculus method tests a value of X for optimality by seeing what happens to the value of the function for other values on either side of X. If X does indeed maximize $Y = F(X)$, then the effect of increasing or decreasing X should be a drop in the value of Y. Calculus gives us a quick way to perform such a test.

Figure 5A.1 illustrates the basic idea. It shows the graph of a function $Y = F(X)$, where we have used a function of the type that fits our application, even though the idea is perfectly general. Start at any point P with coordinates (X, Y) on the graph. Consider a slightly different value of X, say $(X + h)$. Let k be the resulting change in $Y = F(X)$, so the point Q with coordinates $(X + h, Y + k)$ is also on the graph. The slope of the chord joining P to Q is the ratio k/h. If this ratio is positive, then h and k have the same sign; as X increases, so does Y. If the ratio is negative, then h and k have opposite signs; as X increases, Y decreases.

If we now consider smaller and smaller changes h in X, and the corresponding smaller and smaller changes k in Y, the chord PQ will approach the tangent to the graph at P. The slope of this tangent is the limiting value of the ratio k/h. It is called the derivative of the function $Y = F(X)$ at the point X. Symbolically, it is written as $F'(X)$ or dY/dX. Its sign tells us whether the function is increasing or decreasing at precisely the point X.

For the quadratic function in our application, $Y = A + BX - CX^2$ and

$$Y + k = A + B(X + h) - C(X + h)^2.$$

Therefore, we can find an expression for k as follows:

$$k = [A + B(X + h) - C(X + h)^2] - (A + BX - CX^2)$$

$$= Bh - C[(X + h)^2 - X^2]$$

$$= Bh - C(X^2 + 2Xh + h^2 - X^2)$$

$$= (B - 2CX)h - Ch^2.$$

Then $k/h = (B - 2CX) - Ch$. In the limit as h goes to zero, $k/h = (B - 2CX)$. This last expression is then the derivative of our function.

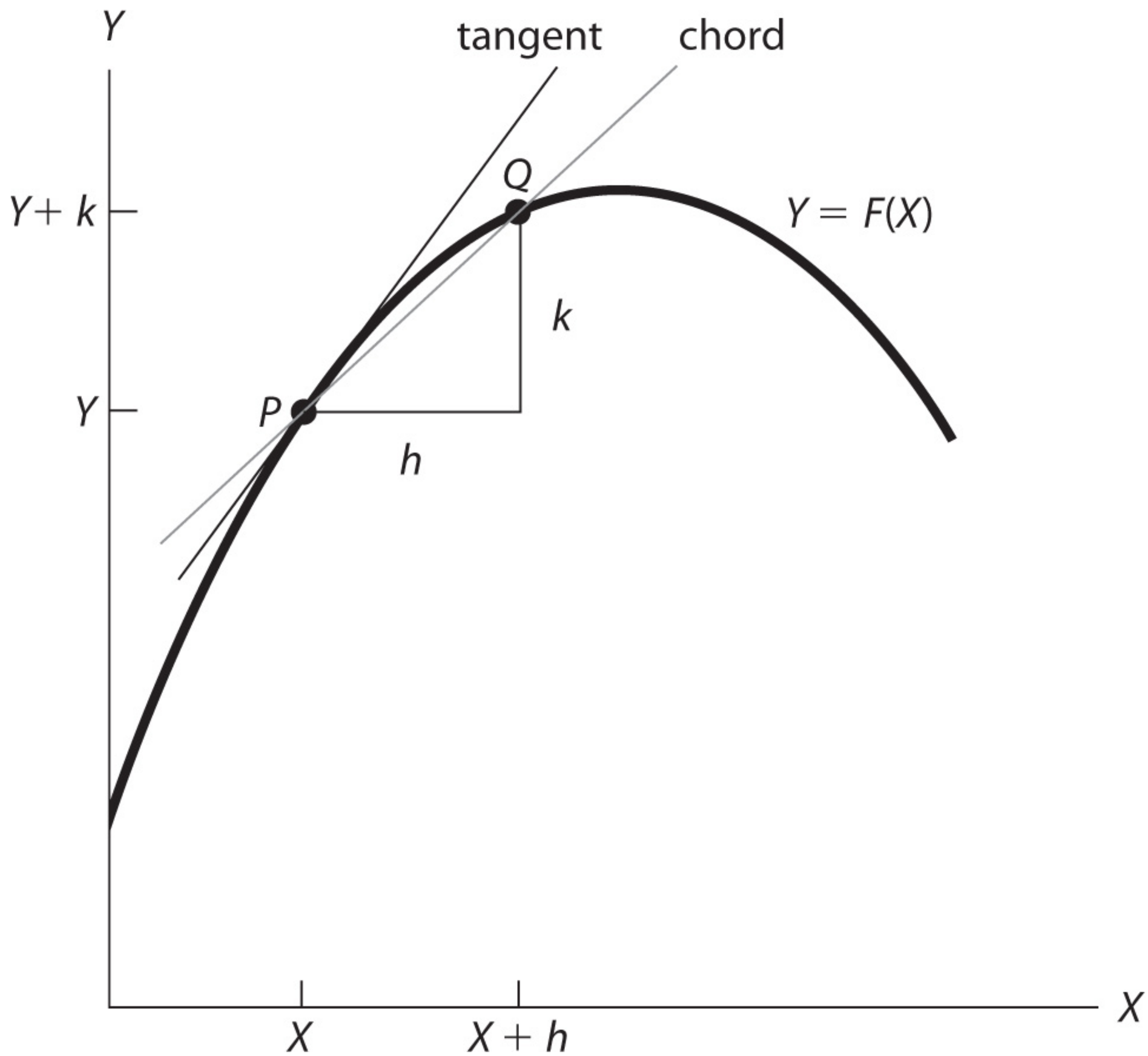

Figure 5A.1 Derivative of a Function Illustrated

Now we use the derivative to find a test for optimality. Figure 5A.2 illustrates the idea. The point M yields the highest value of $Y = F(X)$. The function increases as we approach the point M from the left and decreases after we have passed to the right of M. Therefore, the derivative $F'(X)$ should be positive for values of X smaller than M and negative for values of X larger than M. By continuity, the derivative precisely at M should be 0. In ordinary language, the graph of the function should be flat where it peaks.

In our quadratic example, the derivative is $F'(X) = B - 2CX$. Our optimality test implies that the function is optimized when this is 0, or at $X = B/(2C)$. This is exactly the formula given in the chapter text.

One additional check needs to be performed. If we turn the whole figure upside down, M is the minimum value of the upside-down function, and at this trough the graph will also be flat. So, for a general function $F(X)$, setting $F'(X) = 0$ might yield an X that gives its minimum rather than its maximum. How do we distinguish the two possibilities?

At a maximum, the function will be increasing to its left and decreasing to its right. Therefore, the derivative will be positive for values of X smaller than the purported maximum, and negative for larger values. In other words, the derivative, itself regarded as a function of X, will be decreasing at this point. A decreasing function has a negative derivative. Therefore, the derivative of the derivative, what is called the second derivative of the original function, written as $F'(X)$ or d^2Y/dX^2, should be negative at a maximum. Similar logic shows that the second derivative should be positive at a minimum; that is what distinguishes the two cases.

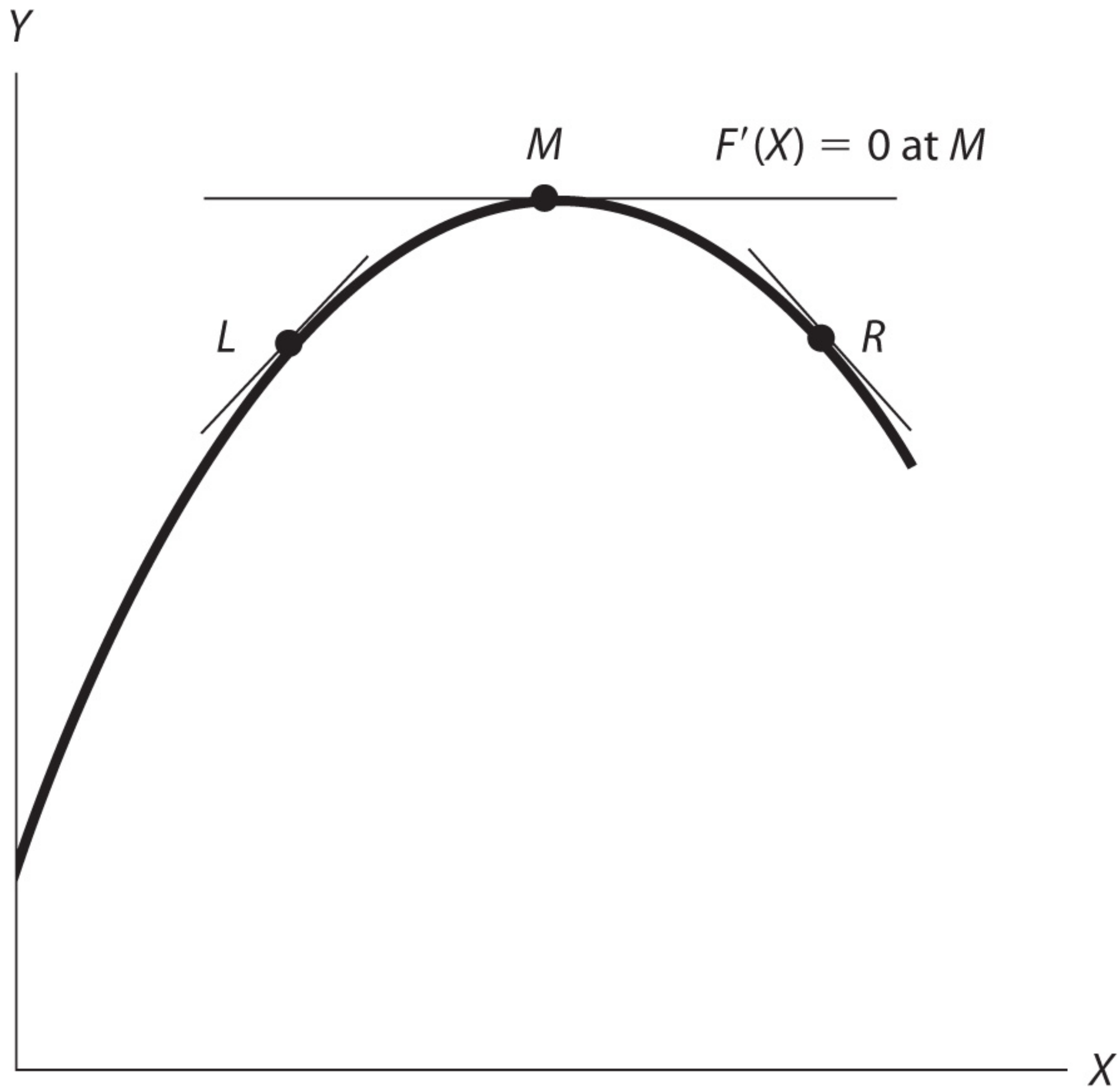

Figure 5A.2 Optimum of a Function

For the derivative $F'(X) = B - 2CX$ of our quadratic example, applying the same h, k procedure to $F'(X)$ as we did to $F(X)$ shows $F'(X) = -2C$. This result is negative so long as C is positive, which we assumed when stating the problem in the chapter text. The test $F'(X) = 0$ is called the first-order condition for maximization of $F(X)$, and $F'(X) < 0$ is the second-order condition.

To fix the idea further, let us apply it to the specific example of Xavier's best response that we considered in the chapter text. We had the expression

$\Pi_x = -8(44 + P_y) + (16 + 44 + P_y)P_x - 2(P_x)^2.$

This is a quadratic function of P_x (holding the other restaurant’s price, P_y, fixed). Our method gives its derivative:

$$\frac{d\Pi_x}{dP_x} = (60 + P_y) - 4P_x.$$

The first-order condition for P_x to maximize Π_x is that this derivative should be 0. Setting it equal to 0 and solving for P_x gives the same equation as derived in Section 1.A. (The second-order condition is $d^2\Pi_x/dP_x^2 < 0$, which is satisfied because the second-order derivative is just -4.)

We hope you will regard the calculus method as simple enough and that you will have occasion to use it again in a few places later—for example, in Chapter 11 on collective action. But if you find it too difficult, here is a noncalculus alternative method that works for quadratic functions. Rearrange terms to write the function as

$$
\begin{aligned}
Y &= A + BX - CX^2 \\
&= A + \frac{B^2}{4C} - \frac{B^2}{4C} + BX - CX^2 \\
&= A + \frac{B^2}{4C} - C\left(\frac{B^2}{4C^2} - 2\frac{B}{C} + X^2\right) \\
&= A + \frac{B^2}{4C} - C\left(\frac{B}{2C} - X\right)^2.
\end{aligned}
$$

In the final form of the expression, X appears only in the last term, where a square involving it is being subtracted (remember $C > 0$). The whole expression is maximized when this subtracted term is made as small as possible, which happens when $X = B/(2C)$. Voila!

This method of "completing the square" works for quadratic functions and therefore will suffice for most of our uses. It also avoids calculus. But we must admit that it smacks of magic. Calculus is more general and more methodical. It repays a little study many times over.

Endnotes

- Needless to say, we give only the briefest, quickest treatment, leaving out all issues of functions that don't have derivatives, functions that are maximized at an extreme point of the interval over which they are defined, and so on. Some readers will know all we say here; some will know much more. Others who want to find out more should refer to any introductory calculus textbook. Return to reference 27

6 ■ Combining Sequential and Simultaneous Moves

IN CHAPTER 3, we considered games with purely sequential moves; Chapters 4 and 5 dealt with games with purely simultaneous moves. We developed concepts and techniques of analysis appropriate to these pure game types: game trees and rollback equilibrium for sequential-move games, game tables and Nash equilibrium for simultaneous-move games. In reality, however, many strategic situations contain elements of both types of interactions. Furthermore, although we used game trees (extensive forms) as the sole method of illustrating sequential-move games and game tables (strategic forms) as the sole method of illustrating simultaneous-move games, we can use either form for any type of game.

In this chapter, we examine many of these possibilities. We begin by showing how games that combine sequential and simultaneous moves can be solved by combining game trees and game tables, and by combining rollback and Nash equilibrium analysis, in appropriate ways. Then we consider the effects of changing the nature of the interactions in a particular game. Specifically, we look at the effects of changing the rules of a game to convert sequential play into simultaneous play, and vice versa, and of changing the order of moves in sequential play. This topic gives us an opportunity to compare the equilibria found by using the concept of rollback, in a sequential-move game, with those found by using the Nash equilibrium concept, in the simultaneous-move version of the same game. From this comparison, we extend the concept of Nash equilibria to sequential-move games. It turns out that the rollback equilibrium is a special case, usually called a refinement, of these Nash equilibria.

1 GAMES WITH BOTH SIMULTANEOUS AND SEQUENTIAL MOVES

As mentioned several times thus far, most real games that you will encounter will be made up of numerous smaller components. Each of these components may entail simultaneous play or sequential play, so that the full game requires you to be familiar with both. The most obvious examples of strategic interactions containing both sequential and simultaneous components are those between two (or more) players over an extended period of time. You may play a number of different simultaneous-move games against your roommate during your year together: Your action in any one of these games is influenced by the history of your interactions up to then and by your expectations about the interactions to come. Many sporting events, interactions between competing firms in an industry, and political relationships are also sequentially linked series of simultaneous-move games. Such games can be analyzed by combining the tools presented in Chapter 3 (game trees and rollback) and in Chapters 4 and 5 (game tables and Nash equilibria).[1] The only difference is that the actual analysis becomes more complicated as the number of moves and interactions increases.

A. Two-Stage Games and Subgames

Our main illustrative example of a game with both simultaneous and sequential components tells a story of two would-be telecom giants, CrossTalk and GlobalDialog. Each firm can choose whether to invest $10 billion in the purchase of a fiber-optic network. The two firms make their investment decisions simultaneously. If neither chooses to make the investment, that is the end of the game. If one invests and the other does not, then the investor has to make a pricing decision for its telecom services. It can choose either a high price, which will attract 60 million customers, from each of whom it will make an operating profit (gross profit not including the cost of its investment) of $400, or a low price, which will attract 80 million customers, from each of whom it will make an operating profit of $200. If both firms acquire fiber-optic networks and enter the market, then their pricing choices become a second simultaneous-move game. Each firm can choose either the high or the low price. If both choose the high price, they will split the total market equally, each will get 30 million customers and an operating profit of $400 from each customer. If both choose the low price, again they will split the total market equally, so each will get 40 million customers and an operating profit of $200 from each customer. If one firm chooses the high price and the other the low price, then the low-price firm will get all the 80 million customers at that price, and the high-price firm will get nothing.

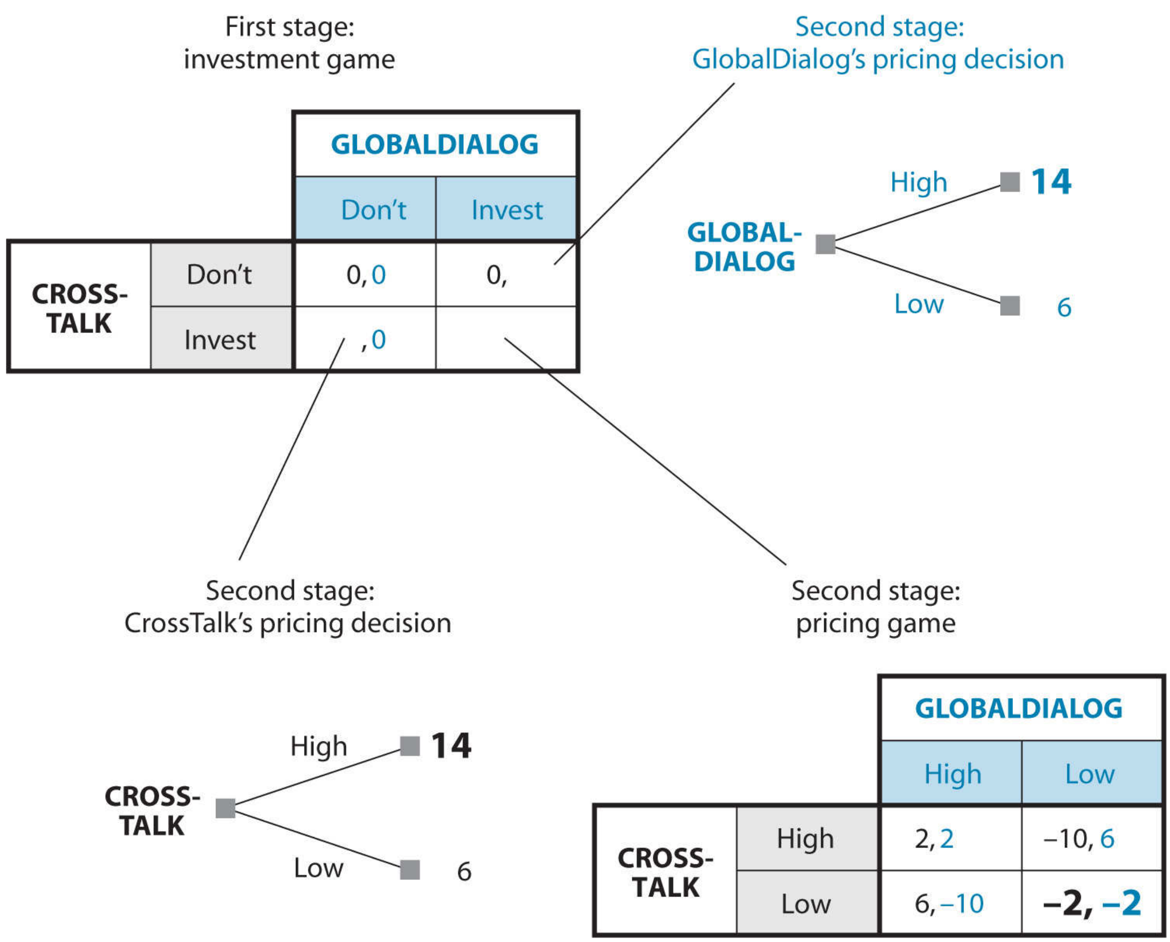

Figure 6.1 Two-Stage Game Combining Sequential and Simultaneous Moves

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The interaction between CrossTalk and GlobalDialog forms a two-stage game. Of the four combinations of simultaneous-move choices at the first (investment) stage, one ends the game, two lead to a second-stage (pricing) decision by just one player, and the last leads to a simultaneous-move (pricing) game at the second stage. We show this game pictorially in Figure 6.1.

Regarded as a whole, Figure 6.1 illustrates a game tree, but one that is more complex than the trees in Chapter 3. You can think of it as an elaborate “tree house” with multiple levels. The levels are shown in different parts of the same two-dimensional figure, as if you are looking down at the tree from a helicopter positioned directly above it.

The first-stage game is represented by the payoff table in the top-left quadrant of Figure 6.1. You can think of it as the first floor of the tree house. It has four “rooms.” The room in the northwest corner corresponds to “Don’t invest” first-stage moves by both firms. If the firms’ decisions take the game to this room, there are no further choices to be made, so we can think of this room as equivalent to a terminal node of a game tree, and we can show the payoffs in that cell of the table: Both firms get 0. However, all the other combinations of actions for the two firms lead to rooms that lead to further choices, so we cannot yet show all the payoffs for those cells. Instead, we show branches leading to the second floor of the tree house. The northeast and southwest rooms show only the payoff to the firm that has not invested; the branches leading from each of these rooms take us to single-firm pricing decisions in the second stage. The branch from the southeast room leads to a multiroom second-floor structure within the tree house, which represents the second-stage pricing game that is played if both firms have invested in the first stage. This second-floor structure has four rooms corresponding to the four combinations of the two firms’ pricing moves. All the second-floor rooms are like terminal nodes of a game tree, so we can show the payoffs in each case. Those payoffs consist of each firm’s net profit—operating profit minus its previous investment costs; payoff values are written in billions of dollars.

To see how the payoff values are calculated, consider first the second-stage pricing decision illustrated in the southwest corner of Figure 6.1. The game arrives in that corner if CrossTalk is the only firm that has invested. Then, if it chooses the high price, its operating profit is $400 × 60 million = $24 billion; after subtracting the $10 billion investment cost, its payoff (net profit) is $14 billion, which we write as 14. In the same corner, if CrossTalk chooses the low price, then its operating profit is $200 × 80 million = $16 billion, yielding the payoff 6 after accounting for its original investment. In this situation, GlobalDialog’s payoff is 0, as shown in the southwest room of the first floor of our tree house. Similar calculations for the case in which GlobalDialog is the only firm to invest give us the payoffs shown in the northeast corner of Figure 6.1; again, the payoff of 0 for CrossTalk is shown in the northeast room of the first-stage game table.

If both firms invest, both play the second-stage pricing game illustrated in the southeast corner of the figure. When both choose the high price in the second stage, each gets operating profit of \$400 × 30 million (half of the market), or \$12 billion; after subtracting the \$10 billion investment cost, each is left with a net profit of \$2 billion, or a payoff of 2. If both firms choose the low price in the second stage, each gets operating profit of \$200 × 40 million = \$8 billion, and after subtracting the \$10 billion investment cost, each is left with a net loss of \$2 billion, or a payoff of −2. Finally, if one firm charges the high price and the other firm the low price, then the low-price firm has operating profit of \$200 × 80 million = \$16 billion, leading to the payoff 6, while the high-price firm gets no operating profit and simply loses its \$10 billion investment, for a payoff of −10.

As with any multistage game in Chapter 3, we must solve this game backward, starting with the second-stage game. In the case of the two single-firm pricing decisions, we see at once that the high-price decision yields the higher payoff. We highlight this by showing that payoff in larger type.

The second-stage pricing game has to be solved by using methods developed in Chapter 4. It is immediately evident, however, that this game is a prisoners' dilemma. Low price is the dominant strategy for each firm, so the outcome is the room in the southeast corner of the second-stage game table: Each firm gets a payoff of −2.[2] Again, we show these payoffs in large type to highlight the fact that they are the payoffs obtained in the second-stage equilibrium.

		GLOBALDIALOG	
		Don't	Invest
CROSSTALK	Don't	0, 0	0, 14
	Invest	14, 0	−2, −2

FIGURE 6.2 First-Stage Investment Game (After Substituting Rolled-Back Payoffs from the Equilibrium of the Second Stage)

Rollback analysis now tells us that each set of first-stage moves should be evaluated by looking ahead to the equilibrium of the second-stage game (or the optimal second-stage decision in cases where only one firm has an action to take) and the resulting payoffs. We can therefore substitute the second-stage payoffs that we have just calculated into the previously empty or partly empty rooms on the first floor of our tree house. This substitution gives us a first floor with known payoffs, shown in Figure 6.2.

Now we can use the methods of Chapter 4 to solve this simultaneous-move game. You should immediately recognize the game in Figure 6.2 as a game of chicken. It has two pure-strategy Nash equilibria, each of which entails one firm choosing Invest and the other choosing Don't. The firm that invests makes a huge profit, so each firm prefers the equilibrium in which it is the investor while the other firm stays out of the market. In Chapter 4, we briefly discussed the ways in which one of the two equilibria might get selected. We also pointed out the possibility that each firm might try to get its preferred outcome, with the result, in this case, that both of them invest and both lose money. In Chapter 7, we will investigate this type of game further, showing that it has a third Nash equilibrium, in mixed strategies.

Analysis of Figure 6.2 shows that the first-stage game in our example does not have a unique Nash equilibrium. This problem is not too serious, because we can leave the solution ambiguous to the extent we did in the preceding paragraph. Matters would be more difficult if the second-stage game did not have a unique equilibrium. Then it would be essential to specify the precise process by which an outcome gets selected so that we could figure out the second-stage payoffs and use them to roll back to the first stage.

The second-stage pricing game shown in the payoff table in the bottom-right quadrant of Figure 6.1 is one part of the complete two-stage game. However, it is also a full-fledged game in its own right, with a fully specified structure of players, strategies, and payoffs. To bring out this dual nature more explicitly, it is called a subgame of the full game.

More generally, a subgame is the part of a multimove game that begins at a particular node of the original game. The tree for a subgame is then just that part of the tree for the full game that takes this node as its root, or initial node. A multimove game has as many subgames as it has decision nodes.

B. Configurations of Multistage Games

In the multistage game illustrated in Figure 6.1, each stage consists of a simultaneous-move game. However, that may not always be the case. Simultaneous and sequential components may be mixed and matched in any way. In this section, we give two more examples to clarify this point and to reinforce the ideas introduced in the preceding section.

The first example is a slight variation of the CrossTalk–GlobalDialog game. Suppose one of the firms—say, GlobalDialog—has already made the $10 billion investment in the fiber-optic network. CrossTalk knows of this investment and now has to decide whether to make its own investment. If CrossTalk does not invest, then GlobalDialog will have a simple pricing decision to make. If CrossTalk invests, then the two firms will play the second-stage pricing game already described. The tree for this multistage game has conventional branches starting at the initial node and has a simultaneous-move subgame starting at one of the nodes to which these initial branches lead. The complete tree is shown in Figure 6.3.

When the tree has been set up, it is easy to analyze the game. We show the rollback analysis in Figure 6.3 by using large type for the equilibrium payoffs that result from the second-stage game or decision and a thicker branch for CrossTalk's first-stage choice. In words, CrossTalk figures out that if it invests, the ensuing prisoners' dilemma of pricing will leave it with payoff −2, whereas staying out will get it 0. Thus, it prefers the latter. GlobalDialog gets 14 instead of the −2 that it would have gotten if CrossTalk had invested, but CrossTalk's concern is to maximize its own payoff and not to ruin GlobalDialog deliberately.

This analysis does raise the possibility, though, that GlobalDialog may try to get its investment done quickly, before CrossTalk makes its decision, so as to ensure its most preferred outcome from the full game. And CrossTalk may try to beat

GlobalDialog to the punch in the same way. In Chapter 8, we study some methods, called strategic moves, that may enable players to secure such advantages.

Our second example comes from football. Before each play, the coach for the offense chooses the play that his team will run; simultaneously, the coach for the defense sends his team out with instructions on how they should align themselves to counter the offense. Thus, these moves are simultaneous. Suppose the offense has just two alternatives, a safe play and a risky play, and the defense can align itself to counter either of them. If the offense has planned to run the risky play and the quarterback sees the defensive alignment that will counter it, he can call for a change in the play at the line of scrimmage. And the defense, hearing the change, can respond by changing its own alignment. Thus, we have a simultaneous-move game at the first stage, and one of the combinations of moves at this stage leads to a sequential-move subgame. Figure 6.4 shows the complete tree.

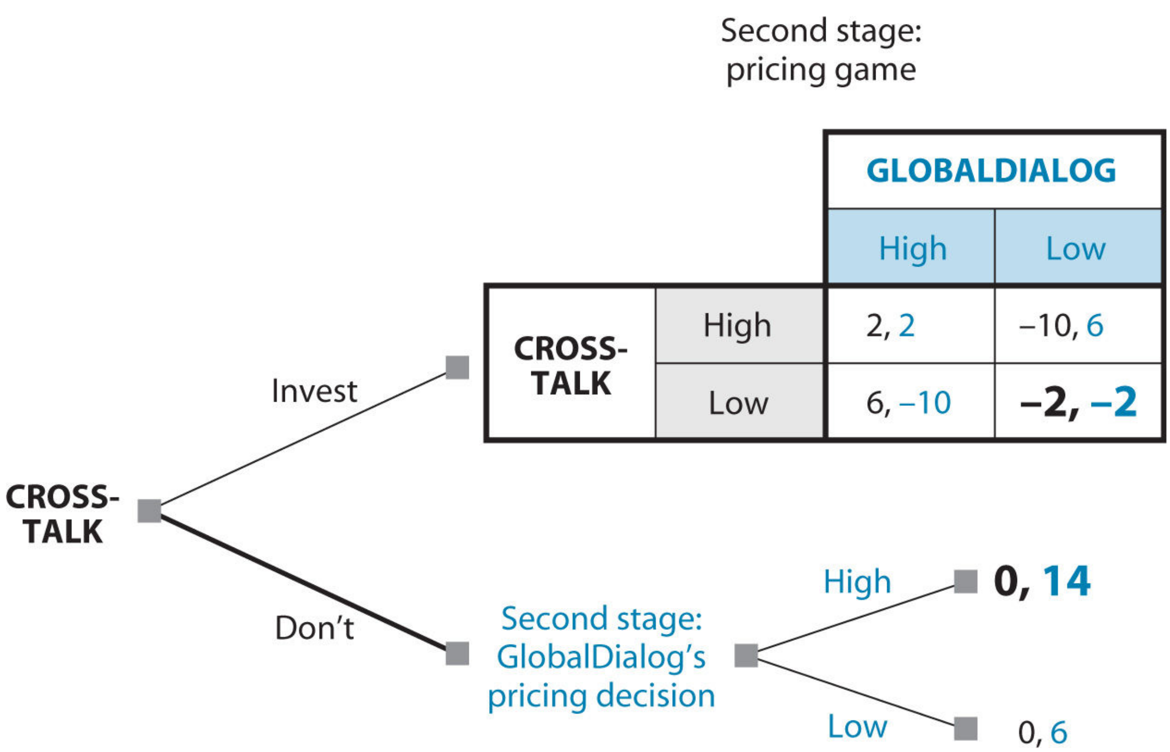


Figure 6.3 Two-Stage Game When One Firm Has Already Invested

This is a zero-sum game in which the offense’s payoffs are measured in the number of yards that it expects to gain, and the defense’s payoffs are exactly the opposite, measured in the number of yards it expects to give up. The safe play for the offense gets it 2 yards, even if the defense is ready for it; if the defense is not ready for it, the safe play does not do much better, gaining 6 yards. The risky play, if it catches the defense unready to cover it, gains 30 yards. But if the defense is ready for the risky play, the offense loses 10 yards. We show this set of payoffs of −10 for the offense and 10 for the defense at the terminal node in the subgame where the offense does not change the play. If the offense does change the play (back to safe), the payoffs are (2, −2) if the defense responds and (6, −6) if it does not; these payoffs are the same as those that arise when the offense plans the safe play from the start.

We show the chosen branches in the sequential subgame as thick lines in Figure 6.4. It is easy to see that if the offense changes its play, the defense will respond to keep its payoff at −2 rather than −6, and that the offense will still prefer to change its play to get 2 rather than −10. Rolling back, we put the resulting set of payoffs, (2, −2), in the bottom-right cell of the simultaneous-move game of the first stage. Then we see that this game has no Nash equilibrium in pure strategies. The reason is the same as that in the tennis-point game of Chapter 4, Section 8: One player (defense) wants to match its opponent’s moves (align to counter the play that the offense is choosing) while the other (offense) wants to unmatch its opponent’s moves (catch the defense in the wrong alignment). In Chapter 7, we show how to calculate the mixed-strategy equilibrium of such a game. It turns out that the offense should choose the risky play with probability ⅛, or 12.5%.

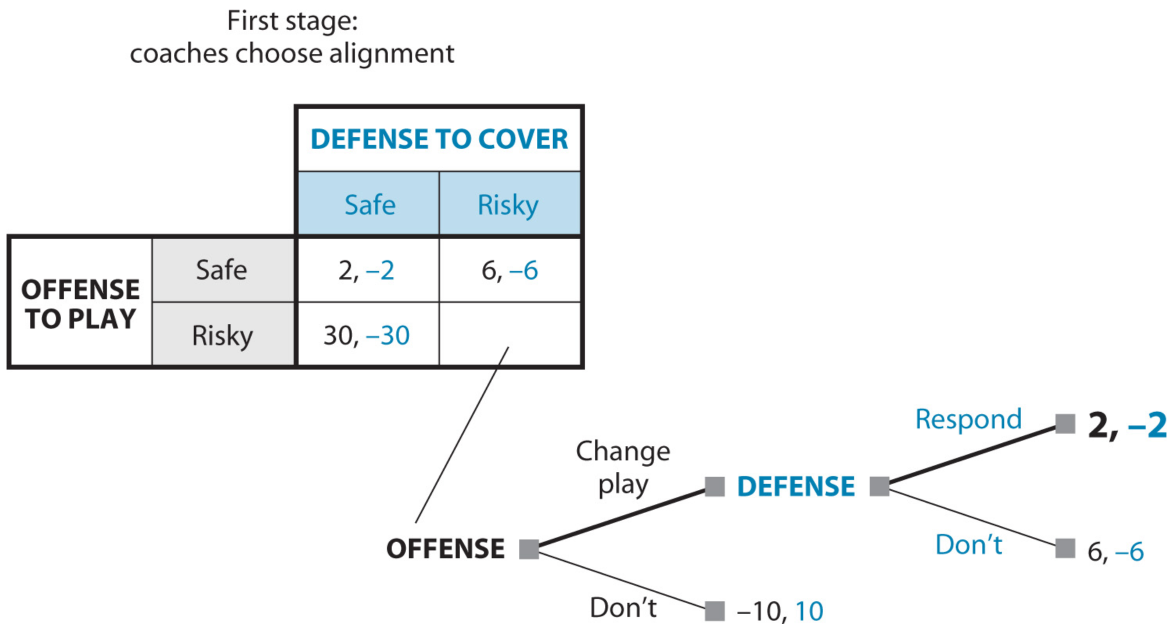


Figure 6.4 Simultaneous-Move First Stage Followed by Sequential Moves

Endnotes

- Sometimes the simultaneous part of the game will have equilibria in mixed strategies; then, the tools we develop in Chapter 7 will be required. We mention this possibility in this chapter where relevant and give you an opportunity to use such methods in exercises for the later chapters. Return to reference 1
- As is usual in a prisoners' dilemma, if the firms could successfully collude and charge high prices, both could get the higher payoff of 2. But this outcome is not an equilibrium because each firm is tempted to cheat to try to get the much higher payoff of 6. Return to reference 2

Glossary

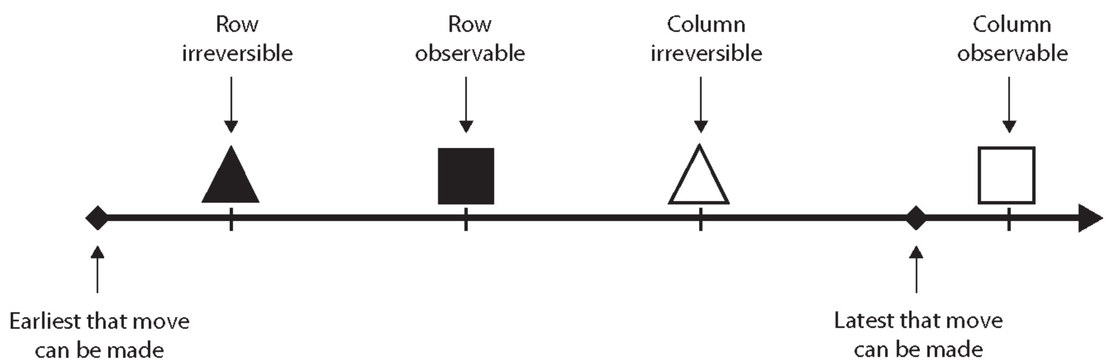
subgame
: A game comprising a portion or remnant of a larger game, starting at a noninitial node of the larger game.

2 CHANGING THE ORDER OF MOVES IN A GAME

Thus far, we have presented every game as having a fixed order of moves that is out of the players’ control. This approach has allowed us to develop the appropriate concepts and methods to analyze different types of games: backward induction to find the rollback equilibrium in sequential-move games (Chapter 3), best-response analysis to find Nash equilibria in simultaneous-move games (Chapters 4 and 5), or a blend of both methods for games with elements of both sequential and simultaneous play (Section 1 of this chapter). In this section, we focus in more detail on what determines the strategic order of moves (or simply “order of moves”) in a game, and we consider a player’s ability and incentive to change the order of moves.

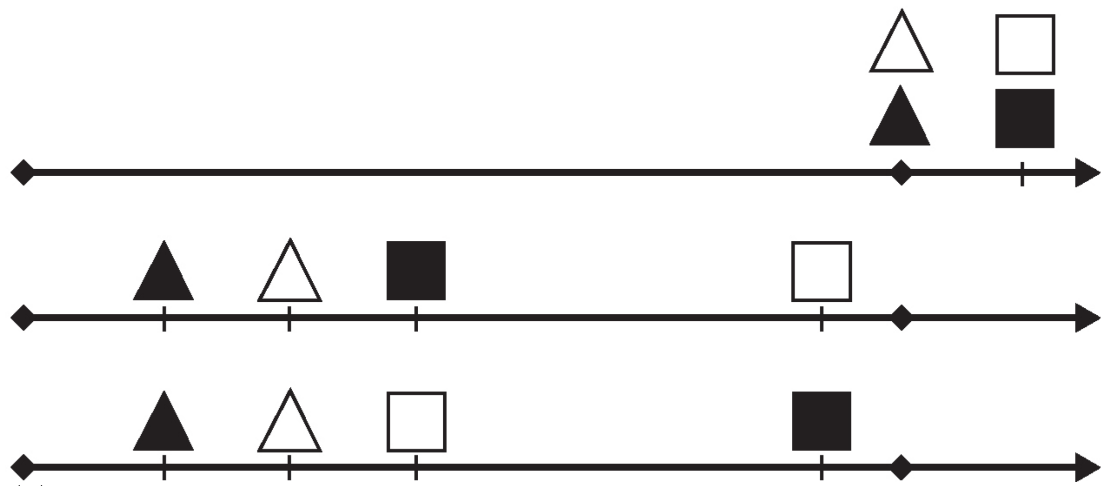
A. What Determines the Order of Moves?

The strategic order of moves in a two-player game depends on two key factors: when each move becomes irreversible and when it becomes observable. To determine the strategic order of moves, we first ask, When does each player’ s move become irreversible? In other words, When does each player “make her move” ? If both players make their moves at the same exact moment in time, then the game obviously has simultaneous moves. But what if one player makes her move before the other player in chronological time? Whether the game has sequential or simultaneous moves depends on when this earlier move becomes observable.



(a) Example: Row moves first


(b) Example: Column moves first



(c) Examples: Simultaneous moves

FIGURE 6.5 Strategic Order of Moves Depends on the Chronological Order in Which Players' Moves Become Irreversible and Observable

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The second key question in determining the strategic order of moves is: When does each player's move become observable to the other player? Suppose that Row's move is the first to become irreversible. If Row's move also becomes observable before Column's move becomes irreversible, then Column is able to choose her move *in reaction* to Row's choice. The game has sequential moves, with Row as first mover. On the other hand, if Row's move is the first to become irreversible but remains unobservable until after Column has moved, then Column must make her choice in ignorance of Row's move. The game has simultaneous moves.

Figure 6.5 provides several illustrative examples, depicting the players' moments of irreversibility as triangles and moments of observability as squares on a chronological timeline. In this visualization, a game has sequential moves whenever one player's triangle and square both appear to the left of the other player's triangle and square, as in Figures 6.5a, which shows Row as the first mover, and 6.5b, which shows Column as the first mover. For any other configuration—including the three examples shown in Figure 6.5c—the game has simultaneous moves.

Consider the pure coordination game between Sherlock Holmes and Dr. Watson that we considered in Chapter 4. As both dashed off on separate errands, Holmes shouted that they should rendezvous at "four o'clock at our meeting place," but both men realized, in retrospect, that “our meeting place” could potentially mean St. Bart’s (a hospital) or Simpson’s (a restaurant). Because of the distance between the two locations and travel time across the city of London, Holmes and Watson each had to commit themselves to just one of these two destinations well before 4:00 p.m. For instance, perhaps Watson had to make his choice at 3:45 p.m., while Holmes made his choice at 3:50 p.m. Watson moved first from a chronological point of view, but because Holmes had no way of observing Watson’s choice until 4:00 p.m. (when Watson would either be in the same place as Holmes or not), the game had simultaneous moves. The order-of-moves visualization for this game is shown in Figure 6.6.

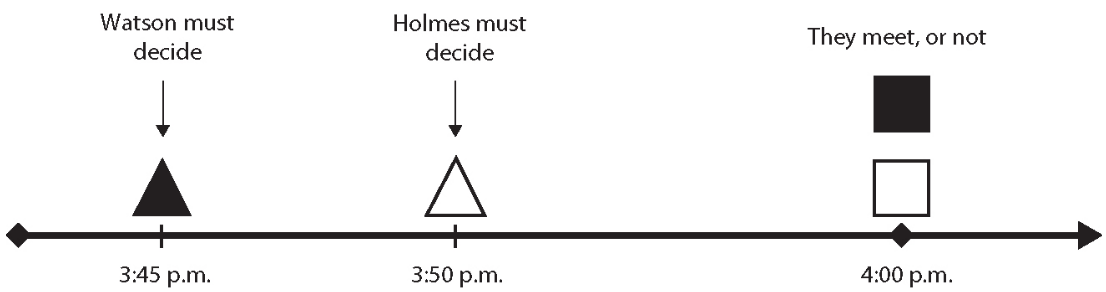


Figure 6.6 Order-of-Moves Visualization for Holmes and Watson Coordination Game

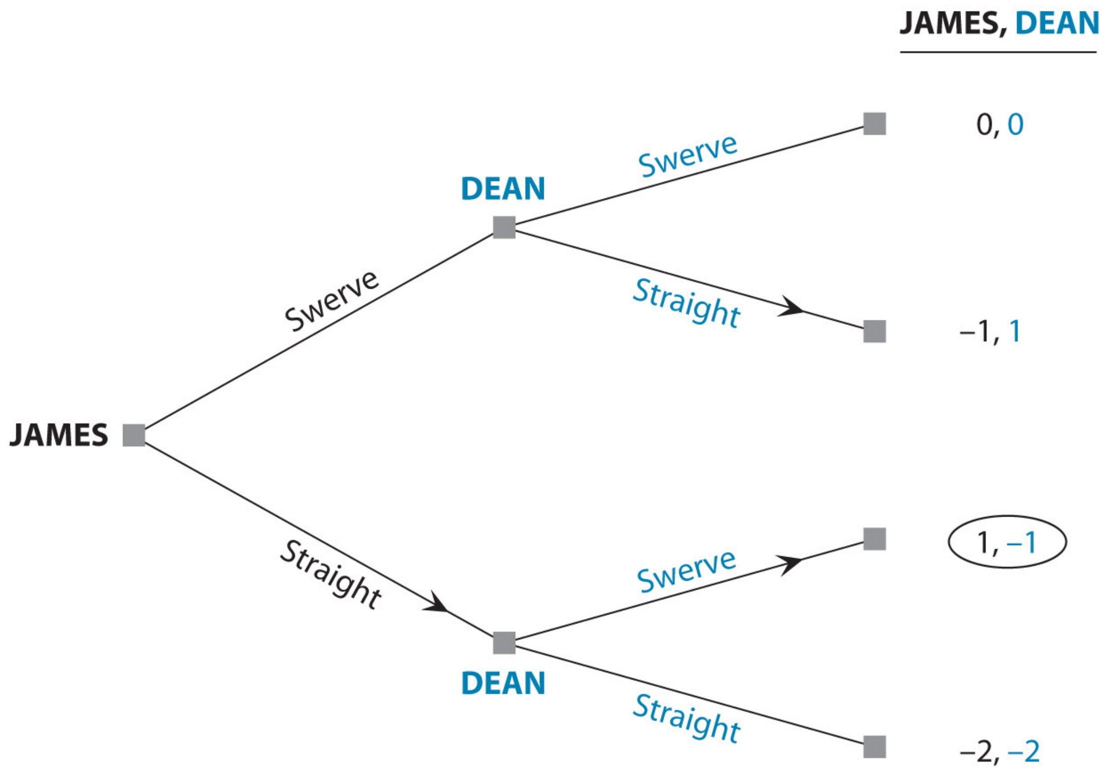
B. Moving First versus Moving Second

I. GAMES WITH A FIRST-MOVER ADVANTAGEThere is a *first-mover advantage* in a sequential-move game if each player gets a better outcome for herself in the rollback equilibrium of the game when she moves first than she gets in the rollback equilibrium of the game when she moves second.

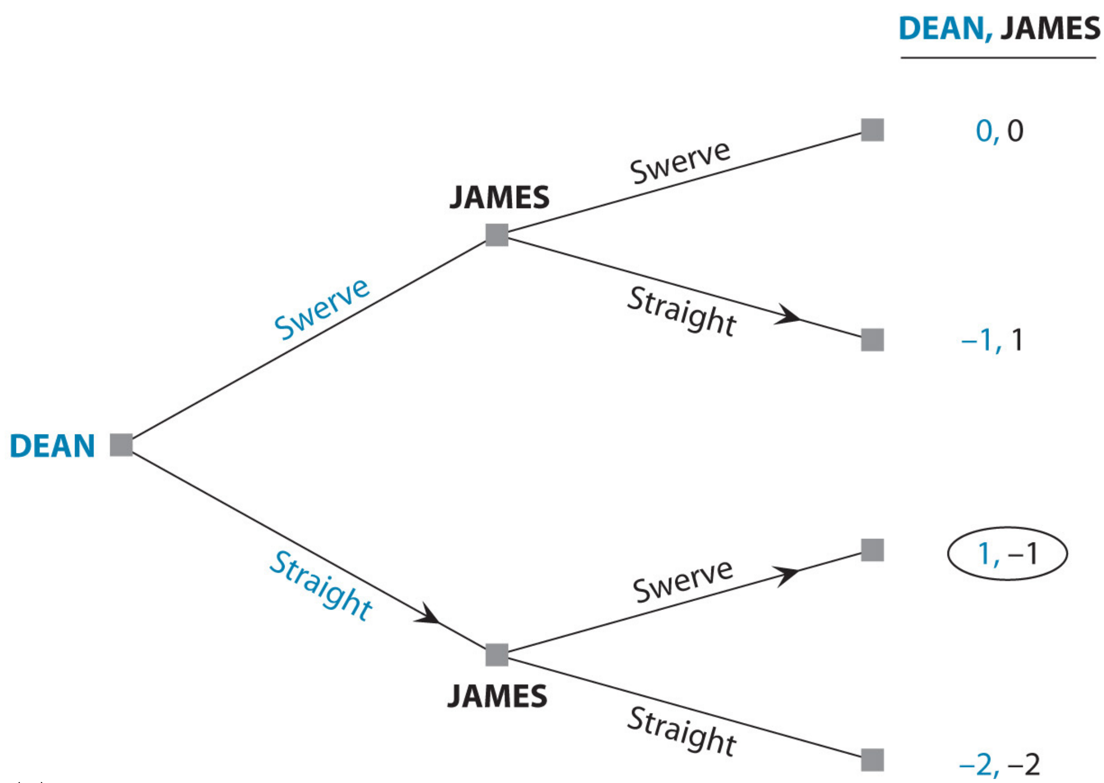
Consider the game of chicken from Chapter 4. We reproduce this game’s strategic form in Figure 6.7a along with two extensive forms, one for each possible ordering of play, in Figures 6.7b and 6.7c. If the game has sequential moves, there is a unique rollback equilibrium in which the first mover goes straight (is “tough”) and the second mover swerves (is “chicken”). Note that the first mover gets his best outcome while the second mover does not. Hence, there is a first-mover advantage in this game.

(a) Simultaneous play

		DEAN	
		Swerve (Chicken)	Straight (Tough)
JAMES	Swerve (Chicken)	0, 0	−1, 1
	Straight (Tough)	1, −1	−2, −2



(b) Sequential play: James moves first



(c) Sequential play: Dean moves first

Figure 6.7 Chicken in Simultaneous-Play and Sequential-Play Versions

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In the game of chicken, moving first is best for each player, but interestingly, moving simultaneously may be even worse than moving last. In Chapter 7, we will return to this game and show that in the simultaneous-move version of the game, there is an equilibrium in mixed strategies in which each player has an expected payoff of -0.5, worse than what each player gets by moving last.

II. GAMES WITH A SECOND-MOVER ADVANTAGEThere is a *second-mover advantage* in a sequential-move game if each player gets a better outcome for herself in the rollback equilibrium of the game when she moves second than in the rollback equilibrium of the game when she moves first.

Consider the tennis-point game described in Chapter 4. Recall that in that game, Evert is planning the location of her passing shot while Navratilova considers where to cover. That simultaneous-move version of the game assumed that both players were skilled at disguising their intended moves until the very last moment, so that they moved at essentially the same time. If Evert’s movement as she goes to hit the ball belies her shot intentions, however, then Evert’s moments of irreversibility and observability both occur before Navratilova makes her move. Navratilova can react, and becomes the second mover in the game. In the same way, if Navratilova leans toward the side that she intends to cover before Evert actually hits her return, then Evert is the second mover.

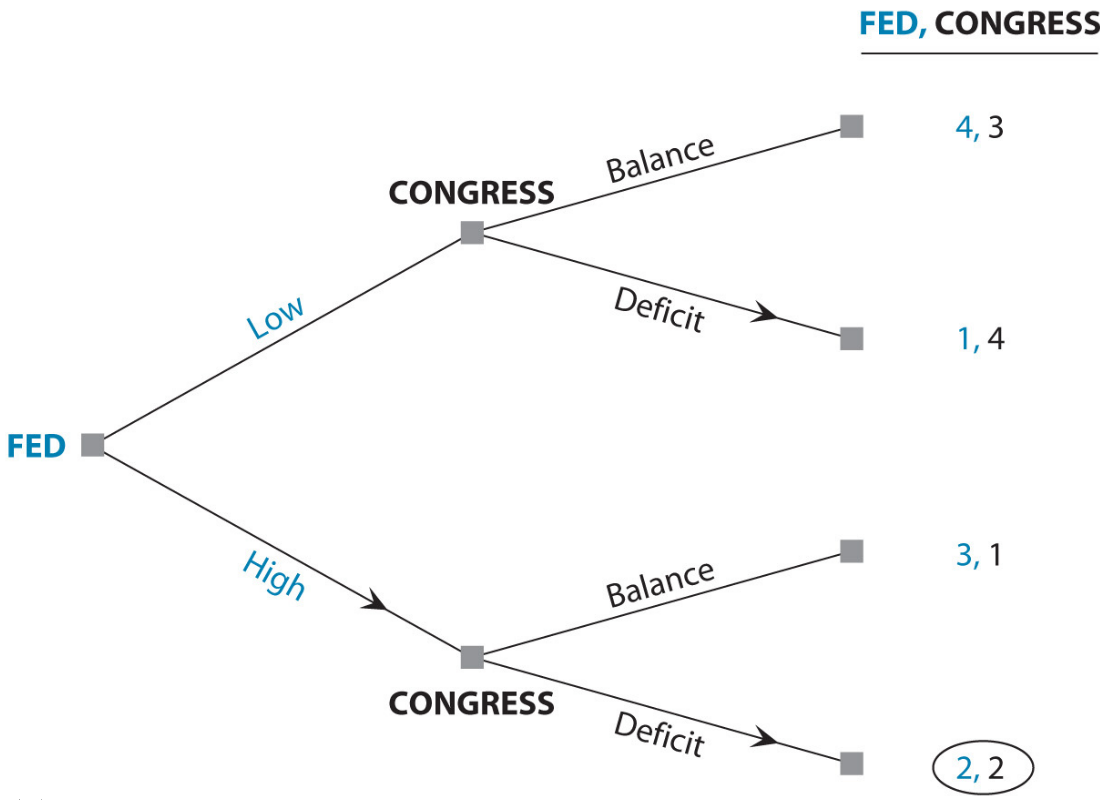
The simultaneous-move version of this game has no equilibrium in pure strategies. In each ordering of the sequential-move version, however, there is a unique rollback equilibrium outcome. There are different outcomes, however, depending on who moves first. If Evert moves first, then Navratilova chooses to cover whichever direction Evert chooses, and Evert should opt for a down-the-line shot. Each player is expected to win the point half the time in this equilibrium. If the order is reversed, Evert chooses to send her shot in the opposite direction from that which Navratilova covers; so Navratilova should move to cover crosscourt. In this case, Evert is expected to win the point 80% of the time. The second mover does better by being able to respond optimally to the opponent’s move. (You should be able to draw game trees similar to those in Figure 6.7b and 6.7c to illustrate this point.)

We return to the simultaneous-move version of this game in Chapter 7, where we show that it does have a Nash equilibrium in mixed strategies. In that equilibrium, Evert succeeds, on average, 62% of the time. Her success rate in the mixed-strategy equilibrium of the simultaneous-move game is thus better than the 50% that she gets by moving first, but is worse than the 80% that she gets by moving second.

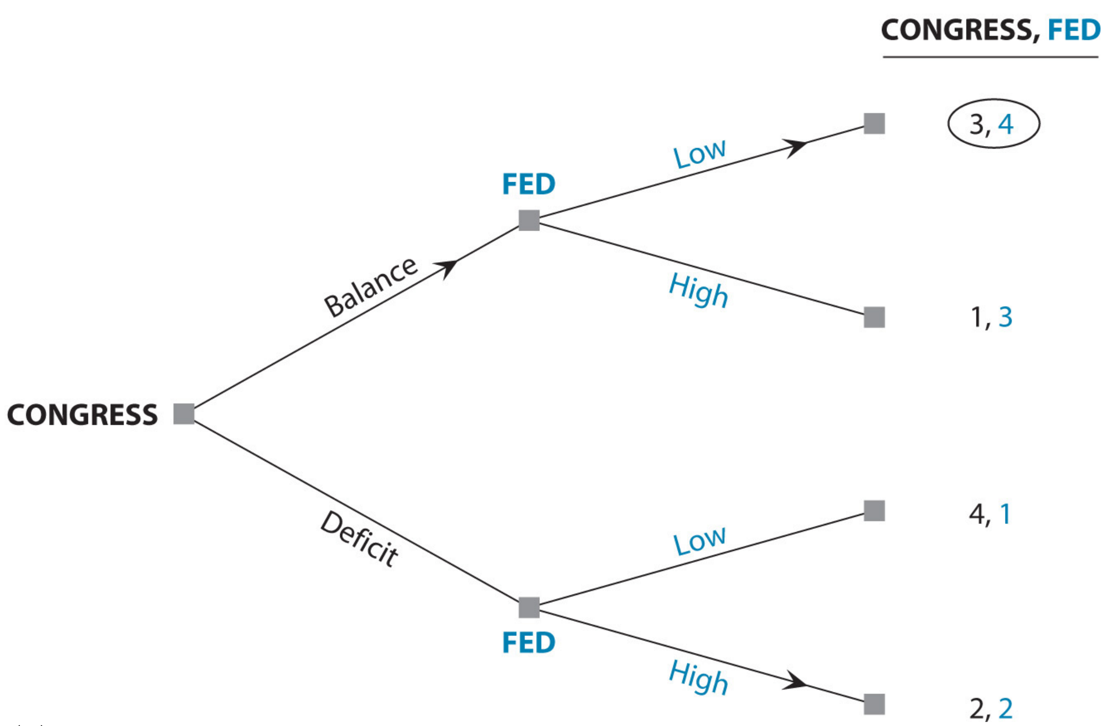
III. GAMES IN WHICH THE PLAYERS' MOVE-ORDER PREFERENCES ARE ALIGNEDIn some games, both players prefer the same ordering of moves. For instance, Row and Column may both prefer that Row move first. In such games, there is neither a first-mover advantage nor a second-mover advantage.

(a) Simultaneous moves

		FEDERAL RESERVE	
		Low interest rates	High interest rates
CONGRESS	Budget balance	3, 4	1, 3
	Budget deficit	4, 1	2, 2



(b) Sequential moves: Fed moves first



(c) Sequential moves: Congress moves first

Figure 6.8 Three Versions of the Fiscal–Monetary Policy Game

Consider the game of fiscal and monetary policies played by Congress and the Federal Reserve. We studied the simultaneous-move version of this game in Chapter 4. We reproduce the game's ordinal payoff table (Figure 4.5) as Figure 6.8a and show the two sequential-move versions as Figure 6.8b and 6.8c. For brevity, we write the strategies in the sequential-move game trees as Balance and Deficit, instead of Budget Balance and Budget Deficit, for Congress and as High and Low, instead of High Interest Rates and Low Interest Rates, for the Fed.

In the simultaneous-move version, Congress has a dominant strategy (Deficit), and the Fed, knowing this, chooses High, resulting in the second-worst outcome (ordinal payoff 2) for both players. The same outcome arises in the unique rollback equilibrium of the sequential-move version of the game in which the Fed moves first. The Fed foresees that, whether it chooses High or Low, Congress will respond with Deficit. High is therefore the better choice for the Fed, yielding an ordinal payoff of 2 instead of 1.

But the sequential-move version in which Congress moves first is different. Now Congress foresees that if it chooses Deficit, the Fed will respond with High, whereas if it chooses Balance, the Fed will respond with Low. Of these two possible outcomes, Congress prefers the latter, as it yields a payoff of 3 instead of 2. Therefore, the rollback equilibrium with this ordering of moves is for Congress to choose a balanced budget and the Fed to respond with low interest rates. The resulting ordinal payoffs, 3 for Congress and 4 for the Fed, are better for both players than when moves are simultaneous or when the Fed moves first.

A surprising aspect of the rollback equilibrium when Congress moves first is that Congress plays Balance, its dominated strategy, rather than Deficit, its dominant strategy. How can this be? The fact that Balance is dominated means that, for any *fixed* interest-rate policy by the Fed, Congress would always prefer Deficit over Balance. Because Congress is the first mover, however, the Fed's interest-rate policy is not fixed, but is dependent on what Congress chooses to do—the Fed will "reward" Congress with low interest rates only if Congress chooses to balance the budget. To change the Fed's behavior and get that reward, Congress is willing to play its dominated strategy, something it would never do if the game had simultaneous moves.

The distinction between dominant and superdominant strategies, discussed originally in Section 4 of Chapter 4, also plays a role here. Deficit is a dominant strategy for Congress and, as such, would be its equilibrium strategy if the game had simultaneous moves. Deficit is not a superdominant strategy, however, because the outcome with a deficit and high interest rates is worse for Congress than the one with a balanced budget and low interest rates. Consequently, when Congress, as first mover, looks ahead to the Fed's choices and sees (Deficit, High) and (Balance, Low) as the only possible outcomes of the game, Congress will choose to balance the budget.

If Congress and the Fed could choose the order of moves in the game, they would agree that Congress should move first. The outcome with that order of moves (a balanced budget and low interest rates) is better for both players than the outcome that occurs under simultaneous moves or when the Fed moves first (a budget deficit and high interest rates). Indeed, when budget deficits and inflation threaten, chairs of the Federal Reserve, in testimony before various congressional committees, often offer such deals: They promise to respond to fiscal discipline by lowering interest rates. But it is often not enough to make a verbal deal with the other player. The technical requirements of a first move—namely, that it be observable to the second mover and irreversible thereafter—must be satisfied. In the context of fiscal and monetary policies, it is fortunate that the legislative process of setting fiscal policy in the United States is both very visible and very slow, whereas monetary policy can be changed quickly in a meeting of the Federal Reserve Board. Therefore, the sequential-play scenario where Congress moves first and the Fed moves second is quite realistic.

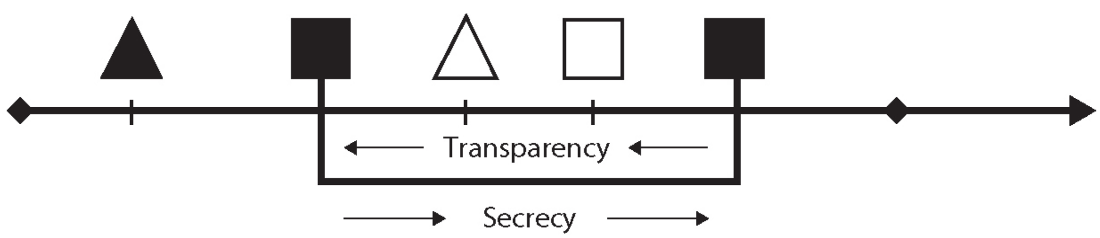
IV. GAMES IN WHICH ORDER OF MOVES DOESN'T MATTERIn some games, equilibrium outcomes are the same no matter what the order of moves. This will always be true whenever one of the players has a superdominant strategy or both players have a dominant strategy.

Consider the prisoners' dilemma game in Chapter 4, in which a husband and wife are being questioned regarding their roles in a crime. The Nash equilibrium of that simultaneous-move game is for each player to confess (that is, to defect from cooperating with the other). But what would happen if one spouse made an irreversible and observable choice before the other chose at all? Using rollback on a game tree similar to that in Figure 6.7b (which you can draw on your own as a check of our analysis), one can show that the second player does best to confess if the first has already confessed (for a payoff of 10 years rather than 25 years in jail), and the second player also does best to confess if the first has denied (1 year rather than 3 years in jail). Given these choices by the second player, the first player does best to confess (10 years rather than 25 years in jail). The equilibrium entails 10 years of jail for both spouses regardless of which one moves first. Thus, the equilibrium is the same in all three versions of this game!

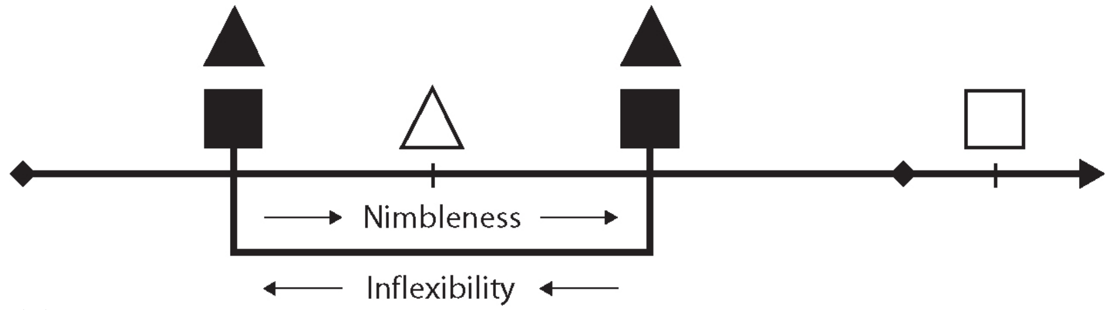
C. How One Player Can Change the Order of Moves

A player can change the *strategic* order of moves by taking steps to change the *chronological* order in which moves become irreversible and/or observable. Whether or how a player would choose to change the order of moves depends on her payoffs for the possible outcomes of the game. Here we highlight some especially simple ways that a player can alter the order of moves and discuss some circumstances in which she might want to do so. We will discuss a wider range of game-changing tactics, especially so-called strategic moves, in more depth in Chapter 8.

I. TRANSPARENCY VERSUS SECRECYSuppose that Row has to make her move far in advance and hence makes her irreversible move first in chronological time, as shown in Figure 6.9a. If Row is *transparent* in making her move, so that Column can easily observe it, Row can ensure that the game has sequential moves and that she is the first mover. In this case (illustrated by the leftmost black square in the figure), Row' s move is observable before Column' s move becomes irreversible. On the other hand, if Row is *secretive* with her move, so that Column cannot observe it, Row' s move becomes observable after Column' s move becomes irreversible (illustrated by the rightmost black square in the figure), and the game has simultaneous moves. Whether Row prefers to be transparent or secretive depends on whether Row gets a better payoff in the game when moving first or when moving simultaneously with Column. For example, in our tennis-point example, each player would want to be secretive, but in a game of chicken, a player choosing Tough would want to be transparent. (A player choosing Swerve in Chicken may want to be secretive in the hope that the other would swerve also.)



(a) Transparency allows Row to move first rather than simultaneously



(b) Nimbleness allows Row to move simultaneously rather than first

Figure 6.9 How Row Can Change the Order of Moves

II. NIMBLENESS VERSUS INFLEXIBILITYSuppose that Column moves secretly, so that her move is not observable until Row’s move is complete, while Row’s move is observable to Column as soon as it becomes irreversible. Row, however, might have the ability—call it *nimbleness*—to wait for some period of time before making her move irreversible (and also observable). This possibility is illustrated in Figure 6.9b. With nimbleness, Row can wait as long as possible before making her move, ensuring that the game has simultaneous moves. (When Row is nimble, Column’s move is the first to become irreversible, but since Column moves secretly, her move does not become observable until after Row’s move becomes irreversible.) If Row cannot be nimble, we say she is *inflexible*, and her inflexibility requires that she move right away, before Column, which makes her the first mover. Whether Row wants to be nimble or inflexible therefore depends on whether she gets a better payoff when moves are simultaneous or when she moves first.

Consider a game of chicken played by two firms, each eyeing a new market that has room for only one firm to operate profitably. Each firm would like to be the first mover because it could then deter the other firm’s entrance while entering profitably itself. Both firms will naturally jockey to make their own moves as inflexible and transparent as possible, but this may be easier for some firms than for others. Suppose that one of the firms is traded on the New York Stock Exchange, while the other is privately held. Because publicly traded firms in the United States must submit quarterly audited financial reports that give investors (and competitors) a window into their operations, they necessarily have high transparency. Moreover, investors demand information about publicly traded firms, and analysts’ scrutiny of these firms’ activities further increases transparency. Such scrutiny could possibly make publicly traded firms more inflexible as well. If the CEO of a publicly traded firm goes on the record in the business press saying that entering the new market is essential to the firm’s long-term growth, she knows that if she were to then back down and cancel her plan to enter that market, investors would swiftly punish her company’s stock. To avoid such a reaction, she is effectively forced to go through with the announced strategy. The fact that publicly held firms must disclose so much information is often viewed as a disadvantage. However, in this context, disclosure requirements benefit the CEO and her firm by allowing it to seize the first-mover advantage.

Glossary

strategic order
: The order of moves from a game-theoretic point of view, determined by considerations of observability and irreversibility. It may differ from the chronological order of actions and, in turn, determine whether the game has sequential or simultaneous moves.

irreversible
: Cannot be undone by a later action. In a sequential-move game, the first mover’s action must be irreversible and *observable* before the second mover’s action is irreversible.

observable
: Known to other players before they make their responding actions. Together with irreversibility, this is an important condition for a game to be sequential-move.

3 ALTERNATIVE METHODS OF ANALYSIS

Game trees are the natural way to display sequential-move games, and payoff tables are the natural representation of simultaneous-move games. However, each technique can be adapted to the other type of game. Here we show how to translate the information contained in one illustration to an illustration of the other type. In the process, we develop some new ideas that will prove useful in subsequent analysis of games.

A. Illustrating Simultaneous-Move Games Using Game Trees

Consider the game of the passing shot in tennis, originally described in Chapter 4, where the action is so quick that moves are truly simultaneous. But suppose we want to show this game in extensive form—that is, by using a tree, rather than a table as in Figure 4.17. We show how this can be done in Figure 6.10.

To draw the tree, we must choose one player—say, Evert—to make her choice at the initial node of the tree. The branches for her two choices, a down-the-line shot (DL) and a crosscourt shot (CC), then end in two action nodes, at each of which Navratilova makes her choice. However, because the moves are actually simultaneous, Navratilova must choose without knowing what Evert has picked. That is, she must choose without knowing whether she is at the node following Evert’ s DL branch or the one following Evert’ s CC branch. Our tree must in some way show this lack of information on Navratilova’ s part.

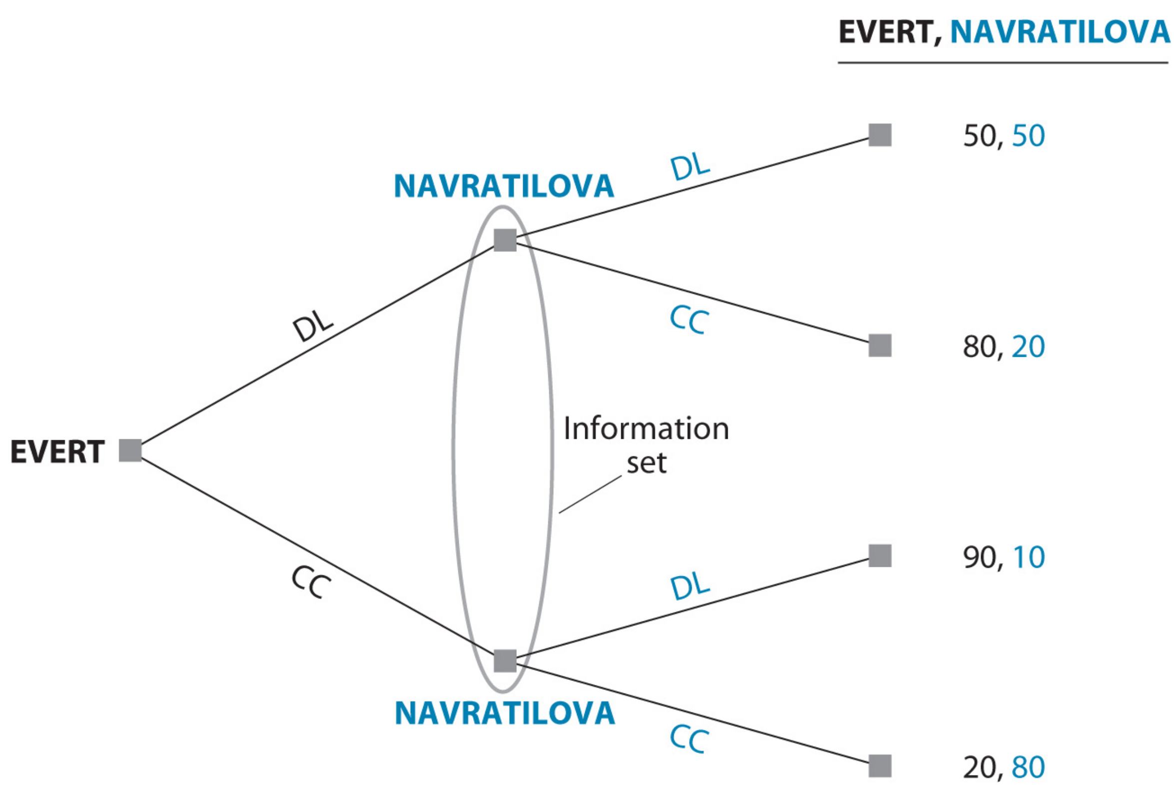


Figure 6.10 Simultaneous-Move Tennis-Point Game Shown in Extensive Form

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We illustrate Navratilova's strategic uncertainty about the node at which her decision is being made by drawing an oval or balloon to surround the two possible nodes. (An alternative is to connect the nodes with a dotted line; the line is dotted to distinguish it from the solid lines that represent the branches of the tree.) The nodes within this oval constitute an information set for the player who makes a move there. An information set consisting of more than one node indicates the presence of imperfect information for the player; she cannot distinguish between the nodes in the set given the available information (because she cannot observe the other player's move before making her own). As such, her choice of strategies from within the information set must specify the same move at all the nodes contained in it. That is, Navratilova must choose either DL at both the nodes in this information set or CC at both of them. She cannot choose DL at one and CC at the other.

Accordingly, we must adapt our definition of *strategy*. In Chapter 3, we clarified that a *complete* strategy must specify the move that a player would make at each *node* where the rules of the game specify that it is her turn to move. We should now more accurately redefine a complete strategy as specifying the move that a player would make at each *information set* at whose nodes the rules of the game specify that it is her turn to move.

The concept of an information set is also relevant when a player faces external uncertainty about some condition other than another player's moves that affects his decision. For example, a farmer planting a crop is uncertain about the weather during the growing season, although he knows the probabilities of various alternative possibilities from past experience or meteorological forecasts. We can regard the weather as a random choice by an outside player, Nature, that has no payoffs but merely chooses according to known probabilities.[3] We can then enclose the various nodes corresponding to Nature's moves in an information set for the farmer, constraining the farmer's choice to be the same at all of these nodes. Figure 6.11 illustrates this situation.

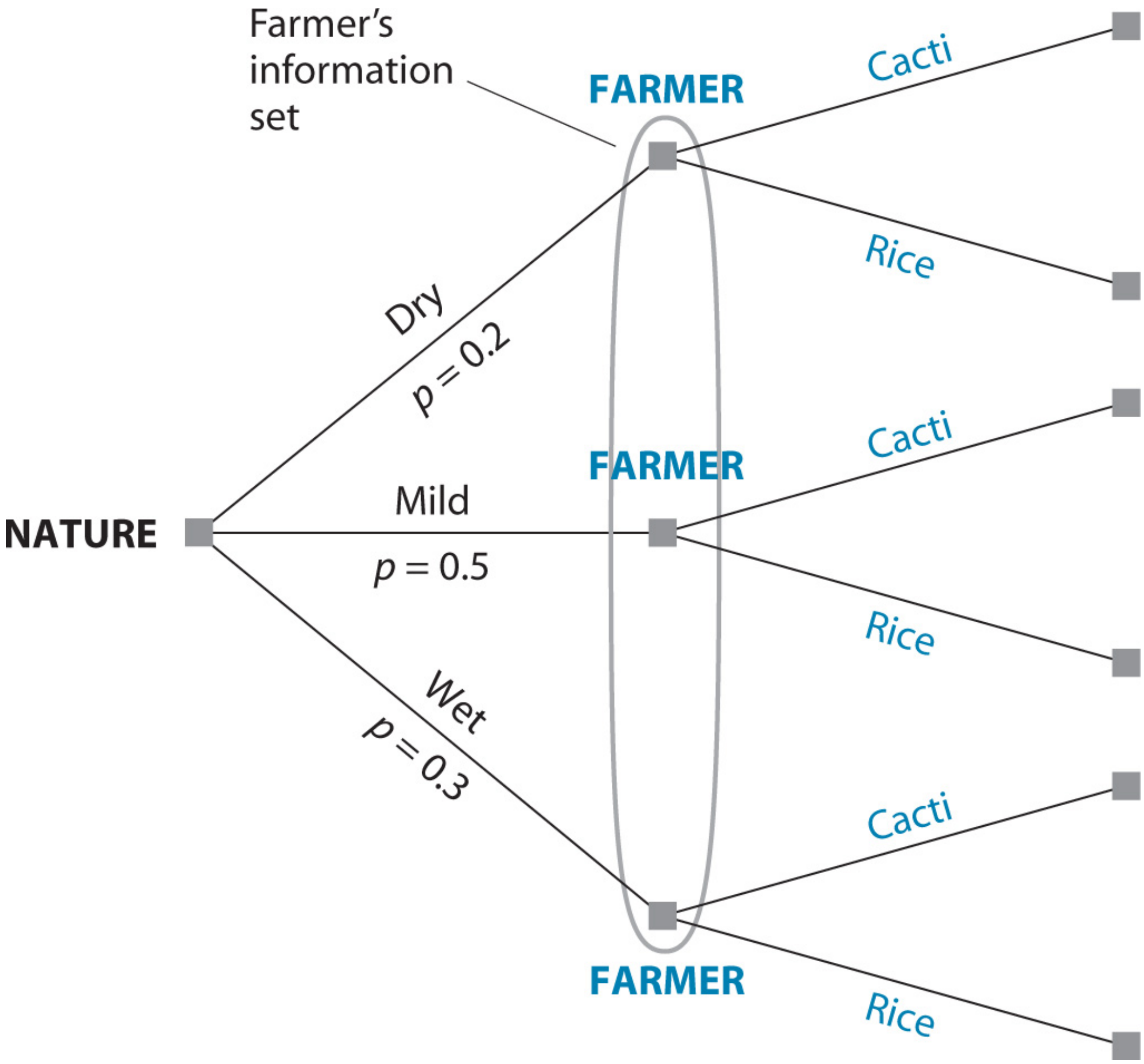

Figure 6.11 Nature and Information Sets

Using the concept of an information set, we can formalize the concepts of perfect and imperfect information in a game, which we introduced in Section 2.D of Chapter 2. A game has perfect information if it has neither strategic nor external uncertainty, which will happen if it has no information sets enclosing two or more nodes. Thus, a game has perfect information if all of its information sets consist of singleton nodes.

Although this representation is conceptually simple, it does not provide any simpler way of solving a simultaneous-move game. Therefore, we use it only occasionally, where it conveys some point more simply than any alternative. Some examples of game trees using information sets can be found in Chapters 9 and 13.

B. Showing and Analyzing Sequential-Move Games Using Game Tables

Consider now the sequential-move game of fiscal and monetary policy from Figure 6.8c, in which Congress has the first move. Suppose we want to show this game in normal or strategic form—that is, by using a game table. The rows and the columns of the table show the strategies of the two players. We must therefore begin by specifying those strategies.

For Congress, the first mover, listing the strategies is easy. There are just two possible moves—Balance and Deficit—and they are also the two strategies. For the Fed—the second mover—matters are more complex. Remember that a strategy is a complete plan of action, specifying the moves to be made at each node where it is a player’s turn to move. Because the Fed gets to move at two nodes (and because we are supposing that this game actually has sequential moves, so that the two nodes are not combined into one information set) and can choose either Low or High at each node, it has four strategies. These strategies are (1) Low if Balance, High if Deficit (we can write this as “L if B, H if D” for short); (2) High if Balance, Low if Deficit (“H if B, L if D” for short); (3) Low always; and (4) High always.

We show the resulting two-by-four payoff matrix in Figure 6.12. The last two columns are no different from those in the two-by-two payoff matrix for this game under simultaneous-move rules (Figure 6.8a). If the Fed chooses a strategy in which it always makes the same move, it is just as if the Fed were moving without taking into account what Congress had done—it is as if the players’ moves were simultaneous. But calculation of the payoffs for the first two columns, where the Fed’s second move does depend on Congress’s first move, needs some care.

To illustrate, let’s first consider the cell in the first row and the second column. Here, Congress chooses Balance and the Fed chooses H if B, L if D. Given Congress’s choice, the Fed’s actual choice under this strategy is High. Then the payoffs are those for the Balance and High combination—namely, 1 for Congress and 3 for the Fed.

Best-response analysis quickly shows that the game has two pure-strategy Nash equilibria, which we show by shading the cells gray. One is in the top-left cell, where Congress' s strategy is Balance and the Fed' s is L if B, H if D, so that the Fed' s actual choice is L. This outcome is just the rollback equilibrium of the sequential-move game. But there is another Nash equilibrium in the bottom-right cell, where Congress chooses Deficit and the Fed chooses High always. As in any Nash equilibrium, neither player has a clear reason to deviate from the strategy that leads to this outcome. Congress would do worse by switching to Balance, and the Fed could do no better by switching to any of its other three strategies, although it could do just as well with L if B, H if D.

This sequential-move game, when analyzed in its extensive form using the game tree in Figure 6.8c, led to a single rollback equilibrium. But when analyzed in its normal or strategic form using the game table in Figure 6.12, it has two Nash equilibria. What is going on?

		FED			
		L if B, H if D	H if B, L if D	Low always	High always
CONGRESS	Balance	3, 4	1, 3	3, 4	1, 3
	Deficit	2, 2	4, 1	4, 1	2, 2

You may need to scroll left and right to see the full figure.

FIGURE 6.12 Sequential-Move Game of Fiscal and Monetary Policy in Strategic Form

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The answer lies in the different nature of the logic of Nash equilibrium and rollback analyses. Nash equilibrium requires that neither player have a reason to deviate from her strategy, given the strategy of the other player. However, rollback does not take the strategies of later movers as given. Instead, it asks what would be optimal to do if the opportunity to move actually arose.

In our example, the Fed' s strategy of High always does not satisfy the criterion of being optimal if the opportunity to move actually arose. If Congress chose Deficit, then High would indeed be Fed' s optimal response. However, if Congress chose Balance, and the Fed had to respond, it would want to choose Low, not High. So High always does not describe the Fed' s optimal response in all possible configurations of play and cannot be a rollback equilibrium. But the logic of Nash equilibrium does not impose such a test, instead regarding the Fed' s High always as a strategy that Congress could legitimately take as given. If it did so, then Deficit would be Congress' s best response. And, conversely, High always is one of the Fed' s best responses to Congress' s Deficit (although it is tied with L if B, H if D). Thus, the pair of strategies Deficit and High always are mutual best responses and constitute a Nash equilibrium, although they do not constitute a rollback equilibrium.

We can therefore think of rollback as a further test, supplementing the requirements of Nash equilibrium and helping to select from among multiple Nash equilibria in the strategic form. In other words, rollback is a refinement of the Nash equilibrium concept.

To state this idea somewhat more precisely, recall the concept of a subgame. At any node of the full game tree, we can think of the part of the game that begins there as a subgame. In fact, as successive players make their choices, the play of the game moves along a succession of nodes, and each move can be thought of as starting a subgame. The equilibrium derived by using rollback corresponds to one particular succession of choices in each subgame and gives rise to the equilibrium path of play. Certainly, other paths of play are consistent with the rules of the game. We call these other paths off-equilibrium paths, and we call any subgames that arise along these paths off-equilibrium subgames.

With this terminology, we can now say that the equilibrium path of play is itself determined by the players' expectations of what would happen if they chose a different action—if they moved the game to an off-equilibrium path and started an off-equilibrium subgame. Rollback requires that all players make their best choices in *every* subgame of the larger game, whether or not the subgame lies along the path to the ultimate equilibrium outcome.

Because strategies are complete plans of action, a player's strategy must specify what she will do in each eventuality, at each and every node of the game, whether on or off the equilibrium path of play, where it is her turn to act. When she reaches a particular decision node, only the plan of action starting there—namely, the part of her full strategy that pertains to the subgame starting at that node—is pertinent. This part of her strategy is called the continuation of the strategy for that subgame. Rollback requires that the optimal (equilibrium) strategy be such that its continuation in every subgame is optimal for the player whose turn it is to act at that node, whether or not the node and the subgame lie on the equilibrium path of play.

Let's return to the fiscal–monetary policy game, with Congress moving first, and consider the second Nash equilibrium, in the lower-right corner of Figure 6.12, that arises in its strategic form. Here, Congress chooses Deficit and the Fed chooses High. On the equilibrium path of play, High is indeed the Fed's best response to Deficit. Congress's choice of Balance, however, would be the start of an off-equilibrium path. That path leads to a node where a rather trivial subgame starts—namely, a decision by the Fed. The Fed's purported equilibrium strategy High always asks it to choose High in this subgame. But that choice is not optimal; this second Nash equilibrium is specifying a nonoptimal choice for an off-equilibrium subgame.

In contrast, the equilibrium path of play for the Nash equilibrium in the upper-left corner of Figure 6.12 is for Congress to choose Balance and the Fed to follow with Low. Here, the Fed would be responding optimally on the equilibrium path of play. The off-equilibrium path would have Congress choosing Deficit, and the Fed, given its strategy of L if B, H if D, would follow with High. It is optimal for the Fed to respond to Deficit with High, so that choice remains optimal off the equilibrium path as well as on it.

The requirement that continuation of a strategy remain optimal under all circumstances is important because the equilibrium path itself is the result of players' thinking strategically about what would happen if they did something different. A later player may try to achieve an outcome that she would prefer by threatening that certain actions by the first mover will be met with dire responses, or by promising that certain other actions will be met with kindly responses. But the first mover will be skeptical of such threats and promises. The only way to remove that doubt is to check whether the stated responses would actually be optimal for the later player. If those responses would not be optimal, then the threats or promises have no credibility, and those responses will not be observed along the equilibrium path of play.

The equilibrium found by using rollback is called a subgame-perfect equilibrium (SPE). It is a set of strategies (complete plans of action), one for each player, such that, at every node of the game tree, whether or not that node lies along the equilibrium path of play, the continuation of the same strategy in the subgame starting at that node is optimal for the player who takes the action there. More simply, an SPE requires players to use strategies that constitute a Nash equilibrium in every subgame of the larger game.

In fact, as a rule, in games with finite trees and perfect information, where players can observe every previous action taken by all other players so that there are no information sets containing multiple nodes, rollback finds the unique (except for trivial and exceptional cases of ties) subgame-perfect equilibrium of the game. Consider: If you look at any subgame that begins at the last decision node for the last player who moves, the best choice for that player is the one that gives her the highest payoff. But that is precisely the action chosen with the use of rollback. As players move backward through the game tree, rollback eliminates all unreasonable strategies, including noncredible threats or promises, so that the collection of actions ultimately selected is the SPE. Therefore, for the purposes of this book, subgame perfectness is just a fancy name for rollback. At more advanced levels of game theory, where games include complex information structures and information sets, subgame-perfectness becomes a richer notion.

Endnotes

- Some people believe that Nature is actually a malevolent player who plays a zero-sum game with us, so that its payoffs are higher when ours are lower. For example, we believe it is more likely to rain if we have forgotten to bring an umbrella. We understand such thinking, but it does not have real statistical support. Return to reference 3

Glossary

information set
: A set of nodes among which a player is unable to distinguish when taking an action. Thus, his strategies are restricted by the condition that he should choose the same action at all points of an information set. For this, it is essential that all the nodes in an information set have the same player designated to act, with the same number and similarly labeled branches emanating from each of these nodes.

off-equilibrium path
: A path of play that does not result from the players' choices of strategies in a subgame-perfect equilibrium.

off-equilibrium subgame
: A subgame starting at a node that does not lie on the equilibrium path of play.

continuation
: The continuation of a strategy from a (noninitial) node is the remaining part of the plan of action of that strategy, applicable to the subgame that starts at this node.

credibility
: A strategy is credible if its continuation at all nodes, on or off the equilibrium path, is optimal for the subgame that starts at that node.

subgame-perfect equilibrium (SPE)
: A configuration of strategies (complete plans of action) such that their continuation in any subgame remains optimal (part of a rollback equilibrium), whether that subgame is on- or off-equilibrium. This ensures credibility of all the strategies.

4 THREE-PLAYER GAMES

We have restricted the discussion so far in this chapter to games with two players and two moves each. But the same methods also work for some larger and more general examples. We now illustrate this by using the street-garden game introduced in Chapter 3. Specifically, we (1) change the rules of the game from sequential to simultaneous moves, and then (2) keep the moves sequential but show and analyze the game in its strategic form. First, we reproduce the tree for the original sequential-move game (Figure 3.5) as Figure 6.13 here and remind you of its rollback equilibrium.

The equilibrium strategy of the first mover (Emily) is simply one move, “Don’t contribute.” The second mover (Nina) chooses from among 4 possible strategies (with a choice of two responses at each of two nodes) and chooses the strategy “Choose Don’t contribute (D) if Emily has chosen Contribute, and choose Contribute (C) if Emily has chosen Don’t contribute,” or, more simply, “D if C, C if D,” or even more simply, “DC.” The third mover (Talia) has 16 available strategies (with a choice of two responses at each of four nodes), and her equilibrium strategy is “Choose D following Emily’s C and Nina’s C, C following their CD, C following their DC, and D following their DD,” or “DCCD” for short.

Remember, too, the reason for these choices. The first mover has the opportunity to choose Don’t, knowing that the other two will recognize that a pleasant garden won’t be forthcoming unless they contribute, and knowing that they prefer a pleasant garden strongly enough that they will contribute.

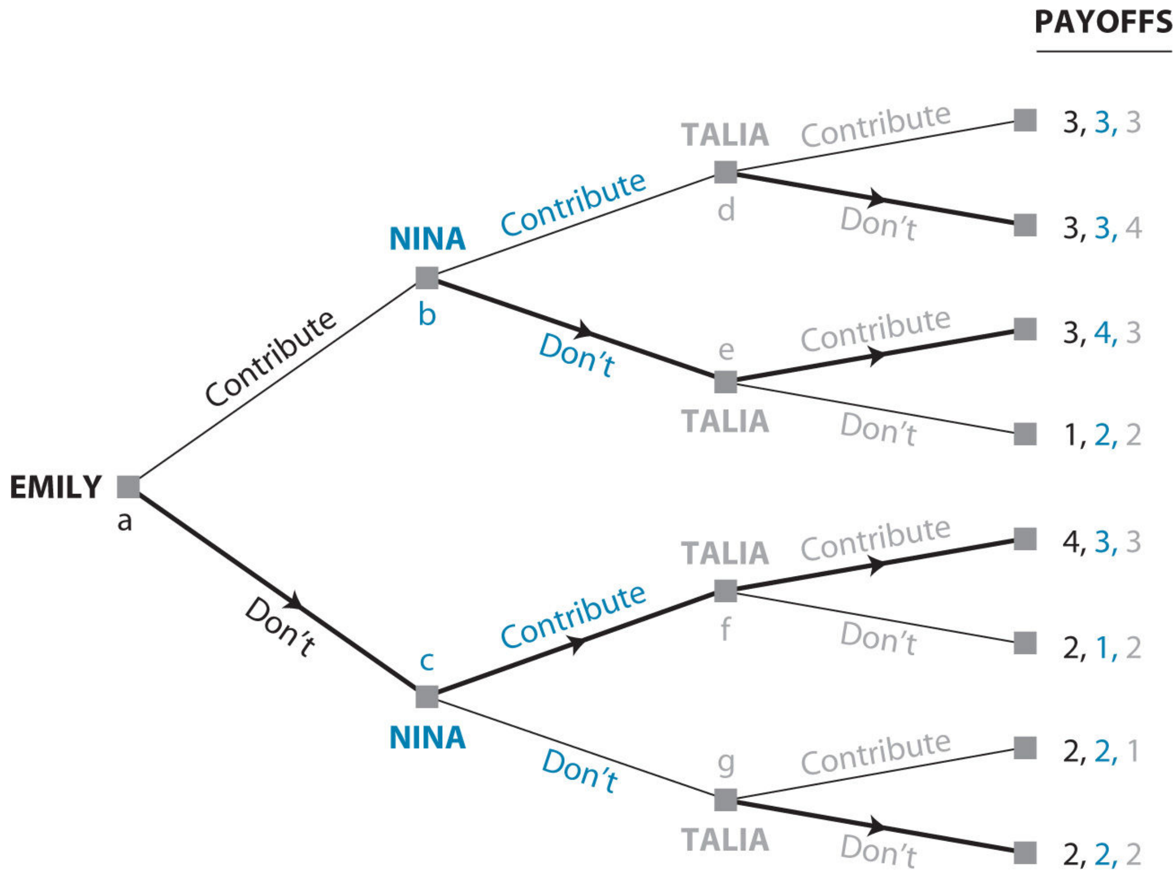

Figure 6.13 The Street-Garden Game with Sequential Moves

Now we change the rules of the game to make it a simultaneous-move game. (In Chapter 4, we solved a simultaneous-move version with somewhat different payoffs; here we keep the payoffs the same as in Chapter 3.) The payoff matrix is shown in Figure 6.14. Best-response analysis shows easily that there are four Nash equilibria.

In three of the Nash equilibria of the simultaneous-move game, two players contribute, while the third does not. These equilibria are similar to the rollback equilibrium of the sequential-move game. In fact, each one corresponds to the rollback equilibrium of the sequential-move game with a particular order of play. Further, any given order of play in the sequential-move version of this game leads to the same simultaneous-move payoff table.

But there is also a fourth Nash equilibrium here, where no one contributes. Given the specified strategies of the other two—namely, Don’t contribute—any one player is powerless to bring about the pleasant garden and therefore chooses not to contribute as well. Thus, in the change from sequential to simultaneous moves, the first-mover advantage has been lost. Multiple equilibria arise, only one of which retains the original first mover’s high payoff.

Next we return to the sequential-move version—Emily first, Nina second, and Talia third—but show the game in its normal or strategic form. In the sequential-move game, Emily has 2 pure strategies, Nina has 4, and Talia has 16, so this means constructing a payoff table that is 2 × 4 × 16. With the use of the same conventions that we used for three-player tables in Chapter 4, this particular game would require a table with 16 “pages” of two-by-four payoff tables. That would look too messy. so we opt instead for a reshuffling of the players. Let Talia be the Row player, Nina be the Column player, and Emily be the Page player. Then “all” that is required to illustrate this game is the 16 × 4 × 2 game table shown in Figure 6.15. The order of payoffs still corresponds to our earlier convention in that they are listed in the order Row player, Column player, Page player; in our example, that means the payoffs are now listed in the order Talia, Nina, and Emily.

TALIA chooses:

Contribute

		NINA	
		Contribute	**Don’t**
EMILY	Contribute	3, 3, 3	3, 4, 3
	Don’t	4, 3, 3	2, 2, 1

You may need to scroll left and right to see the full figure.

Don’t Contribute

		NINA	

		NINA	
		Contribute	Don’ t
EMILY	Contribute	3, 3, 4	1, 2, 2
	Don’ t	2, 1, 2	2, 2, 2

You may need to scroll left and right to see the full figure.

FIGURE 6.14 The Street-Garden Game with Simultaneous Moves

As in the fiscal–monetary policy game between Congress and the Fed, there are multiple Nash equilibria in the simultaneous-move street-garden game. (In Exercise S9, we ask you to find them all.) But there is only one subgame-perfect equilibrium, corresponding to the rollback equilibrium found in Figure 6.13. Although best-response analysis does find all the Nash equilibria, successive elimination of dominated strategies can reduce the number of reasonable equilibria for us here. This process works because elimination identifies those strategies that include noncredible components (such as High always for the Fed in Section 3.B). As it turns out, such elimination can take us all the way to the unique subgame-perfect equilibrium.

In Figure 6.15, we start with Talia and eliminate all of her (weakly) dominated strategies. This step eliminates all but the strategy listed in the eleventh row of the table, DCCD, which we have already identified as Talia’ s rollback equilibrium strategy. Elimination can continue with Nina, for whom we must compare outcomes from strategies across both pages of the table. To compare her CC to CD, for example, we look at the payoffs associated with CC in *both pages* of the table and compare these payoffs with the similarly identified payoffs for CD. For Nina, the elimination process leaves only her strategy DC; again, this is the rollback equilibrium strategy found for her above. Finally, Emily has only to compare the two remaining cells associated with her choice of Don’ t and Contribute; she gets the highest payoff when she chooses Don’ t and so makes that choice. As before, we have identified her rollback equilibrium strategy.

The unique subgame-perfect outcome in the game table in Figure 6.15 thus corresponds to the cell associated with the rollback equilibrium strategies for each player. Note that the process of successive elimination that leads us to this subgame-perfect equilibrium is carried out by considering the players in reverse order of the actual play of the game. This order conforms to the order in which player actions are considered in rollback analysis and therefore allows us to eliminate exactly those strategies, for each player, that are not consistent with rollback. In so doing, we eliminate all the Nash equilibria that are not subgame-perfect.

EMILY								
	Contribute				Don’ t			
	NINA				NINA			
TALIA	CC	CD	DC	DD	CC	CD	DC	DD
CCCC	3, 3, 3	3, 3, 3	3, 4, 3	3, 4, 3	3, 3, 4	1, 2, 2	3, 3, 4	1, 2, 2
CCCD	3, 3, 3	3, 3, 3	3, 4, 3	3, 4, 3	3, 3, 4	2, 2, 2	3, 3, 4	2, 2, 2
CCDC	3, 3, 3	3, 3, 3	3, 4, 3	3, 4, 3	2, 1, 2	1, 2, 2	2, 1, 2	1, 2, 2
CDCC	3, 3, 3	3, 3, 3	3, 2, 1	2, 2, 1	3, 3, 4	1, 2, 2	3, 3, 4	1, 2, 2
DCCC	4, 3, 3	4, 3, 3	3, 4, 3	3, 4, 3	3, 3, 4	1, 2, 2	3, 3, 4	1, 2, 2

CCDD	3，3，3	3，3，3	3，4，3	3，4，3	2，1，2	2，2，2	2，1，2	2，2，2
CDDC	3，3，3	3，3，3	2，2，1	2，2，1	2，1，2	1，2，2	2，1，2	1，2，2
DDCC	4，3，3	4，3，3	2，2，1	2，2，1	3，3，4	1，2，2	3，3，4	1，2，2
CDCD	3，3，3	3，3，3	2，2，1	2，2，1	3，3，4	2，2，2	3，3，4	2，2，2
DCDC	4，3，3	4，3，3	3，4，3	3，4，3	2，1，2	1，2，2	2，1，2	1，2，2
DCCD	4，3，3	4，3，3	3，4，3	3，4，3	3，3，4	2，2，2	3，3，4	2，2，2
CDDD	3，3，3	3，3，3	2，2，1	2，2，1	2，1，2	2，2，2	2，1，2	2，2，2
DCDD	4，3，3	4，3，3	3，4，3	3，4，3	2，1，2	2，2，2	2，1，2	2，2，2
DDCD	4，3，3	4，3，3	2，2，1	2，2，1	3，3，4	2，2，2	3，3，4	2，2，2
DDDC	4，3，	4，3，	2，2，	2，2，	2，1，	1，2，	2，1，	1，2，

	3	3	1	1	2	2	2	2
DDDD	4, 3, 3	4, 3, 3	2, 2, 1	2, 2, 1	2, 1, 2	2, 2, 2	2, 1, 2	2, 2, 2

FIGURE 6.15 The Street-Garden Game in Strategic Form

SUMMARY

Many games include multiple components, some of which entail simultaneous play and others of which entail sequential play. In two-stage (and multistage) games, a “tree house” can be used to illustrate the game; this construction allows the identification of the different stages of play and the ways in which those stages are linked together. Full-fledged games that arise in later stages of play are called *subgames* of the full game. Players’ actions in each subgame are determined by the *continuation* of their strategies for that subgame.

The *strategic order* of moves in a game is determined by when each player’s move becomes irreversible and observable. If the Row player’s move becomes irreversible and observable before the Column player’s move becomes irreversible, the game has *sequential moves* with Row as first mover, and vice versa. On the other hand, if each player’s move becomes irreversible before the other player’s move becomes observable, the game has *simultaneous moves*. Players can influence the order of moves by making a move in a transparent or secretive way or by becoming more or less nimble or inflexible in the making of a move.

Changing the order of moves may or may not alter the equilibrium outcome of a game. Simultaneous-move games that are changed to make moves sequential may have the same outcome (if both players have dominant strategies), may have a first-mover or second-mover advantage, or may lead to better outcomes for both players. The sequential-move version of a simultaneous-move game will generally have a unique rollback equilibrium even if the simultaneous-move version has no equilibrium or multiple equilibria. Similarly, a sequential-move game that has a unique rollback equilibrium may have several Nash equilibria when the rules are changed to make it a simultaneous-move game.

Simultaneous-move games can be illustrated using a game tree by collecting decision nodes in *information sets* when players must make decisions without knowing at which specific node they find themselves. Similarly, sequential-move games can be illustrated using a game table; in this case, each player’s full set of strategies must be carefully identified. Solving a sequential-move game from its strategic form may lead to many possible Nash equilibria. The number of potential equilibria can be reduced by using the criterion of *credibility* to eliminate some strategies as possible equilibrium strategies. This process leads to the *subgame-perfect equilibrium (SPE)* of the sequential-move game. These solution processes also work for games with additional players.

KEY TERMS

continuation (196)

credibility (196)

information set (192)

irreversible (182)

observable (182)

off-equilibrium path (195)

off-equilibrium subgame (195)

subgame (180)

subgame-perfect equilibrium (SPE) (196)

strategic order (182)

Glossary

continuation
: The continuation of a strategy from a (noninitial) node is the remaining part of the plan of action of that strategy, applicable to the subgame that starts at this node.

credibility
: A strategy is credible if its continuation at all nodes, on or off the equilibrium path, is optimal for the subgame that starts at that node.

information set
: A set of nodes among which a player is unable to distinguish when taking an action. Thus, his strategies are restricted by the condition that he should choose the same action at all points of an information set. For this, it is essential that all the nodes in an information set have the same player designated to act, with the same number and similarly labeled branches emanating from each of these nodes.

irreversible
: Cannot be undone by a later action. In a sequential-move game, the first mover’s action must be irreversible and *observable* before the second mover’s action is irreversible.

observable
: Known to other players before they make their responding actions. Together with irreversibility, this is an important condition for a game to be sequential-move.

off-equilibrium path
: A path of play that does not result from the players’ choices of strategies in a subgame-perfect equilibrium.

off-equilibrium subgame
: A subgame starting at a node that does not lie on the equilibrium path of play.

subgame
: A game comprising a portion or remnant of a larger game, starting at a noninitial node of the larger game.

subgame-perfect equilibrium (SPE)
: A configuration of strategies (complete plans of action) such that their continuation in any subgame remains optimal (part of a rollback equilibrium), whether that subgame is on- or off-equilibrium. This ensures credibility of all the strategies.

strategic order
: The order of moves from a game-theoretic point of view, determined by considerations of observability and irreversibility. It may differ from the chronological order of actions and, in turn, determine whether the game has sequential or simultaneous moves.

SOLVED EXERCISES

1. Consider the simultaneous-move tennis-point game with two players that has no Nash equilibrium in pure strategies, illustrated in Figure 4.17. If the game were transformed into a sequential-move game, would you expect that game to exhibit a first-mover advantage, a second-mover advantage, or neither? Explain your reasoning.
2. A Rebel Force of guerilla fighters seeks to inflict damage on a Conventional Army of the government, while the Conventional Army would like to destroy the Rebel Force. The two sides play a game in which each must decide whether to locate their forces in the Hills or in the Valley. The Rebels can inflict the most damage from the Valley, but if the Conventional Army is also in the Valley, it will be able to force the Rebels into open combat, where they are most vulnerable. By contrast, the Rebel Force is safer in the Hills, although it is also less capable of inflicting damage from that location. The ordinal payoff matrix for this game is shown below.

		CONVENTIONAL ARMY	
		Valley	Hills
REBEL FORCE	Valley	1, 4	4, 1
	Hills	3, 2	2, 3

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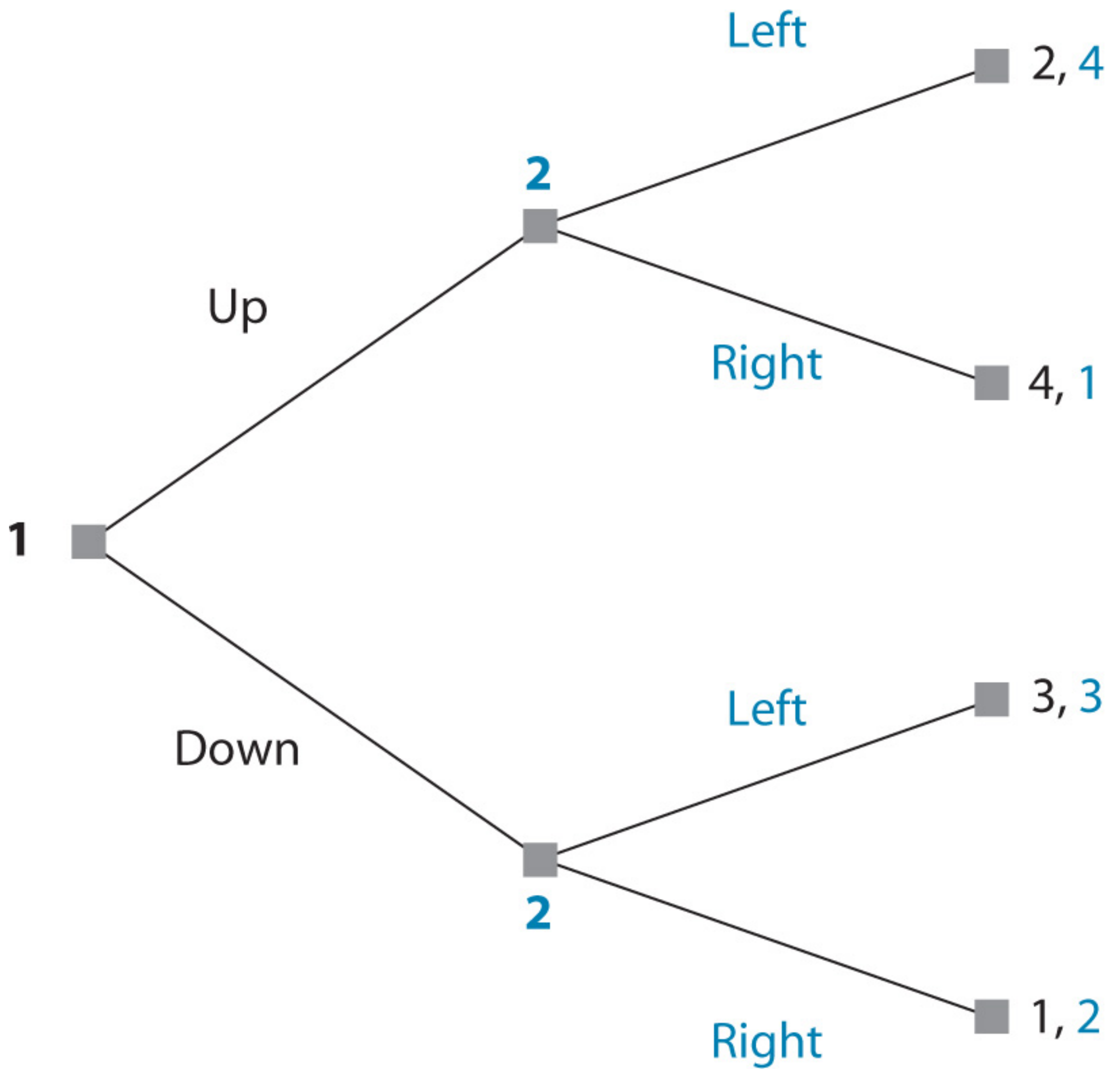
   1. Is this a zero-sum game? Explain your answer. Hint: Remember that a zero-sum game is one in which any change in outcome that makes one player better off must make the other player worse off. To show that a game is *not* zero-sum, it suffices to identify one outcome that is better for both players than some other outcome.
   2. Identify all pure-strategy Nash equilibria of this game (or explain why no such equilibrium exists) in the case when the Rebel Force and Conventional Army move simultaneously.
   3. Draw the game tree for the case when the Rebel Force moves first. What are the rollback equilibrium strategies and outcome?
   4. Draw the game tree for the case when the Conventional Army moves first. What are the rollback equilibrium strategies and

outcome?

5. Does the order of moves matter in this game? If so, does the game have a first-mover advantage, a second-mover advantage, or do both players prefer the same ordering of moves?
6. In the 1961 military handbook *Guerrilla Warfare*, Marxist revolutionary Che Guevara wrote, "The fundamental characteristic of a guerrilla band is mobility." Discuss how the mobility, and therefore the nimbleness, of guerrilla forces might affect the possible ways in which moves could be ordered in this game. In particular, consider whether all the move orders analyzed in parts (b), (c), and (d) would actually be possible with a very mobile Rebel Force.

3. Consider the game represented by the game tree below. The first mover, Player 1, may move either Up or Down, after which Player 2 may move either Left or Right. Payoffs for the possible outcomes are shown in the game tree below. Show this game in strategic form. Then find all the pure-strategy Nash equilibria in the game. If there are multiple equilibria, indicate which one is subgame-perfect. For those equilibria that are not subgame-perfect, identify the reason (the source of their lack of credibility).



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4. Consider the Airbus - Boeing game in Exercise S4 in Chapter 3. Show that game in strategic form and locate all Nash equilibria. Which one of the equilibria is subgame-perfect? For those equilibria that are not subgame-perfect, identify the reason.
5. Return to the two-player game tree in part (a) of Exercise S2 in Chapter 3.
    1. Draw the game in strategic form, making Scarecrow the Row player and Tinman the Column player.
    2. Find the Nash equilibrium.
6. Return to the two-player game tree in part (b) of Exercise S2 in Chapter 3.
    1. Draw the game in strategic form. (Hint: Refer to your answer to Exercise S2 in Chapter 3.) Find all Nash equilibria. There will be many.

2. For those equilibria that you found in part (a) that are not subgame-perfect, identify the reason.

7. Return to the three-player game tree in part (c) of Exercise S2 in Chapter 3.
   1. Draw the game table. Make Scarecrow the Row player, Tinman the Column player, and Lion the Page player. (Hint: Refer to your answer to Exercise S2 in Chapter 3.) Find all Nash equilibria. There will be many.
   2. For those equilibria that you found in part (a) that are not subgame-perfect, identify the reason.
8. Consider a simplified baseball game played between a pitcher and a batter. The pitcher chooses between throwing a fastball or a curve, while the batter chooses which pitch to anticipate. The batter has an advantage if he correctly anticipates the type of pitch. In this constant-sum game, the batter's payoff is the probability that he will get a base hit. The pitcher's payoff is the probability that the batter will fail to get a base hit, which is simply 1 minus the payoff for the batter. There are four potential outcomes:
   1. If the pitcher throws a fastball, and the batter guesses fastball, the probability of a hit is 0.300.
   2. If the pitcher throws a fastball, and the batter guesses curve, the probability of a hit is 0.200.
   3. If the pitcher throws a curve, and the batter guesses curve, the probability of a hit is 0.350.
   4. If the pitcher throws a curve, and the batter guesses fastball, the probability of a hit is 0.150.

   Suppose that the pitcher is "tipping" his pitches—which means that the pitcher is holding the ball, positioning his body, or doing something else in a way that reveals to the batter which pitch he is going to throw. For our purposes, this means that the pitcher–batter game is a sequential-move game in which the pitcher announces his pitch choice before the batter has to choose his strategy.

   1. Draw a game tree for this situation.
   2. Suppose that the pitcher knows he is tipping his pitches but can't stop himself from doing so. Thus, the pitcher and batter are playing the game you just drew. Find the rollback equilibrium of this game.
   3. Now change the timing of the game so that the batter has to reveal his action (perhaps by altering his batting stance)

before the pitcher chooses which pitch to throw. Draw the game tree for this situation, and find the rollback equilibrium.

Now assume that tipping by each player occurs so quickly that neither player can react to them, so that the game is in fact simultaneous.

1. (d) Draw a game tree to represent this simultaneous-move game, indicating information sets where appropriate.
2. (e) Draw the game table for the simultaneous-move game. Is there a Nash equilibrium in pure strategies? If so, what is it?

9. The street-garden game analyzed in Section 4 of this chapter has a 16 × 4 × 2 game table when the sequential-move version of the game is expressed in strategic form, as in Figure 6.15. There are *many* Nash equilibria to be found in this table.
   1. Use best-response analysis to find all Nash equilibria in Figure 6.15.
   2. Identify the subgame-perfect equilibrium from among your set of all Nash equilibria. Other equilibrium outcomes look identical to the subgame-perfect one—they entail the same payoffs for each of the three players—but they arise after different combinations of strategies. Explain how this can happen. Describe the credibility problems that arise in the equilibria that are not subgame-perfect.
10. Figure 6.1 represents the two-stage game between CrossTalk and GlobalDialog with a combination of tables and trees. Instead, represent the entire two-stage game with a single, very large game tree. Be careful to label which player makes the decision at each node, and remember to indicate information sets between nodes where necessary.
11. Recall the mall location game in Exercise S9 in Chapter 3. That three-player sequential-move game has a game tree that is similar to the one for the street-garden game shown in Figure 6.13.
    1. Draw the tree for the mall location game. How many strategies does each store have?
    2. Illustrate the game in strategic form and find all pure-strategy Nash equilibria in the game.
    3. Use successive elimination of dominated strategies to find the subgame-perfect equilibrium. (Hint: Reread the last two paragraphs of Section 4 in this chapter.)

12. The rules of the mall location game, analyzed in Exercise S11 above, specify that when all three stores request space in Urban Mall, the two bigger (more prestigious) stores get the available spaces. The original version of the game also specifies that the firms move sequentially in requesting mall space.
    1. Suppose that the three firms make their location requests simultaneously. Draw the payoff table for this version of the game and find all Nash equilibria. Which one of these equilibria do you think is most likely to be played in the real world? Explain.

    Now suppose that when all three stores simultaneously request Urban Mall, the two spaces are allocated by lottery, giving each store an equal chance of getting into Urban Mall. With such a system, each would have a two-thirds probability (or a 66.67% chance) of getting into Urban Mall when all three have requested space there, and a one-third probability (33.33% chance) of being alone in Rural Mall.

    1. (b) Draw the game table for this new version of the simultaneous-move mall location game. Find all Nash equilibria of the game. Which one of these equilibria do you think is most likely to be played in the real world? Explain.
    2. (c) Compare and contrast the equilibria found in part (b) with the equilibria found in part (a). Did you get the same Nash equilibria? Why or why not?
13. Return to the game between Monica and Nancy in Exercise S10 in Chapter 5. Assume that Monica and Nancy choose their effort levels sequentially instead of simultaneously. Monica commits to her choice of effort level first, and on observing this decision, Nancy commits to her own effort level.
    1. What is the subgame-perfect equilibrium of the game where the joint profits are $4m + 4n + mn$, the costs of their efforts to Monica and Nancy are m^2 and n^2, respectively, and Monica commits to an effort level first?
    2. Compare the payoffs to Monica and Nancy with those found in Exercise S10 in Chapter 5. Does this game have a first-mover or a second-mover advantage? Explain.
14. Extending Exercise S13, Monica and Nancy need to decide which (if either) of them will commit to an effort level first. To do this, each of them simultaneously writes on a separate slip of paper whether or not she will commit first. If they both write Yes or

they both write No, they choose effort levels simultaneously, as in Exercise S10 in Chapter 5. If Monica writes Yes and Nancy writes No, then Monica commits to her move first, as in Exercise S13 above. If Monica writes No and Nancy writes Yes, then Nancy commits to her move first.

1. Use the payoffs to Monica and Nancy in Exercise S13, as well as in Exercise S10 in Chapter 5, to construct the game table for the first-stage paper-slip decision game. (Hint: Note the symmetry of the game.)
2. Find the pure-strategy Nash equilibria of this first-stage game.

UNSOLVED EXERCISES

1. Consider a game in which there are two players, A and B. Player A moves first and chooses either Up or Down. If A chooses Up, the game is over, and each player gets a payoff of 2. If A chooses Down, then B gets a turn and chooses between Left and Right. If B chooses Left, both players get 0; if B chooses Right, A gets 3 and B gets 1.
   1. Draw the tree for this game and find the subgame-perfect equilibrium.
   2. Show this sequential-move game in strategic form, and find all Nash equilibria. Which is or are subgame-perfect? Which is or are not? If any are not, explain why.
   3. What method could be used to find the subgame-perfect equilibrium using the strategic form of the game? (Hint: Refer to the last two paragraphs of Section 4 in this chapter.)
2. A Monopolist faces potential competition from an Entrant in the undifferentiated product market for computer memory disks. The Monopolist currently operates one factory and must decide whether or not to build a second factory. The Entrant must similarly decide whether to build its first factory. A factory would cost either firm \$1.5 billion to build. Total (gross) profit—after subtracting production costs, but not factory construction costs—depends on how many factories have been built and is divided between the two firms in proportion to how many factories they operate. In particular, total (gross) profit takes the form $\Pi(Q) = Q(4 - Q)$, where Q is the total number of factories. Each firm's net profit in each of the four possible outcomes (after accounting for the cost of building factories) is shown in the payoff matrix below, expressed in billions of dollars. (You can verify these payoffs if you so desire. Recall that the Monopolist already has one factory and that each new factory costs \$1.5 billion.)

	ENTRANT	
	Build	Don't Build

You may need to scroll left and right to see the full figure.

		ENTRANT	
		Build	Don’t Build
MONOPOLIST	Build	0.5, −0.5	2.5, 0
	Don’t Build	2, 0.5	3, 0

You may need to scroll left and right to see the full figure.

1. Is this a zero-sum game? Explain your answer. (For a hint, see Exercise S2 above.)
2. Identify all pure-strategy Nash equilibria of this game (or explain why no such equilibrium exists) in the case when the Monopolist and Entrant move simultaneously.
3. Draw the game tree for the case when the Monopolist moves first. What are the rollback equilibrium strategies and outcome?
4. Draw the game tree for the case when the Entrant moves first. What are the rollback equilibrium strategies and outcome?
5. Does the order of moves matter in this game? If so, does the game have a first-mover advantage, a second-mover advantage, or do both players prefer the same ordering of moves?
6. You serve on the Board of Directors of the Entrant firm. At a board meeting, another director makes the following argument: “$1.5 billion is an enormous investment and we don’t want to enter the market if the Monopolist is going to be building another factory. We need to wait to see whether they are going to be building a second factory before deciding what to do.” How would you use game theory to respond to this argument?

3. Return to the two-player game tree in part (a) of Exercise U2 in Chapter 3.
   1. Write the game in strategic form, making Albus the Row player and Minerva the Column player. Find all Nash equilibria.
   2. For those equilibria you found in part (a) of this exercise that are not subgame-perfect, identify the reason.
4. Return to the two-player game tree in part (b) of Exercise U2 in Chapter 3.
   1. Write the game in strategic form. Find all Nash equilibria.
   2. For those equilibria you found in part (a) that are not subgame-perfect, identify the reason.

5. Return to the two-player game tree in part (c) of Exercise U2 in Chapter 3.
   1. Draw the game table. Make Albus the Row player, Minerva the Column player, and Severus the Page player. Find all Nash equilibria.
   2. For those equilibria you found in part (a) that are not subgame-perfect, identify the reason.
6. Consider the cola industry, in which Coke and Pepsi are the two dominant firms. (To keep the analysis simple, just forget about all the others.) The market size is $8 billion. Each firm can choose whether to advertise. Advertising costs $1 billion for each firm that chooses it. If one firm advertises and the other doesn't, then the former firm captures the whole market. If both firms advertise, they split the market 50:50 and pay for the advertising. If neither advertises, they split the market 50:50 but without the expense of advertising.
   1. Draw the payoff table for this game, and find the equilibrium when the two firms move simultaneously.
   2. Draw the game tree for this game (assuming that it is played sequentially), with Coke moving first and Pepsi following.
   3. Is either equilibrium in parts (a) and (b) better from the joint perspective of Coke and Pepsi? How could the two firms do better?
7. Along a stretch of a beach are 500 children in five clusters of 100 each. (Label the clusters A, B, C, D, and E, in that order.) Two ice-cream vendors are deciding simultaneously where to locate. Each vendor must choose the exact location of one of the clusters.

   If there is a vendor in a cluster, all 100 children in that cluster will buy an ice cream. For clusters without a vendor, 50 of the 100 children are willing to walk to a vendor who is one cluster away, only 20 are willing to walk to a vendor two clusters away, and no children are willing to walk the distance of three or more clusters. The ice cream melts quickly, so the walkers cannot buy for the nonwalkers.

   If the two vendors choose the same cluster, each will get a 50% share of the total demand for ice cream. If they choose different clusters, then those children (locals or walkers) for whom one vendor is closer than the other will go to the closer one, and those for whom the two are equidistant will be split 50:50 between them. Each vendor seeks to maximize her sales.

    1. Construct the five-by-five payoff table for the vendor location game. The following list will give you a start and a check on your calculations.
        1. If both vendors choose to locate at A, each sells 85 units.
        2. If the first vendor chooses B and the second chooses C, the first sells 150 and the second sells 170.
        3. If the first vendor chooses E and the second chooses B, the first sells 150 and the second sells 200.
    2. Eliminate dominated strategies as far as possible.
    3. In the remaining table, locate all pure-strategy Nash equilibria.
    4. If the game is altered to one with sequential moves, where the first vendor chooses her location first and the second vendor follows, what are the locations and the sales that result from the subgame-perfect equilibrium? How does the change in the timing of moves here help players resolve the coordination problem in part (c)?
8. Return to the game among the three lions in the Roman Colosseum in Exercise S8 in Chapter 3.
    1. Draw this game in strategic form. Make Lion 1 the Row player, Lion 2 the Column player, and Lion 3 the Page player.
    2. Find all Nash equilibria for the game. How many did you find?
    3. You should have found Nash equilibria that are not subgame-perfect. For each of those equilibria, which lion is making a noncredible threat? Explain.
9. Assume that the mall location game (from Exercises S9 in Chapter 3 and S11 in this chapter) is now played sequentially, but with a different order of play: Big Giant, then Titan, then Frieda's.
    1. Draw the new game tree.
    2. What is the subgame-perfect equilibrium of the game? How does it compare with the subgame-perfect equilibrium for Exercise S9 in Chapter 3?
    3. Now draw the strategic form for this new version of the game.
    4. Find all Nash equilibria of the game. How many are there? How does this number compare with the number of equilibria from Exercise S11 in this chapter?
10. Return to the game between Monica and Nancy in Exercise U10 in Chapter 5. Assume that Monica and Nancy choose their effort levels sequentially instead of simultaneously. Monica commits to her choice of effort level first. On observing this decision, Nancy commits to her own effort level.

1. What is the subgame-perfect equilibrium of the game where the joint profits are $5m + 4n + mn$, the costs of their efforts to Monica and Nancy are m^2 and n^2, respectively, and Monica commits to an effort level first?
2. Compare the payoffs to Monica and Nancy with those found in Exercise U10 in Chapter 5. Does this game have a first-mover or second-mover advantage?
3. Using the same joint profit function as in part (a), find the subgame-perfect equilibrium for the game where *Nancy* must commit first to an effort level.

11. Extending Exercise U10, Monica and Nancy need to decide which (if either) of them will commit to an effort level first. To do this, each of them simultaneously writes on a separate slip of paper whether or not she will commit first. If they both write Yes or they both write No, they choose effort levels simultaneously, as in Exercise U10 in Chapter 5. If Monica writes Yes and Nancy writes No, they play the game in part (a) of Exercise U10 above. If Monica writes No and Nancy writes Yes, they play the game in part (c).
    1. Use the payoffs to Monica and Nancy in parts (b) and (c) in Exercise U10 above, as well as those in Exercise U10 in Chapter 5, to construct the game table for the first-stage paper-slip decision game.
    2. Find the pure-strategy Nash equilibria of this first-stage game.
12. In the faraway town of Saint James, two firms, Bilge and Chem, compete in the soft-drink market (Coke and Pepsi aren't in this market yet). They sell identical products, and since their good is a liquid, they can easily choose to produce fractions of units. Since they are the only two firms in this market, the price of the good (in dollars), P, is determined by $P = (30 - Q_B - Q_C)$, where Q_B is the quantity produced by Bilge and Q_C is the quantity produced by Chem (each measured in liters). At this time, both firms are considering whether to invest in new bottling equipment that will lower their operating costs.
    1. If firm j decides *not* to invest, its total cost will be $C_j = Q_j^2/2$, where j stands for either B (Bilge) or C (Chem).
    2. If firm j decides to invest, its total cost will be $C_j = 20 + Q_j^2/6$. This new cost function reflects the cost of investing in the new machines (20) as well as the lower operating costs associated with those machines.

The two firms make their investment choices simultaneously, but the payoffs in this investment game depend on the subsequent duopoly games that arise. The game is thus really a two-stage game: decide whether to invest, and then play a duopoly game.

1. Suppose both firms decide to invest. Write the profit functions in terms of Q_B and Q_C for the two firms. Use these to find the Nash equilibrium of the quantity-setting game. What are the equilibrium quantities and profits for both firms? What is the market price?
2. Now suppose both firms decide not to invest. What are the equilibrium quantities and profits for both firms? What is the market price?
3. Now suppose that Bilge decides to invest, and Chem decides not to invest. What are the equilibrium quantities and profits for both firms? What is the market price?
4. Draw the two-by-two game table for the first-stage investment game between the two firms. Each firm has two strategies: Invest and Don't Invest. The payoffs are simply the profits found in parts (a), (b), and (c). (Hint: Note the symmetry of the game.)
5. What is the subgame-perfect equilibrium of the overall two-stage game?

13. Two French aristocrats, Chevalier Chagrin and Marquis de Renard, fight a duel. Each has a pistol loaded with one bullet. They start 10 steps apart and walk toward each other at the same pace, 1 step at a time. After each step, either may fire his gun. When one shoots, the probability of scoring a hit depends on the distance. After k steps it is $k/5$, so it rises from 0.2 after the first step to 1 (certainty) after 5 steps, at which point the two are right up against each other. If one player fires and misses while the other has yet to fire, the walk must continue even though the bulletless one now faces certain death; this rule is dictated by the code of the aristocracy. Each player gets a payoff of −1 if he himself is killed and 1 if the other is killed. If neither or both are killed, each gets 0.

    This is a game with five sequential steps and simultaneous moves (shoot or don't shoot) at each step. Find the rollback (subgame-perfect) equilibrium of this game.

Hint: Begin at step 5, when the duelists are right up against each other. Set up the two-by-two table for the simultaneous-move game at this step, and find the Nash equilibrium. Now move back to step 4, where the probability of scoring a hit is 4/5, or 0.8, for each. Set up the two-by-two table for the simultaneous-move game at this step, correctly specifying in the appropriate cell what happens in the future. For example, if one shoots and misses, but the other does not shoot, then the other will wait until step 5 and score a sure hit. If neither shoots, then the game will go to the next step, for which you have already found the equilibrium. Using all this information, find the payoffs in the two-by-two table of step 4, and find the Nash equilibrium at this step. Work backward in the same way through the rest of the steps to find the Nash equilibrium of the full game.

14. Describe an example of business competition that is similar in structure to the duel in Exercise U13.

7 ■ Simultaneous-Move Games: Mixed Strategies

IN OUR STUDY of simultaneous-move games in Chapter 4, we came across some games that the solution methods described there could not solve—games with no Nash equilibria in pure strategies. To predict outcomes for such games, we need to extend our concepts of *strategy* and *equilibrium* to allow for the possibility that players may use strategies in which they make *random* choices among the actions available to them, more commonly known as *mixed strategies*.

Consider the tennis-point game from the end of Chapter 4. This game is zero-sum; the interests of the two tennis players are exactly opposite. Evert wants to hit her passing shot to whichever side—down the line (DL) or crosscourt (CC)—is not covered by Navratilova, whereas Navratilova wants to cover the side to which Evert hits her shot. In other words, Evert wants her choice to differ from Navratilova's, and Navratilova wants her choice to coincide with Evert's. In Chapter 4, we pointed out that in such a situation, any systematic choice by Evert will be exploited by Navratilova to her own advantage and therefore to Evert's disadvantage. Conversely, Evert can exploit any systematic choice by Navratilova. To avoid being thus exploited, each player wants to keep the other guessing, which can be done by acting unsystematically or randomly.

However, randomness doesn't mean choosing each shot half the time or alternating between the two. The latter would itself be a systematic action open to exploitation. A 60:40 or 75:25 random mix may be better than a 50:50 mix depending on the situation. In this chapter, we develop methods for calculating the best mix and discuss how well game theory helps us understand actual play in zero-sum games with equilibria in mixed strategies.

Our method for calculating the best mix can also be applied to non-zero-sum games. However, in such games, the players' interests can partially coincide, so when player B exploits A's systematic choice to her own advantage, it is not necessarily to A's disadvantage. Therefore, the logic of keeping the other player guessing is weaker, or even absent altogether, in non-zero-sum games. We will discuss whether and when mixed-strategy equilibria make sense in such games.

We start this chapter with a discussion of mixed strategies in two-by-two games and describe the most direct method for calculating best responses and finding a mixed-strategy equilibrium. Many of the concepts and methods we develop in Section 2 continue to be valid in more general games, and Sections 6 and 7 extend these methods to games where players may have more than two pure strategies. We conclude with some general observations about how to mix strategies in practice and some evidence on whether mixing is observed in reality.

1 WHAT IS A MIXED STRATEGY?

When players choose to act unsystematically, they choose from among the pure strategies available to them in some random way. Consider the tennis-point game discussed in Section 8 of Chapter 4. In that game, Martina Navratilova and Chris Evert each choose from two initially given pure strategies: down the line (DL) and crosscourt (CC). We call a random mixture of these two pure strategies a *mixed strategy*.

Such mixed strategies cover a continuous range of possibilities. At one extreme, DL could always be chosen (probability 1), meaning that CC is never chosen (probability 0); this “mixture” is simply the pure strategy DL. At the other extreme, DL could be chosen with probability 0 and CC with probability 1; this “mixture” is the same as the pure strategy CC. In between is a continuous range of possibilities: DL chosen with probability 75% (0.75) and CC with probability 25% (0.25); or both chosen with probabilities 50% (0.5) each; or DL with probability ⅓ (0.33 . . .) and CC with probability ⅔ (0.66 . . .); and so on.[1]

The payoff from a mixed strategy is the probability-weighted average of the payoffs from its constituent pure strategies. For example, in the tennis-point game, suppose that Navratilova were to play the pure strategy DL and Evert were to play a mixture of 75% DL and 25% CC. Against Navratilova’s DL, Evert’s payoff from DL is 50, and her payoff from CC is 90. On average, Evert’s mixture (0.75 DL, 0.25 CC) therefore yields $0.75 \times 50 + 0.25 \times 90 = 37.5 + 22.5 = 60$. This result is Evert’s expected payoff from this particular mixed strategy.[2]

The probability of choosing one or the other pure strategy is a continuous variable that ranges from 0 to 1. Therefore, mixed strategies are just special kinds of continuously variable strategies like those we studied in Chapter 5. Each pure strategy is an extreme special case where the probability of choosing that pure strategy equals 1.

The notion of Nash equilibrium also extends easily to include mixed strategies. In this case, Nash equilibrium is defined as the list of mixed strategies, one for each player, such that the choice of each is her best choice, in the sense of yielding the highest expected payoff for her, given the mixed strategies of the other players. Allowing for mixed strategies in a game solves the problem of nonexistent Nash equilibria, which we encountered for pure strategies, automatically and almost entirely. Nash’s celebrated theorem shows that, under very general circumstances (which are broad enough to cover all the games that we meet in this book and many more besides), a Nash equilibrium in mixed strategies exists.

At this broadest level, therefore, incorporating mixed strategies into our analyses does not entail anything different from the general theory of continuous strategies developed in Chapter 5. However, the special case of mixed strategies raises several special conceptual and methodological issues that deserve separate study.

Endnotes

- When a chance event has just two possible outcomes, people often speak of the odds in favor of or against one of the outcomes. If the two possible outcomes are labeled A and B, and the probability of A is p, so that the probability of B is $(1 - p)$, then the ratio $p/(1 - p)$ gives the odds in favor of A, and the reverse ratio $(1 - p)/p$ gives the odds against A. Thus, when Evert chooses CC with probability 0.25 (25%), the odds against her choosing CC are 3 to 1, and the odds in favor of it are 1 to 3. This terminology is often used in betting contexts, so those of you who misspent your youth in that way will be more familiar with it. However, this usage does not readily extend to situations in which three or more outcomes are possible, so we avoid it here. Return to reference 1
- Game theory assumes that players will calculate and try to maximize their expected payoffs when probabilistic mixtures of strategies or outcomes are included. It is important to remember that the word *expected* in *expected payoff* is a technical term from probability and statistics theory. It merely denotes a probability-weighted average. It does not refer to the payoff that a player should expect, in the sense of regarding it as her right or entitlement. Return to reference 2

Glossary

expected payoff
: The probability-weighted average (statistical mean or expectation) of the payoffs of one player in a game, corresponding to all possible realizations of a random choice of nature or mixed strategies of the players.

2 MIXING MOVES

We begin our analysis of games with mixed-strategy equilibria using the tennis-point example from Chapter 4 that we reintroduced above. This game does not have a Nash equilibrium in pure strategies. In this section, we show how extending the set of strategies to include mixed strategies remedies this deficiency, and we interpret the resulting equilibrium as one in which each player keeps the other guessing. In Figure 7.1, we reproduce the payoff matrix of Figure 4.17. For extra clarity, we also indicate Navratilova's payoff numbers (50, 20, 10, and 80) in blue whenever they appear in the text for the remainder of this chapter. We adopt this number-coloring convention as a way of helping you to understand the origin of the numbers in the equations that follow and so that you can more easily reproduce the analysis yourself.

		NAVRATILOVA	
		DL	CC
EVERT	DL	50, 50	80, 20
	CC	90, 10	20, 80

FIGURE 7.1 No Equilibrium in Pure Strategies

A. The Benefit of Mixing

In the tennis-point game illustrated in Figure 7.1, if Evert always chooses DL, Navratilova will then cover DL and hold Evert's payoff down to 50. Similarly, if Evert always chooses CC, Navratilova will choose to cover CC and hold Evert down to 20. If Evert can choose only one of her two basic (pure) strategies and Navratilova can predict that choice, Evert's better (or less bad) pure strategy will be DL, which will yield her a payoff of 50.

But suppose Evert is not restricted to using only pure strategies and can choose a mixed strategy, perhaps one in which her probability of playing DL on any one occasion is 75%, or 0.75; this makes her probability of playing CC 25%, or 0.25. Using the method outlined in Section 1, we can calculate *Navratilova*'s expected payoff against this mixture as

$0.75 \times 50 + 0.25 \times 10 = 37.5 + 2.5 = 40$ if she covers DL, and

$0.75 \times 20 + 0.25 \times 80 = 15 + 20 = 35$ if she covers CC.

If Evert chooses this 75–25 mix, the expected payoffs show that Navratilova can best exploit it by covering DL.

When Navratilova chooses DL to best exploit Evert's 75–25 mix, her choice works to Evert's disadvantage because this is a zero-sum game. Evert's expected payoffs are

$0.75 \times 50 + 0.25 \times 90 = 37.5 + 22.5 = 60$ if Navratilova covers DL, and

$0.75 \times 80 + 0.25 \times 20 = 60 + 5 = 65$ if Navratilova covers CC.

By choosing DL, Navratilova holds Evert down to 60 rather than 65. But notice that Evert's payoff with the mixture is still better than the 50 she would get by playing purely DL or the 20 she would get by playing purely CC.[3]

This 75–25 mix, while improving Evert's expected payoff relative to her pure strategies, does leave her strategy open to some exploitation by Navratilova. By choosing to cover DL, Navratilova can hold Evert down to a lower expected payoff than when she chooses to cover CC. Ideally, Evert would like to find a mix that would be exploitation-proof—a mix that would leave Navratilova no obvious choice of pure strategy to use against it. Evert's exploitation-proof mixture must have the property that Navratilova gets the same expected payoff against it by covering DL as by covering CC; that is, it must keep Navratilova indifferent between her two pure strategies. This property, called the opponent's indifference property, is the key to mixed-strategy equilibria in non-zero-sum games, as we will see later in this chapter.

Finding the exploitation-proof mixture requires taking a more general approach to describing Evert's mixed strategy so that we can solve algebraically for the appropriate probabilities. In this approach, we denote the probability of Evert choosing DL with the algebraic symbol p, so the probability of her choosing CC is $1 - p$. We refer to this mixture as Evert's p-mix for short.

Against the p-mix, Navratilova's expected payoffs are

$50p + 10(1 - p)$ if she covers DL, and

$20p + 80(1 - p)$ if she covers CC.

For Evert's strategy—her p-mix—to be exploitation-proof, these two expected payoffs for Navratilova should be equal. That implies $50p + 10(1 - p) = 20p + 80(1 - p)$; or $30p = 70(1 - p)$; or $100p = 70$; or $p = 0.7$. Thus, Evert's exploitation-proof mix uses DL with probability 70% and CC with probability 30%. With these probabilities, Navratilova gets the same expected payoff from each of her pure strategies and therefore cannot exploit any one of them to her advantage (or to Evert's disadvantage in this zero-sum game). And Evert's expected payoff from this mixed strategy is

$50 \times 0.7 + 90 \times 0.3 = 35 + 27 = 62$ if Navratilova covers DL, and also

$80 \times 0.7 + 20 \times 0.3 = 56 + 6 = 62$ if Navratilova covers CC.

This expected payoff is better than the 50 that Evert would get if she used the pure strategy DL and better than the 60 she would get from the 75–25 mixture. We now know this mixture is exploitation-proof, but is it Evert’s optimal or equilibrium mixture?

B. Best Responses and Equilibrium

To find the equilibrium mixtures in this game, we return to the method of best-response analysis originally described in Chapter 4 and extended to games with continuous strategies in Chapter 5. Our first task is to identify Evert's best response to—her best choice of p for—each of Navratilova's possible strategies. Since those strategies can also be mixed, they are similarly described by the probability with which Navratilova covers DL. We label this probability q, so $1 - q$ is the probability that Navratilova covers CC. We refer to Navratilova's mixed strategy as her q-mix, and we now look for Evert's best choice of p for each of Navratilova's possible choices of q.

Using Figure 7.1, we see that Evert's p-mix gets her the expected payoff

$50p + 90(1 - p)$ if Navratilova covers DL, and

$80p + 20(1 - p)$ if Navratilova covers CC.

Therefore, against Navratilova's q-mix, Evert's expected payoff is

$$[50p + 90(1 - p)]q + [80p + 20(1 - p)](1 - q).$$

Rearranging the terms, Evert's expected payoff becomes

$$[50q + 80(1 - q)]p + [90q + 20(1 - q)]\ (1 - p)$$

$$= [90q + 20(1 - q)] + [50q + 80(1 - q) - 90q - 20(1 - q)]p$$

$$= [20 + 70q] + [60 - 100q]p,$$

and we use this expected payoff to help us find Evert's best-response values of p.

We are trying to identify the p that maximizes Evert's payoff at each value of q, so the key question is how her expected payoff varies with p. What matters is the coefficient on p: $[60 - 100q]$. Specifically, it matters whether that coefficient is positive (in which case Evert's expected payoff increases as p increases) or negative (in which case Evert's expected payoff decreases as p increases). Clearly, the sign of the coefficient depends on q, the critical value of q being the one that makes $60 - 100q = 0$. That q value is 0.6.

Thus, when Navratilova's $q < 0.6$, $[60 - 100q]$ is positive, Evert's expected payoff increases as p increases, and her best choice is $p = 1$, or the pure strategy DL. Similarly, when Navratilova's $q > 0.6$, Evert's best choice is $p = 0$, or the pure strategy CC. If Navratilova's $q = 0.6$, Evert gets the same expected payoff regardless of p, and any mixture between DL and CC is just as good as any other; any p from 0 to 1 can be a best response. We summarize our results for this game for future reference:

If $q < 0.6$, the best response is $p = 1$ (pure DL).

If $q = 0.6$, any p-mix is a best response.

If $q > 0.6$, the best response is $p = 0$ (pure CC).

These expressions should confirm your intuition that when q is low (Navratilova is sufficiently unlikely to cover DL), Evert should choose DL, and when q is high (Navratilova is sufficiently likely to cover DL), Evert should choose CC. The exact sense of *sufficiently*, and therefore the switching point $q = 0.6$, depends, of course, on the specific payoffs in this particular example.

We pointed out earlier that mixed strategies are simply a special kind of continuous strategy, in which the probability is the continuous variable. Now we have found Evert's best p corresponding to each of Navratilova's choices of q. In other words, we have found Evert's best-response rule, and we can graph it exactly as we did for the games in Chapter 5.

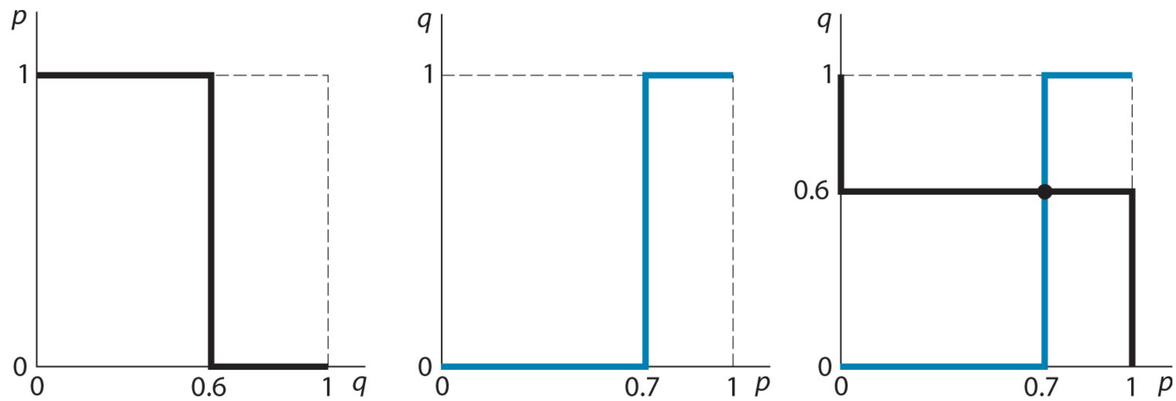

Figure 7.2 Best Responses and Equilibrium in the Tennis-Point Game

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We show this graph in the left-hand panel of Figure 7.2, with q on the horizontal axis and p on the vertical axis. Both variables are probabilities, limited to the range from 0 to 1. For $q < 0.6$, p is at its upper limit of 1; for $q > 0.6$, p is at its lower limit of 0. At $q = 0.6$, all values of p between 0 and 1 are equally "best" for Evert; therefore, the best-response curve is the vertical line between 0 and 1. This is a new flavor of best-response curve; unlike the steadily rising or falling lines or curves of Chapter 5, it is flat over two intervals of q and vertical at the point where those two intervals meet. But conceptually, it is just like any other best-response curve.

Navratilova's best-response rule—her best q-mix corresponding to each of Evert's p-mixes—can be calculated and graphed in the same way; we leave this for you to do so you can consolidate your understanding of the idea and the algebra. You should also check that your intuition regarding Navratilova's choices is consistent with what the calculations indicate, as we did for Evert's. We simply state the result here:

If $p < 0.7$, the best response is $q = 0$ (pure CC).

If $p = 0.7$, any q-mix is a best response.

If $p > 0.7$, the best response is $q = 1$ (pure DL).

This best-response rule for Navratilova is graphed in the middle panel of Figure 7.2.

The right-hand panel of Figure 7.2 combines the left-hand and middle panels by reflecting the left graph across the diagonal ($p = q$ line) so that p is on the horizontal axis and q on the vertical axis, then superimposing this graph on the middle graph. Now the two best-response curves meet at exactly one point, where $p = 0.7$ and $q = 0.6$. Here, each player's mixture is the best response to the other's mixture, so the pair of mixtures constitutes a Nash equilibrium in mixed strategies.

This representation of best-response rules includes pure strategies as special cases corresponding to the extreme values of p and q. We can see that the best-response curves do not have any points in common on any of the edges of the graph where each value of p and q equals either 0 or 1; this shows us that the game does not have any pure-strategy equilibria, as we determined directly in Section 8 of Chapter 4.[4] The mixed-strategy equilibrium in this example is the unique Nash equilibrium of the game.

You can also calculate Navratilova's exploitation-proof choice of q using the same method that we used in Section 2.A for finding Evert's exploitation-proof p. You will get the answer $q = 0.6$. Thus, the two exploitation-proof choices are indeed best responses to each other, and they are the Nash equilibrium mixtures for the two players.

In fact, if all you want to do is to find a mixed-strategy equilibrium of a zero-sum game where each player has just two pure strategies, you don't have to go through the detailed construction of best-response curves, graph them, and look for their intersection. You can write down the equations from Section 2.A for each player's exploitation-proof mixture and solve them. If both probabilities fall between 0 and 1, you have found what you want. If the solution includes a probability that is negative, or greater than 1, then the game does not have a mixed-strategy equilibrium; you should go back and look for a pure-strategy equilibrium. In Sections 6 and 7 we examine solution techniques for games where a player has more than two pure strategies.

Endnotes

- Not every mixed strategy will perform better than its constituent pure strategies. For example, if Evert chooses a 50–50 mixture of DL and CC, Navratilova can hold Evert’s expected payoff down to 50, exactly the same as Evert would get from pure DL. And a mixture that attaches a probability of less than 30% to DL will be worse for Evert than pure DL. We suggest that you verify these statements as a useful exercise to acquire the skill of calculating expected payoffs and comparing strategies. Return to reference 3
- If, in some game you are trying to solve, the best-response curves meet on an edge (but not exactly at a corner) of the graph, one player uses a pure strategy in equilibrium and the other mixes; if the best-response curves meet at a corner of the graph, both players use pure strategies in equilibrium. Return to reference 4

Glossary

opponent’s indifference property
: An equilibrium mixed strategy of one player in a two-person game has to be such that the other player is indifferent among all the pure strategies that are actually used in her mixture.

3 NASH EQUILIBRIUM AS A SYSTEM OF BELIEFS AND RESPONSES

When the moves in a game are simultaneous, neither player can respond to the other's actual choice. Instead, each takes her best action in light of what she thinks the other might be choosing to do at that instant. In Chapter 4, we called such thinking a player's belief about the other's strategy choice. We then interpreted Nash equilibrium as a configuration where such beliefs are correct, so that each player chooses her best response to the actual actions of the other. This concept proved useful for understanding the structures and outcomes of many important types of games, most notably the prisoners' dilemma, coordination games, and games of chicken.

In Chapter 4, however, we considered only pure-strategy Nash equilibria. Therefore, a hidden assumption went almost unremarked—namely, that each player was sure or confident in her belief that the other would choose a particular pure strategy. Now that we are considering mixed strategies, the concept of belief requires a corresponding reinterpretation.

Players may be unsure about what others might be doing. In the coordination game in Chapter 4, in which Sherlock Holmes wanted to meet Dr. Watson, he might be unsure whether his partner would go to St. Bart's or Simpson's, and his belief might be that there was a 50–50 chance that Watson would go to either one. And in the tennis-point example, Evert might recognize that Navratilova was trying to keep her (Evert) guessing and would therefore be unsure which of the available actions Navratilova would play. In Chapter 2, Section 2, we labeled this concept *strategic uncertainty*, and in Chapter 4, we mentioned that such uncertainty can give rise to mixed-strategy equilibria. Now we develop this idea more fully.

It is important, however, to distinguish between being unsure and having incorrect beliefs. For example, in the tennis-point example, Navratilova cannot be sure of what action Evert is choosing on any one occasion. But she can still have correct beliefs about Evert's mixture—namely, about the probabilities with which Evert chooses each of her two pure strategies. Having correct beliefs about mixed strategies means knowing or calculating or guessing the correct probabilities with which the other player chooses each of her underlying basic or pure actions. In our example, it turned out that Evert's equilibrium mixture was 70% DL and 30% CC. If Navratilova believes that Evert will play DL with 70% probability and CC with 30% probability, then her belief, although uncertain, will be correct in equilibrium.

Thus, we have an alternative and mathematically equivalent way to define Nash equilibrium in terms of beliefs: Each player forms beliefs about the probabilities with which the other is choosing his actions and chooses her own best response to those probabilities. A Nash equilibrium in mixed strategies occurs when those beliefs are correct in the sense just explained.

In the next section, we consider mixed strategies and their Nash equilibria in non-zero-sum games. In such games, there is no general reason that one player's pursuit of her own interests should work against her opponent's interests. Therefore, it is not generally the case that she would want to conceal her intentions from her opponent, and there is no general argument in favor of keeping her opponent guessing. However, because moves are simultaneous, each player may still be unsure of what action the other is taking, and may therefore have uncertain beliefs that make her unsure about how she should act. This situation can lead to mixed-strategy equilibria, and their interpretation in terms of subjectively uncertain but correct beliefs proves particularly important.

4 MIXING IN NON-ZERO-SUM GAMES

The same mathematical method used to find mixed-strategy equilibria in zero-sum games—namely, solution of equations derived from the opponent's indifference property—can be applied to non-zero-sum games as well, and it can reveal mixed-strategy equilibria in some of them. However, in such games, the players' interests may coincide to some extent. Therefore, the fact that one player will exploit her opponent's systematic choice of strategy to her advantage need not work out to her opponent's disadvantage, as was the case with zero-sum interactions. In a coordination game of the kind we studied in Chapter 4, for example, the players are better able to coordinate if each can rely on the other's acting systematically; random actions only increase the risk of coordination failure. As a result, mixed-strategy equilibria have a weaker rationale, and sometimes no rationale at all, in non-zero-sum games. Here, we examine mixed-strategy equilibria in some prominent non-zero-sum games and discuss their relevance, or lack thereof.

A. Will Holmes Meet Watson? Assurance, Pure Coordination, and Battle of the Sexes

We illustrate mixing in non-zero-sum games by using the assurance version of the meeting coordination game between Holmes and Watson. For your convenience, we reproduce its game table (Figure 4.14) here as Figure 7.3. We consider the game from Watson’s perspective first. If Watson is confident that Holmes will go to St. Bart’s, he, too, should go to St. Bart’s. If Watson is confident that Holmes will go to Simpson’s, so should he. But if Watson is unsure about Holmes’s choice, what is his own best choice?

To answer this question, we must give a more precise meaning to the uncertainty in Holmes’s mind. (The technical term for uncertainty in a player’s mind, in the theory of probability and statistics, is *subjective uncertainty*. In the context where the uncertainty is about another player’s action in a game, it is also strategic uncertainty, which we discussed in Chapter 2, Section 2.D.) We gain precision by stipulating the probability with which Watson thinks Holmes will choose one meeting place or the other. The probability of Holmes choosing St. Bart’s can be any real number between 0 and 1 (that is, between 0% and 100%). We cover all possible cases by using algebra, letting the symbol p denote the probability (in Watson’s mind) that Holmes will choose St. Bart’s; the variable p can take on any real value between 0 and 1. Then $(1 - p)$ is the probability (again, in Watson’s mind) that Holmes will choose Simpson’s. In other words, we describe Watson’s strategic uncertainty as follows: He thinks that Holmes is using a mixed strategy, mixing the two pure strategies, St. Bart’s and Simpson’s, in proportions or probabilities p and $(1 - p)$, respectively. We call this mixed strategy Holmes' s p-mix, even though for the moment it is purely an idea in Watson' s mind.

		WATSON	
		St. Bart' s	Simpson' s
HOLMES	St. Bart' s	1, 1	0, 0
	Simpson' s	0, 0	2, 2

FIGURE 7.3 Assurance

Given his uncertainty, Watson can calculate the expected payoffs from his own actions when they are played against his belief about Holmes' s p-mix. If Watson chooses St. Bart' s, it will yield him $1 \times p + 0 \times (1 - p) = p$. If he chooses Simpson' s, it will yield him $0 \times p + 2 \times (1 - p) = 2 \times (1 - p)$. When p is high, $p > 2(1 - p)$; so if Watson is fairly sure that Holmes is going to St. Bart' s, then he does better by also going to St. Bart' s. Similarly, when p is low, $p < 2(1 - p)$; if Watson is fairly sure that Holmes is going to Simpson' s, then he does better by going to Simpson' s. If $p = 2(1 - p)$, or $3p = 2$, or $p = ⅔$, the two choices give Watson the same expected payoffs. Therefore, if he believes that $p = ⅔$, he might be unsure about his own choice, so he might dither between the two.

Holmes can figure this out, and that makes him unsure about Watson' s choice. Thus, Holmes also faces subjective strategic uncertainty. Suppose, in Holmes' s mind, Watson will choose St. Bart' s with probability q and Simpson' s with probability $(1 - q)$. Similar reasoning shows that Holmes should choose St. Bart' s if he believes Watson' s $q > ⅔$ and Simpson' s if he believes $q < ⅔$. If Holmes believes $q = ⅔$, he will be indifferent between the two actions and unsure about his own choice.

Now we have the basis for a mixed-strategy equilibrium with $p = 2/3$ and $q = 2/3$. In such an equilibrium, these p and q values are simultaneously the actual mixture probabilities and the subjective beliefs of each player about the other's mixture probabilities. The correct beliefs sustain each player's own indifference between the two pure strategies and therefore each player's willingness to mix the two. This situation matches exactly the concept of a Nash equilibrium as a system of self-fulfilling beliefs and responses, as described in Section 3.

The key to finding the mixed-strategy equilibrium is that Watson is willing to mix his two pure strategies only if his subjective uncertainty about Holmes's choice is just right—that is, if the value of p in Holmes's p-mix is just right. Algebraically, this idea is borne out by solving for the equilibrium value of p by using the equation $p = 2(1 - p)$, which ensures that Watson gets the same expected payoff from each of his two pure strategies when each is matched against Holmes's p-mix. When the equation holds in equilibrium, it is as if Holmes's mixture probabilities are doing the job of keeping Watson indifferent. We emphasize *as if* because in this game, Holmes has no reason to keep Watson indifferent; that outcome is merely a property of the equilibrium. Still, the general idea is worth remembering: In a mixed-strategy Nash equilibrium, each person's mixture probabilities keep the other player indifferent between his pure strategies. We derived the opponent's indifference property in the discussion of zero-sum games above, and now we see that it remains valid even in non-zero-sum games.

However, the mixed-strategy equilibrium has some very undesirable properties in the assurance game. First, it yields both players rather low expected payoffs. Watson's expected payoffs from his two actions, p and $2(1 - p)$, both equal $2/3$ when $p = 2/3$. Similarly, Holmes's expected payoffs against Watson's equilibrium q-mix for $q = 2/3$ are also both

⅔. Thus, each player gets ⅔ in the mixed-strategy equilibrium. In Chapter 4, we found two pure-strategy equilibria for this game; even the worse of them (both choosing St. Bart's) yields the players 1 each, and the better one (both choosing Simpson's) yields them 2 each.

The reason the two players fare so badly in the mixed-strategy equilibrium is that when they choose their actions independently and randomly, they create a significant probability of going to different places; when that happens, they do not meet, and each gets a payoff of 0. Holmes and Watson fail to meet if one goes to St. Bart's and the other goes to Simpson's, or vice versa. The probability of this happening when both are using their equilibrium mixtures is $2 \times (⅔) \times (⅓) = 4/9$.[5] Similar problems exist in the mixed-strategy equilibria of most non-zero-sum games.

A second undesirable property of the mixed-strategy equilibrium here is that it is very unstable. If either player departs ever so slightly from the exact values p = ⅔ or q =⅔, the best choice of the other tips to one pure strategy. Once one player chooses a pure strategy, then the other also does better by choosing the same pure strategy, and play moves to one of the two pure-strategy equilibria. This instability of mixed-strategy equilibria is also common to many non-zero-sum games. However, some important non-zero-sum games do have mixed-strategy equilibria that are not so fragile. One example, considered later in this chapter and in Chapter 12, is the mixed-strategy equilibrium in the game of chicken, which has an interesting evolutionary interpretation.

Given this analysis of the mixed-strategy equilibrium in the assurance version of the meeting coordination game, you can now probably guess the mixed-strategy equilibria for the related non-zero-sum meeting games. In the pure-coordination version (see Figure 4.13), the payoffs from meeting in the two places are the same, so the mixed-strategy equilibrium will have $p = ½$ and $q = ½$. In the battle-of-the-sexes version (see Figure 4.15), Watson prefers to meet at Simpson's because his payoff is 2 rather than the 1 that he gets from meeting at St. Bart's. Watson's decision hinges on whether his subjective probability of Holmes's going to St. Bart's is greater than or less than ⅔. (Watson's payoffs here are similar to those in the assurance version, so the critical p is the same.) Holmes prefers to meet at St. Bart's, so his decision hinges on whether his subjective probability of Watson's going to St. Bart's is greater than or less than ⅓. Therefore, the mixed-strategy Nash equilibrium again has $p = ⅔$ and $q = ⅓$.

		DEAN	
		Swerve (Chicken)	Straight (Tough)
JAMES	Swerve (Chicken)	0, 0	−1, 1
	Straight (Tough)	1, −1	−2, −2

FIGURE 7.4 Chicken

B. Will James Meet Dean? Chicken

The non-zero-sum game of chicken also has a mixed-strategy equilibrium that can be found using the method developed above, although its interpretations are slightly different. Recall that this is a game between James and Dean, who are trying to *avoid* a meeting; the game table, originally introduced in Figure 4.16, is reproduced here as Figure 7.4.

If we introduce mixed strategies into this game, James's p-mix will entail a probability p of swerving and a probability $1 - p$ of going straight. Against that p-mix, Dean gets expected payoffs of $0 \times p - 1 \times (1 - p) = p - 1$ if he chooses Swerve and $1 \times p - 2 \times (1 - p) = 3p - 2$ if he chooses Straight. Comparing the two, we see that Dean does better by choosing Swerve when $p - 1 > 3p - 2$, or when $2p < 1$, or when $p < ½$; that is, when p is low and James is more likely to choose Straight. Conversely, when p is high and James is more likely to choose Swerve, then Dean does better by choosing Straight. If James's p-mix has p exactly equal to ½, then Dean is indifferent between his two pure actions; he is therefore equally willing to use a mixture of the two. Similar analysis of the game from James's perspective when considering his options against Dean's q-mix yields the same results. Therefore, $p = ½$ and $q = ½$ is a mixed-strategy equilibrium of this game.

The properties of this equilibrium have some similarities to, but also some differences from, the mixed-strategy equilibrium of the Holmes–Watson assurance game. Here, each player's expected payoff in the mixed-strategy equilibrium is low ($-½$). This is bad for James and Dean, as was the case with the expected payoff of ⅔ for Holmes and Watson in the assurance game, but in this game of chicken, the mixed-strategy equilibrium payoff is not worse for both players than either of the two pure-strategy equilibrium payoffs. In fact, because player interests are somewhat opposed here, each player will do strictly better in the mixed-strategy equilibrium than in the pure-strategy equilibrium that entails his choosing Swerve.

This mixed-strategy equilibrium is again unstable, however. If James increases his probability of choosing Straight to just slightly above ½, this change tips Dean's choice to pure Swerve. Then (Straight, Swerve) becomes the pure-strategy equilibrium. If James instead lowers his probability of choosing Straight slightly below ½, Dean chooses Straight, and the game goes to the other pure-strategy equilibrium.[6]

In this section, we found mixed-strategy equilibria in several non-zero-sum games by solving the equations that come from the opponent's indifference property. We already know from Chapter 4 that these games also have equilibria in pure strategies. Best-response curves can give us a comprehensive picture, displaying all Nash equilibria at once. As you already know all of the equilibria from the two separate analyses, we do not spend time and space graphing the best-response curves here. We merely note that when there are two pure-strategy equilibria and one mixed-strategy equilibrium, as in the previous examples, you will find that the best-response curves cross in three different places, one for each of the Nash equilibria. We will also invite you to graph best-response curves for similar games in the exercises at the end of this chapter.

Endnotes

- The probability that each chooses St. Bart's in equilibrium is $\frac{2}{3}$. The probability that each chooses Simpson's is $\frac{1}{3}$. The probability that one chooses St. Bart's while the other chooses Simpson's is $(\frac{2}{3}) \times (\frac{1}{3})$. But that can happen two different ways (once when Holmes chooses St. Bart's and Watson chooses Simpson's, and again when the choices are reversed) so the total probability of not meeting is $2 \times (\frac{2}{3}) \times (\frac{1}{3})$. See the appendix to this chapter for more details on the algebra of probabilities. Return to reference 5
- In Chapter 12, we consider a different kind of stability—namely, evolutionary stability. The question in the evolutionary context is whether a stable mix of Straight and Swerve choosers can arise and persist in a population of chicken players. The answer is yes, and the proportions of the two types are exactly equal to the probabilities of playing each action in the mixed-strategy equilibrium. Thus, we derive a new and different motivation for that equilibrium in this game. Return to reference 6

5 GENERAL DISCUSSION OF MIXED-STRATEGY EQUILIBRIA

Now that we have seen how to find mixed-strategy equilibria in both zero-sum and non-zero-sum games, it is worthwhile to consider these equilibria in more detail. In particular, we highlight in this section some general properties of mixed-strategy equilibria and some initially counterintuitive aspects of such equilibria.

A. Weak Sense of Equilibrium

The opponent's indifference property described in Section 2 implies that in a mixed-strategy equilibrium, each player gets the same expected payoff from each of her two pure strategies, and therefore also gets the same expected payoff from any mixture between them. Thus, mixed-strategy equilibria are Nash equilibria only in a weak sense. When one player is choosing her equilibrium mix, the other has no positive reason to deviate from her own equilibrium mix. But she would not do any worse if she chose another mix or even one of her pure strategies. Each player is indifferent between her pure strategies and, indeed, any mixture of them, so long as the other player is playing her correct (equilibrium) mix.

This property may at first seem to undermine the basis for mixed-strategy Nash equilibria as a solution concept for games. Why should a player choose her appropriate mixture when the other player is choosing her own? Why not just do the simpler thing and choose one of her pure strategies? After all, the expected payoff is the same. The answer is that playing a pure strategy would give the other player an incentive to deviate from her own mixture.

For instance, in the tennis-point game played between Evert and Navratilova, imagine that Evert said to herself, "When Navratilova is choosing her best mix (q = 0.6), I get the same payoff from DL, CC, or any mixture. So why bother to mix; why don't I just play DL?" If Evert followed through on this plan, Navratilova would undoubtedly notice Evert's consistent play and switch herself to her pure strategy of covering DL, making Evert worse off than in the mixed-strategy equilibrium. Consequently, Evert has an incentive to mix in order to keep Navratilova guessing.

On the other hand, if Holmes chooses pure St. Bart’s in the assurance game, then Watson’s response (also playing pure St. Bart’s) gives both players a higher payoff than in the mixed-strategy equilibrium, in which both players go to St. Bart’s two-thirds of the time. Moreover, the outcome where both players always go to St. Bart’s is itself a pure-strategy Nash equilibrium. We will return to discuss this difference between the tennis-point game and the assurance game in Chapter 12, where we will introduce the concept of evolutionary stability. As we will show in that analysis, the mixed-strategy equilibrium of the tennis-point game is evolutionarily stable, but the mixed-strategy equilibrium of the assurance game is evolutionarily unstable.

B. Counterintuitive Changes in Mixture Probabilities with Changes in Payoffs

Games with mixed-strategy equilibria may exhibit some features that seem counterintuitive at first glance. The most interesting of them is the change in the equilibrium mixes that follows a change in the structure of a game's payoffs. To illustrate, we return to Evert and Navratilova and their tennis-point game.

Suppose that Navratilova works on improving her skill at covering down the line to the point where Evert's success when using her DL strategy against Navratilova's covering DL drops from 50% to 30%. This improvement in Navratilova's skill alters the payoff table from that in Figure 7.1. We present the new table in Figure 7.5.

		NAVRATILOVA	
		DL	CC
EVERT	DL	30, 70	80, 20
	CC	90, 10	20, 80

FIGURE 7.5 Changed Payoffs in the Tennis-Point Game

The only change from the table in Figure 7.1 has occurred in the upper-left cell of Figure 7.5, where our earlier 50 for Evert is now a 30 and the 50 for Navratilova is now a 70. This change in the payoff table does not lead to a game with a pure-strategy equilibrium because the players still have opposing interests: Navratilova still wants their choices to coincide, and Evert still wants their choices to differ. We still have a game in which mixing will occur.

But how will the equilibrium mixes in this new game differ from those calculated in Section 2? At first glance, many people would argue that Navratilova should cover DL more often now that she has gotten so much better at doing so. Thus, the assumption is that her equilibrium q-mix should be more heavily weighted toward DL, and her equilibrium q should be higher than the 0.6 calculated before.

But when we calculate Navratilova's q-mix that will keep Evert indifferent between her two pure strategies, we get $30q + 80(1 - q) = 90q + 20(1 - q)$, or $q = 0.5$. Thus, the actual equilibrium value for q, 50%, has exactly the opposite relation to the original q of 60% than what many people's intuition predicts.

Although our intuition seems reasonable, it misses an important aspect of the theory of strategy: the interaction between the two players. Evert reassesses her equilibrium mix after the change in payoffs, and Navratilova must take the new payoff structure *and* Evert's behavior into account when determining her new equilibrium mix. Specifically, because Navratilova is now so much better at covering DL, Evert uses CC more often in her mix. To counter that, Navratilova covers CC more often.

We can see this more explicitly by calculating Evert's new mixture. Her equilibrium p must equate Navratilova's expected payoff from covering DL, $70p + 10(1 - p)$, with her expected payoff from covering CC, $20p + 80(1 - p)$. So we have $70p + 10(1 - p) = 20p + 80(1 - p)$, or $90 - 60p = 20 + 60p$, or $120p = 70$. Thus, Evert's p must be 7/12, which is 0.583, or 58.3%. Comparing this new equilibrium p with the original 70% calculated in Section 2 shows that Evert has significantly decreased the number of times she sends her shot DL in response to Navratilova's improved skills. Evert has taken into account the fact that she is now facing an opponent with better DL coverage, and so she does better to play DL less frequently in her mixture. By virtue of this behavior, Evert makes it better for Navratilova to decrease the frequency of her DL play. Evert would now exploit any other choice of mix by Navratilova, particularly a mix heavily favoring DL.

So is Navratilova's skill improvement wasted? No, but we must judge it properly—not by how often one strategy or the other gets used, but by the resulting payoffs. When Navratilova uses her new equilibrium mix with $q = 0.5$, Evert's expected payoff from either of her pure strategies is $(30 \times 0.5) + (80 \times 0.5) = (90 \times 0.5) + (20 \times 0.5) = 55$. This is less than Evert's expected payoff of 62 in the original example. Thus, Navratilova's average expected payoff rises from 38 to 45, and she does benefit by improving her DL coverage.

Unlike the counterintuitive result that we saw when we considered Navratilova's strategic response to the change in payoffs, we see here that her response is absolutely intuitive when considered in light of her expected payoff. In fact, players' expected-payoff responses to changed payoffs can never be counterintuitive, although their strategic responses, as we have seen, can be.[7] The most interesting aspect of this counterintuitive outcome in players' strategic responses is the message that it sends to tennis players and to strategic game players more generally: Navratilova should improve her down-the-line coverage so that she does not have to use it so often.

Next, we present an even more general, and more surprising, result of changes in mixture probabilities. The opponent's indifference property means that each player's equilibrium mixture depends only on the other player's payoffs, not on her own. Consider the assurance game in Figure 7.3. Suppose Watson's payoff from meeting at Simpson's increases from 2 to 3, while all other payoffs remain unchanged. Now, against

Holmes's p-mix, Watson gets $1 \times p + 0 \times (1 - p) = p$ if he chooses St. Bart's, and $0 \times p + 3 \times (1 - p) = 3 - 3p$ if he chooses Simpson's. Watson is indifferent between his two pure strategies when $p = 3 - 3p$, or $4p = 3$, or $p = 3/4$, compared with the value of two-thirds we found earlier for Holmes's p-mix in the original game. The calculation of Holmes's indifference condition is unchanged and yields $q = 2/3$ for Watson's equilibrium strategy. The change in Watson's payoffs changes Holmes's mixture probabilities, not Watson's! In Exercise S13, you will have the opportunity to prove that this is true quite generally: A player's equilibrium mixing proportions do not change with his own payoffs, only with his opponent's payoffs.

C. Risky and Safe Choices in Zero-Sum Games

In sports, some strategies are relatively safe; they do not fail disastrously even if anticipated by the opponent, but they don’t do very much better when unanticipated. Other strategies are risky; they do brilliantly if the opponent is not prepared for them, but fail miserably if the opponent is ready. In American football, on third down with a yard to go, a run up the middle is safe, and a long pass is risky. An interesting question arises because in some third-and-one situations, there is more at stake than in others. For example, making the play from your opponent’s 10-yard line has a much greater effect on your chance of scoring than making the play from your own 20-yard line. The question is, when the stakes are higher, should you play the risky strategy more or less often than when the stakes are lower?

To make this question concrete, consider the success probabilities shown in Figure 7.6. (Note that, whereas in the tennis-point game we used percentages between 0 and 100, here we use probabilities between 0 and 1.) The offense’s safe play is the run; the probability of a successful first down is 0.6 if the defense anticipates a run versus 0.7 if the defense anticipates a pass. The offense’s risky play is the pass because its probability of success depends much more on what the defense does: Its probability of success is 0.8 if the defense anticipates a run, but only 0.3 if it anticipates a pass.

		DEFENSE EXPECTS	
		Run	Pass
OFFENSE PLAYS	Run	0.6	0.7

	DEFENSE EXPECTS		
		Run	Pass
	Pass	0.8	0.3

FIGURE 7.6 Probability of Offense's Success on Third Down with One Yard to Go

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		DEFENSE	
		Run	Pass
OFFENSE	Run	0.6V, −0.6V	0.7V, −0.7V
	Pass	0.8V, −0.8V	0.3V, −0.3V

FIGURE 7.7 The Third-and-One Game

Suppose that when the offense succeeds with its play, it earns a payoff equal to V, and if the play fails, the payoff is 0. The payoff V could be some number of points, such as three for a field goal or seven for a touchdown. Alternatively, it could represent some amount of status or money that the team earns, perhaps $V = 100$ for succeeding in a game-winning play in an ordinary game or $V = 1{,}000{,}000$ for clinching victory in the Super Bowl.[8]

The payoff table for the game between Offense and Defense, illustrated in Figure 7.7, shows expected payoffs for each team. The expected payoff in each case is the weighted average of the success payoff V and the failure payoff 0. For example, the expected payoff to Offense for playing Run when Defense expects Run is $0.6 \times V + 0.4 \times 0 = 0.6V$. The zero-sum nature of the game means that Defense's payoff in the same cell is $-0.6V$. You can similarly compute the expected payoffs for each cell of the table to verify that the payoffs shown below are correct.

In the mixed-strategy equilibrium, Offense's probability p of choosing Run is determined by the opponent's indifference property. The correct p therefore satisfies

$$p[-0.6V] + (1 - p)[-0.8V] = p[-0.7V] + (1 - p)[-0.3V].$$

Notice that we can divide both sides of this equation by V to eliminate V entirely from the calculation for p. The equation becomes $-0.6p - 0.8(1 - p) = -0.7p - 0.3(1 - p)$, or $0.1p = 0.5(1 - p)$.[9] Solving this reduced equation yields $p = 5/6$, so Offense will play Run with high probability in its optimal mixture. This safer play is often called the *percentage play* because it is the normal play in such situations. The risky play (Pass) is played only occasionally to keep the opponent guessing or, in football commentators' terminology, "to keep the defense honest."

The interesting part of this result is that the expression for p is completely independent of V. That is, the theory says that you should mix the percentage play and the risky play in exactly the same proportions on a big occasion as you would on a minor occasion. This result runs against the intuition of many people. They think that the risky play should be engaged in less often when the occasion is more important. Throwing a long pass on third down with a yard to go may be fine on an ordinary Sunday afternoon in October, but doing so in the Super Bowl is too risky.

So which is right: theory or intuition? We suspect that readers will be divided on this issue. Some will think that the sports commentators are wrong about playing it safer in more important games and will be glad to have found a theoretical argument to refute their claims. Others will side with the commentators and argue that bigger occasions call for safer play. Still others may think that bigger risks should be taken when the prizes are bigger, but even they will find no support in the theory, which says that the size of the prize or the loss should make no difference to the mixture probabilities.

In many previous parts of this book where discrepancies between theory and intuition arose, we argued that the discrepancies were only apparent, that they were the result of failing to make the theory sufficiently general or rich enough to capture all the features of the situation that created the intuition, and that improving the theory removed the discrepancy. This situation is different: The problem is fundamental to the calculation of payoffs from mixed strategies as probability-weighted averages or expected payoffs. And almost all of existing game theory has this starting point.[10]

Endnotes

- For a general theory of the effect that changing the payoff in a particular cell of a payoff table has on the equilibrium mixture and the expected payoffs in equilibrium, see Vincent Crawford and Dennis Smallwood, "Comparative Statics of Mixed-Strategy Equilibria in Noncooperative Games," *Theory and Decision*, vol. 16 (May 1984), pp. 225–32. Return to reference 7
- Note that V is not necessarily a monetary amount. It can capture other relevant aspects of payoffs to the team, such as aversion to risk or loss (spelled out and used in greater detail in Chapters 9 and 14), effect of criticism by fans and journalists, and so on. Return to reference 8
- This result comes from the fact that we can eliminate V entirely from the opponent's indifference equation, so it does not depend on the particular success probabilities specified in Figure 7.6. The result is therefore quite general for mixed-strategy games where each payoff equals a success probability times a success value. Return to reference 9
- Vincent P. Crawford, "Equilibrium without Independence," *Journal of Economic Theory*, vol. 50, no. 1 (February 1990), pp. 127–54; and James Dow and Sergio Werlang, "Nash Equilibrium under Knightian Uncertainty," *Journal of Economic Theory*, vol. 64, no. 2 (December 1994), pp. 305–24, are among the few research papers that suggest alternative foundations for game theory. And our exposition of this problem in the first edition of this book inspired an article that uses such new methods on it: Simon Grant, Atsushi Kaji, and Ben Polak, "Third Down and a Yard to Go: Recursive Expected Utility and the Dixit-Skeath Conundrum," *Economic Letters*, vol. 73, no. 3 (December 2001), pp. 275–86. Unfortunately, the article uses concepts more advanced

than those available at the introductory level of this book. Return to reference 10

6 MIXING WHEN ONE PLAYER HAS THREE OR MORE PURE STRATEGIES

Our discussion of mixed strategies up to this point has been confined to games in which each player has available only two pure strategies and mixes between them. In many strategic situations, each player has available a larger number of pure strategies, and we should be ready to calculate equilibrium mixes for those cases as well. However, these calculations get complicated quickly. For truly complex games, we would turn to a computer to find the mixed-strategy equilibrium. But for some small games, it is possible to calculate equilibria by hand quite easily. The calculation process gives us a better understanding of how the equilibrium works than can be obtained just from looking at a computer-generated solution. Therefore, in this section and the next one, we solve some of these larger games.

In this section, we consider zero-sum games in which one of the players has two pure strategies, whereas the other player has more. In such games, we find that the player who has three (or more) pure strategies typically uses only two of them in equilibrium. The others do not figure in his mix; they have probabilities of 0. We must determine which strategies are used and which ones are not.[11]

Our example is that of the tennis-point game, augmented here by giving Evert a third type of passing shot. In addition to passing down the line or crosscourt, she now can consider using a lob (a slower, but higher and longer, passing shot). The equilibrium of the game depends on the payoffs of the lob against each of Navratilova’ s two defensive choices. We begin with the case that is most likely to arise, then consider a coincidental or exceptional case.

A. A General Case

Evert now has three pure strategies in her repertoire: DL, CC, and Lob. We leave Navratilova with just two pure strategies, Cover DL or Cover CC. The payoff table for this new game can be obtained by adding a Lob row to Figure 7.1; the result is shown in Figure 7.8. We have assumed that Evert's payoffs from Lob are between the best and the worst she can get with DL and CC, and that they vary little with Navratilova's choices between DL and CC. We have shown not only the payoffs from all the pure strategies, but also those for Evert's three pure strategies against Navratilova's q-mix. [We do not show a row for Evert's p-mix because we don't need it. It would require two probabilities—say, p_1 for DL and p_2 for CC, and then the probability for Lob would be $(1 - p_1 - p_2)$. We will show you how to solve for equilibrium mixtures of this type in the following section.]

		NAVRATILOVA		
		DL	CC	q-mix
EVERT	DL	50, 50	80, 20	$50q + 80(1 - q)$, $50q + 20(1 - q)$
	CC	90, 10	20, 80	$90q + 20(1 - q)$, $10q + 80(1 - q)$
	Lob	70, 30	60, 40	$70q + 60(1 - q)$, $30q + 40(1 - q)$

You may need to scroll left and right to see the full figure.

FIGURE 7.8 Payoff Table for the Tennis-Point Game with Lob

Technically, before we begin looking for a mixed-strategy equilibrium, we should verify that this game has no pure-strategy equilibrium. This is easy to do, however, so we leave it to you and turn to mixed strategies.

We will use the logic of best responses to consider Navratilova's optimal choice of q. In Figure 7.9, we show Evert's expected payoffs (success percentages) from playing each of her pure strategies DL, CC, and Lob as the q in Navratilova's q-mix varies over its full range from 0 to 1. The payoff lines are just graphs of Evert's payoff expressions in the right-hand column of Figure 7.8. For each q, if Navratilova were to choose that q-mix in equilibrium, Evert's best response would be to choose the strategy that gives her (Evert) the highest payoff. We show this set of best-response outcomes for Evert with the thicker line in Figure 7.9; in mathematical jargon, this is the *upper envelope* of the three payoff lines. Navratilova wants to choose her own best possible q—the q that makes her own payoff as high as possible (thereby making Evert's payoff as low as possible)—from this set of Evert's best responses.

To be more precise about Navratilova's optimal choice of q, we must calculate the coordinates of the kink points in the thicker line showing her worst-case (Evert's best-case) outcomes. The value of q at the leftmost kink in this line makes Evert indifferent between DL and Lob. That q must equate the two payoffs from DL and Lob when used against the q-mix. Setting those two expressions equal gives us $50q + 80(1 - q) = 70q + 60(1 - q)$, or $q = 20/40 = ½ = 50\%$. Evert's expected payoff at this point is $50 \times 0.5 + 80 \times 0.5 = 70 \times 0.5 + 60 \times 0.5 = 65$. At the second (rightmost) kink, Evert is indifferent between CC and Lob. Thus, the q value at this kink is the one that equates the CC and Lob payoff expressions. Setting $90q + 20(1 - q) = 70q + 60(1 - q)$, we find $q = 40/60 = ⅔ = 66.7\%$. Here, Evert's expected payoff is $90 \times 0.667 + 20 \times 0.333 = 70 \times 0.667 + 60 \times 0.333 = 66.67$. Therefore, Navratilova's best (or least bad) choice of q is at the left kink, namely, $q = 0.5$. Evert's expected payoff is 65, so Navratilova's is 35.

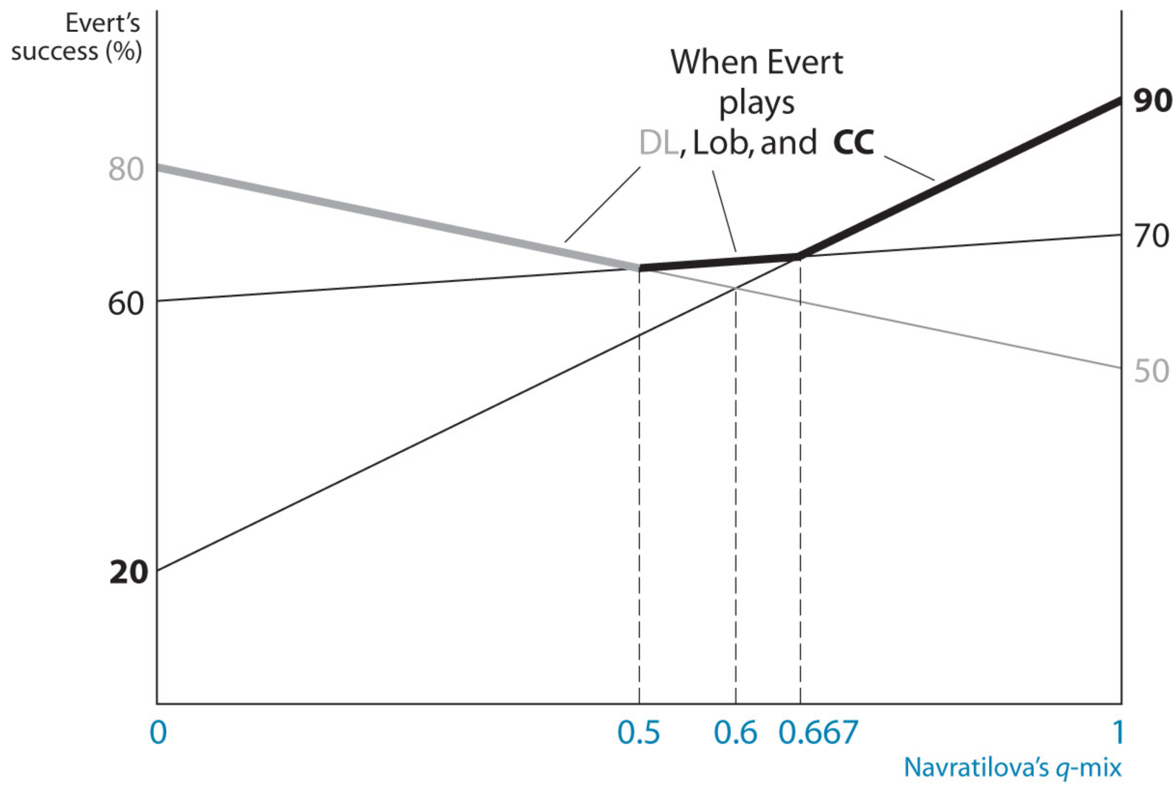

Figure 7.9 Graphical Solution for Navratilova's q-Mix

When Navratilova chooses $q = 0.5$, Evert is indifferent between DL and Lob, and either of these choices gives her a better payoff than does CC. Therefore, Evert will not use CC at all in equilibrium. CC will be an unused strategy in her equilibrium mix.

Now we can proceed with the equilibrium analysis as if this were a game with just two pure strategies for each player: DL and CC for Navratilova, and DL and Lob for Evert. We are back in familiar territory. Therefore, we leave the calculation to you and just tell you the result. Evert's optimal mixture in this game entails her using DL with probability 0.25 and Lob with probability 0.75. Evert's expected payoff from this mixture, against Navratilova's DL and CC, is $50 \times 0.25 + 70 \times 0.75 = 80 \times 0.25 + 60 \times 0.75 = 65$, as of course it should be.

We could not have started our analysis with this two-by-two game because we did not know in advance which of her three strategies Evert would not use. But we can be confident that in the general case, a player in Evert’ s situation will mix between just two strategies rather than mixing among all three. If we consider all the possible combinations of values that the payoff numbers could take, the resulting three expected payoff lines would cross at a single point only in exceptional cases; generally, they will intersect pair by pair. Then the upper envelope has the shape that we see in Figure 7.9. Its lowest point is defined by the intersection of the payoff lines associated with two of the three strategies and these are the two strategies that will be used in the player’ s equilibrium mix. The payoff from the third strategy lies below the intersection at this point, so the player choosing among the three strategies does not use that third one.

B. Exceptional Cases

The positions and intersections of the three lines of Figure 7.9 depend on the payoffs specified for the pure strategies. We chose the payoffs for that particular game to show a general configuration of payoff lines. But if the payoffs stand in other, very specific relationships to each other, we can get some exceptional configurations with different results. We describe the possibilities here, but leave it to you to redraw the graph for these cases.

First, if Evert's payoffs from Lob against Navratilova's DL and CC are equal, then the line for Lob is horizontal, and a whole range of q-values make Navratilova's mixture exploitation-proof. For example, if the two payoffs in the Lob row of the table in Figure 7.8 are 70 each, then it is easy to calculate that the left kink in a revised Figure 7.9 would be at $q = 1/3$ and the right kink at $q = 5/7$. For any q in the range from 1/3 to 5/7, Evert's best response is Lob, and we get an unusual equilibrium in which Evert plays a pure strategy and Navratilova mixes. Further, Navratilova's equilibrium mixture probabilities are indeterminate within the range from $q = 1/3$ to $q = 5/7$.

Second, if Evert's payoffs from Lob against Navratilova's DL and CC are lower than those of Figure 7.8 by just the right amounts (or those of Evert's other two strategies are higher by just the right amounts), all three lines can meet at one point. For example, if the payoffs of Evert's Lob are 66 and 56 against Navratilova's DL and CC, respectively, instead of 70 and 60, then for $q = 0.6$, Evert's expected payoff from Lob becomes $66 \times 0.6 + 56 \times 0.4 = 39.6 + 22.6 = 62$, the same as that from DL and CC when $q = 0.6$. So Evert is indifferent among all three of her strategies when $q = 0.6$ and is willing to mix among all three.

In this special case, Evert's equilibrium mixture probabilities are not fully determinate. Rather, a whole range of mixtures, including some where all three strategies are used, can do the job of keeping Navratilova indifferent between her DL and CC and therefore willing to mix. However, Navratilova must use a mixture with $q = 0.6$. If she does not, Evert's best response will be to switch to one of her pure strategies, and this will work to Navratilova's detriment. We do not dwell on the determination of the precise range over which Evert's equilibrium mixtures can vary, because this case can arise only for exceptional combinations of the payoff numbers and is therefore relatively unimportant.

Note that Evert's payoffs from using her Lob against Navratilova's DL and CC could be even lower than the values that make all three lines intersect at one point (for example, if the payoffs from Lob were 75 and 30 instead of 70 and 60 as in Figure 7.8). Then Lob is never the best response for Evert even though it is not dominated by either DL or CC. This case of Lob being dominated by a *mixture* of DL and CC is explained in the online appendix to this chapter, available at digital.wwnorton.com/gamesofstrategy5.

Endnotes

- Even when a player has only two pure strategies, he may not use one of them in equilibrium. The other player then generally finds one of his strategies to be better against the one that the first player does use. In other words, the equilibrium "mixtures" collapse to the special case of pure strategies. But when one or both players have three or more strategies, we can have a genuinely mixed-strategy equilibrium where some of the pure strategies go unused.
Return to reference 11

7 MIXING WHEN BOTH PLAYERS HAVE THREE STRATEGIES

When we consider two-player games in which both players have three pure strategies and are considering mixing among all three, we need two variables to specify each mix.[12] The row player’s p-mix would put probability p_1 on her first pure strategy and probability p_2 on her second pure strategy. Then the probability of her using the third pure strategy would equal 1 minus the sum of the probabilities of the other two. The same would be true for the column player’s q-mix. So when both players have three strategies, we cannot find a mixed-strategy equilibrium without doing two-variable algebra. In many cases, however, such algebra is still manageable.

A. Full Mixtures of All Strategies

Consider a simplified representation of a penalty kick in soccer. Suppose a right-footed kicker has just three pure strategies: kick to the left, right, or center. (Left and right refer to the goalie's left or right. For a right-footed kicker, the most natural motion would send the ball to the goalie's right.) Then he can mix among these strategies, with probabilities denoted by p_L, p_R, and p_C, respectively. Any two of them can be taken to be the independent variables and the third expressed in terms of them. If p_L and p_R are the two independent variables, then $p_C = 1 - p_L - p_R$. The goalie also has three pure strategies—namely, move to the kicker's left (the goalie's own right), move to the kicker's right, or continue to stand in the center—and can mix among them with probabilities q_L, q_R, and q_C, two of which can be chosen independently.

We again use the opponent's indifference property to focus on the mixture probabilities for one player at a time. Each player's probabilities should be such that the other player is indifferent among all the pure strategies that constitute his mixture. This gives us a set of equations that can be solved for the mixture probabilities. In the soccer example, the kicker's (p_L, p_R) would satisfy two equations expressing the requirement that the goalie's expected payoff from moving to his left should equal that from moving to his right and that the goalie's expected payoff from moving to his right should equal that from staying at the center. (Then the equality of the expected payoffs from left and center follows automatically and is not a separate equation.) With more pure strategies, the number of probabilities to be solved for and the number of equations that they must satisfy also increase.

Figure 7.10 shows the game table for the interaction between Kicker and Goalie, with success percentages as payoffs for each player. (Unlike the evidence we present on European soccer later in this chapter, these are not real data, but similar rounded numbers meant to simplify calculations.) This is a zero-sum game in which the kicker wants to maximize his probability of successfully scoring a goal, and the goalie wants to minimize the probability of a goal. For example, if the kicker kicks to his left while the goalie moves to the kicker's left (the top-left corner cell), we suppose that the kicker still succeeds (in scoring) 45% of the time and the goalie therefore succeeds (in saving a goal) 55% of the time. But if the kicker kicks to his right and the goalie moves to the kicker's left, then the kicker has a 90% chance of scoring; we suppose a 10% probability that he might kick wide or too high, so the goalie is still successful 10% of the time. You can experiment with different payoff numbers that you think might be more appropriate.

		GOALIE		
		Left	Center	Right
KICKER	Left	45, 55	90, 10	90, 10
	Center	85, 15	0, 100	85, 15
	Right	95, 5	95, 5	60, 40
You may need to scroll left and right to see the full figure.				

FIGURE 7.10 Soccer Penalty-Kick Game

It is easy to verify that the game has no equilibrium in pure strategies. So suppose the kicker is mixing with probabilities p_L, p_R, and $p_C = 1 - p_L - p_R$. For each of the goalie's pure strategies, this mixture yields the goalie the following payoffs:

Left: $55p_L + 15p_C + 5p_R = 55p_L + 15(1 - p_L - p_R) + 5p_R$

Center: $10p_L + 100p_C + 5p_R = 10p_L + 100(1 - p_L - p_R) + 5p_R$

Right: $10p_L + 15p_C + 40p_R = 10p_L + 15(1 - p_L - p_R) + 40p_R$.

The opponent's indifference rule says that the kicker should choose p_L and p_R so that all three of these expressions are equal in equilibrium.

Equating the Left and Right expressions and simplifying, we have $45p_L = 35p_R$, or $p_R = (9/7)p_L$. Next, equate the Center and Right expressions and simplify by using the link between p_L and p_R just obtained. This gives

$10p_L + 100[1 - p_L - (9p_L/7)] + 5(9p_L/7) = 10p_L + 15[1 - p_L - (9p_L/7)] + 40(9p_L/7)$, or $[85 + 120(9/7)]p_L = 85$, which yields $p_L = 0.355$.

Then we get $p_R = 0.355(9/7) = 0.457$, and finally $p_C = 1 - 0.355 - 0.457 = 0.188$. The goalie's payoff from any of his pure strategies against this mixture can then be calculated by using any of the preceding three payoff lines; the result is 24.6.

The goalie's mixture probabilities can be found by writing down and solving the equations for the kicker's indifference among his three pure strategies against the goalie's mixture. We will do this in detail for a slight variant of the same game in Section 7.B, so we omit the details here and just give you the answer: $q_L = 0.325$, $q_R = 0.561$, and $q_C = 0.113$. The kicker's payoff from any of his pure strategies when played against the goalie's equilibrium mixture is 75.4. That answer is, of course, consistent with the goalie's payoff of 24.6 that we calculated before.

Now we can interpret the findings. The kicker does better with his pure Right than his pure Left, both when the goalie guesses correctly (60 > 45) and when he guesses incorrectly (95 > 90). Therefore, the kicker chooses Right with the highest probability, and, to counter that, the goalie chooses Right with the highest probability, too. However, the kicker should not and does not choose his pure strategy Right; if he did so, the goalie would then choose his own pure strategy Right, too, and the kicker’s payoff would be only 60, less than the 75.4 that he gets in the mixed-strategy equilibrium.

B. Equilibrium Mixtures with Some Strategies Unused

In the preceding equilibrium, the probabilities of using Center in the mix are quite low for each player. The (Center, Center) combination would result in a sure save, and the kicker would get a really low payoff—namely, 0. Therefore, the kicker puts a low probability on this choice. But then the goalie, too, should put a low probability on it, concentrating on countering the kicker's more likely choices. But if the kicker gets a sufficiently high payoff from choosing Center when the goalie chooses Left or Right, then the kicker will choose Center with some positive probability. If the kicker's payoffs in the Center row were lower, he might then choose Center with probability 0; if so, the goalie would similarly put probability 0 on Center. The game would be reduced to one with just two basic pure strategies, Left and Right, for each player.

We show such a variant of the soccer game in Figure 7.11. The only difference in payoffs between this variant and the original game of Figure 7.10 is that the kicker's payoffs from (Center, Left) and (Center, Right) have been lowered even further, from 85 to 70, causing the goalie's payoffs in these outcomes to rise from 15 to 30. That might be because this kicker has the habit of kicking too high and therefore missing the goal when aiming for the center. Let us try to calculate the equilibrium here by using the same methods as in Section 7.A. This time, we do it from the goalie's perspective: We try to find his mixture probabilities q_L, q_R, and q_C that would make the kicker indifferent among all three of his pure strategies when they are played against the goalie's mixture.

The kicker’s payoffs from his pure strategies are

Left: $45q_L + 90q_C + 90q_R = 45q_L + 90(1 - q_L - q_R) + 90q_R = 45q_L + 90(1 - q_L)$,

Center: $70q_L + 0q_C + 70q_R = 70q_L + 70q_R$, and

Right: $95q_L + 95q_C + 60q_R = 95q_L + 95(1 - q_L - q_R) + 60\ q_R = 95(1 - q_R) + 60q_R$.

		GOALIE		
		Left	Center	Right
KICKER	Left	45, 55	90, 10	90, 10
	Center	70, 30	0, 100	70, 30
	Right	95, 5	95, 5	60, 40
You may need to scroll left and right to see the full figure.				

FIGURE 7.11 Variant of Soccer Penalty-Kick Game

Equating the Left and Right expressions and simplifying, we have $90 - 45q_L = 95 - 35q_R$, or $35q_R = 5 + 45q_L$. Next, equate the Left and Center expressions and simplify to get $90 - 45q_L = 70q_L + 70q_R$, or $115q_L + 70q_R = 90$. Substituting for q_R from the first of these equations (after multiplying through by 2 to get $70q_R = 10 + 90q_L$) into the second yields $205q_L = 80$, or $q_L = 0.390$. Then, using this value for q_L in either of the equations gives $q_R = 0.644$. Finally, we use both of these values to obtain $q_C = 1 - 0.390 - 0.644 = -0.034$. Because probabilities cannot be negative, something has obviously gone wrong.

To understand what happens in this example, start by noting that Center is now a poorer strategy for the kicker than it was in the original version of the game, where his probability of choosing it was already quite low. But the logic of the opponent’s indifference property, expressed in the equations that led to the solution, means that the kicker has to be kept willing to use this poor strategy. That can happen only if the goalie is using his best counter to the kicker’s Center—namely, the goalie’s own Center—sufficiently infrequently. And in this example, that logic has to be carried so far that the goalie’s probability of Center has to become negative.

As pure algebra, the solution that we derived may be fine, but it violates the requirement of probability theory and real-life randomization that probabilities be nonnegative. The best that can be done in reality is to push the goalie’s probability of choosing Center as low as possible—namely, to zero. But that leaves the kicker unwilling to use his own Center. In other words, we get a situation in which each player is not using one of his pure strategies in his mixture—that is, each is using it with probability 0.

Can there then be an equilibrium in which each player is mixing between his two remaining strategies—namely, Left and Right? If we regard this reduced two-by-two game in its own right, we can easily find its mixed-strategy equilibrium. With all the practice that you have had so far, it is safe to leave the details to you and to state the result:

Kicker’s mixture probabilities: $p_L = 0.4375$, $p_R = 0.5625$.

Goalie’s mixture probabilities: $q_L = 0.3750$, $q_R = 0.6250$.

Kicker’s success percentage: 73.13.

Goalie’s success percentage:26.87.

We found this result by simply removing the two players’ Center strategies from consideration on intuitive grounds.

But we must make sure that it is a genuine equilibrium of the full three-by-three game. That is, we must check that neither player finds it desirable to bring in his third strategy, given the mixture of two strategies chosen by the other player.

When the goalie is choosing this particular mixture, the kicker's payoff from pure Center is $0.375 \times 70 + 0.625 \times 70 = 70$. This payoff is less than the 73.13 that he gets from either his pure Left or pure Right, or any mixture between the two, so the kicker does not want to bring his Center strategy into play. When the kicker is choosing the two-strategy mixture with the preceding probabilities, the goalie's payoff from pure Center is $0.4375 \times 10 + 0.5625 \times 5 = 7.2$. This number is (well) below the 26.87 that the goalie would get using his pure Left or pure Right, or any mixture of the two. Thus, the goalie does not want to bring his Center strategy into play either. The equilibrium that we found for the two-by-two game is indeed an equilibrium of the three-by-three game.

To allow for the possibility that some strategies may go unused in an equilibrium mixture, we must modify or extend the opponent's indifference principle: Each player's equilibrium mix should be such that the other player is indifferent among all the strategies *that are actually used in his equilibrium mix*. The other player is not indifferent between these and his unused strategies; he prefers the ones used to the ones unused. In other words, against the opponent's equilibrium mix, all the strategies used in your own equilibrium mix should give you the same expected payoff, which in turn should be higher than what you would get from any of your unused strategies.

Which strategies will go unused in equilibrium? Answering that question requires much trial and error, as in our previous calculation, or leaving it all to a computer program, and once you have understood the concept, it is safe to do the latter. For the general theory of mixed-strategy equilibria when players can have any number of possible strategies, see the online appendix to this chapter.

Endnotes

- More generally, if a player has N pure strategies, then her mix has $(N - 1)$ independent variables, or “degrees of freedom of choice.” Return to reference 12

8 HOW TO USE MIXED STRATEGIES IN PRACTICE

There are several important things to remember when finding or using a mixed strategy in a zero-sum game. First, to use a mixed strategy effectively in such a game, a player needs to do more than calculate the percentages with which to use each of her actions. Indeed, in our tennis-point game, Evert cannot simply play DL seven-tenths of the time and CC three-tenths of the time by mechanically rotating seven shots down the line and three shots crosscourt. Why not? Because mixing your strategies is supposed to help you benefit from the element of surprise against your opponent. If you use a recognizable pattern of moves, your opponent is sure to discover it and exploit it to her advantage.

The lack of a pattern means that, after any history of choices, the probability of choosing DL or CC on the next turn is the same as it always was. That is, if a run of several successive DLs happens by chance, there is no sense in which CC is now “due” on the next turn. In practice, many people mistakenly think otherwise, and therefore they alternate their choices too much compared with what a truly random sequence of choices would require: They produce too few runs of identical successive choices. However, detecting a pattern from observed actions is a tricky statistical exercise that the opponents may not be able to perform while playing the game. As we will see in Section 9, analysis of data from grand-slam tennis finals found that servers alternated their serves too much, but that receivers were not able to detect and exploit this departure from true randomization.

The importance of avoiding predictability is clearest in ongoing interactions of a zero-sum nature. Because of the diametrically opposed interests of the players in such games, your opponent always benefits from exploiting your choice of action to the greatest degree possible. Thus, if you play the same game against each other on a regular basis, she will always be on the lookout for ways to break the code that you are using to randomize your moves. If she can do so, she has a chance to improve her payoffs in future plays of the game. But even in single-meet (sometimes called one-shot) zero-sum games, mixing remains beneficial because of the benefit of tactical surprise.

Predictability may come from a pattern in series of successive plays, or from certain quirks that act as a "tell" to your opponent. A wonderful example comes from top-level men's tennis.[13] For a long time, Andre Agassi found Boris Becker's serve unplayable. Finally, after watching hours of films, he figured out that the way Becker stuck his tongue slightly out of his lips indicated which way he would serve. And Agassi was smart enough not to use this knowledge all the time, which could make Becker suspicious enough to make him change his behavior. Agassi used Becker's tell against him only on important points.

Daniel Harrington, a winner of the World Series of Poker and the author, with Bill Robertie, of an excellent series of books on how to play in Texas Hold 'Em tournaments, notes the importance of randomizing your strategy in poker in order to prevent opponents from reading what cards you're holding and exploiting your behavior.[14] Because humans often have trouble being unpredictable, he gives the following advice about how to implement a mixture between the pure strategies of calling and raising:

> It's hard to remember exactly what you did the last four or five times a given situation appeared, but fortunately

> you don't have to. Just use the little random number generator that you carry around with you all day. What's that? You didn't know you had one? It's the second hand on your watch. If you know that you want to raise 80 percent of the time with a premium pair in early position and call the rest, just glance down at your watch and note the position of the second hand. Since 80 percent of 60 is 48, if the second hand is between 0 and 48, you raise, and if it's between 48 and 60 you just call. The nice thing about this method is that even if someone knew exactly what you were doing, they still couldn't read you![15]

Of course, in using the second hand of a watch to implement a mixed strategy, it is important that your watch not be so accurate and synchronized that your opponent can use the same watch and figure out what you are going to do!

So far, we have assumed that you are interested in implementing a mixed strategy in order to avoid possible exploitation by your opponent. But if your opponent is not playing his equilibrium strategy, you may want to try to exploit his mistake. A simple example is illustrated using an episode of *The Simpsons* in which Bart and Lisa play a game of rock-paper-scissors with each other. (In Exercise S10, we give a full description of this three-by-three game and ask you to derive each player's equilibrium mixture.) Just before they choose their strategies, Bart thinks to himself, "Good ol' Rock. Nothing beats Rock," while Lisa thinks to herself, "Poor Bart. He always plays Rock." Clearly, Lisa's best response is the pure strategy Paper against this naive opponent; she need not use her equilibrium mix.

We have observed a more subtle example of exploitation when pairs of students play a best-of-100 version of the tennis-point game in this chapter. Like professional tennis players, many of our students switch strategies too often, apparently thinking that playing five DLs in a row doesn't look "random" enough. To exploit this behavior, a Navratilova player could predict that after playing three DLs in a row, an Evert player is likely to switch to CC, and she can exploit this by switching to CC herself. She should do this more often than if she were randomizing independently each round, but ideally not so often that the Evert player notices and starts learning to repeat her strategy in longer runs.

Finally, you must understand and accept the fact that the use of mixed strategies guards you against exploitation and gives the best possible expected payoff against an opponent who is making her best choices, but that it is only a probabilistic average. On particular occasions, you can get poor outcomes. For example, the long pass on third down with a yard to go, intended to keep the defense honest, may fail on any specific occasion. If you use a mixed strategy in a situation in which you are responsible to a higher authority, therefore, you may need to plan ahead for this possibility. You may need to justify your use of such a strategy ahead of time to your coach or your boss, for example. They need to understand why you have adopted your mixture and why you expect it to yield you the best possible payoff on average, even though it might yield an occasional low payoff as well. Even such advance planning may not work to protect your reputation, though, and you should prepare yourself for criticism in the face of a bad outcome.[16]

Endnotes

- See https://www.youtube.com/watch?v=3woPuCIk_d8. We thank David Reiley for this example. Return to reference 13
- Poker is a game of incomplete information because each player holds private information about her cards. While we do not analyze the details of such games until Chapter 9, they may involve mixed-strategy equilibria (called *semiseparating equilibria*) in which the random mixtures are specifically designed to prevent other players from using your actions to infer your private information. Return to reference 14
- Daniel Harrington and Bill Robertie, *Harrington on Hold ’Em: Expert Strategies for No-Limit Tournaments*, vol. 1, *Strategic Play* (Henderson, Nev.: Two Plus Two Publishing, 2004), p. 53. Return to reference 15
- In Super Bowl XLIX, played on February 1, 2015, the Seattle Seahawks trailed the New England Patriots by a score of 28–24 with only 26 seconds left. From the 1-yard line of the Patriots, the Seahawks threw the ball instead of letting their star running back, Marshawn Lynch, run up the middle. The pass was intercepted, and New England won the game. The play choice was mercilessly criticized in the media. But it was eminently justifiable on several levels, not just for the general principle of mixing, but also because the run would be expected, and Lynch had been stopped from the 1-yard line in some previous games. Return to reference 16

9 EVIDENCE ON MIXING

A. Zero-Sum Games

Early researchers who performed laboratory experiments were generally dismissive of mixed strategies. To quote Douglas Davis and Charles Holt, “Subjects in experiments are rarely (if ever) observed flipping coins, and when told ex post that the equilibrium involves randomization, subjects have expressed surprise and skepticism.” [17] When the predicted equilibrium entailed mixing two or more pure strategies, experimental results showed some subjects in a group pursuing one of the pure strategies and others pursuing another, but this does not constitute true mixing by an individual player. When subjects played zero-sum games repeatedly, individual players often chose different pure strategies over time. But they seemed to mistake alternation for randomization—that is, they switched their choices more often than true randomization would require.

Later laboratory research has found somewhat better evidence for mixing in zero-sum games. When subjects are allowed to acquire a lot of experience in these games, they do appear to learn mixing. However, departures from equilibrium predictions remain significant. Averaged across all subjects, the empirical probabilities are usually rather close to those predicted by equilibrium, but many individual subjects play proportions far from those predicted by equilibrium. To quote Colin Camerer, “The overall picture is that mixed equilibria do not provide bad guesses about how people behave, on average.” [18]

Evidence gathered from field observations on how people play zero-sum games in sports and other activities suggests that inexperienced and amateur players do sometimes make choices that conform to theoretical predictions, but not always. Professionals, especially top-ranked ones, who have a lot of experience and who play for substantial stakes conform quite closely to the theory of mixed-strategy equilibria. For example, a classic empirical analysis of tennis by Mark Walker and John Wooders examined the serve-and-return play of top-level players at Wimbledon, and a more recent study by Wooders, with two other coauthors, considered the same topic, but based its analysis on a newly available big-data set.[19] These authors modeled the interaction as a game with two players, the server and the receiver, in which each player has two pure strategies. The server can serve to the receiver's forehand or backhand, and the receiver can guess to which side the serve will go and move that way. Because serves are so fast at the top levels (especially in men's singles), the receiver cannot react after observing the actual direction of the serve; rather, the receiver must move in anticipation of the serve's direction. Thus, this game has simultaneous moves. Further, because the receiver wants to guess correctly and the server wants to wrong-foot the receiver, this interaction has a mixed-strategy equilibrium. The research statistically tested one important prediction of the theory: that the correct mix should be exploitation-proof, so that the server should win a point with the same probability whether he serves to the receiver's forehand or backhand.

The big-data set employed in the more recent paper comes from Hawk-Eye, the computerized ball-tracking system used at Wimbledon and other top professional tennis tournaments, and includes 500,000 serves. Each data point includes the precise trajectory and bounce point of the ball. Wooders and his coauthors combined these data with data on the rankings of the players involved in each interaction.[20] Their findings are striking. The *opponent's indifference property*, that the serving player should be indifferent about what direction to serve, is strongly supported in men's play, for both first and second serves. It is also borne out, although less strongly, in women's play.[21] Further, it is the receiver's play that can or cannot exploit any failure of optimization in the server's mix; therefore, it is remarkable that equalization occurs even against top receivers.

Correct mixing should be serially uncorrelated—one serve to the opponent's forehand should neither increase nor decrease the probability that the next serve will also be to the forehand. In practice, people exhibit too much alternation relative to this standard. Tennis players in this study did likewise, but the serial correlation was smaller for better players. And in any case, the excessive negative serial correlation was too small to be exploited.

		GOALIE	
		Left	Right
KICKER	Left	50	95
	Right	93	70

FIGURE 7.12 Soccer Penalty Kick Success Probabilities in European Major Leagues

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As we showed in Section 8, penalty kicks in soccer are another excellent context in which to study mixed strategies. The advantage to analyzing penalty kicks is that one can actually observe the strategies of both the kicker and the goalie: not only where the kicker aims, but also in which direction the goalie dives. This means that one can compute the actual mixing probabilities and compare them with the theoretical prediction. The disadvantage, relative to tennis, is that no two players ever face each other more than a few times in a season. Instead of analyzing specific matchups of players, one must aggregate across all kickers and goalies in order to get enough data. Two studies using exactly this kind of data find firm support for predictions of the theory.

Using a large data set from professional soccer leagues in Europe, Ignacio Palacios-Huerta constructed the payoff table of the kicker’s average success probabilities shown in Figure 7.12.[22] Because the data include both right- and left-footed kickers, and therefore the natural direction of kicking differs between them, they refer to any kicker’s natural side as “Right.” (Kickers usually kick with the inside of the foot. A right-footed kicker naturally kicks to the goalie’s right and a left-footed kicker to the goalie’s left.) The choices are Left and Right for each player. When the goalie chooses Right, it means covering the kicker’s natural side.

Using the opponent’s indifference property, it is easy to calculate that the kicker *should* choose Left 38.3% of the time and Right 61.7% of the time. This mixture achieves a success rate of 79.6% no matter what the goalie chooses. The goalie *should* choose the probabilities of covering her Left and Right to be 41.7 and 58.3, respectively; this mixture holds the kicker down to a success rate of 79.6%.

What actually happens? Kickers choose Left 40.0% of the time, and goalies choose Left 41.3% of the time. These values are startlingly close to the theoretical predictions. The chosen mixtures are almost exploitation-proof. The kicker’s mix achieves a success rate of 79.0% against the goalie’s Left and 80% against the goalie’s Right. The goalie’s mix holds kickers down to 79.3% if they choose Left and 79.7% if they choose Right.

In an earlier paper, Pierre-André Chiappori, Timothy Groseclose, and Steven Levitt used similar data and found similar results.[23] They also analyzed the whole sequence of choices of each kicker and goalie and did not find too much alternation between Left and Right. One reason for this last result could be that most penalty kicks take place as isolated incidents across many games, by contrast with the rapidly repeated points of tennis, so players may find it easier to ignore what happened on the previous kick. Nevertheless, these findings suggest that behavior in soccer penalty kicks is even closer to true mixing than behavior in the tennis serve-and-return game.

With such strong empirical confirmation of the theory, one might ask whether the mixed-strategy skills that players learn in soccer carry over to other game contexts. One study indicated that the answer is yes (Spanish professional soccer players played exactly according to the equilibrium predictions in laboratory experiments with two-by-two and four-by-four zero-sum games). But a second study failed to replicate these results. That study examined American major-league soccer players as well as participants in the World Series of Poker (who, as noted in Section 8 above, also have professional reasons to prevent exploitation by mixing). It found that the professionals' behavior in abstract matrix games was just as far from equilibrium as that of students. As with the results on professional chess players that we discussed in Chapter 3, experience led these professional players to mix according to equilibrium theory in their jobs, but this experience did not automatically lead the players to equilibrium in new and unfamiliar games.[24]

B. Non-Zero-Sum Games

Laboratory experiments on mixed strategies in non-zero-sum games yield even more negative results than experiments involving mixing in zero-sum games. That is not surprising. As we have seen, the opponent's indifference property is a logical property of the equilibrium in zero-sum games. But in contrast, the players in a non-zero-sum game may have no positive or purposive reason to keep the other players indifferent. Thus, the reasoning underlying the mixture calculations is more difficult for players to comprehend and learn, and this shows up in their behavior.

In a group of experimental subjects playing a non-zero-sum game, we may see some pursuing one pure strategy and others pursuing another. This type of mixing in the population, although it does not fit the theory of mixed-strategy equilibria, does have an interesting evolutionary interpretation, which we examine in Chapter 12.

As we saw in Section 5.B, a player's mixture probabilities should not change when that player's own payoffs change. But in fact they do: Players tend to choose an action more often when their own payoffs from that action increase.[25] The players do change their actions from one round to the next in repeated trials with different partners, but not in accordance with equilibrium predictions. The overall conclusion is that you should interpret and use mixed-strategy equilibria in non-zero-sum games with, at best, considerable caution.

Endnotes

- Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton, N.J.: Princeton University Press, 1993), p. 99. Return to reference 17
- For a detailed account and discussion, see Chapter 3 of Colin F. Camerer, *Behavioral Game Theory* (Princeton, N.J.: Princeton University Press, 2003); the quote is from p. 468. Return to reference 18
- Mark Walker and John Wooders, "Minimax Play at Wimbledon," *American Economic Review*, vol. 91, no. 5 (December 2001), pp. 1521–38; Romain Gouriot, Lionel Page, and John Wooders, "Nash at Wimbledon: Evidence from Half a Million Serves," working paper, January 2018, available at http://www.johnwooders.com/papers/NashAtWimbledon.pdf. Return to reference 19
- They also develop and use a more powerful statistical test, but we do not go into those technicalities here. Return to reference 20
- The authors attribute the difference to the fact that serves in women's play are slower, so any departure from optimal mixing is easier to exploit. Return to reference 21
- See Ignacio Palacios-Huerta, "Professionals Play Minimax," *Review of Economic Studies*, vol. 70, no. 20 (2003), pp. 395–415. Return to reference 22
- Pierre-André Chiappori, Timothy Groseclose, and Steven Levitt, "Testing Mixed Strategy Equilibria When Players are Heterogeneous: The Case of Penalty Kicks in Soccer," *American Economic Review*, vol. 92, no. 4 (September 2002), pp. 1138–51. Return to reference 23
- The first study referenced is Ignacio Palacios-Huerta and Oskar Volij, "Experientia Docet: Professionals Play Minimax in Laboratory Experiments," *Econometrica*, vol.

76, no. 1 (January 2008), pp. 71–115. The second is Steven D. Levitt, John A. List, and David H. Reiley, “What Happens in the Field Stays in the Field: Exploring Whether Professionals Play Minimax in Laboratory Experiments,” *Econometrica*, vol. 78, no. 4 (July 2010), pp. 1413–34. Return to reference 24

- Jack Ochs, “Games with Unique Mixed-Strategy Equilibria: An Experimental Study,” *Games and Economic Behavior*, vol. 10, no. 1 (July 1995), pp. 202–17. Return to reference 25

SUMMARY

Zero-sum games in which one player prefers a coincidence of actions and the other prefers the opposite often have no Nash equilibrium in pure strategies. In these games, each player wants to be unpredictable and thus may use a mixed strategy, which specifies a probability distribution over her set of pure strategies. Each player’s equilibrium mixture probabilities are calculated using the *opponent’s indifference property*, which means that the opponent should get equal *expected payoffs* from all her pure strategies when facing the player’s equilibrium mix. Best-response-curve graphs can be used to show all mixed-strategy (as well as pure-strategy) equilibria of a game.

Non-zero-sum games can also have mixed-strategy equilibria that can be calculated from the opponent’s indifference property and illustrated using best-response curves. But here the motivation for keeping the opponent indifferent is weaker or missing; therefore such equilibria have less appeal and are often unstable.

Mixed strategies are a special case of continuous strategies but have additional properties that deserve separate study. Mixed-strategy equilibria can be interpreted as outcomes in which each player has correct beliefs about the probabilities with which the other player chooses from among her underlying pure actions. And mixed-strategy equilibria may have some counterintuitive properties when payoffs for players change.

If one player has three pure strategies and the other has only two, the player with three available strategies will generally use only two in her equilibrium mix. If both players have three (or more) pure strategies, equilibrium mixtures may put positive probability on all pure strategies or only a subset. All strategies that are actively used in the mixture yield equal expected payoffs against the opponent’s equilibrium mix; all the unused ones yield lower expected payoffs. In these large games, equilibrium mixtures may also be indeterminate in some exceptional cases.

When using mixed strategies, players should remember that their system of randomization should not be predictable in any way. Most importantly, they should avoid excessive alternation of actions. Laboratory experiments show only weak support for the use of mixed strategies. But mixed-strategy equilibria give good predictions in many zero-sum situations in sports played by experienced professionals.

KEY TERMS

expected payoff (214)

opponent’s indifference property (216)

Glossary

expected payoff
: The probability-weighted average (statistical mean or expectation) of the payoffs of one player in a game, corresponding to all possible realizations of a random choice of nature or mixed strategies of the players.

opponent's indifference property
: An equilibrium mixed strategy of one player in a two-person game has to be such that the other player is indifferent among all the pure strategies that are actually used in her mixture.

SOLVED EXERCISES

1. Consider the following game:

		COLIN	
		Safe	Risky
ROWENA	Safe	4, 4	4, 1
	Risky	1, 4	6, 6

   1. Which other game does this game most resemble: tennis, assurance, or chicken? Explain.
   2. Find all of this game’s Nash equilibria.
2. The following table illustrates the money payoffs associated with a two-person simultaneous-move game:

		COLIN	
		Left	Right
ROWENA	Up	1, 16	4, 6
	Down	2, 20	3, 40

   1. Find the Nash equilibrium in mixed strategies for this game.
   2. What are the players’ expected payoffs in this equilibrium?
   3. Rowena and Colin jointly get the most money when Rowena plays Down. However, in the equilibrium, she does not always play Down. Why not? Can you think of ways in which a more cooperative outcome could be sustained?
3. Recall Exercise S8 from Chapter 4, about an old lady looking for help crossing the street and two players simultaneously deciding whether to offer help. If you did

that exercise, you also found all pure-strategy Nash equilibria of the game. Now find the mixed-strategy equilibrium of the game.

4. Revisit the tennis-point game in Section 2.A of this chapter. Recall that the mixed-strategy Nash equilibrium found in that section had Evert playing DL with probability 0.7, while Navratilova played DL with probability 0.6. Now suppose that Evert injures herself later in the match, so that her DL shots are much slower and easier for Navratilova to defend against. The payoffs are now given by the following table:

		NAVRATILOVA	
		DL	CC
EVERT	DL	30, 70	60, 40
	CC	90, 10	20, 80

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   1. Relative to the game before her injury (see Figure 7.1), the strategy DL seems much less attractive to Evert. Would you expect Evert to play DL more, less, or the same amount in a new mixed-strategy equilibrium? Explain.
   2. Find each player's equilibrium mixture for this game. What is Evert's expected payoff in equilibrium?
   3. How do the equilibrium mixtures found in part (b) compare with those of the original game and with your answer to part (a)? Explain why each mixture has or has not changed.

5. Exercise S8 in Chapter 6 introduced a simplified version of baseball, and in part (e) you determined that the simultaneous-move game has no Nash equilibrium in pure strategies. This is because pitchers and batters have conflicting goals. Pitchers want to get the ball *past* batters, but batters want to *connect* with pitched balls. The game table is as follows:

		PITCHER	
		Throw fastball	Throw curve
BATTER	Anticipate fastball	0.30, 0.70	0.20, 0.80
	Anticipate curve	0.15, 0.85	0.35, 0.65

---

1. Find the mixed-strategy Nash equilibrium of this simplified baseball game.
2. What is each player's expected payoff for the game?
3. Now suppose that the pitcher tries to improve his expected payoff in the mixed-strategy equilibrium by slowing down his fastball, thereby making it more similar to a curve ball. This changes the payoff to the hitter in the upper-left cell from 0.30 to 0.25, and the pitcher's payoff adjusts accordingly. Can this modification improve the pitcher's expected payoff as desired? Explain your answer carefully and show your work. Also, explain *why* slowing the fastball can or cannot improve the pitcher's expected payoff in the game.

6. Undeterred by their experiences with the game of chicken so far (see Section 4.B), James and Dean decide to increase the excitement (and the stakes) by starting their cars farther apart. This way they can keep the crowd in suspense longer, and they'll be able to accelerate to even higher speeds before they may or may not be involved in a much more serious collision. The new game table thus has a higher penalty for collision.

		DEAN	
		Swerve	Straight
JAMES	Swerve	0, 0	−1, 1
	Straight	1, −1	−10, −10

---

1. What is the mixed-strategy Nash equilibrium for this more dangerous version of chicken? Do James and Dean play Straight more or less often than in the game shown in Figure 7.4?
2. What is the expected payoff to each player in the mixed-strategy equilibrium found in part (a)?
3. James and Dean decide to play this game of chicken repeatedly (say, in front of different crowds of reckless youths). Moreover, because they don't want to collide, they collude and alternate between the two pure-strategy equilibria. Assuming they play an even number of games, what is the average payoff to each of them when they collude in this way? Is this better or worse than what they can expect from playing the mixed-strategy equilibrium? Why?
4. After several weeks of not playing chicken as in part (c), James and Dean agree to play again. However, each of them has entirely forgotten which pure-strategy Nash equilibrium they played last time, and neither realizes this until they're revving their engines moments before starting the game. Instead of playing the mixed-strategy Nash equilibrium, each of them tosses a separate coin to decide which strategy to play. What are the expected payoffs to James and Dean when each mixes 50 – 50 in this way? How do these payoffs compare with their expected payoffs when they play their equilibrium mixtures? Explain why these payoffs are the same or different from those found in part (c).

7. Section 2.B illustrates how to graph best-response curves for the tennis-point game. Section 4.B notes that when there are multiple equilibria, they can be identified from multiple intersections of the best-response curves. For the battle-of-the-sexes game in Figure 4.15, graph the best responses of Holmes and Watson on a $p-q$ coordinate plane. Label all Nash equilibria.
8. Consider the following game:

		COLIN	
		Yes	No
ROWENA	Yes	x, x	0, 1
	No	1, 0	1, 1

---

1. For what values of x does this game have a unique Nash equilibrium? What is that equilibrium?
2. For what values of x does this game have a mixed-strategy Nash equilibrium? With what probability, expressed in terms of x, does each player play Yes in this mixed-strategy equilibrium?
3. For the values of x found in part (b), is the game an example of an assurance game, a game of chicken, or a game similar to tennis? Explain.
4. Let $x = 3$. Graph the best-response curves of Rowena and Colin on a $p - q$ coordinate plane. Label all Nash equilibria in pure and mixed strategies.
5. Let $x = 1$. Graph the best-response curves of Rowena and Colin on a $p - q$ coordinate plane. Label all Nash equilibria in pure and mixed strategies.

9. Consider the following game:

		PROFESSOR PLUM		
		Revolver	Knife	Wrench
MRS. PEACOCK	Conservatory	1, 3	2, −2	0, 6
	Ballroom	3, 1	1, 4	5, 0
You may need to scroll left and right to see the full figure.				

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1. Graph the expected payoffs from each of Professor Plum’ s strategies as a function of Mrs. Peacock’ s p-mix.

2. Over what range of p does Revolver yield a higher expected payoff for Professor Plum than Knife?
3. Over what range of p does Revolver yield a higher expected payoff than Wrench?
4. Which pure strategies will Professor Plum use in his equilibrium mixture? Why?
5. What is the mixed-strategy Nash equilibrium of this game?

10. Many of you will be familiar with the children's game rock-paper-scissors. In rock-paper-scissors, two people simultaneously choose Rock, Paper, or Scissors, usually by putting their hands into the shape of one of the three choices. The game is scored as follows. A player choosing Scissors beats a player choosing Paper (because scissors cut paper). A player choosing Paper beats a player choosing Rock (because paper covers rock). A player choosing Rock beats a player choosing Scissors (because rock breaks scissors). If two players choose the same object, they tie. Suppose that each individual play of the game is worth 10 points. The following matrix shows the possible outcomes in the game:

		LISA		
		Rock	Scissors	Paper
BART	Rock	0, 0	10, −10	−10, 10
	Scissors	−10, 10	0, 0	10, −10
	Paper	10, −10	−10, 10	0, 0

You may need to scroll left and right to see the full figure.

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1. Derive the mixed-strategy equilibrium of the rock-paper-scissors game.
2. Suppose that Lisa announced that she would use a mixture in which her probability of choosing Rock would be 40%, her probability of choosing Scissors

would be 30%, and her probability of choosing Paper would be 30%. What is Bart's best response to this strategy choice by Lisa? Explain why your answer makes sense, given your knowledge of mixed strategies.

11. Recall the game between ice-cream vendors on a beach from Exercise U7 in Chapter 6. In that game, we found two asymmetric pure-strategy equilibria. There is also a symmetric mixed-strategy equilibrium in the game.
    1. Draw the five-by-five table for the game.
    2. Eliminate dominated strategies, and explain why they should not be used in the equilibrium.
    3. Use your answer to part (b) to help you find the mixed-strategy equilibrium of the game.
12. Suppose that the soccer penalty kick game of Section 7.A in this chapter is expanded to include a total of six distinct strategies for the kicker: to shoot high and to the left (HL), low and to the left (LL), high and in the center (HC), low and in the center (LC), high right (HR), and low right (LR). The goalie continues to have three strategies: to move to the kicker's left (L) or right (R) or to stay in the center (C). The players' success percentages are shown in the following table:

		GOALIE		
		L	C	R
KICKER	HL	0.50, 0.50	0.85, 0.15	0.85, 0.15
	LL	0.40, 0.60	0.95, 0.05	0.95, 0.05
	HC	0.85, 0.15	0, 0	0.85, 0.15
	LC	0.70, 0.30	0, 0	0.70, 0.30
	HR	0.85, 0.15	0.85, 0.15	0.50, 0.50
	LR	0.95, 0.05	0.95, 0.05	0.40, 0.60

You may need to scroll left and right to see the full figure.

In this problem, you will verify that the mixed-strategy equilibrium of this game entails the goalie using L and R each 42.2% of the time and C 15.6% of the time, while the kicker uses LL and LR each 37.8% of the time and HC 24.4% of the time.

1. Given the goalie’s proposed mixed strategy, compute the expected payoff to the kicker for each of his six pure strategies. (Use only three significant digits to keep things simple.)
2. Use your answer to part (a) to explain why the kicker’s proposed mixed strategy is a best response to the goalie’s proposed mixed strategy.
3. Given the kicker’s proposed mixed strategy, compute the expected payoff to the goalie for each of his three pure strategies. (Again, use only three significant digits to keep things simple.)
4. Use your answer to part (a) to explain why the goalie’s proposed mixed strategy is a best response to the kicker’s proposed mixed strategy.
5. Using your previous answers, explain why the proposed strategies are indeed a Nash equilibrium.
6. Compute the equilibrium payoff to the kicker.

13. (**Optional**) In Section 5.B, we demonstrated for the assurance game that changing Watson’s payoffs does not change his equilibrium mix—only Holmes’s payoffs determine Watson’s equilibrium mix. In this exercise, you will prove this as a general result for the mixed-strategy equilibria of all two-by-two games. Consider a general two-by-two non-zero-sum game with the payoff table shown below:

		COLIN	
		Left	Right
ROWENA	Up	a, A	b, B

		COLIN	
		Left	Right
	Down	c, C	d, D

---

1. Suppose the game has a mixed-strategy equilibrium. As a function of the payoffs in the table, solve for the probability that Rowena plays Up in equilibrium.
2. Solve for the probability that Colin plays Left in equilibrium.
3. Explain how your results show that each player’s equilibrium mixture depends only on the other player’s payoffs.
4. What conditions must be satisfied by the payoffs in order to guarantee that the game does indeed have a mixed-strategy equilibrium?

14. **(Optional)** Recall Exercise S15 of Chapter 4, which was based on the bar scene from the film *A Beautiful Mind*. Here we consider the mixed-strategy equilibria of that game when played by $n > 2$ young men.
    1. Begin by considering the symmetric case in which each of the n young men independently approaches the solitary blonde with some probability p. This probability is determined by the condition that each young man should be indifferent between the pure strategies Blonde and Brunette, given that everyone else is mixing. What is the condition that guarantees the indifference of each player? What is the equilibrium value of p in this game?
    2. There are also some asymmetric mixed-strategy equilibria in this game. In these equilibria, $m < n$ young men each approach the blonde with probability q, and the remaining $n - m$ young men approach the brunettes. What is the condition that guarantees that each of the m young men is indifferent, given what everyone else is doing? What condition must hold so that the remaining $n - m$ players don’t want to

switch from the pure strategy of approaching a brunette? What is the equilibrium value of q in the asymmetric equilibrium?

UNSOLVED EXERCISES

1. In American football, the offense can either run the ball or pass the ball, whereas the defense can either anticipate (and prepare for) a run or anticipate (and prepare for) a pass. Assume that the expected payoffs (in yards) for the two teams on any given down are as follows:

		DEFENSE	
		Anticipate Run	Anticipate Pass
OFFENSE	Run	1, −1	5, −5
	Pass	9, −9	3, −3

You may need to scroll left and right to see the full figure.

   1. Show that this game has no pure-strategy Nash equilibrium.
   2. Find the unique mixed-strategy Nash equilibrium of this game.
   3. Explain why the mixture used by the offense is different from the mixture used by the defense.
   4. How many yards is the offense expected to gain per down in equilibrium?
2. On the eve of a problem-set due date, a professor receives an e-mail from one of her students, who claims to be stuck on one of the problems after working on it for more than an hour. The professor would rather help the student if he has sincerely been working, but she would rather not render aid if the student is just fishing for hints. Given the timing of the request, she could simply pretend not to have read the e-mail until later. Obviously, the student would rather receive help

whether or not he has been working on the problem. But if help isn't coming, he would rather be working instead of slacking, since the problem set *is* due the next day. Assume the payoffs are as follows:

		STUDENT	
		Work and ask for help	Slack and fish for hints
PROFESSOR	Help student	3, 3	−1, 4
	Ignore e-mail	−2, 1	0, 0

You may need to scroll left and right to see the full figure.

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1. What is the mixed-strategy Nash equilibrium of this game?
2. What is the expected payoff to each of the players?

3. Exercise S14 in Chapter 4 introduced the game Evens or Odds, which has no Nash equilibrium in pure strategies. It does have an equilibrium in mixed strategies.
   1. If Anne plays 1 (that is, she shows one finger) with probability p, what is the expected payoff to Bruce from playing 1, in terms of p? What is his expected payoff from playing 2?
   2. What level of p will make Bruce indifferent between playing 1 and playing 2?
   3. If Bruce plays 1 with probability q, what level of q will make Anne indifferent between playing 1 and playing 2?
   4. What is the mixed-strategy equilibrium of this game? What is the expected payoff of the game to each player?
4. Return again to the tennis rivals Evert and Navratilova, discussed in Section 2.A. Months later, they meet again

in a new tournament. Evert has healed from her injury (see Exercise S4), but during that same time Navratilova has worked very hard on improving her defense against DL serves. The payoffs are now as follows:

		NAVRATILOVA	
		DL	CC
EVERT	DL	25, 75	80, 20
	CC	90, 10	20, 80

---

1. Find each player's equilibrium mixture for this game.
2. What has happened to Evert's *p*-mix compared with the original game presented in Figure 7.1? Why?
3. What is Evert's expected payoff in equilibrium? Why is it different from her expected payoff in the original game in Figure 7.1?

5. Section 4.A of this chapter discussed mixing in the battle-of-the-sexes game between Holmes and Watson.
   1. What do you expect to happen to the equilibrium values of p and q found in Section 4.A if Watson decides he really likes Simpson's a lot more than St. Bart's, so that the payoffs in the (Simpson's, Simpson's) cell of Figure 7.3 are now (1, 3)? Explain your reasoning.
   2. Now find the new mixed-strategy equilibrium values of p and q. How do they compare with those of the original game?
   3. What is the expected payoff to each player in the new mixed-strategy equilibrium?
   4. Do you think Holmes and Watson might play the mixed-strategy equilibrium in this new version of the game? Explain why or why not.
6. Consider the following variant of chicken, in which James's payoff from being "tough" when Dean is "chicken" is 2, rather than 1:

		DEAN	
		Swerve	Straight
JAMES	Swerve	0, 0	−1, 1
	Straight	2, −1	−2, −2

---

1. Find the mixed-strategy equilibrium in this game, including the expected payoffs for the players.
2. Compare the results with those of the original game in Section 4.B of this chapter. Is Dean's probability of playing Straight (being tough) higher now than before? What about James's probability of playing Straight?
3. What has happened to the two players' expected payoffs? Are these differences in the equilibrium outcomes paradoxical in light of the new payoff structure? Explain how your findings can be understood in light of the opponent's indifference principle.

7. For the chicken game in Figure 7.4, graph the best responses of James and Dean on a $p - q$ coordinate plane. Label all Nash equilibria.
8. Consider the following game:

		COLIN			
		L	M	N	R
ROWENA	Up	1, 1	2, 2	3, 4	9, 3
	Down	2, 5	3, 3	1, 2	7, 1
You may need to scroll left and right to see the full figure.					

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1. Find all pure-strategy Nash equilibria of this game.
2. Now find a mixed-strategy equilibrium of the game. What are the players' expected payoffs in the

equilibrium?

9. Consider a revised version of the game from Exercise S9:

		PROFESSOR PLUM		
		Revolver	Knife	Wrench
MRS. PEACOCK	Conservatory	1, 3	2, −2	0, 6
	Ballroom	3, 2	1, 4	5, 0

You may need to scroll left and right to see the full figure.

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   1. Graph the expected payoffs from each of Professor Plum's strategies as a function of Mrs. Peacock's p-mix.
   2. Which strategies will Professor Plum use in his equilibrium mixture? Why?
   3. What is the mixed-strategy Nash equilibrium of this game?
   4. Note that this game is only slightly different from the game in Exercise S9. How are the two games different? Explain why you intuitively think that the equilibrium outcome has changed from Exercise S9.

10. Consider a modified version of rock-paper-scissors in which Bart gets a bonus when he wins with Rock. If Bart picks Rock while Lisa picks Scissors, Bart wins twice as many points as when either player wins in any other way. The new payoff matrix is

		LISA		
		Rock	Scissors	Paper
BART	Rock	0, 0	20, −20	−10, 10
	Scissors	−10, 10	0, 0	10, −10

You may need to scroll left and right to see the full figure.

	LISA		
	Rock	Scissors	Paper
Paper	10, −10	−10, 10	0, 0

You may need to scroll left and right to see the full figure.

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1. What is the mixed-strategy equilibrium in this version of the game?
2. Compare your answer here with your answer for the mixed-strategy equilibrium in Exercise S10. How can you explain the differences in the equilibrium strategy choices?

11. Consider the following game:

		MACARTHUR		
		Air	Sea	Land
PATTON	Air	0, 3	2, 0	1, 7
	Sea	2, 4	0, 6	2, 0
	Land	1, 3	2, 4	0, 3

You may need to scroll left and right to see the full figure.

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1. Does this game have a pure-strategy Nash equilibrium? If so, what is it?
2. Find a mixed-strategy equilibrium for this game.
3. Actually, this game has two mixed-strategy equilibria. Find the one you didn’t find in part (b). (Hint: In one of these equilibria, one of the players plays a mixed strategy, whereas the other plays a pure strategy.)

12. The recalcitrant James and Dean are playing their more dangerous variant of chicken again (see Exercise S6). They’ve noticed that their payoff for being perceived as

tough varies with the size of the crowd. The larger the crowd, the more glory and praise each receives by driving straight when his opponent swerves. Smaller crowds, of course, have the opposite effect. Let $k > 0$ be the payoff for appearing tough. The game may now be represented as

		DEAN	
		Swerve	Straight
JAMES	Swerve	0, 0	−1, k
	Straight	k, −1	−10, −10

---

1. Expressed in terms of k, with what probability does each driver play Swerve in the mixed-strategy Nash equilibrium? Do James and Dean play Swerve more or less often as k increases?
2. In terms of k, what is the expected payoff to each player in the mixed-strategy Nash equilibrium found in part (a)?
3. At what value of k do both James and Dean mix 50 - 50 in the mixed-strategy equilibrium?
4. How large must k be for the average payoff to be positive under the alternating scheme discussed in part (c) of Exercise S6?

13. (**Optional**) Recall the game from Exercise S13 in Chapter 4, where Larry, Moe, and Curly can choose to buy tickets toward a prize worth \$30. We found six pure-strategy Nash equilibria in that game. In this problem, you will find a symmetric equilibrium in mixed strategies.
    1. Eliminate the weakly dominated strategy for each player. Explain why a player would never use this weakly dominated strategy in his equilibrium mixture.
    2. Find the equilibrium in mixed strategies.
14. (**Optional**) Exercises S4 and U4 demonstrate that in zero-sum games such as the Evert - Navratilova tennis-point game, changes in a player' s payoffs can sometimes lead to unexpected or counterintuitive changes to her

equilibrium mixture. But what happens to her expected payoff in equilibrium? Consider the following general form of a two-player zero-sum game:

		COLIN	
		L	R
ROWENA	U	a, −a	b, −b
	D	c, −c	d, −d

Assume that there is no Nash equilibrium in pure strategies, and assume that a, b, c, and d are all greater than or equal to 0. Can an *increase* in any one of a, b, c, or d lead to a *lower* expected payoff for Rowena in equilibrium? If not, prove why not. If so, provide an example.

■ Appendix: Working with Probabilities

To calculate the expected payoffs and mixed-strategy equilibria of games in this chapter, we had to do some simple manipulation of probabilities. Some simple rules govern calculations involving probabilities. Many of you may be familiar with them, but we give a brief statement and explanation of the basics here by way of reminder or remediation, as appropriate. We also explain how to calculate expected values for random numerical amounts.

THE BASIC ALGEBRA OF PROBABILITIES

The basic intuition about the probability of an event comes from thinking about the frequency with which this event occurs by chance among a larger set of possibilities. Usually, any one element of this larger set is just as likely to occur by chance as any other, so finding the probability of the event in which we are interested is simply a matter of counting the elements corresponding to that event and dividing by the total number of elements in the whole large set.[26]

In any standard deck of 52 playing cards, for instance, there are four suits (clubs, diamonds, hearts, and spades) and 13 cards in each suit (ace through 10 and the face cards—jack, queen, king). We can ask a variety of questions about the likelihood that a card of a particular suit or value—or suit *and* value—might be drawn from this deck of cards: How likely are we to draw a spade? How likely are we to draw a black card? How likely are we to draw a 10? How likely are we to draw the queen of spades? and so on. We would need to know something about the calculation and manipulation of probabilities to answer such questions. If we had two decks of cards, one with blue backs and one with green backs, we could ask even more complex questions ("How likely are we to draw one card from each deck and have them both be the jack of diamonds?"), but we would still use the algebra of probabilities to answer them.

In general, a probability measures the likelihood of a particular event or set of events occurring. The likelihood that you will draw a spade from a deck of cards is just the probability of the event "drawing a spade." Here, the larger set of possibilities has 52 elements—the total number of equally likely possibilities—and the event "drawing a spade" corresponds to a subset of 13 particular elements. Thus, you have 13 chances out of the 52 to get a spade, which makes the probability of getting a spade in a single draw equal to 13/52 = ¼= 25%. To see this another way, consider the fact that there are four suits of 13 cards each, so your chance of drawing a card from any particular suit is one out of four, or 25%. If you made 52 such draws (each time from a complete deck), you would not always draw exactly 13 spades; by chance, you might draw a few more or a few less. But the chance averages out over different such occasions—say, over different sets of 52 draws. Then the probability of 25% is the average of the frequencies of drawing spades in a large number of observations.[27]

The algebra of probabilities simply develops such ideas in general terms and obtains formulas that you can then apply mechanically instead of having to do the thinking from scratch every time. We will organize our discussion of these probability formulas around the types of questions that one might ask when drawing cards from a standard deck (or two: blue backed and green backed).[28] This method will allow us to provide both specific and general formulas for you to use later. You can use the card-drawing analogy to help you reason out other questions about probabilities that you encounter in other contexts. One other point to note: In ordinary language, it is customary to write probabilities as percentages, but the algebra requires that they be written as fractions or decimals; thus, instead of 25%, the mathematics works with 13/52, or 0.25. We will use one or the other, depending on the occasion; be aware that they mean the same thing.

A. The Addition Rule

The first questions that we ask are, If we draw one card from the blue deck, how likely are we to draw a spade? And how likely are we to draw a card that is not a spade? We already know that the probability of drawing a spade is 25% because we determined that earlier. But what is the probability of drawing a card that is not a spade? It is the same as the likelihood of drawing a club or a diamond or a heart instead of a spade. It should be clear that the probability in question should be larger than any of the individual probabilities of which it is formed; in fact, the probability is 13/52 (clubs) + 13/52 (diamonds) + 13/52 (hearts) = 0.75. The *or*s in our verbal interpretation of the question are the clue that the probabilities should be added together because we want to know the chances of drawing a card from any of those three suits.

We could more easily have found our answer to the second question by noting that, given that we draw a spade 25% of the time, not drawing a spade is what happens the other 75% of the time. Thus, the probability of drawing "not a spade" is 75% (100% − 25%), or, more formally, 1 − 0.25 = 0.75. As is often the case with probability calculations, the same result can be obtained here by two different routes, entailing different ways of thinking about the event for which we are trying to find the probability. We will see other examples of such alternative routes later in this appendix, where it will become clear that the different methods of calculation can sometimes require vastly different amounts of effort. As you develop experience, you will discover and remember the easy ways or shortcuts. In the meantime, be comforted by the knowledge that each of the different routes, when correctly followed, leads to the same final answer.

To generalize our preceding calculation, we note that, if you divide the set of events, *X*, in which you are interested into some number of subsets, *Y*, *Z*, . . . , none of which overlap (in mathematical terminology, such subsets are said to be disjoint), then the probabilities of each subset occurring must sum to the probability of the full set of events; if that full set of events includes all possible outcomes, then its probability is 1. In other words, if the occurrence of *X* requires the occurrence of *any one* of several disjoint *Y*, *Z*, . . . , then the probability of *X* is the sum of the separate probabilities of *Y*, *Z*, ... Using Prob(*X*) to denote the probability that *X* occurs and remembering the caveats on *X* (that it requires any one of *Y*, *Z*, . . .) and on *Y*, *Z*, . . . (that they must be disjoint), we can write the addition rule in mathematical notation as Prob(*X*) = Prob(*Y*) + Prob(*Z*) + . . . ;.

EXERCISE Use the addition rule to find the probability of drawing two cards, one from each of the two decks, such that the two cards have identical faces.

B. The Multiplication Rule

Now we ask, What is the likelihood that when we draw two cards, one from each of the two decks, both of them will be spades? This event occurs if we draw a spade from the blue deck *and* a spade from the green deck. The switch from *or* to *and* in our verbal interpretation of what we are looking for indicates a switch in mathematical operations from addition to multiplication. Thus, the probability of two spades, one from each deck, is the product of the probabilities of drawing a spade from each deck, or $(13/52) \times (13/52) = 1/16 = 0.0625$, or 6.25%. Not surprisingly, we are much less likely to get two spades than we were in the previous section to get one spade. (Always check to make sure that your calculations accord in this way with your intuition regarding the outcome.)

In much the same way as the addition rule requires events to be disjoint, the multiplication rule requires that they be independent events: If we break down a set of events, X, into some number of subsets Y, Z, . . . , those subsets are independent if the occurrence of one does not affect the probability of the other. Our events—a spade from the blue deck and a spade from the green deck—satisfy this condition of independence; that is, drawing a spade from the blue deck does nothing to alter the probability of getting a spade from the green deck. If we were drawing both cards from the same deck, however, then after we had drawn a spade (with a probability of 13/52), the probability of drawing another spade would no longer be 13/52 (in fact, it would be 12/51); drawing one spade and then a second spade from the *same* deck are not independent events.

The formal statement of the multiplication rule tells us that, if the occurrence of X requires the simultaneous occurrence of *all* the several independent events Y, Z, . . . , then the probability of X is the *product* of the separate probabilities of Y, Z, . . . : $\text{Prob}(X) = \text{Prob}(Y) \times \text{Prob}(Z) \times \cdots$.

EXERCISE Use the multiplication rule to find the probability of drawing two cards, one from each of the two decks, and getting a red card from the blue deck and a face card from the green deck.

C. Expected Values

If a numerical amount (such as money winnings or rainfall) is subject to chance and can take on any one of n possible values X_1, X_2, . . . , X_n with respective probabilities p_1, p_2, . . . , p_n, then the expected value is defined as the weighted average of all its possible values using the probabilities as weights; that is, as $p_1X_1 + p_2X_2 + . . . + p_nX_n$. For example, suppose you bet on the toss of two fair coins. You win \$5 if both coins come up heads, \$1 if one shows heads and the other tails, and nothing if both come up tails. Using the rules for manipulating probabilities discussed earlier in this section, you can see that the probabilities of these events are, respectively, 0.25, 0.50, and 0.25. Therefore, your expected winnings are (0.25 × \$5) + (0.50 × \$1) + (0.25 × \$0) = \$1.75.

Endnotes

- When we say “by chance,” we simply mean that a systematic order cannot be detected in the outcome or that it cannot be determined by using available scientific methods of prediction and calculation. Actually, the motions of coins and dice are fully determined by laws of physics, and highly skilled people can manipulate decks of cards, but for all practical purposes, coin tosses, rolls of dice, or card shuffles are devices of chance that can be used to generate random outcomes. However, randomness can be harder to achieve than you think. For example, a perfect shuffle, where a deck of cards is divided exactly in half and then interleaved by dropping cards one at a time alternately from each, may seem a good way to destroy the initial order of the deck. But Stanford mathematician Persi Diaconis has shown that, after eight of the shuffles, the original order is fully restored. For the slightly imperfect shuffles that people carry out in reality, he finds that some order persists through six, but randomness suddenly appears on the seventh! See “How to Win at Poker, and Other Science Lessons,” *The Economist*, October 12, 1996. For an interesting discussion of such topics, see Deborah J. Bennett, *Randomness* (Cambridge, Mass.: Harvard University Press, 1998), chap. 6–9. Return to reference 26
- Bennett, *Randomness*, chap. 4 and 5, offers several examples of such calculations of probabilities. Return to reference 27
- If you want a more detailed exposition of the following addition and multiplication rules, as well as more exercises to practice these rules, we recommend David Freeman, Robert Pisani, and Robert Purves, *Statistics,*

4th ed. (New York: W. W. Norton & Company, 2007), chap. 13–14. Return to reference 28

Glossary

probability
: The probability of a random event is a quantitative measure of the likelihood of its occurrence. For events that can be observed in repeated trials, it is the long-run frequency with which it occurs. For unique events or other situations where uncertainty may be in the mind of a person, other measures are constructed, such as subjective probability.

disjoint
: Events are said to be disjoint if two or more of them cannot occur simultaneously.

addition rule
: If the occurrence of *X* requires the occurrence of *any one* of several disjoint *Y*, *Z*, . . . , then the probability of *X* is the sum of the separate probabilities of *Y*, *Z*, . . .

independent events
: Events *Y* and *Z* are independent if the actual occurrence of one does not change the probability of the other occurring. That is, the conditional probability of *Y* occurring given that *Z* has occurred is the same as the ordinary or unconditional probability of *Y*.

multiplication rule
: If the occurrence of *X* requires the simultaneous occurrence of *all* the several independent *Y, Z,* . . . , then the probability of *X* is the *product* of the separate probabilities of *Y, Z,* . . .

expected value
: The probability-weighted average of the outcomes of a random variable, that is, its statistical mean or expectation.

SUMMARY

The *probability* of an event is the likelihood of its occurrence by chance from among a larger set of possibilities. Probabilities can be combined by using some rules. The *addition rule* says that the probability of any one of a number of *disjoint* events occurring is the sum of the probabilities of these events. According to the *multiplication rule*, the probability that all of a number of *independent events* will occur is the product of the probabilities of these events. Probability-weighted averages are used to compute *expected values*, such as payoffs in games.

KEY TERMS

addition rule (261)

disjoint (261)

expected value (262)

independent events (262)

multiplication rule (262)

probability (260)

Glossary

addition rule

If the occurrence of *X* requires the occurrence of *any one* of several disjoint *Y*, *Z*, . . . , then the probability of *X* is the sum of the separate probabilities of *Y*, *Z*, . . .

disjoint

Events are said to be disjoint if two or more of them cannot occur simultaneously.

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multiplication rule

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probability

The probability of a random event is a quantitative measure of the likelihood of its occurrence. For events that can be observed in repeated trials, it is the long-run frequency with which it occurs. For unique events or other situations where uncertainty may be in the mind of a person, other measures are constructed, such as subjective probability.

PART THREE

Some Broad Classes of Strategies and Games

8 ■ Strategic Moves

SUPPOSE YOU ARE about to play a game of strategy. You can figure out that the outcome will yield you one of your lower payoffs. And you would like to do something to change the outcome of the game in your favor. Various devices, collectively labeled strategic moves, exist for this purpose. The design and use of strategic moves is often specific to each context, but some general principles can be culled from examples and from game-theoretic thinking. Those principles are the subject of this chapter.

The interaction among players when they are attempting to find strategic moves that can change a game they are about to play constitutes a game of its own. For conceptual clarity, we call this the *pregame*. In the pregame, you make declarations and take actions aimed at changing the upcoming game; we will often refer to this upcoming game simply as *the game*. To use strategic moves effectively, players need to ask (and answer) several key questions. We begin by laying out these questions and then use the answers to guide our subsequent analyses.

What is to be accomplished? If you could improve your outcome in the game just by changing your own action, your original action must not have been a Nash equilibrium choice. So, any strategic move you make must seek to change some *other* player's action from what she would otherwise choose in equilibrium—that is, to change the other player's *default action* to some *favored action* that is better for you.

What changes are feasible? The actual game has specified rules and payoffs. Pregame moves can change these, but only within certain limits. Legislatures often make their own procedural rules, and manipulating these rules to facilitate or hinder the passage of particular legislation is a well-recognized pregame. But if a football team has a weak passing offense, it cannot suddenly change the rules of football to forbid forward passes. As for payoffs, you may be able to alter your own payoffs for certain outcomes—for example, by entering into side bets about those outcomes or by laying your reputation on the line. But in most cases, you cannot directly change others' payoffs. The order of moves in the upcoming game may also limit what sort of strategic moves you can make. In each case, feasibility is an important prerequisite for making a strategic move.

What changes are credible? In many games you play, you would like the other player to believe that you have limited or expanded your own set of available actions, or that you have changed your payoffs in some way. Even when such changes are technically feasible, simply declaring that you have done so may not be credible. For instance, in the game of chicken, you would like the other player to believe that you are unable to swerve, since she will then have an incentive to swerve out of your way. But the other player may not believe you when you say that you cannot swerve. If you are dieting, but love desserts, you can counter temptation by vowing to send $1,000 to a charity every time you order dessert. But unless there is some external agent to observe any cheating and hold you to your resolve, your vow may not be credible. The question of *credibility*, a concept we first introduced in Chapter 6, is crucial to the effective use of strategic moves, and we will examine it in connection with all the moves we discuss in this chapter.

What kinds of strategic moves? A pregame move may specify that you will take some particular action in the upcoming game regardless of what the other player does in that game. For example, you might declare you will definitely go straight in the game of chicken in order to induce the other player to swerve. Or a pregame move may establish a rule for how you will respond to the other player's move in the upcoming game depending on what she does. For example, a firm might announce that it will match a rival’s price in order to induce the rival to set a high price (which is beneficial to both). We will show you how to think about both types of pregame moves in the following section, and we will provide examples illustrating how and when to use such moves in the remainder of the chapter.

Glossary

strategic move
: Action taken at a pregame stage that changes the strategies or the payoffs of the subsequent game (thereby changing its outcome in favor of the player making the move).

feasibility
: Possibility within the (unchangeable) physical and/or procedural restrictions that apply in a given game.

1 A CLASSIFICATION OF STRATEGIC MOVES

In this section, we focus our attention on two-player games, but the ideas involved are perfectly general, and we will occasionally illustrate them with more than two players. We will call the two players in our examples Ann and Bob. As described above, strategic moves can be declared in two ways in the pregame: They can specify an action that you will take in the upcoming game no matter what. Or they can specify a rule you will follow in the upcoming game in which your chosen actions depend on the choices made by the other player. The first type of strategic move is *unconditional*, and the second type is *conditional*. We set out the details of each type of move below.

A. Unconditional Strategic Moves

Suppose that, in the pregame, Ann were to declare, "In the upcoming game, I will take a particular action, X, no matter what you do." This unconditional strategic move is called a commitment. When Ann's commitment is credibly made, Bob will make his move taking into account that Ann's move is essentially already fixed. That is, he will assume that the subsequent game will unfold *as if* Ann is the first mover and has already observably and irreversibly chosen X. In this way, a credible commitment allows a player who would otherwise move simultaneously or move last to become the first mover, seizing any *first-mover advantage* that may exist in the game.

In the street-garden game of Chapter 3, three women play a sequential-move game in which each must decide whether to contribute toward the creation of a public flower garden on their street; two or more contributors are necessary for the creation of a pleasant garden. The rollback equilibrium entails the first player (Emily) choosing not to contribute while the other players (Nina and Talia) do contribute. By making a credible commitment in the pregame not to contribute in the upcoming game, however, Talia (or Nina) could alter the outcome of the game. Even though she does not get her turn to announce her decision until after Emily and Nina have made theirs public, Talia could let it be known that she has sunk all of her savings (and energy) into a large house-renovation project, so she has absolutely nothing left to contribute to the street garden. By doing so, Talia essentially commits herself not to contribute regardless of Emily's and Nina's decisions, before Emily and Nina make those decisions. In other words, Talia changes the game to one in which she is the first mover. You can easily check that the new rollback equilibrium entails Emily and Nina both contributing to the garden, and that the equilibrium payoffs are 3 to each of them but 4 to Talia—the same equilibrium outcome associated with the game when Talia moves first. Several additional detailed examples of commitments are provided later in this chapter.

B. Conditional Strategic Moves

Consider now the use of a conditional strategic move. Suppose Ann declares a response rule in the pregame, stating how she will respond to each of Bob's possible actions in the upcoming game. For this strategic move to be meaningful, Ann must be the second mover in the upcoming game, or arrange in the pregame to become the second mover. (Ann could, for example, manipulate the reversibility or observability of her or Bob's moves, as described in Section 2 of Chapter 6.)

Suppose Ann does move last in the upcoming game and hence could potentially use a conditional strategic move. In order to have any hope of changing Bob's behavior to her own advantage, Ann must change Bob's belief about how she is going to act as second mover in the upcoming game. In any effective strategic move, Ann's response rule must therefore specify at least one response that is different from what she would normally do in the absence of the strategic move. Thus, at least one of Ann's declared responses must *not* be a best response.

In addition, Ann's conditional strategic move must be designed to improve her outcome in the upcoming game. The strategic move should motivate Bob to take an action Ann prefers, called the favored action, rather than the default action that he would otherwise have taken in the rollback equilibrium of the upcoming game.

There are three ways in which Ann might be able to get Bob to change his decision with a conditional strategic move. First, she could declare that she will respond to the *favored* action not with her equilibrium best-response action, but instead in a way that gives Bob a relatively *high* payoff—and, often implicitly, that she will respond to the default action with her best response to that action. This is called *making a promise*; Ann's response rule rewards Bob for doing what she wants. Second, she could declare that she will respond to the *default* action not with her equilibrium best-response action, but instead in a way that gives Bob a relatively *low* payoff—and, often implicitly, that she will respond to the favored action with her best response to that action. This is called *making a threat*; Ann's response rule hurts Bob when he does not act the way she wants. Finally, Ann could declare that she will do *both*: She will respond to the favored action in a way that gives Bob a relatively high payoff, and she will respond to the default action in a way that gives Bob a relatively low payoff. This is called *making a combined threat and promise*; Ann's response rule both rewards Bob for doing what she wants and hurts him when he does not act the way she wants.

To keep things clear in the analysis that follows, we introduce some additional terminology here. Specifically, we give names to each piece of a response rule that constitutes a conditional strategic move. A specified response to the favored action that is not your best response and benefits the other player is called a promise, and a specified response to the default action that is not your best response and hurts the other player is called a threat. An affirmation is a specified response to the favored action that is your best response, and a warning is a specified response to the default action that is your best response. With this terminology, *making a threat* entails declaring a response rule that includes a threat and an affirmation and *making a promise* entails declaring a response rule that includes a promise and a warning. In practice, players often state aloud only the threat or the promise when making those moves, leaving the affirmation or warning implicit rather than explicit. Finally, *making a combined threat and promise* entails explicit statement of both a threat and promise (as the name suggests!).

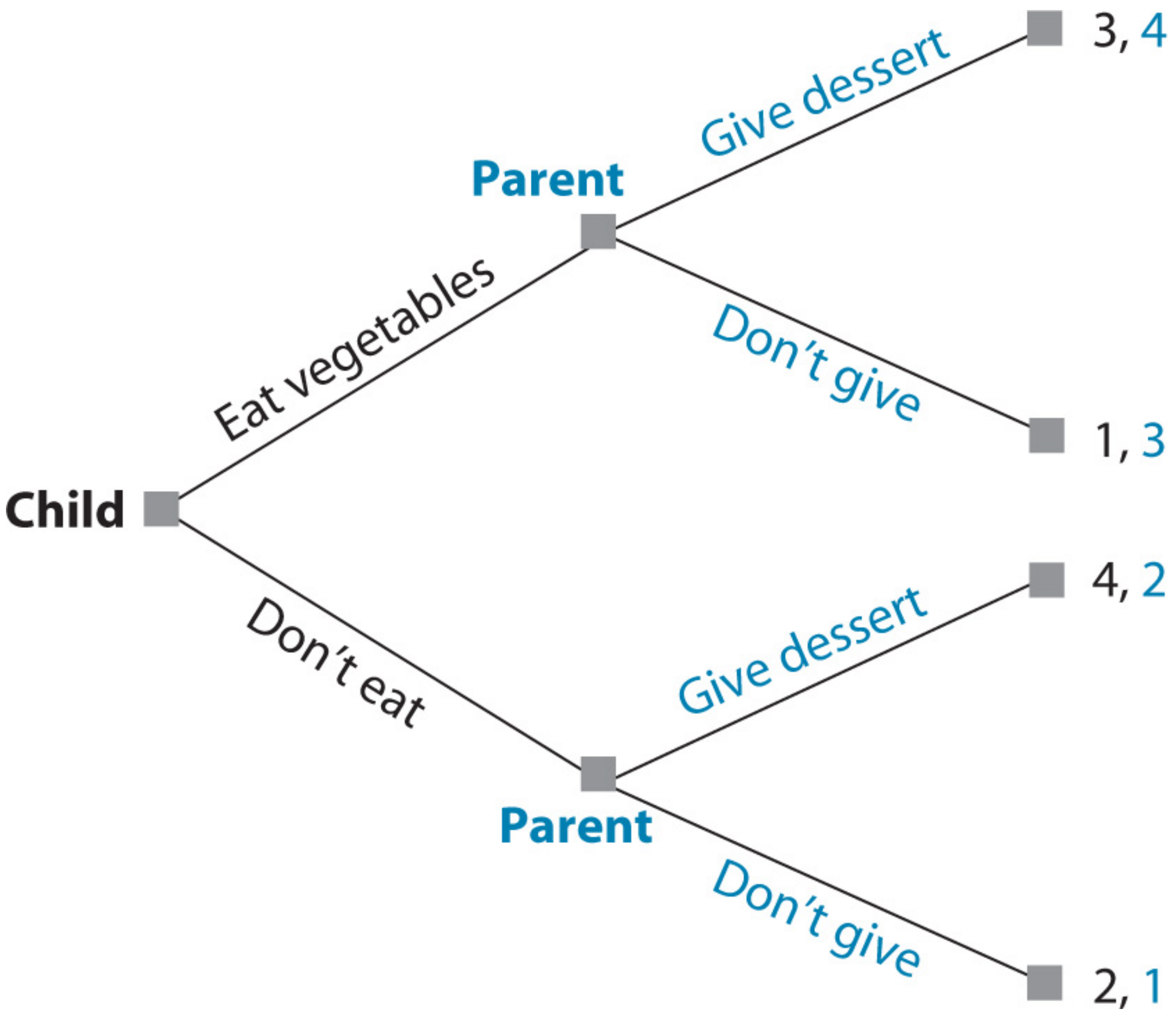

Figure 8.1 Parent–Child Dinner Game

We will provide several in-depth examples of the use of conditional strategic moves later in this chapter, but provide an additional example here to help clarify how to think about strategic moves and how to use the terminology we have just presented. Imagine a game between a parent and child that takes place at the dinner table. In the natural chronological order of moves, the child first decides whether to eat her vegetables, and then the parent decides whether to give the child dessert. So the parent moves last and can feasibly make a conditional strategic move. Figure 8. illustrates the game tree and shows how each player ranks the four possible outcomes. (As usual, 4 is best and 1 is worst.) Rollback analysis tells us what the default equilibrium outcome will be if the parent does nothing to change the game: The child refuses to eat the vegetables, knowing that the parent, unwilling to see the child hungry and unhappy, will give her the dessert. In the equilibrium outcome the child receives her highest payoff (4), but the parent receives one of his lower payoffs (2).

The parent can foresee this outcome and can try to alter it in his favor by making a conditional strategic move in the pregame. He might declare the response rule, “If you eat your vegetables, then I will serve dessert. But if you don’t eat them, then I won’t serve dessert.” More likely, he might say simply, “No dessert unless you eat your veggies!” Such a declaration is a pregame conditional strategic move that, if credibly made, fixes how the parent will react in the upcoming game. If the child believes the parent, this move alters the child’s rollback calculation. The child prunes that branch of the game tree in which the parent serves dessert after the child has not eaten her vegetables and assumes that she cannot achieve that outcome (which gives her payoff 4). This then induces the child to eat her vegetables, as she prefers the outcome in which she eats the vegetables and gets dessert (payoff 3) over the outcome in which she does not eat the vegetables and does not get dessert (payoff 2).

In this example, the parent induces the child to change her action in a way that benefits him. He gets her to change from her default action, Don’t eat, to his favored action, Eat vegetables, by convincing her that he will do something that he normally would not want to do (Don’t give in response to Don’t eat), something that would be harmful to her (because it reduces her payoff). In other words, the parent makes a threat by using a response rule that includes a threat (Don’t give in response to Don’t eat) and an implicit affirmation (Give in response to Eat).

And why should the child believe that the parent really will withhold dessert as he has threatened? After all, if she does not eat the vegetables, her parent will want to serve dessert anyway. This is a reminder that the parent will need to make his strategic move credible if it is to be effective. The parent must somehow motivate himself to follow through on the threatened action even though, normally, he would not want to do so. This process is called *establishing credibility* for the strategic move. For instance, the parent might achieve credibility by making his declaration in front of his other children, putting his authority as a parent on the line and thereby effectively compelling himself to follow through on withholding dessert if the child were to test his resolve. We will examine the issue of credibility in more detail below.

C. The Role of Feasibility, Framing, and Credibility

We conclude this section with some important additional remarks about the use of strategic moves.

First, consider the feasibility of conditional strategic moves. Whether it is feasible for Ann to make a threat or a promise depends on Bob’s payoffs in the game and, in particular, on whether there is anything she can do to make Bob worse off (in the case of making a threat) or better off (in the case of making a promise) than when she plays her best response. If Ann’s best response to the default action is already the most harmful thing that she can do (from Bob’s point of view), then there is no way for her to *threaten* anything further. Ann cannot feasibly make a threat in that type of game. Similarly, if Ann’s best response to the favored action is already the most beneficial thing that she can do (from Bob’s point of view), then there is no way for her to *promise* anything further. She cannot feasibly make a promise in that situation.

For instance, consider again the dinner game shown in Figure 8.1. The child’s default action is Don’t eat, but the parent wants the child to eat her veggies, so his favored action is Eat vegetables. In this game, the parent’s a dominant strategy is to give the child dessert, so his best-response actions are both Give dessert. Whenever the parent chooses a non–best response action (Don’t give), he harms the child; there is no non–best response action that rewards the child. Thus, the parent can potentially make a threat [“If you don’t eat your veggies, I will not give you dessert (and if you do eat them, I will give you dessert)”], but cannot make a promise. His best response of giving dessert when the child eats her veggies is already the best that he can do for her and there is nothing further he can promise.

Second, it sometimes matters how a conditional strategic move is phrased, or framed. When she makes a conditional strategic move, Ann's implied message to Bob can be interpreted as either "Don't take the default action" or "Take the favored action." These two interpretations are classified as, respectively, deterrence and compellence. Depending on the context or framing of the strategic situation, one or the other of these interpretations may be more pertinent, and if multiple strategic moves are available to achieve the same aim, it may be better achieved by one rather than another. Often the best choice depends on the time frame within which Ann wants to achieve her aim. We will illustrate and discuss these issues further in Section 4.B of this chapter.

Finally, because *all* strategic moves entail *not* playing one's best response under certain specific circumstances, establishing *credibility* for these moves is essential. To make effective use of commitments, threats, and promises, you must convince the other players that, if the relevant circumstances arise, you will follow through on your declaration and do something that you normally wouldn't want to do.

There are two basic ways to establish the credibility of a strategic move. First, you can remove the other moves that may tempt you from your own set of future choices. Second, you can change your own payoffs in such a way that it becomes truly optimal for you to act as you have declared you will. This can be done either by *increasing* your own payoff for making the stipulated move or by *reducing* your own payoff for making another move that would otherwise tempt you to deviate from your stipulated move. In the sections that follow, we will first elucidate the mechanics of each type of strategic move, taking as given that credibility can somehow be achieved. We will make some comments about credibility along the way, but will postpone our general analysis of credibility to the last two sections of this chapter.

Glossary

commitment
: An action taken at a pregame stage, stating what action you would take unconditionally in the game to follow.

response rule
: A rule that specifies how you will act in response to various actions of other players.

favored action
: The action that a player making a *strategic move* wants the other player to take, as opposed to the *default action*.

default action
: In the context of strategic moves, the action that the other player (the player not making a strategic move) will take in the absence of a strategic move, as opposed to the *favored action*.

promise
: A response to the *favored action* that benefits the other player and that is not a best response, as specified within a *strategic move*. It is only feasibly made by the second mover in a game. Credibility is required because the specified action is not a best response. The strategic move referred to as "making a promise" entails declaring both a promise and a warning, whereas "making a combined threat and promise" entails declaring both a promise and a threat.

threat
: A response to the default action that harms the other player and that is not a best response, as specified within a strategic move. It can only feasibly be made by the second mover in a game. Credibility is required because the specified action is not a best response. The strategic move referred to as "making a threat" entails declaring both a threat and an affirmation, whereas "making a combined threat and promise" entails declaring both a threat and a promise.

affirmation
: A response to the *favored action* that is a best response, as specified within a strategic move. Only the second mover in a game can feasibly make an affirmation. However, credibility

is not required since the specified action is already the player's best response. The strategic move referred to as "making a threat" entails declaring both a threat and an affirmation.

warning

A response to the default action that is a best response, as specified within a strategic move. Only the second mover in a game can feasibly make a warning. However, credibility is not required since the specified action is already the player's best response. The strategic move referred to as "making a promise" entails declaring both a promise and a warning.

deterrence

An attempt to induce the other player(s) to act to maintain the status quo.

compellence

An attempt to induce the other player(s) to act to change the status quo in a specified manner.

2 COMMITMENTS

We studied the game of chicken in Chapter 4 and found two pure-strategy Nash equilibria: Each player prefers the equilibrium in which he goes straight and the other player swerves.[1] And we saw in Chapter 6 that if the game were to have sequential rather than simultaneous moves, the first mover would choose Straight, leaving the second to make the best of the situation by settling for Swerve rather than causing a crash. Now we can consider the same game from another perspective. Even if the game itself has simultaneous moves, if one player can make a strategic move—create a pregame in which he makes a credible declaration about his action in the upcoming game itself—then he can get the same advantage afforded a first mover—in this case, by making a commitment to act Tough (choose Straight).

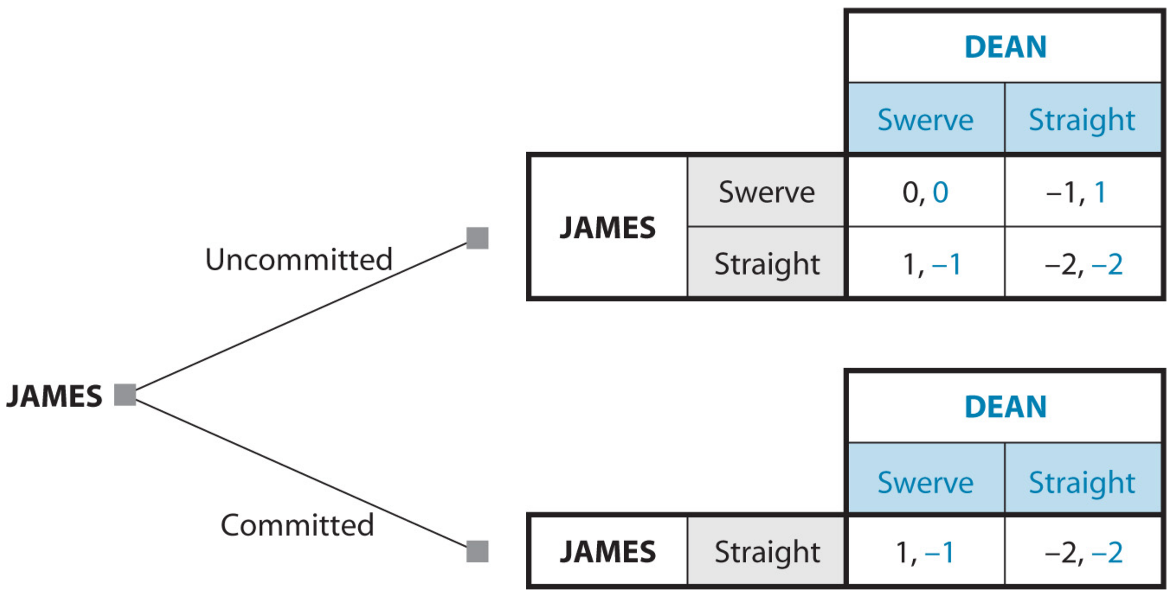

Figure 8.2 Chicken: Commitment by Restricting Freedom to Act

Although the point is simple, we outline the formal analysis here to help you develop your understanding and skill, which will be useful for later, more complex examples. Remember our two players, James and Dean. Suppose James is the one who has the opportunity to make a strategic move. Figure 8.2 shows the tree for the two-stage game. At the first stage (the pregame), James has to decide whether to make a commitment. Along the upper branch emerging from the first node, he does not make a commitment. At the second stage, the original simultaneous-move game is played. Its payoff table is the familiar one shown in Figure 4.16 and Figure 6.7. This second-stage game has multiple equilibria, and James gets his best payoff in only one of them. Along the lower branch emerging from the first node, James makes a commitment. Here, we interpret this commitment to mean giving up his freedom to act in such a way that Straight is the only action available to him at the second stage. Therefore, the second-stage game table has only one row for James, corresponding to his commitment to Straight. In this table, Dean’s best action is Swerve, so the equilibrium outcome gives James his best payoff. Rollback analysis shows that James finds it optimal to make the commitment; this strategic move ensures his best payoff, while not committing leaves the matter uncertain.

How can James make this commitment credibly? Like any first move, the commitment move must be (1) irreversible and (2) observable before the other player makes his choice. People have suggested some extreme and amusing ideas for achieving irreversibility. James can disconnect the steering wheel of the car and throw it out the window so that Dean can see that James can no longer Swerve. Or James could just tie the wheel so that it could no longer be turned, but it would be more difficult to demonstrate to Dean that the wheel was truly tied and that the knot was not a trick one that could be undone quickly. These devices simply remove the Swerve option from the set of choices available to James in the second-stage game, leaving Straight as the only thing he can do.

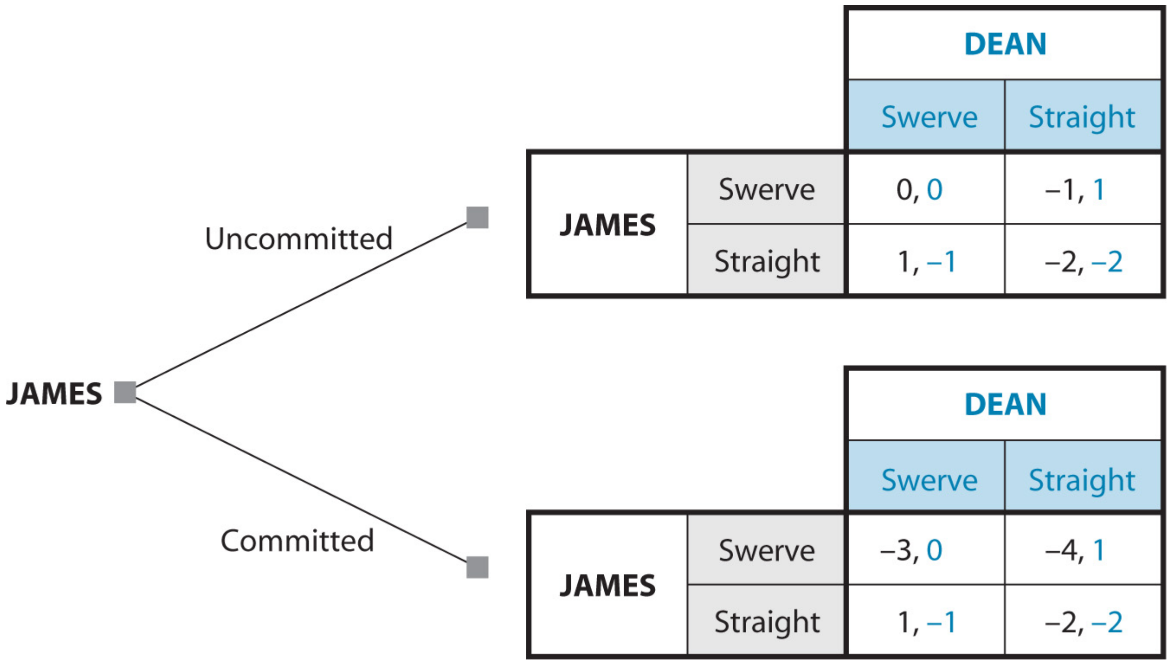

Figure 8.3 Chicken: Commitment by Changing Payoffs

More plausibly, if games of chicken are played every weekend, James can acquire a general reputation for toughness that acts as a guarantee of his action in any one game. In other words, James can alter his own payoff from swerving by subtracting an amount that represents the loss of his reputation. If this amount is large enough—say, 3—then the second-stage game when James has made the commitment has a different payoff table. The complete tree for this version of the game is shown in Figure 8.3.

Now, in the second stage with commitment, Straight has become truly optimal for James; in fact, it is his dominant strategy in that stage. Dean's optimal strategy is then Swerve. Looking ahead to this outcome at the first stage, James sees that he gets payoff 1 by making the commitment (changing his own stage 2 payoffs), while without the commitment, he cannot be sure of 1 and may do much worse. Thus, a rollback analysis shows that James should make the commitment.

Anyone can attempt to make a commitment. Success may depend both on the speed with which you can seize the first move and on the credibility with which you can make that move. If there are lags in observation, the two players may even make incompatible simultaneous commitments: Each disconnects his steering wheel and tosses it out the window just as he sees the other’s wheel come flying out, and the crash becomes unavoidable.

Even if one of the players can gain the advantage by making a commitment, the other player can defeat the first player’s attempt to do so. The second player could demonstrably remove his ability to “see” the other’s commitment—for example, by cutting off communication.

		STUDENT	
		Punctual	Late
TEACHER	Weak	4, 3	2, 4
	Tough	3, 2	1, 1

FIGURE 8.4 Payoff Table for Class Deadline Game

Games of chicken may be a 1950s anachronism, but our second example is perennial and familiar. In a class, the teacher’s enforcement of assignment deadlines can be Weak or Tough, and the students’ submissions can be Punctual or Late. Figure 8.4 shows this game in strategic form. The teacher does not like being tough; for her, the best outcome (with a payoff of 4) is that students are punctual even when she is weak; the worst (1) is that she is tough but students are still late. As for the two intermediate outcomes, she recognizes the importance of punctuality and ranks (Tough, Punctual) better than (Weak, Late). For the students, the best outcome is (Weak, Late), so that they can party all weekend without suffering any penalty for turning in their assignments late. (Tough, Late) is the worst for them, just as it is for the teacher. Of the intermediate outcomes, they prefer (Weak, Punctual) to (Tough, Punctual) because they have higher self-esteem if they can think that they acted punctually of their own volition rather than because of the threat of a penalty.[2]

If this game is played as a simultaneous-move game, or if the teacher moves second, Weak is dominant for the teacher, and the students choose Late. The equilibrium outcome is (Weak, Late), and the payoffs are (2, 4). But the teacher can achieve a better outcome by committing at the outset to the policy of Tough. We do not show a tree for this game, as we did in Figures 8.2 and 8.3, but that tree would be very similar to that for the preceding games of chicken, so we leave it for you to draw. Without a commitment, the second-stage game is as before, and the teacher gets a payoff of 2. However, when the teacher is committed to Tough, the students find it better to respond with Punctual at the second stage, and the teacher gets a payoff of 3. The teacher gains an advantage here by committing to tough enforcement, an action she would not normally take in simultaneous play. Making this strategic move changes the students' expectations, and therefore their actions, in a way that benefits the teacher.

Once the students believe that the teacher is really committed to tough enforcement, they will choose to turn in their assignments punctually. If they test her resolve by being late, the teacher would like to make an exception for them, maybe with an excuse to herself, such as "just this once." The existence of this temptation to shift away from a commitment is what makes its credibility problematic. Like any first move, the teacher's commitment to toughness must be irreversible and observable to students before they make their own decisions whether to be punctual or late. The teacher must establish the ground rules of deadline enforcement right away, before any assignments are due, and the students must know these rules. In addition, the students must know that the teacher cannot, or at any rate will not, change her mind and make an exception for them. A teacher who leaves loopholes and provisions for incompletely specified emergencies is merely inviting imaginative excuses accompanied by fulsome apologies and assertions that "it won't happen again."

The teacher might also achieve credibility by hiding behind general university regulations, as this removes the Weak option from her set of available choices at stage 2. Or, as in the game of chicken, she might establish a reputation for toughness, changing the payoffs she would get from Weak by creating a sufficiently high cost of loss of reputation.

Endnotes

- We saw in Chapter 7, and will see again in Chapter 12, that the game has a third equilibrium, in mixed strategies, in which both players do quite poorly. Return to reference 1
- You may not regard these specific rankings of outcomes as applicable either to you or to your own teachers. We ask you to accept them for this example, whose main purpose is to convey some *general ideas* about commitment in a simple way. The same disclaimer applies to all the examples that follow. Return to reference 2

3 THREATS AND PROMISES

If you were free to do whatever you wanted, you would never follow through when making a threat or a promise, because, by definition, threats and promises specify responses that make you worse off. Nonetheless, threats and promises are powerful strategic tools that allow you to change others' behavior to your advantage. In this section, we provide three in-depth examples of threats and promises in action, along with some discussion of nuances associated with successfully implementing threats and promises in practice.

A. Making a Threat: U.S. – Japan Trade Relations

Our first example is based on economic relations between the United States and its trading partners in the decades after World War II. The United States believed in open markets, both as a general principle and because it hoped they would lead to faster economic growth in the free world and counter Soviet expansionism. Most other countries were *mercantilist*: They liked exports and disliked imports, wanting to restrict foreign access to their own markets, but wanting their producers to have unrestricted access to the large U.S. market. With only a slight caricature, we take Japan to be a representative of such countries at the time.

Figure 8.5 shows the payoff table for the U.S. – Japan trade game. For the United States, the best outcome (with a payoff of 4) comes when both countries' markets are open. This is partly because of its overall commitment to open markets and partly because of the benefits of trade with Japan itself: U.S. consumers get high-quality cars and consumer electronics, and U.S. producers can export their agricultural and high-tech products. Similarly, its worst outcome (payoff 1) occurs when both markets are closed. Of the two outcomes in which only one market is open, the United States prefers that its own market to be open because Japan's market is smaller, and loss of access to it is less important than the loss of access to Hondas and microchips.

		JAPAN	
		Open	Closed
UNITED STATES	Open	4, 3	3, 4
	Closed	2, 1	1, 2

FIGURE 8.5 Payoff Table for U.S. – Japan Trade Game

As for Japan, for the purpose of this example, we accept the protectionist, producer-oriented picture of "Japan, Inc." It gets its best outcome when the U.S. market is open and its own is closed; it gets its worst outcome when its own market is open and the U.S. market is closed; and of the other two outcomes, it prefers that both markets be open rather than that both be closed because its producers then have access to the much larger U.S. market.[3]

Note that each country has a dominant strategy: for the United States, an open market, and for Japan, a closed market. Thus, no matter what the order of moves—simultaneous, sequential with the United States first, or sequential with Japan first—the equilibrium outcome is (Open, Closed). Japan gets its best possible outcome (payoff 4) in this equilibrium and so has no need for strategic moves. The United States, however, gets only its second-best outcome (payoff 3). Can the United States employ a strategic move to get its best outcome in which both markets are open?

An unconditional commitment to open markets will not work to improve the U.S. payoff because Japan's best response will then be to keep its market closed. But suppose that the United States were to make the following threat: "We will close our market if you close yours." (As discussed previously, the explicit statement of only one half of a full conditional response rule generally implies a best-response action as the other half of the rule. In this case, the implied affirmation is, "We will keep our market open if you keep yours open." That action is the U.S. best response to the favored action of Open.) With the threat in place, the situation becomes the two-stage game shown in Figure 8.6. If the United States does not make the threat, the second stage is as before and leads to the equilibrium in which the U.S. market is open and it gets payoff 3, whereas Japan's market is closed and it gets payoff 4. If the United States does make the threat, then at the second stage, only Japan has freedom of choice. Along this branch of the tree, we show only Japan as an active player and write down the payoffs to the two parties: If Japan keeps its market closed, the United States will close its own, leading to payoff 1 for the United States and payoff 2 for

Japan. If Japan keeps its market open, then the United States will open its own, leading to payoff 4 for the United States and payoff 3 for Japan. Faced with (only) these two options, Japan will choose to open its market. Finally, reasoning backward to the very beginning of the two-stage game, the United States will choose to declare its threat, anticipating that Japan will respond by opening its market, and will get its best possible outcome.

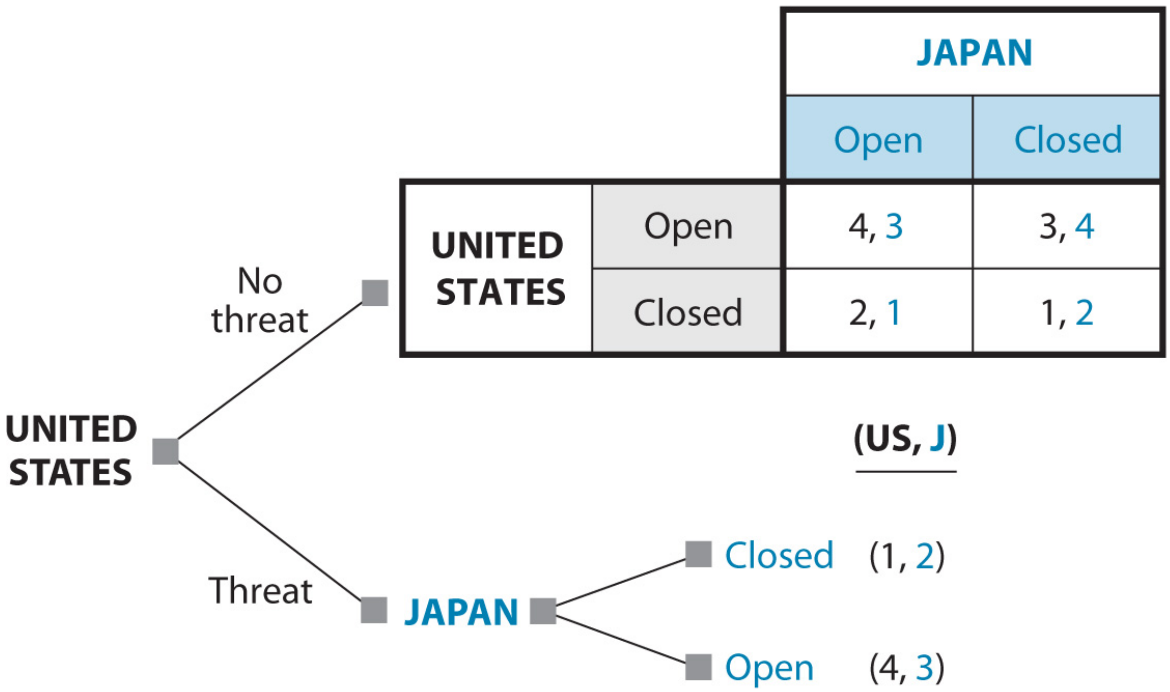

Figure 8.6 Game Tree for the U.S. – Japan Trade Game with Threat

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Having described the mechanics of the threat, we now point out some of its important features, recapitulating and strengthening some points made earlier. First, notice that when the United States deploys its threat credibly, Japan doesn't follow its dominant strategy, Closed. Japan knows that the United States will take actions that depart from *its* dominant strategy and responds accordingly. Japan is looking at a choice between just two cells in the payoff table, the top-left and the bottom-right, and of those two, it prefers the former.

Next, notice that the credibility of the threat is problematic because, if Japan were to put it to the test by keeping its market closed, the United States would face the temptation to refrain from carrying out the threat. If the threatened action were the best U.S. response after the fact, there would have been no need to make the threat in advance (although the United States might have issued a *warning* just to make sure that Japan understood the situation). The strategic move has a special role exactly because it locks a player into doing something other than what it would have wanted to do after the fact.

How can the United States make its threat credible, then? One option is to enact a law that mandates the threatened action under the right circumstances. Such a law would remove the best-response action from the set of available choices at stage 2. Some reciprocity provisions in the World Trade Organization agreements have this effect, but the procedures involved are very slow and uncertain. Another possibility would be to delegate fulfillment of the threat to an agency such as the U.S. Commerce Department, which has been captured by U.S. producers who would like to keep U.S. markets closed so as to reduce the competitive pressure on their firms. This delegation would change the U.S. payoffs in the game—replacing the true U.S. payoffs with those of the Commerce Department—making the threatened action truly optimal. (A danger of this approach is that the Commerce Department might then retain a protectionist stance even if Japan opened its market. In that case, gaining credibility for the threat might cause the implied affirmation to become a *promise* that would then have to be made credible.)

Another key feature of a threat is that if it works to induce the other player to take the favored action, the player making the threat *does not have to carry out* the threatened action. Whether the threat is large or small is immaterial. All that matters is whether the threat is big enough to work. So it might seem that you should always use as big a threat as possible, just to make absolutely sure that the other player will have an incentive to comply. However, there is a danger in using supersized threats. Perhaps you have miscalculated or misunderstood the payoff structure, or maybe the threatened action could take place by mistake even if the other player performs the favored action, or perhaps the other player doesn't believe that you will actually follow through with such a mutually harmful action. For these reasons, it is wise to refrain from using threats that are more severe than necessary. For example, imagine that the United States were to threaten to pull out of its defensive alliances with Japan if Japan didn't buy its rice and semiconductors. Because the U.S.–Japan alliance is of such great military and geopolitical importance to the United States, many in Japan would doubt that the United States would ever follow through on such a threat, undermining its credibility and hence its effectiveness.

If the only available threat appears "too big," a player can reduce its size by making its fulfillment a matter of chance. A threat of this kind, which creates a risk, but not a certainty, of the bad outcome, is called *brinkmanship*. It is an extremely delicate and even dangerous variant of the conditional strategic move. We will discuss brinkmanship in greater detail in Section 5.B and illustrate its use during the Cuban missile crisis in Chapter 13.

Finally, observe that Japan gets a worse outcome when the United States deploys its threat than it would without the threat, so it would like to take strategic action to defeat or disable U.S. attempts to use the threat. For example, suppose its market is currently closed. Japan can agree to open its market in principle, but stall in practice, pleading unavoidable delays for assembling the necessary political consensus to legislate the market opening, then more delays for writing the necessary administrative regulations to implement the legislation, and so on. Because the United States does not want to go ahead with its threatened action, at each point it is tempted to accept the delay. Or Japan can claim that its domestic politics makes it difficult to open all markets fully; will the United States accept the outcome if Japan keeps just a few of its industries protected? It gradually expands this list, and at any point, the next small step is not enough cause for the United States to unleash a trade war. This device of defeating a threat by small steps, or "slice by slice," is called *salami tactics*. We will discuss this and other ways to undermine another's strategic moves in Section 6 of this chapter.

		YVONNE'S BISTRO	
		20 (low)	26 (high)
XAVIER'S TAPAS	20 (low)	288, 288	360, 216
	26 (high)	216, 360	324, 324

FIGURE 8.7 Payoff Table for Restaurant Prisoners' Dilemma (in Hundreds of Dollars per Month)

Our U.S.–Japan trade game example comes close to capturing the reality of the 1980s, when the United States was by far the dominant economy in the free world and, as leader, undertook the responsibility to prevent the world's international trade system from falling into a spiral of protectionism like the one that caused so much harm in the Great Depression of the 1930s.[4] Now, with the economic rise of China and Europe, this leadership role has become much less important, and the United States has become more mercantilist. In the next edition of this book, we may have to re-work this example, reversing the roles!

B. Making a Promise: The Restaurant Pricing Game

Consider the restaurant pricing game of Chapter 5, simplified here to have only two possible prices: the jointly best price of \$26 or the Nash equilibrium price of \$20. We saw in Chapter 5 that this simplified version of the game is a prisoners' dilemma, in which both restaurants (Xavier's Tapas Bar and Yvonne's Bistro) have a dominant strategy—to price low—but both are better off when both price high. Each restaurant's profits can be calculated using the equations in Section 1 of Chapter 5; the results are shown in Figure 8.7. Without any strategic moves, the game has the usual equilibrium in dominant strategies in which both restaurants charge the low price of \$20, and both get lower profits than they would if they both charged the high price of \$26.

If either restaurant owner can make the credible promise, "I will charge a high price if you do" (and, implicitly, "I will charge a low price if you do"), the best joint outcome is achieved. For example, if Xavier makes the promise, then Yvonne knows that her choice of \$26 will be reciprocated, leading to the payoff shown in the lower-right cell of the table, and that her choice of \$20 will bring forth Xavier's best response—namely, matching her price of \$20—leading to the upper-left cell. Of the two payoffs, Yvonne prefers the first, and therefore chooses the high price.

The analysis can be done more properly by drawing a tree for the two-stage game in which Xavier has the choice of making or not making the promise at the first stage. We omit the tree, partly so that you can improve your understanding of the process by constructing it yourself and partly to show that such detailed analysis becomes unnecessary as one becomes familiar with the ideas involved.

The feasibility and credibility of Xavier’s promise is open to doubt. First, in order for Xavier to be able to respond to Yvonne’s move, he must arrange to move second in the pricing game, or Yvonne must arrange to move first. Assuming that the pricing game can be changed so that Yvonne moves first and Xavier moves second, Xavier’s promise will be feasible—but will it be credible? If Yvonne moves first and (irreversibly) sets her price high, Xavier will be tempted to renege on his promise and set his price low. Xavier must somehow convince Yvonne that he will not give in to the temptation to cheat—that is, to charge a low price when Yvonne charges a high price.

How can he do so? Perhaps Xavier can leave the pricing decision in the hands of a local manager, with clear written instructions to reciprocate with the high price if Yvonne charges the high price. Xavier can invite Yvonne to inspect these instructions, after which he can leave on a solo round-the-world sailing trip so that he cannot rescind them. (Even then, Yvonne may be doubtful—Xavier might secretly carry a telephone or a laptop computer on board.) This scenario is tantamount to removing the cheating action from the choices available to Xavier in the second-stage game.

Or Xavier can develop a reputation for keeping his promises, in business and in the community more generally. In a continuing interaction, the promise may work because reneging on the promise once could cause future cooperation to collapse. In essence, an ongoing relationship means splitting the game into smaller segments, in each of which the benefit of reneging is too small to justify the cost. In each such game, then, the payoff from cheating is altered by the cost of a collapse of future cooperation.[5]

Note that, as in the case of the U.S. threat to Japan, the promise “I will price high if you price high” is only half of Xavier’s response rule. The other half, as noted above, is “I will price low if you price low.” This statement is a *warning* (not a threat) because pricing low is already Xavier’s best response; it need not necessarily be made explicit.

Threats and promises differ in a crucially important way. When a threat succeeds in inducing the other player to take the favored action, the player making the threat does not have to carry out the threatened action and does not have to pay any price for having made the threat. On the other hand, when a promise is successful, the player making the promise always has to follow through and deliver what has been promised, which is costly. In the restaurant pricing example, the cost to Xavier of following through on his promise to match a high price is giving up the opportunity to undercut Yvonne and get the highest possible profit; in other instances where the promiser offers an actual gift or an inducement to the other player, the cost may be more tangible. A player making a promise therefore has a clear incentive to keep the cost of the promised action as small as possible.

		CHINA	
		Action	Inaction
UNITED STATES	Action	3, 3	2, 4
	Inaction	4, 1	1, 2

FIGURE 8.8 Payoff Table for U.S.–China Political Action Game

C. Making a Combined Threat and Promise: Joint U.S.–China Political Action

Finally, we consider an example in which a player benefits from making a combined threat and promise. In this example, the United States and China are contemplating whether to take action to compel North Korea to give up its nuclear weapons programs. Figure 8.8 is the payoff table for the United States and China when each must choose between action and inaction.

Each country would like the other to take on the whole burden of taking action against North Korea, so the top-right cell has the best payoff for China (4), and the bottom-left cell has the best payoff for the United States. The worst outcome (payoff 1) for the United States is the one in which no action is taken, because it finds the increased threat of nuclear war in that case to be unacceptable. For China, however, the worst outcome arises when it takes on the whole burden of action, because the costs of its action are so high. Both countries regard a joint action as the second-best outcome (payoff 3). The United States assigns a payoff of 2 to the situation in which it is the only country to act. And China assigns a payoff of 2 to the case in which no action is taken.

Without any strategic moves, this intervention game is dominance solvable. Inaction is the dominant strategy for China, and Action is the best response of the United States when China chooses Inaction. The equilibrium outcome is therefore the top-right cell, with payoffs of 2 for the United States and 4 for China. Because China gets its best outcome, it has no need for strategic moves. But the United States can try to do better.

Can the United States use a strategic move to improve on its equilibrium payoff—to get either its best or its second-best outcome rather than its second-worst? First, consider whether the United States can induce China to intervene alone in North Korea, the very best outcome for the United States (payoff 4). Unfortunately, no strategic move makes this possible. To see why, note that any strategic move ultimately gives China a choice between two outcomes, the one that follows from China choosing Action and the one that follows from China choosing Inaction. Intervening alone is China's very worst outcome here (payoff 1), so there is no way to induce China to make that choice. When choosing between intervening alone and *any other outcome*, China will choose the other outcome.

Can the United States use a strategic move to achieve its second-best outcome, a joint intervention with China? Possibly, but only by declaring a response rule that combines a promise ("If you [China] intervene, then so will we [United States]") and a threat ("If you don't intervene, then neither will we"). This conditional strategic move, if credibly made, forces China to choose between the top-left cell (in which both intervene) and the bottom-right cell (in which neither intervenes) in Figure 8.8. Faced with this choice, China prefers to intervene, and the United States gets its second-best outcome.

Because the threat and the promise both entail taking actions that go against U.S. national interest, the United States has to make both the threat and the promise explicit and find a way to make them each credible. Usually such credibility has to be achieved by means of a treaty that covers the whole relationship, not just with agreements negotiated separately when each incident arises.

Endnotes

- Again, we ask you to accept this payoff structure as a vehicle for conveying some general ideas. You can experiment with the payoff tables to see what difference that would make to the roles of the two countries and the effectiveness of their strategic moves. Return to reference 3
- In Chapter 10, we offer some general analysis of how a big player can resolve a prisoners' dilemma through leadership. Return to reference 4
- In Chapter 10, we will investigate in greater detail the importance of repeated or ongoing relationships in attempts to reach the cooperative outcome in a prisoners' dilemma. Return to reference 5

4 SOME ADDITIONAL TOPICS

A. When Do Strategic Moves Help?

We have seen several examples in which a strategic move brings a better outcome to one player or another than the original game without such moves. What can be said in general about the desirability of such moves?

Making an unconditional strategic move—a commitment—is not always advantageous. This is especially obvious when the original game has a second-mover advantage. In that case, committing oneself to a move in advance is clearly a mistake, as doing so effectively makes one the first mover. In such situations, each player will do whatever he can to avoid being forced to make a commitment in the pregame.

Making a conditional strategic move, on the other hand, can never be disadvantageous. At the very worst, one can declare a response rule that would have been optimal after the fact. However, if such a move brings one an actual gain, it must be because one is choosing a response rule that in some eventualities specifies an action different from what one would find optimal at that later time. Thus, whenever a conditional strategic move brings a positive gain, it does so precisely when (one might say precisely because) its credibility is inherently questionable and must be achieved by some credibility device. We have mentioned some such devices in connection with each earlier example, and we will discuss the topic of achieving credibility in greater generality in the next section of this chapter.

What about the desirability of being on the receiving end of a strategic move? The answer depends on what type of strategic move is being made. It is never desirable to let another player threaten you. If a threat seems likely, you can gain by looking for a different kind of advance action—one that makes the threat less effective or less credible. We will consider some such actions in the final section of this chapter. It is, however, often desirable to let another player make promises to you. In fact, both players may benefit when one can make a credible promise, as in the prisoners' dilemma example of restaurant pricing earlier in this chapter, in which a promise achieved the cooperative outcome. Thus, it may be in the players' mutual interest to facilitate the making of promises in the pregame. Finally, it may or may not be desirable to let the other player make a commitment, as we saw in Chapter 6, Section 2.B. In the monetary–fiscal policy game shown in Figure 6.8, the Fed benefits when Congress is able to make a commitment; but in the game of chicken shown in Figure 6.7, each player gets their best outcome if they alone are able to make a commitment. Thus, in some cases you may benefit by enabling the other player to make a commitment, while in other cases you will be better off blocking his ability to do so.

B. Deterrence versus Compellence

In principle, a player making a threat or making a promise can use these moves to achieve either deterrence or compellence. Recall that deterrence entails sending the message, “Don’t take the default action,” and compellence entails sending the message, “Take the favored action.” These goals can seem very similar. Often there are only two possible actions, and not taking the default action is equivalent to taking the favored one. For example, a parent who wants a child who would normally take the default action of not studying to take the favored action of studying hard can promise a reward (a new mountain bike) for good performance in school or can threaten a punishment (a strict curfew the following term) if the performance is not sufficiently good. Similarly, a parent who wants a child not to let her room get messy, the child’s default action, can try either a reward (promise) or a punishment (threat) in order to get the child to take the favored action. Whether either of these moves is actually deterrent or compellent depends on how the move relates to some status quo in the interaction.

As a rule of thumb, the status quo in an upcoming game is defined by one of two things. The status quo describes either a tangible current state of affairs (the child’s bedroom is currently neat and tidy) or a player’s natural inclination to behave in a particular way (the child doesn’t like to study). Strategic moves that attempt to maintain the status quo are deterrent; those that attempt to change it are compellent. Thus, the promise to reward a child who keeps her room clean (or the threat to punish a child who lets her room get messy) is deterrent. And the promise to reward a child who studies (or the threat to punish a child who doesn’t) is compellent.

In practice, deterrence is better achieved by a threat and compellence by a promise, as we explain in more detail below. That means you might want to manipulate the status quo to your advantage if it is possible to do so. If you do not want to have to make the deterrent promise, “I will reward you for keeping your room clean,” because it isn’t obvious when you should follow through on your promise, you can wait until the child’s room is messy. (Presumably, you wouldn’t need to wait all that long.) Promising to reward the child for cleaning up a currently messy room is compellent and may be more easily put into effect.

Why is deterrence better achieved by making a threat? A deterrent threat can be passive—you don’t need to do anything so long as the other player doesn’t do what you are trying to deter. And it can be static—you don’t have to impose any time limit. Thus, you can set a trip wire and then leave things up to the other player. Consider again the parent who wants the child to keep her currently clean room in its pristine condition. The parent can threaten, “If you ever go to bed without cleaning your room, I will take away your screen privileges for a month.” Then the parent can sit back to wait and watch; only if the child acts contrary to the parent’s wishes does the parent have to act on her threat. Trying to achieve the same deterrence with a promise would require more complex monitoring and continual action: “At the end of each week in which I know that you kept your room tidy each day, I will give you $25.”

Compellence does not entail the passiveness of deterrence. Compellence must have a deadline or it is pointless—the other side can defeat your purpose by procrastinating or by eroding your threat in small steps (salami tactics). This makes a compellent threat harder to implement than a compellent promise. The parent who wants the child to study hard can simply say, “Each term at college that you get an average of B or better, I will give you $500.” The child will then take the initiative in showing the parent each time he has fulfilled the conditions. Trying to achieve the same thing by a threat— “Each term that your average falls below B, I will take away one of your electronic devices” —will require the parent to be much more vigilant and active. The child will postpone bringing home the grade report or will try to hide his devices.

We reiterate that you can change a threat into a promise, or deterrence into compellence, or vice versa, by changing the status quo. And you can use this change to your own advantage when making a strategic move. If you want to achieve compellence, try to choose a status quo such that what you do when the other player acts to comply with your demand becomes a reward, so that you are using a compellent promise. To give a rather dramatic example, a mugger can convert the threat, “If you don’t give me your wallet, I will take out my knife and cut your throat” into the promise, “Here is a knife at your throat; as soon as you give me your wallet I will take it away.” And if you want to achieve deterrence, try to choose a status quo such that, if the other player acts contrary to your wishes, what you do is a punishment, so that you are using a deterrent threat.

Glossary

credibility device

A means by which a player acquires credibility, for instance, when declaring a promise or threat as part of a strategic move.

5 MAKING YOUR STRATEGIC MOVES CREDIBLE

We have emphasized the importance of the credibility of strategic moves throughout this chapter, and we have accompanied each example with some brief remarks about how credibility could be achieved in that particular context. Most devices for achieving credibility are indeed context specific, and there is a lot of art to discovering or developing such devices. Some general principles can help you organize your search.

We pointed out two broad approaches to credibility: (1) reducing your own future freedom of action in such a way that you have no choice but to carry out the action stipulated by your strategic move, and (2) changing your own future payoffs in such a way that it becomes optimal for you to do what you stipulate in your strategic move. We now elaborate some practical methods for implementing each of these approaches.

A. Reduce Your Freedom of Action

I. AUTOMATE YOUR RESPONSE Suppose that, in the pregame, you relinquish your ability to make decisions in the upcoming game and instead hand that authority over to a mechanical device, or similar procedure or mechanism, that is programmed to carry out your committed, threatened, or promised action under the appropriate circumstances. You demonstrate to the other player that you have done so. Then he will be convinced that you have no freedom to change your mind, and your strategic move will be credible (as long as he does not suspect that you control an override button to prevent a catastrophe!). The doomsday device, a nuclear explosive device that would detonate and contaminate the whole world's atmosphere if the enemy launched a nuclear attack, is the best-known example of this kind of mechanical device; it was popularized by the early 1960s movies *Fail Safe* and *Dr. Strangelove*.

II. DELEGATE YOUR DECISIONYou could delegate the power to act to another person, or agent, or to an organization that is required to follow certain pre-set rules or procedures. That is how the United States makes credible its threat to levy countervailing duties (retaliatory tariffs imposed to counter other countries' export subsidies). They are set by two agencies of the U.S. government—the Commerce Department and the International Trade Commission—whose operating procedures are laid down in the general trade laws of the country.

Your agent should not have his own objectives that defeat the purpose of your strategic move. For example, if one player delegates to an agent the task of inflicting threatened punishment, and the agent is a sadist who enjoys inflicting punishment, then he may act even when there is no reason to act—that is, even when the second player has complied. If the second player suspects this, then the threat loses its effectiveness, because his options amount to “damned if you do and damned if you don’t.”

As with automated response mechanisms like the doomsday device, delegation may not provide a complete guarantee of credibility because mandates can always be altered. In fact, the U.S. government has often set aside countervailing duties and reached other forms of agreement with other countries so as to prevent costly trade wars.

III. BURN YOUR BRIDGESMany invaders, from Xenophon in ancient Greece to William the Conqueror in England to Cortés in Mexico, are supposed to have deliberately cut off their own army’s avenue of retreat to ensure that it would fight hard. Some of them literally burned bridges behind them, while others burned ships, but the device has become a cliché. Its most recent users in a military context may have been the Japanese kamikaze pilots in World War II, who carried only enough fuel to reach the U.S. naval ships into which they were to ram their airplanes. The principle even appears in the earliest known treatise on war, in a commentary attributed to Prince Fu Ch’ai: “Wild beasts, when they are at bay, fight desperately. How much more is this true of men! If they know there is no alternative they will fight to the death.”[6]

Related devices are used in other high-stakes games. Although the member countries of Europe’s Economic and Monetary Union (EMU) could have retained separate currencies and merely fixed the exchange rates among them, they adopted a common currency—the euro—precisely to make the process irreversible and thereby give themselves a much greater incentive to make the union a success. (In fact, it was the extent of the necessary commitment that kept some nations, Great Britain in particular, from agreeing to be part of the

EMU.) It would not be totally impossible to abandon the euro and go back to separate national currencies—but it would be very costly. And indeed, plans to leave the eurozone have been seriously discussed—but never implemented—in several member countries, most notably Greece, over the last decade. Like that of automated devices, the credibility of burning bridges is not an all-or-nothing matter, but one of degree.

IV. CUT OFF COMMUNICATIONIf you send the other player a message declaring your commitment to a particular action and at the same time cut off any means for her to communicate with you, then she cannot argue or bargain with you to reverse your action. The danger in cutting off communication is that if both players do so simultaneously, they may make mutually incompatible commitments that can cause great mutual harm. Additionally, cutting off communication is harder to do with a conditional strategic move, because you have to remain open to the one message that tells you whether the other player has complied and therefore whether you need to carry out your threat or promise. In this high-tech age, it is also quite difficult for a person to cut herself off from all contact.

But large teams or organizations can try variants of this device. Consider a labor union that makes its decisions at mass meetings of members. To convene such a meeting takes a lot of planning—reserving a hall, communicating with members, and so forth—and several weeks of time. A meeting is convened to decide on a wage demand. If management does not meet the demand in full, the union leadership is authorized to call a strike, and then it must call a new mass meeting to consider any counteroffer. This process puts management under a lot of time pressure in the bargaining; it knows that the union will not be open to communication for several weeks at a time. Here, we see that cutting off communication for extended periods can establish some degree of credibility, but not absolute credibility. The union’s device does not make communication totally impossible; it only creates several weeks of delay.

B. Change Your Payoffs

I. ESTABLISH A REPUTATION You can change your payoffs by acquiring a reputation for carrying out threats and delivering on promises. Such a reputation is most useful in a repeated game against the same player. It is also useful when playing different games against different players if each of those players can observe your actions in the games that you play with others. The circumstances favorable to the emergence of such a reputation are the same as those for achieving cooperation in the prisoners' dilemma, as we will see in Chapter 10, and for the same reasons. The greater the likelihood that the interaction will continue, and the greater the concern for the future relative to the present, the more likely the players will be to resist current temptations for the sake of future gains. The players will therefore be more willing to acquire and maintain reputations.

In technical terms, reputation is a device that links different games so that the payoffs of actions in one game are altered by the prospects of repercussions in other games. If you fail to carry out your threat or promise in one game, your reputation suffers, and you get a lower payoff in other games. Therefore, when you consider any one of these games, you should adjust your payoffs in that game to take into consideration such repercussions on your payoffs in the linked games.

The benefit of reputation in ongoing relationships explains why your regular car mechanic is less likely to cheat you by doing an unnecessary or excessively costly or shoddy repair than is a random garage that you go to in an emergency. But what would your regular mechanic actually stand to gain from acquiring this reputation if competition forced him to charge a price so low that he would no profit on any transaction? You pay indirectly for his integrity when he fixes your car—you have to be willing to let him charge you a little bit more than the rates that the cheapest garage in the area might advertise. The same reasoning explains why, when you are away from home, you might settle for the known quality of a restaurant chain instead of taking the risk of going to an unknown local restaurant. And a department store that expands into a new line of merchandise can use the reputation that it has acquired for its existing lines to promise its customers the same high quality in the new line.

In games where credible promises by one or both parties can bring mutual benefit, the players can cooperate in fostering the development of their reputations. But if the interaction ends at a known, specific time, there is always the problem of the endgame.

In the Middle East peace process that started in 1993 with the Oslo Accords, the early steps, in which Israel transferred some control over Gaza and small isolated areas of the West Bank to the Palestinian Authority and the latter accepted the existence of Israel and reduced its anti-Israel rhetoric and violence, continued well for a while. But as the final stages of the process approached, neither side trusted the other to deliver on its promises, and by 1998 the process stalled. Sufficiently attractive rewards could have come from the outside; for example, the United States or Europe could have promised both parties economic aid or expanded commerce to keep the process going. The United States offered Egypt and Israel large amounts of aid in this way to achieve the Camp David Accords in 1978. But such promises were not made in the more recent situation, and at the date of this writing, prospects for progress do not look bright.

II. DIVIDE THE GAME INTO SMALL STEPSSometimes a single game can be divided into a sequence of smaller games, thereby allowing reputation effects to come into play. In home construction projects, it is customary for the homeowner to pay the contractor in installments as the work progresses. In the Middle East peace process, Israel would never have agreed to a complete transfer of the West Bank to the Palestinian Authority in one fell swoop in return for a single promise to recognize Israel and stop its attacks. Proceeding in steps enabled the process to go at least part of the way. But this case again illustrates the difficulty of sustaining momentum as the endgame approaches.

III. USE TEAMWORKTeamwork is yet another way to embed one game in a larger game to enhance the credibility of strategic moves. It requires a group of players to monitor one another. If one fails to carry out a threat or a promise, the others are required to inflict punishment on him; failure to do so makes them, in turn, vulnerable to similar punishment by the others, and so on. Thus, a player’s payoffs in the larger game are altered in a way that makes it credible that each individual member’s actions will conform to the team’s norms of behavior.

Many universities have academic honor codes that act as credibility devices for students. Examinations are not proctored by the faculty; instead, students are required to report to a student committee if they see any cheating. That committee holds a hearing and hands out punishment, as severe as suspension for a year or outright expulsion, if it finds the accused student guilty of cheating. Students are very reluctant to place their fellow students in such jeopardy. To stiffen their resolve, such codes include the added twist that failure to report an observed infraction is itself an offense against the code. Even then, the general belief is that these codes work only imperfectly. A survey of 417 undergraduates at Princeton University in 2009 found that “one of every five respondents admitted to violating a professor’s rule for take-home assignments,” and that “of the 85 students who said they had become aware of another student violating the Honor Code, only four said they reported the infraction.” [7]

IV. APPEAR IRRATIONALYour threat may lack credibility if other players know you are rational and that it is too costly for you to follow through with your threatened action. Therefore, those other players may believe that you will not carry out the threatened action if you are put to the test. You can counter this problem by appearing to be irrational so that they will believe that your payoffs are different from what they originally perceived. Apparent irrationality can thus turn into strategic rationality when the credibility of a threat is in question. Similarly, apparently irrational motives, such as honor or saving face, may make it credible that you will deliver on a promise even when tempted to renege.

The other player may see through such rational irrationality. Therefore, if you attempt to make your threat credible by claiming irrationality, he will not readily believe you. You will have to acquire a reputation for irrationality—for example, by acting irrationally in some related game. You could also use one of the strategies that we will discuss in Chapter 9 and do something that is a credible *signal* of irrationality (or delegate your decision to someone who is truly crazy) to achieve an equilibrium in which you can separate from the falsely irrational.

V. WRITE A CONTRACTYou can make it costly to yourself to fail to carry out a threat or to deliver on a promise by signing a contract under which you have to pay a sufficiently large sum in that eventuality. If such a contract is written with sufficient clarity that it can be enforced by a court or some outside authority, the resulting change in payoffs makes it optimal for you to carry out the stipulated action, and your threat or the promise becomes credible.

In the case of a promise, the other player can be the other party to the contract. It is in his interest that you deliver on the promise, so he will hold you to the contract if you fail to fulfill the promise. A contract to enforce a threat is more problematic. The other player does not want you to carry out the threatened action and will not want to enforce the contract, unless he gets some longer-term benefit in associated games by being subjected to a credible threat in this one. To implement a threat, the contract therefore has to include a third party. But when you bring in a third party and a contract merely to ensure that you will carry out your threat if put to the test, the third party does not actually benefit from your failure to act as stipulated. The contract thus becomes vulnerable to any renegotiation that would provide the third-party enforcer with some positive benefit. If the other player puts you to the test, you can say to the third party, “Look, I don’t want to carry out the threat. But I am being forced to do so by the prospect of the penalty in the contract, and you are not getting anything out of all this. Here is a sum of money in exchange for releasing me from the contract.” Thus, the contract itself is not credible; therefore, neither is the threat. The third party must have its own longer-term reasons for holding you to the contract, such as wanting to maintain its reputation, if the contract is to be renegotiation-proof and therefore credible.

Written contracts are usually more binding than verbal ones, but even verbal ones may constitute commitments. When George H. W. Bush said, “Read my lips; no new taxes” in the presidential campaign of 1988, the American public took this promise to be a binding contract; when Bush reneged on it in 1990, the public held that against him in the election of 1992.

VI. USE BRINKMANSHIPIn the U.S.–Japan trade game, we found that a threat can be too big to be credible. If a smaller but effective threat cannot be found in a natural way, the size of the large threat can be reduced to a credible level by making its fulfillment a matter of chance. The United States may not credibly be able to say to Japan, “If you don’t keep your markets open to U.S. goods, we will not defend you if the Russians or the Chinese attack you.” But it can credibly say, “If you don’t keep your markets open to U.S. goods, the relations between our countries will deteriorate, which will create the risk that, if you are faced with an invasion, Congress at that time will not sanction U.S. military involvement in defending you.” As mentioned earlier, such deliberate creation of risk is called brinkmanship. This is a subtle idea and difficult to put into practice. Brinkmanship is best understood by seeing it in operation, and the detailed case study of the Cuban missile crisis in Chapter 13 serves just that purpose.

In this section, we have described several devices for making one’s strategic moves credible and examined how well they work. In conclusion, we want to emphasize a point that runs through the entire discussion: Credibility in practice is not an all-or-nothing matter, but one of degree. Even though the theory is stark—rollback analysis shows either that a threat works or that it does not—the practical application of strategic moves must recognize that between these polar extremes lies a whole spectrum of possibility and probability.

Endnotes

- Sun Tzu, *The Art of War*, trans. Samuel B. Griffith (Oxford: Oxford University Press, 1963), p. 110. Return to reference 6
- The referenced survey was conducted by the *Daily Princetonian*, Princeton University's student-run newspaper. See Michelle Wu and Jack Ackerman, "In honor we trust?" *Daily Princetonian*, April 30, 2009, http://www.dailyprincetonian.com/article/2009/04/in-honor-we-trust, and "Editorial: We don't second that," *Daily Princetonian*, May 7, 2009, http://www.dailyprincetonian.com/article/2009/05/editorial-we-dont-second-that. Return to reference 7

Glossary

doomsday device

An automaton that will under specified circumstances generate an outcome that is very bad for all players. Used for giving credibility to a severe threat.

reputation

Relying on the effect on payoffs in future or related games to make threats or promises credible, when they would not have been credible in a one-off or isolated game.

rational irrationality

Adopting a strategy that is not optimal after the fact, but serves a rational strategic purpose of lending credibility to a threat or a promise.

contract

In this context, a way of achieving credibility for one’s strategic move by entering into a legal obligation to perform the committed, threatened, or promised action in the specified contingency.

6 COUNTERING YOUR OPPONENT’S STRATEGIC MOVES

If your opponent can make a commitment or a threat that works to your disadvantage, then, before he actually does so, you may be able to make a strategic countermove of your own. You can do so by making his future strategic move less effective—for example, by removing its irreversibility or undermining its credibility. In this section, we examine some devices that can help achieve this purpose. Some are similar to the devices that the other side can use for its own purposes.

A. Appear Irrational

Apparent irrationality can work for the intended recipient of a commitment or a threat just as well as it does for the maker. If you are known to be so irrational that you will not give in to any threat and will accept the damage that befalls you when your opponent carries out that threat, then he may as well not make the threat in the first place, because having to carry it out will only end up hurting him, too. Everything that we said earlier about the difficulties of credibly convincing the other side of your irrationality holds true here as well.

B. Cut Off Communication

If you make it impossible for your opponent to convey to you the message that she has made a certain commitment or a threat, then your opponent will see no point in doing so. Thomas Schelling illustrates this possibility with the story of a child who is crying too loudly to hear his parent’s threats.[8] Thus, it is pointless for the parent to make any strategic moves; communication has effectively been cut off.

C. Leave Escape Routes Open

If your opponent can benefit by burning bridges to prevent his own retreat, you can benefit by dousing those fires, or perhaps even by constructing new bridges or roads by which your opponent can retreat. This device was also known to the ancients. Sun Tzu said, “To a surrounded enemy, you must leave a way of escape.” The intent is not actually to allow the enemy to escape. Rather, “show him there is a road to safety, and so create in his mind the idea that there is an alternative to death. Then strike.”[9]

D. Undermine Your Opponent’s Motive to Uphold His Reputation

If the person threatening you says, “Look, I don’t want to carry out this threat, but I must because I want to maintain my reputation with others,” you can respond, “It is not in my interest to publicize the fact that you did not punish me. I am only interested in doing well in this game. I will keep quiet; both of us will avoid the mutually damaging outcome; and your reputation with others will stay intact.” Similarly, if you are a buyer bargaining with a seller and he refuses to lower his price on the grounds that “if I do this for you, I would have to do it for everyone else,” you can point out that you are not going to tell anyone else. This response may not work; the other player may suspect that you would tell a few friends, who would tell a few others, and so on.

E. Use Salami Tactics

Salami tactics are devices used to whittle down the other player's threat in the way that a salami is cut—one slice at a time. You fail to comply with the other player's wishes (whether for deterrence or compellence) to such a small degree that it is not worth the other player's while to carry out the comparatively drastic and mutually harmful threatened action just to counter that small transgression. If that works, you transgress a little more, and a little more again, and so on.

You know this device perfectly well from your own childhood. Thomas C. Schelling[10] gives a wonderful description of the process:

> Salami tactics, we can be sure, were invented by a child. . . . Tell a child not to go in the water and he'll sit on the bank and submerge his bare feet; he is not yet "in" the water. Acquiesce, and he'll stand up; no more of him is in the water than before. Think it over, and he'll start wading, not going any deeper. Take a moment to decide whether this is different and he'll go a little deeper, arguing that since he goes back and forth it all averages out. Pretty soon we are calling to him not to swim out of sight, wondering whatever happened to all our discipline.

Salami tactics work particularly well against compellence because they can take advantage of the *time* dimension. When your parent tells you to clean up your room "or else," you can put off the task for an extra hour by claiming that you have to finish your homework, then for a half day because you have to go to football practice, then for an evening because you can't possibly miss *The Simpsons* on TV, and so on.

To counter the countermove of salami tactics, you must make a correspondingly graduated threat. There should be a scale of punishments that fits the scale of noncompliance or procrastination. This can also be achieved by gradually raising the risk of disaster, another application of brinkmanship.

Endnotes

- Thomas C. Schelling, *The Strategy of Conflict* (Oxford: Oxford University Press, 1960), p. 146. Return to reference 8
- Sun Tzu, *The Art of War*, pp. 109–10. Return to reference 9
- Thomas C. Schelling, *Arms and Influence* (New Haven, Conn.: Yale University Press, 1966), pp. 66–67. Return to reference 10

Glossary

salami tactics
: A method of defusing threats by taking a succession of actions, each sufficiently small to make it nonoptimal for the other player to carry out his threat.

SUMMARY

Actions taken by players to alter the outcome of an upcoming game in their favor are known as *strategic moves*. These pregame moves must be *observable* and *irreversible* to act as first moves. They must also be *feasible* within the rules and order of play in the upcoming game, and they must be credible if they are to have their desired effect of altering the equilibrium outcome of the game. *Commitment* is an unconditional strategic move used to seize a first-mover advantage when one exists. Such a move usually entails committing to a strategy that would not have been one’s equilibrium strategy in the original version of the game.

Conditional strategic moves entail declaring a response rule specifying one’s response to another player’s *default action* and *favored action*. A *threat* specifies a non-best response to the other player’s default action, while a *promise* specifies a non-best response to the other player’s favored action. When making a threat or a promise, players generally make only the threat or promise explicit, leaving the response to the other action implicit because it is a best response. The implied best response is called a *warning* (if it is a best response to the default action) or an *affirmation* (if it is a best response to the favored action).

Strategic moves may be designed either to *deter* rivals’ actions and preserve the status quo or to *compel* rivals’ actions and alter the status quo. Threats carry the possibility of mutual harm, but cost nothing if they work. Promises are costly only to the maker, but only if they are successful. Threats can be arbitrarily large, although excessive size compromises credibility, but promises are usually kept just large enough to be effective. If the implicit affirmation (or warning) that accompanies a threat (or promise) is not credible, players must *make a combined* *threat and promise* and see to it that both components are credible.

Credibility must be established for any strategic move. There are a number of general principles to consider in making moves credible and a number of specific *credibility devices* that can be used to acquire credibility. They generally work either by reducing the maker's future freedom to choose or by altering the maker's payoffs from future actions. Specific devices of this kind include establishing a *reputation*, using teamwork, demonstrating apparent irrationality, burning bridges, and making *contracts*, although the acquisition of credibility is often context specific. Similar devices exist for countering strategic moves made by rival players.

KEY TERMS

affirmation (270)

commitment (269)

compellence (273)

contract (291)

credibility device (285)

default action (270)

deterrence (273)

doomsday device (287)

favored action (270)

feasibility (268)

promise (270)

rational irrationality (291)

reputation (289)

response rule (269)

salami tactics (294)

strategic move (267)

threat (270)

warning (270)

Glossary

strategic move
: Action taken at a pregame stage that changes the strategies or the payoffs of the subsequent game (thereby changing its outcome in favor of the player making the move).

feasibility
: Possibility within the (unchangeable) physical and/or procedural restrictions that apply in a given game.

commitment
: An action taken at a pregame stage, stating what action you would take unconditionally in the game to follow.

response rule
: A rule that specifies how you will act in response to various actions of other players.

favored action
: The action that a player making a *strategic move* wants the other player to take, as opposed to the *default action*.

default action
: In the context of strategic moves, the action that the other player (the player not making a strategic move) will take in the absence of a strategic move, as opposed to the *favored action*.

promise
: A response to the *favored action* that benefits the other player and that is not a best response, as specified within a *strategic move*. It is only feasibly made by the second mover in a game. Credibility is required because the specified action is not a best response. The strategic move referred to as “making a promise” entails declaring both a promise and a warning, whereas “making a combined threat and promise” entails declaring both a promise and a threat.

threat

A response to the default action that harms the other player and that is not a best response, as specified within a strategic move. It can only feasibly be made by the second mover in a game. Credibility is required because the specified action is not a best response. The strategic move referred to as “making a threat” entails declaring both a threat and an affirmation, whereas “making a combined threat and promise” entails declaring both a threat and a promise.

affirmation

A response to the *favored action* that is a best response, as specified within a strategic move. Only the second mover in a game can feasibly make an affirmation. However, credibility is not required since the specified action is already the player’s best response. The strategic move referred to as “making a threat” entails declaring both a threat and an affirmation.

warning

A response to the default action that is a best response, as specified within a strategic move. Only the second mover in a game can feasibly make a warning. However, credibility is not required since the specified action is already the player’s best response. The strategic move referred to as “making a promise” entails declaring both a promise and a warning.

deterrence

An attempt to induce the other player(s) to act to maintain the status quo.

compellence

An attempt to induce the other player(s) to act to change the status quo in a specified manner.

credibility device

A means by which a player acquires credibility, for instance, when declaring a promise or threat as part of a strategic move.

doomsday device

An automaton that will under specified circumstances generate an outcome that is very bad for all players. Used for giving credibility to a severe threat.

reputation

Relying on the effect on payoffs in future or related games to make threats or promises credible, when they would not have been credible in a one-off or isolated game.

rational irrationality

Adopting a strategy that is not optimal after the fact, but serves a rational strategic purpose of lending credibility to a threat or a promise.

contract

In this context, a way of achieving credibility for one's strategic move by entering into a legal obligation to perform the committed, threatened, or promised action in the specified contingency.

salami tactics

A method of defusing threats by taking a succession of actions, each sufficiently small to make it nonoptimal for the other player to carry out his threat.

SOLVED EXERCISES

1. "One could argue that the size of a promise is naturally bounded, while in principle a threat can be arbitrarily severe so long as it is credible (and error free)." First, briefly explain why the statement is true. Despite the truth of the statement, players might find that an arbitrarily severe threat might not be to their advantage. Explain why the latter statement is also true.
2. For each of the following three games, answer these questions:
   1. What is the equilibrium if neither player can use any strategic moves?
   2. Can one player improve his payoff by using a strategic move [commitment, threat (with affirmation), promise (with warning), or combined threat and promise]? If so, which player makes what strategic move?

1.

		COLUMN	
		Left	Right
ROW	Up	0, 0	2, 1
	Down	1, 2	0, 0

You may need to scroll left and right to see the full figure.

2.

		COLUMN	
		Left	Right
ROW	Up	4, 3	3, 4
	Down	2, 1	1, 2

You may need to scroll left and right to see the full figure.

3.

		COLUMN	
		Left	Right

You may need to scroll left and right to see the full figure.

		COLUMN	
		Left	Right
ROW	Up	4, 1	2, 2
	Down	3, 3	1, 4
You may need to scroll left and right to see the full figure.			

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3. In the classic film *Mary Poppins*, the Banks children are players in a strategic game with a number of different nannies. In their view of the world, nannies are inherently harsh, and playing tricks on nannies is great fun. That is, they view themselves as playing a game in which the nanny moves first, showing herself to be either Harsh or Nice, and the children move second, choosing to be either Good or Mischievous. A nanny prefers to have Good children to take care of, but is also inherently Harsh, so she gets her highest payoff of 4 from (Harsh, Good) and her lowest payoff of 1 from (Nice, Mischievous), with (Nice, Good) yielding 3 and (Harsh, Mischievous) yielding 2. The children similarly most prefer to have a Nice nanny and to be Mischievous; they get their highest two payoffs when the nanny is Nice (4 if Mischievous, 3 if Good) and their lowest two payoffs when the nanny is Harsh (2 if Mischievous, 1 if Good).
   1. Draw the game tree for this game and find the subgame-perfect equilibrium in the absence of any strategic moves.
   2. In the film, before the arrival of Mary Poppins, the children write their own ad for a new nanny in which they state, “If you won’t scold and dominate us, we will never give you cause to hate us; we won’t hide your spectacles so you can’t see, put toads in your bed, or pepper in your tea.” Use the tree from part (a) to argue that this statement constitutes a promise. What would the outcome of the game be if the children keep their promise?
   3. What is the implied warning that goes with the promise in part (b)? Does the warning need to be made credible? Explain your answer.
   4. How could the children make the promise in part (b) credible?
   5. Is the promise in part (b) compellent or deterrent? Explain your answer by referring to the status quo in the game—namely, what would happen in the absence of the strategic move.

4. The following exercise is an interpretation of the rivalry between the United States and the Soviet Union for geopolitical influence during the 1970s and 1980s.[11] Each side has the choice of two strategies: Aggressive and Restrained. The Soviet Union wants to achieve world domination, so being Aggressive is its dominant strategy. The United States wants to prevent the Soviet Union from achieving world domination; it will match Soviet aggressiveness with aggressiveness, and restraint with restraint. Specifically, the payoff table is

		SOVIET UNION	
		Restrained	Aggressive
UNITED STATES	Restrained	4, 3	1, 4
	Aggressive	3, 1	2, 2

You may need to scroll left and right to see the full figure.

For each player, 4 is best and 1 is worst.

1. Consider this game when the two countries move simultaneously. Find the Nash equilibrium.
2. Next, consider three different and alternative ways in which the game could be played with sequential moves:
    1. The United States moves first, and the Soviet Union moves second.
    2. The Soviet Union moves first, and the United States moves second.
    3. The Soviet Union moves first, and the United States moves second, but the Soviet Union has a further move in which it can change its first move.

    For each case, draw the game tree and find the subgame-perfect equilibrium.

3. What are the key strategic considerations for the two countries?

5. At a United Nations meeting during the Cuban missile crisis of October 1962, the U.S. ambassador to the U.N., Adlai Stevenson, challenged the Soviet ambassador, Valerian Zorin. Stevenson asked, “Do you . . . deny that the USSR has placed and is

placing . . . missiles and sites in Cuba?" Zorin replied, "I am not in an American courtroom, sir, and therefore I do not wish to reply." Stevenson retorted, "You are in the courtroom of world opinion right now. Yes or no? . . . I am prepared to wait for my answer until hell freezes over."

Comment on this interaction with reference to the theory of strategic moves. Consider specifically what Stevenson was trying to accomplish, what strategic move he was attempting to use, and whether the move was likely to work to achieve Stevenson's preferred outcome.

6. Tabloid favorite Kim Kardashian has to decide one night whether to party at L.A.'s hottest dance club. A paparazzo has to decide whether to stalk the club that night in the hope of taking a photograph of her. Kim wants most to party, but would also prefer not to be bothered by the paparazzo. The paparazzo wants to be at the club to take a photograph if Kim shows up, but would otherwise prefer to go elsewhere. The ordinal payoff matrix for this game is shown below.

		PAPARAZZO	
		Stalk	Don't stalk
KIM KARDASHIAN	Party	3, 4	4, 2
	Don't party	1, 1	1, 2

You may need to scroll left and right to see the full figure.

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1. Identify all pure-strategy Nash equilibria of this game (or explain why no such equilibrium exists) in the case when Kim and the paparazzo move simultaneously.
2. Draw the game tree for the case when the paparazzo moves first. What are the rollback equilibrium strategies and outcome?
3. In the case when the paparazzo moves first, is it possible for Kim to achieve her best possible outcome by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would Kim use, and how might she phrase her declaration?

7. North Korea's leader, Kim Jong-un, must decide whether to keep or dismantle North Korea's nuclear weapons, while China must decide whether or not to provide economic aid to North Korea. China wants most for North Korea to dismantle, in order to avoid having a nuclear-armed neighbor, and has a dominant strategy, to provide aid, in order to prevent a humanitarian crisis on its southern border. For his part, Kim wants most to receive aid, but has a dominant strategy, Don't dismantle, in order to use his nukes to extract more concessions in the future. The ordinal payoff matrix for this game is shown below.

		CHINA	
		Aid	Don't aid
NORTH KOREA [Kim Jong-un]	Dismantle	3, 4	1, 3
	Don't dismantle	4, 2	2, 1

   1. Identify all pure-strategy Nash equilibria of this game (or explain why no such equilibrium exists) in the case when North Korea and China move simultaneously.
   2. Draw the game tree for the case when North Korea moves first. What are the rollback equilibrium strategies and outcome?
   3. In the case when North Korea moves first, is it possible for China to achieve its best possible outcome by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would China use, and how might it phrase its declaration?

8. Dear Old Dad has just waved goodbye to his son, Student, who has enrolled at a prestigious private college, despite Dad's concern that the tuition is too much for the family to pay. Now at school, Student must decide whether to find a job or party during his free time, while Dad must decide whether to contribute toward Student's tuition (Pay) or save for retirement. The ordinal payoff matrix for this game is shown below. Student prefers to party, but most of all, wants Dad to pay. In particular, Student is willing to find a job if doing so will cause Dad to pay. For Dad, the best outcome is that in which Student finds a job and Dad can save, but if Student should choose to party, Dad would prefer to pay so that his son is not left with a ruinous debt-load. Further, should he pay, Dad prefers that his son find a job.

		DEAR OLD DAD	
		Pay	Save
STUDENT	Party	4, 2	2, 1
	Job	3, 3	1, 4

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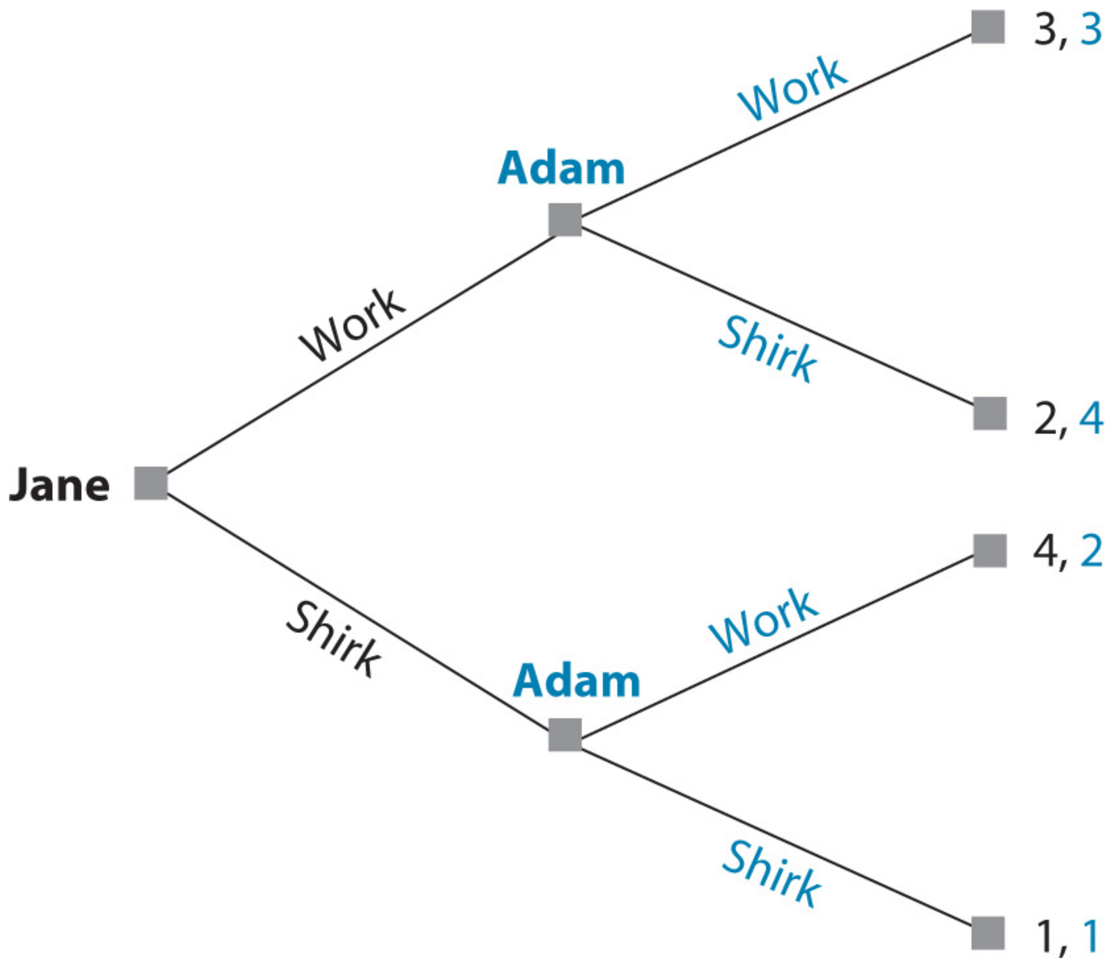
1. Identify all pure-strategy Nash equilibria of this game (or explain why no such equilibrium exists) in the case when Student and Dad move simultaneously.
2. Draw the game tree for the case when Student moves first. What are the rollback equilibrium strategies and outcome?
3. In the case when Student moves first, is it possible for Dad to achieve an outcome that is better than the rollback equilibrium outcome by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would Dad use, and how might he phrase his declaration?

9. Jane and Adam are teammates on a homework assignment that must be completed by noon tomorrow. It is currently midnight, and Adam is already asleep. He is an "early bird" who will wake up at 7:00 a.m., while Jane is a "night owl" who will stay awake until 5:00 a.m. and then sleep until after noon. Either of them can complete the assignment on their own (so that both get a good grade) with an hour of work. Both of them would prefer not to do this work, but most of all want to avoid a situation where no one does the work and both get a bad grade. Jane decides first whether to do the work between midnight and 5:00 p.m., and sends Adam an e-mail before going to sleep to tell him what she has done. Adam then decides whether to do the work himself between 7:00 a.m. and noon. The game tree for this game is shown below.



1. What are the rollback equilibrium strategies and outcome?
2. Is it possible for Adam to achieve his best possible outcome (payoff 4) by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would Adam use, and how might he phrase his declaration?

10. Consider the military game described in Exercise S2 in Chapter 6. The ordinal payoff matrix for that game between a Rebel Force and a Conventional Army is reproduced below. For the purposes of this exercise, assume that the high mobility of the Rebel Force makes it impossible for the Conventional Army to move last.

		CONVENTIONAL ARMY	
		Valley	Hills
REBEL FORCE	Valley	1, 4	4, 1
	Hills	3, 2	2, 3

1. Draw the game tree for this game when the Rebel Force moves second. What are the rollback equilibrium strategies and outcome?
2. Is it possible for the Rebel Force to achieve an outcome that is better than the rollback equilibrium outcome by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would it use, and how would it phrase its declaration?

UNSOLVED EXERCISES

1. In a scene from the movie *Manhattan Murder Mystery*, Woody Allen and Diane Keaton are at a hockey game in Madison Square Garden. She is obviously not enjoying herself, but he tells her, "Remember our deal. You stay here with me for the entire hockey game, and next week I will come to the opera with you and stay until the end." Later, we see them coming out of the Met into the deserted Lincoln Center Plaza while music is still playing inside. Keaton is visibly upset: "What about our deal? I stayed to the end of the hockey game, and so you were supposed to stay till the end of the opera." Allen answers, "You know I can't listen to too much Wagner. At the end of the first act, I already felt the urge to invade Poland." Comment on the strategic choices made here by using your knowledge of the theory of strategic moves and credibility.
2. Consider a game between a parent and a child. The child can choose to be good (G) or bad (B); the parent can punish the child (P) or not (N). The child gets enjoyment worth 1 from bad behavior, but hurt worth −2 from punishment. Thus, a child who behaves well and is not punished gets 0; one who behaves badly and is punished gets 1 − 2 = −1; and so on. The parent gets −2 from the child's bad behavior and −1 from inflicting punishment.
    1. Set up this game as a simultaneous-move game, and find the equilibrium.
    2. Next, suppose that the child chooses G or B first and that the parent chooses P or N after having observed the child's action. Draw the game tree and find the subgame-perfect equilibrium.
    3. Now suppose that before the child acts, the parent can declare any strategic move. What strategic move should the parent make in order to get his best possible outcome (G,N)? Draw the modified game tree for this sequential-move game when the parent has committed to this strategic move, and use this modified game tree to show why the child chooses to be good.
3. Thucydides's history of the Peloponnesian War has been expressed in game-theoretic terms by Professor William Charron of St. Louis University.[12] Athens had acquired a large empire of small coastal cities around the Aegean Sea because of its leadership role in

defending the Greek world from Persian invasions. Sparta, fearing Athenian power, was contemplating war against Athens. If Sparta decided against war, Athens would have to decide whether to retain or relinquish its empire. But Athens, in turn, feared that if it gave independence to the small cities, they could choose to join Sparta in a greatly strengthened alliance against Athens and receive very favorable terms from Sparta for doing so. Thus there are three players, Sparta, Athens, and the small cities, which move in that order. There are four possible outcomes, and the payoffs are as follows (4 being best):

Outcome	Sparta	Athens	Small cities
War	2	2	2
Athens retains empire	1	4	1
Small cities join Sparta	4	1	4
Small cities become independent	3	3	3

1. Draw the game tree and find the rollback equilibrium outcome. Is there another outcome that is better for all players?
2. What strategic move or moves could be used to attain the better outcome? Discuss the credibility of such moves.

4. It is possible to reconfigure the payoffs in the game in Exercise S3 so that the children’ s statement in their ad is a threat, rather than a promise.
   1. Redraw the tree from part (a) of Exercise S3 and fill in payoffs for both players so that the children’ s statement becomes a *threat*.
   2. Define the status quo in your game, and determine whether the threat is deterrent or compellent.
   3. Explain why the threatened action needs to be made credible, given your payoff structure.
   4. Explain why the implied warning does not need to be made credible.

5. Explain why the children would want to make a threat in the first place, and suggest a way in which they might make their threatened action credible.

5. Consider a research funding game in which two government agencies, the U.S. Department of Energy (DoE) and the Defense Advanced Research Projects Agency (DARPA), each decide which of two research projects (“batteries” or “solar”) to fund. DoE has a dominant strategy, investing in batteries, but wants most for DARPA to invest in solar. DARPA prefers to invest in the same project as DoE, but wants most for DoE to invest in batteries. The ordinal payoff matrix for this game is shown below.

		DARPA	
		Batteries	Solar
DoE	Batteries	2, 4	4, 3
	Solar	1, 1	3, 2

---

1. Identify all pure-strategy Nash equilibria of this game (or explain why no such equilibrium exists) in the case when DoE and DARPA move simultaneously.
2. Draw the game tree for the case when DoE moves first. What are the rollback equilibrium strategies and outcome?
3. In the case when DoE moves first, is it possible for DARPA to achieve its best possible outcome by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would DARPA use, and how might it phrase its declaration?
4. Draw the game tree for the case when DARPA moves first. What are the rollback equilibrium strategies and outcome?
5. In the case when DARPA moves first, is it possible for DoE to achieve its best possible outcome by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would DoE use, and how might it phrase its declaration?

6. North Korea’s leader Kim Jong-un plays a game with the United States: He decides whether to keep or dismantle his nuclear weapons, while the United States decides whether or not to provide him with economic aid. Like China in Exercise S7, the United States wants most for North Korea to dismantle its nukes, but unlike China, the United States has a dominant strategy, not to provide economic aid. (Assume that if North Korea dismantled

its nukes, the United States would then feel little incentive to aid North Korea’s economic recovery.) The ordinal payoff matrix for this game is shown below.

		UNITED STATES (U.S.)	
		Aid	Don’t aid
NORTH KOREA [Kim Jong-un]	Dismantle	3, 3	1, 4
	Don’t dismantle	4, 1	2, 2

---

1. Identify all pure-strategy Nash equilibria of this game (or explain why no such equilibrium exists) in the case when North Korea and the United States move simultaneously.
2. Draw the game tree for the case when North Korea moves first. What are the rollback equilibrium strategies and outcome?
3. Draw the game tree for the case when the United States moves first. What are the rollback equilibrium strategies and outcome?
4. In the case when North Korea moves first, is it possible for the United States to achieve its best possible outcome (payoff 4) by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would the United States use, and how might it phrase its declaration?
5. In the case when North Korea moves first, is it possible for the United States to achieve its second-best possible outcome (payoff 3) by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would the United States use, and how might it phrase its declaration?

7. In 1981, Ronald Reagan was a newly elected president with tremendous popular support and a vision for tax reform. But whether he could get the support he needed in Congress to implement his vision depended on the game being played between Democrats and Republicans in Congress. In a pair of articles that year,[13] *New York Times* columnist (and economist) Leonard Silk laid out the essence of this game, with ordinal payoffs as shown below. The Democrats get their best outcome when they attack Reagan’s vision and the Republicans compromise, because the Democrats can then claim credit for fiscal responsibility while

implementing their favored budget. The Republicans prefer to support Reagan completely no matter what Democrats do, and they get their best outcome when Reagan’ s budget gets bipartisan support. When the Democrats attack while the Republicans hold firm, the result is a stalemate, and both parties lose. The Democrats would be willing to moderate their attack if the Republicans would compromise, in which case both parties would get their second-best outcomes.

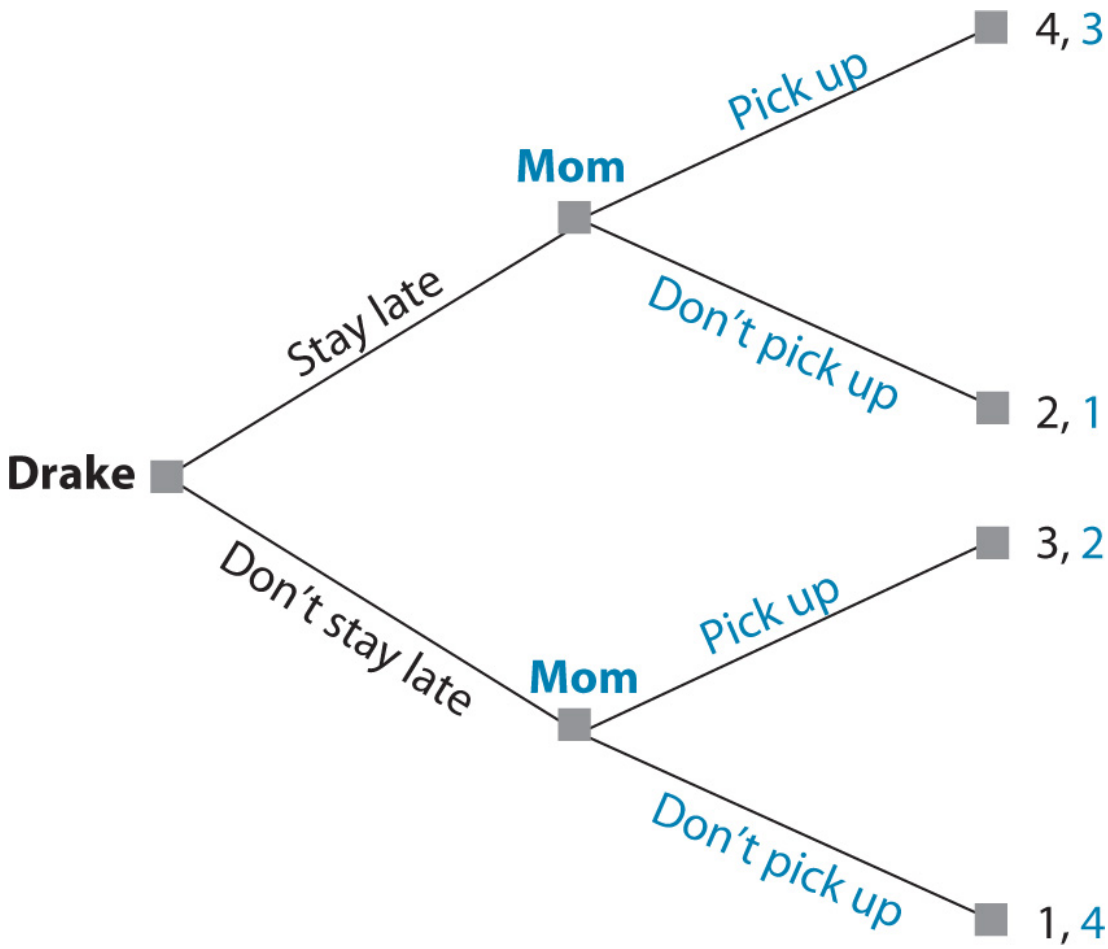
		REPUBLICANS	
		Support Reagan completely	Compromise
DEMOCRATS	Mainly support Reagan	2, 4	3, 3
	Attack Reagan	1, 2	4, 1
You may need to scroll left and right to see the full figure.			

---

1. Does either player in this game have a dominant or superdominant strategy? Thoroughly explain your answer.
2. Identify all pure-strategy Nash equilibria of this game (or explain why no such equilibrium exists) in the case when Democrats and Republicans move simultaneously.
3. Draw the game tree for the case when Republicans move first. What are the rollback equilibrium strategies and outcome?
4. In the case when Republicans move first, is it possible for Democrats to achieve an outcome that is better than the rollback equilibrium outcome by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would the Democrats use, and how might they phrase their declaration?

8. Drake is a high-school student who lives in a tough urban neighborhood. Every day, Drake decides whether he will stay late after school and play basketball with his friends, and then calls his Mom, who decides whether to pick him up from school or ask him to get home another way—either by taking the bus (if he is not staying late) or by walking home (if he is staying late). Drake likes playing basketball, but most of all wants his Mom to pick him up, and least of all wants to walk home after staying late. For her part, Mom’ s best outcome is when Drake takes the

bus home, but most of all, she does not want Drake to need to walk home. The game tree for this game is shown below.



1. What are the rollback equilibrium strategies and outcome?
2. Is it possible for Mom to achieve her best possible outcome (payoff 4) by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would Mom use, and how might she phrase her declaration?

9. Consider the market entry game described in Exercise U2 in Chapter 6. The payoff table for that game between a Monopolist and an Entrant is reproduced below. For the purposes of this exercise, assume that the Entrant moves first.

	ENTRANT	
	Build	Don’t build

You may need to scroll left and right to see the full figure.

		ENTRANT	
		Build	Don’t build
MONOPOLIST	Build	0.5, −0.5	2.5, 0
	Don’t build	2, 0.5	3, 0

You may need to scroll left and right to see the full figure.

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1. Draw the game tree for this game. What are the rollback equilibrium strategies and outcome?
2. Is it possible for the Monopolist (as second mover) to achieve its best possible outcome by credibly declaring a strategic move in the pregame? If not, why not? And if so, what sort of strategic move would the Monopolist use, and how might it phrase its declaration?

10. Write a brief description of a game in which you have participated that entailed strategic moves such as a commitment, threat, or promise, paying special attention to the essential aspect of credibility. Provide an illustration of the game if possible, and explain why the game that you describe ended as it did. Did the players use sound strategic thinking in making their choices?

Endnotes

- We thank political science professor Thomas Schwartz at UCLA for the idea for this exercise. Return to reference 11
- William C. Charron, “Greeks and Games: Forerunners of Modern Game Theory,” *Forum for Social Economics*, vol. 29, no. 2 (Spring 2000), pp. 1–32. Return to reference 12
- “Economic Scene,” *New York Times,* April 10, 1981, p. D2 and April 15, 1981, p. D2. This exercise is based on an example in *Thinking Strategically* by Avinash Dixit and Barry Nalebuff (New York: W.W. Norton, 1991), pp. 131–35. Return to reference 13

9■ Uncertainty and Information

IN CHAPTER 2, we mentioned different ways in which uncertainty (external and strategic) can arise in a game and ways in which players can have limited information about aspects of the game (imperfect and incomplete, and symmetrically or asymmetrically available to the players). We have already encountered and analyzed some of these concepts. Most notably, in simultaneous-move games, no player knows the actions other players are taking; this is a case of strategic uncertainty. In Chapter 6, we saw that strategic uncertainty gives rise to asymmetric and imperfect information because the different actions that could be taken by one player must be lumped into one information set for the other player. In Chapters 4 and 7, we saw how such strategic uncertainty is handled by having each player formulate beliefs about the others' actions (including beliefs about the probabilities with which different actions may be taken when mixed strategies are played) and by applying the concept of Nash equilibrium, in which such beliefs are confirmed. In this chapter, we focus on some further ways in which uncertainty and informational limitations arise in games.

We will begin this chapter by examining various strategies that individuals and societies can use for coping with the imperfect information generated by external uncertainty or risk. Recall that *external uncertainty* is uncertainty about matters outside any player's control, but affecting the payoffs of the game; weather is a simple example. We will describe the basic ideas behind the diversification, or spreading, of risk by an individual player and the pooling of risk by multiple players. Although these strategies can benefit everyone, the division of the total gains among the participants can be unequal; therefore, these situations contain a mixture of common interest and conflict.

We will then consider the informational limitations that often arise in games. Information in a game is *complete* only if all of the rules of the game—the strategies available to all players and the payoffs of each player as functions of the strategies of all players—are fully known by all players and, moreover, are common knowledge among them. By this exacting standard, most games in reality have *incomplete information*. Moreover, the incompleteness is usually *asymmetric*: Each player knows his own capabilities and payoffs much better than he knows those of other players. As we pointed out in Chapter 2, manipulation of information becomes an important dimension of strategy in such games. In this chapter, we will discuss when information can and cannot be communicated verbally in a credible manner. We will also examine other strategies designed to convey or conceal one's own information and to elicit another player's information. We spoke briefly of some such strategies—namely, screening and signaling—in Chapters 1 and 2; here, we study those strategies in more detail.

Of course, players in many games would also like to manipulate the actions of others. Managers would like their workers to work hard and well; insurance companies would like their policyholders to exert care to reduce the risk of an insured-against event occurring. If information were perfect, the actions of those other players would be observable. Workers' pay could be made contingent on the quality and quantity of their effort; payouts to insurance policyholders could be made contingent on the care they exercised. But in reality, these actions are difficult to observe; that creates a situation of imperfect asymmetric information, commonly called moral hazard. Thus, the players in these games have to devise various incentives to influence others' actions in the desired direction. We take up the design of such incentives, termed *mechanism design* or *incentive design*, in Chapter 14.

Information and its manipulation in games has been a topic of active research in recent decades. That research has shed new light on many previously puzzling matters in economics, such as the nature of incentive contracts, the organization of companies, markets for labor and for durable goods, government regulation of business, and myriad others.[1] More recently, political scientists have used the same concepts to explain phenomena such as the relationship of changes in tax and spending policy to elections, as well as the delegation of legislation to committees. These ideas have also spread to biology, where evolutionary game theory explains features such as the peacock's large and ornate tail as a signal. Perhaps even more importantly, you will recognize the role that signaling and screening play in your daily interactions with family, friends, teachers, coworkers, and so on, and you will be able to improve your strategies in these games.

Endnotes

- The pioneers of the theory of asymmetric information in economics have shared in two Nobel Prizes. George Akerlof, Michael Spence, and Joseph Stiglitz were recognized in 2001 for their work on signaling and screening, and Leo Hurwicz, Eric Maskin, and Roger Myerson in 2007 for their work on mechanism design. Return to reference 1

Glossary

moral hazard
: A situation of information asymmetry where one player’s actions are not directly observable to others.

1 STRATEGIES FOR DEALING WITH RISK

Imagine that you are a farmer subject to the vagaries of weather. If the weather is good for your crops, you will have an income of \$160,000. If it is bad for them, your income will be only \$40,000. The two possibilities are equally likely (with a probability of ½, or 0.5, or 50% each). Therefore, your average or expected income is \$100,000 (= ½ × 160,000 + ½ × 40,000), but there is considerable risk around this average value.

What can you do to reduce the risk to your income? You might try growing a crop that is less subject to the vagaries of weather, but suppose you have already done all such things that are under your individual control. Then you might be able to reduce the risk to your income further by getting someone else to accept some of that risk. Of course, you must give the other person something else in exchange. This quid pro quo usually takes one of two forms: a cash payment, or a mutual exchange or sharing of risk.

A. Sharing of Risk

Suppose you have a neighbor who faces a risk similar to yours, but gets good weather exactly when you get bad weather, and vice versa. (You live on opposite sides of an island, and rain clouds visit one side or the other, but not both.) In technical jargon, *correlation* is a measure of alignment between any two uncertain quantities—in this discussion, between one person's risk and another's. Thus, in this example, your neighbor's risk is perfectly negatively correlated with yours. The combined income of you and your neighbor is $200,000, no matter what the weather: It is totally risk free. You can enter into a contract that gets each of you $100,000 for sure: You promise to give him $60,000 in years when you are lucky, and he promises to give you $60,000 in years when he is lucky. You have eliminated your risks by combining them.

Currency swaps provide a good example of negative correlation of risk in real life. A U.S. firm exporting to Europe gets its revenues in euros, but it is interested in its dollar profits, which depend on the fluctuating euro-dollar exchange rate. Conversely, a European firm exporting to the United States faces similar uncertainty about its profits in euros. When the euro falls relative to the dollar, the U.S. firm's euro revenues convert into fewer dollars, and the European firm's dollar revenues convert into more euros. The opposite happens when the euro rises relative to the dollar. Thus, fluctuations in the exchange rate generate negatively correlated risks for the two firms. Both can reduce these risks by contracting for an appropriate swap of their revenues.

Even without such perfect negative correlation, risk sharing has some benefit. Return to your role as an island farmer, and suppose that you and your neighbor face risks that are independent from each other, as if the rain clouds could toss a separate coin to decide whether to visit each of you. Then there are four possible outcomes, each with a probability of ¼. The incomes you and your neighbor earn in these four cases are illustrated in Figure 9.1a. However, suppose the two of you were to make a contract to share risk; then your incomes would be those shown in Figure 9.1b. Although your average (expected) income in each table is $100,000, without the sharing contract, each of you would get $160,000 or $40,000 with probabilities of ½ each. With the contract, each of you would get $160,000 with probability ¼, $100,000 with probability ½, and $40,000 with probability ¼. Thus, for each of you, the contract has reduced the probabilities of the two extreme outcomes from ½ to ¼ and increased the probability of the middle outcome from 0 to ½. In other words, the contract has reduced the risk for each of you.

		NEIGHBOR	
		Lucky	Not
YOU	Lucky	160,000, 160,000	160,000, 40,000
	Not	40,000, 160,000	40,000, 40,000

(a) Without sharing

		NEIGHBOR	
		Lucky	Not
YOU	Lucky	160,000, 160,000	100,000, 100,000
	Not	100,000, 100,000	40,000, 40,000

(b) With sharing

FIGURE 9.1 Sharing Income Risk

In fact, as long as your incomes are not totally positively correlated—that is, as long as your luck and your neighbor's luck do not move in perfect tandem—you can both reduce your risks by sharing them. And if there are more than two of you with some degree of independence in your risks, then the law of large numbers makes possible even greater reduction in the risk of each. That is exactly what insurance companies do: By combining the similar but independent risks of many people, an insurance company is able to compensate any one of them when he suffers a large loss. It is also the basis of portfolio diversification: By dividing your wealth among many different assets with different kinds and degrees of risk, you can reduce your total exposure to risk.

However, such arrangements for risk sharing depend on observability of outcomes and enforcement of contracts. Otherwise, each farmer has the temptation to pretend to have suffered bad luck, or simply to renege on the deal and refuse to share when he has good luck. Similarly, an insurance company may falsely deny claims, but its desire to maintain its reputation for the sake of its ongoing business may check such reneging.

B. Paying to Reduce Risk

Now consider the possibility of trading risk for cash. Suppose you are the same farmer facing the same risk as before. But now your neighbor has a sure income of $100,000. You face a lot of risk, and he faces none. He may be willing to take a little of your risk for a price that is agreeable to both of you.

Suppose you come to an arrangement where the neighbor will give you $10,000 if your luck is bad, and you will give him $10,000 if your luck is good. Thus, your income will be $150,000 if your luck is good and $50,000 if your luck is bad; your neighbor’s income will be $90,000 and $110,000 in those respective events. The spread between your incomes in the two situations has decreased from $120,000 to $100,000—that is, by $20,000; the spread between your neighbor’s two incomes has increased from $0 to $20,000. Although the changes in the two spreads are equal, it is reasonable to suppose that an equal change in spreads is more of a concern when it starts from an already large base. Thus, the change will be less of a concern to your neighbor than to you. For sake of definiteness, suppose that you are willing to pay up to $3,000 up front to your neighbor to enter into such a contract with you; this is, in effect, a contract providing partial insurance for your risk, and the amount you pay up front is an insurance premium. Your neighbor is willing to enter into the contract for much less—say, only $250.[2] There is plenty of room between the neighbor’s willingness to accept and your willingness to pay; somewhere in this range, you two can negotiate a price and strike a deal that benefits you both.[3] At a premium close to $250, you reap almost all the gain from the deal; at a premium close to $3,000, your neighbor does.

Where the price actually settles depends on many things: whether there are many more farmers facing risks like yours than farmers with safe incomes like your neighbor’s (that is, the extent of competition that exists on the two sides of the deal); the attitudes toward risk on the two sides; and so on. If your “neighbor” is actually an insurance company, it can be nearly unconcerned about your risk because it is combining numerous such risks and because it is owned by well-diversified investors, for each of whom this business is only a small part of their total risk. And if insurance companies have to compete fiercely for business, the insurance market can offer you nearly complete insurance at a price that leaves almost all of the gain with you.

Common to all such arrangements is the idea that mutually beneficial deals can be struck whereby, for a suitable price, someone facing less risk takes some risk off the shoulders of someone else who faces more. In fact, the idea that a price and a market for risk exist is the basis for almost all of the financial arrangements in a modern economy. Stocks and bonds, as well as complex financial instruments such as derivatives, are just ways of spreading risk to those who are willing to bear it for the lowest asking price. Many people think these markets are purely forms of gambling. In a sense, they are. But those who start out with the least risk take the gambles, perhaps because they have already diversified in the way that we saw earlier. And the risk is sold or shed by those who are initially most exposed to it. This enables the latter to be more adventurous in their enterprises than they would be if they had to bear all the risk of those enterprises themselves. Thus, financial markets promote entrepreneurship by facilitating risk trading.

Here we have only considered the sharing of a given total risk. In practice, people may be able to take actions to reduce that total risk: A farmer can guard crops against frosts, and a car owner can drive carefully to reduce the risk of an accident. If such actions are not publicly observable, the game will be one of imperfect information, raising the problem of moral hazard that we mentioned in the introduction: People who are well insured will lack the incentive to reduce the risk they face. We will look at such problems, and the design of mechanisms to cope with them, in Chapter 14.

C. Manipulating Risk in Contests

The farmers in our example above faced risk due to the weather, not from any actions of their own or of other farmers. If the players in a game can affect the risk they or other players face, they can use such manipulation of risk strategically. A prime example is contests such as research and development races between companies to develop and market new information technology or biotech products; many sports contests have similar features.

The outcome of a sports event or related contest is determined by a mixture of skill and chance. You win if

Your skill + your luck > rival's skill + rival's luck

or

Your luck − rival's luck > rival's skill − your skill.

We denote the left-hand side of this equation with the symbol *L,* which measures your "luck surplus." *L* is an uncertain magnitude; suppose its probability distribution is a normal curve, or bell curve, as illustrated by the black curve in Figure 9.2. At any point on the horizontal axis, the height of the curve represents the probability that *L* takes on that value. Thus, the area under this curve between any two points on the horizontal axis equals the probability that *L* lies between those points. Suppose your rival has more skill, so you are an underdog in the contest. Your "skill deficit," which equals the difference between your rival's skill and your skill, is therefore positive, as shown by the point *S*. You win if your luck surplus, *L*, exceeds your skill deficit, *S*. Therefore, the area under the curve to the right of the point *S*, which is shown by the gray hatching in Figure 9.2, represents your probability of winning. If you make the situation chancier, the bell curve will be flatter, like the blue curve in Figure 9.2, because the probability of relatively high and low values of *L* increases while the probability of moderate values decreases. Then the area under the curve to the right of S also increases. In Figure 9.2, the larger area under the flatter bell curve is shown by blue hatching. As the underdog, you should therefore adopt a strategy that flattens the curve to increase your probability of winning. Conversely, if you are the favorite, you should try to reduce the element of chance in the contest.

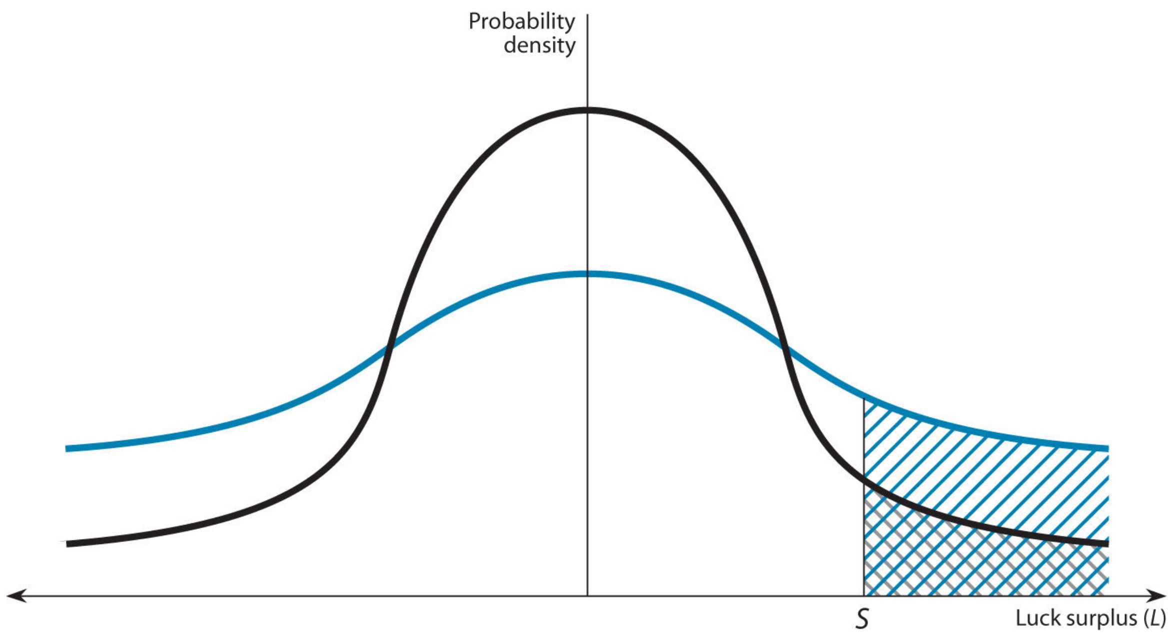

Figure 9.2 The Effect of Greater Risk on the Chances of Winning

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Thus, we should see underdogs, or those who have fallen behind in a long race, try unusual or risky strategies, because it is their only chance to get level or ahead. In contrast, favorites, or those who have stolen a lead, will play it safe. Here is a practical piece of advice based on this principle: If you want to challenge someone who is a better player than you to a game of tennis, choose a windy day.

You may stand to benefit by manipulating not only the amount of risk in your strategy, but also the correlation between your risk and that of other players. The player who is ahead will try to choose a correlation as high and as positive as possible; then, whether his own luck is good or bad, the luck of his opponent will be the same, and his lead will be protected. Conversely, the player who is behind will try to find a risk as uncorrelated with that of his opponent as possible. It is well known that in a two-sailboat race, the boat that is behind should try to steer differently from the boat ahead, and the boat ahead should try to imitate all the tacks of the one behind.[4]

Endnotes

- Here we do not need the rigorous theories of risk aversion that yield such numbers for willingness to avoid risk, so we will refer interested readers to somewhat more advanced textbooks, such as Hal Varian, *Intermediate Microeconomics*, 9th ed. (New York, W. W. Norton, 2014), Chapter 12 . For any advanced students or teachers among the readers, these numbers come from an expected utility calculation with a square root utility function. Return to reference 2
- After it becomes known whether your luck is good or bad, one of you will have to make a unilateral payment to the other and may be tempted to renege. Therefore, it is important to have an external enforcement authority, or a repeated relationship between the two parties that keeps them honest, to sustain such deals. Return to reference 3
- Avinash Dixit and Barry Nalebuff, *Thinking Strategically* (New York: W. W. Norton, 1991), give a famous example of the use of this strategy in sailboat racing. For a more general theoretical discussion, see Luis Cabral, "R&D Competition When the Firms Choose Variance," *Journal of Economics and Management Strategy*, vol. 12, no. 1 (Spring 2003), pp. 139–50. Return to reference 4

Glossary

negatively correlated
: Two random variables are said to be negatively correlated if, as a matter of probabilistic average, when one is above its expected value, the other is below its expected value.

positively correlated
: Two random variables are said to be positively correlated if, as a matter of probabilistic average, when one is above its expected value, the other is also above its expected value, and vice versa.

2 ASYMMETRIC INFORMATION: BASIC IDEAS

In many games, one player, or some of the players, may have the advantage of knowing with greater certainty what has happened or what will happen. Such advantages, or asymmetries of information, are common in actual strategic situations. At the most basic level, each player may know his own preferences or payoffs—for example, his risk tolerance in a game of brinkmanship, his patience in bargaining, or his peaceful or warlike intentions in international relations—quite well, but those of the other players much more vaguely. The same is true for a player's knowledge of his own innate characteristics (such as the skill of a job applicant or the riskiness of an applicant for auto or health insurance). And sometimes the actions available to one player—for example, the weaponry and readiness of a country for war—are not fully known to other players. Finally, some actual outcomes (such as the actual dollar value of loss to an insured homeowner in a flood or an earthquake) may be observed by one player, but not by others.

By manipulating what the other players know about your abilities and preferences, you can affect the equilibrium outcome of a game. Therefore, such manipulation of asymmetric information itself becomes a game of strategy. You might think that each player will always want to conceal the information he has and elicit information from the others, but that is not so. The better-informed player may want to do one of the following:

1. *Conceal information* or *reveal misleading information.* When mixing moves in a zero-sum game, you don't want the other player to see what you have done; for example, you bluff in poker to mislead others about your cards.

2. *Reveal selected information truthfully.* When you make a strategic move, you want others to see what you have done so that they will respond in the way you desire. For example, if you are in a tense situation but your intentions are not hostile, you want others to know this credibly, so that there will be no unnecessary fight.

Similarly, the less informed player may want to do one of the following:

1. *Elicit information* or *filter truth from falsehood.* An employer wants to find out the skill of a prospective employee or the effort of a current employee. An insurance company wants to know an applicant's risk class, the amount of a claimant's loss, and any contributory negligence by the claimant that would reduce its liability.
2. *Remain ignorant.* Being unable to know your opponent's strategic move can immunize you against his commitments and threats. Top-level politicians or managers often benefit from having such "credible deniability."

In most cases, words alone do not suffice to convey credible information; rather, actions speak louder than words. Yet even actions may not convey information credibly, if they are too easily imitated by others who do not possess the information in question. Therefore, less-informed players should pay closer attention to what a better-informed player does than to what he says. And knowing that the others will interpret his actions in this way, the better-informed player should in turn try to manipulate his actions for their information content.

When you are playing a strategic game, you may find that you have information that other players do not have. You may have information that is "good" (for yourself) in the sense that, if the other players knew this information, they would alter their actions in a way that would increase your payoff.

You know that you are a nonsmoker, for example, and should qualify for lower life-insurance premiums, or that you are a very highly skilled worker and should get paid a higher wage. You may also have "bad" information, whose disclosure would cause others to act in a way that would hurt you. You know that you are weak in math, for example, and don't deserve to be admitted to a prestigious graduate program, or that you have been dealt poor cards in a poker game and shouldn't win the hand. Of course, no matter whether your information is good or bad, you would like for others to believe that it is good. Therefore, you try to think of, and take, actions that will induce them to believe that your information is good. For example, you might provide medical records regarding your lung health to your insurance company or bluff in poker by betting as if you had an excellent hand. Such actions are called signals, and the strategy of using them is called signaling. Other players know that you can signal and will interpret your actions accordingly. So signaling and interpreting signals is a game, and how it plays out will be illustrated and analyzed in this chapter.

If other players know more than you do or take actions that you cannot directly observe, you can use strategies that reduce your informational disadvantage. You may be a life-insurance company and need to know the smoking habits of your clients, or a graduate school admissions officer seeking to identify the skills of your applicants. You want to do something to get your clients or applicants to reveal the information they have that can affect your payoffs. The strategy of making another player act so as to reveal his information is called screening, and specific methods used for this purpose are called screening devices.[5]

Because a player's private information often consists of knowledge of his own abilities or preferences, it is useful to think of players who come to a game possessing different private information as different types. When credible signaling works, less informed players will, in equilibrium, be able to infer the information of the more informed ones correctly from their actions; the law school, for example, will admit only the truly qualified applicants. Another way to describe the equilibrium outcome is to say that the different types are correctly revealed, or *separated*. Therefore, we call this equilibrium a separating equilibrium. In some cases, however, one or more types may successfully mimic the actions of other types, so that the less informed players cannot infer types from actions and cannot identify the different types; insurance companies, for example, may offer only one kind of life-insurance policy. In that case, we say the types are pooled together in equilibrium, and we call this equilibrium a pooling equilibrium. A third type of equilibrium, called a semiseparating equilibrium, occurs when one or more types mimic the actions of other types randomly, using a mixed strategy. When studying games of incomplete information, we will see that identifying the kind of equilibrium that occurs is of primary importance.

Endnotes

- A word of warning: In ordinary language, the word *screening* can have different meanings. The one used in game theory is that of testing or scrutinizing. Thus, a less-informed player uses screening to find out what a better-informed player knows. The alternative sense of *screening*—namely, concealing—is when a better-informed player who does not want to disclose his information through his actions chooses to mimic the behavior that would result from information he knows to be false. Such behavior leads to a *semiseparating* equilibrium, as discussed in Section 6.D. Return to reference 5

Glossary

signals

Devices used for signaling.

signaling

Strategy of a more-informed player to convey his “good” information credibly to a less-informed player.

screening

Strategy of a less-informed player to elicit information credibly from a more-informed player.

screening devices

Methods used for screening.

type

Players who possess different private information in a game of asymmetric information are said to be of different types.

separating equilibrium

A perfect Bayesian equilibrium in a game of asymmetric information, where the actions in the equilibrium reveal player type.

pooling equilibrium

A perfect Bayesian equilibrium in a game of asymmetric information, where the actions in the equilibrium cannot be used to distinguish type.

semiseparating equilibrium

A perfect Bayesian equilibrium in a game of asymmetric information, where the actions in the equilibrium convey some additional information about the players’ types, but some ambiguity about these types remains.

3 DIRECT COMMUNICATION, OR “CHEAP TALK”

The simplest way to convey information to others would seem to be to tell them; likewise, the simplest way to elicit information from them would seem to be to ask. But in a game of strategy, players should be aware that others may not tell the truth and, likewise, that their own assertions may not be believed by others. That is, the *credibility* of mere words may be questionable. It is a common saying that talk is cheap; indeed, direct communication has zero or negligible *direct* cost. However, it can *indirectly* affect the outcome and payoffs of a game by changing one player’s beliefs about another player’s actions or by influencing the selection of one equilibrium out of multiple equilibria. Direct communication that has no direct cost has come to be called *cheap talk* by game theorists, and the equilibrium achieved by using direct communication is termed a cheap talk equilibrium.

A. Perfectly Aligned Interests

Direct communication of information works well if the players' interests are well aligned. The assurance game first introduced in Chapter 4 provides the most extreme example of this. We reproduce its payoff table (Figure 4.14) here as Figure 9.3.

		WATSON	
		St. Bart's	Simpson's
HOLMES	St. Bart's	1, 1	0, 0
	Simpson's	0, 0	2, 2

FIGURE 9.3 Assurance Game

The interests of Sherlock Holmes and Dr. Watson are perfectly aligned in this game: They both want to meet, and both prefer meeting at Simpson's. The problem is that the game is played noncooperatively; they are making their choices independently, without knowledge of what the other is choosing. But suppose that Holmes is given an opportunity to send a message to Watson (or Watson is given an opportunity to ask a question and Holmes replies) before their choices are made. If Holmes's message (or reply; we will not keep repeating this) is "I am going to Simpson's," Watson has no reason to think he is lying.[6] If Watson believes Holmes, Watson should choose Simpson's, and if Holmes believes that Watson will believe him, it is equally optimal for Holmes to choose Simpson's, making his message truthful. Thus, direct communication very easily achieves the mutually preferable outcome. This is indeed the reason why, when we considered this game in Chapter 4, we had to construct a scenario in which such communication was infeasible.

Let us examine the outcome of allowing direct communication in the assurance game more precisely in game-theoretic terms. We have created a two-stage game. In the first stage, only Holmes acts, and his action is his message to Watson. In the second stage, the original simultaneous-move game is played. In the full two-stage game, we have a rollback equilibrium where the strategies (complete plans of action) are as follows. The second-stage action plans for both players are "If Holmes's first-stage message was 'I am going to St. Bart's,' then choose St. Bart's; if Holmes's first-stage message was 'I am going to Simpson's,' then choose Simpson's." (Remember that players in sequential-move games must specify *complete* plans of action.) The first-stage action for Holmes is to send the message "I am going to Simpson's." Verification that this is indeed a rollback equilibrium of the two-stage game is easy, and we leave it to you.

However, this equilibrium where cheap talk "works" is not the only rollback equilibrium of this game. Consider the following strategies: The second-stage action plan for each player is to go to St. Bart's regardless of Holmes's first-stage message; and Holmes's first-stage message can be anything. We can verify that this is also a rollback equilibrium. Regardless of Holmes's first-stage message, if one player is going to St. Bart's, then it is optimal for the other player to go there also. Thus, in each of the second-stage subgames that could arise—one after each of the two messages that Holmes could send—both choosing St. Bart's is a Nash equilibrium of the subgame. Then, in the first stage, Holmes, knowing his message is going to be disregarded, is indifferent about which message he sends.

The cheap talk equilibrium—where Holmes's message is not disregarded—yields higher payoffs, and we might normally think that it would be the one selected as a focal point. However, there may be reasons of history or culture that favor the other equilibrium. For example, for some reasons quite extraneous to this particular game, Holmes may have a reputation for being totally unreliable in following through on meetings, perhaps due to his absent-mindedness. Then people might generally disregard statements about meetings from him, and, knowing this to be the usual state of affairs, Watson might not believe this particular one.

Such problems exist in all communication games. They always have alternative equilibria where the communication is disregarded and therefore irrelevant. Game theorists call these babbling equilibria. Having noted that they exist, however, we will focus on cheap talk equilibria, where communication does have some effect.

B. Totally Conflicting Interests

The credibility of direct communication depends on the degree of alignment of players' interests. As a dramatic contrast with the assurance game example, consider a game where the players' interests are totally in conflict—namely, a zero-sum game. A good example is the tennis-point game in Figure 4.17; we reproduce its payoff matrix as Figure 9.4. Remember that the payoffs are Evert's success percentages. Remember, too, that this game has a Nash equilibrium only in mixed strategies (derived in Chapter 7); Evert's expected payoff in that equilibrium is 62.

Now suppose that we construct a two-stage game. In the first stage, Evert is given an opportunity to send a message to Navratilova. In the second stage, the original simultaneous-move game is played. What will be the rollback equilibrium?

		NAVRATILOVA	
		DL	CC
EVERT	DL	50, 50	80, 20
	CC	90, 10	20, 80

FIGURE 9.4 Tennis-Point Game

It should be clear that Navratilova will not believe any message she receives from Evert. For example, if Evert's message is "I am going to play DL," and Navratilova believes her, then Navratilova should choose to cover DL. But if Evert thinks that Navratilova will cover DL, then Evert's best choice is CC. At the next level of thinking, Navratilova should see through this and not believe the assertion of DL.

But there is more. Navratilova should not believe that Evert will do exactly the opposite of what she says either. Suppose Evert’s message is “I am going to play DL,” and Navratilova thinks, “She is just trying to trick me, and so I will take it that she will play CC.” This will lead Navratilova to choose to cover CC. But if Evert thinks that Navratilova will disbelieve her in this simple way, then Evert should choose DL after all. And Navratilova should see through this, too.

Thus, Navratilova’s disbelief should mean that she should totally disregard Evert’s message. Then the full two-stage game has only a babbling equilibrium. The two players’ actions in the second stage will be simply those of the original equilibrium, and Evert’s first-stage message can be anything. This is true of all zero-sum games.

C. Partially Aligned Interests

But what about games in which there is a mixture of conflict and common interest? Whether direct communication is credible in such games depends on how conflict and cooperation mix when players' interests are only partially aligned. Thus, we should expect to see both cheap talk and babbling equilibria in games of this type. More generally, the greater the alignment of interests, the more information should be communicable. We illustrate this intuition with an example.

Consider a situation that you may have already experienced or, if not, soon will when you start to earn and invest. When your financial adviser recommends an investment, she may be doing so as part of developing a long-term relationship with you for the steady commissions that your business will bring her, or she may be a fly-by-night operator who touts a loser, collects the up-front fee, and disappears. The credibility of her recommendation depends on what type of relationship you establish with her.

Suppose you want to invest \$100,000 in the asset recommended by your adviser, and that you can anticipate three possible outcomes. The asset could be a bad investment (B), leading to a 50% loss, or a payoff of −50 (measured in thousands of dollars). Or the asset could be a mediocre investment (M), yielding a 1% return, or a payoff of 1. Or, finally, it could be a good investment (G), yielding a 55% return, or a payoff of 55. If you choose to invest, you pay the adviser a 2% fee up front regardless of the performance of the asset; this fee gives your adviser a payoff of 2 and simultaneously lowers your payoff by 2. Your adviser will also earn 20% of any gain you make, leaving you with a payoff of 80% of the gain, but she will not have to share in any loss.

With no specialized knowledge related to the particular asset that has been recommended to you, you cannot judge which of the three outcomes is most likely. Therefore, you simply assume that all three possibilities—B, M, and G—are equally likely: that there is a one-third chance of each outcome occurring. In this situation, in the absence of any further information, you calculate your expected payoff from investing in the recommended asset as $[(⅓ \times -50) + (⅓ \times 0.8 \times 1) + (⅓ \times 0.8 \times 55)] - 2 = [⅓ \times (-50 + 0.8 + 44)] - 2 = [⅓ \times (-5.2)] - 2 = -1.73 - 2 = -3.73$. This calculation indicates an expected loss of $3,730. Therefore, you do not make the investment, and your adviser does not get any fee. Similar calculations show that you would also choose not to invest, due to a negative expected payoff, if you believed the asset was definitely the B type, definitely the M type, or definitely any probability-weighted combination of the B and M types alone.

Your adviser is in a different situation. She has researched the investment and knows which of the three possibilities—B, M, or G—is the truth. We want to determine what she will do with her information—specifically whether she will truthfully reveal to you what she knows about the asset. We consider the various possibilities below, assuming that you update your belief about the asset's type based on the information you receive from your adviser. For this example, we assume that you simply believe what you are told: You assign probability 1 to the asset being the type stated by your adviser.[7]

I. SHORT-TERM RELATIONSHIPIf your adviser tells you that the recommended asset is type B, you will choose not to invest. Why? Because your expected payoff from that asset is −50, and investing will cost you an additional 2 (in fees to the adviser), for a final payoff of −52. Similarly, if she tells you the asset is M, you will also not invest. In that case, your expected payoff is 80% of the return of 1 minus the 2 in fees, for a total of −1.2. Only if the adviser tells you that the asset is G will you choose to invest. In this situation, your expected payoff is 80% of the 55 return less the 2 in fees, or 42.

What will your adviser do with her knowledge, then? If the truth is G, your adviser will want to tell you the truth in order to induce you to invest. But if she anticipates no long-term relationship with you, she will be tempted to tell you that the truth is G, even when she knows the asset is either M or B. If you decide to invest on the basis of her statement, she simply pockets her 2% fee and flees; she has no further need to stay in touch. Knowing that there is a possibility of getting bad advice, or false information, from an adviser with whom you will interact only once, you should ignore the adviser's recommendation altogether. Therefore, in this asymmetric-information, short-term-relationship game, credible communication is not possible. The only equilibrium is the babbling one in which you ignore your adviser; there is no cheap talk equilibrium in this case.

II. LONG-TERM RELATIONSHIP: FULL REVELATIONNow suppose your adviser works for a firm that you have invested with for years; losing your future business may cost her her job. If you invest in the asset she recommends, you can compare its actual performance to your adviser's forecast. That forecast could prove to have been wrong in a small way (the forecast was M and the truth is B, or the forecast was G and the truth is M) or in a large way (the forecast was G and the truth is B). If you discover such misrepresentations, your adviser and her firm lose your future business. They may also lose business from others if you bad-mouth them to friends and acquaintances. If the adviser attaches a cost to her loss of reputation, she is implicitly concerned about your possible losses, and therefore her interests are *partially aligned* with yours. Suppose the cost to her reputation of a small misrepresentation is 2 (the monetary equivalent of a $2,000 loss) and that of a large misrepresentation is 4 (a $4,000 loss). We can now determine whether the partial alignment of your interests with those of your adviser is sufficient to induce her to be truthful.

As we discussed earlier, your adviser will tell you the truth if the asset is G to induce you to invest. We need to consider her incentives when the truth is *not* G, when the asset is actually B or M. Suppose first that the asset is B. If your adviser truthfully reveals the asset's type, you will not invest, and she will not collect any fee, but she will also suffer no reputation cost: her payoff from reporting B when the truth is B is 0. If she tells you the asset is M (even though it is B), you still will not invest because your expected payoff is −1.2, as we calculated earlier. Then the adviser will still get 0, so she has no incentive to lie and tell you that a B asset is really M.[8] But what if she reports G? If you believe her and invest, she will get the up-front fee of 2, but she will also suffer the reputation cost of the large error, 4.[9] Her payoff from reporting G when the truth is B is negative; your adviser would do better to reveal B truthfully. Thus, in situations when the truth about the asset is G or B, the adviser's incentives are to reveal the type truthfully.

But what if the truth is M? Truthful revelation does not induce you to invest: The adviser's payoff is 0 from reporting M. If she reports G and you believe her, you invest. The adviser gets her fee of 2, plus the 20% of the 1 that is your return from M, and she also suffers the reputation cost of the small misrepresentation, 2. Her payoff is $2 + (0.2 \times 1) - 2 = 0.2 > 0$. Thus, your adviser does stand to benefit by falsely reporting G when the truth is M. Knowing this, you will not believe any report of G.

Because your adviser has an incentive to lie when the asset she is recommending is M, full information cannot be credibly revealed in this situation. The babbling equilibrium, where any report from the adviser is ignored, is still a possible equilibrium. But is it the *only* equilibrium here, or is some partial communication possible? The failure to achieve full revelation occurs because the adviser will misreport M as G, so suppose we lump those two possibilities together into one and label it "not-B." Thus, the adviser asks herself what she should report: "B or not-B?"[10] Now we can consider whether your adviser will choose to report truthfully in this case of partial communication.

III. LONG-TERM RELATIONSHIP: PARTIAL REVELATIONTo determine your adviser's incentives in the "B or not-B" situation, we need to figure out what inference you will draw from the report of not-B, assuming you believe it. Your *prior* (original) belief was that B, M, and G were equally likely, with probabilities ⅓ each. If you are told the asset is not-B, you are left with two possibilities, M and G. You regarded the two as equally likely originally, and there is no reason to change that assumption, so you now give each a probability of ½. These are your new, *posterior*, probabilities, conditioned on the information you receive from your adviser's report. With these probabilities, your expected payoff if you invest when the report is not-B is $[\frac{1}{2}\times (0.8 \times 1)] + [\frac{1}{2} \times (0.8 \times 55)] - 2 = 0.4 + 22 - 2 = 20.4 > 0$. This positive expected payoff is sufficient to induce you to invest when given a report of not-B.

Knowing that you will invest if you are told not-B, we can determine whether your adviser will have any incentive to lie. Will she want to tell you not-B even if the truth is B? When the asset is actually B and the adviser tells the truth (reports B), her payoff is 0, as we calculated earlier. If she reports not-B instead, and you believe her, she gets 2 in fees.[11] She also incurs the reputation cost associated with misrepresentation. Because you assume that M or G is equally likely on the basis of the not-B report, the expected value of the reputation cost in this case is ½ times the cost of 2 for small misrepresentation plus ½ times the cost of 4 for large misrepresentation: The expected reputation cost is then $(½ \times 2) + (½ \times 4) = 3$. Your adviser's net payoff from reporting not-B when the truth is B is $2 - 3 = -1$. Therefore, she does not gain by making a false report to you. Because telling the truth is your adviser's best strategy here, a cheap talk equilibrium with credible *partial revelation* of information is possible.

The concept of the partial-revelation cheap talk equilibrium can be made more precise using the concept of a *partition*. Recall that you can anticipate three possible outcomes or events: B, M, and G. In our example, this set of events is divided, or partitioned, into distinct subsets, and your adviser then reports to you which subset contains the truth. (Of course, the verity of her report remains to be examined as part of the analysis.) Here, our three events are partitioned into two subsets, one consisting of the singleton B, and the other consisting of the pair of events {M, G}. In the partial-revelation equilibrium, these two subsets can be distinguished on the basis of the adviser's report, but the finer distinction between M and G—leading to the finest possible partition into three subsets, each consisting of a singleton—cannot be made. That finer distinction would be possible only in a case in which a full-revelation equilibrium exists.

We advisedly said earlier that a cheap talk equilibrium with credible partial revelation of information is *possible*. This game is one with multiple equilibria because the babbling equilibrium also remains possible. The configuration of strategies and beliefs where you ignore the adviser's report, and the adviser sends the same report (or even a random report) regardless of the truth, is still an equilibrium: Given each player's strategy, the other player has no reason to change her actions or beliefs. In the terminology of partitions, we can think of this babbling equilibrium as having the coarsest possible, and trivial, partition, with just one (sub)set {B, M, G} containing all three possibilities. In general, whenever you find a non-babbling equilibrium in a cheap talk game, there will also be at least one other equilibrium with a coarser or cruder partition of outcomes.

IV. MULTIPLE EQUILIBRIAAs an example of a situation in which coarser partitions are associated with additional equilibria, consider the case in which your adviser's cost of reputation is higher than assumed above. Let the reputation cost be 4 (instead of 2) for a small misrepresentation of the truth and 8 (instead of 4) for a large misrepresentation. Our analysis above showed that your adviser will report G if the truth is G, and that she will report B if the truth is B. These results continue to hold. Your adviser wants you to invest when the truth is G, and she still gets the same payoff from reporting B when the truth is B as she does from reporting M in that situation. The higher reputation cost gives her even less incentive to falsely report G when the truth is B. So if the asset is either B or G, the adviser can be expected to report truthfully.

The problem for full revelation in our earlier example arose because of the adviser's incentive to lie when the asset is M. With our earlier numbers, her payoff from reporting G when the truth is M was higher than that from reporting M truthfully. Will that still be true with the higher reputation costs?

Suppose the truth is M and the adviser reports G. If you believe her and invest in the asset, her expected payoff is 2 (her fee) + 0.2×1 (her share in the actual return from M) $- 4$ (her reputation cost) $= -1.8 < 0$. The truth would get her 0. She no longer has the temptation to exaggerate the quality of the asset. The outcome where she always reports the truth, and you believe her and act upon her report, is now a cheap talk equilibrium with full revelation. This equilibrium has the finest possible partition, consisting of three singleton subsets, {B}, {M}, and {G}.

There are also *three* other equilibria in this case, each with a coarser partition than the full-revelation equilibrium. Both two-subset situations—one with {B, M} and {G} and the other with {B} and {M, G}—and the babbling situation with {B, M, G} are all alternative possible equilibria. We leave it to you to verify this. Which one prevails depends on all the considerations addressed in Chapter 4 in our discussion of games with multiple equilibria.

The biggest practical difficulty associated with attaining a non-babbling equilibrium with credible information communication lies in the players' knowledge about the extent to which their interests are aligned. The extent of alignment of the two players' interests must be common knowledge between them. In the investment example, it is critical that you know from past interactions or other credible sources (for example, a contract) that the adviser has a large reputational concern in your investment outcome. If you did not know to what extent her interests were aligned with yours, you would be justified in suspecting that she was exaggerating the quality of an asset to induce you to invest for the sake of the fee she would earn immediately.

What happens when even richer messages are possible? For example, suppose that your adviser could report a number *g*, representing her estimate of the rate of growth of the asset price, and that *g* could range over a continuum of values. In this situation, as long as the adviser gets some extra benefit if you buy a bad stock that she recommends, she has some incentive to exaggerate *g*. Therefore, a full-revelation cheap talk equilibrium is no longer possible. But a partial-revelation cheap talk equilibrium may be possible. The continuous range of growth rates may be split into intervals—say, from 0% to 1%, from 1% to 2%, and so on—such that the adviser finds it optimal to tell you truthfully into which of these intervals the actual growth rate falls, and you find it optimal to accept this advice and take your optimal action on its basis. The higher the adviser's valuation of her reputation, the finer the possible partition will be—for example, half-percentage points instead of whole or quarter-percentage points instead of half. However, we must leave further explanation of this idea to more advanced treatments of the subject.[12]

Endnotes

- This reasoning assumes that Holmes’s payoffs are as stated, and that this fact is common knowledge between the two. If Watson suspects that Holmes wants Watson to go to Simpson’s so Holmes can go to St. Bart’s to follow up privately on a different case, Watson’s strategy will be different! Analysis of games of asymmetric information thus depends on how many different possible “types” of players are actually conceivable. Return to reference 6
- In the language of probability theory, the probability you assign to a particular event after having observed, or heard, information or evidence about that event is known as the *posterior probability* of the event. You thus assign posterior probability 1 to the stated quality of the asset. *Bayes’ theorem*, which we explain in detail in the appendix to this chapter, provides a formal quantification of the relationship between prior and posterior probabilities. Return to reference 7
- We are assuming that if you do not invest in the recommended asset, you do not find out its actual return, so the adviser can suffer no reputation cost in that case. This assumption fits nicely with the general interpretation of “cheap talk.” No message has any direct payoff consequences for the sender; those consequences arise only if the receiver acts upon the information received in the message. Return to reference 8
- The adviser’s payoff calculation does not include a 20% share of your return here. The adviser knows the truth to be B and so knows you will incur a loss, in which she will not share. Return to reference 9
- Our apologies to William Shakespeare. Return to reference 10

- Again, the adviser’s payoff calculation includes no portion of your gain because you will incur a loss: the truth is B, and the adviser knows the truth. Return to reference 11
- The seminal paper by Vincent Crawford and Joel Sobel, “Strategic Information Transmission,” *Econometrica*, vol. 50, no. 6 (November 1982), pp. 1431–52, developed this theory of partial communication. An elementary exposition and survey of further work is in Joseph Farrell and Matthew Rabin, “Cheap Talk,” *Journal of Economic Perspectives*, vol. 10, no. 3 (Summer 1996), pp. 103–18. Return to reference 12

Glossary

cheap talk equilibrium
: In a game where communication among players (which does not affect their payoffs directly) is followed by their choices of actual strategies, a cheap talk equilibrium is one where the strategies are chosen optimally given the players' interpretation of the communication, and the communication at the first stage is optimally chosen by calculating the actions that will ensue.

babbling equilibrium
: In a game where communication among players (which does not affect their payoffs directly) is followed by their choices of actual strategies, a babbling equilibrium is one where the strategies are chosen ignoring the communication, and the communication at the first stage can be arbitrary.

4 ADVERSE SELECTION, SIGNALING, AND SCREENING

A. Adverse Selection and Market Failure

In many games, one of the players knows something pertinent to the outcomes that the other players don’t know. An employer knows much less about the skills of a potential employee than does the employee himself; vaguer but important characteristics such as work attitude and collegiality are even harder to observe. An insurance company knows much less about the health or the driving skills of someone applying for medical or auto insurance than does the applicant. The seller of a used car knows a lot about the car from long experience; a potential buyer can at best get a little information by inspection.

In such situations, direct communication will not credibly signal information. Unskilled workers will claim to have skills to get higher-paid jobs; people who are bad risks will claim good health or driving habits to get lower insurance premiums; owners of bad cars will assert that their cars run fine and have given them no trouble in all the years they have owned them. The other parties to the transactions will be aware of the incentives to lie and will not trust information conveyed by words alone. There is no possibility of a cheap talk equilibrium of the type described in Section 3.

What if the less informed parties in these transactions have no way of obtaining the pertinent information at all? In other words, to use the terminology introduced in Section 2, suppose that no credible screening devices or signals are available. If an insurance company offers a policy that costs 5 cents for each dollar of coverage, then the policy will be especially attractive to people who know that their own risk (of illness or a car crash) exceeds 5%. Of course, some people who know their risk to be lower than 5% will still buy the insurance because they are risk averse. But the pool of applicants for this insurance policy will have a larger proportion of the high-risk people than the population as a whole. The insurance company will selectively attract an unfavorable, or adverse, group of customers. This phenomenon, which is very common in transactions involving asymmetric information, is known as adverse selection (a term that, in fact, originated within the insurance industry).

The potential consequences of adverse selection for market transactions were dramatically illustrated by George Akerlof in a paper that became the starting point of economic analysis of asymmetric-information situations and won him a Nobel Prize in 2001.[13] We use his example to introduce you to the effects that adverse selection may have.

B. The Market for “Lemons”

Think of the market in 2020 for a specific kind of used car—say, a 2017 Citrus. Suppose that in use, these cars have either proved to be largely trouble free and reliable or have had many things go wrong. The usual slang name for the latter type of car is “lemon,” so for contrast, let us call the former type “orange.”

Suppose that each owner of an orange Citrus values it at $12,500; he is willing to part with it for a price higher than this, but not for a lower price. Similarly, each owner of a lemon Citrus values it at $3,000. Suppose that potential buyers are willing to pay more than these values for each type. If a buyer could be confident that the car he was buying was an orange, he would be willing to pay $16,000 for it; if the car was a known lemon, he would be willing to pay $6,000. Since the buyers value each type of car more than do the original owners, it benefits everyone if all the cars are traded. The price for an orange can be anywhere between $12,500 and $16,000; that for a lemon anywhere between $3,000 and $6,000. For definiteness, we will suppose that there is a limited stock of such cars and a larger number of potential buyers. Then the buyers, competing with one another, will drive the price up to their full willingness to pay. The prices will be $16,000 for an orange and $6,000 for a lemon—if each type can be identified with certainty.

But information about the quality of any specific car is asymmetric between the two parties to the transaction. The owner of a Citrus knows perfectly well whether it is an orange or a lemon. Potential buyers don’t, and the owner of a lemon has no incentive to disclose the truth. For now, we confine our analysis to the private used-car market, in which laws requiring truthful disclosure are either nonexistent or hard to enforce. We also assume away any possibility that the potential buyer can observe something that tells him whether the car is an orange or a lemon; similarly, the car owner has no way to indicate the type of car he owns. Thus, for this example, we consider the effects of the information asymmetry alone, without allowing either side of the transaction to signal or screen.

When buyers cannot distinguish between oranges and lemons, there cannot be distinct prices for the two types in the market. Oranges and lemons, if sold at all, must sell at the same price p, which we will refer to as the "Citrus price." Whether efficient trade is possible under such circumstances will depend on the proportions of oranges and lemons in the population. We suppose that oranges are a fraction f of used Citruses and lemons the remaining fraction $(1 - f)$.

Even though buyers cannot verify the quality of an individual car, they can know the proportion of oranges in the Citrus population as a whole—for example, from newspaper reports—and we assume this to be the case. If all Citruses are being traded, a potential buyer will expect to get a random selection, with probabilities f and $(1 - f)$ of getting an orange and a lemon, respectively. The expected value of the car purchased is $16,000 \times f + 6,000 \times (1 - f) = 6,000 + 10,000 \times f$. He will buy such a car if its expected value exceeds the Citrus price; that is, if $6,000 + 10,000 \times f > p$.

Now consider the point of view of the seller, who knows whether his car is an orange or a lemon. The owner of a lemon is willing to sell it as long as the Citrus price exceeds its value to him; that is, if $p > 3,000$. But the owner of an orange requires $p > 12,500$. If this condition for an orange owner to sell is satisfied, so is the condition for a lemon owner to sell.

To meet the requirements for all buyers and sellers to want to make the trade, therefore, we need $6,000 + 10,000 \times f > p > 12,500$. If the fraction of oranges in the population satisfies $6,000 + 10,000 \times f > 12,500$, or $f > 0.65$, a Citrus price can be found that does the job; otherwise there cannot be efficient trade. If $6,000 + 10,000 \times f < 12,500$ (leaving out the exceptional and unlikely case where the two are equal), owners of oranges are unwilling to sell at the maximum price potential buyers are willing to pay. We then have adverse selection in the set of used cars put up for sale: No oranges will appear in the market at all. The potential buyers will recognize this, will expect to get a lemon, and will pay at most $6,000. The owners of lemons will be happy with this outcome, so lemons will sell. But the market for oranges will collapse completely due to the asymmetry of information. The outcome will be that bad cars drive out the good.

Because the lack of information makes it impossible to get a reasonable price for an orange, the owners of oranges will want a way to convince buyers that their cars are the good type. They will want to *signal* their type. The trouble is that the owners of lemons will want to pretend that their cars, too, are oranges, and to this end they can imitate most of the signals that owners of oranges might attempt to use. Michael Spence, who developed the concept of signaling, summarizes the problems facing our orange owners in his pathbreaking book on signaling:

> Verbal declarations are costless and therefore useless. Anyone can lie about why he is selling the car. One can offer to let the buyer have the car checked. The lemon owner can make the same offer. It's a bluff. If called, nothing is lost. Besides, such checks are costly. Reliability reports from the owner's mechanic are untrustworthy. The clever non-lemon owner might pay for the checkup but let the purchaser choose the inspector.

> The problem for the owner, then, is to keep the inspection cost down. Guarantees do not work. The seller may move to Cleveland, leaving no forwarding address.[14]

In reality, the situation is not as hopeless as Spence implies. People and firms that regularly sell used cars as a business can establish a reputation for honesty and profit from this reputation by charging a markup. (Of course, some used-car dealers are unscrupulous.) Some buyers are knowledgeable about cars; some buy from personal acquaintances and can therefore verify the history of the car they are buying. Or dealers may offer warranties, a topic we will discuss in more detail shortly. And in other markets, it is harder for bad types to mimic the actions of good types, so that credible signaling is possible. For a specific example of such a situation, consider the possibility that education can signal skill. If it is to do so, it must be hard for unskilled people to acquire enough education to be mistaken for highly skilled people. The key requirement for a signal to separate types is that it be sufficiently more costly for the truly unskilled to send the signal than for the truly skilled.

C. Signaling and Screening: Sample Situations

The basic idea of signaling or screening to convey or elicit information is very simple: Players of different *types* (that is, players possessing different information about their own characteristics or about the game and its payoffs more generally) should find it optimal to take different actions so that their actions truthfully reveal their types. Situations of information asymmetry, and signaling and screening strategies to cope with them, are ubiquitous. Here are some additional situations to which the methods of analysis developed throughout this chapter can be applied.

I. INSURANCEThe prospective buyers of an insurance policy vary in their risk classes, or their levels of riskiness to the insurer. For example, among the numerous applicants for an automobile collision insurance policy will be some drivers who are naturally cautious and others who are simply less careful. Each potential customer has a better knowledge of his or her own risk class than does the insurance company. Given the terms of any particular policy, the company will make less profit (or incur a greater loss) on the higher-risk customers. However, the higher-risk customers will be the ones who find the specified policy more attractive. Thus, the company will attract the less favorable group of customers, and we have a situation of adverse selection.[15] Clearly, the insurance company would like to distinguish between the risk classes. They can do so by using a screening device.

Suppose as an example that there are just two risk classes. The company can then offer two policies, from which any individual customer chooses one. The first has a lower premium (in units of cents per dollar of coverage), but covers a lower percentage of any loss incurred by the customer; the second has a higher premium, but covers a higher percentage, perhaps even 100%, of the customer’s loss. (In the case of collision insurance, the loss represents the cost of having an auto body shop complete the needed repairs to the customer’s car.) A higher-risk customer is more likely to suffer a loss and is therefore more willing to pay the higher premium to get more coverage. The company can then adjust the premiums and coverage ratios so that customers of the higher-risk type choose the high-premium, high-coverage policy and customers of the lower-risk type choose the lower-premium, lower-coverage policy. If there are more risk types, there have to be correspondingly more policies in the menu offered to prospective customers; with a continuous spectrum of risk types, there may be a corresponding continuum of policies.

Of course, this insurance company has to compete with other insurance companies for the business of each customer. That competition affects the premiums it can charge and the levels of coverage it can offer. Sometimes competition may even preclude the attainment of an equilibrium, as each offering can be undercut by another.[16] But the general idea behind differential premiums and policies for customers of different risk classes is valid and important.

II. WARRANTIESMany types of durable goods—cars, computers, washing machines—vary in their quality. Any company that has produced such a good will have a pretty good idea of its quality. But a prospective buyer will be much less informed. Can a company that knows its product to be of high quality signal this fact credibly to its potential customers?

The most obvious, and most commonly used, signal of high quality is a good warranty. The cost of providing a warranty is lower for a genuinely high-quality product because its producer is less likely to be called on to provide repairs or replacement than is a company with a shoddier product. Therefore, warranties can serve as credible signals of quality, and consumers are intuitively aware of this fact when they make their purchase decisions.

Typically in such situations, the signal has to be carried to excess in order to make it sufficiently costly to mimic. Thus, the producer of a high-quality car has to offer a sufficiently long or strong warranty to signal its quality credibly. This requirement is especially relevant for any company that is a relative newcomer to the market or one that does not have a previous reputation for offering high-quality products. Hyundai, for example, began selling cars in the United States in 1986, and for its first decade had a low-quality reputation. In the mid-1990s, it invested heavily in better technology, design, and manufacturing. To revamp its image, it offered the then-revolutionary 10-year, 100,000-mile warranty. Now it ranks with consumer groups as one of the better-quality automobile manufacturers.

III. PRICE DISCRIMINATIONThe buyers of most products are heterogeneous in terms of their willingness to pay, their willingness to devote time to searching for a better price, and so on. Companies would like to identify those potential customers with a higher willingness to pay and charge them one—presumably fairly high—price while offering selective good deals to those who are not willing to pay so much (as long as that willingness to pay still exceeds the cost of supplying the product). A company can successfully charge different prices to different types of customers by using screening devices to separate the types. We will discuss such strategies, known as price discrimination, in more detail in Chapter 14. Here we provide just a brief overview.

The example of discriminatory prices best known to most people comes from the airline industry. Business travelers are willing to pay more for their airline tickets than are tourists, often because at least some of the cost of the ticket is borne by the business traveler’s employer. It would be illegal for airlines blatantly to identify each traveler’s type and to charge different types different prices. But the airlines take advantage of the fact that tourists are more willing to commit to an itinerary well in advance, while business travelers need to retain flexibility in their plans. Therefore, airlines charge different prices for nonrefundable versus refundable fares and leave it to the travelers to choose their fare type. This pricing strategy is an example of *screening by self-selection*, which we will investigate more formally in Section 5 of this chapter. Other devices—advance purchase or Saturday night stay requirements; different classes of onboard service (first versus business versus coach)—serve the same screening purpose.

Price discrimination is not specific to high-priced products like airline tickets. Other discriminatory pricing schemes can be observed in many markets where product prices are considerably lower than those for air travel. Coffee and sandwich shops, for example, commonly offer “frequent buyer” discount cards. These cards effectively lower the price of coffee or a sandwich to the shop’s regular customers. The idea is that regular customers are more willing to search for the best deal in the neighborhood, while visitors or occasional users would go to the first coffee or sandwich shop they see without spending the time necessary to determine whether any lower prices might be available. The higher regular price and “free 11th item” discount represent the menu of options from which the two types of customers select, thereby separating themselves by type.

IV. PRODUCT DESIGN AND ADVERTISINGCan an attractive, well-designed product exterior serve the purpose of signaling high quality? The key requirement for a credible signal is that its cost be sufficiently higher for a company trying to mimic high quality than for one that has a truly high-quality product. Typically, the cost of the product exterior is the same regardless of the innate quality that resides within. Therefore, the mimic would face no cost differential, and the signal would not be credible.

But such signals may have some partial validity. Exterior design is a fixed cost that is spread over the whole product run. Buyers do learn about quality from their own experience, from friends, and from reviews and comments in the media. These considerations indicate that a high-quality good can expect to have a longer market life and higher total sales. Therefore, the cost of an expensive exterior is spread over a larger volume, and adds less to the cost of each unit of the product, if that product is of higher innate quality. The firm is in effect making a statement: “We have a good product that will sell a lot. That is why we can afford to spend so much on its design. A fly-by-night firm would find this cost prohibitive for the few units it expects to sell before people find out its poor quality and don’t buy any more from it.” Even expensive, seemingly useless and uninformative product launch and advertising campaigns can have a similar signaling effect.[17]

Similarly, when you walk into a bank and see solid, expensive marble counters and plush furnishings, you may be reassured about the bank’s stability. However, for this particular signal to work, it is important that the building, furnishings, and decor be specific to the bank. If everything could easily be sold to other types of establishments and the space converted into a restaurant, then a fly-by-night operator could mimic a truly solid bank at no higher cost. In that situation, the signal would not be credible.

V. CRIMEThe world of organized crime is full of interactions that require signaling and screening. How do you know whether a prospective new member of your group is a police infiltrator? How do you know whether your partner in an illegal transaction will cheat you when your deal cannot be enforced in a court of law? People in that world have devised many strategies that exemplify some basic principles of games with asymmetric information. Here are just a few examples.[18]

To become a full member of a Mafia family, a candidate must "make his bones" (establish his bona fides or credibility) by committing a serious crime, often murder. More than just a test of toughness and ability, this requirement is a good screening device. Someone who truly wants to belong to the organization will go that far, but a police infiltrator will not. Infiltrators have been able to penetrate close to this level, however, by hanging out with criminals for a long time and mimicking their behavior, dress, and talk closely. Thus, Joseph Pistone was able to pass himself off as the Mafia hanger-on Donnie Brasco in the 1997 film of the same name because "it just was very hard for mobsters to think that, taken together, all the things he did and did not do were not near-perfect discriminating signals."[19]

The process of identifying potential business partners is risky and error-prone for criminals. The recipient of a signal may be a police informer or entrapper. Or he may be an innocent outsider who is horrified when he realizes the criminal intent of the signaler and goes to the police. The strategy for reducing these risks, in theory as well as in reality, is to use a multistage process, starting with a deliberately ambiguous signal. If the recipient reacts adversely, the signal can be explained away as meaning something different and innocent. If it is received favorably, a slightly less ambiguous signal is sent, and so on.

The 1951 British comedy *The Lavender Hill Mob* provides a brilliant example. Henry Holland (played by Alec Guinness) is a lowly bank clerk whose job is to supervise transport of large quantities of gold bars from one bank to another. He would like to steal a consignment and escape to live in luxury in an idyllic tropical spot. But he recognizes the difficulty of getting the bullion out of the country. At his lodging, he meets Alfred Pendlebury (Stanley Holloway), who runs a foundry where lead is melted to cast Eiffel Tower replica paperweights that are exported to France and sold as souvenirs at the actual tower. Holland coyly sounds out Pendlebury on the topic of partnering to steal the bullion by pondering aloud what it would take to do so successfully. As a last hint, he notes that "supposing one had the right sort of partner," one could get the gold to Europe "in the form of, shall we say, Eiffel Tower paperweights." Pendlebury remarks with a laugh, "By Jove, Holland, it's a good job we're both honest men." Holland, with a look of mock solemnity, replies, "It is indeed, Pendlebury." They understand each other's implied agreement and proceed to plan the job, but any allegation of criminality to that point remained deniable.[20]

Another excellent example of signaling comes from *Cogan's Trade*, a novel about Boston mafiosi. A character named Mark Trattman runs a high-stakes poker game under Mafia protection, and he arranges for it to be robbed in exchange for a cut of the proceeds. By the time the bosses find this out, the fuss has died down, and Trattman is well liked, so they do nothing. But then someone else gets the idea that if he robs the game, Trattman will be blamed again. The bosses detect the truth, and now they face a bigger problem: Their reputation as good protectors has been ruined and must be rebuilt. They need an effective signal, and to be credible in the standard game-theoretic sense, they must carry it to overkill—in this context, literally so. Even though Trattman is innocent this time, he has to be killed. Cogan, the up-and-coming enforcer, explains this very clearly to the godfather's consigliere[21]: "It's his responsibility. He did it before and he lied before and he fooled everybody, and I said . . . 'They should have whacked him out before.' . . . Now it happened again. . . . It's his responsibility for what the guys think. . . . Trattman did it before, [they think] Trattman did it again. . . . Trattman's gotta be hit." Of course, Cogan also "whacks out" the two actual perpetrators of the second robbery. First he forms a temporary alliance with one of them to set up the other, promising to spare him in exchange. The other fails to realize that the promise is not credible. Failing to check credibility, to see through cheap talk, and to figure out the proper rollback solution of a game can have life-and-death consequences!

VI. POLITICAL BUSINESS CYCLESIncumbent governments often increase spending to get the economy to boom just before an election, thereby hoping to attract more votes and win the election. But shouldn't rational voters see through this stratagem and recognize that as soon as the election is over, the government will be forced to retrench, perhaps setting off a recession? For pre-election spending to be an effective signal of type, there has to be some uncertainty in the voters' minds about the "competence type" of the government. The likely recession will create a political cost for the government, but this cost will be smaller if the government is competent in its handling of the economy. If the cost differential between competent and incompetent government types is large enough, a sufficiently high expenditure spike can credibly signal competence.[22]

Another similar example relates to inflation controls. Many countries at many times have suffered high inflation, and many governments have piously declared their intention to reduce the rate of inflation. Can a government that truly cares about price stability credibly signal its type? Yes. Governments can issue bonds protected against inflation. The interest rate on such a bond is automatically ratcheted up by the rate of inflation, or the capital value of the bond rises in proportion to the increase in the price level. Issuing government debt in this form is more costly to a government that embraces policies that lead to higher inflation because it has to make good on its contract to pay more interest or increase the value of its debt. Therefore, a government with genuinely anti-inflation preferences can issue inflation-protected bonds as a credible signal, separating itself from the inflation-loving type of government.

VII. EVOLUTIONARY BIOLOGYIn many species of birds, such as peafowl, the males (peacocks) have very elaborate and heavy plumage, which females (peahens) find attractive. One should expect the females to seek genetically superior males so that their offspring will be better equipped to survive to adulthood and to attract mates in their turn. But why does elaborate plumage indicate such desirable genetic qualities? One would think that such plumage might be a handicap, making the male bird more visible to predators (including human hunters) and less able to evade them. Why do females choose these seemingly handicapped males? The answer comes from the conditions for credible signaling. Although heavy plumage is indeed a handicap, it is less of a handicap to a male who is sufficiently genetically superior in qualities such as strength and speed. The weaker the male, the harder it will be for him to produce and maintain plumage of a given quality. Thus, it is precisely the heaviness of the plumage that makes it a credible signal of the male's quality.[23]

VIII. AI AND COMPUTERS PLAYING SIGNALING GAMESIn Chapter 3, Section 5, we discussed how artificial intelligence has revolutionized the ability of computers to play games like chess and Go. However, the games we described there were all games of perfect and complete information. More recently, AI has progressed sufficiently to allow computers to play games involving hidden information and strategies like signaling or screening. By playing no-limit Texas Hold'em poker trillions of times against itself and then improving its strategies further in play against top humans, a computer AI program called Pluribus, designed by Noam Brown of Facebook and Tuomas Sandholm of Carnegie Mellon University, was able to defeat six of the world’s top (human) poker players. During its learning process, Pluribus devised some previously unused strategies, such as dramatically upping the ante of small pots, that human pros are now mimicking.[24]

D. Experimental Evidence

The characterization of and solutions for equilibria in games with signaling and screening entail some subtle concepts and computations. Thus, in each case presented above, formal models must be carefully described in order to formulate reasonable and accurate predictions of player choices. In all such games, players must revise or update their beliefs about the probabilities of other players' types on the basis of their observations of those other players' actions. This updating requires an application of an algebraic formula called Bayes' theorem (also referred to as *Bayes' rule* or *Bayes' formula*), which is explained in the appendix to this chapter. We will carefully analyze an example of a game that requires this kind of updating in Section 6 of this chapter.

You can imagine, without going into any of the details in the appendix, that these probability-updating calculations are quite complex. Should we expect players to perform them correctly? There is ample evidence that people are very bad at performing calculations that include probabilities and are especially bad at updating probabilities on the basis of new information.[25] Therefore, we should be justifiably suspicious of equilibria that depend on the players' doing so.

Relative to this expectation, the findings of economists who have conducted laboratory experiments on signaling games are encouraging. Some surprisingly subtle refinements of *Bayesian Nash* and *perfect Bayesian equilibria* are successfully observed, even though these refinements require not only the updating of information by observing actions along the equilibrium path of play, but also the drawing of inferences from off-equilibrium actions that should never have been taken in the first place. However, the verdict of the experiments is not unanimous; much seems to depend on the precise details of the design of the experiment.[26]

Endnotes

- George Akerlof, “The Market for Lemons: Qualitative Uncertainty and the Market Mechanism,” *Quarterly Journal of Economics*, vol. 84, no. 3 (August 1970), pp. 488–500. Return to reference 13
- A. Michael Spence, *Market Signaling: Information Transfer in Hiring and Related Screening Processes* (Cambridge, Mass.: Harvard University Press, 1974), pp. 93–94. The present authors apologize on behalf of Spence to any residents of Cleveland who may be offended by any unwarranted suggestion that that’s where shady sellers of used cars go! Return to reference 14
- Here we are not talking about the possibility that a well-insured driver will deliberately exercise less care. That is moral hazard, and it can be mitigated using co-insurance schemes similar to those discussed here. For now, our concern is purely adverse selection, where some drivers are careful by nature, and others are equally uncontrollably spaced out and careless when they drive. Return to reference 15
- See Michael Rothschild and Joseph Stiglitz, “Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information,” *Quarterly Journal of Economics*, vol. 90, no. 4 (November 1976), pp. 629–49. Return to reference 16
- Kyle Bagwell and Gary Ramey, “Coordination Economies, Advertising, and Search Behavior in Retail Markets,” *American Economic Review*, vol. 84, no. 3 (June 1994), pp. 498–517. Return to reference 17
- Probably the best serious scholar of this underworld (as opposed to numerous sensational writers) is sociologist Diego Gambetta. We highly recommend his *Codes of the Underworld: How Criminals Communicate* (Princeton, N.J.: Princeton University Press, 2009). Return to reference 18

- Gambetta, *Codes of the Underworld*, p. 22. See footnote 18. Return to reference 19
- Readers will surely recognize from their personal lives that the same graduated revelation of interest is the correct strategy when exploring the possibility of a romantic relationship. Return to reference 20
- George V. Higgins, *Cogan's Trade* (New York: Carroll and Graf, 1974), Chapter 8. Return to reference 21
- These ideas and the supporting evidence are reviewed by Alan Drazen in "The Political Business Cycle after 25 Years," in *NBER Macroeconomics Annual 2000*, ed. Ben S. Bernanke and Kenneth S. Rogoff (Cambridge, Mass.: MIT Press, 2001), pp. 75–117. Return to reference 22
- Matt Ridley, *The Red Queen: Sex and the Evolution of Human Behavior* (New York: Penguin, 1995), p. 148. Return to reference 23
- The exploits of Pluribus and its creators are detailed in "35 Innovators Under 35," *MIT Technology Review*, vol. 122, no. 4 (July/August 2019), p. 9; and "Bet on the Bot: AI Beats the Professionals at 6-Player Texas Hold 'Em," *All Things Considered*, NPR, WBUR, Boston, 11 July 2019. Return to reference 24
- Deborah J. Bennett, *Randomness* (Cambridge, Mass.: Harvard University Press, 1998), pp. 2–3 and Chapter 10. See also Paul Hoffman, *The Man Who Loved Only Numbers* (New York: Hyperion, 1998), pp. 233–40, for an entertaining account of how several probability theorists, as well as the brilliant and prolific mathematician Paul Erdös, got a very simple probability problem wrong and even failed to understand their error when it was explained to them. Return to reference 25
- Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton, N.J.: Princeton University Press, 1995), review and discuss these experiments in their Chapter 7. Return to reference 26

Glossary

adverse selection
: A form of information asymmetry where a player’s type (available strategies, payoffs . . .) is his private information, not directly known to others.

Bayes’ theorem
: An algebraic formula for estimating the probabilities of some underlying event by using knowledge of some consequences of it that are observed.

5 SIGNALING IN THE LABOR MARKET

Many of you expect that when you graduate, you will work for an elite firm in finance or computing. These firms have two kinds of jobs. One kind requires good quantitative and analytical skills and a high capacity for hard work, and offers high pay in return. The other kind of job is a semi-clerical, lower-skill, lower-paying job. Of course, you want the job with higher pay. You know your own qualities and skills far better than your prospective employer does. If you are highly skilled, you want your employer to know this about you, and he also wants to know. He can test and interview you, but what he can find out by these methods is limited by the available time and resources. You can tell him how skilled you are, but mere assertions about your qualifications are not credible. More objective evidence is needed, both for you to offer and for your employer to seek out.

What items of evidence can the employer seek, and what can you offer? Recall from Section 2 of this chapter that your prospective employer will use *screening devices* to identify your qualities and skills. You will use *signals* to convey your information about those same qualities and skills. Sometimes similar or even identical devices can be used for both signaling and screening.

In this instance, if you have selected (and passed) particularly tough and quantitative courses in college, your course choices can be credible evidence of your capacity for hard work in general and of your quantitative and logical skills in particular. Let us consider the role of course choice as a screening device.

A. Screening to Separate Types

To keep things simple, we approach this screening game using intuition and some algebra. Suppose college students are of just two types when it comes to the qualities most desired by employers: A (able) and C (challenged). Potential employers in finance or computing are willing to pay $160,000 a year to a type A and $60,000 to a type C. Other employment opportunities yield the A types a salary of $125,000 and the C types a salary of $30,000. These are the same numbers as in the Citrus car example in Section 4.B above, but multiplied by a factor of 10 better to suit the reality of the job-market example. And just as in the used-car example, where we supposed there was a fixed supply and numerous potential buyers, we suppose here that there are many potential employers who have to compete with one another for a limited number of job candidates, so they have to pay the maximum amount that they are willing to pay. Because employers cannot directly observe any particular job applicant's type, they have to look for other credible means to distinguish among them.[27]

Suppose the types differ in their tolerance for taking tough courses rather than easy ones in college. Each type must sacrifice some party time or other activities to take a tougher course, but this sacrifice is smaller or easier to bear for the A types than it is for the C types. Suppose the A types regard the cost of each such course as equivalent to $3,000 a year of salary, while the C types regard it as equivalent to $15,000 a year of salary. Can an employer use this differential to screen his applicants and tell the A types from the C types?

Consider the following hiring policy: Anyone who has taken a certain number, *n*, or more of tough courses will be regarded as an A and paid $160,000, and anyone who has taken fewer than *n* will be regarded as a C and paid $60,000. The aim of this policy is to create natural incentives whereby only the A types will take tough courses, and the C types will not. Neither wants to take more tough courses than he has to, so the choice is between taking *n* to qualify as an A or giving up and settling for being regarded as a C, in which case he may as well not take any of the tough courses and just coast through college.

To succeed, such a policy must satisfy two kinds of conditions. The first set of conditions requires that the policy give each type of job applicant the incentive to make the choice that the firm wants him to make. In other words, the policy should be compatible with the incentives of the applicants; therefore, conditions of this kind are called incentive-compatibility conditions. The second set of conditions ensures that, by making the incentive-compatible choice, the applicants get a better (at least, no worse) payoff than they would get from their alternative opportunities. In other words, the applicants should be willing to participate in the firm's offer; therefore, conditions of this kind are called participation conditions. We will develop these conditions in the labor market context now. Similar conditions will appear in other examples later in this chapter and again in Chapter 14, where we develop the general theory of incentive design.

I. INCENTIVE COMPATIBILITYThe criterion that the employer devises to distinguish an A from a C—here, the number of tough courses taken—should be sufficiently strict that the C types do not bother to meet it, but not so strict as to discourage even the A types from attempting it. The correct value of *n* must be such that the true C types prefer to settle for being revealed as such and getting $60,000, rather than incurring the extra cost of imitating the A types' behavior.[28] That is, we need the policy to be incentive compatible for the C types, so

$$60{,}000 \geqslant 160{,}000 - 15{,}000n, \text{ or } 15n \geqslant 100, \text{ or } n \geqslant 6.67.$$

Similarly, the condition such that the true A types prefer to prove their type by taking n tough courses is

$$160{,}000 - 3{,}000n \geqslant 60{,}000, \text{ or } 3n \leqslant 100, \text{ or } n \leqslant 33.33.$$

These incentive-compatibility conditions or, equivalently, incentive-compatibility constraints, align a job applicant's incentives with the employer's desires, or make it optimal for the applicant to reveal the truth about his skill through his action. The n satisfying both constraints, because it is required to be an integer, must be at least 7 and at most 33.[29] The higher end of the range is not realistically relevant in this example, as an entire college program is typically 32 courses, but in other contexts it might matter.

What makes it possible to meet both conditions is that the two types have *sufficiently different* costs of taking tough courses: In particular, the "good" type that the employer wishes to identify has a sufficiently lower cost. When the constraints are met, the employer can devise a policy to which the two types will respond differently, thereby revealing their types. In this case, we get screening based on self-selection, which is an example of separation of types.

We did not assume here that the tough courses actually imparted any additional skills or work habits that might convert C types into A types. In our scenario, the tough courses serve only the purpose of identifying those who already possess these attributes. In other words, they have a pure screening function. In reality, education does increase productivity. But it also has the additional screening or signaling function of the kind described here. In our example, we found that education might be undertaken solely for the latter function; in reality, the corresponding outcome is that education is carried further than is justified by increased productivity alone. This extra education carries an extra cost—the cost of the information asymmetry.

II. PARTICIPATIONWhen the incentive-compatibility conditions for the two types of jobs in our firm are satisfied, the A types will take n tough courses and get a payoff of 160,000 − 3,000n, and the C types will take no tough courses and get a payoff of 60,000. For the applicants to be willing to make these choices instead of taking their alternative opportunities, the participation conditions must be satisfied as well. So we need

$$160{,}000 - 3{,}000n \geqslant 125{,}000, \text{ and } 60{,}000 \geqslant 30{,}000.$$

The C types' participation condition is trivially satisfied in this example (although that may not be the case in other examples); the A types' participation condition requires $n \leqslant 11.67$, or, since n must be an integer, $n \leqslant 11$. Here, any n that satisfies the A types' participation condition of $n \leqslant 11$ also satisfies their incentive compatibility condition of $n \leqslant 33$, so the latter condition becomes logically redundant.

The full set of conditions that are required to achieve separation of types and to attain a *separating equilibrium* in this labor market is then $7 \leqslant n \leqslant 11$. This restriction on possible values of n combines the incentive-compatibility condition for the C types and the participation condition for the A types. The participation condition for the C types and the incentive-compatibility condition for the A types in this example are automatically satisfied when the other conditions hold.

When the requirement of taking enough tough courses is used for screening, the A types bear the cost. Assuming that only the minimum number of tough courses needed to achieve separation is used—namely, $n = 7$—the cost to each A type has the monetary equivalent of $7 \times 3{,}000 = 21{,}000$. This is the cost, in this context, of the information asymmetry. It would not exist if an applicant's type could be directly and objectively identified. Nor would it exist if the population consisted solely of A types. The A types have to bear this cost because there are some C types in the population from whom they (or their prospective employers) seek to distinguish themselves.[30]

B. Pooling of Types

Rather than having the A types bear the cost of the information asymmetry, might it be better not to bother with the separation of types at all? With the separation, A types get a salary of \$160,000 but suffer a cost, the monetary equivalent of \$21,000, in taking the tough courses; thus, their net payoff is \$139,000. And C types get a salary of \$60,000. What happens to the two types if they are not separated?

If employers do not use screening devices, they have to treat every applicant as a random draw from the population and pay all the same salary. When there is no way to distinguish different types, we say that there is pooling of types, or simply pooling when the sense is clear. *Pooling of types* is therefore the opposite of *separation of types*, discussed earlier. In a competitive job market, the common salary under pooling will be the population average of what an applicant is worth to an employer, and this average will depend on the proportions of the types in the population. For example, if 65% of the population is type A and 35% is type C, then the common salary under pooling will be

$$0.65 \times 160{,}000 + 0.35 \times 60{,}000 = 125{,}000.$$

The A types will then prefer the situation with separation because it yields \$139,000 instead of the \$125,000 with pooling. But if the proportions are 80% A and 20% C, then the common salary with pooling will be \$140,000, and the A types will be worse off under separation than they would be under pooling. The C types are always better off under pooling. The existence of the A types in the population means that the common salary with pooling will always exceed the C types' salary of \$60,000 under separation.

However, even if both types prefer the outcome under pooling, it cannot be an equilibrium when many employers or applicants compete with one another in the screening or signaling process. Suppose the population proportions are 80–20 and there is an initial situation with pooling where both types are paid $140,000. An employer can announce that he will pay $144,000 for someone who takes just one tough course. Relative to the initial situation, the A types will find this offer worthwhile because their cost of taking the course is only $3,000 and it raises their salary by $4,000, whereas the C types will not find it worthwhile because their cost, $15,000, exceeds the benefit, $4,000. Because this particular employer selectively attracts the A types, each of whom is worth $160,000 to him but is paid only $144,000, he makes a profit by deviating from the pooling salary.

But his deviation starts a process of adjustment by competing employers, and that process causes the old pooling situation to collapse. As A types flock to work for him, the pool that remains available to other employers has a lower average quality, and eventually those employers cannot afford to pay $140,000 anymore. As the salary in the pool is lowered, the differential between that salary and the $144,000 offered by the deviating employer widens to the point where the C types also find it desirable to take that one tough course. But then the deviating employer must raise his requirement to two courses and must increase the salary differential to the point where two courses become too much of a burden for the C types, but the A types find it acceptable. Other employers who would like to hire some A types must use similar policies to attract them. This process continues until the job market reaches the separating equilibrium described earlier.

Even if the employers did not take the initiative to attract As rather than Cs, a type A earning $140,000 in a pooling situation might take a tough course, take his transcript to a prospective employer, and say, “I have a tough course on my transcript, and I am asking for a salary of $144,000. This should be convincing evidence that I am type A; no type C would make you such a proposition.” Given the facts of the situation, the argument is valid, and the employer should find it very profitable to agree: The employee, being type A, will generate $160,000 for the employer but get only $144,000 in salary. Other A types can do the same. This starts the same kind of cascade that leads to the separating equilibrium. The only difference is in who takes the initiative. Now the type A workers choose to get the extra education as credible proof of their type; their actions constitute signaling rather than screening.

The general point is that, even though the outcome under pooling may be better for all, the employers and workers are playing a game in which pursuing their own individual interests leads them to the separating equilibrium. This game is like a prisoners’ dilemma with many players, and therefore there is something unavoidable about the cost of the information asymmetry.

C. Many Types

We have considered an example with only two types to be separated, but the analysis generalizes immediately. Suppose there are several types: A, B, C, . . . , ranked in an order that is at the same time decreasing in their worth to an employer and increasing in the costs of extra education. Then it is possible to set up a sequence of requirements for successively higher levels of education such that the very worst type needs none, the next-worst type needs the lowest level, the type third from the bottom needs the next higher level, and so on, and the types will self-select the level that identifies them.

To finish this discussion, we provide one further point, or perhaps a word of warning, regarding signaling. Suppose you are the informed party and have available an action that would credibly signal good information (information whose credible transmission would work to your advantage). If you fail to send that signal, you will be assumed to have bad information. In this respect, signaling is like playing chicken: If you refuse to play, you have already played and lost.

You should keep this in mind when you have the choice between taking a course for a letter grade or on a pass/fail basis. The whole population in the course spans the whole spectrum of grades; suppose the average is B. A student is likely to have a good idea of his own abilities. Those reasonably confident of getting an A+ have a strong incentive to take the course for a letter grade. Once they have made that choice, the average grade for the remaining students is less than B, say, B−, because the top end has been removed from the distribution. Now, among these students, those expecting an A have a strong incentive to choose the letter-grade option. That in turn lowers the average of the rest. And so on. Finally, the pass/fail option is chosen by only those anticipating Cs and Ds. A strategically smart reader of a transcript (a prospective employer or the admissions officer for a professional graduate school) will be aware that the pass/fail option will be selected mainly by students in the lower portion of the grade distribution; such a reader will therefore interpret a Pass as a C or a D, not as the class-wide average B.

Endnotes

- You may wonder whether the fact that the two types have different opportunities at other firms can be used to distinguish between them. For example, an employer may say, “Show me an offer of a job at $125,000, and I will accept you as type A and pay you $160,000.” However, such a competing offer can be forged or obtained in cahoots with someone else, so it is not reliable. Return to reference 27
- We require merely that the payoff from choosing the option intended for one’s type be at least as high as that from choosing a different option, not that it be strictly greater. However, it is possible to approach the outcome of this analysis as closely as one wants while maintaining a strict inequality, so nothing substantial hinges on this assumption. Return to reference 28
- If in some other context the corresponding choice variable is not required to be an integer—for example, if it is a sum of money or an amount of time—then a whole continuous range will satisfy both incentive-compatibility constraints. Return to reference 29
- In the terminology of economics, the C types in this example inflict a *negative external effect* on the A types. We will develop this concept in Chapter 11. Return to reference 30

Glossary

incentive-compatibility condition (constraint)
: A constraint on an incentive scheme or screening device that makes it optimal for the agent (more-informed player) of each type to reveal his true type through his actions.

participation condition (constraint)
: A constraint on an incentive scheme or a screening device that should give the more-informed player an expected payoff at least as high as he can get outside this relationship.

incentive-compatibility condition (constraint)
: A constraint on an incentive scheme or screening device that makes it optimal for the agent (more-informed player) of each type to reveal his true type through his actions.

self-selection
: Where different types respond differently to a screening device, thereby revealing their type through their own action.

separation of types
: An outcome of a signaling or screening game in which different types follow different strategies and get the different payoffs, so types can be identified by observing actions.

pooling of types
: An outcome of a signaling or screening game in which different types follow the same strategy and get the same payoffs, so types cannot be distinguished by observing actions.

pooling
: Same as **pooling of types**.

6 EQUILIBRIA IN TWO-PLAYER SIGNALING GAMES

Our analysis so far in this chapter has covered the general concept of incomplete information as well as the specific strategies of screening and signaling; we have also seen some of the outcomes—separation and pooling—that can arise when these strategies are being used. We have seen how adverse selection could arise in a market where many car owners and buyers come together, and how signals and screening devices would operate in an environment where many employers and employees meet one another. However, we have not described and solved a game in which just two players with differential information confront each other. Here, we develop an example to show how that can be done using a game tree and payoff table as our tools of analysis. We will see that the outcome can be a *separating equilibrium*, a *pooling equilibrium*, or a third type, called a *semiseparating equilibrium*.

A. Basic Model and Payoff Structure

In this section, we analyze a game of market entry with asymmetric information. The players are two auto manufacturers, Tudor and Fordor. Tudor Auto Corporation currently enjoys a monopoly in the market for a particular kind of automobile—say, a nonpolluting, fuel-efficient compact car. An innovator, Fordor, has a competing concept and is deciding whether to enter the market. But Fordor does not know how tough a competitor Tudor will prove to be. Specifically, Tudor's production cost, unknown to Fordor, may be high or low. If it is high, Fordor can enter the market and compete profitably; if it is low, Fordor's entry and development costs will not be recouped by its subsequent operating profit, and it will incur a net loss if it enters.

The two firms interact in a sequential-move game. In the first stage of the game (period 1), Tudor sets a price (high or low, for simplicity), knowing that it is the only manufacturer in the market (that is, that it has a monopoly). In the next stage (period 2), Fordor makes its entry decision. Payoffs, or profits, are determined by the market price of a car relative to each firm's production costs and, for Fordor, its market entry and development costs as well.

Tudor, of course, would prefer that Fordor not enter the market. It might therefore try to use its price in the first stage of the game as a signal of its cost. A low-cost firm would charge a lower price than would a high-cost firm. Tudor might therefore hope that if it keeps its period 1 price low, Fordor will interpret this as evidence that Tudor's cost is low, and will stay out. (In later periods, once Fordor has given up and is out of the picture, Tudor could jack its price back up.) Just as a poker player might bet on a poor hand, hoping that the bluff will succeed and the opponent will fold, Tudor might try to bluff Fordor into staying out. Of course, Fordor, too, is a strategic player, and is aware of this possibility. The question is whether Tudor can bluff successfully in an equilibrium of this game. The answer depends on the probability that Tudor has a genuinely low production cost and on Tudor's cost of bluffing. We consider different cases below and show the differing equilibria that result.

In all the cases, the per-unit costs and prices are expressed in thousands of dollars, and the numbers of cars sold are expressed in hundreds of thousands, so the profits are measured in hundreds of millions of dollars. This format will help us write the payoffs and tables in a relatively compact form that is easy to read. We calculate those payoffs using the same type of analysis that we used for the restaurant pricing game in Chapter 5, assuming that the underlying relationship between the price charged per car (P) and the quantity of cars demanded (Q) is given by $P = 25 - Q$.[31] To enter the market, Fordor must incur an up-front cost of 40 (this amount is in the same units as profits, or hundreds of millions, so the actual figure is $4 billion) to build its plant, launch an ad campaign, and so on. If it enters the market, its cost for producing and delivering each of its cars to the market will always be 10 (thousand dollars).

Tudor could be either an old, lumbering firm with a high per-unit production cost of 15 or a nimble, efficient producer with a lower per-unit cost. To start, we suppose that the lower cost is 5; this cost is less than what Fordor can achieve. Later, in Sections 6.C and 6.D, we will investigate the effects of other cost levels. For now, suppose further that Tudor can achieve the lower per-unit cost with probability 0.4, or 40% of the time; therefore, it has the higher per-unit cost with probability 0.6, or 60% of the time.[32]

Fordor's choices in the market entry game will depend on how much information it has about Tudor's costs. We assume that Fordor knows Tudor's two possible cost types—high (at 15) and low (at 5)—and therefore can calculate the profits associated with each type (as we do below). In addition, Fordor will form some belief about the probability that Tudor is the low-cost type. We are assuming that the structure of the game is common knowledge to both players. Therefore, although Fordor does not know the type of the specific Tudor it is facing, Fordor's prior belief exactly matches the probability with which Tudor has the lower cost; that is, Fordor's belief is that the probability of facing a low-cost Tudor is 40%.

If Tudor's cost is high, at 15 (thousand), then under conditions of unthreatened monopoly in period 1, it will maximize its profit by pricing its car at 20 (thousand). At that price, it will sell 5 (hundred thousand) units and make a profit of 25 [= 5 × (20 − 15) hundred million, or $2.5 billion]. If Fordor enters and the two compete in period 2, then the Nash equilibrium of their duopoly game will yield a market price of 17 and operating profits of 3 to Tudor and 45 to Fordor. (In all cases, we have rounded to the nearest integer for simplicity.) Fordor's operating profit would exceed its up-front cost of entry (40), so Fordor would choose to enter and earn a net profit of 5 if it knew Tudor to be the high-cost type.

If Tudor's cost is low, at 5, then as a monopoly, it will price its car at 15, selling 10 units and making a profit of 100. In the second-stage equilibrium following the entry of Fordor, the market price of a car will be 13 and operating profits will be 69 for Tudor and 11 for Fordor (again, rounded to the nearest integer). The 11 profit would be less than Fordor's cost of entry of 40. Therefore, it would not enter, and would avoid incurring a loss of 29, if it knew Tudor to be the low-cost type.

B. Separating Equilibrium

If Tudor’ s cost is actually high, but it wants Fordor to think that its cost is low, Tudor must mimic the action of the low-cost type; that is, it must price its car at 15 in the first stage of the game. But that price equals its production cost in this case; Tudor will make a profit of 0. Will this sacrifice of initial profit give Tudor the benefit of scaring Fordor off and enjoying the benefits of being a monopoly in subsequent periods?

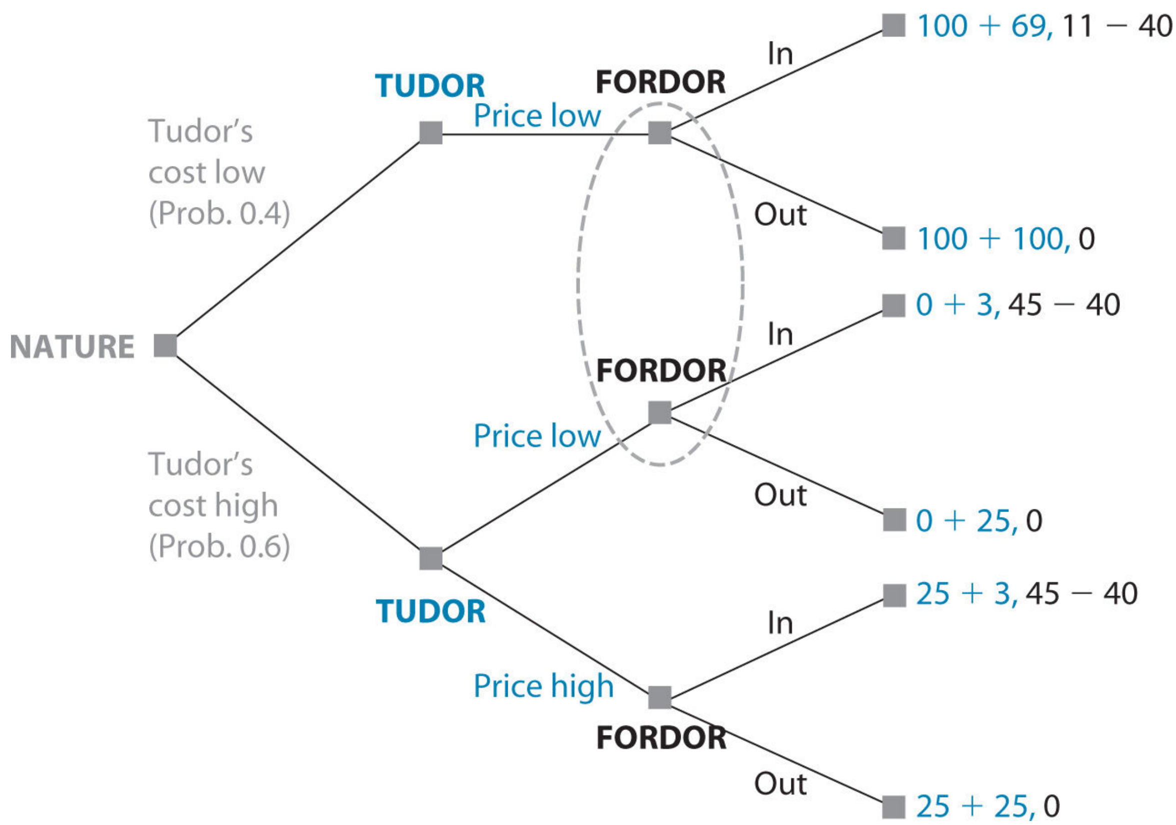

Figure 9.5 Extensive Form of Market Entry Game: Tudor’ s Low Cost Is 5

We show the full game in extensive form in Figure 9.5. Note that we use the fictitious player called Nature, as in Chapter 6, to choose Tudor’ s cost type at the start of the game. Then Tudor makes its pricing decision. We assume that if Tudor’ s cost is low, it will not choose a high price.[33] But if Tudor’ s cost is high, it may choose either the high price or, if it wants to bluff, the low price. Fordor cannot tell apart the two situations in which Tudor prices low; therefore, Fordor's two decision nodes following Tudor's choice of a low price are enclosed in one information set. Fordor must choose either In at both or Out at both.

At each terminal node, the first payoff entry (in blue) is Tudor's profit, and the second entry (in black) is Fordor's profit. Tudor's profit is added over two periods: the first period, when it is the sole producer, and the second period, when it may be a monopolist or a duopolist, depending on whether Fordor enters. Fordor's profit covers only the second period and is non-zero only when it has chosen to enter.

		FORDOR	
		Regardless (II)	Conditional (OI)
TUDOR	Bluff (LL)	169 × 0.4 + 3 × 0.6 = 69.4, −29 × 0.4 + 5 × 0.6 = −8.6	200 × 0.4 + 25 × 0.6 = 95, 0
	Honest (LH)	169 × 0.4 + 28 × 0.6 = 84.4, −29 × 0.4 + 5 × 0.6 = −8.6	200 × 0.4 + 28 × 0.6 = 96.8, 5 × 0.6 = 3

FIGURE 9.6 Strategic Form of Market Entry Game: Tudor's Low Cost Is 5

Using one step of rollback analysis, we see that Fordor will choose In at the bottom node where Tudor has chosen the high price, because $45 - 40 = 5 > 0$. Therefore, we can prune the Out branch at that node. Then each player has just two strategies (complete plans of action). For Tudor, the strategies are Bluff, or choose the low price in period 1 at either cost (low-low, or LL in shorthand notation of the kind presented in Chapter 3); and Honest, or choose the low price in period 1 if its cost is low and the high price if its cost is high (LH). For Fordor, the two strategies are Regardless, or enter irrespective of Tudor's period 1 price (II, for In-In); and Conditional, or enter only if Tudor's period 1 price is high (OI).

We can now show the game in strategic (normal) form. Figure 9.6 shows each player's two possible strategies; payoffs in each cell are the expected profits for each firm, given the probability (40%) that Tudor's cost is low. You may find your calculation of these profits easier if you label the terminal nodes in the tree and determine which ones are relevant for each cell of the table.

This is a simple dominance-solvable game. For Tudor, Honest dominates Bluff. And Fordor's best response to Tudor's dominant strategy of Honest is Conditional. Thus (Honest, Conditional) is the only (subgame-perfect) Nash equilibrium of the game.

The equilibrium found in Figure 9.6 is a separating one. The two cost types of Tudor charge different prices in period 1. This action reveals Tudor's type to Fordor, which then makes its market entry decision appropriately.

The key to understanding why Honest is the dominant strategy for Tudor can be found in the comparison of its payoffs when Fordor plays Conditional. These are the outcomes when Tudor's bluff "works": Fordor enters if Tudor charges the high price in period 1 and stays out if Tudor charges the low price in period 1. If Tudor is truly the low-cost type, then its payoffs when Fordor plays Conditional are the same whether it chooses Bluff or Honest. But when Tudor is actually the high-cost type, the results are different.

If Fordor's strategy is Conditional and Tudor is the high-cost type, Tudor can bluff successfully. However, even a successful bluff will be too costly. If Tudor charged its best monopoly (Honest) price in period 1, it would make a profit of 25; the low bluffing price would reduce this period 1 profit drastically—in this instance, all the way to 0. The higher monopoly price in period 1 would encourage Fordor's entry and reduce period 2 profit for Tudor, from the monopoly level of 25 to the duopoly level of 3. But Tudor's period 2 benefit from charging the low bluffing price and keeping Fordor out (25 − 3 = 22) is less than the period 1 cost imposed by bluffing and giving up its monopoly profit (25 − 0 = 25). As long as there is some positive probability that Tudor is the high-cost type, then the benefits from choosing Honest will outweigh those from choosing Bluff, even when Fordor’s choice is Conditional.

If the low price were not so low, then a truly high-cost Tudor would sacrifice less by mimicking the low-cost type. In such a case, Bluff might be a more profitable strategy for a high-cost Tudor. We consider exactly this possibility next.

C. Pooling Equilibrium

Let us now suppose that the lower of the two possible production costs for Tudor is 10 per car instead of 5. With this cost change, the high-cost Tudor still makes a profit of 25 as a monopoly if it charges its profit-maximizing price of 20. But the low-cost Tudor now charges 17.5 as a monopoly (instead of 15) and makes a profit of 56. If the high-cost type mimics the low-cost type and also charges 17.5, its profit is now 19 (rather than the 0 it earned in this case before); the loss of profit from bluffing is now much smaller: 25 − 19 = 6, rather than 25. If Fordor enters, then the two firms' profits in their duopoly game are 3 for Tudor and 45 for Fordor if Tudor is the high-cost type (as in Section 6.B). If Tudor is the low-cost type and Fordor enters, the equilibrium market price of a car in the duopoly is 15 and profits are now 25 for each firm; in this situation, Fordor and the low-cost Tudor have identical production costs of 10.

Suppose again that the probability of Tudor being the low-cost type is 40% (0.4) and that Fordor's belief about the probability of Tudor being the low-cost type is correct. The tree for this new game is shown in Figure 9.7. Because Fordor will still choose In when Tudor prices High, the game again collapses to one in which each player has exactly two complete strategies; those strategies are the same ones we described in Section 6.B. The payoff table for the normal form of this game is then the one illustrated in Figure 9.8.

This is another dominance-solvable game. Here it is Fordor that has a dominant strategy, however; it will always choose Conditional. And given the dominance of Conditional, Tudor will choose Bluff. Thus, (Bluff, Conditional) is the unique (subgame-perfect) Nash equilibrium of this game. In all other cells of the table, one firm gains by deviating to its other action. We leave it to you to think about the intuitive explanations of why each of these deviations is profitable.

The equilibrium found using Figure 9.8 involves pooling. Both cost types of Tudor charge the same (low) price, and seeing this, Fordor stays out. When both cost types of Tudor charge the same price, observation of that price does not convey any information to Fordor. Its estimate of the probability of Tudor’ s being the low-cost type stays at 0.4, and it calculates its expected profit from entry to be −3 < 0, so it does not enter. Even though Fordor knows full well that Tudor is bluffing in equilibrium, the risk of calling its bluff is too great because the probability of Tudor’ s cost actually being low is sufficiently great.

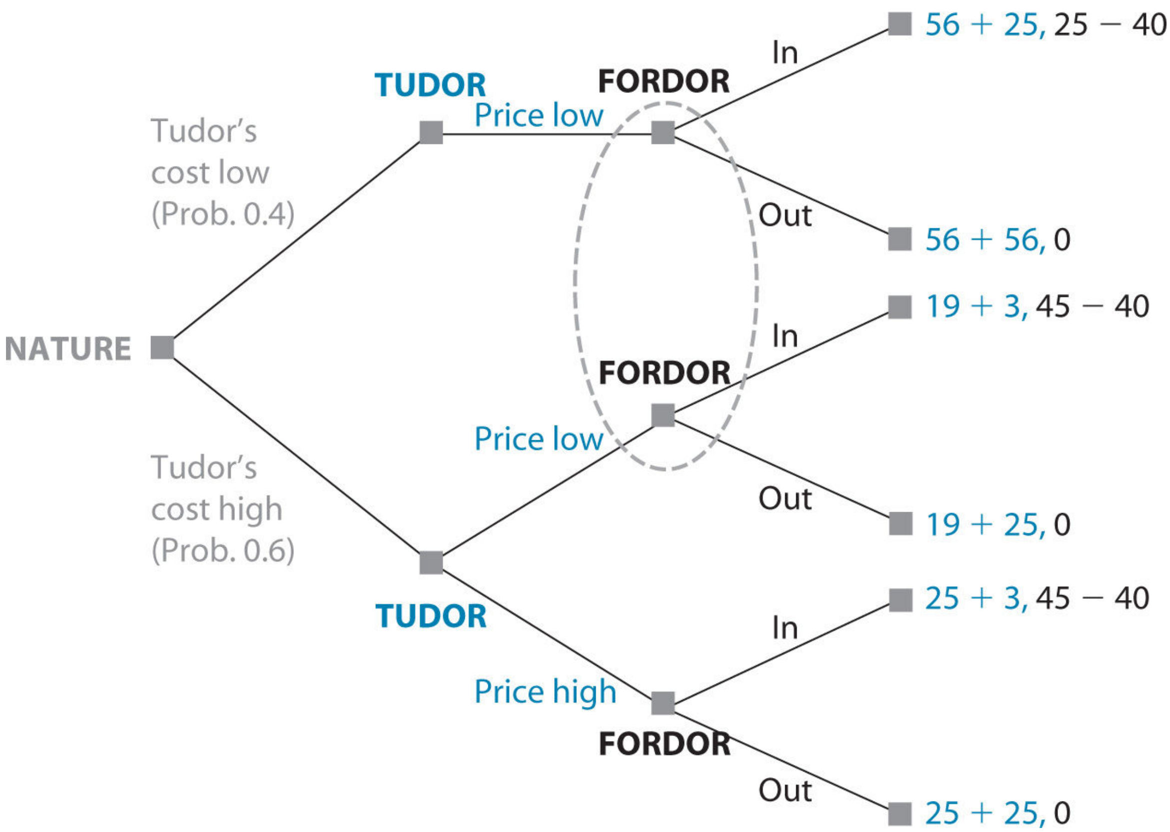

Figure 9.7 Extensive Form of Market Entry Game: Tudor’ s Low Cost Is 10

		FORDOR	
		Regardless (II)	Conditional (OI)
TUDOR	Bluff (LL)	81 × 0.4 + 22 × 0.6 = 45.6, -15 × 0.4 + 5 × 0.6 = -3	112 × 0.4 + 44 × 0.6 = 71.2, 0

	FORDOR	
	Regardless (II)	Conditional (OI)
Honest (LH)	81 × 0.4 + 28 × 0.6 = 49.2, −15 × 0.4 + 5 × 0.6 = −3	112 × 0.4 + 28 × 0.6 = 61.6, 5 × 0.6 = 3

FIGURE 9.8 Strategic Form of Market Entry Game: Tudor's Low Cost Is 10

What if this probability were smaller—say, 0.1—and Fordor was aware of this fact? If all the other numbers remain unchanged, then Fordor's expected profit from its Regardless strategy is $-15 \times 0.1 + 5 \times 0.9 = 4.5 - 1.5 = 3 > 0$. Then Fordor will enter no matter what price Tudor charges, and Tudor's bluff will not work. Such a situation results in a new kind of equilibrium, whose features we consider next.

D. Semiseparating Equilibrium

Here we consider the outcomes in the market entry game when Tudor's probability of achieving the low production cost of 10 is small, only 10% (0.1). All the cost, market price, and profit numbers are the same as in Section 6.C; only the probabilities have changed. Therefore, we do not show the game tree again; we show only the payoff table (Figure 9.9).

		FORDOR	
		Regardless (II)	Conditional (OI)
TUDOR	Bluff (LL)	$81 \times 0.1 + 22 \times 0.9 = 27.9$, $-15 \times 0.1 + 5 \times 0.9 = 3$	$112 \times 0.1 + 44 \times 0.9 = 50.8$, 0
	Honest (LH)	$81 \times 0.1 + 28 \times 0.9 = 33.3$, $-15 \times 0.1 + 5 \times 0.9 = 3$	$112 \times 0.1 + 28 \times 0.9 = 36.4$, $5 \times 0.9 = 4.5$

FIGURE 9.9 Strategic Form of Market Entry Game: Tudor's Low Cost Is 10 with Probability 0.1

In this new situation, the game illustrated in Figure 9.9 has no equilibrium in pure strategies. From (Bluff, Regardless), Tudor gains by deviating to Honest; from (Honest, Regardless), Fordor gains by deviating to Conditional; from (Honest, Conditional), Tudor gains by deviating to Bluff; and from (Bluff, Conditional), Fordor gains by deviating to Regardless. Once again, we leave it to you to think about the intuitive explanations of why each of these deviations is profitable.

So now we need to look for an equilibrium in mixed strategies. We suppose that Tudor mixes Bluff and Honest with probabilities p and $(1 - p)$, respectively. Similarly, Fordor mixes Regardless and Conditional with probabilities q and $(1 - q)$, respectively. Tudor's p-mix must keep Fordor indifferent between its two pure strategies of Regardless and Conditional; therefore, we need

$3p + 3(1 - p) = 0p + 4.5(1 - p)$, or $4.5(1 - p) = 3$, or

$1 - p = ⅔$, or $p = ⅓$.

And Fordor's q-mix must keep Tudor indifferent between its two pure strategies of Bluff and Honest; therefore, we need

$27.9q + 50.8(1 - q) = 33.3q + 36.4(1 - q)$, or $5.4q = 14(1 - q)$, or

$q = 14.4/19.8 = 16/22 = 0.727$.

The mixed-strategy equilibrium of the game, then, entails Tudor playing Bluff one-third of the time and Honest two-thirds of the time, while Fordor plays Regardless sixteen twenty-seconds of the time and Conditional six twenty-seconds of the time.

In this equilibrium, the two Tudor cost types are only partially separated. The low-cost Tudor always prices low in period 1, but the high-cost Tudor uses a mixed strategy and will charge the low price one-third of the time. If Fordor observes a high price in period 1, it can be sure that it is dealing with a high-cost Tudor; in that case, it will always enter. But if Fordor observes a low price, it does not know whether it faces a truly low-cost Tudor or a bluffing, high-cost Tudor. Then Fordor also plays a mixed strategy, entering 72.7% of the time. Thus, a high price conveys full information, but a low price conveys only partial information about Tudor's cost type. Therefore, this kind of equilibrium is labeled a *semiseparating equilibrium.*

		TUDOR'S PRICE		Sum of row
		Low	High	
TUDOR'S PRICE	Low	0.1	0	0.1
	High	0.9 × 1/3 = 0.3	0.9 × 2/3 = 0.6	0.9
Sum of column		0.4	0.6	

You may need to scroll left and right to see the full figure.

FIGURE 9.10 Applying Bayes' Theorem to the Market Entry Game

To understand better the mixed strategies of each firm and the semiseparating equilibrium, consider how Fordor can use the partial information conveyed by Tudor's low price. If Fordor sees the low price in period 1, it will use this observation to update its belief about the probability that Tudor has a low cost; it does this updating using Bayes' theorem. We provide a thorough explanation of this theorem and its use in the appendix to this chapter; here, we simply apply the analysis found there to our market entry game. The table of calculations is shown as Figure 9.10; note that this table is similar to Figure 9A.1 in the appendix.

Figure 9.10 shows the possible Tudor cost types in the rows and the prices Fordor observes in the columns. The values in the cells represent the overall probability that a Tudor of the type shown in the corresponding row chooses the price shown in the corresponding column (incorporating Tudor's equilibrium mixture probability); the final row and column show the total probabilities of each Tudor cost type and of observing each price, respectively.

Using Bayes' theorem, when Fordor observes Tudor charging a low period 1 price, it will revise its belief about the probability of Tudor having a low cost by taking the probability that a low-cost Tudor is charging the low price (the 0.1 in the top-left cell) and dividing that by the total probability of the two Tudor cost types choosing the low price (0.4, the column sum in the left column). This calculation yields Fordor's updated belief that the probability of a low-cost Tudor is $0.1/0.4 = 0.25$. Then Fordor also updates its expected profit from entry to $-15 \times 0.25 + 5 \times 0.75 = 0$. Thus, Tudor's equilibrium mixture is exactly right for making Fordor indifferent between entering and not entering when it sees the low period 1 price. This outcome is exactly what is needed to keep Tudor willing to mix in the equilibrium.

The prior probability of a low-cost Tudor, 0.1, was too low to deter Fordor from entering. Fordor's posterior probability of 0.25, revised after observing the low price in period 1, is higher. Why? Precisely because the high-cost Tudor is not always bluffing. If it were, then the low price would convey no information at all. Fordor's posterior probability would equal 0.1 in that case, whereupon it would enter. But when the high-cost Tudor bluffs only sometimes, a low price is more likely to be indicative of low cost.

We have developed the equilibria in this market entry game in an intuitive way, but we now look back and think systematically about the nature of those equilibria. In each case, we first ensured that each player's (and each type's) strategy was optimal, given the strategies of everyone else; that is, we applied the concept of Nash equilibrium. Second, we ensured that players drew the correct inferences from their observations; this required a probability calculation using Bayes' theorem, most explicitly in the case of a semiseparating equilibrium. The combination of concepts necessary to identify equilibria in such asymmetric information games justifies giving them the label Bayesian Nash equilibria. Finally, although this was a rather trivial part of this example, we did a little bit of rollback, or subgame-perfectness, reasoning. The use of rollback justifies calling such equilibria perfect Bayesian equilibria as well. Our example was a simple instance of all these equilibrium concepts; you will meet some of them again in slightly more sophisticated forms in later chapters, and in much fuller contexts in further studies of game theory.

Endnotes

- We do not supply the full calculations necessary to generate the profit-maximizing prices and the resulting firm profits in each case. You may do so on your own for extra practice, using the methods you learned in Chapter 5. Return to reference 31
- Tudor's probability of having the low per-unit cost could be denoted with an algebraic parameter, *z*. The equilibrium will be the same regardless of the value of *z*, as you will be asked to show in Exercise S4 at the end of this chapter. Return to reference 32
- This seems obvious: Why choose a price different from the profit-maximizing price? Charging the high price when you have low cost not only sacrifices some profit in period 1 (if the low-cost Tudor charges 20, its sales will drop by so much that it will make a profit of only 75 instead of the 100 it gets by charging 15), but also increases the risk of Fordor's entry and so lowers period 2 profits as well (competing with Fordor, the low-cost Tudor would have a profit of only 69 instead of the 100 it gets under monopoly). However, game theorists have found strange equilibria where a high period 1 price for Tudor is perversely interpreted as evidence of low cost, and they have applied great ingenuity in ruling out these equilibria. We leave out these complications, as we did in our analysis of cheap talk equilibria earlier, but refer interested readers to In-Koo Cho and David Kreps, "Signaling Games and Stable Equilibria," *Quarterly Journal of Economics*, vol. 102, no. 2 (May 1987), pp. 179–222. Return to reference 33

Glossary

Bayesian Nash equilibrium
: A Nash equilibrium in an asymmetric information game where players use Bayes' theorem and draw correct inferences from their observations of other players' actions.

perfect Bayesian equilibrium (PBE)
: An equilibrium where each player's strategy is optimal at all nodes given his beliefs, and beliefs at each node are updated using Bayes' rule in the light of the information available at that point including other players' past actions.

7 TWO-WAY ASYMMETRIC INFORMATION AND DYNAMIC CHICKEN

We have just analyzed a two-player sequential-move game in which one player had information about its own type (and therefore its true payoff structure) that the other player did not have. In that analysis, we showed that one player's action at the first stage of the game could be used, in equilibrium, to convey information about its type to the other player or to hide that information (or, indeed, to do either with some probability). Here we extend our thinking about sequential-move games with private information in two ways, both in the context of the game of chicken that we introduced in Chapter 4. We first consider the possibility that the asymmetry of information in a two-player game could be two-way: Each player might know his own payoffs but lack information about the other's. We then consider the effect of (two-way) asymmetric information on a sequential-move interaction in which the sequential nature of the game arises due to a timing factor. In chicken, a player's actual choice may not be simply Straight or Swerve, but rather *when* to take one of those actions as the two cars get closer. To incorporate such possibilities, we introduce a dynamic version of chicken with private information. This model allows us to analyze the strategic move known as *brinkmanship* more thoroughly than was possible in Chapter 8.

		DEAN	
		Swerve	Straight
JAMES	Swerve	0, 0	−1, W_D
	Straight	W_J, −1	−2, −2

FIGURE 9.11 Chicken with Two-Way Private Information

A. Two-Way Asymmetric Information in Chicken

In our earlier study of the game of chicken, both in Chapter 4 and in Chapter 7, we assumed that our players, James and Dean, had complete information about the game they were playing. They were fully informed about each other's strategies and payoffs. In chicken, it is quite reasonable to imagine that the well-rounded certainty we assumed in Chapters 4 and 7 is unlikely. And the one-way asymmetry of information that we have assumed so far in this chapter is also unlikely. Instead, each player would be likely to know his own payoffs, but each would probably be somewhat unsure about the other's payoffs. We would then have a two-way asymmetry of information. Our first task here will be to consider how the game of chicken and its equilibria change when we allow for this two-way, or two-sided, private information.

Suppose that each player is unsure about the value the other places on "winning" (acting tough rather than being chicken, or choosing Straight rather than Swerve). In the version of chicken described in Chapters 4 and 7 and illustrated in Figures 4.16 and 7.4, that "winning payoff" was 1 for each player. Now let us denote that payoff with the algebraic symbol W.[34] The resulting payoff matrix is shown in Figure 9.11.

In Figure 9.11, W_J refers to James's winning payoff and W_D refers to Dean's winning payoff. However, when there can be no confusion, we will drop the J and D subscripts and refer simply to "his W" or "the other's W." We assume that each player knows his own winning payoff, but regards that of the other as unknown. However, each can guess the range and probabilities of the other's W. Suppose each player believes that the other's W is equally likely to be any value between 0 and 2. (That is, each believes that the other's W is distributed uniformly over [0, 2].) Then the probability that the other's W is less than 1, for example, is ½, and the probability that the other's W is less than any number x in the range is $x/2$.

You can form some intuition about the equilibrium strategies and outcomes that will arise in this version of the game without doing any calculations. If you were playing chicken and knew your *W* was close to 0, what choice would you make? Probably Swerve (or concede). And if you were playing and your *W* was close to 2, you would probably choose Straight (or act tough). Presumably you would make the same choices for any "low" or "high" *W*, with "low" and "high" depending on some cutoff or threshold value of *W*. If the other player thought the same way, each of you would, in equilibrium, choose Swerve or Straight by comparing your own *W* with the threshold value.

We can now conjecture that each player in this game will choose his equilibrium strategy by comparing his own value of *W* with a threshold value. Further, in equilibrium, that threshold value of *W*, which we call *W**, will be the same for both players. Here we calculate a precise value for *W** and, in the process, verify our conjecture about the equilibrium.

Consider the game from James's perspective. (The calculation for Dean is identical.) James figures that Dean will Swerve if his *W* is < *W**. As we described above, the probability of that happening is *W**/2. Otherwise, with probability

$$1-\left(\frac{W^*}{2}\right)=\frac{2-W^*}{2},$$

Dean will go Straight.

James's expected payoffs from his two possible actions are then

$$0\times\left(\frac{W^*}{2}\right)+(-1)\times\left(\frac{2-W^*}{2}\right)$$

from Swerve,

and

$$W \times \left(\frac{W^*}{2}\right) + (-2) \times \left(\frac{2-W^*}{2}\right)$$ from Straight.

James chooses Swerve whenever his expected payoff from Swerve exceeds that from Straight. Therefore, James should choose Swerve if

$$0 \times \left(\frac{W^*}{2}\right) + (-1) \times \left(\frac{2-W^*}{2}\right) > W \times \left(\frac{W^*}{2}\right) + (-2) \times \left(\frac{2-W^*}{2}\right)$$ or if

$$W \times \left(\frac{W^*}{2}\right) < \left(\frac{2-W^*}{2}\right),$$ or

$$W < \left(\frac{2-W^*}{W^*}\right).$$

The final statement verifies our intuition that James chooses Swerve when his W is low, in this case below $(2 - W^*)/W^*$. In order for James to prefer Swerve when $W < W^*$ and prefer Straight when $W > W^*$, it must be that James is indifferent between Swerve and Straight when $W = W^*$. This *indifference condition* gives us a formula for the threshold value, W^*:

$$W^* = \left(\frac{2-W^*}{W^*}\right).$$

To solve for W^*, we rewrite the equation above to be

$$(W^*)^2 + W^* = 2.$$

One possible solution to this equation occurs at $W^* = 1$. And for $W^* > 0$ (which must be true here because W lies between 0 and 2), the left-hand side is an increasing function, so it can equal the right-hand side only once. The solution, that $W^* = 1$, is unique.[35]

Thus, each player in the chicken game with two-way private information has an unambiguous equilibrium strategy. Although each is uncertain about the other's W, he should concede (choose Swerve) if his own W is less than 1, but not concede (choose Straight) if it is greater than 1. (Any player with a W exactly equal to 1 is indifferent between the two choices and can flip a coin or otherwise randomize over the available options.)

B. Two-Way Asymmetric Information in a Large Population

There is another way to interpret the uncertainty facing the players here. Rather than thinking of the game as played between just two teens, James and Dean, consider the whole population of teenagers in their town as potential players. Each teen is naturally endowed with a different degree of toughness, so each has his own (known only to him) value of W. Assume that the distribution of all of their individual W s is uniform over the range [0, 2].

At each of their evening encounters, the group of teens draws straws, and the two drawing the short straws are matched to play the version of chicken illustrated in Figure 9.11. Each player perceives the other as having an unknown W chosen randomly from the full range of values from 0 to 2 and, knowing his own W, chooses his optimal strategy as calculated above. Over many such pairwise matches occurring during a year or more of evening adventures, half of the teens who play (those with $W < 1$) choose Swerve and the other half (those with $W > 1$) choose Straight. (As above, those with W exactly equal to 1 can flip a coin to decide which action to choose.)

Interestingly, the fraction of players choosing Swerve in our large-population description of chicken (½) is the same as the probability of choosing Swerve in the mixed-strategy equilibrium for chicken with complete and symmetric information that we found in Chapter 7. The probabilities in the mixed-strategy equilibrium, then, can be interpreted alternatively as the fractions of a population choosing the different pure strategies when members of the population have heterogeneous, privately known payoffs and are randomly matched to play the game.

This interpretation of the probabilities in mixed-strategy equilibria, originally offered by John Harsanyi,[36] has come to be known by the somewhat misleading name *purification*. It is not the case that we are getting rid of mixed strategies, or that we are replacing them with pure strategies, but rather that we are thinking about them differently. Instead of thinking about the equilibrium probabilities as referring to particular actions for each player, we are thinking about them as referring to the observed fractions of a particular behavior when multiple games are played within large populations. We examine this interpretation of mixed strategies more explicitly in Chapter 12, on evolutionary games.

C. Dynamic Chicken: Wars of Attrition and Brinkmanship

Our previous analysis of chicken is too simplistic in another respect: The game is modeled as a simultaneous-move one in which each driver has a pure either-or choice of Straight or Swerve. In reality, as the cars are driving toward each other, the true choice is one of timing. Each player must decide *when* to swerve (if at all). The longer both drivers keep going straight, the higher the risk that the cars will get too close and a crash will occur even if one or both then choose to swerve. So we should model the game as if it were a sequential-move one, with a simultaneous-move chicken game played at each step of the sequential-move game. Such games are known as *dynamic games*. In dynamic chicken, if both players choose Straight at any given step, the game proceeds to the next step, with a higher risk of collision.

Analyzing chicken as a dynamic game with two-way private information opens up a whole new range of applications of greater importance than teenagers' contests. In a management–labor negotiation, the workers cannot be sure how much the company can afford to raise wages or benefits, and the company cannot be sure how determined the workers are to hold out or go on strike. As the talks proceed, tensions and tempers rise, making a miscalculation that inflicts economic damage on both parties more likely. Each has to decide how far to push this risk, so each is testing out the other's toughness.

Perhaps the most dramatic and dangerous instance of dynamic chicken (or "chicken in real time") was the Cuban missile crisis, which we will discuss in greater detail as a case study in Chapter 13. As the crisis unfolded over 13 days, the political and military stances of the U.S. and USSR created ever-increasing risk of nuclear war.

Such games, in which each player is testing the other’s tolerance for risk while assessing how far to push its own, are often called wars of attrition. They also exemplify Thomas Schelling’s strategy of *brinkmanship*, which we discussed briefly in Chapter 8. Recall the basic idea behind brinkmanship: If your threatened action would inflict an intolerably large cost on yourself, then your threat is not credible. However, that cost can be scaled down by making your threatened action probabilistic rather than guaranteed to occur. The use of such a probabilistic threat is what is termed brinkmanship. But that simple description of the strategic move leaves unanswered the question of how far to scale down your threat in your attempt to make the risk simultaneously tolerable to you but too big for the other player (so that he backs down and your threat works). The dynamic version of chicken provides a framework for determining how far to scale down your threat. Imagine starting at a greatly scaled-down level and gradually raising the risk (that is, raising the probability that you will follow through on your threat). If the opponent’s risk-tolerance threshold is reached first, he concedes, and you win the game of brinkmanship. If your threshold arrives and your opponent is revealed to be tougher than you are, you concede, and your opponent wins. Of course, in the meantime there is a positive probability that the mutually damaging bad outcome happens (and both players fall off the “brink”). The concession thresholds are simply the levels where the risk of that bad outcome becomes too much to bear for each of the players.

Dynamic chicken is not a game in which just one player threatens the other, however. Rather, both are trying brinkmanship simultaneously. We present a simple analytical model of dynamic chicken here, with just two steps to the interaction. The math gets complicated with more steps, but the intuition emerges clearly from analyzing just two.

Consider a two-step chicken game, with simultaneous moves within each step but sequential moves from one step to the next. At each of the two steps, James and Dean each have the choice of whether to concede (choose Swerve). At the first step, if one or both concede, the game ends, with the payoffs as shown in Figure 9.11.

If neither concedes, the mutual disaster (a car crash) happens with some externally specified probability, in which case the game ends and each player gets a payoff of −2. To capture the idea of increasing risk inherent in the use of brinkmanship, we assume here that the probability of that disaster is proportional to the number of steps and increases linearly from one step to the next,. Thus, the probability of mutual disaster after the first step of a two-step game is ½.[37] With the remaining probability (also ½), the game goes on to the second step. At that step, if neither concedes, the calamity is sure to occur, and the payoff matrix is exactly as in Figure 9.11. The game starts with a uniform distribution of each player's W over the range [0, 2]. In the first step, some players, those with W values below some threshold, will concede. Then, if the game goes to the next step, only the players with W values above that threshold remain. Of course, this first-step threshold will be determined as part of the process of finding the solution to the game.

As usual in sequential-move games, we find the solution using rollback analysis. Start at the end of the game, with the second step. Denote the range of W values that remain at this stage as $[X, 2]$, where X represents the threshold of concession at the previous step. Its value will be determined in the process of finding the game's equilibrium. Let Y denote the concession threshold at the second step; its value will also be determined as we work toward a solution.

Then, as we did in the two-way private information calculations above, we consider the game from James's perspective. James knows that, in the second step of the game, Dean will concede if his W lies between X and Y, but act tough if his W is above Y. He then figures that the probability that Dean concedes (chooses Swerve) is $q = (Y - X)/(2 - X)$, where $Y - X$ is the range of possible W values at which Dean concedes and $2 - X$ is the range of all possible W values. Similarly, the probability that Dean acts tough (chooses Straight) is $1 - q = (2 - Y)/(2 - X)$. Therefore, James's expected payoffs from his own two choices are

$$0 \times \left(\frac{Y-X}{2-X}\right) + (-1) \times \left(\frac{2-Y}{2-X}\right)$$ from conceding (Swerve), and

$$W \times \left(\frac{Y-X}{2-X}\right) + (-2) \times \left(\frac{2-Y}{2-X}\right)$$ from acting tough (Straight).

Therefore, James should concede if

$$0 \times \left(\frac{Y-X}{2-X}\right) + (-1) \times \left(\frac{2-Y}{2-X}\right) > W \times \left(\frac{Y-X}{2-X}\right) + (-2) \times \left(\frac{2-Y}{2-X}\right),$$

---

which is equivalent to $$W \times \left(\frac{Y-X}{2-X}\right) < \left(\frac{2-Y}{2-X}\right)$$

or, even more simply, $$W < \left(\frac{2-Y}{Y-X}\right).$$

As before, this calculation verifies the intuition that a player with a low W should concede (choose Swerve). It also yields a formula for the threshold of concession in the second step, depending on the first-step threshold X; this is the value we called Y above. Thus,

$$Y = \left(\frac{2-Y}{Y-X}\right). \quad (9.1)$$

Now roll back to the first step, where the full range of W values, [0, 2], is in play. In this step, players with W values below X concede (Swerve). Here, James calculates the probability of Dean conceding as $X/2$, as in Section 7.A. above, and his own expected payoff if he concedes (Swerves) as

$$0 \times \left(\frac{X}{2}\right) + (-1) \times \left(\frac{2-X}{2}\right).$$

James's expected payoff if he does not concede (chooses Straight) is a little more complicated.[38] If Dean concedes, James gets his W. If Dean does not concede, the disaster happens with probability ½, and James gets −2. But there is also the chance, with probability ½, that the game continues to the second step. To determine James's expected payoff from not conceding, we need to know James's payoff at that second step. This calculation is most critical to the James who is at the threshold in the first step—that is, to the James whose $W = X$. Because $X < Y$, this James is sure to concede at the second step.[39] Therefore, we should use the second-step payoff from concession as James's payoff when the game continues. We have already calculated that payoff as

$$0 \times \left(\frac{Y-X}{2-X}\right) + (-1) \times \left(\frac{2-Y}{2-X}\right) = -\left(\frac{2-Y}{2-X}\right).$$

Putting all the pieces together shows that James's expected payoff from acting tough (choosing Straight) at the first step is

$$\frac{X}{2} \times W + \frac{2-X}{2} \times \left(\frac{1}{2} \times (-2) - \frac{1}{2} \times \left(\frac{2-Y}{2-X} \right) \right).$$

So James should concede at the first step if this expected payoff from Straight is smaller than his expected payoff from Swerve, calculated just above. James should concede if

$$\frac{X}{2} \times W + \frac{2-X}{2} \times \left(\frac{1}{2} \times (-2) - \frac{1}{2} \times \left(\frac{2-Y}{2-X} \right) \right) < -\left(\frac{2-X}{2} \right),$$

which, after canceling terms and simplifying, reduces to

$$W < \frac{2-Y}{2X}.$$

This final expression indicates that the threshold for concession at the first step of the game, which we called *X* above, satisfies

$$X = \frac{2-Y}{2X}. \qquad (9.2)$$

To fully describe the equilibrium of the two-step game of chicken, we have to solve equations (9.1) and (9.2) for *X* and *Y*. In general, these equations depend on the payoff values and the probabilities of disaster at each step, so one would need to solve them numerically. (We provide such a numerical solution in our application of this analysis to the Cuban missile crisis in Chapter 13.) But in the two-step case, and with the specific numbers we have chosen here, we can find a simple, explicit solution.

Each of the equations for X and Y includes a $(2 - Y)$ term that allows us to write the equation for X as $(2 - Y) = (2X) \times X$ and the equation for Y as $(2 - Y) = Y \times (Y - X)$. Then it follows that

$$2 \times X^2 = Y \times (Y - X) = 2 - Y. \quad (9.3)$$

Consider the first equality in equation (9.3) and complete the square as follows:

$$2X^2 = Y^2 - YX, \text{ so } 2.25X^2 = Y^2 - YX + 0.25X^2 = (Y - 0.5X)^2.$$

Then taking the square root yields $1.5X = Y - 0.5X$, or $Y = 2X$. Substitute this expression for Y into the second equality in equation (9.3) and complete the square again to find

$$2 - 2X = 2X^2, \text{ or } X^2 + X = 1, \text{ so } X^2 + X + 0.25 = 1.25 \text{ or } (X + 0.5)^2 = 1.25.$$

Then the final solution for X and Y is

$$X = \sqrt{1.25} - 0.5 = 0.618,$$ and $Y = 1.236$.

The remaining numbers required to fully specify a solution to the two-step game follow easily. At the first step, the probability that each player concedes is 0.618/2 = 0.309. The probability that neither concedes is $(1 - 0.309)^2 = 0.477$. In the case that neither concedes in the first step, the calamity occurs with probability 0.477/0.5 = 0.2385. Otherwise (with probability 0.2385), the game goes to the second step. At that point, the distribution of the remaining Ws is uniform over [0.618, 2], and the resulting probability that each concedes is

$$\frac{1.236 - 0.618}{2 - 0.618} = 0.447.$$

The probability that neither concedes (and the calamity then occurs) is $(1 - 0.447)^2 = 0.306$. Note that the calamity may occur at two times: after the first stage, which happens with probability 0.2385, or after the second stage, which happens with probability $0.2385 \times 0.306 = 0.0730$ (because the second stage is reached with probability 0.2385 and then calamity occurs with probability 0.306). Overall, then, the likelihood of calamity in the two-stage game is $0.2385 + 0.0730 = 0.3115$, or 31.15%.

Observe that the first-step threshold for W, beyond which a player would concede, is 0.618 here, but the threshold for concession in the one-step game above was 1.000. This result should feel intuitively reasonable. In the two-step game, there is only a 50% chance that the calamity occurs after both players act tough in the first step. But the second-step threshold for concession (1.236) is higher than that in the one-step game. Again, there is an intuitive explanation for the result. At the second step of the two-step game, even though the probability of mutual disaster rises to 1 should both players act tough, only the relatively tough players (those with $W > 0.618$) are left in the game!

Endnotes

- It is technically possible for a player's payoffs in all four cells to be private information and unknown to the other player. That level of uncertainty is too hard to handle mathematically, but analyzing the effect of uncertainty about just one of the other player's payoffs suffices to convey the implications of such limited information. To reinforce your understanding, you should try to do a similar analysis with private information about another of the payoff numbers. We provide an example of private information about the payoff associated with disaster (Straight, Straight) in Chapter 13. Return to reference 34
- Alternatively, we could complete the square. Then $(W^*)^2 + W^* + 0.25 = 2.25$, or $(W^* + 0.5)^2 = (1.5)^2$, so $W^* = 1.5 - 0.5 = 1$, and we reject the negative root as economically irrelevant. Return to reference 35
- Harsanyi shared the Nobel Prize in Economics with John Nash and Reinhard Selten in 1994. Return to reference 36
- In other applications, one might want the probability of mutual disaster to increase quadratically, exponentially, or in some other way that best fits the situations being considered. We provide an example of a "situation-specific" pattern for the probabilities of disaster in our application of dynamic chicken to the Cuban missile crisis in Chapter 13. Return to reference 37
- The calculation that follows is similar to the one for the CrossTalk–GlobalDialog game in Chapter 6, Section 1.A, where we calculated payoffs for some cells of the first-stage payoff matrix using the results of the second-stage equilibrium. Return to reference 38
- You may then wonder: If this James knows he is going to concede at the next step anyway, why doesn't he concede right away at the first step and get it over with? The answer, of course, is because of the possibility that Dean concedes at the first step, in which event James will get his W. The calculation we make takes into account all these

possibilities and their probabilities and payoffs. Return to reference 39

Glossary

dynamic chicken
: A game of chicken in which the choice to play Weak may be made at any time, the game ends as soon as either player chooses Weak, and the risk of the mutually worst outcome increases gradually over time if neither player has played Weak; a special case of the *war of attrition*.

war of attrition
: A contest between multiple players in which each player decides when to retreat, the victor is whoever remains the longest, and choosing to remain longer is costly for each player.

SUMMARY

When facing imperfect or incomplete information, game players with different attitudes toward risk or different amounts of information can engage in strategic behavior to control and manipulate the risk and information in a game. Players can reduce their risk with payment schemes or by sharing the risk with others, although the latter approach is complicated by *moral hazard* and *adverse selection*. Risk can sometimes be manipulated to a player’s benefit, depending on the circumstances within the game.

Players with private information may want to conceal or reveal that information, while those without that information may try to elicit it or avoid it. In some cases, mere words may be sufficient to convey information credibly, and then a *cheap talk equilibrium* can arise. The extent to which player interests are aligned plays an important role in achieving such equilibria. When the information content of a player’s words is ignored, the game has a *babbling equilibrium*.

More generally, specific actions taken by players convey information. *Signaling* works only if the signal action entails different costs to players of different *types*. To elicit information, a *screening device* that looks for a specific action may be required. A screening device works only if *it* induces others to reveal their *types* truthfully; there must be both *incentive compatibility* and *participation conditions (constraints).* At times, credible signaling or screening may not be possible; then the equilibrium can entail *pooling,* or there can be a complete collapse of the market or transaction for one of the types. Many examples of signaling and screening games can be found in ordinary situations such as the labor market or the provision of insurance.

In the equilibrium of a game with asymmetric information, players must not only use their best actions given their information, but must also draw correct inferences (update their information) by observing the actions of others. This type of equilibrium is known as a *Bayesian Nash equilibrium*. When the further requirement of optimality at all nodes (as in rollback analysis) is imposed, the equilibrium becomes a *perfect Bayesian equilibrium*. The outcome of such a game may entail pooling, separation, or semi-separation, depending on the specifics of the payoff structure and the specified updating processes used by players. In some parameter ranges, such games may have multiple types of perfect Bayesian equilibria. The evidence on players' abilities to achieve perfect Bayesian equilibria seems to suggest that, despite the difficult probability calculations necessary, such equilibria are often observed. Different experimental results appear to depend largely on the design of the experiment.

Some games of asymmetric information have multiple stages (are *dynamic*), and information is partially revealed over time as the game proceeds. A good example is the *war of attrition*, or *dynamic chicken*, where the strategic decision is when to swerve rather than whether to swerve, and the players are testing each other's levels of toughness.

KEY TERMS

adverse selection (326)

babbling equilibrium (319)

Bayesian Nash equilibrium (350)

Bayes’ theorem (335)

cheap talk equilibrium (317)

dynamic chicken (354)

incentive-compatibility condition (constraint) (337)

moral hazard (309)

negatively correlated (310)

participation condition (constraint) (337)

perfect Bayesian equilibrium (350)

pooling (339)

pooling equilibrium (317)

pooling of types (339)

positively correlated (311)

screen (316)

screening device (316)

self-selection (338)

semiseparating equilibrium (317)

separating equilibrium (317)

separation of types (338)

signal (316)

signaling (316)

type (317)

war of attrition (354)

Glossary

moral_hazard
: A situation of information asymmetry where one player’ s actions are not directly observable to others.

negatively_correlated
: Two random variables are said to be negatively correlated if, as a matter of probabilistic average, when one is above its expected value, the other is below its expected value.

positively_correlated
: Two random variables are said to be positively correlated if, as a matter of probabilistic average, when one is above its expected value, the other is also above its expected value, and vice versa.

signals
: Devices used for signaling.

signaling
: Strategy of a more-informed player to convey his “good” information credibly to a less-informed player.

screening
: Strategy of a less-informed player to elicit information credibly from a more-informed player.

screening_devices
: Methods used for screening.

type
: Players who possess different private information in a game of asymmetric information are said to be of different types.

separating_equilibrium
: A perfect Bayesian equilibrium in a game of asymmetric information, where the actions in the equilibrium reveal player type.

pooling_equilibrium

A perfect Bayesian equilibrium in a game of asymmetric information, where the actions in the equilibrium cannot be used to distinguish type.

semiseparating equilibrium

A perfect Bayesian equilibrium in a game of asymmetric information, where the actions in the equilibrium convey some additional information about the players' types, but some ambiguity about these types remains.

cheap talk equilibrium

In a game where communication among players (which does not affect their payoffs directly) is followed by their choices of actual strategies, a cheap talk equilibrium is one where the strategies are chosen optimally given the players' interpretation of the communication, and the communication at the first stage is optimally chosen by calculating the actions that will ensue.

babbling equilibrium

In a game where communication among players (which does not affect their payoffs directly) is followed by their choices of actual strategies, a babbling equilibrium is one where the strategies are chosen ignoring the communication, and the communication at the first stage can be arbitrary.

adverse selection

A form of information asymmetry where a player's type (available strategies, payoffs . . .) is his private information, not directly known to others.

Bayes' theorem

An algebraic formula for estimating the probabilities of some underlying event by using knowledge of some consequences of it that are observed.

incentive-compatibility condition (constraint)

A constraint on an incentive scheme or screening device that makes it optimal for the agent (more-informed player) of each type to reveal his true type through his actions.

participation condition (constraint)

A constraint on an incentive scheme or a screening device that should give the more-informed player an expected payoff at least as high as he can get outside this relationship.

self-selection

Where different types respond differently to a screening device, thereby revealing their type through their own action.

separation of types

An outcome of a signaling or screening game in which different types follow different strategies and get the different payoffs, so types can be identified by observing actions.

pooling of types

An outcome of a signaling or screening game in which different types follow the same strategy and get the same payoffs, so types cannot be distinguished by observing actions.

pooling

Same as **pooling of types**.

Bayesian Nash equilibrium

A Nash equilibrium in an asymmetric information game where players use Bayes' theorem and draw correct inferences from their observations of other players' actions.

perfect Bayesian equilibrium (PBE)

An equilibrium where each player's strategy is optimal at all nodes given his beliefs, and beliefs at each node are updated using Bayes' rule in the light of the information available at that point including other players' past actions.

dynamic chicken

A game of chicken in which the choice to play Weak may be made at any time, the game ends as soon as either player chooses Weak, and the risk of the mutually worst outcome increases gradually over time if neither player has played Weak; a special case of the *war of attrition*.

war of attrition

A contest between multiple players in which each player decides when to retreat, the victor is whoever remains the longest, and choosing to remain longer is costly for each player.

SOLVED EXERCISES

1. A local charity has been given a grant to serve free meals to the homeless in its community, but it is worried that its program might be exploited by nearby college students, who are always on the lookout for a free meal. Both a homeless person and a college student receive a payoff of 10 for a free meal. The cost of standing in line for the meal is $t^2/320$ for a homeless person and $t^2/160$ for a college student, where t is the amount of time in line measured in minutes. Assume that the staff of the charity cannot observe the true type of those coming for free meals.
   1. What is the minimum wait time t that will achieve separation of types?
   2. After a while, the charity finds that it can successfully identify and turn away college students half of the time. College students who are turned away receive no free meal and, further, incur a cost of 5 for their time and embarrassment. Will the partial identification of college students reduce or increase the answer in part (a)? Explain.
2. Consider the used-car market for the 2017 Citrus described in Section 4.B. There is now a surge in demand for used Citruses; buyers would now be willing to pay up to \$18,000 for an orange and \$8,000 for a lemon. All else remains identical to the example in Section 4.B.
   1. What price would buyers be willing to pay for a 2017 Citrus of unknown type if the fraction of oranges in the population, f, were 0.6?
   2. Will there be a market for oranges if $f = 0.6$? Explain.
   3. What price would buyers be willing to pay if f were 0.2?

4. Will there be a market for oranges if $f = 0.2$? Explain.
5. What is the minimum value of f such that the market for oranges does not collapse?
6. Explain why the increase in the buyers' willingness to pay changes the threshold value of f, where the market for oranges collapses.

3. Suppose electricians come in two types: competent and incompetent. Both types of electricians can get certified, but for the incompetent types, certification takes extra time and effort. Competent ones have to spend C months preparing for the certification exam; incompetent ones take twice as long. Certified electricians can earn 100 (thousand dollars) each year working on building sites for licensed contractors. Uncertified electricians can earn only 25 (thousand dollars) each year in freelance work; licensed contractors won't hire them. Each type of electrician gets a payoff equal to $\sqrt{S} - M,$ where S is the salary measured in thousands of dollars and M is the number of months spent getting certified. What is the range of values of C for which a competent electrician will choose to use certification as a signaling device but an incompetent one will not?
4. Return to the Tudor–Fordor example in Section 6.A, when Tudor's low per-unit production cost is 5. Let z be the probability that Tudor actually has a low per-unit cost.
   1. Rewrite the table in Figure 9.6 in terms of z.
   2. How many pure-strategy equilibria are there when $z = 0$? Explain.
   3. How many pure-strategy equilibria are there when $z = 1$? Explain.
   4. Show that the Nash equilibrium of this game is always a separating equilibrium for any value of z between 0 and 1 (inclusive).

5. Consider the Tudor–Fordor interaction again, but in a situation where Tudor has a low per-unit production cost of 6 (instead of 5 or 10 as in Section 6). If Tudor's cost is low (6), then it will earn 90 in a profit-maximizing monopoly. If Fordor enters the market, Tudor will earn 59 in the resulting duopoly, while Fordor earns 13. If Tudor's cost is actually high (that is, its per-unit cost is 15) and it prices its cars as if its cost were low (that is, as if it had a per-unit cost of 6), then it will earn 5 in a monopoly situation.
    1. Draw a game tree for this game equivalent to Figure 9.5 or 9.7 in the text, changing the appropriate payoffs.
    2. Write the normal form of this game, assuming that the probability that Tudor's cost is low is 0.4.
    3. What is the equilibrium of the game? Is it separating, pooling, or semiseparating? Explain why.
6. Felix and Oscar are playing a simplified version of poker. Each makes an initial bet of $8. Then each separately draws a card, which may be High or Low with equal probabilities. Each sees his own card, but not that of the other.

   Then Felix decides whether to Pass or to Raise (bet an additional $4). If he chooses to Pass, the two cards are revealed and compared. If the outcomes are different, the one who has the High card collects the whole pot. The pot has $16, of which the winner himself contributed $8, so his winnings are $8. The loser's payoff is −$8. If the outcomes are the same, the pot is split equally, and each gets his $8 back (payoff 0).

   If Felix chooses Raise, then Oscar has to decide whether to Fold (concede) or See (match with his own additional $4). If Oscar chooses Fold, then Felix collects the pot irrespective of the cards. If Oscar chooses See, then the cards are revealed and compared. The procedure at that point is the same as that in the preceding paragraph, but the pot is now bigger.

1. Show the game in extensive form. (Be careful about information sets.) If the game is instead written in the normal form, Felix has four strategies: (1) Pass always (PP for short), (2) Raise always (RR), (3) Raise if his own card is High and Pass if it is Low (RP), and (4) the other way round (PR). Similarly, Oscar has four strategies: (1) Fold always (FF), (2) See always (SS), (3) See if his own card is High and Fold if it is Low (SF), and (4) the other way round (FS).
2. Show that the table of payoffs to Felix is as follows:

		OSCAR			
		FF	SS	SF	FS
FELIX	PP	0	0	0	0
	RR	8	0	1	7
	RP	2	1	0	3
	PR	6	−1	1	4

You may need to scroll left and right to see the full figure.

(In each case, you will have to take an expected value by averaging over the consequences for each of the four possible combinations of the card draws.)

3. Eliminate dominated strategies as far as possible. Find the mixed-strategy equilibrium in the remaining table and the expected payoff to Felix in the equilibrium.

    4. Use your knowledge of the theory of signaling and screening to explain intuitively why the equilibrium has mixed strategies.

7. Felix and Oscar are playing another simplified version of poker called Stripped-Down Poker. Both make an initial bet of $1. Felix (and only Felix) draws one card, which is either a King or a Queen with equal probability (there are four Kings and four Queens). Felix then chooses whether to Fold or to Bet. If Felix chooses to Fold, the game ends, and Oscar receives Felix's $1 in addition to his own. If Felix chooses to Bet, he puts in an additional $1, and Oscar chooses whether to Fold or to Call.

   If Oscar Folds, Felix wins the pot (consisting of Oscar's initial bet of $1 and $2 from Felix). If Oscar Calls, he puts in another $1 to match Felix's bet, and Felix's card is revealed. If the card is a King, Felix wins the pot ($2 from each of them). If it is a Queen, Oscar wins the pot.

    1. Show the game in extensive form. (Be careful about information sets.)
    2. How many strategies does each player have?
    3. Show the game in strategic form, where the payoffs in each cell reflect the expected payoffs given each player's respective strategy.
    4. Eliminate dominated strategies, if any. Find the equilibrium in mixed strategies. What is the expected payoff to Felix in equilibrium?

8. Wanda works as a waitress and consequently has the opportunity to earn cash tips that are not reported by her employer to the Internal Revenue Service. Her tip income is rather variable. In a good year (G), she earns a high income, so her tax liability to the IRS is $5,000. In a bad year (B), she earns a low income, and her tax liability to the IRS is $0. The IRS knows that the probability of her having a good year is 0.6, and the

probability of her having a bad year is 0.4, but it doesn’t know for sure which outcome has resulted for her in any particular tax year.

In this game, first Wanda decides how much income to report to the IRS. If she reports high income (H), she pays the IRS $5,000. If she reports low income (L), she pays the IRS $0. Then the IRS has to decide whether to audit Wanda. If she reports high income, they do not audit, because they automatically know they’re already receiving the tax payment Wanda owes. If she reports low income, then the IRS can either audit (A) or not audit (N). When the IRS audits, it costs the IRS $1,000 in administrative costs, and it also costs Wanda $1,000 in the opportunity cost of the time she spends gathering bank records and meeting with the auditor. If the IRS audits Wanda in a bad year (B), then she owes nothing to the IRS, although she and the IRS have each incurred the $1,000 auditing cost. If the IRS audits Wanda in a good year (G), then she has to pay the $5,000 she owes to the IRS, and she and the IRS each incur the auditing cost.

1. Suppose that Wanda has a good year (G), but she reports low income (L). Suppose the IRS then audits her (A). What is the total payoff to Wanda, and what is the total payoff to the IRS?
2. Which of the two players has an incentive to bluff (that is, to give a false signal) in this game? What would bluffing consist of?
3. Show this game in extensive form. (Be careful about information sets.)
4. How many pure strategies does each player have in this game? Explain your reasoning.
5. Draw the payoff matrix for this game. Find all Nash equilibria. Identify whether the equilibria you find are separating, pooling, or semiseparating.
6. Let x equal the probability that Wanda has a good year. In the original version of this problem, we had

$x = 0.6$. Find a value of x such that Wanda always reports low income in equilibrium.

7. What is the full range of values of x for which Wanda always reports low income in equilibrium?

9. The design of a health-care system concerns matters of information and strategy at several points. The users—potential and actual patients—have better information about their own state of health, lifestyle, and so forth than the insurance companies can find out. The providers—doctors, hospitals, and so forth—know more about what the patients need than do either the patients themselves or the insurance companies. Doctors also know more about their own skills and efforts, and hospitals about their own facilities. Insurance companies may have some statistical information about outcomes of treatments or surgical procedures from their past records. But outcomes are affected by many unobservable and random factors, so the underlying skills, efforts, or facilities cannot be inferred perfectly from observation of the outcomes. The pharmaceutical companies know more about the efficacy of drugs than do the others. As usual, the parties do not have natural incentives to share their information fully or accurately with others. The design of the overall scheme must try to face these matters and find the best feasible solutions.

   Consider the relative merits of various payment schemes—fee for service versus capitation fees to doctors, comprehensive premiums per year versus payment for each visit for patients, and so forth—from this strategic perspective. Which are likely to be most beneficial to those seeking health care? To those providing health care? Think also about the relative merits of private insurance and coverage of costs from general tax revenues.

10. In a television commercial for a well-known brand of instant cappuccino, a gentleman is entertaining a lady

friend at his apartment. He wants to impress her and offers her cappuccino with dessert. When she accepts, he goes into the kitchen to make the instant cappuccino—simultaneously tossing take-out boxes into the trash and faking the noises made by a high-class (and expensive) espresso machine. As he is doing so, a voice comes from the other room: "I want to see the machine. . . ."

Use your knowledge of games of asymmetric information to comment on the actions of these two people. Pay attention to their attempts to use signaling and screening, and point out specific instances of each strategy. Offer an opinion about which player is the better strategist.

11. (**Optional**, requires appendix) In the genetic test example in the appendix to this chapter, suppose the test comes out negative (*Y* is observed). What is the probability that the person tested does not have the genetic defect (*B* exists)? Calculate this probability by applying Bayes' theorem, and then check your answer by doing an enumeration of the 10,000 members of the population.
12. (**Optional**, requires appendix) Return to the example of the 2017 Citrus in Section 4.B. The two types of Citrus—the reliable orange and the hapless lemon—are outwardly indistinguishable to a buyer. In the example, if the fraction *f* of oranges in the Citrus population is less than 0.65, the seller of an orange will not be willing to part with the car for the maximum price buyers are willing to pay, so the market for oranges will collapse.

    But what if a seller has a costly way to signal her car's type? Although oranges and lemons are in nearly every respect identical, the defining difference between the two is that lemons break down much more frequently. Knowing this, owners of oranges make the following proposal. On a buyer's request, the seller will in one day take a 500-mile round-trip drive in the car. (Assume this trip will be verifiable via odometer readings and a time-stamped receipt from a gas station 250 miles away.) For the sellers of both types of Citrus, the cost of the trip in fuel and time is $0.50 per mile (that is, $250 for the 500-mile trip). However, with probability q, a lemon attempting the journey will break down. If a car breaks down, the cost is $2 per mile of the total length of the attempted road trip (that is, $1,000). Additionally, breaking down will be a sure sign that the car is a lemon, so a Citrus that does so will sell for only $6,000.

Assume that the fraction of oranges in the Citrus population, f, is 0.6. Also, assume that the probability of a lemon breaking down, q, is 0.5.

1. Use Bayes' theorem to determine f_{updated}, the fraction of Citruses that have successfully completed a 500-mile road trip that are oranges. Assume that all Citrus owners attempt the trip. Is f_{updated} greater than or less than f? Explain why.
2. Use f_{updated} to determine the price, p_{updated}, that buyers are willing to pay for a Citrus that has successfully completed the 500-mile road trip.
3. Will an owner of an orange be willing to make the road trip and sell her car for p_{updated}? Why or why not?
4. What is the expected payoff of attempting the road trip to the seller of a lemon?
5. Would you describe the outcome of this market as pooling, separating, or semiseparating? Explain.

UNSOLVED EXERCISES

1. Consider again the case of the 2017 Citrus. Almost all cars depreciate over time, and so it is with the Citrus. With every month that passes, all sellers of Citruses—regardless of type—are willing to accept $100 less than they were the month before. Also, with every passing month, buyers are maximally willing to pay $400 less for an orange than they were the previous month and $200 less for a lemon. Assume that the example in the text takes place in month 0. Eighty percent of the Citruses are oranges, and this proportion never changes.
   1. Fill out three versions of the following table for month 1, month 2, and month 3:

	Willingness to accept of sellers	Willingness to pay of buyers
Orange		
Lemon		

   2. Graph the price that sellers of oranges will be willing to accept over the next 12 months. On the same figure, graph the price that buyers will be willing to pay for a Citrus of unknown type (given that the proportion of oranges is 0.8). (Hint: Make the vertical axis range from 10,000 to 14,000.)
   3. Is there a market for oranges in month 3? Why or why not?
   4. In what month does the market for oranges collapse?
   5. If owners of lemons experienced no depreciation (that is, they were never willing to accept anything less than $3,000), would this affect the timing of the collapse of the market for oranges? Why or why not?

In what month would the market for oranges collapse in this case?

6. If buyers experienced no depreciation in the price a lemon (that is, they were always willing to pay up to \$6,000 for a lemon), would this affect the timing of the collapse of the market for oranges? Why or why not? In what month would the market for oranges collapse in this case?

2. An economy has two types of jobs, Good and Bad, and two types of workers, Qualified and Unqualified. The population consists of 60% Qualified and 40% Unqualified workers. In a Bad job, either type of worker produces 10 units of output. In a Good job, a Qualified worker produces 100 units, and an Unqualified worker produces 0. There is enough demand for workers that for each type of job, companies must pay what they expect the appointee to produce.

   Companies must hire each worker without observing his type and pay him before knowing his actual output. But Qualified workers can signal their qualification by getting educated. For a Qualified worker, the cost of getting educated to level n is $n^2/2$, whereas for an Unqualified worker, it is n^2. These costs are measured in the same units as output, and n must be an integer.

   1. What is the minimum level of n that will achieve separation?
   2. Now suppose the signal is made unavailable. Which kinds of jobs will be filled by which kinds of workers, and at what wages? Who will gain and who will lose from this change?

3. You are the Dean of the Faculty at St. Anford University. You hire assistant professors for a probationary period of seven years, after which they come up for tenure and are either promoted and gain a job for life or turned down, in which case they must find another job elsewhere.

Your assistant professors come in two types, Good and Brilliant. Any types worse than Good have already been weeded out in the hiring process, but you cannot directly distinguish between Good and Brilliant types. Each individual assistant professor knows whether he or she is Brilliant or merely Good. You would like to tenure only the Brilliant types.

The payoff from a tenured career at St. Anford is $2 million; think of this as the expected value today of salaries, consulting fees, and book royalties, plus the monetary equivalent of the pride and joy that the faculty member and his or her family would get from being tenured at St. Anford. Anyone denied tenure at St. Anford will get a faculty position at Boondocks College, and the value today of that career is $0.5 million.

Your faculty can do research and publish the findings. But each publication requires effort and time and causes strain on the family; all these are costly to the faculty member. The monetary equivalent of this cost is $30,000 per publication for a Brilliant assistant professor and $60,000 per publication for a Good one. You can set a minimum number N of publications that an assistant professor must produce in order to achieve tenure.

1. Without doing any math, describe, as completely as you can, what would happen in a separating equilibrium to this game.
2. There are two potential types of pooling outcomes to this game. Without doing any math, describe what they would look like, as completely as you can.
3. Now please go ahead and do some math. What is the set of possible N that will accomplish your goal of screening the Brilliant professors out from the merely Good ones?

4. Return to the Tudor–Fordor problem from Section 6.C, when Tudor's low per-unit production cost is 10. Let z be the probability that Tudor actually has a low per-unit cost.
   1. Rewrite the table in Figure 9.8 in terms of z.
   2. How many pure-strategy equilibria are there when z = 0? What type of equilibrium (separating, pooling, or semiseparating) occurs when z = 0? Explain.
   3. How many pure-strategy equilibria are there when z = 1? What type of equilibrium (separating, pooling, or semiseparating) occurs when z = 1? Explain.
   4. What is the lowest value of z such that there is a pooling equilibrium?
   5. Explain intuitively why the pooling equilibrium cannot occur when the value of z is too low.
5. Return to the situation in Exercise S5, where Tudor's low per-unit production cost is 6.
   1. Write the normal form of this game in terms of z, the probability that Tudor has a low per-unit cost.
   2. What is the equilibrium when z = 0.1? Is it separating, pooling, or semiseparating?
   3. Repeat part (b) for z = 0.2.
   4. Repeat part (b) for z = 0.3.
   5. Compare your answers in parts (b), (c), and (d) of this problem with part (d) of Exercise U4. When Tudor's low cost is 6 instead of 10, can pooling equilibria be achieved at lower values of z? Or are higher values of z required for pooling equilibria to occur? Explain intuitively why this is the case.
6. Corporate lawsuits may sometimes be signaling games. Here is one example: In 2003, AT&T filed suit against eBay, alleging that its Billpoint and PayPal electronic payment systems infringed on AT&T's 1994 patent on "mediation of transactions by a communications system."

   Let's consider this situation from the point in time when the suit was filed. In response to this suit, as in most patent-infringement suits, eBay can offer to settle with AT&T without going to court. If AT&T accepts eBay's settlement offer, there will be no trial. If AT&T rejects eBay's settlement offer, the outcome will be determined by the court.

The amount of damages claimed by AT&T is not publicly available. Let's assume that AT&T is suing for $300 million. In addition, let's assume that if the case goes to trial, the two parties will incur court costs (for lawyers and consultants) of $10 million each.

Because eBay is actually in the business of processing electronic payments, we might think that eBay knows more than AT&T does about its probability of winning the case. For simplicity, let's assume that eBay knows for sure whether it will be found innocent (i) or guilty (g) of patent infringement. From AT&T's point of view, there is a 25% chance that eBay is guilty (g) and a 75% chance that eBay is innocent (i).

Let's also suppose that eBay has two possible actions: a generous settlement offer (G) of $200 million or a stingy settlement offer (S) of $20 million. If eBay offers a generous settlement, assume that AT&T will accept, thus avoiding a costly trial. If eBay offers a stingy settlement, then AT&T must decide whether to accept (A) and avoid a trial or reject and take the case to court (C). In the trial, if eBay is found guilty, it must pay AT&T $300 million in addition to paying all the court costs. If eBay is found innocent, it will pay AT&T nothing, and AT&T will pay all the court costs.

1. Show the game in extensive form. (Be careful to label information sets correctly.)
2. Which of the two players has an incentive to bluff (that is, to give a false signal) in this game? What would bluffing consist of? Explain your reasoning.

3. Draw the payoff matrix for this game. Find all Nash equilibria. What are the expected payoffs to each player in equilibrium?

7. For the Stripped-Down Poker game that Felix and Oscar play in Exercise S7, what does the mix of Kings and Queens have to be for the game to be fair? That is, what fraction of Kings will make the expected payoff of the game $0 for both players?
8. Bored with Stripped-Down Poker, Felix and Oscar now make the game more interesting by adding a third card type: Jack. Four Jacks are added to the deck of four Kings and four Queens. All rules remain the same as before, except for what happens when Felix Bets and Oscar Calls. When Felix Bets and Oscar Calls, Felix wins the pot if he has a King, they tie and each gets his money back if Felix is holding a Queen, and Oscar wins the pot if the card is a Jack.
    1. Show the game in extensive form. (Be careful to label information sets correctly.)
    2. How many pure strategies does Felix have in this game? Explain your reasoning.
    3. How many pure strategies does Oscar have in this game? Explain your reasoning.
    4. Represent this game in strategic form. This should be a matrix of *expected* payoffs for each player, given a pair of strategies.
    5. Find the unique pure-strategy Nash equilibrium of this game.
    6. Would you call this equilibrium a pooling equilibrium, a separating equilibrium, or a semiseparating equilibrium?
    7. In equilibrium, what is the expected payoff to Felix of playing this game? Is it a fair game?
9. Consider Michael Spence's job-market signaling model with the following specifications.[40] There are two types

of workers, 1 and 2. The productivities of the two types, as functions of their level of education E, are

$W_1(E) = E$ and $W_2(E) = 1.5E$.

The costs of education for the two types, as functions of the level of education, are

$C_1(E) = E^2/2$ and $C_2(E) = E^2/3$.

Each worker’s payoff equals his or her income minus the cost of education. Companies that seek to hire these workers are perfectly competitive in the labor market.

1. If types are public information (observable and verifiable), find expressions for the levels of education, incomes, and utilities of the two types of workers.

   Now suppose each worker’s type is his or her private information.

2. Suppose employers offer two kinds of employment packages: anyone educated to level x_1 will get salary y_1, and anyone educated to level x_2 will get salary y_2, where (x_1, y_1) and (x_2, y_2) are the education and salary levels for type 1 and type 2 that you found in part (a). Verify that type 2 workers do not prefer to get educated to level x_1 to get salary y_1, but type 1’s do prefer to get educated to level x_2 to get salary y_2. That is, verify that separation by self-selection cannot achieve the outcome that one would obtain under public information about types.
3. If we leave the education-wage pair for type 1 as in part (a), what is the range of education–wage pairs for type 2 that can achieve separation?

4. Of the possible separating education-wage packages, which one do you expect to prevail? Give a verbal, but not a formal, explanation for your answer.
5. Who gains or loses from the information asymmetry? How much?

10. Consider the question raised in the following quotation:

    Mr. Robinson pretty much concludes that business schools are a sifting device—M.B.A. degrees are union cards for yuppies. But perhaps the most important fact about the Stanford business school is that all meaningful sifting occurs before the first class begins. No messy weeding is done within the walls. “They don’t want you to flunk. They want you to become a rich alum who’ll give a lot of money to the school.” But one wonders: If corporations are abdicating to the Stanford admissions office the responsibility for selecting young managers, why don’t they simply replace their personnel departments with Stanford admissions officers, and eliminate the spurious education? Does the very act of throwing away a lot of money and two years of one’s life demonstrate a commitment to business that employers find appealing? (From Michael Lewis, review of *Snapshots from Hell: The Making of an MBA*, by Peter Robinson, *New York Times*, May 8, 1994, Book Review section.)

    What answer to Lewis’s question can you give, based on our analysis of strategies in situations of asymmetric information?

11. (**Optional**, requires appendix) An auditor for the IRS is reviewing Wanda’s latest tax return (see Exercise S8), on which she reports having had a bad year. Assume that Wanda is playing this game according to her equilibrium strategy, and that the auditor knows this.
    1. Using Bayes’ theorem, find the probability that Wanda had a good year given that she reports having

had a bad year.
2. Explain why the answer in part (a) is more or less than the baseline probability of having a good year, 0.6.

12. (**Optional**, requires appendix) Return to Exercise S12. Assume, reasonably, that the probability of a lemon's breaking down increases over the length of the road trip. Specifically, let $q = m/(m + 500)$, where m is the number of miles in the round trip.
    1. Find the minimum integer number of miles, m, necessary to avoid the collapse of the market for oranges. That is, what is the smallest m such that the seller of an orange is willing to sell her car at the market price for a Citrus that has successfully completed the road trip? (Hint: Remember to calculate f_{updated} and p_{updated}.)
    2. What is the minimum integer number of miles, m, necessary to achieve complete separation between functioning markets for oranges and lemons? That is, what is the smallest m such that the owner of a lemon will never decide to attempt the road trip?

Endnotes

- See Michael Spence, “Job Market Signaling,” *The Quarterly Journal of Economics*, vol. 87, no. 3 (August 1973), pp. 355–374. Return to reference 40

■ Appendix: Inferring Probabilities from Observing Consequences

When players have different amounts of information in a game, they will try to use some device to ascertain their opponents' private information. As we saw in Section 3 of this chapter, it is sometimes possible for direct communication to yield a cheap talk equilibrium. But more often, players will need to determine one another's information by observing one another's actions. Seeing those actions (or their observed consequences), players can draw inferences about the underlying information possessed by others. For instance, if another player might have "good" or "bad" information, one can use their actions to form an updated belief about the probability that their information is "good" or "bad." Such belief updating requires some relatively sophisticated manipulation of the rules of probability, and we examine this process in detail here. The rules given in the appendix to Chapter 7 for manipulating and calculating the probability of events prove useful in our calculations of payoffs when individual players are differently informed.

The best way to understand this idea is by giving an example. Suppose 1% of the population has a genetic defect that can cause a disease. A test that can identify this genetic defect has 99% accuracy: When the defect is present, the test will fail to detect it 1% of the time, and the test will also falsely find a defect when none is present 1% of the time. We are interested in determining the probability that a person with a positive test result really has the defect. That is, we cannot directly observe the person's genetic defect (underlying condition), but we can observe the results of the test for that defect (consequences)—except that the test is not a perfect indicator of the defect. How certain can we be, given our observations, that the underlying condition does in fact exist?

We can do a simple numerical calculation to answer the question for our particular example. Consider a population of 10,000 persons in which 100 (1%) have the defect and 9,900 do not. Suppose they all take the test. Of the 100 persons with the defect, the test will be (correctly) positive for 99. Of the 9,900 without the defect, it will be (wrongly) positive for 99. So we have 198 positive test results, of which one-half are right and one-half are wrong. If a random person receives a positive test result, it is just as likely to be because the test is right as because the test is wrong, so the risk that the defect is truly present for a person with a positive result is only 50%. (That is why tests for rare conditions must be designed to have especially low rates of false positives.)

For general questions of this type, we use an algebraic formula called *Bayes' theorem* to help us set up the problem and do the calculations. To do so, we generalize our example, allowing for two alternative underlying conditions, A and B (genetic defect or not), and two observable consequences, X and Y (positive or negative test result). Suppose that, in the absence of any information (over the whole population), the probability that A exists is p, so the probability that B exists is $(1 - p)$. When A exists, the chance of observing X is a, so the chance of observing Y is $(1 - a)$. [In the language of probability theory, a is the probability of X conditional on A, and $(1 - a)$ is the probability of Y conditional on A.] Similarly, when B exists, the chance of observing X is b, so the chance of observing Y is $(1 - b)$.

This description shows us that four alternative combinations of events could arise: (1) A exists and X is observed, (2) A

exists and Y is observed, (3) B exists and X is observed, and (4) B exists and Y is observed. The likelihood of these combined events can be determined using what is known in probability theory as the *modified multiplication rule*: the probability of two events both happening is equal to the probability of the first event times the probability of the second event conditional on the first event having happened. Using this rule, we find the probabilities of the four combinations to be, respectively, (1) pa, (2) $p(1 - a)$, (3) $(1 - p)b$, and (4) $(1 - p)(1 - b)$.

Now suppose that X is observed: A person has the test for the genetic defect and gets a positive result. Then we restrict our attention to a subset of the four preceding possibilities—namely, the first and the third, both of which include the observation of X. These two possibilities have a total probability of $pa + (1 - p)b$; this is the probability that X is observed. Within this subset of outcomes in which X is observed, the probability that A *also* exists is just pa, as we have already seen. So we know how likely we are to observe X alone and how likely it is that both X and A exist.

But we are more interested in determining how likely it is that A exists, given that we have observed X—that is, the probability that a person has the genetic defect, given that the test is positive. This calculation is the trickiest one. Using the modified multiplication rule, we know that the probability of both A and X happening equals the product of the probability that X alone happens times the probability of A conditional on X; it is this last probability that we are after. Using the formulas for "A and X" and for "X alone," which we just calculated, the general formula

Prob(A and X) = Prob(X alone) × Prob(A conditional on X)
becomes

$pa = [pa + (1 - p)b] \times$ Prob(A conditional on X); so, Prob(A conditional on X) = $\dfrac{pa}{pa+(1-p)b}$.

This formula gives us an assessment of the probability that A has occurred, given that we have observed X (and have therefore conditioned everything on this fact); it is known as *Bayes' theorem* (or rule or formula).

In our example of testing for the genetic defect, we had Prob(A) = p = 0.01, Prob(X conditional on A) = a = 0.99, and Prob(X conditional on B) = b = 0.01. We can use Bayes' theorem to compute Prob(A conditional on X), the probability that a genetic defect exists when the test comes back positive:

Prob(A conditional on X)

$$
\begin{aligned}
&= \frac{(0.01)(0.99)}{(0.01)(0.99)+(1-0.01)(0.01)} \\
&= \frac{0.0099}{0.0099+0.0099} \\
&= 0.5
\end{aligned}
$$

OBSERVATION		Sum of row
X	Y	

You may need to scroll left and right to see the full figure.

		OBSERVATION		Sum of row
		X	Y	
TRUE CONDITION	A	pa	$p(1-a)$	p
	B	$(1 - p)b$	$(1 - p)(1 - b)$	$1 - p$
Sum of column		$pa + (1 - p)b$	$p(1 - a) + (1 - p)(1 - b)$	
You may need to scroll left and right to see the full figure.				

The probability algebra employing Bayes' theorem confirms the arithmetical calculation that we used earlier, which was based on an enumeration of all the possible cases. The advantage of the formula is that once we have it, we can apply it mechanically; this saves us the lengthy and error-susceptible task of enumerating every possibility and determining each of the necessary probabilities.

We show Bayes' theorem in Figure 9A.1 in tabular form, which may be easier to remember and to use than the preceding formula. The rows of the table show the alternative true conditions that might exist—for example, "genetic defect" and "no genetic defect." Here, we have just two, *A* and *B*, but the method generalizes immediately to any number of possibilities. The columns show the observed events—for example, "test positive" and "test negative."

Each cell in the table shows the overall probability of that combination of the true condition and the observation; these are just the probabilities for the four alternative combinations listed above. The last column on the right shows the sum across the first two columns for each of the top two rows. This sum is the total probability of each true condition (so, for instance, *A*'s probability is *p*, as we have seen). The last row shows the sum of the first two rows in each column. This sum gives the probability that each observation occurs. For example, the entry in the last row of the X column is the total probability that X is observed, either when A is the true condition (a true positive in our genetic test example) or when B is the true condition (a false positive).

To find the probability of a particular condition, given a particular observation, then, Bayes' theorem says that we should take the entry in the cell corresponding to the combination of that condition and that observation and divide it by the column sum in the last row for that observation. As an example, Prob(B conditional on X) = $(1 - p)b/[pa + (1 - p)b]$.

10■ The Prisoners’ Dilemma and Repeated Games

IN THIS CHAPTER, we turn our attention to an in-depth analysis of the prisoners’ dilemma game. As a classic example of the theory of strategy and its implications for predicting the behavior of game players, the prisoners’ dilemma is familiar to anyone who has learned even just a little bit of game theory, and many who have never formally studied the subject know the basic story behind the game. The prisoners’ dilemma is a game in which each player has a dominant strategy, but the equilibrium that arises when all players use their dominant strategies provides a worse outcome for every player than would arise if they all used their dominated strategies instead. The paradoxical nature of this equilibrium leads to several more complex questions about the nature of these interactions that only a more thorough analysis, like the one we provide in this chapter, can begin to address.

We introduced you to the prisoners’ dilemma in Section 3 of Chapter 4. There, we took note of the curious nature of an equilibrium that is actually a bad outcome for the players. The “prisoners” can find another outcome that both prefer to the equilibrium outcome, but they find it difficult to bring about. The focus of this chapter is the potential for achieving that better outcome. That is, we consider whether and how the players in a prisoners’ dilemma can attain and sustain a mutually beneficial cooperative outcome, overcoming their separate incentives to defect for individual gain. We first review the standard prisoners’ dilemma game, then develop some broad categories of solutions. One group of solutions involves changes in the order of moves, or in the way moves are made, that lead to increased opportunities for cooperation. Another group of solutions involves mechanisms or forces outside the game that can change the payoffs for one or both players and can therefore change the equilibrium outcome. The use of strategic moves—specifically, promises (with warnings)—and repetition of the game are solutions of the first type. Mechanisms such as penalty (or reward) schemes, often offered by entities that are not players in the game, are solutions of the second type. We consider both types, but begin with the most studied solution, repeated play.

Prisoners' dilemma payoff structures are commonly observed in games that repeat on a regular basis. The pricing game between Xavier's Tapas Bar and Yvonne's Bistro, introduced in Chapter 5, is an example. Firms typically set prices each week or month, so their interaction is a repeated one. The general theory of such repeated games was the contribution for which Robert Aumann was awarded the 2005 Nobel Prize in economics (jointly with Thomas Schelling). As usual at this introductory level, we look at only a few simple examples of the general theory, but use them to motivate broader conclusions about behavior in repeated games.

This chapter concludes with a discussion of some experimental evidence on player behavior in prisoners' dilemma games as well as several examples of actual dilemmas in action. Experiments that put live players in a variety of prisoners' dilemma games show some perplexing as well as some more predictable behavior; experiments conducted with the use of computer simulations yield additional interesting outcomes. Our examples of real-world dilemmas that end the chapter are provided to give a sense of the diversity of situations in which prisoners' dilemmas arise and to show at least one application of a solution method in action.

1 THE BASIC GAME (REVIEW)

Before we consider methods for avoiding the bad outcome in the prisoners' dilemma, we briefly review the basics of the game. Recall our example from Chapter 4 of the husband and wife suspected of murder. Each is interrogated separately and can choose to confess to the crime or to deny any involvement. The payoff matrix that they face was originally presented as Figure 4.4 and is reproduced here as Figure 10.1. The numbers shown indicate years in jail; therefore, lower numbers are better for both players.

Both players here have a dominant strategy: Each does better to confess, regardless of what the other player does. The equilibrium outcome entails both players deciding to confess and each getting 10 years in jail. If they had both chosen to deny any involvement, however, they would both have been better off, with only 3 years of jail time to serve.

In a general prisoners' dilemma game, the two available strategies are often labeled as *Cooperate* and *Defect*. In Figure 10.1, Deny is the cooperative strategy and could be labeled Cooperate (with each other, not with the police); both players using that strategy yields a better outcome for both players than when both players use the other strategy. Confess is the defecting strategy and could be labeled Defect; when the players do not cooperate with each other, they choose to Confess in the hope of attaining individual gain at the other player's expense. Thus, players in a prisoners' dilemma can always be labeled, according to their choice of strategy, as either *defectors* or *cooperators*. We will use this labeling system throughout the discussion of potential solutions to the dilemma.

	WIFE

		WIFE	
		Confess (Defect)	Deny (Cooperate)
HUSBAND	Confess (Defect)	10 yr, 10 yr	1 yr, 25 yr
	Deny (Cooperate)	25 yr, 1 yr	3 yr, 3 yr

You may need to scroll left and right to see the full figure.

FIGURE 10.1 Payoffs for the Standard Prisoners' Dilemma

We want to emphasize that, although we label one of the strategies here as Cooperate, the prisoners' dilemma game is *noncooperative*, in the sense explained in Chapter 2, because players make their decisions and implement their choices individually. If the two players could discuss, choose, and play their strategies jointly—as if, for example, the prisoners were in the same room and could give a joint answer to the question of whether they were both going to confess—there would be no difficulty about their achieving the outcome that both would prefer. The essence of the questions of whether, when, and how a prisoners' dilemma can be resolved is the difficulty of achieving a cooperative (jointly preferred) outcome through noncooperative (individual) actions.

2 CHANGING THE WAY MOVES ARE MADE: REPETITION

Of all the mechanisms that can sustain cooperation in the prisoners' dilemma, the best known and the most natural is repeated play of the game. Repeated or ongoing relationships among players impart special characteristics to the games that they play against one another. In the prisoners' dilemma, this result manifests itself in the fact that each player fears that one instance of defecting will lead to a collapse of cooperation in the future. If the value of future cooperation is large and exceeds what can be gained in the short term by defecting, then the long-term individual interests of the players can automatically and tacitly keep them from defecting, without the need for any additional punishments or enforcement by third parties.

Consider the meal-pricing dilemma faced by the two restaurants, Xavier's Tapas Bar and Yvonne's Bistro, introduced in Chapter 5. For our purposes here, we have chosen to simplify that game by supposing that only two choices of price are available: the jointly best (collusive) price of \$26 or the Nash equilibrium price of \$20. The payoffs (profits measured in hundreds of dollars per month) for each restaurant can be calculated by using the profit expressions derived in Section 1.A of Chapter 5; these payoffs are shown in Figure 10.2. As in any prisoners' dilemma, each restaurant has a dominant strategy to defect and price its meals at \$20, although both restaurants would prefer the outcome in which each cooperates and charges the higher price of \$26 per meal.

	YVONNE'S BISTRO	
	\$20 (Defect)	\$26 (Cooperate)

You may need to scroll left and right to see the full figure.

		YVONNE’S BISTRO	
		$20 (Defect)	$26 (Cooperate)
XAVIER’S TAPAS	$20 (Defect)	288, 288	360, 216
	$26 (Cooperate)	216, 360	324, 324

You may need to scroll left and right to see the full figure.

FIGURE 10.2 Payoff Table for Restaurant Prisoners’ Dilemma (in Hundreds of Dollars per Month)

Let us start our analysis by supposing that the two restaurants are initially in the cooperative mode, each charging the higher price of $26. If one restaurant owner—say, Xavier—deviates from this pricing strategy, he can increase his profit from 324 to 360 (from $32,400 to $36,000) for one month. But then cooperation will have dissolved, and Xavier’s rival, Yvonne, will see no reason to cooperate with him from then on. Once cooperation has broken down, presumably permanently, Xavier’s profit is 288 ($28,800) each month instead of the 324 ($32,400) it would have been if he had not defected. By gaining 36 ($3,600) in one month of defecting, he gives up 36 ($3,600) each month thereafter by destroying cooperation. Even if the two restaurants’ relationship lasts as little as three months, it seems that defecting is not in Xavier’s best interest. A similar argument can be made for Yvonne’s. Thus, if the two restaurants compete on a regular basis for at least three months, it seems that we might see cooperative behavior and high prices rather than the defecting behavior and low prices predicted by theory for the one-shot game.

A. Finite Repetition

But the solution of the dilemma is not actually that simple. What if the relationship did last exactly three months? Then strategic restaurants would want to analyze the full three-month game and choose their optimal pricing strategies. Each would use rollback to determine what price to charge each month. Starting their analyses with the third month, they would realize that, at that point, there was no future relationship to consider. Each restaurant would find that it had a dominant strategy: to defect. Given that, there would be effectively no future to consider in the second month either. Each player would know that there would be mutual defecting in the third month, and therefore both would defect in the second month; defecting would become the dominant strategy in that month, too. Then the same argument would apply to the first month as well. Knowing that both would defect in the second and third months anyway, neither player would see any future value in cooperation in the first month. Both players defect right from the start, and the dilemma is alive and well.

This result is a very general one. As long as the relationship between the two players in a prisoners' dilemma game lasts a fixed and known length of time, the Nash equilibrium with defecting should prevail in the last period of play. When the players arrive at the end of the game, there is never any value to continued cooperation, and so they defect. Then rollback predicts mutual defecting all the way back to the very first period of play. In practice, however, players in finitely repeated prisoners' dilemma games show a lot of cooperation; more on this to come.

B. Infinite Repetition

Analysis of the finitely repeated prisoners' dilemma shows that even repetition of the game cannot guarantee the players a solution to their dilemma. But what happens if the relationship does not have a predetermined length? What if the two restaurants in our example expect to continue competing with each other indefinitely? Then our analysis must change to incorporate this new aspect of their interaction, and we will see that the incentives of the players change too.

In repeated games of any kind, the sequential nature of the relationship means that players can adopt strategies that depend on behavior in preceding periods of play. Such strategies are known as contingent strategies, and several specific examples are used frequently in the theory of repeated games. Of special note are the contingent strategies known as trigger strategies. A player using a trigger strategy plays cooperatively as long as her rival(s) do so, but any defection on their part triggers a period of punishment, of specified length, in which she defects in response. Two of the best-known trigger strategies are the grim strategy and tit-for-tat. The grim strategy entails cooperating with your rival until such time as she defects from cooperation; once she has done so, you punish her (by choosing the Defect strategy) on every play for the rest of the game.[1] Tit-for-tat (TFT) is not so harshly unforgiving as the grim strategy and is well known for its ability to solve the prisoners' dilemma without requiring permanent punishment. Playing TFT involves cooperating in the first period of play and then choosing, in each future period, the action chosen by your rival in the preceding period of play. Thus, when playing TFT, you cooperate with your rival if she cooperated during the most recent play of the game and you defect (as punishment) if she defected. The punishment phase lasts only as long as your rival continues to defect; you will return to cooperation one period after she chooses to do so.

Let us consider how play might proceed in the infinitely repeated restaurant pricing game if one of the players uses the contingent strategy tit-for-tat. We have already seen that if Xavier defects in one month, he can add 36 to his profits (which will be 360 instead of 324). But if Yvonne is playing TFT, then Xavier's defection induces Yvonne to punish him the next month in retaliation. At that point, Xavier has two choices. One option is to continue to defect by pricing his meals at $20 and to endure Yvonne's continued punishment according to TFT; in this case, Xavier loses 36 (288 rather than 324) in every month thereafter in the foreseeable future. This option appears quite costly. But Xavier *can* get back to cooperation, too, if he so desires. By reverting to the cooperative price of $26 after one month's defection, Xavier will incur only one month's punishment from Yvonne. During that month, Xavier will suffer a loss in profit of 108 (earning 216 rather than the 324 that he would have earned without any defection). In the second month after Xavier's defection, both restaurants could be back at the cooperative price, earning 324 each month. This one-time defection yields Xavier an extra 36 in profit, but costs him an additional 108 during the punishment.

It is important to realize here, however, that Xavier's extra 36 from defecting is gained in the first month, and his losses are incurred in the future. Therefore, the relative importance of the two depends on the relative importance of the present and the future. Here, because payoffs are calculated in dollar terms, an objective comparison can be made. Generally, money (or profit) that is earned today is better than money that is earned later, because even if you do not need (or want) the money until later, you can invest it now and earn a return on it until you need it. So Xavier should be able to calculate whether it is worthwhile to defect on the basis of the total rate of return on his investment (including interest and/or capital gains and/or dividends, depending on the type of investment). We use the symbol r to denote this rate of return. Thus, one dollar invested generates a return of r dollars, and 100 dollars generates $100r$; therefore, the rate of return is sometimes also said to be $(100r)$%.

Note that we can calculate whether it is in Xavier's interest to defect because the restaurants' payoffs are given in dollar terms, rather than as ordinal rankings of outcomes. This means that payoff values in different cells are directly comparable: A payoff of 4 (dollars) is twice as good as a payoff of 2 (dollars) here, whereas in any two-by-two game in which the four possible outcomes are ranked from 1 (worst) to 4 (best), a payoff of 4 is not necessarily exactly twice as good as a payoff of 2. As long as the payoffs to the players are given in measurable units, we can calculate whether defecting in a prisoners' dilemma game is worthwhile.

I. IS IT WORTHWHILE TO DEFECT ONLY ONCE AGAINST A RIVAL PLAYING TFT?One of Xavier's options when playing repeatedly against a rival using TFT is to defect just once and then return to cooperating in subsequent periods. This particular strategy gains him 36 in the first month (the month during which he defects) but loses him 108 in the second month. By the third month, cooperation is restored. Is defecting for only one month worth it?

We cannot directly compare the 36 gained in the first month with the 108 lost in the second month because the monetary value of time must be incorporated into the calculation. That is, we need a way to determine how much the 108 lost in the second month is worth during the first month. Then we can compare that number with 36 to see whether defecting once is worthwhile. What we are looking for is the present value (PV) of 108, or how much profit earned this month (in the present) is equivalent to (has the same value as) 108 earned next month. We need to determine the number of dollars earned this month that, with interest, would give us 108 next month; we call that number PV, the present value of 108.

Given that the (monthly) total rate of return is r, getting PV this month and investing it until next month yields a total next month of PV + rPV, where the first term is the principal being paid back and the second term is the return. When the total is exactly 108, then PV equals the present value of 108. Setting PV + rPV = 108 yields a solution for PV:

$$PV = \frac{108}{1 + r}.$$

For any value of r, we can now determine the exact number of dollars that, earned this month, would be worth 108 next month.

From Xavier's perspective, the question remains whether the gain of 36 this month is offset by the loss of 108 next month. The answer depends on the value of PV. Xavier must compare the gain of 36 with the PV of the loss of 108. To defect once (and then return to cooperation) is worthwhile only if $36 > 108/(1 + r)$. This is the same as saying that defecting once is beneficial only if $36(1 + r) > 108$, which reduces to $r > 2$. Thus, Xavier should choose to defect once against a rival playing TFT only if the monthly total rate of return exceeds 200%. This outcome is very unlikely; for example, prime lending rates rarely exceed 12% per year, which translates into a monthly interest rate of no more than 1% (compounded annually, not monthly)—well below the 200% just calculated. So, if Yvonne is playing TFT, then it is better for Xavier to continue cooperating than to try a single instance of defecting.

II. IS IT WORTHWHILE TO DEFECT FOREVER AGAINST A RIVAL PLAYING TFT?What about the possibility of defecting once and then continuing to defect forever? This second option of Xavier's gains his restaurant 36 in the first month, but loses it 36 in every month thereafter into the future if the rival restaurant plays TFT. Whether such a strategy is in Xavier's best interest again depends on the present value of the losses incurred. But this time the losses are incurred over an infinite horizon of future months of competition.

Xavier's option of defecting forever against a rival playing TFT yields a payoff (profit) stream equivalent to what Xavier would get if he were to defect against a rival using the *grim strategy.* Recall that the grim strategy requires players to punish any defection with retaliatory defection in all future periods. In that case, it is not worthwhile for Xavier to attempt any return to cooperation after his initial defection because the rival firm will be choosing to defect, as punishment, forever. Any defection on Xavier's part against a rival playing the grim strategy would then lead to a gain of 36 in the first month and a loss of 36 in all future months, exactly the same outcome as if he defected forever against a rival playing TFT. The analysis that follows is therefore the same analysis one would complete to assess whether it is worthwhile to defect at all against a rival playing the grim strategy.

To determine whether a defection of this type is worthwhile, we need to figure out the present value of all of the 36s that are lost in future months, add them all up, and compare them with the 36 gained during the month of defecting. The PV of the 36 lost during the first month of punishment and continued defecting on Xavier's part is just $36/(1 + r)$; the calculation is identical to that used in Section 2.B.I to find that the PV of 108 was $108/(1 + r)$. For the next month, the PV must be the dollar amount needed this month that, with two months of compound interest, would yield 36 in two months. If the PV is invested now, then in one month the investor would have that principal amount plus a return of rPV, for a total of PV + rPV, as before; leaving this total amount invested for the second month means that at the end of two months, the investor has the amount invested at the beginning of the second month (PV + rPV) plus the return on that amount, which would be r(PV + rPV). The PV of the 36 lost two months from now must then solve the equation PV + rPV + r(PV + rPV) = 36. Working out the value of PV here yields $\text{PV}(1 + r)^2 = 36$, or $\text{PV} = 36/(1 + r)^2$. You should see a pattern developing. The PV of the 36 lost in the third month of continued defecting is $36/(1 + r)^3$, and the PV of the 36 lost in the fourth month is $36/(1 + r)^4$. In fact, the PV of the 36 lost in the nth month of continued defecting is just $36/(1 + r)^n$. Xavier loses an infinite sum of 36s, and the PV of each of them gets smaller each month.

More precisely, Xavier loses the sum, from $n = 1$ to $n = \infty$ (where n is the number of months of continued defecting after the initial month, which is month 0), of $36/(1 + r)^n$. Mathematically, it is written as the sum of an infinite number of terms:[2]

$$\frac{36}{1+r} + \frac{36}{(1+r)^2} + \frac{36}{(1+r)^3} + \frac{36}{(1+r)^4} + \cdots.$$

Because r is a rate of return and presumably a positive number, the ratio $1/(1 + r)$ will be less than 1; this ratio is generally called the discount factor and is denoted by the Greek letter δ. With $\delta = 1/(1 + r) < 1$, the mathematical rule for infinite sums tells us that this sum converges to a specific value, in this case $36/r$. (The appendix to this chapter contains a detailed discussion of the solution of infinite sums.)

It is now possible to determine whether Xavier will choose to defect forever. He compares his restaurant's gain of 36 with the PV of all the lost 36s, or $36/r$. Then he defects forever only if $36 > 36/r$, or $r > 1$; defecting forever is beneficial in this particular game only if the monthly rate of return exceeds 100%, another unlikely event. Thus, we would not expect Xavier to defect against a cooperative rival when both are playing tit-for-tat. (Nor would we expect defection against a cooperative rival when both are playing the grim strategy.) When both Yvonne and Xavier play TFT, the cooperative outcome in which both set a high price is a Nash equilibrium outcome. Both playing TFT is a Nash equilibrium, and use of this contingent strategy solves the prisoners' dilemma for the two restaurant owners.

Remember that tit-for-tat is only one of many trigger strategies that players could use in repeated prisoners' dilemmas. And it is one of the "nicer" ones. Thus, if TFT can be used to solve the dilemma for the two restaurant owners, other, harsher trigger strategies should be able to do the same. As noted, the grim strategy can also be used to sustain cooperation in this infinitely repeated game, and in others as well.

C. Games of Unknown Length

In addition to considering games of finite or infinite length, we can incorporate a more sophisticated tool to deal with games of unknown length. It is possible that, in some repeated games, players will not know for certain exactly how long their interaction will continue. They may, however, have some idea of the *probability* that the game will continue for another period. For example, our restaurant owners might believe that their repeated competition will continue only as long as their customers find prix fixe menus to be the dining-out experience of choice; if there were some probability each month that à la carte dinners would take over that role, then the nature of the game would be altered.

Recall that the present value of a loss next month is already worth only $\delta = 1/(1 + r)$ times the amount earned. If, in addition, there is only a probability p (less than 1) that the relationship will actually continue to the next month, then next month's loss is worth only p times δ times the amount lost. For Xavier's Tapas Bar, this means that the PV of the 36 lost with continued defecting is $36 \times \delta$ [the same as $36/(1 + r)$] when the game is assumed to be continuing with certainty, but is only $36 \times p \times \delta$ when the game is assumed to be continuing with probability p. Incorporating the probability that the game may end next period means that the present value of the lost 36 is smaller, because $p < 1$, than it is when the game is definitely expected to continue (when p is *assumed* to equal 1).

The effect of incorporating p is that we now effectively discount future payoffs by the factor $p \times \delta$ instead of simply by δ. We can then generate an effective rate of return, R, where $1/(1 + R) = p \times \delta$, and R depends on p and δ as shown:[3]

$$\frac{1}{1+R} = p\delta$$

$$1 = p\delta(1+R)$$

$$R = \frac{1-p\delta}{p\delta}.$$

With a 5% actual rate of return on investments ($r = 0.05$, and so $\delta = 1/1.05 = {}^{20}/_{21}$) and a 50% chance that the game continues for an additional month ($p = 0.5$), then $R = [1 - (0.5)(^{20}/_{21})]/(0.5)(^{20}/_{21}) = 1.1$, or 110%.

The high rates of return required to destroy cooperation (encourage defection) in these examples now seem more realistic, if we interpret them as effective rather than actual rates of return. It becomes conceivable that defecting forever, or even once, might actually be to one's benefit if there is a large enough probability that the game will end in the near future. Consider Xavier's decision whether to defect forever against a TFT-playing rival. Our earlier calculations showed that permanent defecting is beneficial only when r exceeds 1, or 100%. If Xavier's faces the 5% actual rate of return and the 50% chance that the game will continue for an additional month, as we assumed in the preceding paragraph, then the effective rate of return of 110% will exceed the critical value needed for him to continue defecting. Thus, the cooperative behavior sustained by the TFT strategy can break down if there is a sufficiently large chance that the repeated game might be over by the end of the next period of play—that is, when the value of p is sufficiently small.

D. General Theory

We can easily generalize these ideas about when it is worthwhile to defect against TFT-playing rivals so that you can apply them to any prisoners' dilemma game that you encounter. To do so, we can use a table with general payoffs (delineated in appropriately measurable units) that satisfy the standard structure of payoffs in a prisoners' dilemma, as in Figure 10.3. The payoffs in the table must satisfy the relation $H > C > D > L$ for the game to be a prisoners' dilemma, where C is the payoff in the *cooperative* outcome, D is the payoff when both players *defect* from cooperation, H is the *high* payoff that goes to the defector when one player defects while the other cooperates, and L is the *low* payoff that goes to the cooperator in the same situation.

		COLUMN	
		Defect	Cooperate
ROW	Defect	D, D	H, L
	Cooperate	L, H	C, C

FIGURE 10.3 General Version of the Prisoners' Dilemma

In this general version of the prisoners' dilemma, a player's one-time gain from defecting is $(H - C)$. The single-period loss for being punished while you return to cooperation is $(C - L)$, and the per-period loss for perpetual defecting is $(C - D)$. To be as general as possible, we will allow for situations in which there is a probability $p < 1$ that the game continues beyond the next period, and so we will discount payoffs using an effective rate of return of R per period. If $p = 1$, as would be the case when the game is guaranteed to continue, then $R = r$, the rate of return used in our preceding calculations. Replacing r with R, we find that the results attained earlier generalize almost immediately.

We found earlier that a player defects exactly once against a rival playing TFT only if the one-time gain from defecting $(H -$

C) exceeds the present value of the single-period loss from being punished (the PV of $C - L$). In this general game, that means that a player defects once against a TFT-playing opponent only if $(H - C) > (C - L)/(1 + R)$, or $(1 + R)(H - C) > C - L$, or

$$R > \frac{C - L}{H - C} - 1.$$

Similarly, we found that a player defects forever against a rival playing TFT only if the one-time gain from defecting exceeds the present value of the infinite sum of the per-period losses from perpetual defecting (where the per-period loss is $C - D$). In the general game, then, a player defects forever against a TFT-playing opponent, or defects at all against a grim strategy-playing opponent, only if $(H - C) > (C - D)/R$, or

$$R > \frac{C - D}{H - C}.$$

The three critical elements in a player's decision to defect, as seen in these two expressions, are the immediate gain from defection ($H - C$), the future losses from punishment ($C - L$ or $C - D$ per period of punishment), and the effective rate of return (R, which measures the importance of the present relative to the future). Defecting from cooperation becomes more attractive when the immediate gain from defection is higher, the future losses from punishment are smaller, and the effective rate of return is higher. But how high or low do these three quantities need to be, exactly, in order for players to prefer defecting?

First, assume that the values of the gains and losses from defecting are fixed. Then changes in R determine whether a player defects, and defection is more likely when R is large. Large values of R are associated with small values of p and small values of δ (and large values of r), so defection is more likely when the probability of continuation is low or the discount factor is low (or the interest rate is high). Another way to think about it is that defection is more likely when the future is less important than the present, or when there is little future to consider; that is, defection is more likely when players are impatient or when they expect the game to end quickly.

Second, consider the case in which the effective rate of return is fixed, as is the one-period gain from defecting. Then changes in the per-period loss associated with punishment determine whether defecting is worthwhile. Here, it is smaller values of $C - L$ or $C - D$ that encourage defection. In this case, defection is more likely when punishment is not very severe.[4]

Finally, assume that the effective rate of return and the per-period loss associated with punishment are held constant. Now players are more likely to defect when the gains, $H - C$, are high. This situation is more likely when defecting garners a player large and immediate benefits.

This discussion also highlights the importance of the detection of defecting. Decisions about whether to continue along a cooperative path depend on how long defecting might be able to go on before it is detected, on how accurately it is detected, and on how long any punishment can be made to last before an attempt is made to revert to cooperation. Although our model does not incorporate these considerations explicitly, if defecting can be detected accurately and quickly, its benefit will not last long, and the subsequent cost will have to be paid more surely. Therefore, the success of any trigger strategy in resolving a repeated prisoners' dilemma depends on how well (in terms of both speed and accuracy) players can detect defecting. This is one reason why the TFT strategy is often considered dangerous: Slight errors in the execution of actions or in the perception of those actions can send players into continuous rounds of punishment from which they may not be able to escape for a long time, until a slight error of the opposite kind occurs.

You can use all of these ideas to guide you in when to expect more cooperative behavior between rivals and when to expect more defecting and cutthroat actions. If times are bad and an entire industry is on the verge of collapse, for example, so that businesses feel that there is no future, competition may become fiercer (less cooperative behavior may be observed) than in normal times. Even if times are temporarily good, but good conditions are not expected to last, firms may want to make a quick profit while they can, so cooperative behavior may again break down. Similarly, in an industry that emerges temporarily because of a quirk of fashion and is expected to collapse when fashion changes, we should expect less cooperation. Thus, a particular beach resort might become the place to go, but all the hotels there will know that such a situation cannot last, and that they cannot afford to collude on pricing. If, in contrast, the shifts in fashion are among products made by an unchanging group of companies in long-term relationships with one another, cooperation might persist. For example, even if all the children want cuddly bears one year and Transformers Rescue Bots the next, collusion in pricing may occur if the same small group of manufacturers makes both items.

In Chapter 11, we will look in more detail at prisoners' dilemmas that arise in games with many players. We will examine when and how players can overcome such dilemmas and achieve outcomes better for them all.

Endnotes

- Defecting as retaliation under the requirements of a trigger strategy is often termed *punishing* to distinguish it from the original decision to defect. Return to reference 1
- The key formula provided in the appendix to this chapter states that, for any x, the infinite sum
$$\frac{x}{1+r}+\frac{x}{(1+r)^2}+\frac{x}{(1+r)^3}+\cdots=\frac{x}{r}.$$
Return to reference 2
- We could also express R in terms of r and p, in which case $R = (1 + r)/p - 1$. Return to reference 3
- The costs associated with defection may also be smaller if information transmission is not perfect, as might be the case if there are many players, so that difficulties might arise in identifying the defector and in coordinating a punishment scheme. Similarly, gains from defection may be larger if rivals cannot identify a defection immediately. Return to reference 4

Glossary

repeated play
: A situation where a one-time game is played repeatedly in successive periods. Thus, the complete game is mixed, with a sequence of simultaneous-move games.

contingent strategy
: In repeated play, a plan of action that depends on other players’ actions in previous plays. (This is implicit in the definition of a strategy; the adjective “contingent” merely reminds and emphasizes.)

trigger strategy
: In a repeated game, this strategy cooperates until and unless a rival chooses to defect, and then switches to noncooperation for a specified period.

punishment
: We reserve this term for costs that can be inflicted on a player in the context of a repeated relationship (often involving termination of the relationship) to induce him to take actions that are in the joint interests of all players.

grim strategy
: A strategy of noncooperation forever in the future, if the opponent is found to have cheated even once. Used as a threat of punishment in an attempt to sustain cooperation.

tit-for-tat (TFT)
: In a repeated prisoners’ dilemma, this is the strategy of [1] cooperating on the first play and [2] thereafter doing each period what the other player did the previous period.

present value (PV)
: The total payoff over time, calculated by summing the payoffs at different periods each multiplied by the appropriate discount factor to make them all comparable with the initial period’s payoffs.

infinite horizon
: A repeated decision or game situation that has no definite end at a fixed finite time.

compound interest

When an investment goes on for more than one period, compound interest entails calculating interest in any one period on the whole accumulation up to that point, including not only the principal initially invested but also the interest earned in all previous periods, which itself involves compounding over the period previous to that.

discount factor

In a repeated game, the fraction by which the next period’s payoffs are multiplied to make them comparable with this period’s payoffs.

effective rate of return

Rate of return corrected for the probability of noncontinuation of an investment to the next period.

3 CHANGING THE ORDER OF MOVES: PROMISES

A key aspect of the classic prisoners' dilemma game is that players make their moves simultaneously, deciding whether to cooperate (by not confessing, by setting a high price, etc.) without being able to observe the other player's choice. However, many games with a prisoners' dilemma structure unfold sequentially, with one player irreversibly and observably making a move before the other. In this section, we focus on such *sequential-move prisoners' dilemma games*, showing how players can achieve the mutually preferred outcome in which both cooperate as long as either of them can be trusted.

Stanford economist Avner Greif studied the sequential-move prisoners' dilemma (also referred to as a "one-sided prisoners' dilemma") in the historical context of *exchange* in the medieval world at marketplaces known as bazaars.[5] Once a buyer and seller found each other and agreed on the terms of trade, who would be the first to complete their side of the bargain? If the seller first hands over the product, the buyer might run away without paying. On the other hand, if the buyer first hands over the money, the seller might refuse to hand over the product, denying that any money was paid or, even more insidiously, might hand over a defective product whose defects the buyer will discover only after it is too late to return for a refund.

Greif refers to this strategic challenge as "the fundamental problem of exchange" and argues that societies' abilities to resolve this problem (or not) contributed significantly to their growth and prosperity (or lack thereof) during the medieval period.[6] To understand the challenge, consider the sequential-move game shown in Figure 10.4, in which the buyer (Bob) first decides whether to pay, and then the seller (Ann) decides whether to hand over the product. Each benefits individually by choosing to *break the trust*—not following through on their own end of the bargain—but both are better off when both *keep the trust*. The players' payoffs in this game are exactly the same as in a standard prisoners' dilemma, but now one player moves first. (You can compare the payoff structure here with the ordinal payoffs from Figure 4.10 to confirm that this game is a prisoners' dilemma.) As the second mover, Ann's dominant strategy is breaking the trust; anticipating this, Bob will choose to break the trust as well. Thus, in the rollback equilibrium, both players break the trust, and no trade gets made; this is exactly what players do in the Nash equilibrium in the standard simultaneous-move version of the prisoner's dilemma.

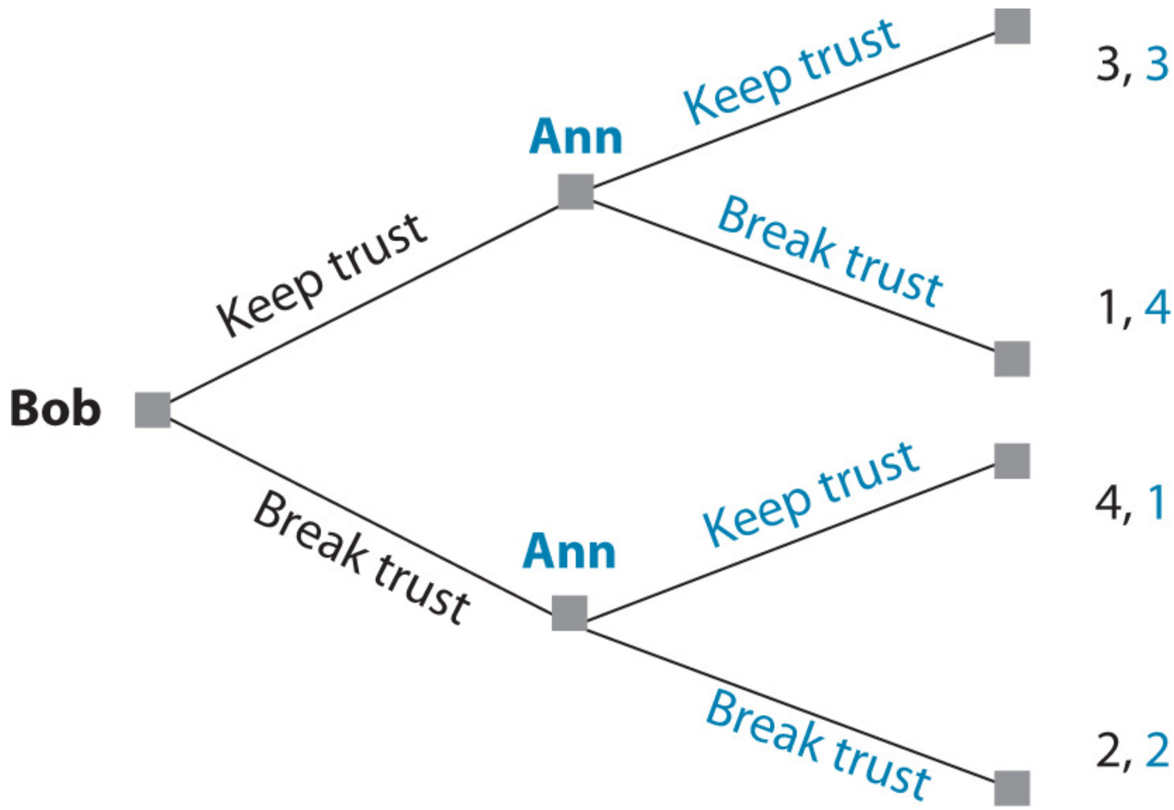

Figure 10.4 Game Tree for a Sequential-Move Prisoners' Dilemma

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The problem here is less severe than in the simultaneous-move game, however, because only the second mover needs to establish her trustworthiness in order to solve the dilemma. In particular, suppose that Ann can *make a promise*, declaring that she will keep the trust (as the second mover) if Bob first keeps the trust himself. (Ann also implicitly warns Bob that she will break the trust if he does.) So long as Ann's promise is credibly made, Bob will prune the branch of the tree in which Ann breaks the trust after he keeps the trust. Bob's choice is then between the outcomes in which both keep the trust or both break it, of which he prefers for both to keep the trust.

To summarize, the fundamental problem of exchange can be solved, so long as *either* side in an exchange can be trusted, by allowing the trusted party to move last in the upcoming game and to make a credible promise in the pregame. Establishing a reputation for trustworthiness is therefore enormously valuable for sellers, as it allows them to make profitable trades where other less trusted sellers cannot. Viewed this way, trust is an asset—and one that can be loaned as well. In medieval bazaars, merchants would typically appoint one of their own to serve as judge to resolve disputes and impose penalties for misbehavior. So long as the judge had a reputation for fairness (to punish wrongdoers) and discernment (to learn the truth in any situation), no *other* merchant would dare to cheat a buyer, knowing that he would face punishment far more costly than whatever gain he could have gotten by cheating. In this way, the trustworthiness of one merchant (appointed as judge) could be extended to all of them.

Such solutions worked in medieval times, allowing trade to flourish even on the frontiers of civilization, but what about today, in the new frontier of electronic commerce? On eBay, for example, buyer *feedback* provides a means by which sellers can establish their trustworthiness. Unfortunately, unscrupulous sellers can easily create false positive feedback for themselves and, even after getting kicked off the site, can purchase on the dark Web a new "verified, ready to sell" eBay identity with a sterling *false* history of positive transactions. (For more on the market for false eBay identities, see Exercise S9.)

To give consumers confidence that they will not be cheated, eBay offers a limited money-back guarantee for buyers who don't receive an item or receive one that is not as described. Such a guarantee ensures that consumers don't suffer from the most obvious sorts of fraud, such as when a man in Texas fleeced 46 customers of $191,000 in a "hot-tub scam" in which he never delivered the promised tubs.[7] However, in subtler cases, when the seller has delivered an item that is not *quite* as described, or is defective in some way that is itself difficult to prove, the buyer may be unable to prove her case—especially if the seller claims that the buyer is herself attempting to commit fraud (or extortion) by submitting a false claim. Clearly, trust on eBay and other e-commerce marketplaces is a work in progress; the fundamental problem of exchange remains.

Endnotes

- Avner Greif, *Institutions and the Path to the Modern Economy* (New York: Cambridge University Press, 2006). See also Avinash Dixit, *Lawlessness and Economics: Alternative Modes of Governance* (Princeton, N.J.: Princeton University Press, 2007). Return to reference 5
- Thomas Hobbes considered the same problem in 1651, writing in his classic treatise *Leviathan*, "If a Covenant be made, wherein neither of the parties performe presently, but trust one another . . . upon any reasonable suspition, it is Voyd. . . . For he that performeth first, has no assurance that the other will performe after; because the bonds of words are too weak to bridle mens . . . avarice." See Thomas Hobbes, *Leviathan* (Urbana, Ill.: Project Gutenberg, 2009), Chapter 14, Section 17; retrieved April 30, 2019, from www.gutenberg.org/ebooks/3207. Return to reference 6
- Mary Flood, "Prison for Houston Man Who Ran eBay Hot Tub Scam," *Houston Chronicle*, May 19, 2010. Return to reference 7

4 CHANGING PAYOFFS: PENALTIES AND REWARDS

Changes in the way moves are made and changes in the order of moves are the major vehicles for the solution of the prisoners' dilemma. But there are several other mechanisms that can achieve the same purpose by changing payoffs for the players. One of the simplest such mechanisms, which averts the prisoners' dilemma in the one-shot version of the game, arises when an agent or entity outside the game has the power to inflict some direct penalty on the players when they defect. Once the payoffs have been altered to incorporate the cost of the penalty, players may find that the dilemma has been resolved.

		WIFE	
		Confess	Deny
HUSBAND	Confess	10 yr, 10 yr	21 yr, 25 yr
	Deny	25 yr, 21 yr	3 yr, 3 yr

FIGURE 10.5 Prisoners' Dilemma with Penalty for the Lone Defector

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Consider the husband-wife dilemma from Section 1 of this chapter. If only one player defects, the game's outcome entails 1 year in jail for the defector and 25 years for the cooperator (see Figure 10.1). The defector, though, upon getting out of jail early, might find the cooperator's friends waiting outside the jail. The physical harm caused by those friends might be equivalent to an additional 20 years in jail. If so, and if the players account for the possibility of this harm, then the payoff structure of the original game has changed.

The new game, with the physical penalty included in the payoffs, is illustrated in Figure 10.5. With the 20 years in jail added to each player’s sentence when that player confesses while the other denies, the game is completely different. Best-response analysis shows that there are now two pure-strategy Nash equilibria: One of them is (Confess, Confess); the other is (Deny, Deny). Now each player finds that it is in his or her best interest to cooperate if the other is going to do so. The game has changed from a prisoners’ dilemma to an assurance game, which we also studied in Chapter 4. Solving the new game requires selecting an equilibrium from the two that exist. One of them—the cooperative outcome—is clearly better than the other from the perspective of both players. Therefore, it may be easy to sustain it as a focal point if some convergence of expectations can be achieved.

Notice that the penalty in this scenario is inflicted on a defector only when his or her rival does *not* defect. However, stricter penalty mechanisms can be incorporated into the prisoners’ dilemma, including ones in which there are penalties for *any* defection. Such discipline typically must be imposed by a third party with some power over *both* players, rather than by the other player’s friends, because the friends would have little authority to penalize the first player for defecting when their associate also defects. If both prisoners are members of a special organization (such as a gang or a crime mafia), and the organization has a standing rule of never confessing to the police under penalty of extreme physical harm, then the game changes again to the one illustrated in Figure 10.6.

		WIFE	
		Confess	Deny
HUSBAND	Confess	30 yr, 30 yr	21 yr, 25 yr
	Deny	25 yr, 21 yr	3 yr, 3 yr

FIGURE 10.6 Prisoners' Dilemma with Penalty for Any Defecting

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Now, the equivalent of an additional 20 years in jail is added to *all* payoffs associated with the Confess strategy (compare Figure 10.6 with Figure 10.1). In the new game, each player has a dominant strategy, as in the original game. The difference is that the change in the payoffs makes Deny the dominant strategy for each player. And (Deny, Deny) becomes the unique pure-strategy Nash equilibrium. The stricter penalty scheme achieved with the third-party enforcement mechanism makes defecting so unattractive to players that the cooperative outcome becomes the new equilibrium of the game.

Larger prisoners' dilemma games are more difficult to solve with mechanisms that change payoffs by exacting penalties for defection. In particular, if there are many players and some uncertainty exists, penalty schemes may be more difficult to maintain. It becomes harder to decide whether actual defecting is taking place or if it's just bad luck or a mistaken move. In addition, if there really is defecting, it is often difficult to determine the identity of the defector from among the larger group. And if the game is a one-shot one, there is no opportunity in the future to correct a penalty that is too severe or to inflict a penalty once a defector has been identified. Thus, penalties may be less successful in large one-shot games than in the two-person games we consider here. We study prisoners' dilemmas with a large number of players in greater detail in Chapter 11.

A further interesting possibility arises when a prisoners' dilemma that has been solved with a penalty mechanism is considered in the context of the larger society in which the game is played. It might be the case that, although the equilibrium outcome is bad for the players, it is actually good for the rest of society, or for some subset of persons within the rest of society. If so, social or political pressures might arise to try to minimize the ability of players to break out of the dilemma. When third-party penalties are the solution to a prisoners' dilemma, as is the case with crime mafias that enforce a no-confession rule, society can come up with its own strategy to reduce the effectiveness of the penalty mechanism. The U.S. Federal Witness Protection Program, in which the government removes the threat of penalty in return for confessions and testimony in court, is an example of a system that has been set up for just this purpose.

Similar opportunities for changing payoffs can be seen in other prisoners' dilemmas, such as the pricing game between our two restaurants. The equilibrium in that case entails both restaurants charging the low price of $20, even though they would enjoy higher profits when charging the higher price of $26. Although the restaurants want to break out of this bad equilibrium—and we have already seen how the use of trigger strategies could help them do so—their customers are happier with the low price offered in the Nash equilibrium of the one-shot game. The customers, then, have an incentive to try to destroy the efficacy of any enforcement mechanism or solution process the restaurants might use. For example, because some firms facing pricing games attempt to solve the dilemma through the use of price-matching campaigns, customers might want to press for legislation banning such practices. We analyze the effects of price-matching strategies in Section 7.B.

Just as a prisoners' dilemma can be resolved by way of mechanisms that reduce the payoffs for defectors, it can also be resolved by mechanisms that increase the payoffs for, or reward, cooperators. Because such solutions are more difficult to implement in practice, we mention them only briefly.

The most important question is who is to pay the rewards. If it is a third party, that person or group must have sufficient interest of its own in the cooperation achieved by the players to make it worth its while to pay out the rewards. A rare example of this solution approach was used when the United States brokered the Camp David Accords between Israel and Egypt by offering large promises of aid to both. If the rewards are to be paid by the players themselves to each other, the trick is to make the rewards contingent (paid out only if the other player cooperates) and credible (guaranteed to be paid if the other player cooperates). Meeting these criteria may require some unusual arrangements; Exercise U6 shows one such example.

Glossary

penalty
: We reserve this term for one-time costs (such as fines) introduced into a game to induce the players to take actions that are in their joint interests.

5 CHANGING PAYOFFS: LEADERSHIP

The final type of soution we consider involves a prisoners' dilemma in which the standard payoff structure is changed due to differences in size between the two players. These situations are frequently ones in which one player takes on the role of leader in the interaction. In most examples of the prisoners' dilemma, the game is assumed to be symmetric; that is, all the players stand to lose (or gain) the same amount from defecting (or cooperating). However, in actual strategic situations, one player may be relatively "large" (a leader) and the other "small." If the sizes of the payoffs are unequal enough, so much of the harm from defecting may fall on the larger player that she acts cooperatively, even while knowing that the other will defect. Saudi Arabia, for example, played such a role as the "swing producer" in OPEC (Organization of Petroleum Exporting Countries) for many years; to keep oil prices high, it cut back on its output when one of the smaller producers, such as Libya, expanded its production.

		SOPORIA	
		Participate	Not
DORMINICA	Participate	−1, −1	−2, 0
	Not	0, −2	−1.6, −1.6

You may need to scroll left and right to see the full figure.

FIGURE 10.7 Payoffs for Equal-Population SANE Research Game (in Billions of Dollars)

As with the OPEC example, leadership tends to be observed more often in games between nations than in games between firms or individual persons. Thus, our example of a game in which leadership may be used to solve the prisoners' dilemma is one played between countries. Imagine that the populations of two countries, Dorminica and Soporia, are threatened by a disease, Sudden Acute Narcoleptic Episodes (SANE). This disease strikes 1 person in every 2,000, or 0.05% of the population, and causes the victim to fall into a deep-sleep state for a year.[8] There are no aftereffects from the disease, but the cost of a worker being removed from the economy for a year is $32,000. Each country has a population of 100 million workers, so the expected number of cases in each is 50,000 (0.0005 × 100,000,000), and the expected cost of the disease is $1.6 billion to each (50,000 × 32,000). The total expected cost of the disease worldwide—that is, in both Dorminica and Soporia—is then $3.2 billion.

Scientists are confident that a crash research program costing $2 billion will lead to a vaccine that is 100% effective in preventing the disease. Comparing the cost of the research program with the worldwide cost of the disease shows that, from the perspective of the entire population, the research program is clearly worth pursuing ($2 billion cost < $3.2 billion savings). However, the government in each country must consider whether to participate in this research program. If both participate, they will share the cost ($1 billion each), and both will get the benefit (avoiding the $1.6 billion cost). If only one government chooses to participate, it must fund the entire research program (at a cost of $2 billion), while the population of the other country can make and use the vaccine for its own population without incurring the research cost.

The payoff matrix for the noncooperative game between Dorminica and Soporia is shown in Figure 10.7. Each country chooses from two strategies, Participate or Not; the payoff matrix shows the costs to the countries, in billions of dollars, of the various strategy combinations. It is straightforward to verify that each country has a dominant strategy not to participate ($0 > -1$ and $-1.6 > -2$). In addition, the equilibrium payoffs are −1.6 to each, but the two countries could each get −1 if they both participate. So the game is a prisoners' dilemma.

		SOPORIA	
		Participate	Not
DORMINICA	Participate	−1.5, −0.5	−2, 0
	Not	0, −2	−2.4, −0.8

You may need to scroll left and right to see the full figure.

FIGURE 10.8 Payoffs for Unequal-Population SANE Research Game (in Billions of Dollars)

But now suppose that the two countries have unequal populations of workers, with 150 million in Dorminica and 50 million in Soporia. Then, if no research is funded by either government, the cost to Dorminica of SANE will be $2.4 billion ($0.0005 \times 150{,}000{,}000 \times 32{,}000$), and the cost to Soporia will be $0.8 billion ($0.0005 \times 50{,}000{,}000 \times 32{,}000$). If both choose Participate, they will share the cost of research in proportion to their populations, $1.5 billion for Dorminica and $0.5 billion for Soporia. The payoff matrix changes to the one illustrated in Figure 10.8.

In this version of the game, Soporia still has a dominant strategy not to participate. But Dorminica's best response to that strategy is now Participate. What has happened to change Dorminica's choice of strategy? Clearly, the answer lies in the unequal distribution of the population in this revised version of the game. Dorminica now stands to suffer such a large portion of the total cost of the disease that it finds it worthwhile to do the research on its own. This is true even though Dorminica knows full well that Soporia is going to be a free rider and get a share of the full benefit of the research.

The research game in Figure 10.8 is no longer a prisoners' dilemma. Here we see that the dilemma has, in a sense, been solved by the size asymmetry between the countries. The larger country chooses to take on a leadership role and provide the benefit for the whole world.

Examples of leadership in what would otherwise be prisoners' dilemma games are common in international diplomacy. The role of leader often falls naturally to the biggest or best established of the players, a phenomenon labeled "the exploitation of the great by the small."[9] For instance, the United States has carried a disproportionate share of the expenditures of our defense alliances, such as NATO, and has maintained a policy of relatively free international trade even while our trading partners, such as Japan and China, have been much more protectionist. In such situations, our theory suggests that the large player would actually act more cooperatively, carrying the burden and tolerating free riding by smaller players. NATO expects each member country to spend 2% of its gross domestic product (GDP) on defense. But our example shows that even when the cost is shared proportionately, small players may still find it optimal not to carry their fair share of the burden and to free ride instead. Whether President Trump's threats in such situations will prove more credible than previous presidents' exhortations to allies to carry their share of the burden is uncertain. Theory suggests not, but only time will tell.

Endnotes

- Think of Rip Van Winkle or of Woody Allen in the movie *Sleeper*, but the duration is much shorter. Return to reference 8
- Mancur Olson, *The Logic of Collective Action* (Cambridge, Mass.: Harvard University Press, 1965), p. 29. Return to reference 9

Glossary

leadership

In a prisoners' dilemma with asymmetric players, this is a situation where a large player chooses to cooperate even though he knows that the smaller players will cheat.

6 EXPERIMENTAL EVIDENCE

Numerous people have conducted experiments in which subjects compete in prisoners' dilemma games against each other.[10] Such experiments show that cooperation can and does occur in such games, even in repeated versions of known and finite length. Many players start off by cooperating and continue to cooperate for quite a while, as long as the rival player reciprocates. Only in the last few plays of a finite game does defecting seem to creep in. Although this behavior goes against the reasoning of rollback, it can be "profitable" if sustained for a reasonable length of time. The cooperating pairs get higher payoffs than would rational, calculating strategists who defect from the very beginning.

The idea that some level of cooperation may constitute rational—that is, equilibrium—behavior has theoretical backing. Consider the fact that when asked about their reasons for cooperating in the early rounds, players will usually say something such as, "I was willing to try and see if the other player was nice, and when this proved to be the case, I continued to cooperate until the time came to take advantage of the other's niceness." Of course, the other player may not have been genuinely nice, but thinking along similar lines. A rigorous analysis of a finitely repeated prisoners' dilemma shows that this type of asymmetric information can actually be another solution to the dilemma. As long as there is some chance that players are nice rather than selfish, even a selfish player can gain by pretending to be nice. She can reap the higher payoffs from cooperation for a while and then also hope to exploit the gains from double-crossing her opponent near the end of the sequence of plays. For a thorough explication of the case in which just one of the players has the choice between being selfish and being nice, see the online appendix to this chapter.[11] Such cooperative behavior in lab experiments can also be rationalized without relying on this type of asymmetric information. Perhaps the players are not sure that the relationship will actually end at the stated time. Perhaps they believe that their reputations for cooperation will carry over to other similar games against the same opponent or other opponents. Perhaps they think it possible that their opponents are naive cooperators, and they are willing to risk a little loss in testing this hypothesis for a couple of plays. If successful, this experiment will lead to higher payoffs for a sufficiently long time.

In some laboratory experiments, players engage in multiple-round games, each round consisting of a given finite number of repetitions. All the repetitions in any one round are played against the same rival, but each new round is played against a new opponent. Thus, there is an opportunity to develop cooperation with an opponent in each round and to learn from preceding rounds when devising one's strategy against new opponents as the rounds continue. These situations have shown that cooperation lasts longer in early rounds than in later rounds. This result suggests that the theoretical argument on the unraveling of cooperation, based on the use of rollback, is being learned from experience of the play itself over time as players begin to understand the benefits and costs of their actions more fully. Another possibility is that players learn simply that they want to be the first to defect, and so the timing of the initial defection occurs earlier as the number of rounds played increases.

Suppose you were playing a game with a prisoners' dilemma structure and found yourself in a cooperative mode with the known end of the relationship approaching. When should you decide to defect? You do not want to do so too early, while a lot of potential future gains remain. But you also do not want to leave it until too late in the game, because then your opponent might preempt you and leave you with a low payoff for the periods in which she defects. Similar calculations are relevant when you are in a finitely repeated relationship with an uncertain end date. Your decision about when to defect cannot be deterministic. If it were, your opponent would figure it out and defect in the period before you planned to do so. If no deterministic choice is feasible, then the unraveling of cooperation must include some element of uncertainty, such as mixed strategies, for both players. Many thrillers whose plots hinge on tenuous cooperation among criminals, or between informants and police, acquire their suspense precisely because of this uncertainty.

Examples of the collapse of cooperation as players near the end of a repeated game are observed in numerous situations in the real world, as well as in the laboratory. The story of a long-distance bicycle (or foot) race is one such example. There may be a lot of cooperation for most of the race, as players take turns leading and letting others ride in their slipstreams; nevertheless, as the finish line looms, each participant will want to make a dash for the tape. Similarly, signs saying “No checks accepted” often appear in stores in college towns each spring near the end of the semester.

Computer-simulation experiments have matched a range of very simple to very complex contingent strategies against each other in two-player prisoners’ dilemmas. The most famous of them were conducted by Robert Axelrod at the University of Michigan. He invited people to submit computer programs that specified a strategy for playing a prisoners’ dilemma repeated a finite but large number (200) of times. There were 14 entrants. Axelrod held a “league tournament” that pitted pairs of these programs against each other, in each case for a run of the 200 repetitions. The point scores for each pairing and its 200 repetitions were recorded, and each program’s scores over all its runs against different opponents were added up to see which program did best in the aggregate against all other programs. Axelrod was initially surprised when “nice” programs did well; none of the top eight programs were ever the first to defect. The winning strategy turned out to be the simplest program: tit-for-tat, submitted by the Canadian game theorist Anatole Rapoport. Programs that were eager to defect in any particular run got the defecting payoff early, but then suffered repetitions of mutual defections and poor payoffs. In contrast, programs that were always nice and cooperative were badly exploited by their opponents. Axelrod explains the success of TFT in terms of four properties: It is at once forgiving, nice, provocable, and clear.

In Axelrod’s words, one does well in a repeated prisoners’ dilemma to abide by these four simple rules: “Don’t be envious. Don’t be the first to defect. Reciprocate both cooperation and defection. Don’t be too clever.”[12] The TFT strategy embodies each of these four ideals for a good, repeated prisoners’ dilemma strategy. It is not envious; it does not continually strive to do better than the opponent, only to do well for itself. In addition, TFT clearly fulfills the admonitions not to be the first to defect and to reciprocate, defecting only in retaliation to the opponent’s preceding defection and always reciprocating in kind. Finally, TFT does not suffer from being overly clever; it is simple and understandable to the opponent. In fact, it won the tournament not because it helped players achieve high payoffs in any individual game—the contest was not about “winner takes all”—but because it was always close; it simultaneously encourages cooperation and avoids exploitation, whereas other strategies cannot.

Axelrod then announced the results of his tournament and invited submissions for a second round. Here, people had a clear opportunity to design programs that would beat TFT. The result: TFT won again! The programs that were cleverly designed to beat it could not beat it by very much, and they did poorly against one another. Axelrod also arranged a tournament of a different kind. Instead of having each program face each other program exactly once, he ran a tournament intended to simulate the evolutionary dynamics underlying the “survival of the fittest.” Each program submitted to the tournament had a certain number of copies included in a larger population, and met an opponent randomly chosen from this population. Those programs that did well were then given a larger proportion of the population in the next round; those that did poorly had their proportion reduced. This was a game of evolution and natural selection, which we will study in greater detail in Chapter 12. But the idea is simple in this context, and the results are fascinating. At first, nasty programs did well at the expense of nice ones. But as the population became nastier and nastier, each nasty program met other nasty programs more and more often, and they began to do poorly and fall in numbers. Then TFT started to do well and eventually triumphed.

However, TFT has some flaws. Most importantly, it assumes no errors in execution of the strategy. If there is some risk that a player intends to play the cooperative action but plays the defecting action in error, then this action can initiate a sequence of retaliatory defecting actions that locks two TFT programs playing one another into a bad outcome; another error is required to rescue them from this sequence. When Axelrod ran a third variant of his tournament, which provided for such random mistakes, TFT could be beaten by even “nicer” programs that tolerated an occasional episode of defecting to see whether it was a mistake or a consistent attempt to exploit them, and retaliated only when convinced that it was not a mistake.[13]

Interestingly, a twentieth-anniversary competition modeled after Axelrod’s original contest and run in 2004 and 2005 generated a new winning strategy.[14] Actually, the winner was a set of strategies designed to recognize one another during play so that one would become docile in the face of the other’s continued defections. (The authors likened their approach to a situation in which prisoners manage to communicate with each other by tapping on their cell walls.) This collusion meant that some of the strategies submitted by the winning team did very poorly, whereas others did spectacularly well, a testament to the value of working together. Of course, Axelrod’s original contest did not permit multiple submissions, so such strategy sets were ineligible, but the winners of the recent competition argue that with no way to preclude coordination, strategies such as those they submitted should have been able to win the original competition as well.

Endnotes

- The literature on experiments involving the prisoners' dilemma game is vast. A brief overview is given by Alvin Roth in *The Handbook of Experimental Economics* (Princeton, N.J.: Princeton University Press, 1995), pp. 26–28. Journals in both psychology and economics can be consulted for additional references. For some examples of the outcomes that we describe, see Kenneth Terhune, "Motives, Situation, and Interpersonal Conflict Within Prisoners' Dilemmas," *Journal of Personality and Social Psychology Monograph Supplement*, vol. 8, no. 30 (1968), pp. 1–24; R. Selten and R. Stoecker, "End Behavior in Sequences of Finite Prisoners' Dilemma Supergames," *Journal of Economic Behavior and Organization*, vol. 7 (1986), pp. 47–70; and Lisa V. Bruttel, Werner Güth, and Ulrich Kamecke, "Finitely Repeated Prisoners' Dilemma Experiments Without a Commonly Known End," *International Journal of Game Theory*, vol. 41 (2012), pp. 23–47. Robert Axelrod's *Evolution of Cooperation* (New York: Basic Books, 1984) presents the results of his computer-simulation tournament for the best strategy in an infinitely repeated dilemma. Return to reference 10
- The corresponding version in which both players can choose between selfish and nice is solved in full in the original article: David Kreps, Paul Milgrom, John Roberts, and Robert Wilson, "Rational Cooperation in a Finitely Repeated Prisoner's Dilemma," *Journal of Economic Theory*, vol. 27 (1982), pp. 245–52. Return to reference 11
- Axelrod, *Evolution of Cooperation*, p. 110. Return to reference 12
- For a description and analysis of Axelrod's computer simulations from the biological perspective, see Matt Ridley, *The Origins of Virtue* (New York: Penguin Books,

1997), pp. 61, 75. For a discussion of the difference between computer simulations and experiments using human players, see John K. Kagel and Alvin E. Roth, *Handbook of Experimental Economics* (Princeton, N.J.: Princeton University Press, 1995), p. 29. Return to reference 13

- See Wendy M. Grossman, “New Tack Wins Prisoner’s Dilemma,” *Wired*, October 13, 2004, available at http://archived.wired.com/culture/lifestyle/news/2004/10/65317 (accessed August 1, 2014). Return to reference 14

7 REAL-WORLD DILEMMAS

Games with the prisoners’ dilemma structure arise in a surprisingly varied number of contexts in the real world. Although we would be foolish to try to show you every possible instance in which the dilemma can arise, we take the opportunity in this section to consider in detail three specific examples from a variety of fields of study. One example comes from evolutionary biology, a field that we will study in greater detail in Chapter 12. A second example describes the policy of “price matching” as a solution to a prisoners’ dilemma pricing game. And a final example concerns international environmental policy and the potential for repeated interactions to mitigate the prisoners’ dilemma in this situation.

A. Evolutionary Biology

In our first example, we consider a game known as the bowerbirds' dilemma, from the field of evolutionary biology.[15] Male bowerbirds attract females by building intricate nesting spots called bowers, and female bowerbirds are known to be particularly choosy about the bowers built by their prospective mates. For this reason, male bowerbirds often go out on search-and-destroy missions aimed at ruining other males' bowers. While they are out, however, they run the risk of losing their own bowers to the beak of another male. The ensuing competition between male bowerbirds, in which they can choose whether to maraud others' bowers or guard their own, has the structure of a prisoners' dilemma game.

Ornithologists have constructed a table that shows the payoffs in a two-bird game with two possible strategies, Maraud and Guard (Figure 10.9). The GG payoff represents the benefits associated with Guarding when the rival bird also Guards; GM represents the payoff from Guarding when the rival bird is a Marauder. Similarly, MM represents the benefits associated with Marauding when the rival bird also is a Marauder; MG represents the payoff from Marauding when the rival bird Guards. Careful scientific study of bowerbird matings led to the discovery that MG > GG > MM > GM. In other words, the payoffs in the bowerbird game have exactly the same structure as the prisoners' dilemma. The birds' dominant strategy is to Maraud, but when both choose that strategy, they end up in equilibrium with each worse off than if they had both chosen to Guard.

In reality, the strategy used by any particular bowerbird is not actually the result of a process of rational choice on the part of the bird. Rather, in evolutionary games, strategies are assumed to be genetically “hardwired” into individual organisms, and payoffs represent reproductive success for the different types. Then equilibria in such games define the proportions of types that naturalists can expect to observe in the population—all Marauders, for instance, if Maraud is a dominant strategy, as in Figure 10.9. This equilibrium outcome is not the best one, however, given the existence of the dilemma. In constructing a solution to the bowerbirds’ dilemma, we can appeal to the repetitive nature of the interaction in the game. In the case of the bowerbirds, repeated play against the same or different opponents in the course of several breeding seasons can allow a bird to choose a flexible strategy based on his opponent’s last move. Contingent strategies such as tit-for-tat can be, and often are, adopted in evolutionary games to solve exactly this type of dilemma. We will return to the idea of evolutionary games and provide detailed discussions of their structure and equilibrium outcomes in Chapter 12.

		BIRD 2	
		Maraud	Guard
BIRD 1	Maraud	MM, MM	MG, GM
	Guard	GM, MG	GG, GG

FIGURE 10.9 Bowerbirds’ Dilemma

B. Price Matching

Now we return to a pricing game, in which we consider two specific stores engaged in price competition with each other, using identical price-matching policies. The stores in question, Staples and Office Depot, are both national chains that regularly advertise their prices for name-brand office supplies. In addition, each store maintains a published policy that guarantees customers that it will match the advertised price of any competitor on a specific item (model and item numbers must be identical) as long as the customer provides the competitor's printed advertisement.[16]

For the purposes of this example, we assume that the firms have only two possible prices that they can charge for a particular package of printer ink (Low or High). In addition, we use hypothetical profit numbers, and we further simplify the analysis by assuming that Staples and Office Depot are the only two competitors in the office supplies market in a particular city—Billings, Montana, for example.

		OFFICE DEPOT	
		Low	High
STAPLES	Low	2,500, 2,500	5,000, 0
	High	0, 5,000	3,400, 3,400

FIGURE 10.10 Staples and Office Depot Ink Pricing Game

		OFFICE DEPOT		
		Low	High	Match
STAPLES	Low	2,500, 2,500	5000, 0	2,500, 2,500

You may need to scroll left and right to see the full figure.

	OFFICE DEPOT		
	Low	High	Match
High	0, 5,000	3,400, 3,400	3,400, 3,400
Match	2,500, 2,500	3,400, 3,400	3,400, 3,400

You may need to scroll left and right to see the full figure.

FIGURE 10.11 Ink Pricing Game with Price Matching

Suppose, then, that the basic structure of the game between the two firms can be illustrated as in Figure 10.10. If both firms advertise low prices, they split the available customer demand, and each earns $2,500. If both advertise high prices, they split a market with lower sales, but their markups end up being large enough to let them each earn $3,400. Finally, if they advertise different prices, then the one advertising a high price gets no customers and earns nothing, whereas the one advertising a low price earns $5,000.

The game illustrated in Figure 10.10 is clearly a prisoners' dilemma. Advertising and selling at a low price is the dominant strategy for each firm, although both would be better off if each advertised and sold at the high price. But, as mentioned earlier, each firm actually makes use of a third pricing strategy: a price-matching guarantee to its customers. How does the inclusion of such a policy alter the prisoners' dilemma that would otherwise exist between these two firms?

Consider the effects of allowing firms to choose among pricing low, pricing high, and price matching. The Match strategy entails advertising a high price, but promising to match any lower price advertised by a competitor. A firm using Match then benefits from advertising high if the rival firm does so also, but it does not suffer any harm from advertising a high price if the rival advertises a low price.

We can see this in the payoff structure for the new game, shown in Figure 10.11. In that table, we see that a combination of one firm playing Low while the other plays Match is equivalent to both playing Low, while a combination of one firm playing High while the other plays Match (or both playing Match) is equivalent to both playing High.

Using our standard tools for analyzing simultaneous-move games shows that High is weakly dominated by Match for both players, and that once High is eliminated, Low is weakly dominated by Match also. The resulting Nash equilibrium entails both firms using the Match strategy. In equilibrium, both firms earn $3,400—the profit level associated with both firms pricing high in the original game. The addition of the Match strategy has allowed the firms to emerge from the prisoners' dilemma that they faced when they had only the choice between two simple pricing strategies, Low or High.

How did this happen? The Match strategy acts as a penalty mechanism. By guaranteeing to match Office Depot's low price, Staples substantially reduces the benefit that Office Depot achieves by advertising a low price while Staples is advertising a high price. Promising to match Office Depot's low price hurts Staples, too, because Staples has to accept the lower profit associated with the low price. Thus, the price-matching guarantee is a method of penalizing both players whenever either one defects. This penalty scheme is just like that in the crime mafia example discussed in Section 4, except that this scheme—and the higher equilibrium prices that it supports—is observed in markets in virtually every city.

Actual empirical evidence of the effects of price-matching policies on prices is available but limited, and some research has found evidence of lower prices in markets with such policies.[17] However, more recent experimental evidence does support the collusive effect of price-matching policies.

This result should put all customers on alert.[18] Even though stores that match prices promote their policies in the name of competition, the ultimate outcome when all firms use such policies can be better for the firms than if there were no price matching at all, and so customers can be the ones who are hurt.

C. International Environmental Policy

Our final example of a real-world prisoners' dilemma pertains to international climate control agreements. In 2003, the American Geophysical Union stated that "humanity is the major influence on the global climate change observed over the past 50 years," but that "rapid societal responses can significantly lessen negative outcomes." Since then, scientists have only grown more certain that emissions of carbon dioxide and other greenhouse gases (GHGs) are causing atmospheric temperatures to rise to levels that threaten the well-being, and perhaps the very existence, of the human species on earth.[19] Except for a nuclear Armageddon, it is hard to think of a worse "global bad."

		THEM	
		Cut emissions	Don't cut
US	Cut emissions	−1, −1	−20, 0
	Don't cut	0, −20	−12, −12

FIGURE 10.12 Greenhouse Gas Emissions Game

The difficulty in achieving a global reduction in GHG emissions comes in part from the nature of the interaction among the nations of the world: It is a prisoners' dilemma. No individual country has any incentive to reduce its own emissions, knowing that if it does so alone, it will bear significant costs with little benefit to overall climate change reduction. And if other countries do reduce their emissions, the first country cannot be stopped from enjoying the benefits of the others' actions.

Consider the emissions reduction problem as a game played between two countries, Us and Them. Estimates generated by the British government's Office on Climate Change suggest that coordinated action may come at a cost of about 1% of GDP per nation, whereas coordinated inaction could cost each nation between 5% and 20% of its GDP, perhaps 12% on average.[20] By extension, the cost to one country of cutting emissions on its own may be at the high end of the inaction estimate (20%), but holding back and letting the other country cut emissions could entail virtually no cost at all. We can then summarize the situation between Us and Them using the game table in Figure 10.12, where payoffs represent changes in GDP for each country.

The game in Figure 10.12 is indeed a prisoners' dilemma. Both countries have a dominant strategy to refuse to cut their emissions. The single Nash equilibrium occurs when neither country cuts emissions, but they suffer as a group as a result of the ensuing climate change.

Some attempts have been made to resolve the dilemma. The two that have progressed farthest toward agreement and implementation are the Kyoto Protocol and the Paris Agreement. However, both have major shortcomings that are already apparent. We outline these agreements briefly here and also discuss some ideas and solutions proposed by economists.

The Kyoto Protocol was negotiated by the United Nations Framework Convention on Climate Change (UNFCCC) in 1997, and went into effect in 2005 to last until 2012. More than 170 countries signed on and more joined later, although the United States was noticeably absent from the list. The protocol was extended in December 2012 to last through 2020. Of 192 member countries, only 124 accepted the extension. Canada withdrew from the protocol in 2012, and others, including Belarus, Ukraine, and Kazakhstan, have stated their intention to withdraw. Thus the future of this agreement is unclear at best.

The Kyoto Protocol had some commendable features. First, it was based on the principle of "common but differentiated responsibilities." This meant that developed countries, which had historically contributed more to GHG accumulation, and at the same time had greater economic capability to combat climate change, should bear a greater share of the burden. Second, the protocol imposed binding targets for emissions reduction. Unfortunately, some countries did not take on the new targets in the post-2012 phase, and the protocol lacked any effective provisions to enforce participation.

Discussions in the UNFCCC on measures to be taken after 2020 led to the separate Paris Agreement of 2015, signed by 195 countries and ratified by 185. Its goal is to keep the global average temperature rise in the twenty-first century below 2° C, and preferably below 1.5° C. For this purpose, each member country is to regularly report on its GHG emission reduction target and on the actions it will take to achieve the target. Unfortunately, these targets and actions are purely voluntary, and despite many ambitious statements, especially from European countries, actual performance has been disappointing. Recent studies, summarized by the American Geophysical Union,[21] find that the commitments of the United States, European Union, and China would not achieve the 1.5° C goal, even if the rest of the world reduced its emissions to zero. President Trump has announced his intention to withdraw the United States from the Paris Agreement, and this action may start a cascade of other withdrawals or relaxation of targets. We saw in Section 5 that large players can serve an important leadership role to resolve prisoners' dilemmas and should be willing to bear a disproportionately large share of the burden for this purpose. In this instance, the opposite seems to be happening: The largest players are lagging with their targets and actions. That does not bode well for the outcome.

Of course, this prisoners' dilemma is not a one-period game, but a repeated one. That observation implies that there are ways to sustain a good outcome, as Michael Liebriech has suggested.[22] He argues that the repeated nature of this game makes it amenable to solution by way of contingent strategies, and that countries should use strategies that embody the four critical properties of TFT, as outlined by Axelrod and described in Section 6. Specifically, countries are encouraged to employ strategies that are "nice" (signing on to the protocol and beginning emissions reductions), "retaliatory" (employing mechanisms to punish those that do not do their part), "forgiving" (welcoming those newly accepting the protocol), and "clear" (specifying actions and reactions).

Liebriech assesses the actions of current players, including the European Union, the United States, and developing countries (as a group), and provides some suggestions for improvements. He explains that the European Union does well with nice, forgiving, and clear, but not with retaliatory, so other countries will do best to defect when interacting with the European Union. One solution would be for the European Union to institute carbon-related import taxes or another retaliatory policy for dealing with recalcitrant trade partners. The United States, in contrast, ranks high on retaliatory and forgiving, given its history of such behavior following the end of the Cold War. But it has not been nice or clear, at least on the national level (individual states may behave differently), giving other countries an incentive to retaliate against it quickly and painfully, if possible. The solution is for the United States to make a meaningful commitment to GHG emission reductions, a standard conclusion in most policy circles. Developing countries are described as not nice (negotiating no carbon limits for themselves), retaliatory, unclear, and quite unforgiving. A more beneficial strategy, argues Liebriech, would be for these countries—particularly China, India, and Brazil—to make clear their commitment to sharing in international efforts to reduce climate change; this approach would leave them less subject to retaliation and more likely to benefit from a global improvement in climatic outlook.

The general conclusion is that the process of international GHG emissions reduction fits the profile of a prisoners' dilemma game. But the future of the global climate should not be considered a lost cause simply because of this aspect of the nations' one-time interaction. Repeated play among the nations involved in the Kyoto Protocol negotiations makes the game amenable to solutions by way of contingent (nice, clear, and forgiving, but also retaliatory) strategies.

Endnotes

- Larry Conik, “Science Classics: The Bowerbird’s Dilemma,” *Discover*, October 1994. Return to reference 15
- The Staples policy applies to in-store purchases only, while Office Depot includes online purchases in its guarantee. Both stores give customers 14 days after purchase to find a lower price, and both include details of their policies on their Web sites. Similar policies exist in many industries, including that for credit cards, where “interest rate matching” has been observed. See Aaron S. Edlin, “Do Guaranteed-Low-Price Policies Guarantee High Prices, and Can Antitrust Rise to the Challenge?” *Harvard Law Review*, vol. 111, no. 2 (December 1997), pp. 529–75. Return to reference 16
- J. D. Hess and Eitan Gerstner present evidence of increased prices as a result of price-matching policies in “Price-Matching Policies: An Empirical Case,” *Managerial and Decision Economics*, vol. 12 (1991), pp. 305–15. Contrary evidence is provided by Arbatskaya, Hviid, and Shaffer, who find that the effect of matching policies is to lower prices; see Maria Arbatskaya, Morten Hviid, and Greg Shaffer, “Promises to Match or Beat the Competition: Evidence from Retail Tire Prices,” *Advances in Applied Microeconomics*, vol. 8, *Oligopoly* (New York: JAI Press, 1999), pp. 123–38. Return to reference 17
- See Subhasish Dugar, “Price-Matching Guarantees and Equilibrium Selection in a Homogeneous Product Market: An Experimental Study,” *Review of Industrial Organization*, vol. 30 (2007), pp. 107–19. Return to reference 18
- As NASA atmospheric scientist Kate Marvel explained during an expert-panel discussion in September 2018, “We are more sure that greenhouse gas is causing climate change than we are that smoking causes cancer.” See Abel Gustafson and Matthew Goldberg, “Even Americans Highly

Concerned about Climate Change Dramatically Underestimate the Scientific Consensus," Yale Program on Climate Change Communication, October 18, 2018. Return to reference 19

- See Nicholas Stern, *The Economics of Climate Change: The Stern Review* (Cambridge: Cambridge University Press, 2007). Return to reference 20
- See https://eos.org/scientific-press/new-studies-highlight-challenge-of-meeting-paris-agreement-climate-goals. Return to reference 21
- Michael Liebriech presents his analysis of the Kyoto Protocol as a repeated prisoners' dilemma in his paper "How to Save the Planet: Be Nice, Retaliatory, Forgiving and Clear," New Energy Finance White Paper, September 11, 2007, available for download from www.bnef.com/InsightDownload/7080/pdf/ (accessed August 1, 2014). Return to reference 22

SUMMARY

The prisoners’ dilemma is probably the most famous game of strategy. Each player has a dominant strategy (Defect), but the equilibrium outcome is worse for all players than when each uses her dominated strategy (Cooperate). The dilemma can be solved in some cases when the way the game is played changes, or when mechanisms exist that can change the payoff structure.

The best-known solution to the dilemma is *repeated play.* In a game with a finite number of periods, the value of future cooperation is eventually zero, and rollback yields an equilibrium with no cooperative behavior. With infinite play (or an uncertain end date), cooperation can be achieved with the use of an appropriate *trigger strategy* such as *tit-for-tat (TFT)* or the *grim strategy;* in either case, cooperation is possible only if the *present value* of cooperation exceeds the present value of defecting. More generally, the prospect of no future relationship or of a short-term relationship leads to decreased cooperation among players.

The dilemma can also be solved with a change in the order of moves in which the second mover is trustworthy and makes a promise, or with *punishment* mechanisms that alter the payoffs for players who defect from cooperation when their rivals are cooperating or when others are also defecting. Another solution mechanism, *leadership*, exists when a large or strong player’s loss from defecting is greater than the available gain from cooperative behavior on that player’s part.

Experimental evidence suggests that players often cooperate longer than theory might predict. Such behavior can be explained by incomplete knowledge of the game on the part of the players or by their views regarding the benefits of cooperation. Tit-for-tat has been observed to be a simple, nice, provocable, and forgiving strategy that performs very well on the average in repeated prisoners' dilemmas.

Prisoners' dilemmas arise in a variety of contexts. Specific examples from evolutionary biology, product pricing, and international environmental policy show how to explain and predict actual behavior by using the framework of the prisoners' dilemma.

KEY TERMS

compound interest (382)

contingent strategy (379)

discount factor (383)

effective rate of return (384)

grim strategy (379)

infinite horizon (381)

leadership (393)

penalty (390)

present value (PV) (381)

punishment (379)

repeated play (377)

tit-for-tat (TFT) (379)

trigger strategy (379)

Glossary

repeated play
: A situation where a one-time game is played repeatedly in successive periods. Thus, the complete game is mixed, with a sequence of simultaneous-move games.

contingent strategy
: In repeated play, a plan of action that depends on other players' actions in previous plays. (This is implicit in the definition of a strategy; the adjective "contingent" merely reminds and emphasizes.)

trigger strategy
: In a repeated game, this strategy cooperates until and unless a rival chooses to defect, and then switches to noncooperation for a specified period.

punishment
: We reserve this term for costs that can be inflicted on a player in the context of a repeated relationship (often involving termination of the relationship) to induce him to take actions that are in the joint interests of all players.

grim strategy
: A strategy of noncooperation forever in the future, if the opponent is found to have cheated even once. Used as a threat of punishment in an attempt to sustain cooperation.

tit-for-tat (TFT)
: In a repeated prisoners' dilemma, this is the strategy of [1] cooperating on the first play and [2] thereafter doing each period what the other player did the previous period.

present value (PV)
: The total payoff over time, calculated by summing the payoffs at different periods each multiplied by the

appropriate discount factor to make them all comparable with the initial period' s payoffs.

infinite horizon
: A repeated decision or game situation that has no definite end at a fixed finite time.

compound interest
: When an investment goes on for more than one period, compound interest entails calculating interest in any one period on the whole accumulation up to that point, including not only the principal initially invested but also the interest earned in all previous periods, which itself involves compounding over the period previous to that.

discount factor
: In a repeated game, the fraction by which the next period' s payoffs are multiplied to make them comparable with this period' s payoffs.

effective rate of return
: Rate of return corrected for the probability of noncontinuation of an investment to the next period.

penalty
: We reserve this term for one-time costs (such as fines) introduced into a game to induce the players to take actions that are in their joint interests.

leadership
: In a prisoners' dilemma with asymmetric players, this is a situation where a large player chooses to cooperate even though he knows that the smaller players will cheat.

SOLVED EXERCISES

1. “If a prisoners’ dilemma is repeated 100 times, and both players know how many repetitions to expect, they are sure to achieve their cooperative outcome.” True or false? Explain and give an example of a game that illustrates your answer.
2. Consider a two-player game between Child’s Play and Kid’s Korner, each of which produces and sells wooden swing sets for children. Each firm can set either a high or a low price for a standard two-swing, one-slide set. If they both set a high price, each receives profits of $64,000 per year. If one sets a low price and the other sets a high price, the low-price firm earns profits of $72,000 per year, while the high-price firm earns $20,000. If they both set a low price, each receives profits of $57,000.
   1. Verify that this game has a prisoners’ dilemma structure by looking at the ranking of the payoffs associated with the different strategy combinations (both cooperate, both defect, one defects, and so on). What are the Nash equilibrium strategies and payoffs in the simultaneous-move game if the players make price decisions only once?
   2. If the two firms decide to play this game for a fixed number of one-year periods—say, for four years—what will each firm’s total profits be at the end of the game? (Don’t discount.) Explain how you arrived at your answer.
   3. Suppose that the two firms play this game repeatedly forever. Let each of them use a grim strategy in which they both price high unless one of them defects, in which case they price low for the rest of the game. What is the one-time gain from defecting against an opponent playing such a strategy? How much

does each firm lose, in each future period, after it defects once? If $r = 0.25$ ($\delta = 0.8$), will it be worthwhile for them to cooperate? Find the range of values of r (or δ) for which this strategy is able to sustain cooperation between the two firms.

4. Suppose the firms play this game repeatedly year after year, with neither expecting any change in their interaction. If the world were to end after four years, without either firm having anticipated this event, what would each firm's total profits (not discounted) be at the end of the game? Compare your answer here with your answer in part (b). Explain why the two answers are different, if they are different, or why they are the same, if they are the same.
5. Suppose now that the firms know that there is a 10% probability that one of them will go bankrupt in any given year. If bankruptcy occurs, the repeated game between the two firms ends. Will this knowledge change the firms' actions when $r = 0.25$? What if the probability of a bankruptcy increases to 35% in any year?

3. A firm has two divisions, each of which has its own manager. These managers are paid according to their effort in promoting productivity in their divisions. The payment scheme is based on a comparison of their two outcomes. If both managers have expended "high effort," each earns \$150,000 a year. If both have expended "low effort," each earns "only" \$100,000 a year. But if one of the two managers shows high effort whereas the other shows low effort, the high-effort manager is paid \$150,000 plus a \$50,000 bonus, but the second (low-effort) manager gets a reduced salary (for subpar performance in comparison with her competition) of \$80,000. The managers make their effort decisions independently and without knowledge of the other manager's choice.

1. Assume that expending effort is costless to the managers and draw the payoff table for this game. Find the Nash equilibrium of the game and explain whether the game is a prisoners' dilemma.
2. Now suppose that expending high effort is costly to the managers (i.e., that it is a costly signal of quality). In particular, suppose that high effort costs an equivalent of $60,000 a year to a manager who chooses this effort level. Draw the game table for this new version of the game and find the Nash equilibrium. Explain whether the game is a prisoners' dilemma and how it has changed from the game in part (a).
3. If the cost of high effort is equivalent to $80,000 a year, how does the game change from that described in part (b)? What is the new equilibrium? Explain whether the game is a prisoners' dilemma and how it has changed from the games in parts (a) and (b).

4. You have to decide whether to invest $100 in a friend's enterprise, where in a year's time the money will increase to $130. You have agreed that your friend will then repay you $120, keeping $10 for himself. But instead he may choose to run away with the whole $130. Any of your money that you don't invest in your friend's venture you can invest elsewhere safely at the prevailing rate of interest r and get $100(1 + r)$ next year.
    1. Draw the game tree for this situation and show the rollback equilibrium. Next, suppose this game is played repeatedly for an infinite number of years. That is, each year you have the opportunity to invest another $100 in your friend's enterprise, and you agree to split the resulting $130 in the manner already described. From the second year onward, you get to make your decision of whether to invest with your friend in the light of whether he made the agreed-upon repayment the preceding year. The rate of interest between any two successive periods is r, the

same as the outside rate of interest, and the same for you and your friend.

2. For what values of r can there be an equilibrium outcome of the infinitely repeated game in which each period you invest with your friend and he repays you as agreed?
3. If the rate of interest is 10% per year, can there be an alternative profit-splitting agreement that is an equilibrium outcome of the infinitely repeated game, where each period you invest with your friend and he repays as agreed?

5. Recall the example from Exercise S3 in which two division managers' choices of high or low effort levels determine their salary payments. In part (b) of that exercise, the cost of exerting high effort is assumed to be $60,000 a year. Suppose now that the two managers play the game in part (b) of Exercise S3 repeatedly for many years. Such repetition allows scope for an unusual type of cooperation in which one is designated to choose high effort while the other chooses low effort. This cooperative agreement requires that the high-effort manager make a side payment to the low-effort manager so that their payoffs are identical.
   1. What size of side payment guarantees that the final payoffs for the two managers are identical? How much does each manager earn in a year in which the cooperative agreement is in place?
   2. Cooperation in this repeated game entails each manager's choosing her assigned effort level and the high-effort manager making the designated side payment. Defection entails refusing to make the side payment. Under what values of the rate of return can this agreement sustain cooperation in the managers' repeated game?
6. Consider the game of chicken in Chapter 4, with slightly more general payoffs (Figure 4.16 shows an example in which $k = 1$):

		DEAN	
		Swerve	Straight
JAMES	Swerve	0, 0	−1, k
	Straight	k, −1	−2, −2

Suppose this game is played every Saturday evening. If $k < 1$, the two players stand to benefit by cooperating to play (Swerve, Swerve) all the time, whereas if $k > 1$, they stand to benefit by cooperating so that one plays Swerve and the other plays Straight, taking turns to go Straight in alternate weeks. Can either type of cooperation be sustained?

7. Recall the example from Exercise S8 in Chapter 5, where South Korea and Japan compete in the market for production of VLCCs. As in parts (a) and (b) of that exercise, the cost of building the ships is \$30 (million) in each country, and the demand for ships is $P = 180 - Q$, where $Q = q_{\text{Korea}} + q_{\text{Japan}}$.
   1. Previously, we found the Nash equilibrium for the game. Now find the collusive outcome. What total quantity should be set by the two countries in order to maximize their joint profit?
   2. Suppose the two countries produce equal quantities of VLCCs, so that they earn equal shares of this collusive profit. How much profit would each country earn? Compare this profit with the amount they would earn in the Nash equilibrium.
   3. Now suppose the two countries are in a repeated relationship. Once per year, they choose production quantities, and each can observe the amount its rival produced in the previous year. They wish to cooperate to sustain the collusive profit levels you found in part (b). In any one year, one of them can defect from the agreement. If one of them holds the quantity

    at the agreed-upon level, what is the best defecting quantity for the other? What are the resulting profits?
    4. Write down a matrix that represents this game as a prisoners' dilemma.
    5. At what interest rates will collusion be sustainable when the two countries use grim (defect forever) strategies?
8. In the (imaginary) game show *Roll It!!*, two players decide at the end of the game whether to Roll or Steal, given a prize pot worth $\$X$ before them. If both players choose Steal, the game ends, and each gets $\$X/2$. If only one chooses Steal, the game ends, and the person who chose Steal gets $\$X$ while the person who chose Roll gets nothing. Finally, if both choose Roll, an enormous six-sided die is rolled; if it comes up 1, the game ends, and both players lose everything; if it comes up 2 through 6, that many thousands of dollars are added to the prize pot, and the game continues with another choice to Roll or Steal, now with the bigger prize pot.
    1. First suppose that Player 2 always chooses Roll, allowing Player 1 to steal the whole prize pot whenever she wants. Verify that, in order to maximize her expected winnings, player 1 should choose Roll whenever the prize pot is less than $20,000 and choose Steal whenever it exceeds $20,000. Hint: Compare Player 1's expected payoff when stealing a pot worth $\$X(+\$X)$ with her expected payoff when rolling *just one more time* and then stealing the pot, if possible.
    2. Suppose that the prize pot has grown to $20,000 or more. Verify that each player has a dominant strategy to choose Steal, and hence that the game will end with each player getting half of the prize pot.
    3. Suppose that the prize pot has grown to $18,000 and, as in part (b), that both players are certain to choose Steal once the prize pot reaches $20,000 or

more. Verify that the resulting game is a prisoners' dilemma.
4. Suppose that the game begins with a prize pot of $1,000. Does a rollback equilibrium exist in which both players decide to Roll at least one time?
5. **(Optional)** What is the highest prize pot the viewers of *Roll It!!* would ever see on the show, assuming that rollback equilibrium choices are always made by the players?

9. People who are kicked off eBay for committing fraud can get back on again by buying a new eBay username on the Aspkin Suspensions Forum (www.aspkin.com). For example, in April 2019, an "Aged, Verified, High-Limit USA eBay/PayPal Seller Account [with] 500/$25,000 Limits" could be had for about $200.[23] But what stops sellers on the Aspkin Forum from cheating the cheaters who come to them looking to get back onto eBay? For this question, assume that it costs $100 to create a new "ready to sell" eBay username, that buyers are willing to pay $300 for such usernames, that the Aspkin Forum is the only place in the world where such usernames can be safely traded, and that the going price is $200 per username. Assume that sellers discount future payoffs using an annual discount factor of $\delta = 2/3$
    1. Suppose that a seller in good standing on the Aspkin Forum will be able to sell one eBay username per year forever. What is the present value of the seller's total profits, if the seller never cheats and hence always remains in good standing?
    2. Suppose that the game of purchasing a new username works as follows: First, a buyer decides whether to pay $200; then, the seller decides whether to deliver (providing the login details for a ready-to-sell eBay username) or cheat the buyer (delivering nothing). Assume that any cheating seller will be detected and permanently barred from the Aspkin Forum. Draw the

game tree for this sequential-move game. What are the rollback equilibrium strategies and payoffs?

3. Imagine that eBay takes steps to make it less lucrative to commit fraud on its site. As a result, the going price for a fraudulent eBay username falls to $120. Verify that a seller with a discount factor of $\delta = 2/3$ now has an incentive to cheat.
4. In the context of part (c), what can the Aspkin Forum do so that sellers on the site no longer have an incentive to cheat buyers?

UNSOLVED EXERCISES

1. Two people, Baker and Cutler, play a game in which they choose and divide a prize. Baker decides how large the total prize should be; she can choose either $10 or $100. Cutler chooses how to divide the prize chosen by Baker; Cutler can choose either an equal division or a split where she gets 90% and Baker gets 10%. Draw the payoff table of the game and find its equilibria for each of the following situations:
   1. When the moves are simultaneous.
   2. When Baker moves first.
   3. When Cutler moves first.
   4. Is this game a prisoners' dilemma? Why or why not?
2. Sociologist Diego Gambetta begins his book *The Sicilian Mafia*[24] by quoting a cattle breeder he interviewed: "When the butcher comes to me to buy an animal, he knows that I want to cheat him [on quality]. But I know that he wants to cheat me [on payment]. So we need Peppe [the mafioso] to make us agree. And we both pay Peppe a percentage of the deal." Relate this situation to the various methods of resolving prisoners' dilemmas in this chapter. Consider the following questions in your answer: (i) Who has a repeated relationship with whom? (ii) Why are both parties in the trade willing to pay Peppe? (iii) What keeps Peppe honest—that is, what stops him from double-crossing a trader who has behaved honestly and extorting him for more payment? Related to this, what determines Peppe's percentage? (iv) Peppe can enforce honesty between the traders either by trashing a cheater's reputation, or by direct physical punishment. In which mode will Peppe's fee be higher?
3. Consider a small town that has a population of dedicated pizza eaters but is able to accommodate only two pizza shops, Donna's Deep Dish and Pierce's Pizza Pies. Each

seller has to choose a price for its pizza, but for simplicity, assume that only two prices are available: high and low. If a high price is set, the sellers can achieve a profit margin of $12 per pie; the low price yields a profit margin of $10 per pie. Each store has a loyal captive customer base that will buy 3,000 pies per week, no matter what price is charged by either store. There is also a floating demand of 4,000 pies per week. The people who buy these pies are price conscious and will go to the store with the lower price; if both stores charge the same price, this demand will be split equally between them.

1. Draw the game table for the pizza-pricing game, using each store’s profits per week (in thousands of dollars) as payoffs. Find the Nash equilibrium of this game and explain why it is a prisoners’ dilemma.
2. Now suppose that Donna’s Deep Dish has a much larger loyal clientele that guarantees it the sale of 11,000 (rather than 3,000) pies a week. Profit margins and the size of the floating demand remain the same. Draw the payoff table for this new version of the game and find the Nash equilibrium.
3. How does the existence of the larger loyal clientele for Donna’s Deep Dish help solve the pizza stores’ dilemma?

4. Six friends stop at a burger joint for lunch. There are two items on the menu: (i) a regular burger that costs $4 and (ii) a deluxe burger that costs $8. Each friend feels that eating a regular burger is worth $5 while eating a deluxe burger is worth $6.
    1. The friends have agreed to split equally the overall cost of the meal; if the total cost is T, then each friend will pay $T/6$. Verify that the resulting game is a prisoners’ dilemma in which each friend has a dominant strategy to order a deluxe burger. Hint: If D is the number of friends who order a deluxe burger,

then the total cost is $T(D) = 8D + 4(6 - D) = 24 + 4D$.

2. One day, the burger joint decides to raise the price of a deluxe burger to \$12. What is the unique Nash equilibrium in the resulting game? Are the friends better off when deluxe burgers cost \$12 or when they cost \$8?

5. A town council consists of three members who vote every year on their own salary increases. Two Yes votes are needed to pass the increase. Each member would like a higher salary but would like to vote against it herself because that looks good to the voters. Specifically, the payoffs of each outcome are as follows:

   Raise passes, own vote is No: 10

   Raise fails, own vote is No: 5

   Raise passes, own vote is Yes: 4

   Raise fails, own vote is Yes: 0

   Voting is simultaneous. Draw the (three-dimensional) payoff table, and show that in Nash equilibrium the raise fails unanimously. Examine how a repeated relationship among the members can secure them salary increases every year if (1) every member serves a three-year term, (2) every year in rotation, one of them is up for reelection, and (3) the townspeople have short memories, remembering only the members' votes on the salary increase motion of the current year and not those of past years.

6. Consider the following game, which comes from James Andreoni and Hal Varian at the University of Michigan.[25] A neutral referee runs the game. There are two players, Row and Column. The referee gives two cards to each: 2 and 7 to Row and 4 and 8 to Column. Who gets what cards is common knowledge. Then, playing simultaneously and

independently, each player is asked to hand over to the referee either his High card or his Low card. The referee hands out payoffs—which come from a central kitty, not from the players' pockets—that are measured in dollars and depend on the cards that he collects. If Row chooses his Low card, 2, then Row gets \$2; if he chooses his High card, 7, then Column gets \$7. If Column chooses his Low card, 4, then Column gets \$4; if he chooses his High card, 8, then Row gets \$8.

1. Show that the complete payoff table is as follows:

		COLUMN	
		Low	High
ROW	Low	2, 4	10, 0
	High	0, 11	8, 7

2. What is the Nash equilibrium? Verify that this game is a prisoners' dilemma.

Now suppose the game has the following stages. The referee hands out cards as before; who gets what cards is common knowledge. At stage 1, each player, out of his own pocket, can hand over a sum of money, which the referee is to hold in an escrow account. This amount can be zero but cannot be negative. When both players have made their stage 1 choices of sums to had over, these choices are publicly disclosed. Then, at stage 2, the two players make their choices of cards, again simultaneously and independently. The referee hands out payoffs from the central kitty in the same way as in the single-stage game before, but in addition, he disposes of the escrow account as follows: If Column chooses his High card, the referee hands over to Column the sum that Row put into the escrow account; if Column chooses his Low card, Row's sum reverts back to him. The disposition of the sum that Column deposited depends similarly on Row’ s card choice. All these rules are common knowledge.

1. (c) Find the rollback (subgame-perfect) equilibrium of this two-stage game. Does it resolve the prisoners’ dilemma? What is the role of the escrow account?

7. Glassworks and Clearsmooth compete in the local market for windshield repairs. The market size (total available profits) is $10 million per year. Each firm can choose whether to advertise on local television. If a firm chooses to advertise in a given year, it costs that firm $3 million. If one firm advertises and the other doesn’ t, then the former captures the whole market. If both firms advertise, they split the market 50:50. If both firms choose not to advertise, they also split the market 50:50.
   1. Suppose the two windshield-repair firms know they will compete for just one year. Draw the payoff matrix for this game. Find the Nash equilibrium strategies.
   2. Suppose the firms play this game for five years in a row, and they know that at the end of five years, both firms plan to go out of business. What is the subgame-perfect equilibrium for this five-period game? Explain.
   3. What would be a tit-for-tat strategy in the game described in part (b)?
   4. Suppose the firms play this game repeatedly forever, and suppose that future profits are discounted with an interest rate of 20% per year. Can you find a subgame-perfect equilibrium that involves higher annual payoffs than the equilibrium in part (b)? If so, explain what strategies are involved. If not, explain why not.
8. Consider the pizza stores introduced in Exercise U3, Donna’ s Deep Dish and Pierce’ s Pizza Pies. Suppose that they are not constrained to choose from only two possible

prices, but that each can choose a specific price to maximize profits. Suppose further that it costs $3 to make each pizza (for both stores), and that experience or market surveys have shown that the relation between sales (Q) and price (P) for each firm is as follows:

$Q_{\text{Pierce}} = 12 - P_{\text{Pierce}} + 0.5\ P_{\text{Donna}}.$

Then profits per week (Y, in thousands of dollars) for each firm are

$Y_{\text{Pierce}} = (P_{\text{Pierce}} - 3)\,Q_{\text{Pierce}} = (P_{\text{Pierce}} - 3)(12 - P_{\text{Pierce}} + 0.5P_{\text{Donna}}),$

$Y_{\text{Donna}} = (P_{\text{Donna}} - 3)\,Q_{\text{Donna}} = (P_{\text{Donna}} - 3)(12 - P_{\text{Donna}} + 0.5P_{\text{Pierce}}).$

1. Use these profit functions to determine each firm's best-response rule, as in Chapter 5, and use these best-response rules to find the Nash equilibrium of this pricing game. What prices do the firms choose in equilibrium? How much profit per week does each firm earn?
2. If the firms work together and choose a joint best price, P, then the profit of each will be

   $Y_{\text{Donna}} = Y_{\text{Pierce}} = (P - 3)(12 - P + 0.5P) = (P - 3)(12 - 0.5P).$

   What price do they choose to maximize joint profits?

3. Suppose the two stores are in a repeated relationship, trying to sustain the joint profit-maximizing prices calculated in part (b). They print new menus each month and thereby commit themselves to prices for the whole month. In any one month, one of them can defect from the agreement. If one of them holds the price at the agreed-upon level, what is the

best defecting price for the other? What are its resulting profits? For what interest rates will their collusion be sustainable if both are using the grim strategy?

9. Now we extend the analysis in Exercise S7 to allow for defecting in a collusive triopoly. Exercise S9 in Chapter 5 finds the Nash equilibrium outcome of a VLCC triopoly of Korea, Japan, and China.
    1. Now find the collusive outcome of the triopoly. That is, what total quantity should be set by the three countries collectively in order to maximize their joint profit?
    2. Assume that under the collusive outcome found in part (a), the three countries produce equal quantities of VLCCs, so that each earns an equal share of the collusive profit. How much profit would each country earn? Compare this profit with the amount each earns in the Nash equilibrium outcome.
    3. Now suppose the three countries are in a repeated relationship. Once per year, they choose production quantities, and each can observe the quantity its rivals produced in the previous year. They wish to cooperate to sustain the collusive profit levels found in part (b). In any one year, one of them can defect from the agreement. If the other two countries are expected to produce their share of the collusive outcome found in parts (a) and (b), what is the best defecting quantity for the third to produce? What is the resulting profit for a defecting country when it produces the optimal defecting quantity while the other two produce their collusive quantities?
    4. Of course, the year after one country defects, both of its rivals will also defect. They will all find themselves back at the Nash equilibrium outcome (permanently, if they use the grim strategy). How much does the defecting country stand to gain in one year of defecting from the collusive outcome? How

much will the defecting country then lose in every subsequent year from earning the Nash equilibrium profit instead of the collusive profit?

5. For what interest rates will collusion be sustainable if the three countries are using the grim strategy? Is this set of interest rates larger or smaller than that found in the duopoly case discussed in Exercise S7, part (e)? Why?

Endnotes

- See https://www.aspkin.com/forums/ebay-accounts-sale/77998-aged-verified-high-limit-usa-ebay-paypal-seller-accounts-sale-delivered-fast.xhtml(accessed April 18, 2019). Return to reference 23
- Diego Gambetta, *The Sicilian Mafia* (Cambridge, Mass.: Harvard University Press, 1993), p.15. Return to reference 24
- James Andreoni and Hal Varian, “Preplay Contracting in the Prisoners’ Dilemma,” *Proceedings of the National Academy of Sciences*, vol. 96, no. 19 (September 14, 1999), pp. 10933–38. Return to reference 25

■ Appendix: Infinite Sums

The computation of present values requires us to determine the current value of a sum of money that is paid to us in the future. As we saw in Section 2, the present value of a sum of money—say, x—that is paid to us n months from now is just $x/(1 + r)^n$, where r is the appropriate monthly rate of return. But the present value of a sum of money that is paid to us next month and every following month in the foreseeable future is more complicated to determine. In that case, the payments continue infinitely, so there is no defined end to the sum of present values that we need to compute. To compute the present value of this flow of payments requires some knowledge of the mathematics of the summation of infinite series.

Consider a player who stands to gain \$36 this month from defecting in a prisoners' dilemma, but who will then lose \$36 every month in the future as a result of her choice to continue defecting while her opponent (using the tit-for-tat, or TFT, strategy) punishes her. In the first of the future months—the first for which there is a loss and the first for which values need to be discounted—the present value of her loss is $36/(1 + r)$; in the second future month, the present value of the loss is $36/(1 + r)^2$; in the third future month, the present value of the loss is $36/(1 + r)^3$. That is, in each of the n future months that she incurs a loss from defecting, that loss equals $36/(1 + r)^n$.

We could write out the total present value of all of her future losses as a large sum with an infinite number of components,

$$PV = \frac{36}{1+r} + \frac{36}{(1+r)^2} + \frac{36}{(1+r)^3} + \frac{36}{(1+r)^4} + \frac{36}{(1+r)^5} + \frac{36}{(1+r)^6} + \cdots,$$

or we could use summation notation as a shorthand device and instead write

$$\mathrm{PV} = \sum_{n=1}^{\infty} \frac{36}{(1+r)^n}.$$

This expression, which is equivalent to the preceding one, is read as "the sum, from n equals 1 to n equals infinity, of 36 over $(1 + r)$ to the nth power." Because 36 is a common factor—it appears in each term of the sum—it can be pulled out to the front of the expression. Thus, we can write the same present value as

$$\mathrm{PV} = 36 \times \sum_{n=1}^{\infty} \frac{1}{(1+r)^n}.$$

We now need to determine the value of the sum within this present-value expression to calculate the actual present value. To do so, we simplify our notation by using the *discount factor* δ in place of $1/(1 + r)$. Then the sum that we are interested in evaluating is

$$\sum_{n=1}^{\infty} \delta^n.$$

It is important to note here that $\delta = 1/(1 + r) < 1$ because r is strictly positive.

An expert on infinite sums would tell you, after inspecting this last sum, that it converges to the finite value $\delta/(1 - \delta)$.[26] Convergence is guaranteed because increasingly large powers of a number less than 1, δ in this case, become smaller and smaller, approaching zero as n approaches infinity. The later terms in our present value, then, decrease in size until they get sufficiently small that the series approaches (but technically never exactly reaches) the particular value of the sum. Although a good deal of more sophisticated mathematics is required to deduce that the convergent value of the sum is $\delta/(1 - \delta)$, proving that this is the correct answer is relatively straightforward.

We use a simple trick to prove our claim. Consider the sum of the first m terms of the series, and denote it by S_m. Thus

$$S_m = \sum_{n=1}^{\infty} \delta^n = \delta + \delta^2 + \delta^3 + \cdots + \delta^{m-1} + \delta^m.$$

Now we multiply this sum by $(1 - \delta)$ to get

$$\begin{aligned}(1-\delta)S_m &= \delta + \delta^2 + \delta^3 + \cdots + \delta^{m-1} + \delta^m \\ &\quad - \delta^2 - \delta^3 - \delta^4 - \cdots - \delta^m - \delta^{m-1} \\ &= \delta - \delta^{m-1}.\end{aligned}$$

Dividing both sides by $(1 - \delta)$, we have

$$S_m = \frac{\delta - \delta^{m+1}}{1 - \delta}.$$

Finally, we take the limit of this sum as m approaches infinity to evaluate our original infinite sum. As m goes to infinity, the value of δ^{m+1} goes to zero because very large and increasing powers of a number less than 1 get increasingly small, but stay nonnegative. Thus, as m goes to infinity, the right-hand side of the preceding equation goes to $\delta/(1 - \delta)$, which is therefore the limit of S_m as m approaches infinity. This completes the proof.

We need only convert back into r to be able to use our answer in the calculation of present values in our prisoners' dilemma games. Because $\delta = 1/(1 + r)$, it follows that

$$\frac{\delta}{1-\delta} = \frac{1/(1+r)}{r/(1+r)} = \frac{1}{r}.$$

The present value of an infinite stream of $36s earned each month, starting next month, is then

$$36 \times \sum_{n=1}^{\infty} \frac{1}{(1+r)^n} = \frac{36}{r}.$$

This is the value that we use in Section 2 to determine whether a player should defect forever. Notice that incorporating a probability of continuation, $p \leqslant 1$, into the discounting calculations changes nothing in the summation procedure used here. We could easily substitute R for r in the preceding calculations, and $p\delta$ for the discount factor, δ.

Remember that you need to find present values only for losses (or gains) incurred (or accrued) *in the future*. The present value of $36 lost today is just $36. So if you wanted the present value of a stream of losses, all of them $36, that begins *today*, you would take the $36 lost today and add it to the present value of the stream of losses in the future. We have just calculated that present value as $36/r$. Thus, the present value of the stream of lost $36s, including the $36 lost today, would be $36 + 36/r$, or $36[(r + 1)/r]$, which equals $36/(1 - \delta)$. Similarly, if you wanted to look at a player's stream of profits under a particular contingent strategy in a prisoners' dilemma, you would not discount the profit amount earned in the very first period; you would discount only those profit figures that represent money earned in future periods.

Endnotes

- An infinite series *con*verges if the sum of the values in the series approaches a specific value, getting closer and closer to that value as additional components of the series are included in the sum. The series *di*verges if the sum of the values in the series gets increasingly larger (more negative) with each addition to the sum. Convergence requires that the components of the series get progressively smaller. Return to reference 26

11■ Collective-Action Games

THE GAMES AND STRATEGIC SITUATIONS considered in the preceding chapters have usually included only two or three players interacting with each other. Such games are common in our own academic, business, political, and personal lives and so are important to understand and analyze. But many social, economic, and political interactions are strategic situations in which numerous players participate at the same time. Strategies for career paths, investment plans, rush-hour commuting routes, and even studying have associated benefits and costs that depend on the actions of many other people. If you have been in any of these situations, you probably thought something was wrong—too many students, investors, and commuters crowding just where you wanted to be, for example. If you have tried to organize fellow students or your community in some worthy cause, you probably faced frustration of the opposite kind—too few willing volunteers. In other words, multiple-person games in a society often seem to produce outcomes that are not deemed satisfactory by many, or even all, of the people in that society. In this chapter, we examine such games from the perspective of the theory that we have already developed. We present an understanding of what goes wrong in such situations and what can be done about it.

In their most general form, such many-player games concern problems of collective action. In these cases, the aims of the whole society, or *collective*, are best served if its members take some particular action or actions, but those actions are not in the best private interests of those individual members. In other words, the socially optimal outcome is not automatically achievable as the Nash equilibrium of the game. Therefore, we must examine how the game can be modified to lead to the optimal outcome, or at least to improve on an unsatisfactory Nash equilibrium. To do so, we must first understand the nature of such games. We find that they come in three forms, all of them familiar to you by now: prisoners' dilemmas, games of chicken, and assurance games. Although our main focus in this chapter is on situations where numerous individuals play such games at the same time, we build on familiar ground by beginning with games between just two players.

Glossary

collective action
: A problem of achieving an outcome that is best for society as a whole, when the interests of some or all individuals will lead them to a different outcome as the equilibrium of a noncooperative game.

1 COLLECTIVE-ACTION GAMES WITH TWO PLAYERS

Imagine that you are a farmer. A neighboring farmer and you can both benefit by constructing an irrigation and flood-control project. The two of you can join together to undertake this project, or one of you can do so on your own. However, after the project has been constructed, both of you will automatically benefit from it. Therefore, each is tempted to leave the work to the other. That is the essence of your strategic interaction and the difficulty of securing collective action.

In Chapter 4, we encountered a game of this kind: Three neighbors were each deciding whether to contribute to a street garden that all of them would enjoy. That game became a prisoners' dilemma in which all three shirked. Our analysis here will include an examination of a broader range of possible payoff structures. Also, in the street-garden game, we rated the possible outcomes on a scale of 1 to 6; when we describe more general collective-action games, we will have to consider more general forms of benefits and costs for each player.

Our irrigation project has two important characteristics. First, its benefits are nonexcludable: A person who has not contributed to paying for it cannot be prevented from enjoying the benefits. Second, its benefits are nonrival: Any one person's benefits are not diminished by the mere fact that someone else is also getting those benefits. Economists call such a project a pure public good; national defense is often given as an example. In contrast, a pure *private* good is fully excludable and rival: Nonpayers can be excluded from its benefits, and if one person gets those benefits, no one else does. A loaf of bread is a good example of a pure private good. Most goods fall somewhere on the two-dimensional spectrum of varying degrees of excludability and rivalness. We will not go any deeper into this taxonomy, but we mention it to help you relate our discussion to what you may encounter in other courses and books.[1]

A. Collective Action as a Prisoners' Dilemma

The costs and the benefits associated with building the irrigation project depend, as do those associated with all collective actions, on which players participate. In turn, the relative sizes of the costs and benefits determine the structure of the game that is played. Suppose either of you acting alone could complete the project in 7 weeks, whereas if the two of you acted together, it would take only 4 weeks of time from each. The two-person project is also of better quality; each of you gets benefits worth 6 weeks of work from a one-person project (whether constructed by you or by your neighbor) and 8 weeks' worth of benefits from a two-person project.

More generally, we can write benefits and costs, and therefore player payoffs, as functions of the number of players participating. So the cost to you of choosing to build the project depends on whether you build it alone or with help. Cost can be written as $C(n)$, where cost, C, depends on the number, n, of players participating in the project. Then $C(1)$ would be the cost to you of building the project alone and $C(2)$ would be the cost *to you* of building the project with your neighbor. Here, $C(1) = 7$ and $C(2) = 4$. Similarly, benefits, B, from the completed project may vary depending on how many players (n) participate in its completion. In our example, $B(1) = 6$ and $B(2) = 8$. Note that these benefits are the same for each of you, regardless of participation, due to the public-good nature of this particular project.

In this game, each of you has to decide whether to work toward the construction of the project or not—that is, to shirk. (Presumably, there is a short window of time in which the work must be done, and you could pretend to be called away on some very important family matter at the last minute, as could your neighbor.) Figure 11.1 shows the payoffs for this game, measured in units equivalent to the value of a week of work. Payoffs are determined by the difference between the cost and the benefit associated with each action. So the payoff function for choosing Build can be written as $P(n) = B(n) - C(n)$, with payoff for participating, P, depending on the number of participants, n; $n = 1$ if you build alone and $n = 2$ if your neighbor also chooses Build. The payoff function for choosing Not is $S(n) = B(n)$, with payoff for shirking, S, also depending on the number of participants n; with $n = 1$ if your neighbor chooses Build. In this case, you incur no cost because you do not participate in the project.

		NEIGHBOR	
		Build	Not
YOU	Build	4, 4	−1, 6
	Not	6, −1	0, 0

FIGURE 11.1 Collective Action as a Prisoners' Dilemma: Version I

		NEIGHBOR	
		Build	Not
YOU	Build	2.3, 2.3	−1, 6
	Not	6, −1	0, 0

FIGURE 11.2 Collective Action as a Prisoners' Dilemma: Version II

Given the payoff structure in Figure 11.1, your best response if your neighbor does not participate is not to participate either: Your benefit from completing the project by yourself

(6) is less than your cost (7), for a net payoff of −1, whereas you can get 0 by not participating. Similarly, if your neighbor does participate, then you can reap the benefit (6) from his work at no cost to yourself, which is better for you than working yourself to get the larger benefit of the two-person project (8) while incurring the cost of the work (4), for a net payoff of 4. The key feature of the game is that it is better for you not to participate no matter what your neighbor does; the same logic holds for him. (In this case, each farmer is said to be a free rider on his neighbor's effort if he lets the other do all the work and then reaps the benefits all the same.) Thus, not building is the dominant strategy for each of you. But both of you would be better off if you were to work together to build (payoff 4) than if neither of you builds (payoff 0). Therefore, the game is a prisoners' dilemma.

We see in this prisoners' dilemma one of the main difficulties that arises in games of collective action. Individually optimal choices—in this case, a farmer choosing not to build regardless of what the other farmer chooses—may not be optimal from the perspective of society as a whole, even if the society is made up of just two farmers. The social optimum in a collective-action game is achieved when the sum total of the players' payoffs is maximized; in this prisoners' dilemma, the social optimum is the (Build, Build) outcome. Nash equilibrium behavior by the players does not consistently bring about the socially optimal outcome, however. Hence, the study of collective-action games has focused on methods to improve on observed (generally Nash) equilibrium behavior to move outcomes toward the socially best ones. As we will see, the divergence between Nash equilibrium and socially optimal outcomes appears in every version of collective-action games.

Now consider what the game would look like if the numbers were to change slightly. Suppose the two-person project yields benefits that are not much better than those from the one-person project: 6.3 weeks' worth of work to each farmer. Then each of you gets a payoff of 6.3 − 4 = 2.3 when both of you build. The resulting payoff table is shown in Figure 11.2. The game is still a prisoners' dilemma and still leads to the equilibrium (Not, Not). However, when both of you build, the total payoff for both of you is only 4.6. The social optimum occurs when one of you builds and the other does not, in which case the total payoff is 6 + (−1) = 5. There are two possible ways to get this outcome: You build and your neighbor shirks, or your neighbor builds and you shirk. Achieving the social optimum in this case then poses a new problem: Who should build and suffer the payoff of −1 while the other is allowed to be a free rider and enjoy the payoff of 6?

		NEIGHBOR	
		Build	Not
YOU	Build	5, 5	2, 6
	Not	6, 2	0, 0

FIGURE 11.3 Collective Action as Chicken: Version I

B. Collective Action as Chicken

Yet another variation in the numbers of the original prisoners' dilemma game of Figure 11.1 changes the nature of the game. Suppose the cost of the work is reduced so that it becomes better for you to build your own project if your neighbor does not. Specifically, suppose the one-person project requires 4 weeks of work, so $C(1) = 4$, and the two-person project takes 3 weeks from each, so $C(2) = 3$ (to each); the benefits are the same as before. Figure 11.3 shows the payoff matrix resulting from these changes. Now your best response is to shirk when your neighbor works and to work when he shirks. In form, this game is just like a game of chicken, where shirking is the Straight or Tough strategy and working is the Swerve or Weak strategy.

If the outcome of this game is one of its pure-strategy equilibria, your and the neighbor's payoffs sum to 8; this total is less than 10, the total of the payoffs that you could get if both of you build. That is, neither of the Nash equilibria provides as much benefit to the pair of you as a whole as that of your joint optimum where both of you choose to build. If the outcome of the chicken game is its mixed-strategy equilibrium, both of you will fare even worse than in either of the pure-strategy equilibria: The two expected payoffs will add up to something less than 8 (4, to be precise).

The collective-action game of chicken has another possible structure if we make some additional changes to the benefits associated with the project. As with version II of the prisoners' dilemma, suppose the two-person project is not much better than the one-person project. Then each farmer's benefit from the two-person project, $B(2)$, is only 6.3, whereas each still gets a benefit of $B(1) = 6$ from the one-person project. We ask you to practice your skill by constructing the payoff table for this game. You will find that it is still a game of chicken—call it chicken II. It still has two pure-strategy Nash equilibria, in each of which only one farmer builds, but the sum of the payoffs when both build is only 6.6, whereas the sum when only one farmer builds is 8. The social optimum is for only one farmer to build, but each farmer prefers the equilibrium in which the other builds. This may lead to a new dynamic game in which each waits for the other to build. Or the original game might yield its mixed-strategy equilibrium, with its low expected payoffs.

		NEIGHBOR	
		Build	Not
YOU	Build	4, 4	−4, 3
	Not	3, −4	0, 0

FIGURE 11.4 Collective Action as an Assurance Game

C. Collective Action as Assurance

Finally, let us change the payoffs for version I of the prisoners' dilemma in a different way altogether, leaving the benefits of the two-person project and the costs of building as originally set out, but reducing the benefit of a one-person project to $B(1) = 3$. This change reduces your benefit as a free rider so much that now if your neighbor chooses Build, your best response is Build. Figure 11.4 shows the payoff table for this version of the game. It is now an assurance game with two pure-strategy equilibria: one where both of you participate and the other where neither of you does.

As in the chicken II version of the game, the socially optimal outcome here is one of the two Nash equilibria. But there is a difference. In chicken II, the two players differ in their preferences between the two equilibria, either of which achieves the social optimum. In the assurance game, both of them prefer the same equilibrium, and that is the sole socially optimal outcome. Therefore, achieving the social optimum should be easier in the assurance game than in chicken.

D. Collective Inaction

Many games of collective action have payoff structures that differ somewhat from those in our irrigation project example. Our farmers find themselves in a situation in which the social optimum entails that at least one, if not both, of them participates in the project. Thus the game is one of collective *action*. Other multiplayer games might better be called games of collective *inaction*. In such games, society as a whole prefers that some or all of the individual players do *not* participate or do *not* act. Examples of this type of interaction include choices among rush-hour commuting routes, investment plans, or fishing grounds.

All of these games have the attribute that players must decide whether to take advantage of some common resource, be it a freeway, a high-yielding stock fund, or an abundantly stocked pond. These “collective inaction” games are better known as *common-resource* games; the total payoff to all players reaches its maximum when players refrain from overusing the common resource. The difficulty associated with not being able to reach the social optimum in such games is known as the “tragedy of the commons,” a phrase coined by Garrett Hardin in his paper of the same name.[2]

We supposed that the irrigation project yielded equal benefits to both farmers. But what if the outcome of both farmers’ building was that the project used so much water that the farms had too little water for their livestock? Then each player’s payoff could be negative when both choose Build, lower than when both choose Not. This would be yet another variant of the prisoners’ dilemma we encountered in Section 1.A, in which the socially optimal outcome entails neither farmer’s building even though each one still has an individual incentive to do so. Or suppose that one farmer’s activity causes harm to the other, as would happen if the only way to prevent one farm from being flooded is to divert the water to the other. Then each player's payoffs could be negative if his neighbor chooses Build. Thus, another variant of chicken could also arise. In this variant, each farmer wants to build when the other does not, whereas it would be collectively better if neither of them did.

Just as the problems pointed out in these examples of both collective action and collective inaction are familiar, the various ways of tackling these problems also follow the general principles discussed in earlier chapters. Before turning to solutions, let us see how the problems manifest themselves in the more realistic setting where several players interact simultaneously in such games.

Endnotes

- Public goods are studied in more detail in textbooks on *public economics*, such as Jonathan Gruber, *Public Finance and Public Policy*, 5th ed. (New York: Worth, 2015); Harvey Rosen and Ted Gayer, *Public Finance*, 10th ed. (Chicago: Irwin/McGraw-Hill, 2014); and Joseph Stiglitz and Jay Rosengard, *Economics of the Public Sector*, 4th ed. (New York: W. W. Norton, 2015). Return to reference 1
- Garrett Hardin, "The Tragedy of the Commons," *Science*, vol. 162 (1968), pp. 1243–48. Return to reference 2

Glossary

nonexcludable
: Benefits that are available to each individual, regardless of whether he has paid the costs that are necessary to secure the benefits.

nonrival
: Benefits whose enjoyment by one person does not detract anything from another person’s enjoyment of the same benefits.

pure public good
: A good or facility that benefits all members of a group, when these benefits cannot be excluded from a member who has not contributed efforts or money to the provision of the good, and the enjoyment of the benefits by one person does not significantly detract from their simultaneous enjoyment by others.

free rider
: A player in a collective-action game who intends to benefit from the positive externality generated by others’ efforts without contributing any effort of his own.

social optimum
: In a collective-action game where payoffs of different players can be meaningfully added together, the social optimum is achieved when the sum total of the players’ payoffs is maximized.

2 COLLECTIVE-ACTION PROBLEMS IN LARGE GROUPS

In this section, we extend our irrigation project example to a situation in which a population of N farmers must each decide whether to participate. Here we make use of the notation we introduced above, with $C(n)$ representing the cost each participant incurs when n of the N total farmers have chosen to participate. Similarly, the benefit to each farmer, regardless of participation, is $B(n)$. Each participant's payoff function is $P(n) = B(n) - C(n)$, whereas the payoff function for each nonparticipant, or shirker, is $S(n) = B(n)$.

Suppose you are contemplating whether to participate or to shirk. Your decision will depend on what the other $(N - 1)$ farmers in the population are doing. In general, you will have to make your decision when the other $(N - 1)$ players consist of n participants and $(N - 1 - n)$ shirkers. If you decide to shirk, the number of participants in the project is still n, so you get a payoff of $S(n)$. If you decide to participate, the number of participants becomes $n + 1$, so you get $P(n + 1)$. Therefore, your final decision depends on the comparison of these two payoffs; you will participate if $P(n + 1) > S(n)$, and you will shirk if $P(n + 1) < S(n)$. This comparison holds true for every version of the collective-action game analyzed in Section 1; the differences in player behavior in the different versions arise because the changes in the payoff structure alter the values of $P(n + 1)$ and $S(n)$.

		NEIGHBOR	
		Build	Not
YOU	Build	$P(2)$, $P(2)$	$P(1)$, $S(1)$
	Not	$S(1)$, $P(1)$	$S(0)$, $S(0)$

FIGURE 11.5 General Form of a Two-Person Collective-Action Game

We can relate the two-player examples in Section 1 to this more general framework. If there are just two players, then $P(2)$ is the payoff to one from building when the other also builds, $S(1)$ is the payoff to one from shirking when the other builds, and so on. Therefore, we can generalize the payoff tables of Figures 11.1 through 11.4 into an algebraic form. This general payoff structure is shown in Figure 11.5.

The game illustrated in Figure 11.5 is a prisoners' dilemma if the inequalities

$P(2) < S(1)$, $P(1) < S(0)$, $P(2) > S(0)$

all hold at the same time. The first says that the best response to Build is Not, the second says that the best response to Not also is Not, and the third says that (Build, Build) is jointly preferred to (Not, Not). The game is equivalent to version I of the prisoners' dilemma if $2P(2) > P(1) + S(1)$, so that the total payoff is higher when both build than when only one builds. You can establish similar inequalities among these payoffs that yield the other types of games in Section 1.

Return now to the multiplayer version of the game with a general n. Given the payoff functions for the two actions, $P(n + 1)$ and $S(n)$, we can use graphs to help us determine which type of game we have encountered and find its Nash equilibrium. We can also then compare the Nash equilibrium with the game's socially optimal outcome.

A. Multiplayer Prisoners' Dilemma

Take a specific version of our irrigation project example in which an entire village of 100 farmers must decide which action to take. Suppose that the project raises the productivity of each farmer's land in proportion to the size of the project; specifically, suppose the benefit to each farmer when n farmers work on the project is $P(n) = 2n$. Suppose also that if you are not working on the project, you can enjoy this benefit and use your extra four weeks of time in some other occupation, so $S(n) = 2n + 4$. Remember that your decision about whether to participate in the project depends on the relative magnitudes of $P(n + 1) = 2(n + 1)$ and $S(n) = 2n + 4$. We draw the two separate graphs of these functions for an individual farmer in Figure 11.6, showing n over its full range from 0 to $(N - 1)$ along the horizontal axis and the payoff to the farmer along the vertical axis. If there are currently very few participants (thus mostly shirkers), your choice will depend on the relative locations of $P(n + 1)$ and $S(n)$ on the left end of the graph. Similarly, if there are already many participants, your choice will depend on the relative locations of $P(n + 1)$ and $S(n)$ on the right end of the graph.

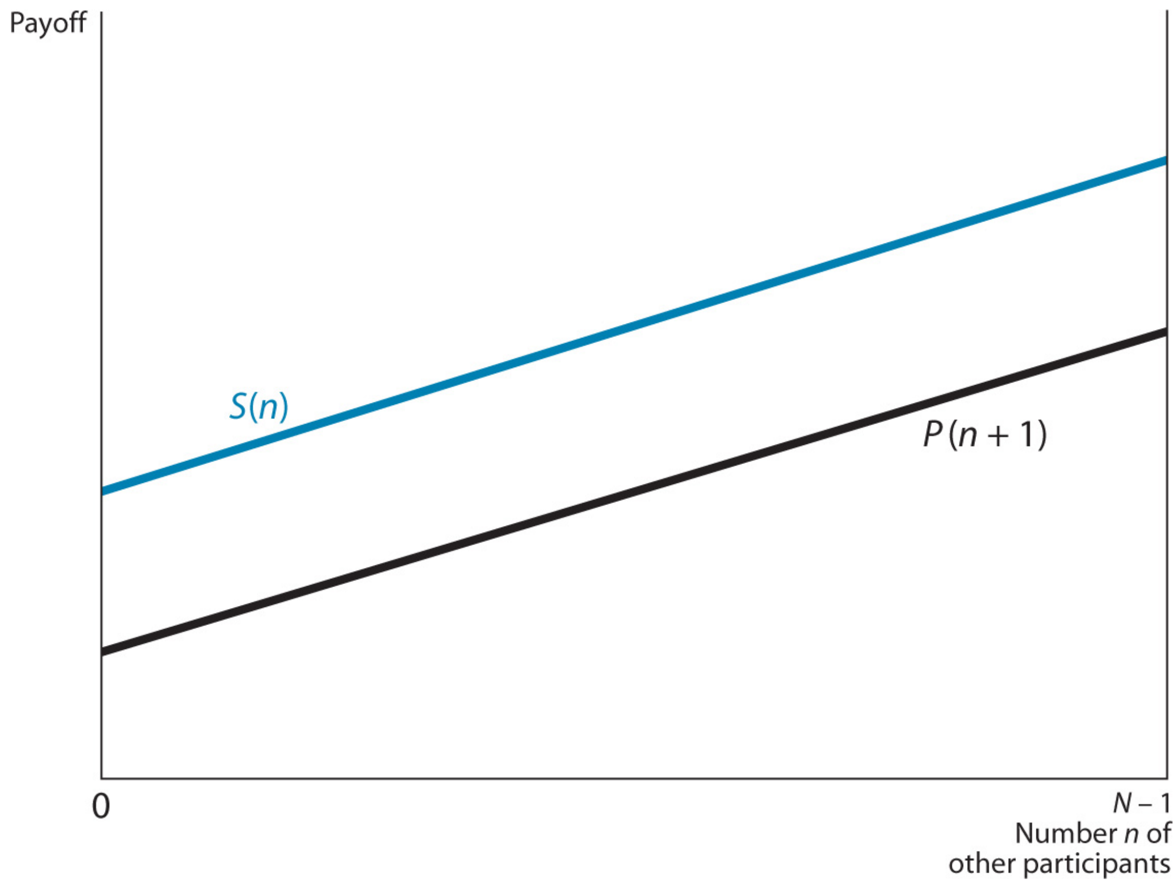


FIGURE 11.6 Multiplayer Prisoners' Dilemma Payoff Graph

Because n can have only integer values, each function $P(n + 1)$ and $S(n)$ technically consists only of a discrete set of points, rather than a continuous set as implied by our smooth curves (which in this instance happen to be straight lines). But when N is large, the discrete points are sufficiently close together that we can connect the successive points and show each payoff function as a continuous curve. We use linear $P(n + 1)$ and $S(n)$ functions in this section to bring out the basic considerations involved in collective-action games in the simplest possible way; we will discuss more complicated possibilities later.

Recall that you determine your choice of action by considering the number of current participants in the project, n, and the payoffs associated with each action at that n. Figure 11.6 illustrates a case in which the curve $S(n)$ lies entirely above the curve $P(n + 1)$. Therefore, no matter how many others participate (that is, no matter how large n gets), your payoff is higher if you shirk than if you participate; shirking is your dominant strategy. These payoffs are identical for all players, so everyone has a dominant strategy to shirk. Therefore, the Nash equilibrium of the game entails everyone shirking, and the project is not built.

Note that both curves rise as n increases. Thus, for each action you can take, you are better off if more of the others participate in the project. And the left intercept of the $S(n)$ curve is below the right intercept of the $P(n + 1)$ curve, so $S(0) = 4 < P(N) = 102$. This inequality says that if everyone, including you, shirks, your payoff is less than if everyone, including you, participates. Everyone would be better off than they are in the Nash equilibrium of the game if the outcome in which everyone participates could be sustained. This makes the game a prisoners' dilemma.

How does the Nash equilibrium found using the curves in Figure 11.6 compare with the social optimum of this game? To answer this question, we need a way to describe the total social payoff at each value of n; we do that by using the payoff functions $P(n)$ and $S(n)$ to construct a third function $T(n)$, showing the total payoff to society as a function of n. The total payoff to society when there are n participants consists of the value $P(n)$ for each of the n participants and the value $S(n)$ for each of the $(N - n)$ shirkers:

$$T(n) = nP(n) + (N - n)\ S(n).$$

The social optimum occurs when the allocation of people between participants and shirkers maximizes the total payoff $T(n)$, or at the number of participants—that is, the value of n—that maximizes $T(n)$. To get a better understanding of where this optimum might be, it is convenient to write $T(n)$ differently, rearranging the expression above to get

$$T(n) = NS(n) - n[S(n) - P(n)].$$

This version of the total social payoff function shows that we can calculate it as if we gave every one of the N farmers the shirker's payoff, but then removed the shirker's extra benefit $[S(n) - P(n)]$ from each of the n participants.

In collective-action games, as opposed to common-resource games, we normally expect $S(n)$ to increase as n increases. Therefore, the first term in this expression, $NS(n)$, also increases as n increases. If the second term does not increase too fast as n increases—as would be the case if the shirker's extra benefit, $[S(n) - P(n)]$, is small and constant—then the effect of the first term dominates in determining the value of $T(n)$.

This is exactly what happens with the total social payoff function for our current 100-farmer example. Here $T(n) = nP(n) + (N - n)S(n)$ becomes $T(n) = n(2n) + (100 - n)(2n + 4) = 2n^2 + 200n - 2n^2 + 400 - 4n = 400 + 196n$. In this case, $T(n)$ increases steadily with n and is maximized at $n = N$ when no one shirks.

Thus, the large-group version of this game holds the same lesson as our two-person example: Society as a whole would be better off if all of the farmers participated in building the irrigation project (if $n = N$). But payoffs are such that each farmer has an individual incentive to shirk. The Nash equilibrium of the game, at $n = 0$, is not socially optimal. Figuring out how to achieve the social optimum is one of the most important topics in the study of collective action, to which we return later in this chapter.

In other situations, $T(n)$ could be maximized for a different value of n, not just at $n = N$. That is, society's aggregate payoff could be maximized by allowing some shirking. Even in the prisoners' dilemma case, it is not automatic that the total payoff function is maximized when n is as large as possible. If the gap between $S(n)$ and $P(n)$ widens sufficiently fast as n increases, then the negative effect of the second term in the expression for $T(n)$ outweighs the positive effect of the first term as n approaches N. Then it may be best to let some people shirk—that is, the socially optimal value for n may be less than N. This result mirrors that of our prisoners' dilemma version II in Section 1.

This type of outcome would arise in our village if $S(n)$ were $4n + 4$, rather than $2n + 4$. Then $T(n) = -2n^2 + 396n + 400$, which is no longer a linear function of n. In fact, a graphing calculator or some basic calculus shows that this $T(n)$ is maximized at $n = 99$, rather than at $n = 100$ as was true before. The change to the payoff structure has created an inequality in the payoffs—the shirkers fare better than the participants—which adds another dimension of difficulty to society's attempts to resolve the dilemma. How, for example, would the village designate exactly one farmer to be the shirker?

B. Multiplayer Chicken

Now we consider some of the other configurations that can arise in the payoffs. For example, when $P(n) = 4n + 36$, so that $P(n + 1) = 4n + 40$, and $S(n) = 5n$, the two payoff curves will cross in the graph. This case is illustrated in Figure 11.7. Here, for small values of n, $P(n + 1) > S(n)$, so if few others are participating, your optimal choice is to participate. For large values of n, $P(n + 1) < S(n)$, so if many others are participating, your optimal choice is to shirk. Note the equivalence of these two statements to the idea in the two-person chicken game that you shirk if your neighbor works and you work if he shirks. This case is indeed a game of chicken. More generally, a collective-action game of chicken occurs when you are given a choice between two actions, and you prefer to do the one that most others are *not* doing.

We can also use Figure 11.7 to determine the location of the Nash equilibrium of this version of the game. Because you choose to participate when n is small and to shirk when n is large, the equilibrium must be some intermediate value of n. Only at that n where the two curves intersect are you indifferent between your two choices. This location represents the equilibrium value of n. In our graph, $P(n + 1) = S(n)$ when $4n + 40 = 5n$ or when $n = 40$; that is the Nash equilibrium number of farmers from the village who will participate in the irrigation project.

If the two curves intersect at a point corresponding to an integer value of n, then that is the Nash equilibrium number of participants. If that is not the case, then strictly speaking, the game has no Nash equilibrium. But in practice, if the current value of n in the population is the integer just to the left of the point of intersection, then one more farmer will just want to participate, whereas if the current value of n is the integer just to the right of the point of intersection, one farmer will want to switch to shirking. Therefore, the number of participants will stay within a small range around the point of intersection, and we can justifiably speak of the intersection as the equilibrium in some approximate sense.

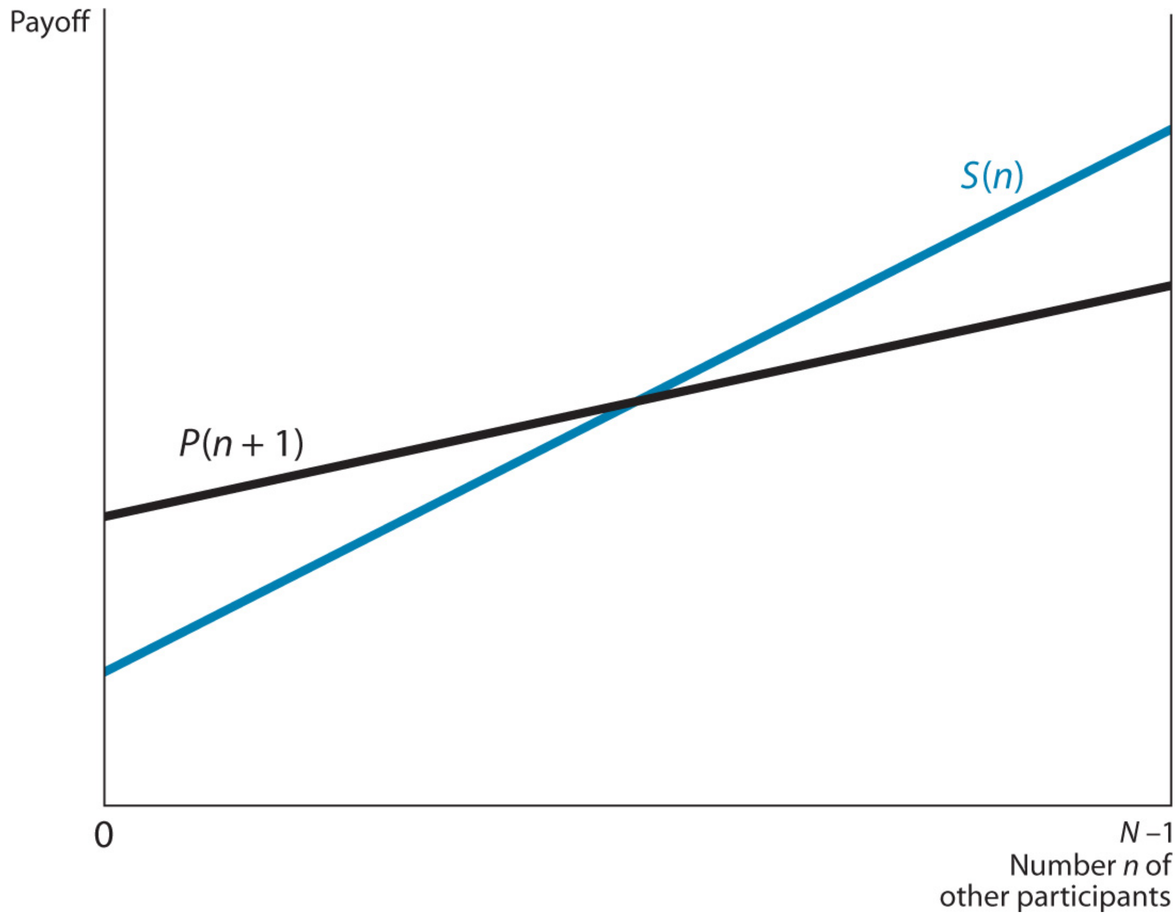


FIGURE 11.7 Multiplayer Chicken Payoff Graph

The payoff structure illustrated in Figure 11.7 shows both curves positively sloped, although they don’t have to be. It is conceivable that the benefit for each person is smaller when more people participate, so the curves could be negatively sloped instead. The important feature of the collective-action game of chicken is that when few people are taking one action, it is better for any one person to take that action; and when many people are taking one action, it is better for any one person to take the other action.

What is the socially optimal outcome in the collective-action game of chicken? If each participant’s payoff $P(n)$ increases as the number of participants increases, and if each shirker’s payoff $S(n)$ does not become too much greater than the $P(n)$ of each participant, then the total social payoff is maximized when everyone participates. This is the outcome in our example, where $T(n) = 536n - n^2$; total social payoff continues to increase for values of n beyond N (100 here), so $n = N$ is the social optimum.

But more generally, some cases of collective-action chicken entail social optima in which it is better to let some shirk. If our group of farmers numbered 300 instead of 100, our example here would yield such an outcome. The socially optimal number of participants, found on a graphing calculator or using calculus, would be 268. The idea that the social optimum can sometimes be attained when only some people work and the rest shirk emerged in the difference between versions I and II of chicken in our examples in Section 1; here, we see an example of the result in a larger population. For an exercise, you may try generating a payoff structure that leads to such an outcome for our village of 100 farmers. By changing the payoff functions, $P(n)$ and $S(n)$, one can find games of chicken where the socially optimal number of participants could even be smaller than that in the Nash equilibrium. We return to examine the question of the social optima of all of these versions of the game in greater detail in Section 3.

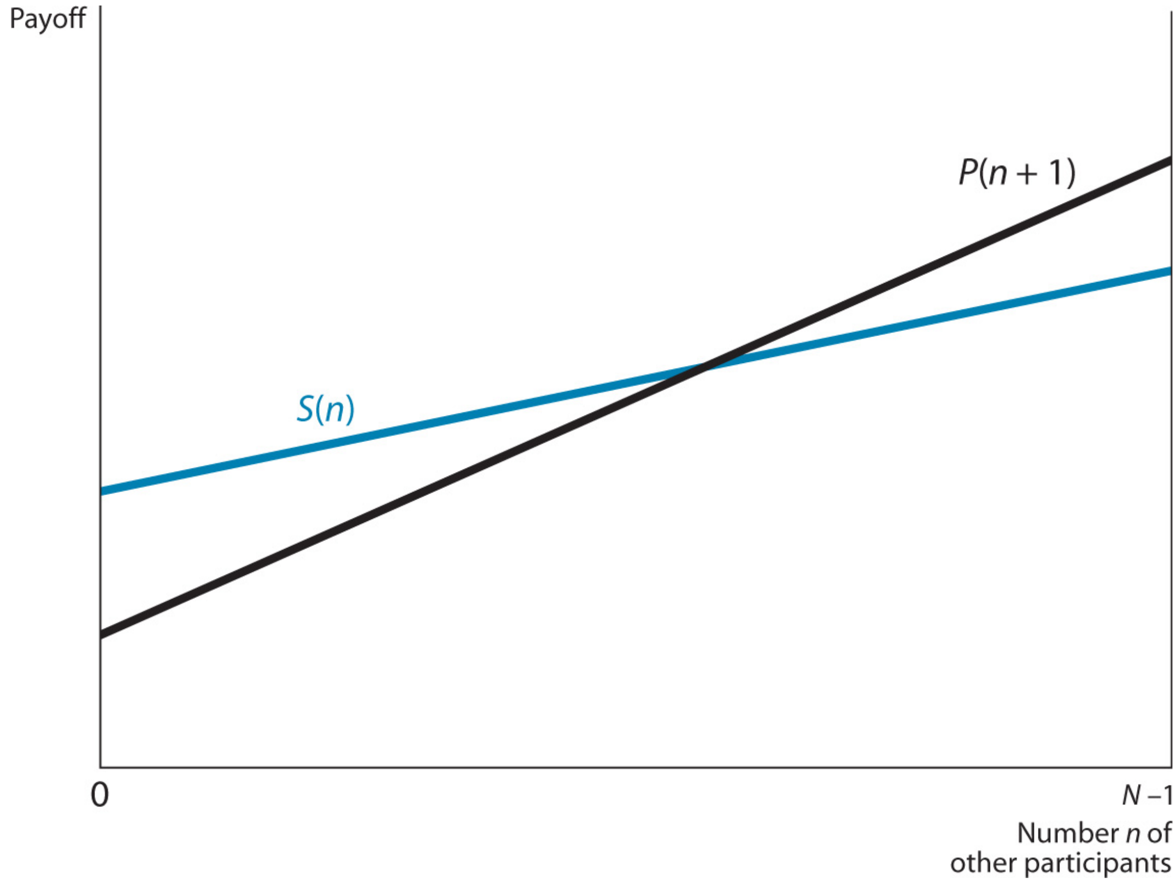


FIGURE 11.8 Multiplayer Assurance Payoff Graph

C. Multiplayer Assurance

Finally, we consider the third possible type of collective-action game: assurance. Figure 11.8 shows the payoff curves for the assurance case, where we suppose that the village farmers get $P(n + 1) = 4n + 4$ and $S(n) = 2n + 100$. Here, $S(n) > P(n + 1)$ for small values of n, so if few others are participating, then you want to shirk, too. But $P(n + 1) > S(n)$ for large values of n, so if many others are participating, then you want to participate, too. In other words, unlike chicken, assurance is a collective-action game in which you want to make the choice that the others are making.

Figure 11.8 looks nearly identical to Figure 11.7, except that the S and P curves have swapped positions. In Figure 11.7, the S curve starts lower and is steeper than the P curve, but in Figure 11.8, the S curve starts higher and is flatter than the P curve. These details of curve placement might seem inconsequential, but they illustrate the essence of the difference between chicken games and assurance games. In a chicken game, each farmer wants to participate when others shirk (graphically, the P curve starts out higher than the S curve), but each has less to gain from participating as more participate (graphically, the P curve rises more slowly than the S curve). Eventually, when the number of other participants in a chicken game is sufficiently large, each farmer prefers to shirk (graphically, the S curve is higher than the P curve once n becomes sufficiently large). On the other hand, in an assurance game, each farmer wants to shirk when others shirk also (graphically, the S curve starts out higher than the P curve), but each has more to gain from participating as more participate (graphically, the P curve rises more quickly than the S curve). Eventually, when the number of other participants in an assurance game is sufficiently large, each farmer prefers to participate (graphically, the P curve is higher than the S curve once n becomes sufficiently large). More simply put, in a multiplayer chicken game, players prefer to “go against the crowd,” while in a multiplayer assurance game, they prefer to “go with the crowd.”

What are the Nash equilibria of the assurance game in Figure 11.8? For any initial value of n to the left of the intersection, each farmer will want to shirk; so there is a Nash equilibrium at $n = 0$, where everyone shirks. But the opposite is true to the right of the intersection. In that portion of the graph, each farmer will want to participate, yielding a second Nash equilibrium at $n = N$, where everyone participates.

Technically, there is also a third Nash equilibrium of this game if the value of n at the intersection is an integer value, as it is in our example. There we find that $P(n + 1) = 4n + 4 = 2n + 100 = S(n)$ when $n = 48$. Thus, if n were exactly 48, we would see an outcome in which there were some participants and some shirkers. This situation could be an equilibrium only if the value of n were exactly right. Even then, it would be a highly unstable situation. If any one farmer accidentally joined the wrong group, his choice would alter the incentives for everyone else, driving the game to one of the (stable) endpoint equilibria. (We will discuss the concept of equilibrium "stability" in detail in Chapter 12, in the context of evolutionary games.)

The social optimum in this game is fairly easy to see in Figure 11.8. Because both curves are rising—so that each person is better off if more people participate—the right-hand extreme equilibrium is clearly the better one for society. This is confirmed in our example by noting that $T(n) = 2n^2 + 100n + 10,000$, which is an increasing function of n for all positive values of n; thus, the socially optimal value of n is the largest one possible, or $n = N$. In the assurance case, then, the socially optimal outcome is actually one of the stable Nash equilibria of the game. As such, it may be easier to achieve than in some of the other cases. The critical question regarding the social optimum, regardless of whether it represents a Nash equilibrium of the underlying game, is how to bring it about.

So far, our examples have focused on relatively small groups of 2 or 100 persons. When the total number of people in the population, N, is very large, however, and any one person makes only a very small difference, then $P(n + 1)$ is almost the same as

$P(n)$. Thus, the condition under which any one person chooses to shirk is $P(n) < S(n)$. Expressing this inequality in terms of the benefits and costs of the common project in our example—namely, $P(n) = B(n) - C(n)$ and $S(n) = B(n)$—we see that $P(n)$ [unlike $P(n + 1)$ in our preceding calculations] is *always* less than $S(n)$; individual persons will *always* want to shirk when N is very large. That is why problems of collective provision of public projects in a large group almost always manifest themselves as prisoners' dilemmas. But as we have seen, this result is not necessarily true for smaller groups. Neither is it true for large groups in other contexts such as traffic congestion, a case we discuss later in this chapter.

In general, we must allow for a broader interpretation of the payoffs $P(n)$ and $S(n)$ than we did in the specific case involving the benefits and the costs of a project. We cannot assume, for example, that the payoff functions will be linear. In fact, in the most general case, $P(n)$ and $S(n)$ can be any functions of n and can intersect many times. Then there can be multiple equilibria, although each can be thought of as representing one of the types described so far.[3] And some games will be of the common-resource type as well, so when we allow for completely general games, we will speak of two actions labeled P and S, which have no necessary connotation of "participation" and "shirking," but allow us to continue with the same symbols for the payoffs. Thus, when n players are taking the action P, $P(n)$ becomes the payoff of each player taking the action P, and $S(n)$ becomes that of each player taking the action S.

Endnotes

- Several exercises at the end of this chapter present examples of simple situations with nonlinear payoff curves and multiple equilibria. For a more general analysis and classification of such diagrams, see Thomas Schelling, *Micromotives and Macrobehavior* (New York: W. W. Norton & Company, 1978), Chapter. 7. The theory can be taken further by allowing each player a continuous choice (for example, the number of hours of participation) instead of just a binary choice of whether to participate. Many such situations are discussed in more specialized books on collective action, for example, Todd Sandler, *Collective Action: Theory and Applications* (Ann Arbor: University of Michigan Press, 1993), and Richard Cornes and Todd Sandler, *The Theory of Externalities, Public Goods, and Club Goods*, 2nd ed. (New York: Cambridge University Press, 1996). Return to reference 3

3 SPILLOVER EFFECTS, OR EXTERNALITIES

So far, we have seen that collective-action games occur in prisoners’ dilemma, chicken, and assurance forms. We have also seen that the Nash equilibria in such games rarely yield the socially optimal level of participation (or nonparticipation). And even when the social optimum is a Nash equilibrium, it is usually only one of several equilibria that may arise. Here we delve further into the differences between the individual (or private) incentives in such games and the group (or social) incentives. We also describe more carefully the effects of each individual’s decision on other individuals as well as on the collective. This analysis makes explicit why differences in incentives exist, how they are manifested, and how one might go about achieving socially better outcomes than those that arise in Nash equilibrium.

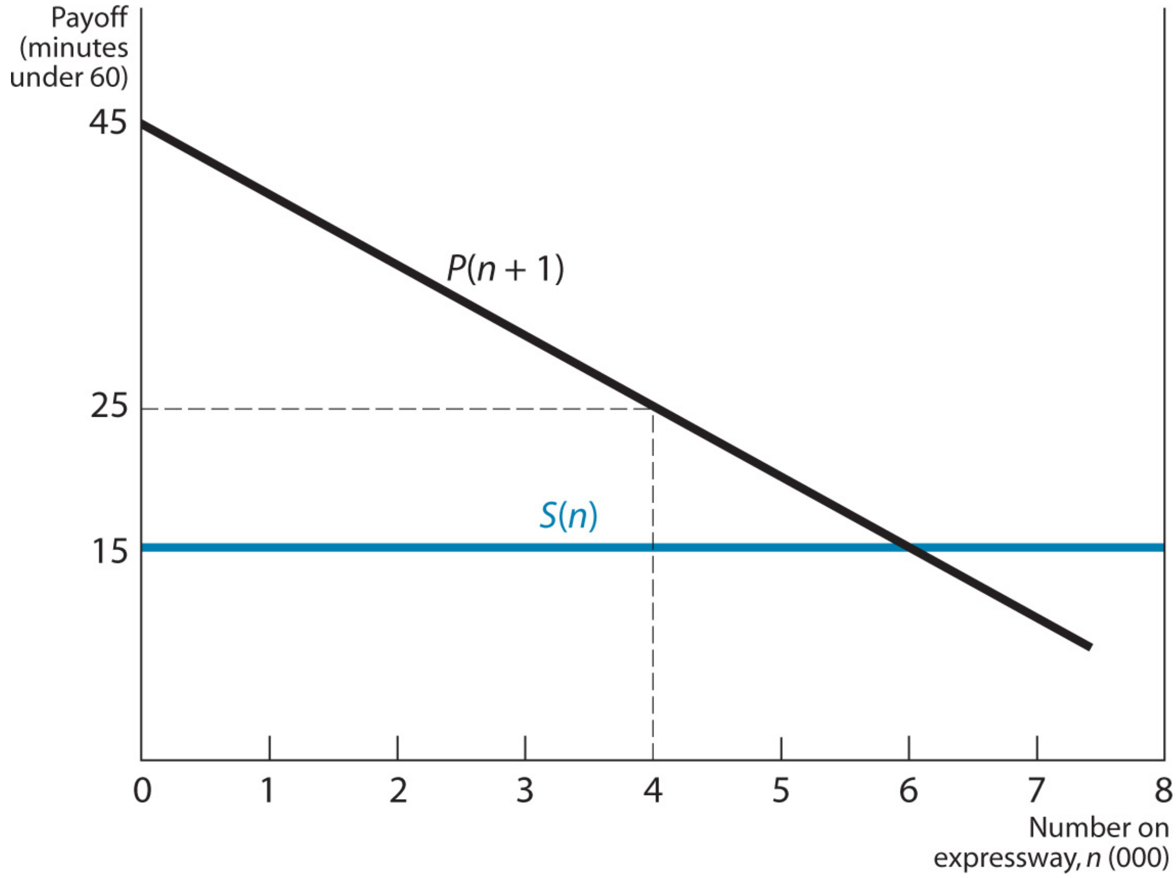


FIGURE 11.9 Commuting Route-Choice Game

A. Commuting and Spillover Effects

We start by thinking about a large population of 8,000 commuters who drive every day from a suburb to the city and back. As one of these commuters, you can take either the expressway (action *P*) or a network of local roads (action *S*). The local-roads route takes a constant 45 minutes, no matter how many cars are going that way. The expressway takes only 15 minutes when uncongested. But every driver who chooses the expressway increases the time for every other driver on the expressway by 0.005 minutes (about one-quarter of a second).

We can measure the payoffs in minutes of time saved—by how much the commute time is less than 1 hour, for instance. Then the payoff to drivers on the local roads, $S(n)$, is a constant $60 - 45 = 15$, regardless of the value of n. But the payoff to drivers on the expressway, $P(n)$, depends on n; in particular, $P(n) = 60 - 15 = 45$ for $n = 0$, but $P(n)$ decreases by 5/1,000 (or 1/200) for every commuter on the expressway. Thus, $P(n) = 45 - 0.005n$. We graph the two payoff curves in Figure 11.9.

Suppose that initially 4,000 cars are on the expressway; $n = 4{,}000$. With so many cars on that road, it takes each of them $15 + 4{,}000 \times 0.005 = 15 + 20 = 35$ minutes to commute to work; each gets a payoff of $P(n) = 25$ [which is $60 - 35$, or $P(4{,}000)$]. As shown in Figure 11.9, that payoff is better than what local-road drivers obtain. You, a local-road driver, might therefore decide to switch from driving the local roads to driving on the expressway. Your switch would increase by 1 the value of n and would thereby affect the payoffs of all the other commuters. There would now be 4,001 drivers (including you) on the expressway, and the commute time for each would be 35 and 1/200, or 35.005, minutes; each would now get a payoff of $P(n + 1) = P(4{,}001) = 24.995$. This payoff is still higher than the 15 from driving on the local roads. Thus, you have a private incentive to make the switch, because for you, $P(n + 1) > S(n)$ $(24.995 > 15)$.

Your switch yields you a *private* gain—because it is privately enjoyed by you—equal to the difference between your payoffs before and after the switch; this private gain is $P(n + 1) - S(n) = 9.995$ minutes. Because you are only one person and therefore a small part of the whole group, the gain in payoff that you receive in relation to the total group payoff is small, or *marginal.* Thus, we call your gain the marginal private gain associated with your switch.

But now the 4,000 other drivers on the expressway each take 0.005 minutes more as a result of your decision to switch; the payoff to each changes by $P(4,001) - P(4,000) = -0.005$. Similarly, the drivers on the local roads face a payoff change of $S(4,001) - S(4,000)$, but this is zero in our example. The cumulative effect on all these other drivers is $4,000 \times -0.005 = -20$ (minutes). Your action, switching from local roads to expressway, has caused this effect on the others' payoffs. Such an effect of one person's action on others' payoffs is called a spillover effect, external effect, or externality. Again, because you are only a very small part of the whole group, we should actually call your effect on others the *marginal spillover effect.*

Taken together, the marginal private gain and the marginal spillover effect are the full effect of your switch on the population of commuters, or the overall marginal change in the whole society's payoff. We call this the marginal social gain associated with your switch. This "gain" may actually be positive or negative, so the use of the word *gain* is not meant to imply that all switches will benefit society as a whole. In fact, in our commuting example, the marginal social gain is $9.995 - 20 = -10.005$ (minutes). Thus, the overall social effect of your switch is negative; the total social payoff is reduced by just over 10 minutes.

B. Spillover Effects: The General Case

We can describe the effects we observe in the commuting example more generally by returning to our total social payoff function, $T(n)$, where n represents the number of people choosing P, so $N - n$ is the number of people choosing S. Suppose that initially n people have chosen P and that one person switches from S to P. Then the number choosing P increases by 1 to $(n + 1)$, and the number choosing S decreases by 1 to $(N - n - 1)$, so the total social payoff becomes

$$T(n + 1) = (n + 1)\ P(n + 1) + [N - (n + 1)]\ S(n + 1).$$

The increase in the total social payoff is the difference between $T(n)$ and $T(n + 1)$:

$$T(n + 1) - T(n) = (n + 1)\ P(n + 1) + [N - (n + 1)]\ S(n + 1) - nP(n) + (N - n)\ S(n) = [P(n + 1) - S(n)] + n[P(n + 1) - P(n)] + [N - (n + 1)]\ [S(n + 1) - S(n)]$$

(11.1)

after collecting and rearranging terms.

Equation (11.1) describes mathematically the various different effects of one person's switch from S to P that we saw earlier in the commuting example. The equation shows how the marginal social gain is divided into the marginal changes in payoffs for the subgroups of the population.

The first of the three terms in equation (11.1)—namely, $[P(n + 1) - S(n)]$—is the marginal private gain enjoyed by the person who switches. As we saw above, this term is what drives a person's choice, and all such individual choices then determine the Nash equilibrium.

The second and third terms in equation (11.1) are just the quantifications of the spillover effects of one person's switch on the others in the population. For the n other people choosing

P, each sees her payoff change by the amount $[P(n + 1) - P(n)]$ when one more person switches to P; this spillover effect is seen in the second group of terms in equation (11.1). There are also $N - (n + 1)$ (or $N - n - 1$) others still choosing S after the one person switches, and each of these players sees her payoff change by $[S(n + 1) - S(n)]$; this spillover effect is shown in the third group of terms in the equation. Of course, the effect that one driver's switch has on commuting time for any other driver on either route is very small, but when there are numerous other drivers (that is, when N is large), the full spillover effect can be substantial.

Thus, we can rewrite equation (11.1) for a switch by one person either from S to P or from P to S as

Marginal social gain = marginal private gain + marginal spillover effect.

For an example in which one person switches from S to P, we have

Marginal social gain = $T(n + 1) - T(n)$,

Marginal private gain = $P(n + 1) - S(n)$, and

Marginal spillover effect = $n[P(n + 1) - P(n)] + [N - (n + 1)][S(n + 1) - S(n)]$.

USING CALCULUS FOR THE GENERAL CASEBefore examining some spillover situations in more detail to see what can be done to achieve socially better outcomes, we restate the general concepts of the analysis of collective-action games in the language of calculus. If you do not know this language, you can omit the remainder of this section without loss of continuity; if you do know it, you will find the alternative statement much simpler to grasp and to use than the algebra employed earlier.

If the total number N of people in the group is very large—say, in the hundreds or thousands—then one person can be regarded as a very small, or infinitesimal, part of this whole. This allows us to treat the number n as a continuous variable. If $T(n)$ is the total social payoff, we calculate the effect of changing n by considering an increase of an infinitesimal marginal quantity dn, instead of a full unit increase from n to $(n + 1)$. To the first order, the change in payoff is $T'(n)\,dn$, where $T'(n)$ is the derivative of $T(n)$ with respect to n. Using the expression for the total social payoff,

$$T(n) = nP(n) + (N - n)\,S(n),$$

and differentiating, we have

$$T'(n) = P(n) + nP'(n) - S(n) + (N - n)\,S'(n)$$

$$= [P(n) - S(n)] + nP'(n) + (N - n)\,S'(n). \quad (11.2)$$

This equation is the calculus equivalent of equation (11.1). $T'(n)$ represents the marginal social gain. The marginal private gain is $P(n) - S(n)$, which is just the change in the payoff of the person making the switch from S to P. In equation (11.1), we had $P(n + 1) - S(n)$ for this change in payoff; now we have $P(n) - S(n)$. This is because the infinitesimal addition of dn to the group of the n people choosing P does not change the payoff to any one of them by a significant amount. However, the total change in their payoff, $nP'(n)$, is sizable, and is recognized in the calculation of the spillover effect [it is the second term in equation (11.2)], as is the change in the payoff of the $(N - n)$ people choosing S [namely, $(N - n)\,S'(n)$], the third term in equation (11.2). These last two terms constitute the marginal spillover effect in equation (11.2).

In the commuting example, we had $P(n) = 45 - 0.005n$, and $S(n) = 15$. Then, with the use of calculus, we see that the private marginal gain for each driver who switches to the expressway when n drivers are already using it is $P(n) - S(n) = 30 - 0.005n$. Because $P'(n) = -0.005$ and $S'(n) = 0$, the spillover effect is $n \times (-0.005) + (N - n) \times 0 = -0.005n$, which equals -20 when $n = 4{,}000$. The answer is the same as before, but calculus simplifies the derivation and helps us find the optimum directly.

C. Commuting Revisited: Negative Externalities

A negative externality exists when the action of one person *lowers* others' payoffs—that is, when it imposes some extra costs on the rest of society. We saw this in our commuting example, where the marginal spillover effect of one person's switch to the expressway was negative, entailing an extra 20 minutes of drive time for other commuters. But the individual who changes her route to work does not take the spillover—the externality—into account when making her choice. She is motivated only by her own payoffs. (Remember that any guilt that she may suffer from harming others should already be reflected in her payoffs.) She will change her action from S to P as long as this change has a positive marginal *private* gain. She is then made better off by the change.

But society would be better off if the commuter's decision were governed by the marginal *social* gain. In our example, the marginal social gain is negative (−10.005), but the marginal private gain is positive (9.995), so the individual driver makes the switch, even though society as a whole would be better off if she did not do so. More generally, in situations with negative externalities, the marginal social gain will be smaller than the marginal private gain due to the existence of the negative spillover effect. Individuals will make decisions based on a cost-benefit calculation that is the wrong one from society's perspective. As a result, individuals will choose actions with negative spillover effects more often than society would like them to do.

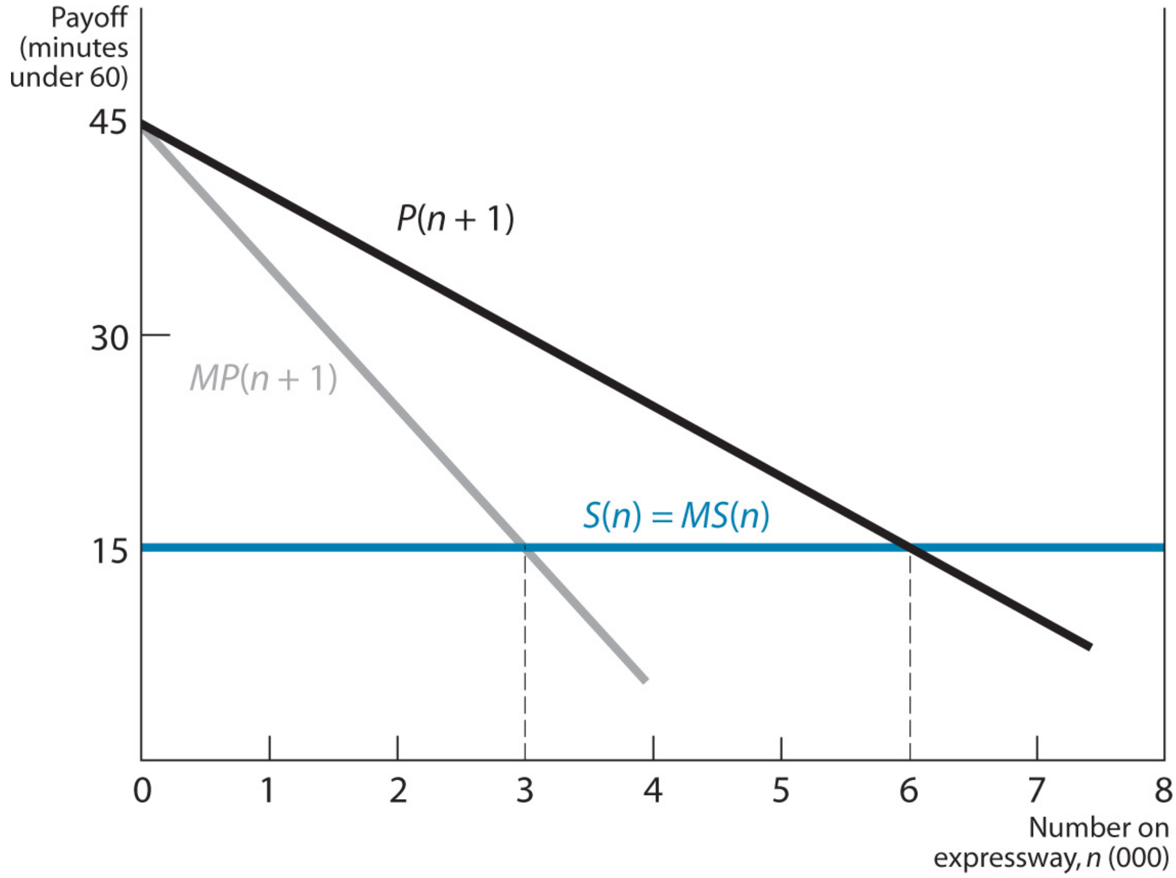


FIGURE 11.10 Equilibrium and Optimum in Commuting Game

We can use equation (11.1) to calculate the precise conditions under which a switch will be beneficial for a particular person versus for society as a whole. Recall that if n people are already using the expressway and another driver is contemplating switching from the local roads to the expressway, she stands to gain from this switch if $P(n + 1) > S(n)$, whereas the total social payoff increases if $T(n + 1) - T(n) > 0$. The private gain is positive if

$$45 - (n + 1) \times 0.005 > 15$$

$$44.995 - 0.005n > 15$$

$$n < 200\ (44.995 - 15) = 5{,}999,$$

whereas the condition for the social gain to be positive is

$$45 - (n + 1) \times 0.005 - 15 - 0.005n > 0$$

$$29.995 - 0.01n > 0$$

$$n < 2{,}999.5.$$

Thus, if given the free choice, commuters will crowd onto the expressway until there are almost 6,000 of them, but all crowding beyond 3,000 reduces the total social payoff. Society as a whole would be best off if the number of commuters on the expressway were kept down to 3,000.

We show this result graphically in Figure 11.10; this figure replicates Figure 11.9, with the addition of marginal private and social gain curves. The two curves indicating $P(n + 1)$ and $S(n)$ meet at $n = 5{,}999$; that is, at the value of n for which $P(n + 1) = S(n)$ or for which the marginal private gain is just zero. Everywhere to the left of this intersection point, any one driver on the local roads calculates that she will get a positive gain by switching to the expressway. As some drivers make this switch, the numbers on the expressway increase—the value of n in society rises, as was the case in our example in Section 3.A. Conversely, to the right of the intersection point (that is, for $n > 5{,}999$), $S(n) > P(n + 1)$; so each of the $(n + 1)$ drivers on the expressway stands to gain by switching to the local road. As some do so, the numbers on the expressway decrease, and n falls. From the left of the intersection point, this process converges to $n = 5{,}999$, and from the right, it converges to $n = 6{,}000$.

If we had used the calculus approach, we would have regarded 1 as a very small increment in relation to n and graphed $P(n)$ instead of $P(n + 1)$. Then the intersection point would have been at $n = 6{,}000$ instead of at $n = 5{,}999$. As you can see, it makes very little difference in practice. What this means is that we can call $n = 6{,}000$ the Nash equilibrium of the route choice game when choices are governed by purely individual considerations. Given a free choice, 6,000 of the 8,000 commuters will choose the expressway, and only 2,000 will drive on the local roads.

But we can also interpret the outcome in this game from the perspective of the whole society of commuters. Society gains from an increase in the number of commuters, n, on the expressway when $T(n + 1) - T(n) > 0$ and loses from an increase in n when $T(n + 1) - T(n) < 0$. To figure out how to show this on the graph, we express the idea somewhat differently. We rearrange equation (11.1) into two pieces, one depending only on P and the other depending only on S:

$$T(n + 1) - T(n) = (n + 1)\ P(n + 1) + [N - (n + 1)]\ S(n + 1) - nP(n) - [N - n]\ S(n)$$

$$= \{P(n + 1) + n[P(n + 1) - P(n)]\} - \{S(n) + [N - (n + 1)][S(n + 1) - S(n)]\}.$$

The expression in the first set of braces is the effect on the payoffs of the set of commuters who choose P; this expression includes the $P(n + 1)$ of the switcher and the spillover effect, $n[P(n + 1) - P(n)]$, on all the other n commuters who choose P. We call this expression the marginal social payoff for the P-choosing subgroup when their number increases from n to $n + 1$, or $MP(n + 1)$ for short. Similarly, the expression in the second set of braces is the marginal social payoff for the S-choosing subgroup, or $MS(n)$ for short. Then, the full expression for $T(n + 1) - T(n)$ tells us that the total social payoff increases when one person switches from S to P (or decreases if the switch is from P to S) if $MP(n + 1) > MS(n)$. The total social payoff decreases when one person switches from S to P (or increases when the switch is from P to S) if $MP(n + 1) < MS(n)$.

Using our expressions for $P(n + 1)$ and $S(n)$ in the commuting example, we have

$$MP(n + 1) = 45 - (n + 1) \times 0.005 + n \times (-0.005) = 44.995 - 0.01n$$

while $MS(n) = 15$ for all values of n. Figure 11.10 illustrates $MP(n + 1)$ and $MS(n)$. Note that $MS(n)$ coincides with $S(n)$ everywhere because the local roads are never congested. But the $MP(n + 1)$ curve lies below the $P(n + 1)$ curve. Because of the negative spillover effect, the social gain from one person's switching to the expressway is less than the private gain to the switcher.

The $MP(n + 1)$ and $MS(n)$ curves meet at $n = 2{,}999$, or approximately 3,000. To the left of this intersection, $MP(n + 1) > MS(n)$, and society stands to gain by allowing one more person on the expressway. To the right, the opposite is true, and society stands to gain by shifting one person from the expressway to the local roads. Thus, the socially optimal allocation of drivers is 3,000 on the expressway and 3,000 on the local roads.

If you wish to use calculus, you can write the total payoff for the expressway drivers as $nP(n) = n(45 - 0.005n) = 45n - 0.005n^2$. Then $MP(n + 1)$ is the derivative of this with respect to n—namely, $45 - 0.005 \times 2n = 45 - 0.01n$. The rest of the analysis can proceed as before.

How might this society achieve this optimum allocation of its drivers? Different cultures and political groups approach such problems in different ways, each with its own merits and drawbacks. The society could simply restrict access to the expressway to 3,000 drivers. But how would it choose those 3,000? It could adopt a first-come, first-served rule, but then drivers would race one another to get there early and waste a lot of time. A bureaucratic society could set up criteria based on complex calculations of needs and merits as defined by civil servants; then everyone would undertake some costly activities to meet these criteria. In a politicized society, the important "swing voters" or organized pressure groups or contributors might be favored. In a corrupt society, those who bribe the officials or the politicians might get the preference. A more egalitarian society could allocate the rights to drive on the expressway by lottery or rotate them from one month to the next. A scheme that lets you drive only on certain days, depending on the last digit of your car's license plate, is an example. But such a scheme is not so egalitarian as it seems, because the rich can have two cars and choose license-plate numbers that will allow them to drive every day.

Many economists prefer a more open system of charges. Suppose each driver on the expressway is made to pay a tax t, measured in units of time. Then the private benefit from using the expressway becomes $P(n) - t$, and the number n in the Nash equilibrium will be determined by $P(n) - t = S(n)$. [Here, we are ignoring the tiny difference between $P(n)$ and $P(n + 1)$, which is possible when N is very large.] We know that the socially optimal value of n is 3,000. Using the expressions $P(n) = 45 - 0.005n$ and $S(n) = 15$, and plugging in 3,000 for n, we find that $P(n) - t = S(n)$—that is, drivers are indifferent between the expressway and the local roads—when $45 - 15 - t = 15$, or $t = 15$. If we value time at a minimum wage of about \$10 an hour, 15 minutes comes to \$2.50. This is the tax or toll that, when charged, will keep the numbers on the expressway down to what is socially optimal.

Note that when 3,000 drivers are on the expressway, the addition of one more increases the time spent by each of them by 0.005 minutes, for a total of 15 minutes. This is exactly the tax that each driver is being asked to pay. In other words, each driver is made to pay the cost of the negative spillover effect that she imposes on the rest of society. This tax brings home to each driver the extra cost of her action and therefore induces her to take the socially optimal action; economists say that she is being made to internalize the externality. This idea, that people whose actions hurt others should pay for the harm that they cause, adds to the appeal of this approach. But the proceeds from the tax are not used to compensate the others directly. If they were, then each expressway user would count on receiving from others just what she pays, and the whole purpose would be defeated. Instead, the proceeds of the tax go into general government revenues, where they may or may not be used in a socially beneficial manner.

Those economists who prefer to rely on markets argue that if the expressway has a private owner, his profit motive will induce him to charge just enough for its use to reduce the number of users to the socially optimal level. An owner knows that if he charges a tax t for each user, the number of users n will be determined by $P(n) - t = S(n)$. His revenue will be $tn = n[P(n) - S(n)]$, and he will act in such a way as to maximize this revenue. In our example, the revenue is $n[45 - 0.005n - 15] = n[30 - 0.005n] = 30n - 0.005n^2$. It is easy to see that this revenue is maximized when $n = 3,000$. But in this case, the revenue goes into the owner's pocket; most people regard that as a bad solution.

D. Positive Spillover Effects

Many aspects of positive spillover effects or positive externalities can be understood simply as mirror images of those of negative spillover effects. A person’s marginal private gain from undertaking activities with positive spillover effects is less than society’s marginal gain from such activities. Therefore, such actions will be underused and their benefits underprovided in the Nash equilibrium. A better outcome can be achieved by augmenting people’s incentives; providing those persons whose actions create positive spillover effects with a reward just equal to the spillover benefit will achieve the social optimum.

Indeed, the distinction between positive and negative spillover effects is to some extent a matter of semantics. Whether a spillover effect is positive or negative depends on which choice you call *P* and which you call *S*. In the commuting example, suppose we called choosing the local roads *P* and choosing the expressway *S*. Then one commuter’s switch from *S* to *P* will reduce the time taken by all the others who choose *S*, so this action will have a positive spillover effect on them. In another example, consider vaccination against some infectious disease. Each person getting vaccinated reduces his own risk of catching the disease (marginal private gain) and reduces the risk of others’ getting the disease through him (spillover effect). If being unvaccinated is called the *S* action, then getting vaccinated has a positive spillover effect. If remaining unvaccinated is called the *P* action, then the act of remaining unvaccinated has a negative spillover effect. These two ways of labeling actions here suggest two sorts of policies that might be used to bring individual actions into conformity with the social optimum: Society can either reward those who get vaccinated or penalize those who fail to do so.

But actions with positive spillover effects can have one very important new feature that distinguishes them from actions with negative spillover effects—namely, positive feedback. Suppose the spillover effect of your choosing *P* is to increase the payoff to the others who are also choosing *P*. Then your choice increases the attractiveness of that action (*P*) and may induce some others to take it also, setting in train a process that culminates in everyone's taking that action. Conversely, if very few people are choosing *P*, then it may be so unattractive that they, too, give it up, leading to a situation in which everyone chooses *S*. In other words, positive feedback can give rise to multiple Nash equilibria, which we now illustrate by using a very real example.

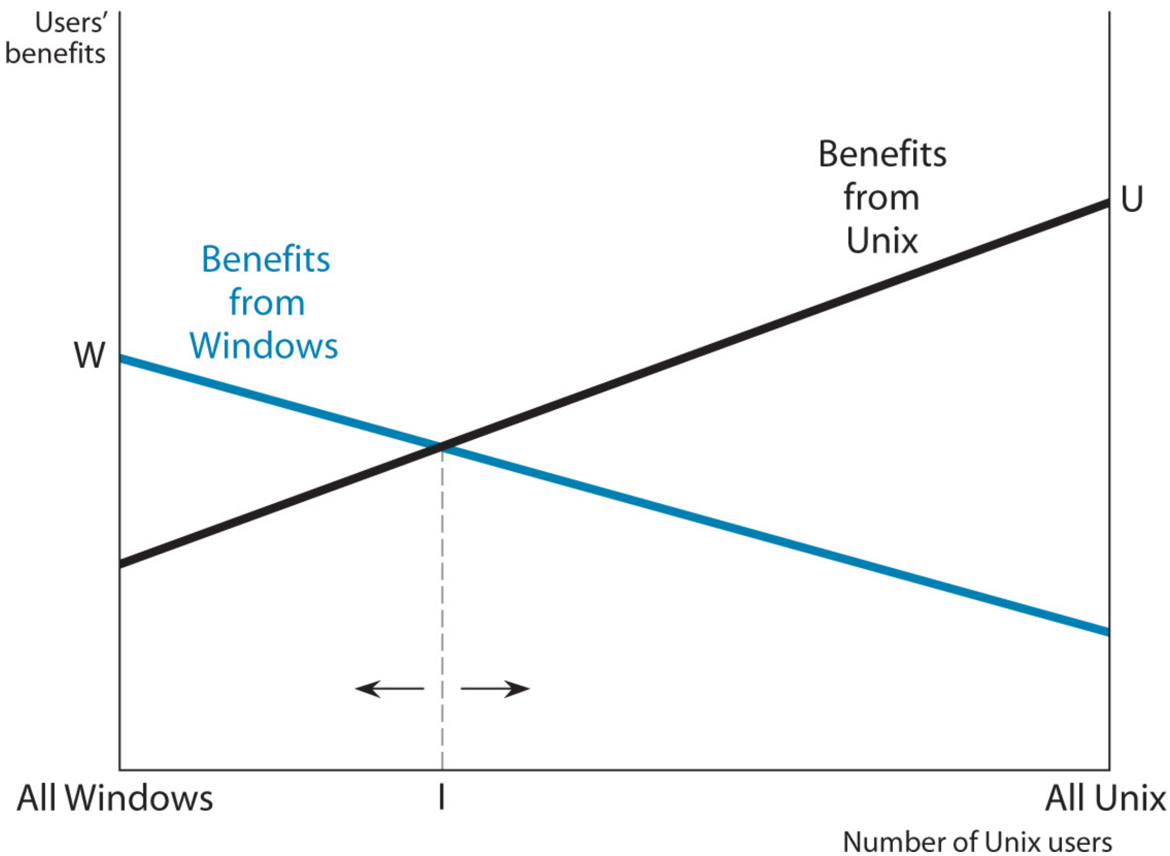


FIGURE 11.11 Payoffs in Operating System–Choice Game

When you buy a computer, you have to choose between one with a Windows operating system and one with an operating system based on Unix, such as Linux. The higher the number of Unix users rises, the better it will be to purchase such a computer. The system will have fewer bugs because more users will have detected those that exist, more application software will be available, and more experts will be available to help with any problems that arise. Similarly, a Windows-based computer will be more attractive the more Windows users there are. In addition, many computing aficionados would argue that the Unix system is superior. Without necessarily taking a position on that matter, we show what will happen if that is the case. Will individual choice lead to the socially best outcome?

A graph similar to Figures 11.6 through 11.8 can be used to show the payoffs to an individual computer purchaser of the two strategies, Unix and Windows. As shown in Figure 11.11, the Unix payoff rises as the number of Unix users rises, and the Windows payoff rises as the number of Unix owners falls (and the number of Windows users rises). As already explained, the graph is drawn assuming that the payoff to Unix users when everyone in the population is a Unix user (at the point labeled U) is higher than the payoff to Windows users when everyone in the population is a Windows user (at W).

If the current population has only a small number of Unix users, then the situation is represented by a point to the left of the intersection of the two payoff curves at I, and each individual user finds it better to choose Windows. When there is a larger number of Unix users in the population, placing the society to the right of I, it is better for each user to choose Unix. Thus, a mixed population of Unix and Windows users is sustainable as an equilibrium only when the current population has exactly I Unix users; only then will no member of the population have any incentive to switch operating systems. And even that situation is unstable. Suppose just one person accidentally makes a different decision. If he switches to Windows, his choice will push the population to the left of I, in which case others will have an incentive to switch to Windows, too. If he switches to Unix, the population moves to the right of I, creating an incentive for more people to switch to Unix. The cumulative effect of these switches will eventually push the society to an all-Unix or an all-Windows outcome; these are the two stable equilibria of the game.[4]

But which of the two stable equilibria will be achieved in this game? The answer depends on where the game starts. If you look at the configuration of today's computer users, you will see a heavily Windows-oriented population. Thus, it seems that because there are so few Unix users (or so many Windows users), the world is moving toward the all-Windows equilibrium. Schools, businesses, and private users have become locked in to this particular equilibrium as a result of an accident of history. If it is indeed true that Unix provides more benefits to society when used by everyone, then the all-Unix equilibrium should be preferred over the all-Windows one that we are approaching. Unfortunately, although society as a whole might be better off with the change, no individual computer user has an incentive to make a change. Only coordinated action can swing the pendulum toward Unix. A critical mass of individual users, more than I in Figure 11.11, must use Unix before it becomes individually rational for others to choose the same operating system.

There are many examples of similar choices of convention being made by different groups of people. The most famous cases are those in which it has been argued, in retrospect, that a wrong choice was made. Advocates claim that electric cars could have been developed for greater efficiency than gasoline; it certainly would have been cleaner. Proponents of the Dvorak typewriter/computer keyboard configuration claim that it would be better than the QWERTY keyboard if used everywhere. Many engineers agree that Betamax had more going for it than VHS in the video recorder market. In such cases, the whims of the public or the genius of advertisers help determine the ultimate equilibrium and may lead to a "bad" or "wrong" outcome from society's perspective. Other situations do not suffer from such difficulties. Few people concern themselves with fighting for a reconfiguration of traffic-light colors, for example.[5]

The ideas of positive feedback and lock-in find an important application in macroeconomics. Production is more profitable the higher the level of demand in the economy, which happens when national income is higher. In turn, income is higher when firms are producing more and are therefore hiring more workers. This positive feedback creates the possibility of multiple equilibria, of which the high-production, high-income one is better for society, but individual decisions may lock the economy into the low-production, low-income equilibrium. The better equilibrium could be turned into a focal point by public declaration—such as "The only thing we have to fear is fear itself"—but the government can also inject demand into the economy to the extent necessary to move it to the better equilibrium. In other words, the possibility of unemployment due to a deficiency of aggregate demand—as discussed in the supply-and-demand language of economic theory by the British economist John Maynard Keynes in his well-known 1936 book titled *Employment, Interest, and Money*—can be seen from a game-theoretic perspective as the result of a failure to solve a collective-action problem.[6]

Endnotes

- The term *positive feedback* may create the impression that it is a good thing, but in technical language "positive" merely connotes "reinforcing" and includes no general value judgment about the outcome. In this example, the same positive feedback mechanism could lead to either an all-Unix outcome or an all-Windows outcome; one outcome could be worse than the other. Return to reference 4
- Not everyone agrees that the Dvorak keyboard and the Betamax video recorder were clearly superior alternatives. See two articles by S. J. Liebowitz and Stephen E. Margolis, "Network Externality: An Uncommon Tragedy," *Journal of Economic Perspectives*, vol. 8 (Spring 1994), pp. 146–49, and "The Fable of the Keys," *Journal of Law and Economics*, vol. 33 (April 1990), pp. 1–25. Return to reference 5
- John Maynard Keynes, *Employment, Interest, and Money* (London: Macmillan, 1936). See also John Bryant, "A Simple Rational-Expectations Keynes-type Model," *Quarterly Journal of Economics*, vol. 98 (1983), pp. 525–28, and Russell Cooper and Andrew John, "Coordination Failures in a Keynesian Model," *Quarterly Journal of Economics*, vol. 103 (1988), pp. 441–63, for formal game-theoretic models of unemployment equilibria. Return to reference 6

Glossary

marginal private gain

The change in an individual's own payoff as a result of a small change in a continuous-strategy variable that is at his disposal.

spillover effect

Same as **external effect**.

external effect

When one person's action alters the payoff of another person or persons. The effect or spillover is *positive* if one's action raises others' payoffs (for example, network effects) and *negative* if it lowers others' payoffs (for example, pollution or congestion). Also called **externality** or **spillover effect**.

externality

Same as **external effect**.

marginal social gain

The change in the aggregate social payoff as a result of a small change in a continuous-strategy variable chosen by one player.

internalize the externality

To offer an individual a reward for the external benefits he conveys on the rest of society, or to inflict a penalty for the external costs he imposes on the rest, so as to bring his private incentives in line with social optimality.

positive feedback

When one person's action increases the payoff of another person or persons taking the same action, thus increasing their incentive to take that action too.

locked in

A situation where the players persist in a Nash equilibrium that is worse for everyone than another Nash equilibrium.

4 A BRIEF HISTORY OF IDEAS

A. The Classics

The problem of collective action has been recognized by social philosophers and economists for a very long time. The seventeenth-century British philosopher Thomas Hobbes argued that society would break down in a “war of all against all” unless it was ruled by a dictatorial monarch, or *Leviathan* (the title of his book). One hundred years later, the French philosopher Jean-Jacques Rousseau described the problem of a prisoners’ dilemma in his *Discourse on Inequality.* A stag hunt needs the cooperation of the whole group of hunters to encircle and kill the stag, but any individual hunter who sees a hare may find it better for himself to leave the circle to chase the hare. But Rousseau thought that such problems were the product of civilization and that people in the natural state lived harmoniously as “noble savages.” At about the same time, two Scots pointed out some dramatic solutions to such problems. David Hume, in his *Treatise on Human Nature,* argued that the expectations of future returns of favors can sustain cooperation. Adam Smith’s *Wealth of Nations* developed a grand vision of an economy in which the production of goods and services motivated purely by private profit could result in an outcome that was best for society as a whole.[7]

The optimistic interpretation persisted, especially among many economists and even several political scientists, to the point where it was automatically assumed that if an outcome was beneficial to a group as a whole, the actions of its members would bring that outcome about. This belief received a necessary rude shock in the mid-1960s, when Mancur Olson published *The Logic of Collective Action.* He pointed out that the best collective outcome would not prevail unless it was in each individual’s private interest to perform the assigned action—that is, unless it was a Nash equilibrium.

However, Olson did not specify the collective-action game very precisely. Although it looked like a prisoners' dilemma, Olson insisted that it was not necessarily so, and we have already seen that the problem can also take the form of a game of chicken or an assurance game.[8]

Another major class of collective-action problems—namely, those concerning the depletion of common-access resources—received attention at about the same time. If a resource such as a fishery or a meadow is open to all, each user will exploit it as much as he can, because any self-restraint on his part will merely make more available for the others to exploit. As we mentioned earlier, Garrett Hardin wrote a well-known article on this subject titled "The Tragedy of the Commons." Common-resource problems are unlike our irrigation project game, in which each person has a strong private incentive to free ride on the efforts of others. In regard to a common resource, each person has a strong private incentive to exploit it to the full, making everyone else pay the cost that results from the degradation of the resource.

B. Modern Approaches and Solutions

Until recently, many social scientists and most physical scientists took a Hobbesian line on the common-resource problem, arguing that it can be solved only by a government that forces everyone to behave cooperatively. Others, especially economists, retained their Smithian optimism. They argued that placing the resource in proper private ownership, where its benefits can be captured in the form of profit by the owner, will induce the owner to restrain its use in a socially optimal manner. He will realize that the value of the resource (fish or grass, for example) may be higher in the future because less will be available, and therefore he can make more profit by saving some of it for that future.

Nowadays, thinkers on all sides have begun to recognize that collective-action problems come in diverse forms and that there is no uniquely best solution to all of them. They also understand that groups or societies do not stand helpless in the face of such problems, but devise various ways to cope with them. Much of this work has been informed by game-theoretic analysis of repeated prisoners’ dilemmas and similar games.[9]

Solutions to collective-action problems of all types must induce individuals to act cooperatively or in a manner that would be best for the group, even though their interests may best be served by doing something else—in particular, taking advantage of the others’ cooperative behavior.[10] Humans exhibit much in the way of cooperative behavior. The act of reciprocating gifts and the skill of detecting cheating, for example, are so common in all societies and throughout history that there is reason to argue that they may be instincts.[11] But human societies generally rely heavily on purposive social and cultural customs, norms, and sanctions in inducing cooperative behavior from their individual members. These methods are conscious, deliberate attempts to design the game in order to solve the collective-action problem.[12]

We approach the matter of solution methods from the perspective of the type of game being played.

A solution is easiest if the collective-action problem takes the form of an assurance game. Then it is in every person's private interest to take the socially best action if he expects all other persons to do likewise. In other words, the socially optimal outcome is a Nash equilibrium. The only problem is that the same game has other, socially worse, Nash equilibria. Then all that is needed to achieve the best Nash equilibrium, and thereby the social optimum, is to make it a focal point—that is, to ensure the convergence of the players' expectations on it. Such a convergence can result from a social custom or convention—namely, a mode of behavior that finds automatic acceptance because it is in everyone's interest to follow it so long as others are expected to do likewise. For example, if all the farmers, herders, weavers, and other producers in an area want to get together to trade their wares, all they need is the assurance of finding others with whom to trade. Then the custom that the market is held in village X on day Y of every week makes it optimal for everyone to be there on that day.[13]

One complication remains. For the desired outcome to be a focal point, each person must have confidence that all others understand it, which in turn requires that those others have confidence that all others understand it . . . In other words, the point must be common knowledge. Usually, some prior social action is necessary to ensure that this is true. Publication in a medium that is known by everyone to be sufficiently widely read, and discussion in an inward-facing circle so that everyone knows that everyone else was present and paying attention, are some methods used for this purpose.[14]

Our analysis in Section 2 suggested that individual payoffs are often configured in such a way that collective-action problems, particularly of large groups, take the form of a prisoners' dilemma. Not surprisingly, the methods for coping with such problems have received the most attention.

The simplest solution method attempts to change people's preferences so that the game is no longer a prisoners' dilemma. If individuals get sufficient pleasure from cooperating, or suffer enough guilt or shame when they cheat, they will cooperate to maximize their own payoffs. If the extra payoff from cooperation is conditional—one gets pleasure from cooperating or guilt or shame from cheating if, but only if, many others are cooperating—then the game can turn into an assurance game. In one of its equilibria, everyone cooperates because everyone else does, and in the other, no one cooperates because no one else does. Then the collective-action problem is the simpler one of making the better equilibrium the focal point. If the extra payoff from cooperation is unconditional—one gets pleasure from cooperating or guilt or shame from cheating regardless of what the others do—then the game can have a unique equilibrium where everyone cooperates. In many situations, it is not even necessary for everyone to have such payoffs. If a substantial proportion of the population does, that may suffice for the desired collective outcome.

Some such prosocial preferences may be innate, hardwired by a biological evolutionary process. But they are more likely to be social or cultural products. Most societies make deliberate efforts to instill prosocial thinking in children during the process of socialization in families and schools. The growth of prosocial preferences is seen in experiments using ultimatum and dictator games of the kind we discussed in Chapter 3. When these experiments are conducted on children of different ages, very young children behave selfishly. By age eight, however, they develop a significant sense of equality. True prosocial preferences develop gradually thereafter, with some relapses, into an adult fair-mindedness. Thus, a long process of education and experience instills *internalized norms* into people's preferences.[15]

However, people do differ in the extent to which they internalize prosocial preferences, and the process may not go far enough to solve many collective-action problems. Most people have sufficiently broad understanding of what the socially cooperative action is in most situations, but individuals retain the personal temptation to cheat. Therefore, a system of external sanctions or punishments is needed to sustain cooperative action. We call these widely understood, but not automatically followed, rules of behavior *enforced norms*.

In Chapter 10, we described in detail several methods for achieving a cooperative outcome in prisoners' dilemma games, including repetition, penalties (or rewards), and leadership. In that discussion, we were mainly concerned with two-person dilemmas. The same methods apply to enforcement of norms in collective-action problems in large groups, with some important modifications or innovations.

We saw in Chapter 10 that repetition was the most prominent of these methods, so we focus the most attention on it here. Repetition can achieve cooperative outcomes as equilibria of individual actions in a repeated two-person prisoners' dilemma by holding up the prospect that cheating will lead to a breakdown of cooperation. More generally, what is needed to maintain cooperation is the expectation in the mind of each player that his personal benefits from cheating will be transitory and that they will quickly be replaced by a payoff lower than that associated with cooperative behavior. If players are to believe that cheating is not beneficial from a long-term perspective, cheating should be detected quickly, and the punishment that follows (reduction in future payoffs) should be sufficiently swift, sure, and painful.

A group has one advantage in this respect over a pair of individual persons. The same pair may not have occasion to interact all that frequently, but each of them is likely to interact with *someone* in the group all the time. Therefore, B’s temptation to cheat A can be countered by his fear that others, such as C, D, and so on, whom he meets in the future will punish him for this action. An extreme case where bilateral interactions are not repeated and punishment must be inflicted on one’s behalf by a third party is, in Yogi Berra’s well-known saying, “Always go to other people’s funerals. Otherwise they won’t go to yours.”

But a group has some offsetting disadvantages over direct bilateral interaction when it comes to sustaining good behavior in repeated interactions. The required speed and certainty of detection and punishment of cheating suffer as the numbers in the group increase. One sees many instances of successful cooperation in small village communities that would be unimaginable in a large city or state.

Start with the detection of cheating, which is never easy. In most real situations, payoffs are not completely determined by the players’ actions, but are subject to some random fluctuations. Even with two players, if one gets a low payoff, he cannot be sure that the other cheated; it may have been that his low payoff resulted from some random event. With more people, an additional question enters the picture: If someone cheated, who was it? Punishing someone without being sure of his guilt beyond a reasonable doubt is not only morally repulsive, but also counterproductive. The incentive to cooperate gets blunted if even cooperative actions are susceptible to punishment by mistake.

Next, with multiple players, even when cheating is detected and the cheater identified, this information has to be conveyed sufficiently quickly and accurately to others. For this, the group must be small, or else it must have a good communication or gossip network. Also, members should not have much reason to accuse others falsely.

Finally, even after cheating is detected and the information spread to the whole group, the cheater's punishment—enforcement of the social norm—has to be arranged. A third person often has to incur some personal cost to inflict such punishment. For example, if C is called on to punish B, who had previously cheated A, C may have to forgo some profitable business that he could have transacted with B. Then the inflicting of punishment is itself a collective-action game and suffers from the same temptation to shirk—that is, not to participate in the punishment. A society could construct a second-round system of punishments for shirking, but that in turn may be yet another collective-action problem! However, humans seem to have evolved an instinct whereby people get some personal pleasure from punishing cheaters even when they have not themselves been the victims of a particular act of cheating.[16] Interestingly, the notion that "one should impose sanctions, even at personal cost, on violators of enforced social norms" seems itself to have become an internalized norm.[17]

Norms are reinforced by observation of society's general adherence to them, and they lose their force if they are frequently seen to be violated. Before the advent of the welfare state, when those who fell on hard economic times had to rely on help from family or friends or their immediate small social group, the work ethic constituted a norm that held in check the temptation to slacken one's own efforts and become a free rider on the support of others. As government took over the supporting role and unemployment compensation or welfare became an entitlement, this norm of the work ethic weakened. After the sharp increases in unemployment in Europe in the late 1980s and early 1990s, a significant fraction of the population became users of the official support system, and the norm weakened even further.[18]

Different societies or cultural groups may develop different conventions and norms to achieve the same purpose. At the trivial level, each culture has its own set of good manners—ways of greeting strangers, indicating approval of food, and so on. When two people from different cultures meet, misunderstandings can arise. More importantly, each company or office has its own ways of getting things done. The differences among these customs and norms are subtle and difficult to pin down, but many mergers fail because of a clash of these "corporate cultures."

Next, consider the chicken form of collective-action games. Here, the nature of the remedy depends on whether the largest total social payoff is attained when everyone participates (what we called chicken version I in Section 1.B) or when some cooperate and others are allowed to shirk (chicken II). For chicken I, where everyone has the individual temptation to shirk, the problem is much like that of sustaining cooperation in the prisoners' dilemma, and all the earlier remarks on that game apply here, too. Chicken II is different—easier in one respect and harder in another. Once an assignment of roles between participants and shirkers is made, no one has the private incentive to switch: If the other driver is assigned the role of going straight, then you are better off swerving, and the other way around. Therefore, if a custom creates the expectation of an equilibrium, it can be maintained without further social intervention such as sanctions. However, in this equilibrium, the shirkers get higher payoffs than the participants do, and this inequality can create its own problems for the game; the conflicts and tensions, if they are major, can threaten the whole fabric of the society. If the interaction is repeated, the inequality problem can be solved by rotating the roles of participants and shirkers to equalize payoffs over time.

Sometimes the problem of differential payoffs in version II of the prisoners’ dilemma or chicken is solved not by restoring equality, but by oppression or coercion, which forces a dominated subset of society to accept the lower payoff and allows the dominant subgroup to enjoy the higher payoff. In many societies throughout history, the work of handling animal carcasses was forced on particular groups or castes in this way. And the history of the maltreatment of racial and ethnic minorities and of women provides vivid examples of such practices. Once such a system becomes established, no one member of the oppressed group can do anything to change the situation. The oppressed must get together as a group and act to change the whole system, though how to do so is itself another problem of collective action.

Finally, consider the role of leadership in solving collective-action problems. In Chapter 10, we pointed out that if the players are of very unequal “size,” the prisoners’ dilemma may disappear because it may be in the private interest of the larger player to continue cooperation and to accept the cheating of the smaller player. Here we recognize the possibility of a different kind of bigness—namely, having a “big heart.” People in most groups differ in their preferences, and many groups have one or a few who take genuine pleasure in expending personal effort to benefit the whole. If there are enough such people for the task at hand, then the collective-action problem disappears. Most schools, churches, local hospitals, and other worthy causes rely on the work of such willing volunteers. This solution, like others before it, is more likely to work in small groups, where the fruits of their actions are more closely and immediately visible to the benefactors, who are therefore encouraged to continue.

C. Applications

In her book *Governing the Commons*, Elinor Ostrom describes several examples of resolution of common-resource problems at local levels. Most of them require taking advantage of features specific to the context in order to set up systems of detection and punishment. A fishing community on the Turkish coast, for example, assigns and rotates locations to its members; the person who is assigned a good location on any given day will naturally observe and report any intruder who tries to usurp his place. Many other users of common resources, including the grazing commons in medieval England, actually restricted access and controlled overexploitation by allocating complex, tacit, but well-understood rights to individual persons. In one sense, this solution bypasses the common-resource problem by dividing up the resource into a number of privately owned subunits.

The most striking feature of Ostrom's range of cases is their immense variety. Some of the prisoners' dilemmas involving the exploitation of common-property resources that she examined were solved by private initiative by the group of people actually in the dilemma; others were solved by external public or governmental intervention. In some instances, the dilemma was not resolved at all, and the group remained trapped in the all-shirk outcome. Despite this variety, Ostrom identifies several common features that make prisoners' dilemmas of collective action easier to solve: (1) it is essential to have an identifiable and stable group of potential participants; (2) the benefits of cooperation have to be large enough to make it worth paying all the costs of monitoring and enforcing the rules of cooperation; and (3) it is very important that the members of the group can communicate with one another. This last feature accomplishes several things. First, it makes the norms clear—everyone knows what behavior is expected, what kind of cheating will not be tolerated, and what sanctions will be imposed on cheaters. Next, it spreads information about the efficacy of the cheating-detection mechanism, thereby building trust and removing the suspicion that each participant might hold that he is abiding by the rules while others are getting away with breaking them. Finally, it enables the group to monitor the effectiveness of the existing arrangements and to improve on them as necessary. All these requirements look remarkably like those identified in Chapter 10 from our theoretical analysis of the prisoners' dilemma and from the observations of Axelrod's tournaments.

Ostrom's study of the fishing village also illustrates what can be done if the social optimum requires different persons to do different things, in which case some get higher payoffs than others. In a repeated relationship, the advantageous position can rotate among the participants, thereby maintaining some sense of equality over time.

Ostrom finds that an external enforcer of cooperation may not be able to detect cheating or impose punishment with sufficient clarity and swiftness. Thus, the frequent call for centralized or government policy to solve collective-action problems is often proved wrong. Another example comes from village communities or "communes" in late-nineteenth-century Russia. These communities solved many collective-action problems of irrigation, crop rotation, management of woods and pastures, and road and bridge construction and repair in just this way. "The village . . . was not the haven of communal harmony. . . . It was simply that the individual interests of the peasants were often best served by collective activity." Reforms by early-twentieth-century czarist governments and by Soviet revolutionaries of the 1920s alike failed, partly because the old system had such a hold on the peasants' minds that they resisted anything new, but also because the reformers failed to understand the role that some of the prevailing practices played in solving collective-action problems and thus failed to replace them with equally effective alternatives.[19]

The difference between small and large groups is well illustrated by Avner Greif's comparison of two groups of traders in countries around the Mediterranean Sea in medieval times. The Maghribis were Jewish traders who relied on extended family and social ties. If one member of this group cheated another, the victim informed all the others by writing letters. When guilt was convincingly proved, no one in the group would deal with the cheater. This system worked well on a small scale of trade. But as trade expanded around the Mediterranean, the group could not find sufficiently close or reliable insiders to go to the countries with the new trading opportunities.

In contrast, the Genoese traders established a more official legal system. A contract had to be registered with the central authorities in Genoa. The victim of any cheating or violation of the contract had to take a complaint to the authorities, who carried out the investigation and imposed the appropriate fines on the cheater. This system, with all its difficulties of detection, could be more easily expanded with the expansion of trade.[20] As economies have grown and world trade has expanded, we have seen a similar shift from tightly linked groups to more arm's-length trading relationships, and from enforcement based on repeated interactions to official law.

The idea that smaller groups are more successful at solving collective-action problems, which forms the major theme of Olson's *Logic of Collective Action* (see footnote 8), has led to an insight important in political science. In a democracy, all voters have equal political rights, and the majority's preference should prevail. But we see many instances in which this does not happen. The effects of policies are generally good for some groups and bad for others. To get its preferred policy adopted, a group has to take political action—lobbying, publicity, campaign contributions, and so on. To do these things, the group must solve a collective-action problem, because each member of the group may hope to shirk and enjoy the benefits that the others' efforts have secured. If small groups are better able to solve this problem, then the policies resulting from the political process will reflect *their* preferences, even if other groups who fail to organize are more numerous and suffer losses greater than the successful groups' gains.

The most dramatic example of policies reflecting the preferences of an organized group comes from the arena of trade policy. A country's restrictions on imports help domestic producers whose goods compete with these imports, but they hurt the consumers of the imported goods and the domestic competing goods alike, because prices for these goods are higher than they would be otherwise. The domestic producers are few in number, and the consumers are almost the whole population of the country; thus, the total dollar amount of the consumers' losses is typically far bigger than the total dollar amount of the producers' gains. Political considerations based on constituency membership numbers and economic considerations of dollar gains and losses alike would lead us to expect a consumer victory in this policy arena; we would expect to see at least a push for the idea that import restrictions should be abolished, but we don't. The smaller and more tightly knit associations of producers are better able to organize for political action than are the numerous, dispersed consumers.

More than 70 years ago, the American political scientist E. E. Schattschneider provided the first extensive documentation and discussion of how pressure politics drives trade policy. He recognized that "the capacity of a group for organization has a great influence on its activity," but he did not develop any systematic theory of what determines this capacity.[21] The analysis of Olson and others has improved our understanding of the issue, but the triumph of pressure politics over economics persists in trade policy to this day. For example, in the late 1980s, the U.S. sugar trade policy that severely limited the amount of imported sugar cost each of the 240 million people in the United States about $11.50 per year, for a total of about $2.75 billion per year, while it increased the annual incomes of about 10,000 sugar-beet farmers by about $50,000 each, and those of 1,000 sugarcane farmers by as much as $500,000 each, for a total of about $1 billion. The net loss to the U.S. economy was $1.75 billion.[22] Each of the unorganized consumers continues to bear his small share of the cost in silence; many of them are not even aware that each is paying $11.50 a year too much for his sweet tooth.

If this overview of the theory and practice of solving collective-action problems seems diverse and lacking a neat summary statement, that is because the problems are equally diverse, and the solutions depend on the specifics of each problem. The one general lesson that we can provide is the importance of letting the participants themselves devise solutions by using their local knowledge of the situation, their advantage of proximity in monitoring the cooperative or shirking actions of others in the community, and their ability to impose sanctions on shirkers by exploiting various ongoing relationships within the social group.

Finally, a word of caution. You might be tempted to come away from this discussion of collective-action problems with the impression that individual freedom always leads to harmful outcomes that can and must be improved by social norms and sanctions. Remember, however, that societies face problems other than those of collective action, some of which are better solved by individual initiative than by joint efforts. Societies can often get hidebound and autocratic, becoming trapped in their norms and customs and stifling the innovation that is so often the key to economic growth. Collective action can become collective inaction.[23]

Endnotes

- The great old books cited in this paragraph have been reprinted many times in many different versions. For each, we list the year of original publication and the details of one relatively easily accessible reprint. In each case, the editor of the reprinted version provides an introduction that conveniently summarizes the main ideas. Thomas Hobbes, *Leviathan; or the Matter, Form, and Power of Commonwealth Ecclesiastical and Civil*, 1651 (Everyman Edition, London: J. M. Dent, 1973); David Hume, *A Treatise of Human Nature*, 1739 (Oxford: Clarendon Press, 1976); Jean-Jacques Rousseau, *A Discourse on Inequality*, 1755 (New York: Penguin Books, 1984); Adam Smith, *An Inquiry into the Nature and Causes of the Wealth of Nations*, 1776 (Oxford: Clarendon Press, 1976). Return to reference 7
- Mancur Olson, *The Logic of Collective Action* (Cambridge, Mass.: Harvard University Press, 1965). Return to reference 8
- Prominent in this literature are Michael Taylor, *The Possibility of Cooperation* (New York: Cambridge University Press, 1987); Elinor Ostrom, *Governing the Commons* (New York: Cambridge University Press, 1990); and Matt Ridley, *The Origins of Virtue* (New York: Viking Penguin, 1996). Return to reference 9
- The problem of the need to attain cooperation and its solutions are not unique to human societies. Examples of cooperative behavior in the animal kingdom have been explained by biologists in terms of the advantage of the gene and of the evolution of instincts. For more, see Chapter 12 and Ridley, *Origins of Virtue*. Return to reference 10
- See Ridley, *Origins of Virtue*, Chapters 6 and 7. Return to reference 11

- The social sciences do not have precise and widely accepted definitions of terms such as *custom* and *norm*; nor are the distinctions among such terms always clear and unambiguous. We set out some definitions in this section, but be aware that you may find different usage in other books. Our approach is similar to those found in Richard Posner and Eric Rasmusen, "Creating and Enforcing Norms, with Special Reference to Sanctions," *International Review of Law and Economics*, vol. 19, no. 3 (September 1999), pp. 369–82, and in David Kreps, "Intrinsic Motivation and Extrinsic Incentives," *American Economic Review*, Papers and Proceedings, vol. 87, no. 2 (May 1997), pp. 359–64; Kreps uses the term *norm* for all the concepts that we classify under different names.

  Sociologists have a taxonomy of norms that is different from that of economists. It is based on the importance of the norms (those pertaining to trivial matters such as table manners are called *folkways*, and those pertaining to weightier matters are called *mores*), and on whether the norms are formally codified as *laws*. They also maintain a distinction between *values* and norms, recognizing that some norms may run counter to persons' values and therefore require sanctions to enforce them. This distinction corresponds to ours between customs, internalized norms, and enforced norms. The conflict between individual values and social goals arises for enforced norms, but not for customs or *conventions*, as we label them, or for internalized norms. See Donald Light and Suzanne Keller, *Sociology*, 4th ed. (New York: Knopf, 1987), pp. 57–60.

  Return to reference 12
- In his study of the emergence of cooperation, *Cheating Monkeys and Citizen Bees* (New York: Free Press, 1999), the evolutionary biologist Lee Dugatkin labels this case

"selfish teamwork." He argues that such behavior is likelier to arise in times of crisis, because each person is pivotal at those times. In a crisis, the outcome of the group interaction is likely to be disastrous for everyone if even one person fails to contribute to the group's effort to get out of the dire situation. Thus, each person is willing to contribute so long as the others do. We will mention Dugatkin's full classification of alternative approaches to cooperation in Chapter 12 on evolutionary games. Return to reference 13

- See Michael Chwe, *Rational Ritual: Culture, Coordination, and Common Knowledge* (Princeton, N.J.: Princeton University Press, 2001), for a discussion of this issue and numerous examples and applications of it. Return to reference 14
- Colin Camerer, *Behavioral Game Theory* (Princeton, N.J.: Princeton University Press, 2003), pp. 65–67. See also pp. 63–75 for an account of differences in prosocial behavior along different dimensions of demographic characteristics and across different cultures. Return to reference 15
- For evidence of such altruistic punishment instinct, see Ernst Fehr and Simon Gächter, "Altruistic Punishment in Humans," *Nature*, vol. 415 (January 10, 2002), pp. 137–40. Return to reference 16
- Our distinction between internalized norms and enforced norms is similar to Kreps's distinction between functions (iii) and (iv) of norms (Kreps, "Intrinsic Motivation and Extrinsic Incentives," p. 359). Society can also reward desirable actions just as it can punish undesirable ones. Again, the rewards, financial or otherwise, can be given externally, or players' payoffs can be changed so that they take pleasure in doing the right thing. The two types of rewards can interact; for example, the peerages and knighthoods given to British philanthropists and others who do good deeds for British

society are external rewards, but individual persons value them only because respect for knights and peers is a British social norm. Return to reference 17

- Assar Lindbeck, "Incentives and Social Norms in Household Behavior," *American Economic Review*, Papers and Proceedings, vol. 87, no. 2 (May 1997), pp. 370–77. Return to reference 18
- Orlando Figes, *A People's Tragedy: The Russian Revolution 1891–1924* (New York: Viking Penguin, 1997), pp. 89–90, 240–41, 729–30. See also Ostrom, *Governing the Commons*, p. 23, for other instances where external, government-enforced attempts to solve common-resource problems actually made them worse. Return to reference 19
- Avner Greif, "Cultural Beliefs and the Organization of Society: A Historical and Theoretical Reflection on Collectivist and Individualist Societies," *Journal of Political Economy*, vol. 102, no. 5 (October 1994), pp. 912–50. Return to reference 20
- E. E. Schattschneider, *Politics, Pressures, and the Tariff* (New York: Prentice-Hall, 1935); see especially pp. 285–86. Return to reference 21
- Stephen V. Marks, "A Reassessment of the Empirical Evidence on the U.S. Sugar Program," in *The Economics and Politics of World Sugar Policies*, ed. Stephen V. Marks and Keith E. Maskus (Ann Arbor: University of Michigan Press, 1993), pp. 79–108. Return to reference 22
- David Landes, *The Wealth and Poverty of Nations* (New York: W. W. Norton, 1998), Chapters 3 and 4, makes a spirited case for this effect. Return to reference 23

Glossary

norm
: A pattern of behavior that is established in society by a process of education or culture, to the point that a person who behaves differently experiences a negative psychic payoff.

sanction
: Punishment approved by society and inflicted by others on a member who violates an accepted pattern of behavior.

custom
: Same as **convention**.

convention
: A mode of behavior that finds automatic acceptance as a focal point, because it is in each individual’s interest to follow it when others are expected to follow it too (so the game is of the assurance type). Also called **custom**.

oppression
: In this context, same as **coercion**.

coercion
: In this context, forcing a player to accept a lower payoff in an asymmetric equilibrium in a collective action game, while other favored players are enjoying higher payoffs. Also called **oppression** in this context.

5 “HELP!”: A GAME OF CHICKEN WITH MIXED STRATEGIES

In our discussion of the chicken variant of collective-action games in earlier sections of this chapter, we looked only at the pure-strategy equilibria. But we know from Chapter 7 that such games have mixed-strategy equilibria, too. In collective-action problems, where each participant is thinking, “It is better if I wait for enough others to participate so that I can shirk; but then again, maybe they won’t, in which case I should participate,” mixed strategies nicely capture the spirit of such vacillation. Our last story is a dramatic, even chilling, application of such a mixed-strategy equilibrium.

In 1964 in New York City (in Kew Gardens, Queens), a woman named Kitty Genovese was killed in a brutal attack that lasted more than half an hour. She screamed through it all, and although her screams were heard by many people, and at least three actually witnessed some part of the attack, no one went to help her, or even called the police.

The story created a sensation, and commentators found several ready theories to explain it. The press and most of the public saw this episode as a confirmation of their belief that New Yorkers—or big-city dwellers, or Americans, or people more generally—were apathetic or didn’t care about their fellow human beings.

However, even a little introspection or observation will convince you that people do care about the well-being of other humans, even strangers. Social scientists offered a different explanation for what happened, which they labeled pluralistic ignorance. The idea behind this explanation is that no one can be sure about what is happening or whether help is really needed and how much. People look to one another for clues or guidance about these matters and try to interpret other people’s behavior in this light. If they see that no one else is doing anything to help, they interpret it as meaning that help is probably not needed, and so they don't do anything either. This explanation has some intuitive appeal, but is unsatisfactory in the Kitty Genovese context. There is a very strong presumption that a screaming woman needs help. What did the onlookers think—that a movie was being shot in their obscure neighborhood? If so, where were the lights, the cameras, the director, other crew?

A better explanation would recognize that although each onlooker might experience strong personal distress from Kitty's suffering and might get genuine personal pleasure if she were saved, each must balance that against the cost of getting involved. You might have to identify yourself if you call the police; you might then have to appear in court as a witness, and so on. Thus, we see that each person may prefer to wait for someone else to call and hope to get for himself the free rider's benefit of the pleasure of a successful rescue.

Social psychologists have a slightly different version of this idea of free riding, which they label diffusion of responsibility. In this version, the idea is that everyone might agree that help is needed, but they are not in direct communication with one another and so cannot coordinate on who should help. Each person may believe that help is someone else's responsibility. And the larger the group, the more likely it is that each person will think that someone else will probably help, and that therefore he can save himself the trouble and the cost of getting involved.

Social psychologists conducted some experiments to test this hypothesis. They staged situations in which someone needed help of different kinds in different places and with different-sized crowds present. Among other things, they found that the larger the size of the crowd, the less likely help was to come forth.

The concept of diffusion of responsibility seems to explain this finding, but not completely. It claims that the larger the crowd, the less likely any one person is to help. But there are more people in a larger crowd, and only one person is needed to act and call the police to secure help. To make it less likely that even one person helps, the chance of any one person helping has to decrease sufficiently fast with the increase in the total number of potential helpers to offset that increase. To find out whether it does so requires game-theoretic analysis, which we now supply.[24]

We consider only the aspect of diffusion of responsibility in which action is not consciously coordinated, and we leave aside all other complications of information and inference. Thus, we assume that everyone believes that taking action is necessary and is worth the cost.

Suppose N people are in the group. The action brings each of them a benefit B. Only one person is needed to take the action; more are redundant. Anyone who acts bears the cost C. We assume that $B > C$; so it is worth any one person's while to act even if no one else is acting. Thus, the action is justified in a very strong sense.

The problem is that anyone who takes the action gets the benefit B and pays the cost C for a net payoff of $(B - C)$, whereas he would get the higher payoff B if someone else took the action. Thus, each person has the temptation to let someone else take the action and to become a free rider on another's effort. When all N people are thinking thus, what will be the equilibrium outcome?

If $N = 1$, the single person has a simple decision problem rather than a game. He gets $B - C > 0$ if he takes the action and 0 if he does not. Therefore, he goes ahead and helps.

If $N > 1$, we have a game of strategic interaction with several equilibria. Let us begin by ruling out some possibilities. With $N > 1$, there cannot be a pure-strategy Nash equilibrium in which all people act, because then any one of them would do better by switching to free riding. Likewise, there cannot be a pure-strategy Nash equilibrium in which no one acts, because *given that no one else is acting* (remember that in Nash equilibrium, each player takes the others' strategies as given), it pays any one person to act.

There *are* Nash equilibria in which exactly one person acts; in fact, there are N such equilibria, one corresponding to each member of the group. But when everyone is making the decision individually in isolation, there is no way to coordinate and designate who is to act. Even if members of the group were to attempt such coordination, they might try to negotiate over the responsibility and not reach a conclusion, at least not in time to be of help. Therefore, it is of interest to examine equilibria in which all members have identical strategies.

We already saw that there cannot be an equilibrium in which all N people follow the same pure strategy. Therefore, we should see whether there can be an equilibrium in which they all follow the same mixed strategy. Actually, mixed strategies are quite appealing in this context. The people are isolated, and each is trying to guess what the others will do. Each is thinking, Perhaps I should call the police . . . but maybe someone else will . . . but what if they don't . . . ? Each breaks off this process at some point and does the last thing that he thought of in this chain, but we have no good way of predicting what that last thing is. A mixed strategy carries the flavor of this idea of a chain of guesswork being broken at a random point.

So suppose P is the probability that any one person will not act. If one particular person is willing to mix strategies, he must be indifferent between the two pure strategies of acting and not acting. Acting gets him $(B - C)$ for sure. Not acting will get him 0 if none of the other $(N - 1)$ people act and B if at least one of them does act. Because the probability that any one person fails to act is P, and because they are deciding independently, the probability that none of the $(N - 1)$ others acts is P^{N-1}, and the probability that at least one does act is $(1 - P^{N-1})$. Therefore, the expected payoff to the one person when he does not act is

$$0 \times P^{N-1} + B(1 - P^{N-1}) = B(1 - P^{N-1}).$$

And that one person is indifferent between acting and not acting when

$$B - C = B(1 - P^{N-1}) \quad \text{or when} \quad P^{N-1} = \frac{C}{B} \quad \text{or} \quad P = \left(\frac{C}{B}\right)^{1/(N-1)}.$$

Note how this indifference condition for *one* selected player determines the probability with which the *other* players mix their strategies.

Having obtained the equilibrium mixture probabilities, we can now see how they change as the group size N changes. Remember that $C/B < 1$. As N increases from 2 to infinity, the power $1/(N - 1)$ decreases from 1 to 0. Then C/B raised to this power—namely, P—increases from C/B to 1. Remember that P is the probability that any one person does not take the action. Therefore, the probability of action by any one person—namely, $(1 - P)$—falls from $1 - C/B = (B - C)/B$ to 0.[25]

In other words, the more people there are, the less likely any one of them is to act. This is intuitively true, and in good conformity with the idea of diffusion of responsibility. But it does not yet give us the conclusion that help is less likely to be forthcoming in a larger group. As we said before, help requires action by only one person. Just because there are more and more people, each of whom is less and less likely to act, we cannot conclude immediately that the probability of *at least one* of them acting gets smaller. More calculation is needed to see whether this is the case.

Because the N persons are randomizing independently in the Nash equilibrium, the probability Q that *not even one* of them helps is

$$Q = P^N = \left(\frac{C}{B}\right)^{N/(N-1)}.$$

As N increases from 2 to infinity, $N/(N-1)$ decreases from 2 to 1, and then Q increases from $(C/B)^2$ to C/B. Correspondingly, the probability that *at least one* person helps—namely, $(1-Q)$—decreases from $1-(C/B)^2$ to $1-C/B$.[26]

So our exact calculation does bear out the hypothesis: The larger the group, the *less* likely help is to be given at all. The probability of help does not, however, reduce to zero even in very large groups; instead, it levels off at a positive value—namely, $(B-C)/B$—which depends on the benefit and cost of action to each individual.

We see how game-theoretic analysis sharpens the ideas from social psychology with which we started. The diffusion of responsibility theory takes us part of the way—namely, to the conclusion that any one person is less likely to act when he is part of a larger group. But the desired conclusion—that larger groups are less likely to provide help at all—needs further and more precise probability calculation based on the analysis of individual mixing and the resulting interactive (game) equilibrium.

And now we ask, did Kitty Genovese die in vain? Do the theories of pluralistic ignorance, diffusion of responsibility, and free riding still play out in the decreased likelihood of individual action within increasingly large cities? Perhaps not. John Tierney of the *New York Times* has publicly extolled the virtues of "urban cranks,"[27] people who encourage the civility of the group through prompt punishment of those who exhibit unacceptable behavior—including litterers, noise polluters, and the generally obnoxious boors of society. Such cranks are essentially enforcers of a cooperative norm for society. And as Tierney surveys the actions of known cranks, he reminds the rest of us that "new cranks must be mobilized! At this very instant, people are wasting time reading while norms are being flouted out on the street. . . . You don't live alone in this world! Have you enforced a norm today?" In other words, we need social norms, as well as some people who have internalized the norm of enforcing norms.

Endnotes

- For a fuller account of the Kitty Genovese story and for the analysis of such situations from the perspective of social psychology, see John Sabini, *Social Psychology*, 2nd ed. (New York: W. W. Norton, 1995), pp. 39–44. Our game-theoretic model is based on Thomas Palfrey and Howard Rosenthal, "Participation and the Provision of Discrete Public Goods," *Journal of Public Economics*, vol. 24 (1984), pp. 171–93. Many purported facts of the story have been recently challenged in *Kitty Genovese: The Murder, the Bystanders, and the Crime that Changed America* by Kevin Cook (New York: W. W. Norton, 2014), but the power and impact of the originally reported story on American thinking about urban crime remains, and it is still a good example for game-theoretic analysis. Return to reference 24
- Consider the case in which B = 10 and C= 8. Then P equals 0.8 when n = 2, rises to 0.998 when n = 100, and approaches 1 as N continues to rise. The probability of action by any one person is 1 − P, which falls from 0.2 to 0 as N rises from 2 toward infinity. Return to reference 25
- With the same sample values for B (10) and C (8), this result implies that increasing N from 2 to infinity increases the probability that not even one person helps from 0.64 to 0.8. And the probability that at least one person helps falls from 0.36 to 0.2. Return to reference 26
- John Tierney, "The Boor War: Urban Cranks, Unite—Against All Uncivil Behavior. Eggs Are a Last Resort," *New York Times Magazine*, January 5, 1997. Return to reference 27

Glossary

pluralistic ignorance
: A situation of collective action where no individual knows for sure what action is needed, so everyone takes the cue from other people’s actions or inaction, possibly resulting in persistence of wrong choices.

diffusion of responsibility
: A situation where action by one or a few members of a large group would suffice to bring about an outcome that all regard as desirable, but each thinks it is someone else’s responsibility to take this action.

SUMMARY

Multiplayer games generally concern problems of *collective action*. The general structure of collective-action games may be manifested as a prisoners’ dilemma, a game of chicken, or an assurance game. The critical difficulty with such games in any form is that the Nash equilibrium arising from individually rational choices may not be the *socially optimal* outcome—the outcome that maximizes the sum of the payoffs of all the players.

In collective-action games, when a person’s action has some effect on the payoffs of all the other players, we say that there are *spillover effects*, or *externalities.* They can be positive or negative and can lead to outcomes driven by private interests that are not socially optimal. When actions create negative spillover effects, they are overused from the perspective of society; when actions create positive spillover effects, they are underused. The additional possibility of *positive feedback* exists when there are positive spillover effects; in such a case, the game may have multiple Nash equilibria.

Problems of collective action have been recognized for many centuries and discussed by scholars from diverse fields. Several early works professed no hope for the situation, but others offered up dramatic solutions. The most recent treatments of the subject acknowledge that collective-action problems arise in diverse areas and that there is no single optimal solution. Social scientific analysis suggests that social *custom*, or *convention*, can lead to cooperative behavior. Other possibilities for solutions come from the creation of *norms* of acceptable behavior. Some of these norms are *internalized* in individuals’ payoffs; others must be *enforced* by the use of *sanctions* in response to the uncooperative behavior. Much of the literature agrees that small groups are more successful at solving collective-action problems than large ones.

In large-group games, *diffusion of responsibility* can lead to behavior in which individual persons wait for others to take action and *free ride* on the benefits of that action. If help is needed, it is less likely to be given at all as the size of the group available to provide it grows.

KEY TERMS

coercion (452)

collective action (420)

convention (448)

custom (448)

diffusion of responsibility (457)

external effect (436)

externality (436)

free rider (423)

internalize the externality (441)

locked in (444)

marginal private gain (436)

marginal social gain (436)

nonexcludable (421)

nonrival (421)

norm (447)

oppression (452)

pluralistic ignorance (456)

positive feedback (443)

pure public good (421)

sanction (447)

social optimum (423)

spillover effect (436)

Glossary

collective action
: A problem of achieving an outcome that is best for society as a whole, when the interests of some or all individuals will lead them to a different outcome as the equilibrium of a noncooperative game.

nonexcludable
: Benefits that are available to each individual, regardless of whether he has paid the costs that are necessary to secure the benefits.

nonrival
: Benefits whose enjoyment by one person does not detract anything from another person's enjoyment of the same benefits.

pure public good
: A good or facility that benefits all members of a group, when these benefits cannot be excluded from a member who has not contributed efforts or money to the provision of the good, and the enjoyment of the benefits by one person does not significantly detract from their simultaneous enjoyment by others.

free rider
: A player in a collective-action game who intends to benefit from the positive externality generated by others' efforts without contributing any effort of his own.

social optimum
: In a collective-action game where payoffs of different players can be meaningfully added together, the social optimum is achieved when the sum total of the players' payoffs is maximized.

marginal private gain
: The change in an individual's own payoff as a result of a small change in a continuous-strategy variable that is

at his disposal.

spillover effect

Same as **external effect**.

external effect

When one person's action alters the payoff of another person or persons. The effect or spillover is *positive* if one's action raises others' payoffs (for example, network effects) and *negative* if it lowers others' payoffs (for example, pollution or congestion). Also called **externality** or **spillover effect**.

externality

Same as **external effect**.

marginal social gain

The change in the aggregate social payoff as a result of a small change in a continuous-strategy variable chosen by one player.

internalize the externality

To offer an individual a reward for the external benefits he conveys on the rest of society, or to inflict a penalty for the external costs he imposes on the rest, so as to bring his private incentives in line with social optimality.

positive feedback

When one person's action increases the payoff of another person or persons taking the same action, thus increasing their incentive to take that action too.

locked in

A situation where the players persist in a Nash equilibrium that is worse for everyone than another Nash equilibrium.

norm

A pattern of behavior that is established in society by a process of education or culture, to the point that a person who behaves differently experiences a negative psychic payoff.

sanction

Punishment approved by society and inflicted by others on a member who violates an accepted pattern of behavior.

custom

Same as **convention**.

convention

A mode of behavior that finds automatic acceptance as a focal point, because it is in each individual’s interest to follow it when others are expected to follow it too (so the game is of the assurance type). Also called **custom**.

oppression

In this context, same as **coercion**.

coercion

In this context, forcing a player to accept a lower payoff in an asymmetric equilibrium in a collective action game, while other favored players are enjoying higher payoffs. Also called **oppression** in this context.

pluralistic ignorance

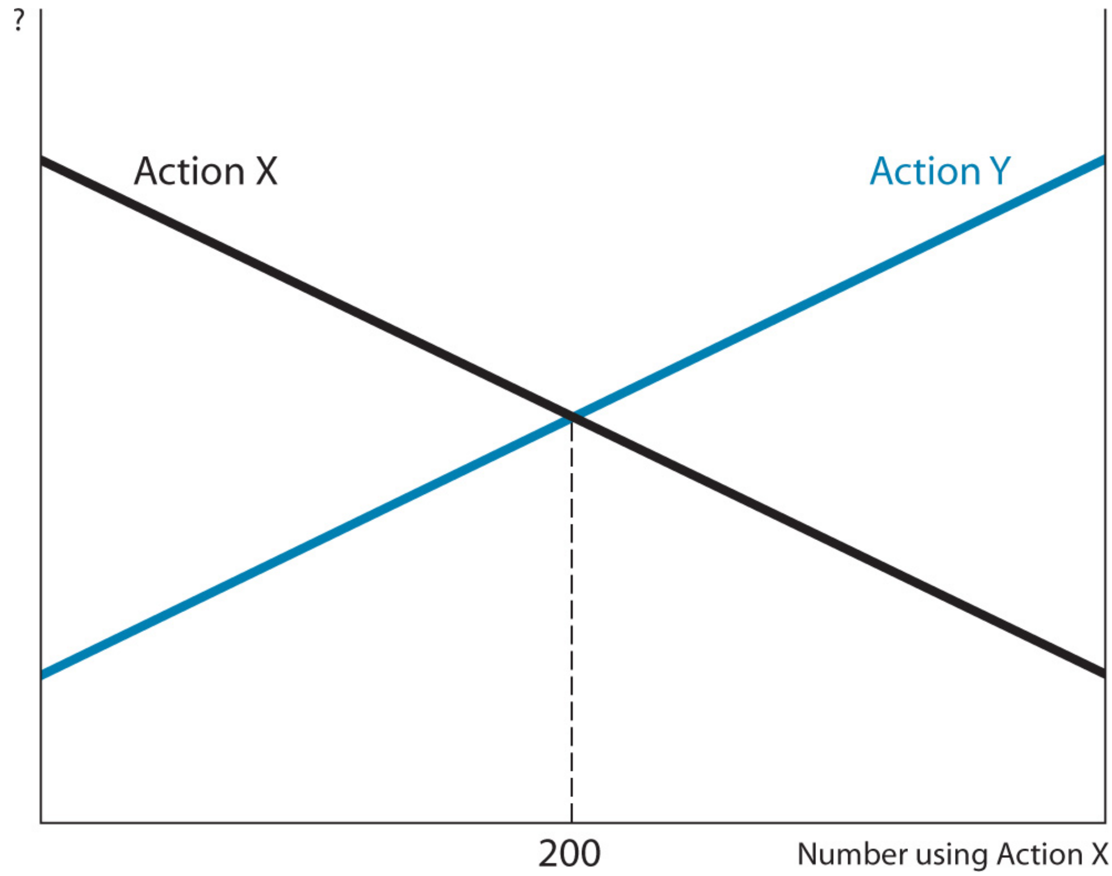
A situation of collective action where no individual knows for sure what action is needed, so everyone takes the cue from other people’s actions or inaction, possibly resulting in persistence of wrong choices.

diffusion of responsibility

A situation where action by one or a few members of a large group would suffice to bring about an outcome that all regard as desirable, but each thinks it is someone else’s responsibility to take this action.

SOLVED EXERCISES

1. Suppose that 400 people are choosing between Action X and Action Y. The relative payoffs of the two actions depend on how many of the 400 people choose Action X and how many choose Action Y. The payoffs are as shown in the following graph, but the vertical axis is not labeled, so you do not know whether the curves show the benefits or the costs of the two actions.



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1. You are told that the outcome in which 200 people choose Action X is an *un*stable equilibrium. If 100 people are currently choosing Action X, would you expect the number of people choosing Action X to increase or decrease over time? Why?
2. For the graph to be consistent with the behavior that you described in part (a), should the curves be labeled as

indicating the *costs* or *benefits* of Action X and Action Y? Explain your answer.

2. A group has 100 members. Each person can choose to participate or not participate in a common project. If n of them participate in the project, then each participant derives the benefit $p(n) = n$, and each of the $(100 - n)$ shirkers derives the benefit $s(n) = 4 + 3n$.
    1. Is this an example of a prisoners' dilemma, a game of chicken, or an assurance game?
    2. Write the expression for the total social payoff for the group.
    3. Show, either graphically or mathematically, that the maximum total social payoff for the group occurs when $n = 74$.
    4. What difficulties will arise in trying to get exactly 74 participants and allowing the remaining 26 to shirk?
    5. How might the group try to overcome the difficulties identified in part (d)?
3. Consider a small geographic region with a total population of 1 million. There are two towns, Alphaville and Betaville, in which each person can choose to live. For each person, the benefit of living in a town increases for a while with the size of the town (because larger towns have more amenities and so on), but after a point it decreases (because of congestion and so on). If x is the fraction of the population that lives in the same town as you do, your payoff is given by

    x if $0 \leqslant x \leqslant 0.4$

    $0.6 - 0.5x$ if $0.4 < x \leqslant 1$.

    1. Draw a graph like Figure 11.11, showing the benefits of living in the two towns, as the fraction of the population living in one versus the other varies continuously from 0 to 1.
    2. Equilibrium is reached either when both towns are populated and their residents have equal payoffs or when one town—say, Betaville—is totally depopulated, and the residents of the other town (Alphaville) get a higher payoff than would the very first person who seeks to populate Betaville. Use your graph to find all such equilibria.
    3. Now consider a dynamic process of adjustment whereby people gradually move toward the town whose residents currently enjoy a larger payoff than do the residents of the other

town. Which of the equilibria identified in part (b) will be stable with these dynamics? Which ones will be unstable?

4. Suppose an amusement park is being built in a city with a population of 100. Voluntary contributions are being solicited to cover the cost. Each citizen is being asked to give \$100. The more people who contribute, the larger the park will be, and the greater the benefit to each citizen. But it is not possible to keep out the noncontributors; they get their share of this benefit anyway. Suppose that when there are n contributors in the population, where n can be any whole number between 0 and 100, the benefit to each citizen in monetary unit equivalents is n^2 dollars.
   1. Suppose that initially no one is contributing. You are the mayor of the city. You would like everyone to contribute, and you can use persuasion on some people. What is the minimum number whom you need to persuade before everyone else will join in voluntarily?
   2. Find the Nash equilibria of the game where each citizen is deciding whether to contribute.
5. In the Italian province of Tuscany, enterprising “hunters” search through the woods each day for the most expensive food in the world, truffles. These delicate subterranean fungi are beloved[28] for their strong pungency and are literally worth their weight in gold, at a cost up to \$200 per ounce. The most prized truffles grow only in the wild and must be found by smell (by dog or pig), making them very hard to find (and harder still if many others are also out looking for them). Consider a game played by 99 residents of San Miniato, a Tuscan town famous for its white truffles, deciding each day whether to hunt for truffles or do some other work. If H people go hunting, each will, on average, find $10 - (H/10)$ ounces of truffles, which sell in the San Miniato market for \$200/ounce. (For simplicity, assume that there is unlimited demand at this price.) Those who do not go hunting will do work that earns them \$50 for the day.
   1. Confirm that each resident prefers to go truffle hunting if no one else does, but prefers not to go truffle hunting if everyone else does.
   2. Suppose that the residents decide sequentially whether to go truffle hunting. How many residents go truffle hunting in the resulting rollback equilibrium? How much daily income is generated in the town (adding up the income of all residents)?

3. One day, the mayor of San Miniato decides to limit the number of people who are allowed to go truffle hunting each day. How many people should be allowed to go truffle hunting each day in order to maximize total town income?
4. Residents who are not selected to go truffle hunting under the mayor's plan will have an incentive to try to sneak away and go hunting anyway. How might the townspeople address this challenge?

6. In the early 1950s, Henry O. Bakken struck oil on his farm near Tioga, North Dakota, but it wasn't until the 1990s that geologists appreciated the massive scale of his discovery: an enormous formation underlying parts of Montana, North Dakota, Saskatchewan, and Manitoba and holding an estimated 413 billion barrels of oil. Little of that oil could be recovered prior to recent advances in drilling technology, including hydraulic fracturing (known as "fracking") and horizontal drilling. In 2015, the North Dakota Department of Natural Resources estimated that the break-even price for drilling "the Bakken" was $40 per barrel, meaning that landowners can earn a profit from drilling their land if oil is selling for more than $40/barrel, but not otherwise. Because the Bakken formation is so large, drilling activity in the area can affect global oil prices. For this exercise, suppose that the global oil price will equal $\$(60 - 25X)$ per barrel, where $0 \leqslant X \leqslant 1$ is the fraction of landowners over the Bakken who actively drill their land. Suppose that the number of landowners is extremely large, so that any fraction X is possible. Assume also that each landowner who decides to drill will extract exactly 100,000 barrels of oil. Finally, for simplicity, assume that drilling is a once-and-for-all decision and that all landowners decide simultaneously whether to drill.[29]
   1. Express each landowner's profit (or loss) when drilling as a function of X. Confirm that each landowner prefers to drill if no one else does (when $X = 0$), but prefers not to drill if everyone else does (when $X = 1$).
   2. Describe the Nash equilibrium of the (simultaneous-move) drilling game. What fraction of landowners choose to drill in the Nash equilibrium?

   One day, leaders in the United States and Canada reach an agreement to impose a new "drilling tax" on landowners who drill anywhere over the Bakken. The tax system collects $10 in tax revenue per barrel of oil extracted and distributes that money back to all owners of land over the formation (equally) in the form of a "carbon dividend."

3. Express each landowner's profit (or loss), when drilling and when not drilling, as functions of X, taking into account the drilling tax and the carbon dividend. Confirm that each landowner prefers to drill if no one else does (when $X = 0$), but prefers not to drill if everyone else does (when $X = 1$).
4. Describe the Nash equilibrium of the drilling game after the drilling tax and carbon dividend are introduced. What fraction of landowners choose to drill in this Nash equilibrium?
5. Do landowners over the Bakken earn more or less profit after the drilling tax and carbon dividend are introduced? Explain your answer in terms of how this policy impacts the collective-action problem.

7. Put the Keynesian idea of unemployment described at the end of Section 3.D into a game with a properly specified set of payoffs, and show the multiple equilibria in a graph. Show the level of production (national product) on the vertical axis as a function of a measure of the level of demand (national income) on the horizontal axis. Equilibrium is reached when national product equals national income—that is, when the function relating the two cuts the 45° line. For what shapes of the function can there be multiple equilibria? Why might you expect such shapes in reality? Suppose that income increases when current production exceeds current income, and that income decreases when current production is less than current income. In this dynamic process, which equilibria are stable and which ones unstable?
8. Write a brief description of a strategic game that you have witnessed or participated in that includes a large number of players and in which individual players' payoffs depend on the number of other players and their actions. Try to illustrate your game with a graph if possible. Discuss the outcome of the actual game in light of the fact that many such games do not have socially optimal outcomes. Do you see evidence of such an outcome in your game?

UNSOLVED EXERCISES

1. Figure 11.5 illustrates the payoffs in a general, two-person collective-action game. There we showed various inequalities in the algebraic payoffs [$p(1)$, etc.] that made the game a prisoners' dilemma. Now you are asked to find similar inequalities corresponding to other kinds of games.
   1. Under what type of payoff structure(s) is the two-person game a chicken game? What further condition(s) on payoffs make the game version I of chicken (as in Figure 11.3)?
   2. Under what type of payoff structure(s) is the two-person game an assurance game?
2. A group has 100 members. Each person can choose to participate or not participate in a common project. If n of them participate in the project, then each participant derives the benefit $p(n) = n$, and each of the $(100 - n)$ shirkers derives the benefit $s(n) = 3n - 180$.
   1. Is this collective-action problem an example of a prisoners' dilemma, a game of chicken, or an assurance game?
   2. Write the expression for the total social payoff of the group.
   3. Show, either graphically or mathematically, that the maximum total social payoff for the group occurs when $n = 100$.
   4. What difficulties will arise in trying to get all 100 group members to participate?
   5. How might the group try to overcome the difficulties identified in part (d)?
3. A class with 30 students enrolled is given a homework assignment with five questions. The first four are the usual kinds of problems, worth a total of 90 points. But the fifth is an interactive game for the class. The question reads: "You can choose whether to answer this question. If you choose to do so, you merely write, 'I hereby answer Question 5.' If you choose not to answer Question 5, your score for the assignment will be based on your performance on the first four questions. If you choose to answer Question 5, then your scoring will be as follows: If fewer than half of the students in the class answer Question 5, you get 10 points for Question 5; that is, 10 points will be added to your score on the other four questions to get your total score for the assignment. If half or more than half of

the students in the class answer Question 5, you get −10 points; that is, 10 points will be subtracted from your score on the other questions."

1. Draw a graph illustrating the payoffs from the two possible strategies, Answer Question 5 and Don't Answer Question 5, in relation to the number of other students who answer it. Find the Nash equilibrium of the game.
2. What would you expect to see happen in this game if it were actually played in a college classroom? Why? Consider two cases: (i) The students make their choices individually, with no communication. (ii) The students make their choices individually, but can discuss these choices ahead of time in a discussion forum available on the class Web site.

4. There are two routes for driving from A to B. One is a freeway, and the other consists of local roads. The benefit of using the freeway is constant and equal to 1.8, irrespective of the number of people using it. Local roads get congested when too many people use this alternative, but if not enough people use it, the few isolated drivers run the risk of becoming victims of crimes. Suppose that when a fraction x of the population is using the local roads, the benefit of this mode to each driver is given by

   $1 + 9x - 10x^2$.

   1. Draw a graph showing the benefits of the two driving routes as functions of x, regarding x as a continuous variable that can range from 0 to 1.
   2. Identify all possible equilibrium traffic patterns from your graph in part (a). Which equilibria are stable? Which ones are unstable? Why?
   3. What value of x maximizes the total social payoff to the whole population?

5. Suppose a class of 100 students is comparing two careers—lawyer and engineer. An engineer gets take-home pay of $100,000 per year, irrespective of the numbers who choose this career. Lawyers make work for one another, so as the total number of lawyers increases, the income of each lawyer increases—up to a point. Ultimately, the competition among them drives down the income of each. Specifically, if there are N lawyers, each will get $100N - N^2$ thousand dollars a year. The annual cost of running a legal practice (office space, secretary, paralegals, access to online reference services, and so forth) is $800,000. Therefore, each

lawyer takes home $100N - N^2 - 800$ thousand dollars a year when there are N of them.

1. Draw a graph showing the take-home income of each lawyer on the vertical axis and the number of lawyers on the horizontal axis. (Plot a few points—say, for 0, 10, 20, . . . , 90, 100 lawyers. Fit a curve to the points, or use a computer graphics program if you have access to one.)
2. When career choices are made in an uncoordinated way, what are the possible equilibrium outcomes?
3. Now suppose the whole class decides how many should become lawyers, aiming to maximize the total take-home income of the whole class. What will be the number of lawyers? (If you can, use calculus, regarding N as a continuous variable. Otherwise, you can use graphical methods or a spreadsheet.)

6. A group of 12 countries is considering whether to form a monetary union. They differ in their assessments of the costs and benefits of this move, but each stands to gain more from joining, and lose more from staying out, when more of the other countries choose to join. The countries are ranked in order of their liking for joining, 1 having the highest preference for joining and 12 the least. Each country has two actions, IN and OUT. Let

$$B(i, n) = 2.2 + n - i$$

be the payoff to a country with ranking i when it chooses IN and n others have chosen IN. Let

$$S(i, n) = i - n$$

be the payoff to a country with ranking i when it chooses OUT and n others have chosen IN.

1. Show that for country 1, IN is the dominant strategy.
2. Having eliminated OUT for country 1, show that IN becomes the dominant strategy for country 2.
3. Continuing in this way, show that all countries will choose IN.
4. Contrast the payoffs in this outcome with those where all choose OUT. How many countries are made worse off by the formation of the union?

Endnotes

- The French gastronome J.A. Brillat-Savarin famously referred to truffles as “the diamond of the kitchen” in his 1825 treatise on food and cooking. (Brillat-Savarin, one of the world’s first gastronomic essayists, also wrote, “Tell me what you eat, and I will tell you what you are.” Certainly, truffle lovers are a distinct breed.) See Jean Anthelme Brillat-Savarin, *The Physiology of Taste; or, Meditations on Transcendental Gastronomy* (Urbana, IL: Project Gutenberg, 2004), Meditation 6, Section 7 and Aphorism 4. Retrieved April 30, 2019, from www.gutenberg.org/ebooks/5434. Return to reference 28
- In reality, landowners have a valuable option: to wait and drill later. For more on the value of options, see Avinash Dixit and Robert Pindyck, *Investment under Uncertainty* (Princeton, N.J.: Princeton University Press, 1994). Return to reference 29

12■ Evolutionary Games

WE HAVE SO FAR STUDIED GAMES with many different features—simultaneous and sequential moves, zero-sum and non-zero-sum payoffs, strategic moves to manipulate rules of games to come, one-shot and repeated play, and even games of collective action in which a large number of people play simultaneously. In all of these games, we maintained the ground rules of conventional game theory—namely, that every player in these games has an internally consistent value system, can calculate the consequences of her strategic choices, and makes the choices that best favor her interests. We recognized the possibility that players' value systems include regard for others, and occasionally—for example, in our discussion of quantal-response equilibrium in Chapter 5—we allowed that the players recognize the possibility of errors. But we maintained the assumption that each player makes a conscious and calculated choice from her available strategies.

In our presentation of the empirical evidence on strategic choice in several of the earlier chapters, we pointed out several "behavioral" departures from the theory of rational decision making. The most cogent and best-developed theory of such behavior comes from the psychologist and 2002 Nobel laureate in economics Daniel Kahneman.[1] He argues that people have two different systems of decision making: System 1 is instinctive and fast, System 2 is calculating and slow. The fast, instinctive system may be partly hardwired into the brain by evolution, but it is also the result of extensive experience and practice, which build intuition. This system is valuable because it saves much mental effort and time, and it is often the first to be deployed when making a decision. Given enough time and attention, it may be supplemented or supplanted by the more calculating and slower System 2. When the instinctive System 1 is used on any one occasion, the outcome constitutes an addition to the stock of experience and may lead to a gradual modification of the instinct.

This theory suggests a very different mode of game playing and analysis of games. Players come to a game with the instinctive System 1 and play the strategy it indicates. This strategy may or may not be optimal for the occasion. Its outcome, if good, reinforces the instinct; otherwise, it contributes to gradual change in the instinct. Of course, the outcome depends on what strategies the other player or players deploy, which depends on the state of their instinctive systems, which in turn depends on their experience, and so forth. We need to find out where this process of interactive dynamics of instincts goes. In particular, we need to determine whether it converges to some fixed strategy choices and, if so, whether those choices correspond to what the calculating slow system would have dictated. The biological theory of evolution and evolutionary dynamics offers one approach to this analysis, which we develop in this chapter.

Endnotes

- Daniel Kahneman, *Thinking*, *Fast and Slow* (New York: Farrar, Straus and Giroux, 2011). Return to reference 1

1 THE FRAMEWORK

The biological theory of evolution rests on three fundamental concepts: heterogeneity, fitness, and selection. The starting assumption is that a significant part of animal behavior is genetically determined; a complex of one or more genes (a genotype) governs a particular pattern of behavior, called a behavioral phenotype. Natural diversity in the gene pool ensures heterogeneity of phenotypes in the population. Some behaviors are better suited than others to the prevailing conditions, and the success of a phenotype is given a quantitative measure called its fitness. People are used to thinking of this success in terms of the common but misleading phrase “survival of the fittest”; however, the ultimate test of biological fitness is not mere survival, but reproductive success. That is what enables an animal to pass on its genes to the next generation and perpetuate its phenotype. The fitter phenotypes then become relatively more numerous in the next generation compared with the less fit phenotypes. This process of selection is the dynamic that changes the mix of genotypes and phenotypes and may lead eventually to a stable state.

From time to time, chance produces new genetic mutations. Many of these mutations produce behaviors (that is, phenotypes) that are ill suited to the environment, and they die out. But occasionally a mutation leads to a new phenotype that is fitter. Then such a mutant gene can successfully invade a population—that is, spread to become a significant proportion of the population.

At any time, a population may contain some or all of its biologically conceivable phenotypes. Those that are fitter than others will increase in proportion, some unfit phenotypes may die out, and other phenotypes not currently in the population may try to invade it. Biologists call a configuration of a population and its current phenotypes evolutionarily stable if the population cannot be invaded successfully by any mutant. This criterion is a static test of evolutionary stability, but often a more dynamic criterion is applied: A configuration is evolutionarily stable if the dynamic process of evolution, starting from any arbitrary mixture of phenotypes in the population, converges to that configuration.[2]

The fitness of a phenotype depends on the relationship of the individual organism to its environment; for example, the fitness of a particular bird depends on the aerodynamic characteristics of its wings. It also depends on the whole complex of the proportions of different phenotypes that exist in the environment—how aerodynamic the bird's wings are relative to those of the rest of its species. Thus, the fitness of a particular animal—with its particular behavioral traits, such as aggression or sociability—depends on whether other members of its species are predominantly aggressive or passive, crowded or dispersed, and so on. For our purposes, this interaction among phenotypes within a species is the most interesting aspect of the story. Sometimes an individual member of a species interacts with members of another species; then the fitness of a particular type of sheep, for example, may depend on the traits that prevail in the local population of wolves. We consider this type of interaction as well.

All of this evolutionary theory finds a ready parallel in game theory. The behavior of a phenotype can be thought of as a *strategy* of the animal in its interactions with others—for example, to fight or to retreat. The difference is that the choice of strategy is not a purposive calculation, as it would be in standard game theory; rather, it is a genetically predetermined fixture of the phenotype. These interactions lead to *payoffs* to the phenotypes. In biology, the payoffs to an animal measure its fitness; when we apply these ideas outside of biology, they can have other connotations of success in the social, political, or economic games in question.

The players' payoffs or fitness numbers can be shown in a payoff table just like that for a standard game, with all conceivable phenotypes of one animal arrayed along the rows of the matrix and those of the other along the columns. If more than two animals interact simultaneously—which is called playing the field in biology—the payoffs can be shown by functions like those for collective-action games described in Chapter 11. We will consider pair-by-pair matches for most of this chapter and will look at many-player interactions briefly in Section 2.F.

Because the population contains a mix of phenotypes, different pairs selected from it will bring to their interactions different combinations of strategies. The actual quantitative measure of the fitness of a phenotype is the average payoff that it gets in all its interactions with others in the population. Those animals with higher fitness will have greater evolutionary success. The eventual outcome of the population dynamics will be an evolutionarily stable configuration of the population.

Biologists have used this approach very successfully. Combinations of aggressive and cooperative behavior, locations of nesting sites, and many more phenomena that elude more conventional explanations can be understood as the stable outcomes of an evolutionary process of selection of fitter strategies. Interestingly, biologists developed the idea of evolutionary games by using the preexisting body of game theory, drawing from its language but modifying the assumption of conscious maximizing to suit their needs. Now game theorists, in turn, are using insights from the research on biological evolutionary games to enrich their own subject.[3]

Indeed, the theory of evolutionary games provides a ready-made framework for studying Kahneman's two systems of decision making.[4] The idea that animals play genetically fixed strategies can be interpreted more broadly in applications of the theory other than in biology. In human interactions, a strategy may be embedded in a player's mind for a variety of reasons—not only by genetics, but also (and probably more importantly) by socialization, cultural background, education, or a rule of thumb based on past experience. All of these can be captured in Kahneman's instinctive, fast System 1. The population can consist of a mixture of different people with different backgrounds or experiences that embed different System 1 strategies in their minds. Thus, for example, some politicians may be motivated to adhere to certain moral or ethical codes even at the cost of electoral success, whereas others are mainly concerned with their own reelection; similarly, some firms may pursue profit alone, whereas others are motivated by social or ecological objectives. We can call each logically conceivable strategy that can be embedded in this way a phenotype for the population of players in the context being studied.

From a population, with its heterogeneity of embedded strategies, pairs of phenotypes are repeatedly randomly selected to interact (play the game) with others of the same or different "species." In each interaction, the payoff of each player depends on the strategies of both; this dependence is governed by the usual rules of the game and can be illustrated in a game table or tree. The *fitness* of a particular strategy is defined as its aggregate or average payoff in its pairings with all the strategies in the population. Some strategies have higher level of fitness than others; in the next generation—that is, the next round of play—these higher-fitness strategies will be used by more players and will proliferate. Strategies with lower fitness will be used by fewer players and will decay or die out. Occasionally, someone may experiment with or adopt a previously unused strategy from the collection of those that are logically conceivable. This corresponds to the emergence of a mutant.

Although we use the biological analogy, the reason that the fitter strategies proliferate and the less fit ones die out in socioeconomic games differs from the strict genetic mechanism of biology: Players who fared well in the last round will transmit information to their friends and colleagues playing the next round, those who fared poorly in the last round will observe which strategies succeeded better and will try to imitate them, and some purposive thinking and revision of previous rules of thumb will take place between successive rounds. Such "social" and "educational" mechanisms of transmission are far more important in most strategic games than any biological genetics; indeed, they are responsible for reinforcing the reelection orientation of legislators and the profit-maximization motive of firms. Finally, conscious experimentation with new strategies substitutes for the accidental mutation in biological games. Gradual modification in the light of outcomes, experience, observation, and experiment constitute the dynamics of Kahneman's calculating, slower System 2.

Evolutionarily stable configurations of biological games can be of two kinds. First, a single phenotype may prove fitter than any other, and the population may come to consist of it alone. Such an evolutionarily stable outcome is called monomorphism—that is, the population contains a single (mono) form (morph). In that case, the unique prevailing strategy is called an evolutionarily stable strategy (ESS). The other possibility is that two or more phenotypes are equally fit (and fitter than some others not played), so they may be able to coexist in certain proportions. Then the population is said to exhibit polymorphism—that is, a multiplicity (poly) of forms (morph). Such a state will be stable if no new phenotype or feasible mutant can achieve a fitness higher than the fitnesses of the types that are already present in the polymorphic population. Polymorphism comes close to the game-theoretic notion of a mixed strategy. However, there is an important difference: To get polymorphism, no individual player need follow a mixed strategy. Each can follow a pure strategy, but the population exhibits a mixture because different individual players pursue different pure strategies.

The whole setup—the population, its conceivable collection of phenotypes, the payoff matrix in the interactions of the phenotypes, and the rule for the evolution of proportions of the phenotypes in relation to their fitness—constitutes an evolutionary game. An evolutionarily stable configuration of the population can be called an *equilibrium* of the evolutionary game.

Some evolutionary games are symmetric, with the two players on similar footing—for example, two members of the same species competing with each other for food or mates; in a social science interpretation, they could be two elected officials competing for the right to continue in public office. In the payoff table for the game, each can be designated as the row player or the column player with no difference in outcome. Other evolutionary games are asymmetric; such games involve two species, such as a predator and a prey in biology, or a firm and a customer in economics. We develop the analysis of evolutionary games and their stable equilibria, as usual, through a series of illustrative examples.

Endnotes

- The dynamics of phenotypes is driven by an underlying dynamics of genotypes, but, at least at the elementary level, evolutionary biology focuses its analysis at the phenotype level and conceals the genetic aspects of evolution. We will do likewise in our exposition of evolutionary games. Some theories at the genotype level can be found in the materials cited in footnote 3. Return to reference 2
- Robert Pool, “Putting Game Theory to the Test,” *Science*, vol. 267 (March 17, 1995), pp. 1591–93, is a good article for general readers and has many examples from biology. John Maynard Smith deals with such games in biology in his *Evolutionary Genetics*, 2nd ed. (Oxford: Oxford University Press, 1998), Chapter 7, and *Evolution and the Theory of Games* (Cambridge: Cambridge University Press, 1982); the former also gives much background on evolution. Recommended for advanced readers are Peter Hammerstein and Reinhard Selten, “Game Theory and Evolutionary Biology,” in *Handbook of Game Theory*, vol. 2, ed. R. J. Aumann and S. Hart (Amsterdam: North Holland, 1994), pp. 929–93; and Jorgen Weibull, *Evolutionary Game Theory* (Cambridge, Mass.: MIT Press, 1995). Return to reference 3
- Indeed, applications of the evolutionary perspective need not stop with game theory. The following joke offers an “evolutionary theory of gravitation” as an alternative to Newton’s or Einstein’s physical theories:

  Question: Why does an apple fall from the tree to the earth?

  Answer: Originally, apples that came loose from trees went in all directions. But only those that were genetically predisposed to fall to the earth could reproduce.

Return to reference 4

Glossary

evolutionarily stable
: A population is evolutionarily stable if it cannot be successfully invaded by a new mutant phenotype.

evolutionarily stable strategy (ESS)
: A phenotype or strategy that can persist in a population, in the sense that all the members of a population or species are of that type; the population is evolutionarily stable (static criterion). Or, starting from an arbitrary distribution of phenotypes in the population, the process of selection will converge to this strategy (dynamic criterion).

fitness
: The expected payoff of a phenotype in its games against randomly chosen opponents from the population.

genotype
: A gene or a complex of genes, which give rise to a phenotype and which can breed true from one generation to another. (In social or economic games, the process of breeding can be interpreted in the more general sense of teaching or learning.)

phenotype
: A specific behavior or strategy, determined by one or more genes. (In social or economic games, this can be interpreted more generally as a customary strategy or a rule of thumb.)

selection
: The dynamic process by which the proportion of fitter phenotypes in a population increases from one generation to the next.

invasion
: The appearance of a small proportion of mutants in the population.

mutation

Emergence of a new genotype.

playing the field
: A many-player evolutionary game where all animals in the group are playing simultaneously, instead of being matched in pairs for two-player games.

monomorphism
: All members of a given species or population exhibit the same behavior pattern.

polymorphism
: An evolutionarily stable equilibrium in which different behavior forms or phenotypes are exhibited by subsets of members of an otherwise identical population.

2 SOME CLASSIC GAMES IN AN EVOLUTIONARY SETTING

In earlier chapters, especially Chapters 4 and 7, we introduced and analyzed several games that have become classics of the theory and have been given memorable stories and names—prisoners' dilemma, chicken, and so on. What happens if we replace the assumption of calculated rational choice in these games by specifying that players come from populations of phenotypes with given strategies, and that the population evolves by selection of the fitter types? Here, we reexamine those games one by one from this evolutionary perspective.

A. Prisoners' Dilemma

Suppose the population is made up of two phenotypes. One type consists of players who are natural-born cooperators; they always work toward the outcome that is jointly best for all players. The other type consists of defectors; they work only for themselves. As an example, we use the restaurant pricing game described in Chapter 5 and presented in a simplified version in Chapter 10. Here, we use the simpler version in which only two pricing choices are available: the jointly best price of $26 and the Nash equilibrium price of $20. A cooperator restaurateur would always choose $26, whereas a defector would always choose $20. The payoffs (profits) for each phenotype in a single play of this discrete dilemma are shown in Figure 12.1. This figure is the same as Figure 10.2, except that here we call the players simply Row and Column, because each can be any individual restaurateur in the population who is chosen at random to compete against a random rival. Remember that under the evolutionary scenario, no one has the choice between defecting and cooperating; each player is "born" with one trait or the other. Which is the more successful (fitter) trait in the population?

A defecting restaurateur gets a payoff of 288 ($28,800 a month) if matched against another defecting type, and a payoff of 360 ($36,000 a month) if matched against a cooperating type. A cooperating type gets 216 ($21,600 a month) if matched against a defecting type, and 324 ($32,400 a month) if matched against another cooperating type. No matter what the type of the matched rival, the defecting type does better than the cooperating type. Therefore, the defecting type has a better expected payoff (and is thus fitter) than the cooperating type, irrespective of the proportions of the two types in the population.

		COLUMN	
		$20 (Defect)	$26 (Cooperate)
ROW	$20 (Defect)	288, 288	360, 216
	$26 (Cooperate)	216, 360	324, 324

FIGURE 12.1 Payoff Table for Restaurant Prisoners' Dilemma (in Hundreds of Dollars per Month)

A little more formally, let x be the proportion of cooperators in the population. Consider any one particular cooperator. In a random pairing, the probability that she will meet another cooperator (and get 324) is x, and the probability that she will meet a defector (and get 216) is $(1 - x)$. Therefore, a typical cooperator's expected payoff is $324x + 216(1 - x)$. For a defector, the probability of meeting a cooperator (and getting 360) is x, and that of meeting another defector (and getting 288) is $(1 - x)$. Therefore, a typical defector's expected profit is $360x + 288(1 - x)$. Now it is immediately apparent that

$360x + 288(1 - x) > 324x + 216(1 - x)$ for all x between 0 and 1.

Therefore, a defector has a higher expected payoff, and is fitter, than a cooperator. This outcome will lead to an increase in the proportion of defectors (a decrease in x) from one "generation" of players to the next, until the population consists entirely of defectors.

Moreover, once a population consists entirely of defectors, it cannot be invaded by mutant cooperators. To see why, consider any very small value of x, meaning that the proportion of cooperators in the population is very small. The cooperators will be less fit than the prevailing defectors, and their proportion in the population will not increase, but will be driven to zero; the mutant strain will die out.

B. Comparing the Evolutionary and Rational-Player Models

The preceding analysis shows that in the prisoners' dilemma, defectors have higher fitness than cooperators, and that an all-defector population cannot be invaded by mutant cooperators. Thus, the evolutionarily stable configuration of the population is monomorphic, consisting of the single strategy or phenotype Defect. We therefore call Defect the *evolutionarily stable strategy* for this population. Of course, Defect is also a strictly dominant strategy when this game is played by rational players. This is not a coincidence. If a game has a strictly dominant strategy, that strategy will also be the (unique) ESS.

More generally, even in games where players do not have a strictly dominant strategy, every ESS must correspond to a Nash equilibrium. To see why, suppose the contrary for the moment. If the use of some strategy (call it S) by all players is not a Nash equilibrium, then some other strategy (call it R) must yield a higher payoff for one player when played against S. A mutant playing R would achieve greater fitness in a population playing S and so would invade successfully. Thus, S cannot be an ESS. In other words, if the use of S by all players is not a Nash equilibrium, then S cannot be an ESS. This is the same as saying that if S is an ESS, it must be a Nash equilibrium for all players to use S.

The evolutionary approach therefore provides a backdoor justification for the rational approach. Even when players are not consciously maximizing, if the more successful strategies get played more often and the less successful ones die out, and if this process converges eventually to a stable strategy, then the outcome must be the same as that resulting from consciously rational play.

Although an ESS must be a Nash equilibrium of the corresponding rational-play game, the converse is not true. There may be multiple Nash equilibria, not all of which are ESS. The concept of an ESS therefore gives us a justification, based on a stability argument, for selecting among multiple Nash equilibria. The examples considered next (chicken, assurance, and penalty kicks) are especially useful for developing your understanding of and intuition for when and why Nash equilibria fail to be evolutionarily stable.

C. Chicken

Remember our 1950s youths racing their cars toward each other and seeing who will be the first to swerve to avoid a collision? Now suppose the players have no choice in the matter: Each is genetically hardwired to be either a Wimp (who always swerves) or a Macho (who always goes straight). The population consists of a mixture of the two types. Pairs are picked at random every week to play the game. Figure 12.2 shows the payoff table for any two such players—say, A and B. (The numbers replicate those we used in Figure 4.16.)

How will the two types fare? The answer depends on the initial proportions in the population. If the population is almost all Wimps, then a Macho mutant will win and score 1 lots of times, whereas the Wimps will get mostly zeroes because they will mostly meet their own kind. But if the population is mostly Macho, then a Wimp mutant scores − 1, which may look bad but is better than the −2 that all the Machos get. You can think of this outcome appropriately in terms of the biological context and the sexism of the 1950s: In a population of Wimps, a Macho newcomer will show all the rest to be chickens and so will impress all the girls. But if the population consists mostly of Machos, they will be in the hospital most of the time and the girls will have to go for the few Wimps who are healthy.

		B	
		Wimp	**Macho**
A	Wimp	0, 0	−1, 1
	Macho	1, −1	−2, −2

FIGURE 12.2 Payoff Table for Chicken

In other words, each type is fitter when it is relatively rare in the population. Therefore, each can successfully invade a population consisting of the other type. We should expect to see both types in the population in equilibrium; that is, we should expect an ESS with a mixture of phenotypes, or polymorphism.

To find the proportions of Wimps and Machos in such an ESS, let us calculate the fitness of each type in a general mixed population. Let

x be the fraction of Machos and $(1 - x)$ be the proportion of Wimps. A Wimp meets another Wimp and gets 0 for a fraction $(1 - x)$ of pairings, and meets a Macho and gets -1 for a fraction x of pairings. Therefore, the fitness of a Wimp is $0 \times (1 - x) - 1 \times x = -x$. Similarly, the fitness of a Macho is $1 \times (1 - x) - 2x = 1 - 3x$. The Macho type is fitter if

$$1 - 3x > -x$$

$$2x < 1$$

$$x < ½.$$

If the population is less than half Macho, then the Machos will be fitter, and their proportion will increase. In contrast, if the population is more than half Macho, then the Wimps will be fitter, and the Macho proportion will fall. Either way, the population proportion of Machos will tend toward ½, and this 50–50 mix will be the stable polymorphic ESS.

Figure 12.3 shows this outcome graphically. Each straight line shows the fitness (the expected payoff in a match against a random member of the population) for one type in relation to the proportion x of Machos.[5] The fitness of the Wimp type as a function of the proportion of the Machos is $-x$, as we saw two paragraphs ago; it is shown by the gently falling line that starts at 0 where $x = 0$ and goes to -1 where $x = 1$. The corresponding function for the Macho type is $1 - 3x$; its fitness is shown by the rapidly falling line that starts at 1 where $x = 0$ and falls to -2 where $x = 1$. The Macho line lies above the Wimp line for $x < ½$ and below it for $x > ½$, showing that the Macho type is fitter when the value of x is small, and the Wimp type is fitter when x is large.

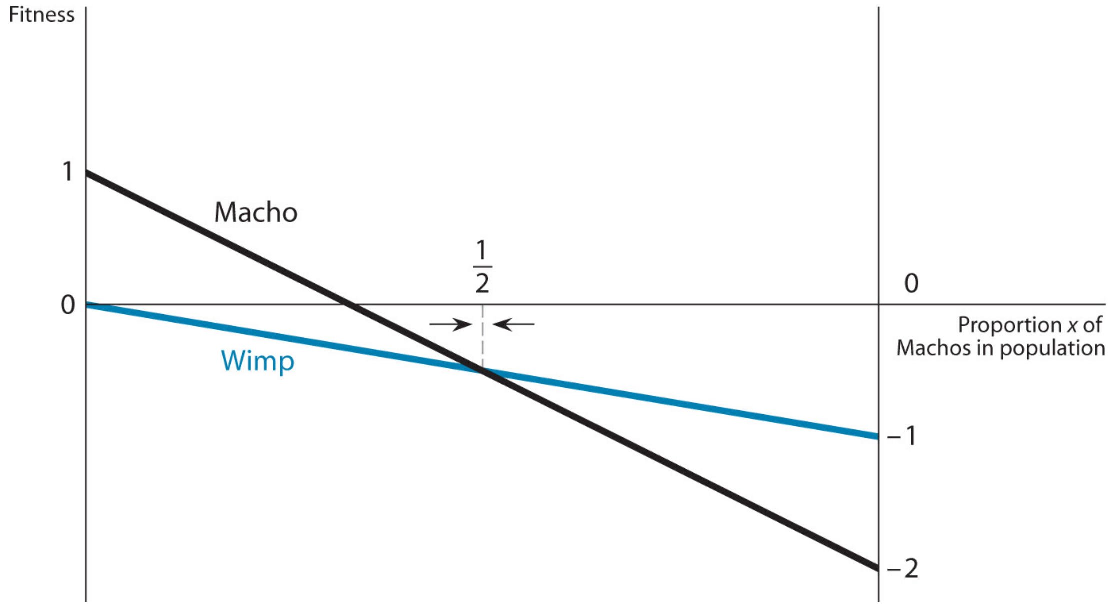

FIGURE 12.3 Fitness Graphs and Polymorphic Equilibrium for Chicken

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Now we can compare and contrast the evolutionary theory of this game with our earlier theory of Chapters 4 and 7, which was based on the assumption that the players were conscious, rational calculators of strategies. There, we found three Nash equilibria: two in pure strategies, where one player goes straight and the other swerves, and one in mixed strategies, where each player goes straight with a probability of ½ and swerves with a probability of ½.

If the population is truly 100% Macho, then all players are equally fit (or equally unfit, in their hospital beds!). Similarly, in a population of nothing but Wimps, all are equally fit. But these monomorphic configurations are unstable. In an all-Macho population, a Wimp mutant will outscore them and invade successfully.[6] Once some Wimps get established, no matter how few, our analysis shows that their proportion will rise inexorably toward ½. Similarly, an all-Wimp population is vulnerable to a successful invasion of mutant Machos, and the process again goes to the same polymorphism. Thus, the polymorphic configuration is the only evolutionarily stable outcome.

Most interesting is the connection between the mixed-strategy equilibrium of the rationally played game and the polymorphic equilibrium of the evolutionary game. The mixture proportions in the equilibrium strategy of the former game are *exactly the same* as the population proportions in the latter game, a 50 – 50 mixture of Wimp and Macho. But the interpretations of these outcomes differ. In the rational framework, each player mixes his own strategies; in the evolutionary framework, every member of the population uses a pure strategy, but different members use different strategies, and so we see a mixture of strategies in the population.[7]

Once again we see a correspondence between Nash equilibria in a rationally played game and stable outcomes in a game with the same payoff structure when played according to the evolutionary rules. We also get better understanding of the mixed-strategy equilibrium, which seemed puzzling when we looked at chicken from the rational perspective. Rational chicken left open the possibility of costly mistakes. Each player went straight one time in two, so one time in four, the players collided. The pure-strategy equilibria avoided the collisions. At that time, this may have led you to think that there was something undesirable about the mixed-strategy equilibrium, and you may have wondered why we were spending time on it. Now you see the reason. That seemingly strange equilibrium emerges as the stable outcome of a natural dynamic process in which each player tries to improve his payoff against the population that he confronts.

D. Assurance

We illustrated assurance games in Chapter 4 with the story of Holmes and Watson deciding where to meet to compare notes at the end of a busy day. We could imagine the same payoff structure applying to a pair of friends deciding where to meet for coffee, at Starbucks or the local diner, for example. In the evolutionary context, we assume that each player in such a coffeehouse-choice game is born liking either Starbucks or the local diner, and that the population includes some of each type. We also assume that pairs of players are chosen at random each day to play the game. We denote the strategies here with S (for Starbucks) and L (for local diner). Figure 12.4 shows the payoff table for a random pairing in this game. The payoffs are the same as those illustrated earlier in Figure 4.14; only the names of the players and the actions have changed.

If we reframe this game as one played by rational strategy-choosing players, we find two equilibria in pure strategies, (S, S) and (L, L), of which the latter is better for both players. If they communicate and coordinate explicitly, they can settle on it quite easily. But if they are making the choices independently, they need to coordinate through a convergence of expectations—that is, by finding a focal point.

The rationally played assurance game has a third equilibrium, in mixed strategies, that we found in Chapter 7. In that equilibrium, each player chooses Starbucks with a probability of $\frac{2}{3}$ and the local diner with a probability of $\frac{1}{3}$; the expected payoff for each player is $\frac{2}{3}$. As we showed in Chapter 7, this payoff is worse than the one associated with the less attractive of the two pure-strategy equilibria, (S, S), because independent mixing leads the players to make clashing or bad choices quite a lot of the time. Here, the bad outcome (a payoff of 0) has a probability of $^4/_9$: The two players go to different meeting places almost half the time.

		FRIEND 2	
		S	L
FRIEND 1	S	1, 1	0, 0
	L	0, 0	2, 2

FIGURE 12.4 Payoff Matrix for the Assurance Game

What happens when this is an evolutionary game? In the population at large, each member is hardwired, either to choose S or to choose L. Randomly chosen pairs of such people are assigned to attempt a meeting. Suppose x is the proportion of S types in the population and $(1 - x)$ is that of L types. Then the fitness of a particular S type individual—her expected payoff in a random encounter of this kind—is $x \times 1 + (1 - x) \times 0 = x$. Similarly, the fitness of each L type is $x \times 0 + (1 - x) \times 2 = 2(1 - x)$. Therefore, the S type is fitter when $x > 2(1 - x)$, or when $x > ⅔$. The L type is fitter when $x < ⅔$. At the balancing point $x = ⅔$, the two types are equally fit.

Once again, as in chicken, the probabilities associated with the mixed-strategy equilibrium that would obtain under rational choice seem to reappear under evolutionary rules as the population proportions in a polymorphic equilibrium. But now this mixed equilibrium is not stable. The slightest chance departure of the proportion x from the balancing point ⅔ will set in motion a cumulative process that takes the population mix farther away from the balancing point. If x increases from ⅔, the S type becomes fitter and propagates faster, increasing x even more. If x falls from ⅔, the L type becomes fitter and propagates faster, lowering x even more. Eventually x will either rise all the way to 1 or fall all the way to 0, depending on which disturbance occurs. The difference relative to the game of chicken is that in chicken, each type is fitter when it is rarer, so the population proportions tend to move away from the extremes and toward a mid-range balancing point. In contrast, in the assurance game, each type is fitter when it is more numerous, because the more of the population that is the same type as you, the lower the risk of failing to meet, so population proportions tend to move toward the extremes.

Figure 12.5 illustrates the fitness graphs and equilibria for the assurance game. This diagram is very similar to Figure 12.3 in that the two lines show the fitnesses of the two types in relation to their proportion in the population, and the intersection of the lines gives the balancing point. The only difference is that, away from the balancing point, the more numerous type is the fitter, whereas in Figure 12.3, it was the less numerous type.

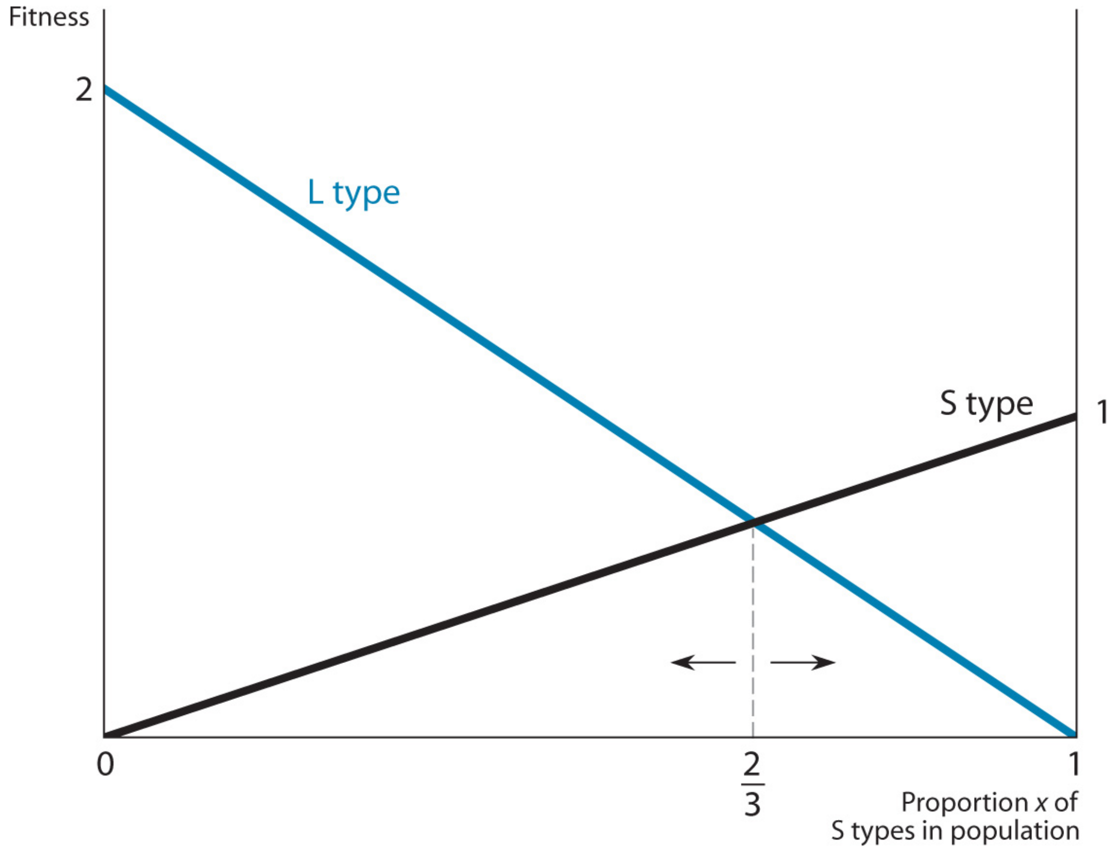

FIGURE 12.5 Fitness Graphs and Equilibria for the Assurance Game

Because each type is less fit when it is rare, the two extreme monomorphic configurations of the population are the only possible evolutionarily stable states. It is easy to check that both outcomes are ESSs according to the static test: An invasion by a small mutant population of the other type will die out because the mutants, being rare, will be less fit. Thus, in assurance or coordination games, in contrast to chicken, the evolutionary process does not preserve the bad equilibrium where there is a positive probability that the players choose clashing strategies. Finally, evolutionary dynamics do not guarantee convergence to the better of the two equilibria when starting from an arbitrary initial mixture of phenotypes—where the population ends up depends on where it starts.

E. Soccer Penalty Kicks

In Chapter 7, we studied mixed-strategy equilibria in the game of penalty kicking in soccer. Some new features of this game emerge when we consider it through the lens of evolutionary games. Most important is its asymmetry: Kickers and goalies have very different roles. In the terminology of this chapter, we may as well regard them as different *species* playing an asymmetric game. We have a population of goalies and a population of kickers, and for each interaction one representative from each population is picked to play. Instead of the goalie making a deliberate choice of whether to go left or right as each kick is taken, and the kicker deciding on each occasion whether to kick to the goalie's left or right, each goalie is either a left-leaner or a right-leaner, and likewise each kicker is either a left-sider or a right-sider. The success of each type of kicker will depend on what type of goalie he faces, and vice versa.

Figure 12.6 is an illustrative matrix of payoffs for such a game. The kicker's payoff is the percentage of times he succeeds in scoring a goal, and the goalie's payoff is the percentage of times he saves the kick (or the kick misses). Remember that for the kicker, left means the goalie's left.

		Goalie	
		Left	Right
Kicker	Left	30, 70	90, 10
	Right	90, 10	50, 50

FIGURE 12.6 Payoff Matrix for the Soccer Penalty-Kick Game

Suppose the kicker population consists of a fraction x of left-siders (and $1 - x$ of right-siders), and the goalie population has a fraction y of left-leaners (and $1 - y$ of right-leaners). Facing a goalie chosen randomly from this population, a left-side kicker will have greater fitness than a right-side kicker if

$30y + 90(1 - y) > 90y + 50(1 - y)$, or $90 - 60y > 40y + 50$, or $y < 0.4$.

If y is small—that is, if the population of goalies is predominantly right-leaning—then left-side kickers are more successful, which agrees with intuition. Conversely, facing a kicker chosen randomly from the kicker population, a left-leaning goalie will have greater fitness than a right-leaning goalie if

$70x + 10(1 - x) > 10x + 50(1 - x)$, or $60x + 10 > 50 - 40x$, or $x > 0.4$.

If x is large—that is, if the population of kickers has predominantly left-siders—then a left-leaning goalie will be more successful, as is again quite intuitive.

Experience based on historical success rates will gradually change behaviors of the current populations and newcomers. Will the dynamics converge to a stable configuration of population proportions?

If $x = 0.4$ and $y = 0.4$, both types of each species will be equally fit, and there will be no inducement to change behavior. This suggests (0.4, 0.4) as the obvious candidate for an ESS, and, confirming the correspondence we found earlier in other games, it is also the unique mixed-strategy Nash equilibrium of the conventional, rationally played game.[8] Let us examine its stability.

If $y < 0.4$, then left-side kickers are more successful, so x will tend to increase; if $y > 0.4$, x will tend to decrease. Conversely, if $x < 0.4$, then right-leaning goalies are more successful, so y will tend to decrease; if $x > 0.4$, y will tend to increase. We show these dynamics graphically in Figure 12.7, using arrows pointing right to show a tendency for x to increase, pointing up to show a tendency for y to increase, and so on.

We see no tendency to converge to (0.4, 0.4). Indeed, if the rates at which x and y change when away from this point have just the right magnitudes, the dynamics can cycle around this point, as shown by the circular loop in Figure 12.7 with arrows indicating the direction of motion. This closed orbit could be small, wound tightly around the ESS, or it could be a bigger loop, depending on the starting point. So an ESS need not always be dynamically stable in the sense that all paths converge to it!

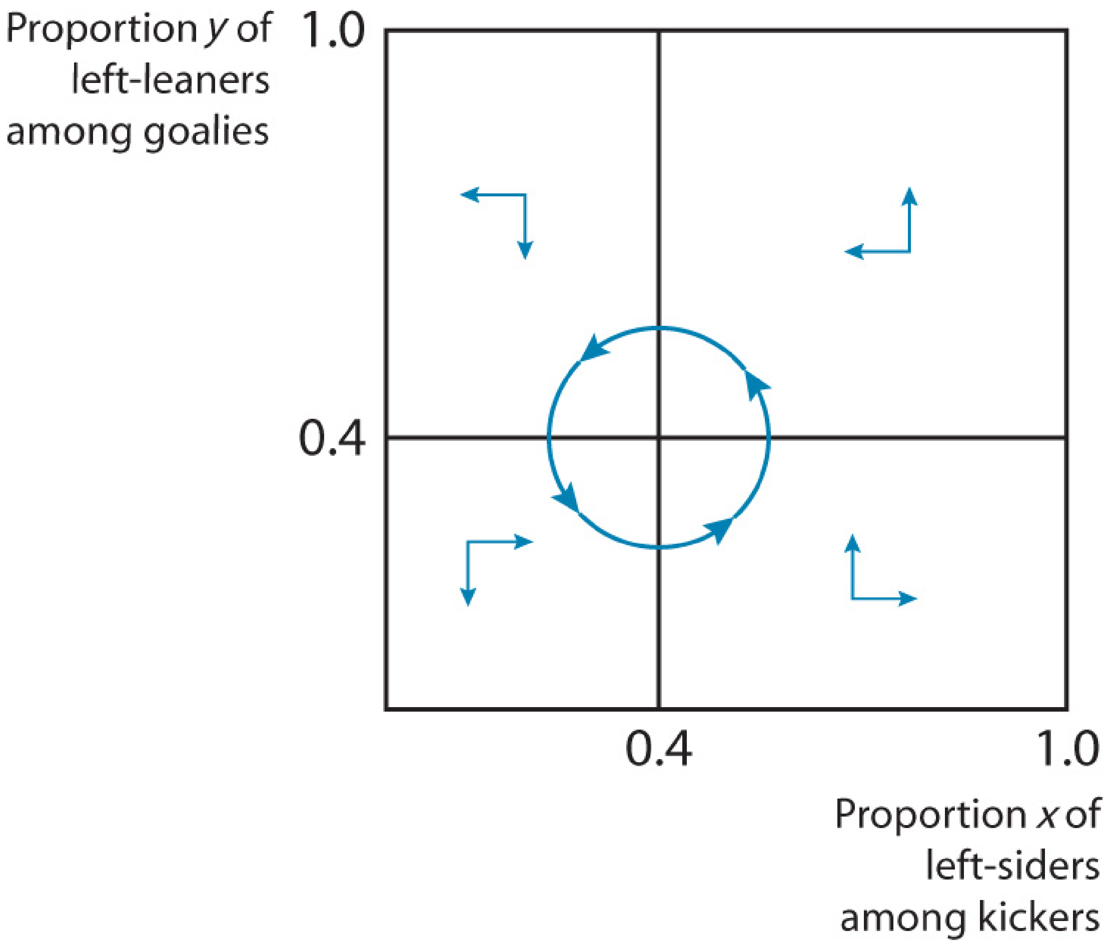

FIGURE 12.7 Dynamics for the Soccer Penalty-Kick Game

Such cyclical outcomes are well-known in some biological contexts. Consider two populations, one a predator species and the other its prey. If predators are numerous, the number of prey will decline. But when there are so few prey that the predators have little food, their numbers will also decline. Then prey can flourish again. But the resulting abundance of food enables the predators to thrive and multiply again. And so on. Such dynamics were observed and mathematically modeled by Alfred Lotka and Vito Volterra in the early twentieth century, and are named after them.[9]

F. Multiplayer Evolutionary Games

It is possible that evolutionary interactions may entail an entire population playing at once, rather than being matched in pairs. Such simultaneous interactions among more than two individuals are referred to as *playing the field*. In biology, all members of a flock of animals with a mixture of genetically determined behaviors may compete among themselves for some resource or territory. In such settings, the fitness of each animal depends on the strategy (phenotype) mix in the whole population. In economics or business, many firms in an industry, each following the strategy dictated by its corporate culture, may similarly compete all with all.

Such evolutionary games stand in the same relation to the rationally played collective-action games of Chapter 11 as do the pair-by-pair evolutionary games of the preceding sections to the rationally played two-person games of Chapters 4 through 7. Just as we converted the expected payoff tables of those chapters into the fitness graphs in Figures 12.3 and 12.5, we can convert the graphs for collective-action games (Figures 11.6–11.8) into fitness graphs for evolutionary games.

For example, consider an animal species whose members all come to a common feeding ground. There are two phenotypes: One fights for food aggressively, and the other hangs around and sneaks what it can. If the proportion of aggressive individuals is small, they will do better, but if there are too many of them, the sneakers will do better by ignoring the ongoing fights. The result will be a collective game of chicken whose fitness diagram will be exactly like Figure 11.7. Because no new principles or techniques are required, we leave it to you to pursue this idea further.

Endnotes

- Literally, the fraction of any particular type in the population can only take on values such as 1/1,000,000, 2/1,000,000, and so on. But if the population is sufficiently large and we show all such values as points on a straight line, as in Figure 12.3, then these points are very tightly packed together, and we can regard them as forming a continuous line. This approach amounts to letting the fractions take on any real value between 0 and 1, and it allows us to talk of the population *proportion* of a certain behavioral type, as we do throughout this chapter. By the same reasoning, if one individual member goes to jail and is removed from the population, her removal does not change the population's proportions of the various phenotypes. Return to reference 5
- *The Invasion of the Mutant Wimps* could be an interesting science-fiction comedy movie. Return to reference 6
- There can also be evolutionarily stable mixed outcomes in which each member of the population adopts a mixed strategy. We investigate this idea further in Section 5.E. Return to reference 7
- We leave this as a simple exercise, as you are by now experienced game theorists. Return to reference 8
- For readers who are good at calculus, we offer a quick proof for the cyclical behavior of the Lotka-Volterra predator-prey system. Suppose the speeds of change in each of x and y are proportional to the distance from the level of the other at which the two types are equally fit. That is, $dx/dt = a(0.4 - y)$ and $dy/dt = b(x - 0.4)$, where a and b are positive constants. Then

$$\frac{d}{dt}\left[\frac{(x-0.4)^2}{a}+\frac{(y-0.4)^2}{b}\right]=\frac{2(x-0.4)}{a}a(0.4-y)+\frac{2(y-0.4)}{b}b(x-0.4)=0.$$

Therefore, the point (x, y) moves along a curve where $(x-0.4)^2/a + (y-0.4)^2/b$ is constant. This movement forms an ellipse centered on the ESS (0.4, 0.4). The value of the constant, and therefore the size of the ellipse, depends on the initial positions of x and y.

Return to reference 9

3 THE REPEATED PRISONERS' DILEMMA

We saw in Chapter 10 that a repetition of the prisoners' dilemma permitted consciously rational players to sustain cooperation for their mutual benefit. Let us see if a similar possibility exists in the evolutionary story. We return to the prisoners' dilemma in pricing analyzed in Section 2.A. From the population of restaurateurs, we choose a pair of players to play the dilemma multiple times in succession. The overall payoff to a player (restaurateur) from such an interaction is the sum of what she gets across all rounds.

Each individual player is still programmed to play just one strategy, but that strategy has to be a complete plan of action. In a game with three rounds, for instance, a strategy can stipulate an action in the second or third play that depends on what happened in the first or second play. For example, "I will always cooperate no matter what" and "I will always defect no matter what" are valid strategies. But "I will begin by cooperating in the first round and cooperate in any later round if you cooperated in the preceding round, but defect in any later round if you defected in the preceding round" is also a valid strategy; in fact, this last strategy is tit-for-tat (TFT).

To keep the initial analysis simple, we suppose in this section that there are just two types of strategies that can possibly exist in the population of restaurateurs: always defect (A) and tit-for-tat (T). Pairs are randomly selected from the population, and each selected pair plays the game a specified number of times. The fitness of each player is simply the sum of her payoffs from all the repetitions played against her specific opponent. We examine what happens with two, three, and more generally, n such repetitions in each pair.

	COLUMN	
	A	T

		COLUMN	
		A	T
ROW	A	576, 576	648, 504
	T	504, 648	648, 648

FIGURE 12.8 Outcomes in the Twice-Repeated Restaurant Prisoners' Dilemma (in Hundreds of Dollars per Month)

A. Twice-Repeated Play

Figure 12.8 shows the payoff table for the game in which two members of the restaurateur population meet and play against each other exactly twice. If both players are A types, both defect both times, and we can refer back to Figure 12.1 to see that each then gets 288 each time, for a total of 576. If both are T types, defection never starts, and each gets 324 each time, for a total of 648. If one is an A type and the other a T type, then on the first play, the A type defects and the T type cooperates, so the former gets 360 and the latter 216. On the second play, both defect and get 288. So the A type's total payoff is 360 + 288 = 648, and the T type's total is 216 + 288 = 504.

In the twice-repeated dilemma, we see that A is weakly dominant. If the population is all A, then T-type mutants cannot invade, and A is an ESS. But if the population is all T, then A-type mutants cannot do any better than the T types. Does this mean that T must be another ESS, just as it would be a Nash equilibrium in the rational-player game-theoretic analysis of this game? The answer is no. If the population were initially all T types and a few A mutants entered, then the mutants would meet the predominant T types most of the time and would do as well as a T would do against another T. But occasionally an A mutant would meet another A mutant, and in this match she would do better than would a T against an A. Thus, the mutants would have just *slightly* higher fitness than would a member of the predominant phenotype. This advantage would lead to an increase, albeit a slow one, in the proportion of mutants in the population. Therefore, an all-T population *could* be invaded successfully by A mutants; T is not an ESS.

The static test for an ESS thus has two parts. First, we see if the mutant does better or worse than the predominant phenotype when each is matched against the predominant type. If this *primary criterion* gives a clear answer, that settles the matter. But if the primary criterion gives a tie, then we use a tie-breaking, or secondary, criterion: Does the mutant fare better or worse than the predominant phenotype when each is matched against a mutant? Ties are exceptional, and most of the time we do not need the *secondary criterion*, but it is there in reserve for situations such as the one illustrated in Figure 12.8.[10]

		COLUMN	
		A	T
ROW	A	864, 864	936, 792
	T	792, 936	972, 972

FIGURE 12.9 Outcomes in the Thrice-Repeated Restaurant Prisoners' Dilemma (in Hundreds of Dollars per Month)

B. Threefold Repetition

Now suppose each matched pair from the (A, T) population plays the game three times. Figure 12.9 shows the fitness outcomes, summed over the three meetings, for each type of player when matched against a rival of each type.

To see how these fitness numbers arise, consider a couple of examples. When two T players meet each other, both cooperate the first time, and therefore both cooperate the second time and the third time as well; both get 324 each time, for a total of 972 each over 3 months. When a T player meets an A player, the latter does well the first time (360 for the A type versus 216 for the T player), but then the T player also defects the second and third times, and each gets 288 in both of those plays (for totals of 936 for A and 792 for T).

The relative fitnesses of the two types depend on the composition of the population. If the population is almost wholly A types, then A is fitter than T (because A types meeting mostly other A types earn 864 most of the time, but T types most often get 792). But if the population is almost wholly T types, then T is fitter than A (because T types earn 972 most of the time when they meet mostly other Ts, but A types earn 936 in such a situation). Each type is fitter when it already predominates in the population. Therefore, T cannot invade successfully when the population is all A, and vice versa. Now there are two possible evolutionarily stable configurations of the population: In one configuration, A is the ESS, and in the other, T is the ESS.

Next, consider the evolutionary dynamics when the initial population is made up of a mixture of the two types. How will the composition of the population evolve over time? Suppose a fraction x of the population is T and the rest, $(1 - x)$, is A. An individual A player, pitted against various opponents chosen from such a population, gets 936 when confronting a T player, which happens a fraction x of the times, and 864 against another A player, which happens a fraction $(1 - x)$ of the times. This gives an average expected payoff of

$$936x + 864(1 - x) = 864 + 72x$$

for each A player. Similarly, an individual T player gets an average expected payoff of

$$972x + 792(1 - x) = 792 + 180x.$$

Then a T player is fitter than an A player if the former earns more on average; that is, if

$$792 + 180x > 864 + 72x$$

$$108x > 72$$

$$x > \tfrac{2}{3}.$$

In other words, if more than two-thirds (67%) of the population is already T, then T players will be fitter, and their proportion will grow until it reaches 100%. If the population starts with less than 67% T, then A players will be fitter, and the proportion of T players will go on declining until there are 0% of them, or 100% A players. The evolutionary dynamics move the population toward one of the two extremes, each of which is a possible ESS. These dynamics lead to the same conclusion as the static test of mutant invasion. Many, but not all, evolutionary games share this feature, that the static and dynamic tests lead to the same conclusions about ESS.

Thus, we have identified two evolutionarily stable configurations of the population. In each one, the population is all of one type (monomorphic). For example, if the population is initially 100% T, then even after a small number of mutant A types arise, the population mix will still be more than 66.66 . . . % T; T will remain the fitter type, and the mutant A strain will die out. Similarly, if the population is initially 100% A, then a small number of T-type mutants will leave the population mix with less than 66.66 . . . % T, so the A types will be fitter and the mutant T strain will die out.

If the initial population has exactly 66.66 . . . % T players (and 33.33 . . . % A players), then the two types are equally fit. However, such a polymorphism is not evolutionarily stable. The population can sustain this delicately balanced outcome only until a mutant of either type surfaces. By chance, such a mutant must arise sooner or later. The mutant's arrival will tip the fitness calculation in favor of the mutant type, and the advantage will accumulate until the ESS with 100% of that type is reached. Thus, this configuration does not meet the secondary criterion for evolutionary stability. We will sometimes loosely speak of such a configuration as an *unstable equilibrium*, so as to maintain the parallel with ordinary game theory, where mutations are not a consideration and a delicately balanced equilibrium can persist. But in the strict logic of the biological process, it is not an equilibrium at all.

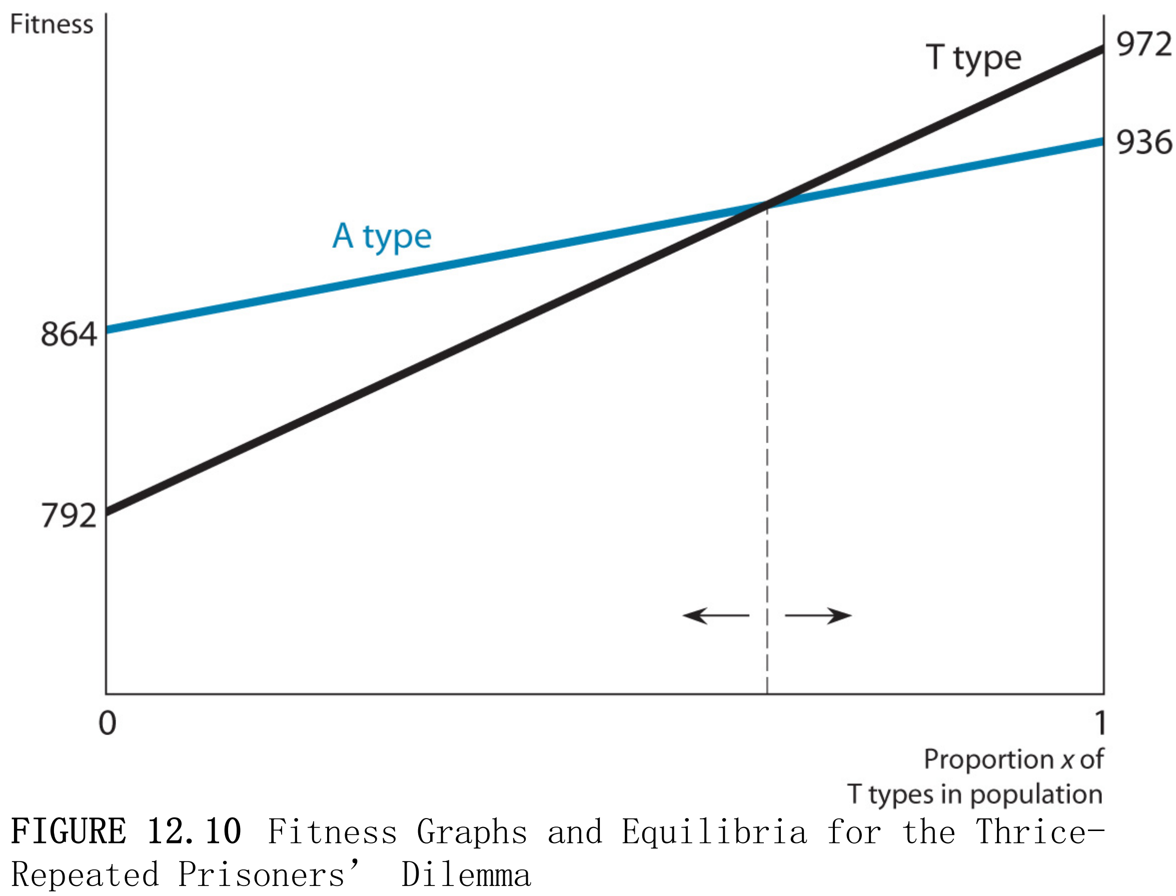

FIGURE 12.10 Fitness Graphs and Equilibria for the Thrice-Repeated Prisoners' Dilemma

This reasoning can be shown in a simple graph that closely resembles the graphs that we drew when calculating the proportions in a mixed-strategy equilibrium with consciously rational players. The only difference is that in the evolutionary context, the proportion in which the separate strategies are played is not a matter of choice by any individual player, but a property of the whole population, as shown in Figure 12.10. Along the horizontal axis, we measure the proportion x of T players in the population, which can range from 0 to 1. We measure fitness along the vertical axis. Each line shows the fitness of one type. The line for the T type starts lower (at 792, compared with 864 for the A-type line) and ends higher (972 against 936). The two lines cross where $x = 0.66$ . . . . To the right of this point, the T type is fitter, so its population proportion increases over time, and x *increases* toward 1. Similarly, to the left of this point, the A type is fitter, so its population proportion increases over time, and x *decreases* toward 0.[11]

C. Multiple Repetitions

What if each pair plays some unspecified number of repetitions of the game? Let us focus on a population consisting of A and T types in which interactions between random pairs occur n times (where $n > 2$). Figure 12.11 shows the total payoffs from playing n repetitions. When two A types meet, they defect, and each earns 288, every time, so each gets $288n$ in n plays. When two T types meet, they begin by cooperating, and neither is the first to defect, so each earns 324 every time, for a total of $324n$. When an A type meets a T type, on the first play the T type cooperates and the A type defects, so the A type gets 360 and the T type gets 216; thereafter, the T type retaliates against the preceding defection of the A type for all of the remaining plays, and each gets 288 in each of the remaining $(n - 1)$ plays. Thus, the A type earns a total of $360 + 288(n - 1) = 288n + 72$ in n plays against a T type, whereas the T type gets $216 + 288(n - 1) = 288n - 72$ in n plays against an A type.

		COLUMN	
		A	T
ROW	A	$288n$, $288n$	$288n + 72$, $288n - 72$
	T	$288n - 72$, $288n + 72$	$324n$, $324n$

FIGURE 12.11 Outcomes in the N-fold-Repeated Dilemma

If the proportion of T types in the population is x, then a typical A type gets $x(288n + 72) + (1 - x)288n$ on average, and a typical T type gets $x(324n) + (1 - x)(288n - 72)$ on average. Therefore, the T type is fitter if

$$x(324n) + (1 - x)(288n - 72) > x(288n + 72) + (1 - x)288n$$

$$36xn > 72$$

$$x > \frac{72}{36n} = \frac{2}{n}.$$

Once again we have two monomorphic ESSs, one all T (or $x = 1$, to which the process converges starting from any $x > 2/n$) and the other all A (or $x = 0$, to which the process converges starting from any $x < 2/n$). As in Figure 12.10, there is also an unstable polymorphic equilibrium at the balancing point $x = 2/n$.

Notice that the proportion of T at the balancing point depends on n; it is smaller when n is larger. When $n = 10$, it is $^2/_{10}$, or 0.2. So if the population is initially 20% T players, and each pair plays 10 repetitions, the proportion of T types will grow until they reach 100% of the population. Recall that when pairs played three repetitions ($n = 3$), the T players needed an initial strength of 67% or more to achieve the same outcome, and only two repetitions meant that T types needed to begin with 100% of the population to survive. (We see the reason for this outcome in our expression for the balancing point for x, which shows that when $n = 2$, x must be above 1 before the T types are fitter.) Remember, too, that a population consisting of all T players achieves cooperation. Thus, cooperation emerges from a larger range of initial conditions when the game is repeated more times. In this sense, with more repetition, cooperation becomes more likely. What we are seeing is the result of the fact that the value of establishing cooperation increases as the length of the interaction increases.

Endnotes

- This game is just one example of a twice-repeated dilemma. With other payoffs in the basic game, twofold repetition may not produce ties. That is so in the Husband–Wife prisoners' dilemma of Chapter 4. If both the primary and secondary tests yield ties, neither phenotype satisfies our definition of ESS, and additional conceptual tools are needed that, unfortunately, lie beyond the scope of our introductory treatment of the subject. Return to reference 10
- You should now be able to draw a similar graph for the twice-repeated case. You will see that the A line is above the T line for all values of $x < 1$, but the two meet on the right-hand edge of the graph where $x = 1$. Return to reference 11

4 THE HAWK–DOVE GAME

The hawk–dove game was the first example biologists studied in their development of the theory of evolutionary games. It has instructive parallels with our analyses so far of the prisoners' dilemma and chicken, so we describe it here to reinforce and improve your understanding of the concepts involved.

The game is played not by birds of these two species, but by two animals of the same species, and Hawk and Dove are merely the names for their strategies. The context is competition for a resource. The Hawk strategy is aggressive and fights to try to get the whole resource of value V. The Dove strategy is to offer to share and avoid a fight. When two Hawk types meet each other, they fight. Each animal is equally likely (probability one-half) to win and get V or to lose, be injured, and get $-C$. Thus, the expected payoff for each is $(V - C)/2$. When two Dove types meet, they share without a fight, so each gets $V/2$. When a Hawk type meets a Dove type, the latter retreats and gets 0, whereas the former gets V. Figure 12.12 shows the payoff table for this game.

The analysis of the hawk–dove game is similar to that for the prisoners' dilemma and chicken, except that the numerical payoffs have been replaced with algebraic symbols. Here we compare the equilibria of this game when each player has a specified strategy, and success is rewarded with faster reproduction of that strategy in the population.

A. Rational Strategic Choice and Equilibrium

If $V > C$, then the game is a prisoners' dilemma in which the Hawk strategy corresponds to Defect and the Dove strategy corresponds to Cooperate. Hawk is the dominant strategy for each player, but (Dove, Dove) is the jointly better outcome.

If $V < C$, then it's a game of chicken. Now $(V - C)/2 < 0$, and so Hawk is no longer a dominant strategy. Rather, there are two pure-strategy Nash equilibria: (Hawk, Dove) and (Dove, Hawk). There is also a mixed-strategy equilibrium, where B's probability p of choosing Hawk is such as to keep A indifferent, as defined by

$$\frac{p(V-C)}{2} + (1-p)V = p \times 0 + \frac{(1-p)V}{2}$$

$$p = \frac{V}{C}.$$

B. Evolutionary Stability for V > C

We start with a population in which Hawks predominate and test whether it can be invaded by mutant Doves. Following the convention used in analyzing such games, we could write the population proportion of the mutant phenotype as m, for mutant, but for clarity in our case we will use d for mutant Dove. The population proportion of Hawks is then $(1 - d)$. Then, in a match against a randomly drawn opponent, a Hawk will meet a Dove a proportion d of the time and get V on each of those occasions, and will meet another Hawk a proportion $(1 - d)$ of the time and get $(V - C)/2$ on each of those occasions. Therefore, the fitness of a Hawk is $dV + (1 - d)(V - C)/2$. Similarly, the fitness of a mutant dove is $d(V/2) + (1 - d) \times 0$. Because $V > C$, it follows that $(V - C)/2 > 0$. Also, $V > 0$ implies that $V > V/2$. Then, for any value of d between 0 and 1, we have

$$dV + \frac{(1-d)(V-C)}{2} > d\frac{V}{2} + (1-d) \times 0,$$

and so the Hawk type is fitter. The Dove mutants cannot successfully invade. The Hawk strategy is evolutionarily stable, and the population is monomorphic (all Hawk).

		B	
		Hawk	Dove
A	Hawk	$(V - C)/2$, $(V - C)/2$	V, 0
	Dove	0, V	$V/2$, $V/2$

FIGURE 12.12 Payoff Table for the Hawk-Dove Game

The same holds true for any population proportion of Doves—that is, for all values of d. Therefore, from any initial mix, the proportion of Hawks will grow, and they will predominate. In addition, if the population is initially all Doves, mutant Hawks can invade and take over. Thus, the dynamics confirm that the Hawk strategy is the only ESS. This algebraic analysis affirms and generalizes our earlier finding for the numerical example of the prisoners' dilemma of restaurant pricing (see Figure 12.1).

C. Evolutionary Stability for V < C

If the initial population is again predominantly Hawks, with a small proportion d of Dove mutants, then each has the same fitness function derived in Section 4.B. When $V < C$, however, $(V - C)/2 < 0$. We still have $V > 0$, and so $V > V/2$. But because d is very small, the comparison of the terms with $(1 - d)$ is much more important than that of the terms with d, so

$$d\frac{V}{2} + (1-d) \times 0 > dV + \frac{(1-d)(V-C)}{2}.$$

Thus, the Dove mutants are fitter than the predominant Hawks and can invade successfully.

But if the initial population is almost all Doves, then we must consider whether a small proportion h of Hawk mutants can invade. (Note that because the mutant is now a Hawk, we have used h for the proportion of the mutant invaders.) The Hawk mutants have a fitness of $h(V - C)/2 + (1 - h)V$ compared with $h\times 0 + (1 - h)(V/2)$ for the Doves. Again, $V < C$ implies that $(V - C)/2 < 0$, and $V > 0$ implies that $V > V/2$. But when h is small, we get

$$\frac{h(V-C)}{2} + (1-h)V > h \times 0 + (1-h)\frac{V}{2}.$$

This inequality shows that Hawks are fitter and will successfully invade a Dove population. Thus, mutants of each type can invade populations of the other type. The population cannot be monomorphic, and neither pure phenotype can be an ESS. The algebra again confirms our earlier finding for the numerical example of chicken (see Figures 12.2 and 12.3).

What happens in the population, then, when $V < C$? There are two possibilities. In one, every player follows a pure strategy, but the population has a stable mix of players following different strategies. This is the polymorphic equilibrium developed for chicken in Section 2.C. The other possibility is that every player uses a mixed strategy. We begin with the polymorphic case.

D. V < C: Stable Polymorphic Population

When the population proportion of Hawks is h, the fitness of a Hawk is $h(V - C)/2 + (1 - h)V$, and the fitness of a Dove is $h \times 0 + (1 - h)(V/2)$. The Hawk type is fitter if

$$\frac{h(V - C)}{2} + (1 - h)V > (1 - h)\frac{V}{2},$$

---

which simplifies to

$$\frac{h(V - C)}{2} + (1 - h)\frac{V}{2} > 0$$

$$V - hC > 0$$

$$h < \frac{V}{C}.$$

The Dove type is then fitter when $h > V/C$, or when $(1 - h) < 1 - (V/C) = (C - V)/C$. Thus, each type is fitter when it is rarer. Therefore, we have a stable polymorphic equilibrium at the balancing point, where the proportion of Hawks in the population is $h = V/C$. This is exactly the probability with which each individual member plays the Hawk strategy in the mixed-strategy Nash equilibrium of the game under the assumption of rational behavior, as calculated in Section 4.A. Again, we have an evolutionary “justification” for the mixed-strategy outcome in chicken.

We leave it to you to draw a graph similar to that in Figure 12.3 for this case. Doing so will require you to determine the dynamics by which the population proportions of each type converge to the stable equilibrium mix.

E. V < C: Each Player Mixes Strategies

Recall the equilibrium mixed strategy of the rational-play game calculated in Section 4.A, in which $p = V/C$ was the probability of choosing to be a Hawk, while $(1 - p)$ was the probability of choosing to be a Dove. Is there a parallel in the evolutionary version, with a phenotype playing a mixed strategy? Let us examine this possibility. We still have types who play the pure Hawk strategy, which we call H, and types who play the pure Dove strategy, called D. But now a third phenotype, called M, can exist; such a type plays a mixed strategy in which it is a Hawk with probability $p = V/C$ and a Dove with probability $1 - p = 1 - (V/C) = (C - V)/C$.

When an H or a D meets an M, their expected payoffs depend on p, the probability that M is playing H, and on $(1 - p)$, the probability that M is playing D. Then each player gets p times her payoff against an H, plus $(1 - p)$ times her payoff against a D. So when an H type meets an M type, she gets the expected payoff

$$\begin{aligned} p\,\frac{V-C}{2} + (1-p)\,V &= \frac{V}{C}\,\frac{V-C}{2} - \frac{C-V}{C}\,V \\ &= -\frac{1}{2}\,\frac{V}{C}(C-V) + \frac{V}{C}(C-V) \\ &= V\,\frac{C-V}{2C}\,. \end{aligned}$$

And when a D type meets an M type, she gets

$$p \times 0 + (1-p)\,\frac{V}{2} = \frac{C-V}{V}\,\frac{V}{2} = \frac{V(C-V)}{V}\,.$$

The two fitnesses are equal. This should not be a surprise; the proportions of the mixed strategy are determined to achieve exactly this equality. So an M type meeting another M type also gets the same expected payoff. For brevity of future reference, we call this common payoff K, where $K = V(C - V)/2C$.

But these equalities create a problem when we test M for evolutionary stability. Suppose the population consists entirely of M types and that a few mutants of the Hawk type, constituting a very small proportion h of the total population, invade. Then the typical mutant gets the expected payoff $h(V - C)/2 + (1 - h)K$. To calculate the expected payoff of an M type, note that she faces another M type in a fraction $(1 - h)$ of the interactions and gets K in each instance. She then faces an H type in a fraction h of the interactions; in these interactions, she plays H a fraction p of the time and gets $(V - C)/2$, and she plays D a fraction $(1 - p)$ of the time and gets 0. Thus, the M type's total expected payoff (fitness) is

$$\frac{hp(V-C)}{2} + (1 - h)K.$$

Because h is very small, the fitnesses of the M type and the mutant H type are almost equal. The point is that when there are very few mutants, both the H type and the M type meet only M types most of the time, and in this interaction the two have equal fitness, as we just saw.

Evolutionary stability hinges on whether the M type is fitter than the mutant H type when each is matched against one of the few mutants. Algebraically, M is fitter than H against other mutant H types when $pV(C - V)/2C = pK > (V - C)/2$. In our example here, this condition holds because $V < C$ [so $(V - C)$ is negative] and because K is positive. Intuitively, this condition tells us that an H-type mutant will always do badly against another H-type mutant because of the high cost of fighting, but the M type fights only part of the time and therefore suffers this cost only a fraction p of the time. Overall, the M type does better than the H type when matched against a mutant.

Similarly, the success of a Dove invasion of the M population depends on the comparison between a mutant Dove’s fitness and the fitness of an M type. As before, the mutant faces another D in a fraction d of the interactions and faces an M in a fraction $(1 - d)$ of the interactions. An M type faces another M type in a fraction $(1 - d)$ of the interactions, but in a fraction d of the interactions, the M faces a D; she plays H a fraction p of those times, thereby gaining pV, and plays D a fraction $(1 - p)$ of those times, thereby gaining $(1 - p)V/2$. The Dove’s fitness is then $dV/2 + (1 - d)K$, while the fitness of the M type is $d \times [pV + (1 - p)V/2] + (1 - d)K$. The final term in each fitness expression is the same, so a Dove invasion is successful only if $V/2$ is greater than $pV + (1 - p)V/2$. This condition does not hold; the latter expression includes a weighted average of V and $V/2$ that must exceed $V/2$ whenever $V > 0$. Thus, the Dove invasion cannot succeed either.

This analysis tells us that M is an ESS. Thus, if $V < C$, the population can exhibit either of two evolutionarily stable outcomes. One entails a mixture of types (a stable polymorphism), and the other entails a single type that mixes its strategies in the same proportions that define the polymorphism.

F. Some General Theory

We can now generalize the ideas illustrated in this section to get a theoretical framework and a set of tools that can then be applied further. This generalization unavoidably requires some slightly abstract notation and a bit of algebra. Therefore, we cover only monomorphic equilibria in a single species. Readers who are adept at this level of mathematics can readily develop the polymorphism cases with two species by analogy. Readers who are either unprepared for this material or uninterested in it can omit this section without loss of continuity.[12]

We consider random pairings from a single species whose population has available strategies I, J, K . . . . Some of them may be pure strategies; some of them may be mixed. Each individual member of the population is hardwired to play just one of these strategies. We let E(I, J) denote the payoff to an I player in a single encounter with a J player. The payoff of an I player meeting another of her own type is E(I, I) in the same notation. We write W(I) for the fitness of an I player. This is simply her expected payoff in encounters with randomly picked opponents, when the probability of her meeting a type is simply the proportion of that type in the population.

Suppose the population is all I type. We consider whether this can be an evolutionarily stable configuration. To do so, we imagine that the population is invaded by a few J-type mutants, so the proportion of mutants in the population is a very small number, m. Now the fitness of an I type is

$$W(\mathrm{I}) = mE(\mathrm{I}, \mathrm{J}) + (1 - m)E(\mathrm{I}, \mathrm{I}),$$

and the fitness of a mutant is

$$W(\mathrm{J}) = mE(\mathrm{J}, \mathrm{J}) + (1 - m)E(\mathrm{J}, \mathrm{I}).$$

Therefore, the difference in fitness between the population's predominant type and its mutant type is

$W(\mathrm{I}) - W(\mathrm{J}) = m[E(\mathrm{I}, \mathrm{J}) - E(\mathrm{J}, \mathrm{J})] + (1 - m)[E(\mathrm{I}, \mathrm{I}) - E(\mathrm{J}, \mathrm{I})]$.

Because m is very small, the predominant type's fitness will be higher than the mutant's if the second half of the preceding expression is positive; that is,

$W(\mathrm{I}) > W(\mathrm{J})$ if $E(\mathrm{I}, \mathrm{I}) > E(\mathrm{J}, \mathrm{I})$.

Then the population meets the primary criterion for evolutionary stability: It cannot be invaded, because the predominant type is fitter than the mutant type when each is matched against a member of the predominant type. Conversely, if $W(\mathrm{I}) < W(\mathrm{J})$, owing to $E(\mathrm{I}, \mathrm{I}) < E(\mathrm{J}, \mathrm{I})$, the J-type mutants can invade successfully, and an all-I population cannot be evolutionarily stable.

However, it is possible that $E(\mathrm{I}, \mathrm{I}) = E(\mathrm{J}, \mathrm{I})$, as indeed happens if the population initially consists of a single phenotype that plays a strategy of mixing between the pure strategies I and J (a monomorphic equilibrium with a mixed strategy), as was the case in our final variant of the hawk–dove game (see Section 4.E). Then the difference between $W(\mathrm{I})$ and $W(\mathrm{J})$ is governed by how each type fares against the mutants.[13] When $E(\mathrm{I}, \mathrm{I}) = E(\mathrm{J}, \mathrm{I})$, we get $W(\mathrm{I}) > W(\mathrm{J})$ if $E(\mathrm{I}, \mathrm{J}) > E(\mathrm{J}, \mathrm{J})$, which indicates that the population meets the secondary criterion for the evolutionary stability of I. This criterion is invoked only if the primary criterion is inconclusive—that is, only if $E(\mathrm{I}, \mathrm{I}) = E(\mathrm{J}, \mathrm{I})$.

If the secondary criterion is invoked—because $E(\mathrm{I}, \mathrm{I}) = E(\mathrm{J}, \mathrm{I})$—there is the additional possibility that it may also be inconclusive. That is, it may also be the case that $E(\mathrm{I}, \mathrm{J}) = E(\mathrm{J}, \mathrm{J})$. If both the primary and secondary criteria for the evolutionary stability of I are inconclusive, then I is considered a *neutral ESS*. Thorough analysis of this special case is beyond the scope of our introductory text, so we leave it to those with a deeper interest in the topic to explore more advanced treatments of the theory.

The primary criterion says that if the strategy I is evolutionarily stable, then for all other strategies J that a mutant might try, $E(\mathrm{I}, \mathrm{I}) \geqslant E(\mathrm{J}, \mathrm{I})$. This means that I is the best response to itself. In other words, if the members of this population suddenly started playing as rational calculators, all members playing I would be a Nash equilibrium. We explained this in the context of earlier examples; here we see the result in its general theoretical definition.

This is a remarkable result. If you were dissatisfied with the rational-play assumption underlying the theory of Nash equilibria given in earlier chapters and you came to the theory of evolutionary games looking for a better explanation, you will find that it yields the same results. The very appealing biological description—fixed nonmaximizing behavior, but selection in response to resulting fitness—does not yield any new outcomes. If anything, it provides a backdoor justification for Nash equilibrium. When a game has several Nash equilibria, the evolutionary dynamics may even provide a good argument for choosing among them.

However, your reinforced confidence in Nash equilibrium should be cautious. Our definition of evolutionary stability is static rather than dynamic. It requires only that the configuration of the population (monomorphic, or polymorphic in just the right proportions) that we are testing for equilibrium cannot be successfully invaded by a small proportion of mutants. It does not test whether, starting from an arbitrary initial population mix, all the types other than the fittest will die out and the equilibrium configuration will be reached. And the test is carried out for those particular classes of mutants that are deemed logically possible; if the theorist has not specified this classification correctly and some type of mutant that she overlooked could actually arise, that mutant might invade successfully and destroy the supposed equilibrium. Finally, as we showed in the soccer penalty kick example of Section 2.E, evolutionary dynamics can fail to converge at all.

Endnotes

- Conversely, readers who want more details can find them in Maynard Smith, *Evolution and the Theory of Games*, especially pp. 14–15. John Maynard Smith is a pioneer in the theory of evolutionary games. Return to reference 12
- If the initial population is polymorphic and m is the proportion of J types, then m may not be "very small" any more. The size of m is no longer crucial, however, because the second term in $W(\mathrm{I}) - W(\mathrm{J})$ is now assumed to be zero. Return to reference 13

Glossary

hawk - dove game
: An evolutionary game where members of the same species or population can breed to follow one of two strategies, Hawk and Dove, and depending on the payoffs, the game between a pair of randomly chosen members can be either a prisoners' dilemma or chicken.

primary criterion
: Comparison of the fitness of a mutant with that of a member of the dominant population, when each plays against a member of the dominant population.

secondary criterion
: Comparison of the fitness of a mutant with that of a member of the dominant population, when each plays against a mutant.

5 EVOLUTION OF COOPERATION AND ALTRUISM

Evolutionary game theory rests on two fundamental ideas: first, that individual organisms are engaged in games with others of their own species or with members of other species, and second, that the genotypes that lead to higher-payoff (fitter) strategies proliferate and increase in their proportions of the population while the rest decline. These ideas suggest a vicious struggle for survival like that depicted by some interpreters of Darwin, who understood "survival of the fittest" in a literal sense and who conjured up images of a "nature red in tooth and claw." In fact, nature shows many instances of cooperation (in which individual animals behave in a way that yields the greatest benefit to everyone in a group) and even altruism (in which individual animals incur significant costs in order to benefit others). Beehives and ant colonies are only the most obvious examples. Can such behavior be reconciled with the perspective of evolutionary games?

Biologists use a fourfold classification of the ways in which cooperation and altruism can emerge among selfish animals (or phenotypes or genes): (1) family dynamics, (2) reciprocal altruism, (3) selfish teamwork, and (4) group altruism.[14] The behavior of ants and bees is probably the easiest to understand as an example of family dynamics. All the individual members of an ant colony or a beehive are closely related and have genes in common to a substantial extent. Most worker ants in a colony are full sisters and therefore have half their genes in common; the survival and proliferation of one ant's genes is helped just as much by the survival of two of her sisters as by her own survival. Most worker bees in a hive are half-sisters and therefore have a quarter of their genes in common. An individual ant or bee does not make a fine calculation of whether it is worthwhile to risk her own life for the sake of two or four sisters, but the underlying genes of those groups whose members exhibit such behavior (phenotype) will proliferate. The idea that evolution ultimately operates at the level of the gene has had enormous implications for biology, although it has been misapplied by many people, just as Darwin's original idea of natural selection has been misapplied.[15] The interesting idea is that a "selfish gene" may prosper by behaving unselfishly in a larger organization of genes, such as a cell. Similarly, a cell and its genes may prosper by participating cooperatively and accepting their allotted tasks in a body.

Reciprocal altruism can arise among unrelated individual members of the same or different species. This behavior is essentially an example of the resolution of prisoners' dilemmas through repetition, in which the players use strategies that are remarkably like tit-for-tat. For example, some small fish and shrimp thrive on parasites that collect in the mouths and gills of some large fish; the large fish let the small ones swim unharmed through their mouths in return for this "cleaning service." Another fascinating, although more gruesome, example is that of vampire bats, who share blood with those who have been unsuccessful in their own hunting. In an experiment in which bats from different home sites were brought together and selectively starved, "only bats that were on the verge of starving (that is, would die within 24 hours without a meal) were given blood by any other bat in the experiment. But, more to the point, individuals were given a blood meal only from bats they already knew from their site. . . . Furthermore, vampires were much more likely to regurgitate blood to the specific individual(s) from their site that had come to their aid when they needed a bit of blood."[16] Once again, it is not to be supposed that each animal consciously calculates whether it is in its individual interest to continue the cooperation or to defect. Instead, its behavior is instinctive.

Selfish teamwork arises when it is in the interest of each individual organism to choose cooperation when all others are doing so. In other words, this concept of cooperative behavior applies to the selection of the good outcome in assurance games; for example, populations are more likely to engage in selfish teamwork in harsh environments than in mild ones. When conditions are bad, shirking by any one animal in a group could bring disaster to the whole group, including the shirker. In such conditions, each animal is crucial for the group's survival, and none shirk so long as others are also pulling their weight. In milder environments, each may hope to become a free rider on the others' efforts without thereby threatening the survival of the whole group, including itself.

The next step goes beyond biology and into sociology: A body (and its cells and, ultimately, its genes) may benefit by behaving cooperatively in a collection of bodies—namely, a society. This idea suggests that cooperation can arise even among individual members of a group who are not close relatives. We do indeed find instances of such behavior, which falls into the final category, group altruism. Groups of predators such as wolves are a case in point, and groups of apes often behave like extended families. Even among species of prey, such cooperation arises, as when individual fish in a school take turns looking out for predators. And cooperation can also extend across species. The general idea is that a group whose members behave cooperatively is more likely to succeed in its interactions with other groups than one whose members seek the benefit of free riding within the group. If, in a particular context of evolutionary dynamics, between-group selection is a stronger force than within-group selection, then we will see group altruism.[17]

An instinct is hardwired into an individual organism's brain by genetics, but reciprocity and cooperation can arise from more purposive thinking or experimentation within the group and can spread by socialization—through explicit instruction or observation of the behavior of elders—instead of genetics. The relative importance of the two channels—nature and nurture—will differ from one species to another and from one situation to another. One would expect socialization to be relatively more important among humans, but there are instances of its role among other animals. We cite a remarkable one. The expedition that Robert F. Scott led to the South Pole in 1911–1912 used teams of Siberian sled dogs. This group of dogs, brought together and trained for this specific purpose, developed within a few months a remarkable system of cooperation and sustained it by using punishment schemes. "They combined readily and with immense effect against any companion who did not pull his weight, or against one who pulled too much . . . their methods of punishment always being the same and ending, if unchecked, in what they probably called justice, and we called murder."[18]

This is an encouraging account of how cooperative behavior can be compatible with evolutionary game theory, and one that suggests that dilemmas of selfish action can be overcome. Indeed, scientists investigating altruistic behavior have recently reported experimental support for the existence of *altruistic punishment*, or *strong reciprocity* (as distinguished from reciprocal altruism), in humans. Their evidence suggests that people are willing to punish those who don't pull their own weight in a group setting, even when it is costly to do so and when there is no expectation of future gain. This tendency toward strong reciprocity may even help to explain the rise of human civilization if groups with this trait were better able to survive the traumas of war and other catastrophic events.[19] Despite these findings, strong reciprocity may not be widespread in the animal world. "Compared to nepotism, which accounts for the cooperation of ants and every creature that cares for its young, reciprocity has proved to be scarce. This, presumably, is due to the fact that reciprocity requires not only repetitive interactions, but also the ability to recognize other individuals and keep score."[20] In other words, precisely those conditions that our theoretical analysis in Section 2.D of Chapter 10 identified as being necessary for a successful resolution of the repeated prisoners' dilemma are seen to be relevant in the context of evolutionary games.

Endnotes

- For an excellent exposition, see Lee Dugatkin's *Cheating Monkeys and Citizen Bees: The Nature of Cooperation in Animals and Humans* (Cambridge, Mass.: Harvard University Press, 2000). Return to reference 15
- In this very brief account, we cannot begin to do justice to these debates or the issues involved. An excellent popular account, and the source of many examples cited in this section, is Matt Ridley, *The Origins of Virtue* (New York: Penguin, 1996). We should also point out that we do not examine the connection between genotypes and phenotypes, or the role of sex in evolution, in any detail. Another book by Ridley, *The Red Queen* (New York: Penguin, 1995), gives a fascinating treatment of this subject. Return to reference 15
- Dugatkin, *Cheating Monkeys*, p. 99. Return to reference 16
- Group altruism used to be thought impossible according to the strict theory of evolution that emphasizes selection at the level of the gene, but the concept is being revived in more sophisticated formulations. See Dugatkin, *Cheating Monkeys*, pp. 141–145, for a fuller discussion. Return to reference 17
- Apsley Cherry-Garrard, *The Worst Journey in the World* (London: Constable, 1922; reprint, New York: Carroll and Graf, 1989), pp. 485–86. Return to reference 18
- For the evidence on altruistic punishment, see Ernst Fehr and Simon Gachter, "Altruistic Punishment in Humans," *Nature*, vol. 415 (January 10, 2002), pp. 137–40. Return to reference 19
- Ridley, *Origins of Virtue*, p. 83. Return to reference 20

SUMMARY

The biological theory of evolution parallels the theory of games used by social scientists. Evolutionary games are played by behavioral *phenotypes* with genetically predetermined, rather than rationally chosen, strategies. In an evolutionary game, phenotypes with higher *fitness* survive repeated interactions with others to reproduce and thus increase their representation in the population. A population containing one or more phenotypes in certain proportions is called *evolutionarily stable* if it cannot be *invaded* successfully by other, *mutant* phenotypes or if it is the endpoint of the dynamics of proliferation of fitter phenotypes. If one phenotype maintains its dominance in the population when faced with an invading mutant type, that phenotype is said to be an *evolutionarily stable strategy (ESS)*, and a population consisting of that phenotype alone is said to be *monomorphic*. If two or more phenotypes coexist in an evolutionarily stable population, it is said to be *polymorphic*.

When the theory of evolutionary games is applied more generally to nonbiological games, the strategies followed by individual players are understood to be standard operating procedures or rules of thumb, instead of being genetically fixed. The process of reproduction stands for more general methods of transmission, including socialization, education, and imitation; mutations represent experimentation with new strategies.

Evolutionary games may have payoff structures similar to those analyzed in Chapters 4 and 7, including the prisoners' dilemma, games of chicken, and assurance games. In each case, the evolutionarily stable strategy mirrors either the pure-strategy Nash equilibrium of a game with the same structure played by rational players or the proportions of the equilibrium mixture in such a game. In one-time play of the prisoners’ dilemma, Defect is the evolutionarily stable strategy; in chicken; types are fitter when rare, so there is a polymorphic equilibrium. In the assurance game, types are less fit when rare, so the polymorphic configuration is unstable and the equilibria are at the extremes. When play is between two different types of members of each of two different species, a more complex, but similarly structured, analysis is used to determine equilibria. And in repeated play of the prisoners’ dilemma, Always defect is evolutionarily stable.

The *hawk–dove game* is the classic biological example of an evolutionary game. Analysis of this game parallels that of the prisoners’ dilemma and chicken; evolutionarily stable strategies depend on the specifics of the payoff structure.

KEY TERMS

evolutionary stability (471)

evolutionarily stable strategy (ESS) (473)

fitness (470)

genotype (470)

hawk–dove game (490)

invasion (470)

monomorphism (473)

mutation (470)

phenotype (470)

playing the field (471)

polymorphism (473)

primary criterion (495)

secondary criterion (495)

selection (470)

Glossary

evolutionarily stable
: A population is evolutionarily stable if it cannot be successfully invaded by a new mutant phenotype.

evolutionarily stable strategy (ESS)
: A phenotype or strategy that can persist in a population, in the sense that all the members of a population or species are of that type; the population is evolutionarily stable (static criterion). Or, starting from an arbitrary distribution of phenotypes in the population, the process of selection will converge to this strategy (dynamic criterion).

fitness
: The expected payoff of a phenotype in its games against randomly chosen opponents from the population.

genotype
: A gene or a complex of genes, which give rise to a phenotype and which can breed true from one generation to another. (In social or economic games, the process of breeding can be interpreted in the more general sense of teaching or learning.)

phenotype
: A specific behavior or strategy, determined by one or more genes. (In social or economic games, this can be interpreted more generally as a customary strategy or a rule of thumb.)

selection
: The dynamic process by which the proportion of fitter phenotypes in a population increases from one generation to the next.

invasion
: The appearance of a small proportion of mutants in the population.

mutation

Emergence of a new genotype.

playing the field

A many-player evolutionary game where all animals in the group are playing simultaneously, instead of being matched in pairs for two-player games.

monomorphism

All members of a given species or population exhibit the same behavior pattern.

polymorphism

An evolutionarily stable equilibrium in which different behavior forms or phenotypes are exhibited by subsets of members of an otherwise identical population.

hawk‐dove game

An evolutionary game where members of the same species or population can breed to follow one of two strategies, Hawk and Dove, and depending on the payoffs, the game between a pair of randomly chosen members can be either a prisoners' dilemma or chicken.

primary criterion

Comparison of the fitness of a mutant with that of a member of the dominant population, when each plays against a member of the dominant population.

secondary criterion

Comparison of the fitness of a mutant with that of a member of the dominant population, when each plays against a mutant.

SOLVED EXERCISES

1. Two travelers buy identical handcrafted souvenirs and pack them in their respective suitcases for their return flight. Unfortunately, the airline manages to lose both suitcases. Because the airline doesn't know the value of the lost souvenirs, it asks each traveler to report a value independently. The airline agrees to pay each traveler an amount equal to the minimum of the two reports. If one report is higher than the other, the airline takes a penalty of $20 away from the traveler with the higher report and gives $20 to the traveler with the lower report. If the two reports are equal, there is no reward or penalty. Neither traveler remembers exactly how much the souvenir cost, so that value is irrelevant; each traveler simply reports the value that her type determines she should report.

   There are two types of travelers. The High type always reports $100, and the Low type always reports $50. Let *h* represent the proportion of High types in the population.

   1. Draw the payoff table for the game played between two travelers selected at random from the population.
   2. Graph the fitness of the High type, with *h* on the horizontal axis. On the same figure, graph the fitness of the Low type.
   3. Describe all the equilibria of this game. For each equilibrium, state whether it is monomorphic or polymorphic and whether it is stable.
2. Consider a population in which there are two phenotypes: natural-born cooperators (who do not confess under questioning) and natural-born defectors (who confess readily). If two members of this population are drawn at random to play a prisoners' dilemma game, their payoffs in a single play are the same as those in the Husband–Wife prisoners' dilemma game of Chapter 4, as shown below.

   1.

		COLUMN	
		Confess (Defect)	Not (Cooperate)
ROW	Confess (Defect)	10 yr, 10 yr	1 yr, 25 yr
	Not (Cooperate)	25 yr, 1 yr	3 yr, 3 yr
You may need to scroll left and right to see the full figure.			

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   1. Suppose that a pair of players plays this dilemma twice in succession. Draw the payoff table for the twice-repeated dilemma

as in Figure 12.8, with A (always defect) and T (play tit-for-tat, starting with not defecting) as the two available strategies.

2. Find all of the ESS in the two-type twice-repeated dilemma in part (a).
3. Now, suppose that there is a third phenotype, N, which never defects. Draw an expanded three-by-three payoff table for the twice-repeated dilemma, with A, T, and N as the three available strategies.
4. Suppose that the population consists of an equal mix of tit-for-tat (T) and never-defect (N) types, but there are no always-defect (A) types. Can such a population be successfully invaded by A mutants?
5. Building on part (d), suppose that fraction q_T of the population are T types and the rest are N types. Verify that there is a balancing point, q^*, $0 < q^* < 1$, such that the population can be invaded by A mutants if $q_T < q^*$, but not if $q_T > q^*$. What is q^*?

3. Consider the thrice-repeated restaurant pricing prisoners' dilemma we studied in Section 3.B, but now suppose that there are three phenotypes in the population: type A, which always defects; type T, which plays tit-for-tat; and type N, which never defects. For your convenience, the expanded payoff table for this thrice-repeated three-phenotype evolutionary game is provided below. For each of parts (a) - (e), below, be specific and explicit in your answers; make sure you use the payoff numbers in the table provided.

		COLUMN		
		A	T	N
ROW	A	864, 864	936, 792	1080, 648
	T	792, 936	972, 972	972, 972
	N	648, 1080	972, 972	972, 972

You may need to scroll left and right to see the full figure.

---

1. Explain why a population that is 100% type A cannot be invaded by either type N or type T mutants.
2. Explain why a population that is 100% type N can be invaded by type A mutants.
3. Explain why a population that is 100% type T cannot be invaded by type A mutants.
4. Suppose that the population initially consists of an equal mix of the three phenotypes. Which of the three phenotypes will have the highest fitness initially and grow in number the fastest? Which

will have the lowest fitness and hence fall in number the fastest?
5. (**Optional**) Given the initial evolutionary dynamics described in part (d), how will population dynamics continue over time? In particular, will the population eventually consist of only one phenotype? If so, which one?

4. In the assurance (meeting coordination) game described in Section 2.D, the payoffs could be thought of as describing the value of something material that the players gained in the various outcomes; they could be prizes given for a successful meeting, for example. Then other individuals in the population might observe the expected payoffs (fitnesses) of the two types, see which was higher, and gradually imitate the fitter strategy. Thus, the proportions of the two types in the population would change. But we can make a more biological interpretation. Suppose the column players are always female and the row players always male. When two players of the same type meet successfully, they pair off, and their children are of the same type as the parents. Therefore, the types would proliferate or die off as a result of successful or unsuccessful meetings. The formal mathematics of this new version of the game makes it a "two-species game" (although the biology of it does not). Thus, the proportion of S-type females in the population—call this proportion x—need not equal the proportion of S-type males—call this proportion y.
   1. Examine the dynamics of x and y by using methods similar to those used in the chapter for the penalty kick game.
   2. Find the stable outcome or outcomes of this dynamic process.
5. Recall from Exercise S1 the travelers reporting the value of their lost souvenirs. Assume that a third traveler phenotype exists in the population. The third traveler type is a mixer; she plays a mixed strategy, sometimes reporting a value of \$100 and sometimes reporting a value of \$50.
   1. Use your knowledge of mixed strategies in rationally played games to posit a reasonable mixture for the mixer phenotype to use in this game.
   2. Draw the three-by-three payoff table for this game when the mixer type uses the mixed strategy that you found in part (a).
   3. Determine whether the mixer strategy is an ESS of this game. (Hint: Test whether a mixer population can be invaded successfully by either the High type or the Low type.)
6. Consider a simplified model in which everyone gets electricity either from solar power or from fossil fuels, which are both in fixed supply. (In the case of solar power, think of the required equipment as being in fixed supply.) The up-front costs of using solar power are high, so when the price of fossil fuels is low (that is, when few

people are using fossil fuels and there is a high demand for solar equipment), the cost of solar power can be prohibitive. In contrast, when many individuals are using fossil fuels, the demand for them (and thus the price) is high, whereas the demand (and thus the price) for solar power is relatively lower. Assume the payoff table for the two types of energy consumers to be as follows:

		COLUMN	
		Solar	Fossil fuels
ROW	Solar	2, 2	3, 4
	Fossil fuels	4, 3	2, 2

---

1. Does this evolutionary game have a monomorphic ESS?
2. Verify that this evolutionary game has a polymorphic ESS. What is the share $s*$ of the population that adopts solar in this ESS? [Hint: Construct a fitness graph like Figure 12.3 or Figure 12.5 for the two types. The balancing point (corresponding to the mixed-strategy Nash equilibrium of the game with rational players) is an ESS if, away from the balancing point, the more numerous type is less fit (as in Figure 12.3) but is not an ESS if the more numerous type is more fit (as in Figure 12.5).]
3. Suppose there are important economies of scale in producing solar equipment, such that the cost savings increase the payoffs in the (Solar, Solar) cell of the table to (y, y), where $y > 2$. How large would y need to be for the polymorphic ESS to have $s* = 0.75$?

7. There are two types of racers—tortoises and hares—who race against each other in randomly drawn pairs. In this world, hares beat tortoises every time without fail. If two hares race, they tie, and they are completely exhausted by the race. When two tortoises race, they also tie, but they enjoy a pleasant conversation along the way. The payoff table is as follows (where $c > 0$):

		COLUMN	
		Tortoise	Hare
ROW	Tortoise	c, c	−1, 1
	Hare	1, −1	0, 0

---

1. Assume that the proportion of tortoises in the population, t, is 0.5. For what values of c will tortoises have greater fitness than hares?

2. For what values of c will tortoises be fitter than hares if t = 0.1?
3. If c = 1, can a single hare successfully invade a population of tortoises? Explain why or why not.
4. In terms of t, how large must c be for tortoises to have greater fitness than hares?
5. In terms of c, what is the level of t in a polymorphic equilibrium? For what values of c will such an equilibrium exist? Explain.

8. Consider a population with two phenotypes, X and Y, with a payoff table as follows:

		COLUMN	
		X	Y
ROW	X	2, 2	5, 3
	Y	3, 5	1, 1

---

1. Find the fitness for X as a function of x, the proportion of X in the population, and the fitness for Y as a function of x.

   Assume that the population dynamics from generation to generation conform to the following model:

$$x_{t+1} = \frac{x_t \times F_{Xt}}{x_t \times F_{Xt} + (1 - x_t) \times F_{Yt}},$$

   where x_t is the proportion of X in the population in period t, x_{t+1} is the proportion of X in the population in period $t + 1$, F_{Xt} is the fitness of X in period t, and F_{Yt} is the fitness of Y in period t.

2. Assume that x_0, the proportion of X in the population in period 0, is 0.2. What are F_{X0} and F_{Y0}?
3. Find x_1, using x_0, F_{X0}, F_{Y0}, and the model given above.
4. What are F_{X1} and F_{Y1}?
5. Find x_2 (rounded to five decimal places).
6. What are F_{X2} and F_{Y2} (rounded to five decimal places)?

9. Consider an evolutionary game between a green type and a purple type with a payoff table as follows:

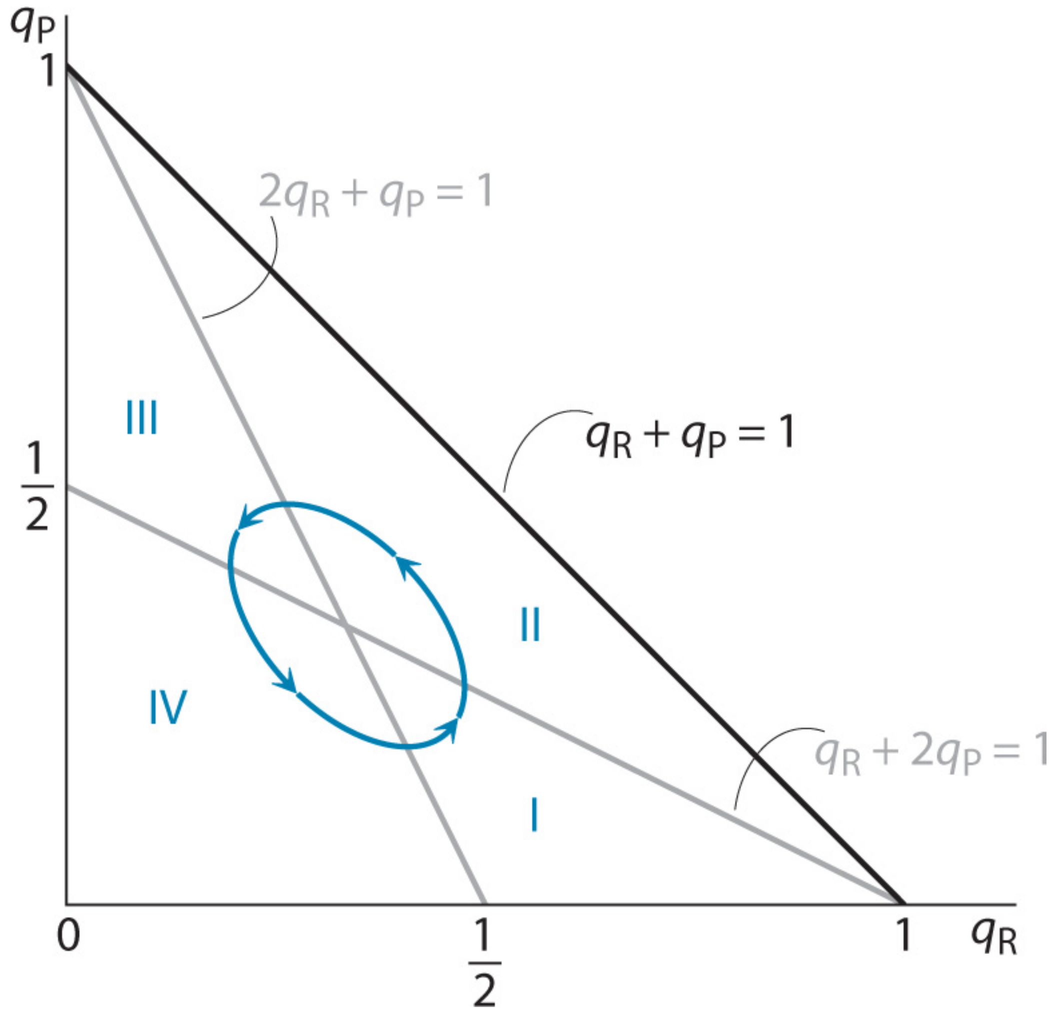
		COLUMN	
		Green	Purple
ROW	Green	a, a	4, 3
	Purple	3, 4	2, 2

Let g be the proportion of greens in the population.

1. In terms of g, what is the fitness of the purple type?
2. In terms of g and a, what is the fitness of the green type?
3. Graph the fitness of the purple types against the fraction g of greens in the population. On the same diagram, show three lines for the fitnesses of the green type when $a = 2$, 3, and 4. What can you conclude from this graph about the range of values of a that guarantees a stable polymorphic equilibrium?
4. Assume that a is in the range found in part (c). In terms of a, what is the proportion of greens, g, in the stable polymorphic equilibrium?

10. At the World RPS Society,[21] people take the children's game of rock-paper-scissors (RPS) very seriously. You might think it's just a matter of chance who wins—Rock beating Scissors, Scissors beating Paper, Paper beating Rock, and a tie if both players make the same choice—but, for professional RPS players, "you can't just rely on luck" to win. For instance, men appear hardwired to start more often with Rock, while women are more likely to start with Paper. To maximize your chances of winning your next RPS throwdown, you should therefore start with Paper against a man or with Scissors against a woman. Of course, once enough people start following that advice, players will benefit by adapting again, and yet again. Here and in Exercise U10, you will examine the dynamics of that evolutionary dance in more detail.
    1. Consider a population with three phenotypes: One always plays Rock (R type), one always plays Paper (P type), and one always plays Scissors (S type). Draw a three-by-three payoff table for a once-played RPS game between two friends, with R, P, and S as the three available phenotypes. Assign payoff 1 to any winning player, payoff −1 to any losing player, and payoff 0 to both players if there is a tie.
    2. Can there be a stable polymorphic configuration of the population with only R and P types? Use your answer to argue that, in any stable configuration, the population must include some of all three types.

Because $q_R+q_P+q_S = 1$, one can visualize all possible (q_R, q_P, q_S) configurations using a triangle as shown in the following figure. In the figure, each point in the triangle is a pair (q_R, q_P) with $q_R+q_P \leqslant 1$; the value for q_S is simply the difference between q_R+q_P and 1 at each point. In this figure, we have highlighted (i) the point (⅓, ⅓) corresponding to the mixed-strategy Nash equilibrium (found in Exercise S10 in [Chapter 7](#)), (ii) the lines consisting of all pairs (q_R, q_P) such that $2q_R+q_P = 1$ (line 1) and such that $q_R+ 2q_P = 1$ (line 2), and (iii) the four regions of the triangle (labeled I, II, III, IV) created by these lines.



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3. Verify that a player of type R gets a positive fitness whenever the population configuration corresponds to a point below line 1, and that a player of type P gets a positive fitness whenever the population configuration corresponds to a point to the right of line 2.

4. Suppose that the initial population configuration is in region I, and suppose that each type grows more numerous if it is getting a positive fitness, or grows less numerous if it is getting a negative fitness. Given these population dynamics, which of the following statements is true?
    1. The population will remain in region I forever.
    2. The population will converge to the (⅓, ⅓) configuration on a path that remains entirely inside region I.
    3. The population will transition to region II (moving counterclockwise).
    4. The population will transition to region IV (moving clockwise).

   Explain your answer.

5. Prove the following statement: “If a strategy is strictly dominated in the payoff table of a game played by rational players, then in the evolutionary version of the same game it will die out, no matter what the initial population mix. If a strategy is weakly dominated, it may coexist with some other types, but not in a mixture of all types.”

UNSOLVED EXERCISES

1. Consider a survival game in which members of a large population of animals meet in pairs and either fight over or share a food source. There are two phenotypes in the population: One always fights, and the other always shares. For the purposes of this question, assume that no other mutant types can arise in the population. Suppose that the value of the food source is 200 calories, and that caloric intake determines each player’s fitness. If two sharing types meet one another, they each get half the food, but if a sharer meets a fighter, the sharer concedes immediately, and the fighter gets all the food.
   1. Suppose that the cost of a fight is 50 calories (for each fighter) and that when two fighters meet, each is equally likely to win the fight and the food or to lose and get no food. Draw the payoff table for the game played between two random players from this population. Find all of the ESSs in the population. What type of game is being played in this case?
   2. Now suppose that the cost of a fight is 150 calories for each fighter. Draw the new payoff table and find all of the ESSs for the population in this case. What type of game is being played here?
   3. Using the notation of the hawk–dove game of Section 12.4, indicate the values of *V* and *C* in parts (a) and (b), and confirm that your answers to those parts match the analysis presented in the chapter.
2. Suppose that a single play of a prisoners’ dilemma has the following payoffs:

		PLAYER 2	
		Cooperate	Defect
PLAYER 1	Cooperate	3, 3	1, 4
	Defect	4, 1	2, 2
You may need to scroll left and right to see the full figure.			

In a large population in which each member’s behavior is genetically determined, each player will be either a defector (that is, always defects in any play of a prisoners’ dilemma game) or a tit-for-tat player (in multiple rounds of a prisoners’ dilemma, she cooperates on the first play, and on any subsequent play she does whatever her opponent did on the preceding play). Pairs of randomly chosen players from this population will play rounds consisting of n single plays of this dilemma (where $n \geqslant 2$). The payoff to each player in one whole round (of n plays) is the sum of her payoffs in the n plays.

Let the population proportion of defectors be p and the proportion of tit-for-tat players be $(1 - p)$. Each member of the population plays rounds of the dilemma repeatedly, matched against a new, randomly chosen opponent for each new round. A tit-for-tat player always begins each new round by cooperating on the first play.

1. Show in a two-by-two table the payoffs to a player of each type when, in one round of plays, each player meets an opponent of each of the two types.
2. Find the fitness (average payoff in one round against a randomly chosen opponent) for a defector.
3. Find the fitness for a tit-for-tat player.
4. Use the answers to parts (b) and (c) to show that, when $p > (n - 2)/(n - 1)$, the defector type has greater fitness and that, when $p < (n - 2)/(n - 1)$, the tit-for-tat type has greater fitness.
5. If evolution leads to a gradual increase in the proportion of the fitter type in the population, what are the possible eventual equilibrium outcomes of this process for the population described in this exercise? (That is, what are the possible equilibria, and which are evolutionarily stable?) Use a diagram with the fitness graphs to illustrate your answer.
6. In what sense does more repetition (larger values of n) facilitate the evolution of cooperation?

3. Consider the thrice-repeated restaurant pricing prisoners' dilemma we studied in Section 3.B, but now suppose that there are three phenotypes in the population: type A, which always defects; type T, which plays tit-for-tat; and type S, which never defects on the first play but always defects on the second play of each round of two successive plays against the same opponent.
    1. Draw the three-by-three fitness table for the game, similar to the fitness table that we provided in Exercise S3.

       For each of parts (b) - (d), be specific and explicit in your answers; make sure you use payoff numbers from the fitness table you draw to answer part (a).

    2. Can a population that is 100% type A be successfully invaded by type S mutants?
    3. Can a population that is 100% type T be successfully invaded by type S mutants?

    4. Is the newly conceived type S strategy an ESS of this game?
4. Following the pattern of Exercise S4, analyze an evolutionary version of the tennis point game (see Figure 4.17). Regard servers and receivers as separate species, and construct a figure like Figure 12.7. What can you say about the ESS and its dynamics?
5. Recall from Exercise U1 the population of animals fighting over a food source worth 200 calories. Assume that, as in part (b) of that exercise, the cost of a fight is 150 calories per fighter. Assume also that a third phenotype exists in the population. That phenotype is a mixer; it plays a mixed strategy, sometimes fighting and sometimes sharing.
    1. Use your knowledge of mixed strategies in rationally played games to posit a reasonable mixture for the mixer phenotype to use in this game.
    2. Draw the three-by-three payoff table for this game when the mixer phenotype uses the mixed strategy that you found in part (a).
    3. Determine whether the mixer strategy is an ESS of this game. (Hint: Test whether a mixer population can be invaded successfully by either the fighting type or the sharing type.)
6. Consider an evolutionary version of the game between Baker and Cutler, from Exercise U1 of Chapter 10. This time, Baker and Cutler are not two individuals, but two separate species. Each time a Baker meets a Cutler, they play the following game. The Baker chooses the total prize to be either \$10 or \$100. The Cutler chooses how to divide the prize chosen by the Baker: The Cutler can choose either a 50:50 split or a 90:10 split in the Cutler's own favor. The Cutler moves first, and the Baker moves second.

    There are two types of Cutlers in the population: Type F chooses a fair (50:50) split, whereas type G chooses a greedy (90:10) split. There are also two types of Bakers: Type S simply chooses the large prize (\$100) no matter what the Cutler has done, whereas type T chooses the large prize (\$100) if the Cutler chooses a 50:50 split, but the small prize (\$10) if the Cutler chooses a 90:10 split.

    Let f be the proportion of type F in the Cutler population, so that $(1 - f)$ represents the proportion of type G. Let s be the proportion of type S in the Baker population, so that $(1 - s)$ represents the proportion of type T.

    1. Find the fitnesses of the Cutler types F and G in terms of s.
    2. Find the fitnesses of the Baker types S and T in terms of f.
    3. For what value of s are types F and G equally fit?
    4. For what value of f are types S and T equally fit?

5. Use the answers above to sketch a graph displaying the population dynamics. Assign f as the horizontal axis and s as the vertical axis.
6. Describe all equilibria of this evolutionary game and indicate which ones are stable.

7. Recall Exercise S7. Hares, it turns out, are very impolite winners. Whenever hares race tortoises, they mercilessly mock their slow-footed (and easily defeated) rivals. The poor tortoises leave the race not only in defeat, but with their tender feelings crushed by the oblivious hares. The payoff table is thus

		COLUMN	
		Tortoise	Hare
ROW	Tortoise	c, c	−2, 1
	Hare	1, −2	0, 0

---

1. For what values of c are tortoises fitter than hares if t, the proportion of tortoises in the population, is 0.5? How does this compare with your answer in Exercise S7, part (a)?
2. For what values of c are tortoises fitter than hares if $t = 0.1$? How does this compare with your answer in Exercise S7, part (b)?
3. If $c = 1$, will a single hare successfully invade a population of tortoises? Explain why or why not.
4. In terms of t, how large must c be for tortoises to be fitter than hares?
5. In terms of c, what is the value of t in a polymorphic equilibrium? For what values of c will such an equilibrium exist? Explain.
6. Will the polymorphic equilibria found to exist in part (e) be stable? Why or why not?

8. **(Use of spreadsheet software recommended)** This problem explores more thoroughly the generation-by-generation population dynamics seen in Exercise S8. Since the math can quickly become very complicated and tedious, it is much easier to do this analysis with the aid of a spreadsheet.

   Again, consider a population with two types, X and Y, with a payoff table as follows:

		COLUMN	
		X	Y
ROW	X	2, 2	5, 3

	COLUMN		
		X	Y
	Y	3, 5	1, 1

Recall that the population dynamics from generation to generation are given by

$$x_{t+1} = \frac{x_t \times F_{Xt}}{x_t \times F_{Xt} + (1 - x_t) \times F_{Yt}},$$

where x_t is the proportion of X in the population in period t, x_t+_1 is the proportion of X in the population in period t + 1, F_{Xt} is the fitness of X in period t, and F_{Yt} is the fitness of Y in period t.

Use a spreadsheet to extend these calculations to many generations. [Hint: Assign three horizontally adjacent cells to hold the values of x_t, F_{Xt}, and F_{Yt}, and have each successive row represent a different period (t = 0, 1, 2, 3, . . .). Use spreadsheet formulas to relate F_{Xt} and F_{Yt} to x_t and x_t+_1 to x_t, F_{Xt}, and F_{Yt} according to the population model given above.]

1. If there are initially equal proportions of X and Y in the population in period 1 (that is, if x_0 = 0.5), what is the proportion of X in the next generation, x_1? What are F_{X1} and F_{Y1}?
2. Use a spreadsheet to extend these calculations to the next generation, and the next, and so on. To four decimal places, what is the value of x_{20}? What are F_{X20} and F_{Y20}?
3. What is x*, the equilibrium level of x? How many generations does it take for the population to be within 1% of x*?
4. Answer the questions in part (b), but with a starting value of x_0 = 0.1.
5. Repeat part (b), but with x_0 = 1.
6. Repeat part (b), but with x_0 = 0.99.
7. Are monomorphic equilibria possible in this model? If so, are they stable? Explain.

9. Consider an evolutionary game between a green type and a purple type, with a payoff table as follows:

COLUMN

		Green	COLUMN Purple
		Green	Purple
ROW	Green	*a*, *a*	*b*, *c*
	Purple	*c*, *b*	*d*, *d*

In terms of the parameters *a*, *b*, *c*, and *d*, find the conditions that will guarantee a stable polymorphic equilibrium.

10. **(Optional**, for mathematically trained students**)** Common side-blotched lizards (*Uta stansburiana*), residents of the California desert, play an unusual evolutionary game. During mating season, males display colorful throat patches in three shades—orange, blue, and yellow—with each color corresponding to a different mating strategy. Orange-throats (type O) tend to be somewhat larger, are aggressive, and can defend large breeding territories that host two or more females. Yellow-throats (type Y) are smaller, generally unable to defend much territory at all, and mate mainly by sneaking into other males' territories. Finally, blue-throats (type B) are monogamous, sticking close to a single mate. These differences in color and in behavior, all seemingly determined by a single gene (!), result in orange-throats beating blue-throats in competition over females, but often being bested by "sneaky" yellow-throats, who in turn can be defeated by blue-throats, who guard their mates more carefully.[22]

    The game played among these lizard phenotypes thus has essentially the same payoff structure as rock-paper-scissors (RPS) in Exercise S10, with type O as Rock, type Y as Paper, and type B as Scissors. The actual reproductive payoffs of these phenotypes are not exactly the same as in RPS, but we suppose for simplicity that they are—with payoffs of +1, 0, or −1, depending on which phenotypes are matched to play the game—and that each phenotype grows in number over time if its fitness is positive, or declines in number if its fitness is negative.

    1. Let q_Y, q_B denote the proportion of lizards in the population that are yellow-throats and blue-throats, respectively, and let $q_O = 1 - q_Y - q_B$ be the remaining proportion of orange-throats. Express the fitnesses of phenotypes Y, B, and O as functions of q_Y, q_B, and q_O, respectively. Verify that (i) type Y grows in number if and only if $q_O > q_B$, (ii) type B grows in number if and only if $q_Y > q_O$, and (iii) type O grows in number if and only if $q_B > q_Y$.

2. Consider the dynamics of this evolutionary system more explicitly. Let the speed of change in a variable x at time t be denoted by the derivative dx/dt. Now consider the following expressions:

$$\frac{dq_{\mathrm{Y}}}{dt} = q_{\mathrm{O}} - q_{\mathrm{B}}, \quad \frac{dq_{\mathrm{B}}}{dt} = q_{\mathrm{Y}} - q_{\mathrm{O}}, \quad \text{and} \quad \frac{dq_{\mathrm{O}}}{dt} = q_{\mathrm{B}} - q_{\mathrm{Y}}.$$

Verify that these derivatives conform to the findings of part (a).

3. Define $X = (q_{\mathrm{Y}})^2 + (q_{\mathrm{B}})^2 + (q_0)^2$. Using the chain rule of differentiation, show that $dX/dt = 0$; that is, show that X remains constant over time.
4. From the definitions of these variables, we know that $q_{\mathrm{Y}}+q_{\mathrm{B}}+q_0= 1$. Combining this fact with the result from part (c), show that over time, in three-dimensional space, the point $(q_{\mathrm{Y}},\ q_{\mathrm{B}},\ q_0)$ moves along a circle.
5. What does the answer to part (d) indicate regarding the stability of the evolutionary dynamics in the side-blotched lizard population? Will the population ever converge to a stable mixture of the three phenotypes?

Endnotes

- The World RPS Society hosts the website www.wrpsa.com. The “facts” presented in this exercise are discussed in that site’s “Rock Paper Scissors Beginner Strategies,” available at https://www.wrpsa.com/rock-paper-scissors-beginner-strategies (accessed May 1, 2019). Return to reference 21
- For more information about the side-blotched lizards, see Kelly Zamudio and Barry Sinervo, “Polygyny, Mate-Guarding, and Posthumous Fertilizations as Alternative Mating Strategies,” *Proceedings of the National Academy of Sciences*, vol. 97, no. 26 (December 19, 2000), pp. 14427–32. Return to reference 22

PART FOUR

Applications to Specific Strategic Situations

13■ Brinkmanship: The Cuban Missile Crisis

IN CHAPTER 1, we explained that our basic approach to strategic games was neither pure theory nor pure case study, but a combination in which theoretical ideas would be developed by using features of particular cases or examples. Thus, we ignored those aspects of each case that were incidental to the concept being developed. However, after you have learned the theoretical ideas, a richer mode of analysis becomes available to you in which factual details of a particular case are more closely integrated with game-theoretic analysis to achieve a fuller understanding of what has happened and why. Such *theory-based case studies* have begun to appear in diverse fields—business, political science, and economic history.[1]

In this chapter, we offer an case from political and military history—namely, the Cuban missile crisis of 1962. Our choice is motivated by the sheer drama of the episode, the wealth of factual information that has become available, and the applicability of the important concept of brinkmanship from game theory.

The crisis, when the world came as close to a nuclear war as it ever has, is often offered as the classic example of brinkmanship. You may think that the risk of nuclear war ended with the dissolution of the Soviet Union, making this case a historical curiosity. But nuclear arms races continue in many parts of the world, and such rivals as India and Pakistan, or Iran and Israel, may find use for the lessons taken from the Cuban missile crisis. Even major-power rivalries, including those between the United States and China, or between the United States and Russia, may heat up. More importantly for many of you, brinkmanship must be practiced in many more common situations, from political negotiations to business–labor relations to marital disputes. Although the stakes in such games are lower than those in a nuclear confrontation between superpowers, the same principles of strategy apply.

In Chapter 8, we introduced the concept of brinkmanship as a strategic move; here is a quick reminder of that analysis. A *threat* is a response rule, and the threatened action inflicts a cost on both the player making the threat and the player whose action the threat is intended to influence. However, if the threat succeeds in its purpose, this action is not actually carried out. Therefore, there is no apparent upper limit to the cost of the threatened action. But the risk of *errors*—that is, the risk that the threat may fail to achieve its purpose, or that the threatened action may occur by accident—forces the strategist to use the minimal threat that achieves its purpose. If a smaller threat is not naturally available, you can scale down a large threat by making its fulfillment probabilistic. You can do something in advance that creates a probability, but not a certainty, that the mutually harmful outcome will happen if the opponent defies you. If the need actually arose, you would not take that bad action if you had the full freedom to choose. Therefore, you must arrange in advance to let things get out of your control to some extent. Brinkmanship is the creation and deployment of such a probabilistic threat; it consists of a deliberate loss of control.

In our extended case study of the Cuban missile crisis, we will explain the concept of brinkmanship in detail. In the process, we will find that many popular interpretations and analyses of the crisis are simplistic. A deeper analysis reveals brinkmanship to be a subtle and dangerous strategy. It also shows that many detrimental outcomes in business and personal interactions—such as strikes and breakups of relationships—are examples of brinkmanship gone wrong.

Therefore, a clear understanding of the strategy, as well as its limitations and risks, is very important to all game players, which includes just about everyone.

Endnotes

- Two excellent examples of theory-based case studies are Pankaj Ghemawat, *Games Businesses Play: Cases and Models* (Cambridge, Mass.: MIT Press, 1997), and Robert H. Bates, Avner Greif, Margaret Levi, Jean-Laurent Rosenthal, and Barry Weingast, *Analytic Narratives* (Princeton, N.J.: Princeton University Press, 1998). A broader analysis of the approach can be found in Alexander L. George and Andrew Bennett, *Case Studies and Theory Development in the Social Sciences* (Cambridge, Mass.: MIT Press, 2005). Return to reference 1

Glossary

brinkmanship
: A threat that creates a risk but not certainty of a mutually bad outcome if the other player defies your specified wish as to how he should act, and then gradually increases this risk until one player gives in or the bad outcome happens.

1 A BRIEF NARRATIVE OF EVENTS

We begin with a brief story of the unfolding of the crisis. Our account draws on several books, including some that were written with the benefit of documents and statements released since the collapse of the Soviet Union.[2] We cannot hope to do justice to the detail, let alone the drama, of the events. We urge you to read the books that tell the story in vivid detail and to talk to any relatives who lived through it to get their firsthand memories.[3]

In late summer and early fall of 1962, at the height of the Cold War, the Soviet Union (USSR) started to place medium- and intermediate-range ballistic missiles (MRBMs and IRBMs) armed with nuclear weapons in Cuba. The MRBMs had a range of 1,100 miles and could hit Washington, D.C.; the IRBMs, with a range of 2,200 miles, could hit most major U.S. cities and military installations. The missile sites were guarded by the latest Soviet SA-2 surface-to-air missiles (SAMs), which could shoot down U.S. high-altitude U-2 reconnaissance planes. They were also defended by IL-28 bombers and tactical nuclear weapons called Luna by the Soviets and FROGs (free rockets over ground) by the United States, which could be used against invading troops.

This was the first time that the Soviets had ever attempted to place their missiles and nuclear weapons outside Soviet territory. Had they been successful, it would have increased their offensive capability against the United States manyfold. It is now believed that the Soviets had very few (U.S. aerial reconnaissance showed only four) operational intercontinental ballistic missiles (ICBMs) in their own country capable of reaching the United States (*War*, 464, 509–10; *Doomsday*, 158). Their initial installation in Cuba had about 40 MRBMs and IRBMs, which was a substantial increase. But the United States would still have retained vast superiority in the nuclear balance between the superpowers. Also, as the Soviets built up their fleet of nuclear-armed submarines, the relative importance of land-based missiles near the United States would have decreased. But the missiles had more than direct military value to the Soviets. Successful placement of missiles so close to the United States would have been an immense boost to Soviet prestige throughout the world, especially in Asia and Africa, where the superpowers were competing for political and military influence. Finally, the Soviets had come to think of Cuba as a poster child for socialism. The opportunity to deter a feared U.S. invasion of Cuba, and to counter Chinese influence in Cuba, weighed importantly in the calculations of the Soviet leader and premier, Nikita Khrushchev. (See *Gamble*, 182–83, for an analysis of Soviet motives.)

The whole operation was attempted in utmost secrecy, and the Soviets hoped to conceal the missiles under palm trees (*Gamble*, Chapter 10)! But this attempt did not work. U.S. surveillance of Cuba and of shipping lanes during the late summer and early fall of 1962 had indicated some suspicious activity. When questioned about it by U.S. diplomats, the Soviets denied any intentions to place missiles in Cuba. Later, faced with irrefutable evidence, they said that their intention was defensive, to deter the United States from invading Cuba. It is hard to believe this, although we know that an offensive weapon *can* serve as a defensive deterrent threat.

On Sunday and Monday, October 14 and 15, an American U-2 spy plane took photographs over western Cuba that showed unmistakable signs of construction on MRBM launching sites. (Evidence of IRBMs was found later, on October 17.) They were shown to U.S. President John F. Kennedy the following day (October 16). He immediately convened an ad hoc group of top-level advisers, later called the Executive Committee of the

National Security Council (ExComm), to discuss the available options.[4] He decided to keep the matter totally secret until he was ready to act, mainly because if the Soviets knew that the Americans knew about the missiles, they might speed up their installation and deployment before the Americans were ready to act, but also because spreading the news without announcing a clear response would create panic in the United States. During the rest of that week (October 16–21), ExComm met numerous times. To preserve secrecy, the president continued his normal schedule, including travel to speak for Democratic candidates in the upcoming midterm congressional elections. He kept in constant touch with ExComm. He dodged press questions about Cuba and persuaded one or two trusted media owners or editors to preserve the facade of business as usual.

Different members of ExComm had widely differing assessments of the situation and supported different actions. The military Joint Chiefs of Staff thought that the missile placement changed the balance of military power substantially; Defense Secretary Robert McNamara thought it had changed "not at all," but regarded the problem as politically important nonetheless (*Tapes*, 89). Kennedy pointed out that the first placement, if ignored by the United States, could grow into something much bigger, and that the Soviets could use the threat of missiles so close to the United States to try to force the withdrawal of U.S., British, and French forces from West Berlin. Kennedy was also aware that the placement of the missiles was a part of the *geopolitical* struggle between the United States and the Soviet Union (*Tapes*, 92).

It now appears that Kennedy was very much on the mark in this assessment. The Soviets planned to expand their presence in Cuba into a major military base (*Tapes*, 677). They expected to complete the missile placement by mid-November. Khrushchev had planned to sign a treaty with Cuba's prime minister, Fidel Castro, in late November, then travel to New York to address the United Nations and issue an ultimatum for a settlement of the Berlin issue (*Tapes*, 679; *Gamble*, 182), using the missiles in Cuba as a threat for this purpose. Khrushchev thought Kennedy would accept the missile placement as a *fait accompli.* Khrushchev appears to have made these plans on his own. Some of his top advisers privately thought them too adventurous, but the top governmental decision-making body of the Soviet Union, the Presidium, supported him, although it acted largely as a rubber stamp for the premier’s decisions (*Gamble*, 180). Castro was at first reluctant to accept the missiles, fearing that they would trigger a U.S. invasion (*Tapes*, 676–78), but in the end he, too, accepted them. The prospect gave him great confidence and lent some swagger to his statements about the United States (*Gamble*, 186–87, 229–30).

In all ExComm meetings up to and including the one on the morning of Thursday, October 18, all the participants appear to have assumed that the U.S. response would be purely military. The only options that they discussed seriously during this time were (1) an air strike directed exclusively at the missile sites and (probably) the SAM sites nearby, (2) a wider air strike including Soviet and Cuban aircraft parked at airfields, and (3) a full-scale invasion of Cuba. If anything, attitudes hardened when the evidence of the presence of the longer-range IRBMs arrived. In fact, at the Thursday meeting, Kennedy discussed a timetable for air strikes to commence that weekend (*Tapes*, 148).

McNamara had first mentioned a blockade of Cuba toward the end of the meeting on Tuesday, October 16, and he developed the idea (in a form uncannily close to the course of action actually taken) in a small group after the formal meeting had ended (*Tapes*, 86, 113). George Ball argued that an air strike without warning would be a “Pearl Harbor” and that the United States should not do it (*Tapes*, 115); he got important support for this view from Robert Kennedy (*Tapes*, 149). The civilian members of ExComm further shifted toward the blockade option when they found that what the military Joint Chiefs of Staff wanted was a massive air strike; the military regarded a limited strike aimed only at the missile sites as so dangerous and ineffective that "they would prefer taking no military action than to take that limited strike" (*Tapes*, 97).

Between October 18 and Saturday, October 20, the majority opinion within ExComm gradually coalesced around the idea of starting with a blockade, simultaneously issuing an ultimatum with a short deadline (periods from 48 to 72 hours were mentioned), and proceeding to military action if necessary after this deadline expired. International law required a declaration of war to set up a blockade, but this problem was ingeniously resolved by calling it a "naval quarantine" of Cuba (*Tapes*, 190–96).

Some people held the same positions throughout these ExComm discussions (October 16–21)—for example, the military Joint Chiefs of Staff constantly favored a major air strike—but others shifted their views, at times dramatically. National Security Adviser McGeorge Bundy initially favored doing nothing (*Tapes*, 172) and then switched toward a preemptive surprise air attack (*Tapes*, 189). President Kennedy's own position also shifted away from an air strike toward a blockade. He wanted the U.S. response to be firm. Although his reasons undoubtedly were mainly military and geopolitical, as a good politician he was also fully aware that a weak response would hurt the Democratic Party in the imminent congressional elections. On the other hand, the responsibility of starting an action that might lead to nuclear war weighed very heavily on him. He was impressed by the CIA's assessment that some of the missiles were already operational, which increased the risk that any air strike or invasion could lead the Soviets to fire those missiles and produce large U.S. civilian casualties (*Gamble*, 235). In the second week of the crisis (October 22 – 28), his decisions seemed constantly to favor the lowest-key options discussed by ExComm.

By the end of the first week’s discussions, the choice lay between a blockade and an air strike. In a straw vote on October 20, the blockade won 11 to 6 (*War*, 516). Kennedy made the decision to impose a blockade and announced it in a television address to the nation on Monday, October 22. He demanded a halt to the shipment of Soviet missiles to Cuba and a prompt withdrawal of those already there.

Kennedy’s speech brought all of the drama and tension of the crisis into the public arena. The United Nations held several dramatic but unproductive debates. Other world leaders and the usual policy wonks of international affairs offered advice and mediation.

Between October 23 and October 25, the Soviets at first tried bluster and denial; Khrushchev called the blockade “banditry, a folly of international imperialism,” and said that his ships would ignore it. The Soviets, in the United Nations and elsewhere, claimed that their intentions were purely defensive and issued statements of defiance. In secret, they explored ways to end the crisis. This exploration included some direct messages from Khrushchev to Kennedy. It also included some very indirect and lower-level approaches by the Soviets. In fact, as early as Monday, October 22—before Kennedy’s TV address—the Soviet Presidium had decided not to let this crisis lead to war. By Thursday, October 25, its members had decided that they were willing to withdraw from Cuba in exchange for a promise by the United States not to invade Cuba, but they had also agreed to “look around” for better deals (*Gamble*, 241, 259). The United States was not aware of any of this Soviet thinking.

In public as well as in private communications, the USSR suggested a swap: withdrawal of U.S. missiles from Turkey and of Soviet ones from Cuba. This possibility had already been discussed by ExComm. The missiles in Turkey were obsolete; the United States wanted to remove them anyway and replace them with a Polaris submarine stationed in the Mediterranean Sea. But it was thought that Turkey would regard the presence of U.S. missiles as a matter of prestige and that it might be difficult to persuade it to accept the change. (Turkey might also correctly regard missiles, fixed on Turkish soil, as a firmer signal of the U.S. commitment to its defense than an offshore submarine, which could move away on short notice; see *Tapes*, 568.)

The blockade went into effect on Wednesday, October 24. Despite their public bluster, the Soviets were cautious in testing it. Apparently, they were surprised that the United States had discovered the missiles in Cuba before the whole installation program was completed; Soviet personnel in Cuba had observed the U-2 overflights but had not reported them to Moscow (*Tapes*, 681). The Soviet Presidium ordered the ships carrying the most sensitive materials (actually, the IRBM missiles) to stop or turn around. But it also ordered General Issa Pliyev, the commander of the Soviet troops in Cuba, to get his troops combat-ready and to use all means except nuclear weapons to meet any attack (*Tapes*, 682). In fact, the Presidium twice prepared (then canceled without sending) orders authorizing him to use tactical nuclear weapons in the event of a U.S. invasion (*Gamble*, 242–43, 272, 276). The U.S. side saw only that several Soviet ships (which were actually carrying oil and other nonmilitary cargo) continued to sail toward the blockade zone. The U.S. Navy showed some moderation in its enforcement of the blockade; a tanker was allowed to pass without being boarded, and the tramp steamer *Marucla*, carrying industrial cargo, was boarded but allowed to proceed after only a cursory inspection. But tension was mounting, and neither side's actions were as cautious as the top-level politicians on both sides would have liked.

On the morning of Friday, October 26, Khrushchev sent Kennedy a conciliatory private letter offering to withdraw the missiles in exchange for a U.S. promise not to invade Cuba. But later that day he toughened his stance. It seems that he was emboldened by two items of evidence. First, he saw that the U.S. Navy was not being excessively aggressive in enforcing the blockade. Second, some dovish statements had appeared in U.S. newspapers. Most notable among them was an article by the influential and well-connected syndicated columnist Walter Lippman, who suggested the swap whereby the United States would withdraw its missiles in Turkey in exchange for the USSR's withdrawing its missiles in Cuba (*Gamble*, 275). Khrushchev sent another letter to Kennedy on Saturday, October 27, offering this swap, and this time he made the letter public. The new letter was presumably a part of the Presidium's strategy of "looking around" for the best deal. Members of ExComm concluded that the first letter expressed Khrushchev's own thoughts, but that the second was written under pressure from hard-liners in the Presidium—or was even evidence that Khrushchev was no longer in control (*Tapes*, 498, 512–13). In fact, both of Khrushchev's letters were discussed and approved by the Presidium (*Gamble*, 263, 275).

ExComm continued to meet, and opinions within it hardened. There was a growing feeling that the blockade by itself would not work. Kennedy's TV speech had imposed no firm deadline; in the absence of a deadline, a compellent threat is vulnerable to the opponent's procrastination.[5] Kennedy had seen this quite clearly, and as early as Monday, October 22, he commented, "I don't think we're gonna be better off if they're just sitting there" (*Tapes*, 216). But a hard, short deadline was presumably thought to be too rigid. By Thursday, others in ExComm were realizing the problem; Bundy, for example, said, “A plateau here is the most dangerous thing” (*Tapes*, 423). The hardening of the Soviet position, as shown by the public “Saturday letter” that followed the conciliatory private “Friday letter,” was another concern. More ominously, that Friday, U.S. surveillance had discovered the presence of tactical nuclear weapons (FROGs) in Cuba (*Tapes*, 475). This discovery showed the Soviet presence there to be vastly greater than thought before, but it also made an invasion more dangerous to U.S. troops. Also on Saturday, a U.S. U-2 plane was shot down over Cuba, and Cuban anti-aircraft defenses fired at lower-flying U.S. reconnaissance planes. The grim mood in ExComm throughout that Saturday was well encapsulated by Douglas Dillon: “We haven’t got but one more day” (*Tapes*, 534).

On that Saturday, U.S. plans leading to escalation were being put in place. An air strike was planned for the following Monday, or Tuesday at the latest, and Air Force reserves were called up (*Tapes*, 612–13). Invasion was seen as the inevitable culmination of events (*Tapes*, 537–38). A tough private letter to Khrushchev from President Kennedy was drafted and was handed over by Robert Kennedy to the Soviet ambassador in Washington, Anatoly Dobrynin. In it, Kennedy made the following offer: (1) The Soviet Union withdraws its missiles and IL-28 bombers from Cuba with adequate verification (and ships no new ones). (2) The United States promises not to invade Cuba. (3) The U.S. missiles in Turkey will be removed after a few months, but this offer is void if the Soviets mention it in public or link it to the Cuban deal. An answer was required within 12 to 24 hours; otherwise “there would be drastic consequences” (*Tapes*, 605–7).

On the morning of Sunday, October 28, just as prayers and sermons for peace were being offered in many churches in the United States, Soviet radio broadcast the text of a letter that Khrushchev was sending to Kennedy, in which he announced that construction of the Cuban missile sites was being halted immediately and that the missiles already installed would be dismantled and shipped back to the Soviet Union. Kennedy immediately sent a reply welcoming this decision, which was broadcast to Moscow by Voice of America radio. It now appears that Khrushchev's decision to back down was made before he received Kennedy's letter through Dobrynin, but that the letter only reinforced it (*Tapes*, 689).

That did not quite end the crisis. The U.S. Joint Chiefs of Staff remained skeptical of the Soviets and wanted to go ahead with their air strike (*Tapes*, 635). In fact, construction activity at the Cuban missile sites continued for a few days. Verification of the missiles' withdrawal by the United Nations proved problematic. The Soviets tried to make the Turkey part of the deal semipublic. They also tried to keep the IL-28 bombers in Cuba out of the withdrawal. Not until November 20 was the deal finally clinched and the withdrawal begun (*Tapes*, 663–65; *Gamble*, 298–310).

Endnotes

- Our sources (which we cite in the text using the word underlined here for each book, followed by the relevant page numbers) include Robert Smith Thompson, *The Missiles of October* (New York: Simon & Schuster, 1992); James G. Blight and David A. Welch, *On the Brink : Americans and Soviets Reexamine the Cuban Missile Crisis* (New York: Hill and Wang, 1989); Richard Reeves, *President Kennedy: Profile of Power* (New York: Simon & Schuster, 1993); Donald Kagan, *On the Origins of War and the Preservation of Peace* (New York: Doubleday, 1995); Aleksandr Fursenko and Timothy Naftali, *One Hell of a Gamble : The Secret History of the Cuban Missile Crisis* (New York: W. W. Norton, 1997); *The Kennedy Tapes : Inside the White House during the Cuban Missile Crisis*, ed. Ernest R. May and Philip D. Zelikow (Cambridge, Mass.: Harvard University Press, 1997); Michael Dobbs, *One Minute to Midnight : Kennedy, Khrushchev and Castro on the Brink of Nuclear War* (New York: Knopf, 2008); and Daniel Ellsberg, *The Doomsday Machine: Confessions of a Nuclear War Planner* (New York: Bloomsbury Publishing, 2017). Graham T. Allison's *Essence of Decision: Explaining the Cuban Missile Crisis* (Boston: Little Brown, 1971) remains important not only for its narrative, but also for its analysis and interpretation. Our view differs from his in some important respects, but we remain in debt to his insights. We follow and extend the ideas in Avinash Dixit and Barry Nalebuff, *Thinking Strategically* (New York: W. W. Norton, 1991), Chapter 8. Return to reference 2
- For those of you with no access to firsthand information, or those who seek a beginner's introduction to both the details and the drama of the missile crisis, we recommend the film *Thirteen Days* (2000, New Line Cinema). A relatively short book by Sheldon Stern uses the evidence

from the Kennedy administration tapes to present as accurate a view of the crisis and its later analysis as possible. His book is perhaps the best short read for interested parties. See Sheldon Stern, *The Cuban Missile Crisis in American Memory: Myths versus Reality* (Stanford, Calif.: Stanford University Press, 2012). Return to reference 3

- Members of ExComm who figured most prominently in the discussions were Secretary of Defense Robert McNamara; National Security Adviser McGeorge Bundy; the chairman of the Joint Chiefs of Staff, General Maxwell Taylor; Secretary of State Dean Rusk and Undersecretary George Ball; Attorney General Robert Kennedy (who was also the president’s brother); Secretary of the Treasury Douglas Dillon (also the only Republican in the cabinet); and Llewellyn Thompson, who had recently returned from being U.S. ambassador in Moscow. During the two weeks that followed, they would be joined by or would consult with several others, including the U.S. ambassador to the United Nations, Adlai Stevenson; Dean Acheson, former secretary of state and a senior statesman of U.S. foreign policy; and the chief of the U.S. Air Force, General Curtis LeMay. Return to reference 4
- See the discussion of salami tactics in Chapter 8, section 6.E. Return to reference 5

2 A SIMPLE GAME-THEORETIC EXPLANATION

At first sight, the game-theoretic aspect of the Cuban missile crisis looks very simple. The United States wanted the Soviet Union to withdraw its missiles from Cuba; thus the U.S. objective was to achieve compellence. For this purpose, the United States deployed a threat: Soviet failure to comply would lead to a nuclear war. This was sufficiently frightening to Khrushchev that he complied. The prospect of nuclear annihilation was equally frightening to Kennedy, but that is in the nature of a threat. All that is needed is that the threat be sufficiently costly to the other side to induce it to act in accordance with our wishes; then we don’t have to carry out the bad action anyway.

A somewhat more formal statement of this argument proceeds by drawing a game tree like that shown in Figure 13.1. The Soviets have installed the missiles, and now the United States has the first move. It chooses between doing nothing and issuing a threat. If the United States does nothing, this is a military and political achievement for the Soviets; so we score the payoffs as −1 for the United States and 1 for the Soviets. If the United States issues its threat, the Soviets get to move, and they can either withdraw or defy. Withdrawal is a humiliation for the Soviets and a reaffirmation of U.S. military superiority, so we score it 1 for the United States and −1 for the Soviets. If the Soviets defy the U.S. threat, there will be a nuclear war. This outcome is terrible for both, so we score this −10 for each. The numbers are (intentionally) chosen to be same as those in the game of chicken (see Figure 4.16) except that the disaster payoff is much worse. The conclusions do not depend on the precise numbers that we have chosen, however. If you disagree with our choice, you can substitute other numbers you think to be a more accurate representation; as long as the *relative* ranking of the outcomes is the same, you will get the same subgame-perfect equilibrium.

Now we can easily find that equilibrium. If faced with the U.S. threat, the Soviets get −1 from withdrawal and −10 by defiance; so they prefer to withdraw. Looking ahead to this outcome, the United States reckons on getting 1 if it issues the threat and −1 if it does not; therefore it is optimal for the United States to make the threat. The outcome gives payoffs of 1 to the United States and −1 to the Soviets.

But a moment's further thought shows this interpretation to be unsatisfactory. One might start by asking why the Soviets would deploy the missiles in Cuba at all, when they could look ahead to this unfolding of the subsequent game in which they would come out the losers. But even more importantly, several facts about the situation and several events in the course of its unfolding do not fit into this picture of a simple threat.

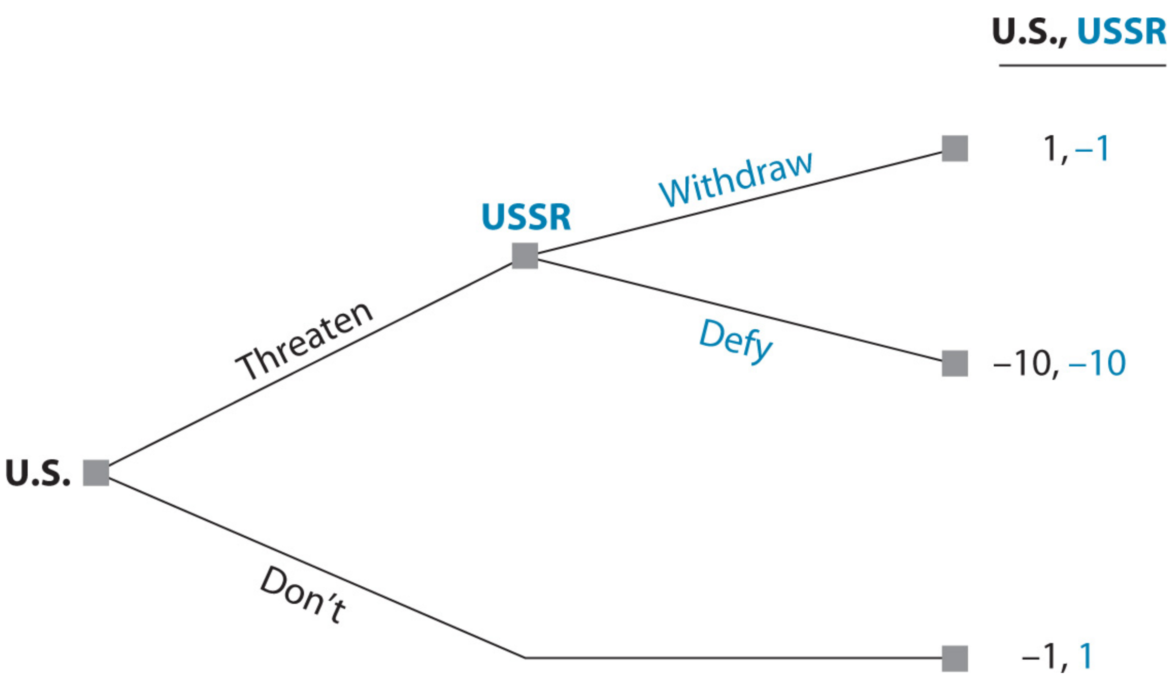

FIGURE 13.1 The Simple-Threat Model of the Crisis

So, let us develop a more satisfactory game-theoretic argument. As we pointed out before, the idea that a threat has only a lower limit on its size—namely, that it be large enough to frighten the opponent—is correct only if the threatener can be absolutely sure that everything will go as planned. But almost all games have some element of uncertainty. You cannot be certain about your opponent's value system, and you cannot be completely sure that any player's intended actions will be accurately implemented. Therefore, a threat carries a twofold risk. Your opponent may defy it, requiring you to carry out the costly threatened action; or your opponent may comply, but the threatened action may occur by mistake anyway. When such risks exist, the cost of a threatened action to oneself becomes an important consideration.

The Cuban missile crisis was replete with such uncertainties. Neither side could be sure of the other's payoffs—that is, of its relative valuations of winning or losing the Cold War nor the costs of a hot war. Also, the choices of "blockade" and "air strike" were much more complex than those simple phrases suggest, and there were many weak links and random effects between an order given in Washington or Moscow and its implementation in the Atlantic Ocean or in Cuba.

Graham Allison's excellent book *Essence of Decision* brings out all of these complexities and uncertainties. They led him to conclude that the Cuban missile crisis cannot be explained in game-theoretic terms. He considers two alternatives: one explanation based on the fact that bureaucracies have their set rules and procedures, and another based on the internal politics of U.S. and Soviet governance and military apparatuses. He concludes that the political explanation is best.

We broadly agree, but interpret the Cuban missile crisis differently. It is not the case that game theory is inadequate for understanding and explaining the crisis; rather, the crisis was *not a two-person game*—United States versus USSR, or Kennedy versus Khrushchev. Each of the two "sides" was itself a complex coalition of players with differing objectives, information, actions, and means of communication. The players within each side were engaged in other games, and some of those players were also directly interacting with their counterparts on the other side. In other words, the crisis can be seen as a complex multiplayer game with players aligned into two broad coalitions. Kennedy and Khrushchev can be regarded as the top-level players in this game, but each was constrained by others in his own coalition with divergent views and information, and neither had full control over the actions of those others. We argue that this more subtle game-theoretic perspective is not only a good way to look at the crisis, but also essential in understanding how to practice brinkmanship. We begin with some items of evidence that Allison emphasizes, as well as others that emerge from other writings.

First, there are several indications of divisions of opinion on each side. On the U.S. side, as already noted, there were wide differences within ExComm. In addition, Kennedy found it necessary to consult others, such as former president Eisenhower and leading members of Congress. Some of them had very different views; for example, Senator William Fulbright said in a private meeting that the blockade "seems to me the worst alternative" (*Tapes*, 271). The media and the political opposition would not give the president unquestioning support for too long either. Kennedy could not have continued on a moderate course if the opinion among his advisers and the public became decisively hawkish.

Individuals also *shifted* positions in the course of the two weeks. For example, McNamara was at first quite dovish, arguing that the missiles in Cuba were not a significant increase in the Soviet threat (*Tapes*, 89) and favoring a blockade and negotiations (*Tapes*, 191), but ended up more hawkish, claiming that Khrushchev's conciliatory letter of Friday, October 26, was "full of holes" (*Tapes*, 495, 585) and urging an invasion (*Tapes*, 537). Most importantly, the U.S. military chiefs always advocated a far more aggressive response. Even after the crisis was over and nearly everyone thought the United States had won a major round in the Cold War, Air Force General Curtis LeMay remained dissatisfied and wanted action: "We lost! We ought to just go in there today and knock 'em off," he said (*Essence*, 206; *Profile*, 425).

Even though Khrushchev was the dictator of the Soviet Union, he was not in full control of the situation. Differences of opinion on the Soviet side are less well documented, but for what it is worth, later memoirists have claimed that Khrushchev made the decision to install the missiles in Cuba almost unilaterally, and that when he informed the members of the Presidium, they thought it a reckless gamble (*Tapes*, 674; *Gamble*, 180). There were limits to how far he could count on the Presidium to rubber-stamp his decisions. Indeed, two years after the crisis, the disastrous Cuban adventure was one of the main charges leveled against Khrushchev when the Presidium dismissed him from office (*Gamble*, 353–55). It has also been claimed that Khrushchev wanted to defy the U.S. blockade, and that only the insistence of First Deputy Premier Anastas Mikoyan led to the Soviets' cautious response (*War*, 521).

Various parties on the U.S. side had very different information and a very different understanding of the situation, and at times these differences led to actions that were inconsistent with the intentions of the leadership, or even against their explicit orders. The concept of an "air strike" to destroy the missiles is a good example. The nonmilitary people in ExComm thought this would be a very narrowly targeted attack that would not cause significant Cuban or Soviet casualties, but the Air Force intended a much broader attack. Luckily, this difference came out in the open early, leading ExComm to decide against an air strike and the president to turn down an appeal by the Air Force (*Essence*, 123, 209). As for the blockade, the U.S. Navy had set procedures for such an action. The political leadership wanted a different and softer process: form the ring closer to Cuba to give the Soviets more time to reconsider, allow obviously nonmilitary cargo ships to pass unchallenged, cripple but do not sink the ships that defy challenge. Despite McNamara's explicit instructions, the Navy mostly followed its standard procedures (*Essence*, 130–32).

There was a similar lack of information and communication, as well as weakness in the chain of command and control, on the Soviet side. For example, the construction of the missile sites was done according to standard bureaucratic procedures. The Soviets, used to constructing ICBM sites in their own country, where they did not face significant risk of air attack, laid out the sites in Cuba, where they would have been much more vulnerable, in a similar way.

All these factors made the outcome of any decision by the top-level commander on each side somewhat *unpredictable.* This gave rise to a substantial risk of the "threat going wrong." And this risk was rising as the crisis continued. On the day the blockade went into effect, Kennedy thought that the chances of war were 20% (*Midnight*, 107); others attribute to him the higher estimate of "between one out of three and even" (*Essence*, 1).

Brinkmanship can use such uncertainty to strategic advantage to make one's threat probabilistic and credible. In effect, Kennedy was saying to Khrushchev, "Neither of us wants nuclear war. But I can't accept the missiles as a fait accompli. The quarantine I have set in motion creates a risk of war. You can end the game, and the risk, by withdrawing the missiles." To achieve the advantage, Kennedy's brinkmanship had to generate a risk high enough that Khrushchev would comply with his wishes, but low enough that he could tolerate it himself.

Neither the precise calculation of these limits nor the implementation of brinkmanship in practice is easy. We will develop two distinct models to bring out different aspects of the use of brinkmanship. In Section 3, we will consider how the United States could calculate the limits of players' tolerance for well-controlled risk. In Section 4, we will examine some of the more practical difficulties of controlling risk. Then we will build a model in which risk rises gradually over time, and each player has to decide how long it is willing to accept the rising risk of disaster rather than conceding the game.

3 BRINKMANSHIP WITH WELL-CONTROLLED RISK

In this section, we present a model that considers one particular form of the uncertainty inherent in the Cuban missile crisis—namely, the United States' lack of knowledge of the Soviets' true motives in installing their missiles on the island. We analyze the effect of this type of uncertainty formally, and we draw conclusions about when and how President Kennedy could have hoped to use brinkmanship successfully. Similar analyses and conclusions can be drawn for other forms of uncertainty involved in this case.

A. When Is a Simple Threat Too Large?

Reconsider the game tree shown in Figure 13.1. Suppose the Soviet payoffs from withdrawal and defiance are the opposite of what they were before: −10 for withdrawal and −1 for defiance. In this alternative scenario, the Soviets are hard-liners. They prefer nuclear annihilation to the prospect of a humiliating withdrawal and the prospect of living in a world dominated by the capitalist United States; their slogan is "Better dead than red-white-and-blue." We show the game tree for this scenario in Figure 13.2. Now, if the United States makes the threat, the Soviets defy it. So the United States stands to get −10 from the threat, but only −1 if it makes no threat and accepts the presence of the missiles in Cuba. It takes the lesser of the two evils. In the subgame-perfect equilibrium of this version of the game, the Soviets "win," and the U.S. threat does not work.

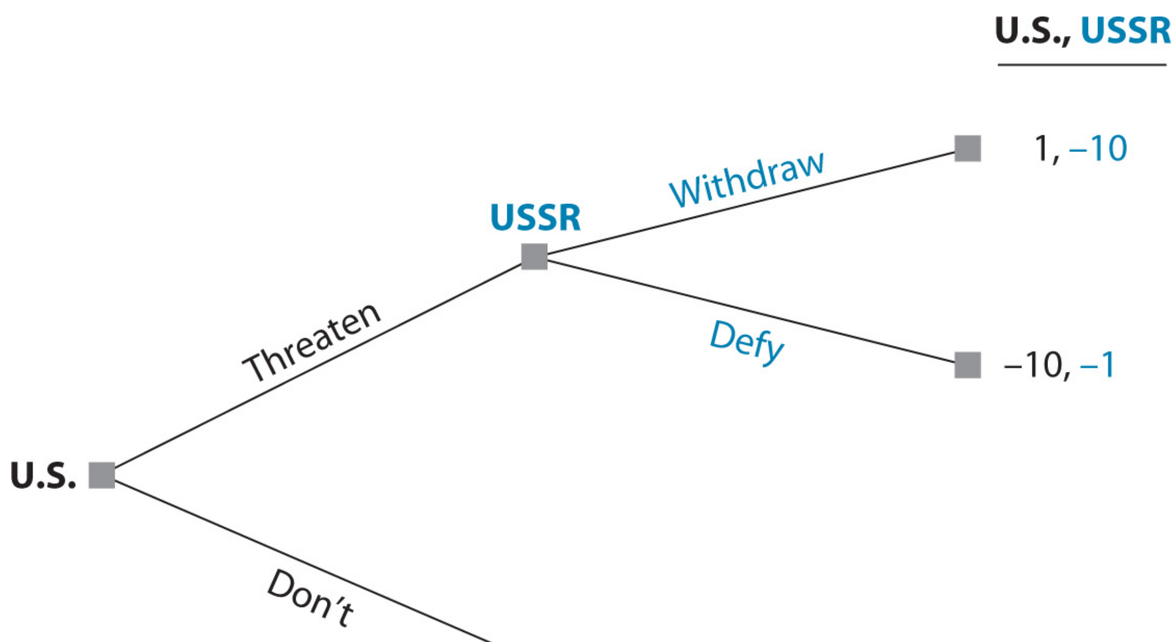

FIGURE 13.2 The Game with Hard-Line Soviets

In reality, when the United States makes its move, it does not know whether the Soviets' stance is hard-line, as in Figure

13.2, or softer, as in Figure 13.1. The United States can try to estimate the probabilities of the two scenarios—for example, by studying past Soviet actions and reactions in different situations. We can regard Kennedy's statement that the probability of the blockade leading to war was between one-third and one-half as his estimate of the probability that the Soviets were hard-line. Because the estimate is imprecise over a range, we work with a general symbol, *p*, for the probability, and we examine the consequences of different values of *p*.

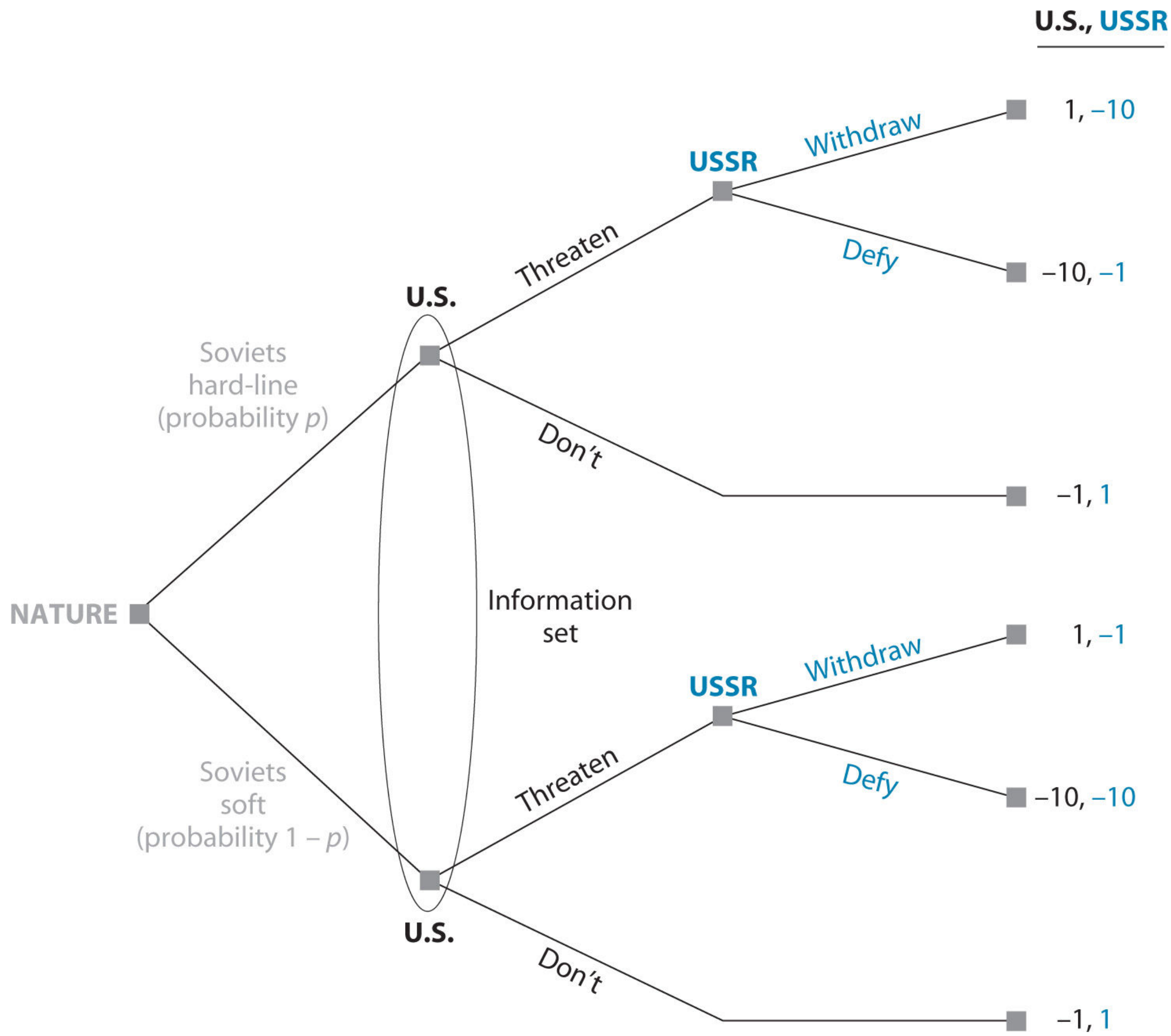

FIGURE 13.3 The Game with Unknown Soviet Payoffs

The tree for this more complex game is shown in Figure 13.3. The game starts with an outside force (here labeled Nature) determining the Soviets' type. Along the upper branch leading from Nature's choice, the Soviets are hard-line. This branch leads to the upper node where the United States makes its decision whether to issue its threat, and the rest of the tree is exactly like that for the game in Figure 13.2. Along the lower branch leading from Nature's choice, the Soviets are soft. This branch leads to the lower node where the United States makes its decision whether to issue its threat, and the rest of the tree is exactly like that for the game in Figure 13.1. But the United States does not know at which node it is making its choice. Therefore, the two U.S. nodes constitute an *information set*, as indicated by the oval that encloses them. Its significance is that the United States cannot take different actions at the two nodes within the set, such as issuing the threat only if the Soviets are soft. It must take the same action at both nodes, either threatening at both nodes or not threatening at both. It must make this decision in light of the probabilities that the game is located at one node or the other—that is, by calculating the *expected* payoffs of the two actions.

The Soviets themselves know what type they are. So we can do some rollback near the end of the game. Along the upper path of play, the hard-line Soviets will defy a U.S. threat, and along the lower path, the soft Soviets will withdraw in the face of the threat. Therefore, the United States can look ahead and calculate that a threat will yield a -10 if the game is actually moving along the upper path (a probability of p) and a 1 if it is moving along the lower path (a probability of $1 - p$). The expected U.S. payoff from making the threat is therefore $-10p + (1 - p) = 1 - 11p$.

If the United States does not make the threat, it gets a -1 along either path; so its expected payoff is -1. Comparing the expected payoffs of the two actions, we see that the United States should make the threat if $1 - 11p > -1$, or $11p < 2$, or $p < ^2/_{11} = 0.18$.

If the threat were sure to work, the United States would not care how bad its payoff might be if the Soviets defied it, whether -10 or even far more negative. But the risk that the Soviets might be hard-liners and might thus defy a threat makes the -10 relevant in the U.S. calculations. Only if the probability, p, of the

Soviets being hard-line is small enough will the United States find it acceptable to make the threat. Thus, the upper limit of $^2/_{11}$ on p is also the upper limit of this U.S. tolerance, given the specific numbers that we have chosen. If we choose different numbers, we will get a different upper limit; for example, if we rate a nuclear war as −100 for the United States, then the upper limit on p will be only $^2/_{101}$. But the idea of a large threat being "too large to make" if the probability of its going wrong is above a critical limit holds in general.

In this instance, Kennedy's estimate was that p lay somewhere in the range from ⅓ to ½. The lower end of this range, 0.33, is unfortunately above our upper limit 0.18 for the risk that the United States is willing to tolerate. The simple bald threat "If you defy us, there will be nuclear war" is too large, too risky, and too costly for the United States to make.

B. The Probabilistic Threat

Suppose Kennedy makes a different kind of threat, one that reduces the large, but simple, threat described above by creating merely a probability, rather than a certainty, that the Soviets will incur dire consequences if they do not comply. With a probabilistic threat of this type, Kennedy is effectively declaring, "If you defy us, there is a probability q (< 1) that nuclear war will result." It is important that the random mechanism that generates this outcome is out of Kennedy's control after the fact, but he can set the probability q in advance. This game is like Russian roulette (a perfect metaphor in this context?). In that potentially lethal game of chance, you load one of the six chambers in a handgun, creating a one-sixth probability that a bullet will actually be fired. But when you pull the trigger after having spun the cylinder containing the chambers, you have no control over whether the chamber that gets fired is actually loaded.

Brinkmanship, the making and deployment of a probabilistic threat, requires creating and controlling a suitable risk of this kind. It requires two apparently inconsistent things. On the one hand, you must let matters get enough out of your control that you will not have full freedom after the fact to refrain from taking the dire action, so that your threat will remain credible. On the other hand, you must retain sufficient control to keep the risk of the action from becoming too large and your threat too costly. Such "controlled lack of control" looks difficult to achieve, and it is. We will consider in Section 5 how to attempt the trick when you need to. Just one hint: All the complex differences of judgment, asymmetries of information, and difficulties of enforcing orders that made a simple threat too risky are exactly the forces that make it possible to create a risk of war and therefore make brinkmanship credible. The real difficulty is not how to lose control, but how to do so in a controlled way.

In the case of the U.S.–Soviet tensions over the Cuban missile sites, we need to ask what levels of q (the probability of nuclear war in Kennedy's brinkmanship threat) will be both *effective* in compelling Khrushchev to withdraw and tolerable (*acceptable*) to Kennedy given his estimated range of p (the probability that he faces hard-line Soviet opponents). To answer this question, we slightly alter the game of Figure 13.3 to get Figure 13.4. Here, if the Soviets defy the United States, war will occur with probability q. With the remaining probability, $(1 - q)$, the United States will give up and accept the presence of Soviet missiles in Cuba. Remember that if the game gets to the point where the Soviets defy the United States, the latter does not have a choice in the matter. The Russian-roulette revolver has been set for the probability q, and chance determines whether the firing pin hits a loaded chamber (that is, whether nuclear war actually happens).

Thus, nobody knows the precise outcome or the payoffs that will result if the Soviets defy this brinkmanship threat, but they know the probability, q, and can calculate expected values. For the United States, the outcome is -10 with the probability q and -1 with the probability $(1 - q)$, so the expected value is $-10q - (1 - q) = -1 - 9q$. For the Soviets, the expected payoff depends on whether they are hard-line or soft (and only they know their own type). If hard-line, they get -1 from war, which happens with probability q, and 1 if the United States gives up, which happens with probability $(1 - q)$. The hard-line Soviets' expected payoff is $-q + (1 - q) = 1 - 2q$. If they were to withdraw, they would get a -10, which is clearly worse no matter what value q takes. Thus, the hard-line Soviets will defy the brinkmanship threat.

The calculation is different if the Soviets are soft. Reasoning as before, we see that they get the expected payoff $-10q + (1 - q) = 1 - 11q$ from defiance and the sure payoff -1 if they withdraw. For them, withdrawal is better if $-1 > 1 - 11q$, or $11q > 2$, or $q > 0.18$. Thus, U.S. brinkmanship must contain at least an 18% probability of war; otherwise, it will not deter the Soviets, even if they are the soft type. We call this lower limit on the probability q the effectiveness condition.

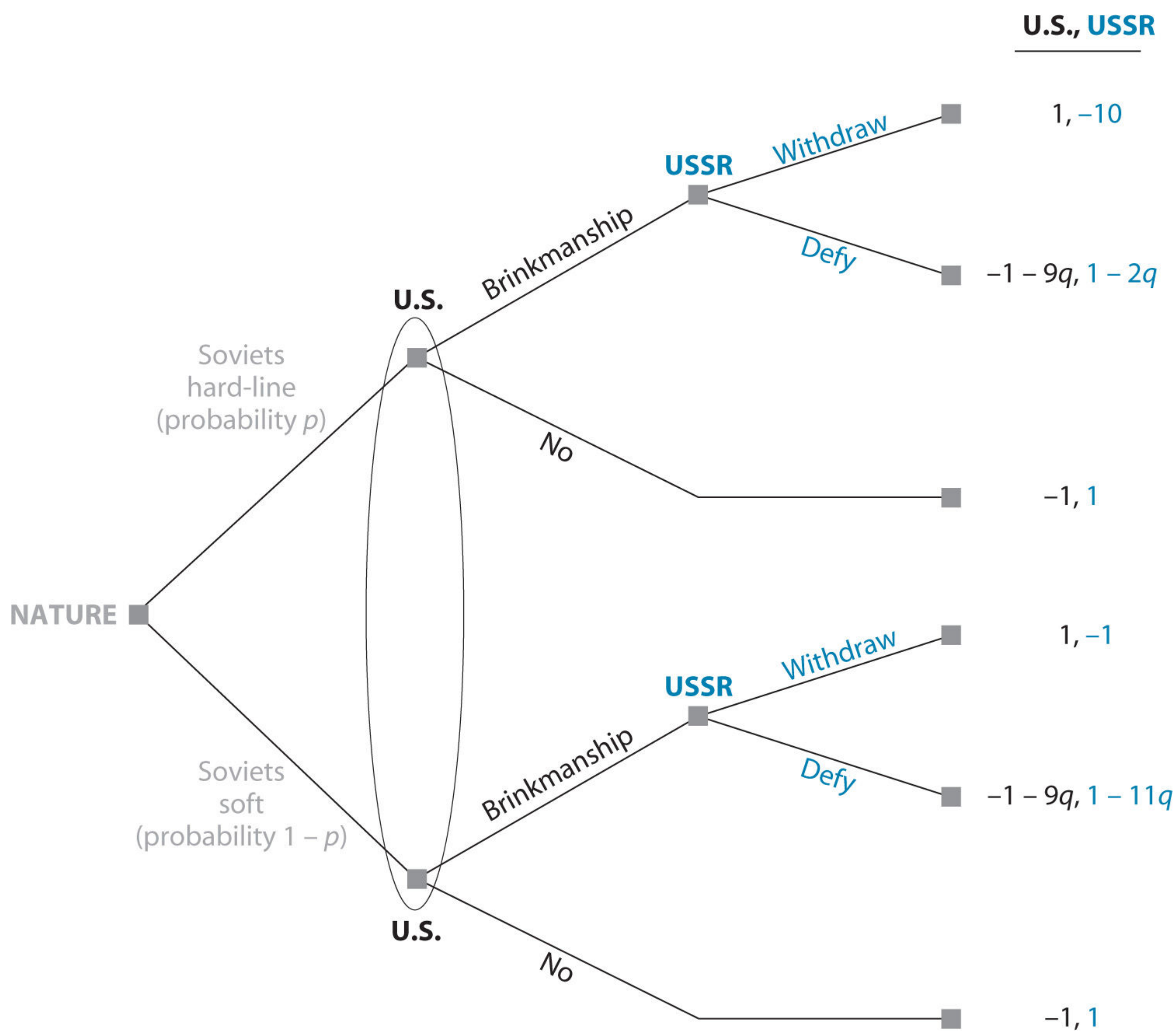

FIGURE 13.4 The Brinkmanship Model of the Crisis

Observe how the expected payoffs for U.S. brinkmanship and Soviet defiance shown in Figure 13.4 relate to the simple-threat model of Figure 13.3; the latter can now be thought of as a special case of the general brinkmanship-threat model of Figure 13.4, corresponding to the extreme value $q = 1$.

We can solve the game shown in Figure 13.4 in the usual way. We have already seen that along the upper path, the Soviets, being hard-line, will defy the United States, and that along the lower path, the soft Soviets will comply with U.S. demands if the effectiveness condition is satisfied. If the effectiveness condition is not satisfied, then both types of Soviets will defy the United States—in which case the United States would do better never to make this threat at all. So let us proceed by assuming that the soft Soviets will comply; we now look at the U.S. choices. Basically, how risky can the threat be and still remain tolerable to the United States?

If the United States makes the threat, it runs the risk p that it will encounter the hard-line Soviets, who will defy the threat. Then the expected U.S. payoff will be $-1 - 9q$, as calculated before. The probability is $(1 - p)$ that the United States will encounter the soft Soviets. We are assuming that they comply; then the United States gets a 1. Therefore, the expected payoff to the United States from the probabilistic threat, assuming that it is effective against the soft Soviets, is

$(-1 - 9q) \times p + 1 \times (1 - p) = -9pq - 2p + 1.$

If the United States refrains from making a threat, it gets a -1. Therefore, the condition for the United States to make the threat is

$$-9pq - 2p + 1 > -1, \text{ or}$$

$$q < \frac{2}{9}\,\frac{1-p}{p} = \frac{0.22(1-p)}{p}.$$

That is, the probability of war must be small enough to satisfy this expression, or the United States will not make the threat at all. We call this upper limit on q the acceptability condition. Note that p enters the formula for the maximum value of q that will be acceptable to the United States; the larger the chance that the Soviets will not give in, the smaller the risk of mutual disaster that the United States finds acceptable.

For the probabilistic threat to benefit the United States, it should satisfy both the effectiveness condition and the acceptability condition. We can determine the resulting probability of war by using Figure 13.5. The horizontal axis is the probability, p, that the Soviets are hard-line, and the vertical axis is the probability, q, that war will occur if they defy the U.S. threat. The horizontal line $q = 0.18$ gives the effectiveness condition. The probabilistic threat will be made only if its associated (p, q) combination is above this line, since otherwise, it will not deter even the soft-type Soviets. The curve $q = 0.22(1 - p)/p$ gives the acceptability condition. The probabilistic threat will be made only if (p, q) is below this curve, since otherwise, the resulting risk of war will be intolerable to the United States, even if soft-type Soviets are successfully deterred. Therefore, an effective and acceptable threat should fall somewhere between these two lines, above and to the left of their point of intersection at $p = 0.55$ and $q = 0.18$ (shown as a gray wedge in Figure 13.5).

The curve reaches $q = 1$ when $p = 0.18$. For values of p less than 0.18, the dire threat (certainty of war) is acceptable to the United States and is effective against the soft-type Soviets. This simply confirms our analysis in Section 3.A.

For values of p in the range from 0.18 to 0.55, the dire threat with $q = 1$ puts (p, q) above the acceptability condition and is too large to be tolerable to the United States. But a scaled-down threat can be found. For this range of values of p, some values of q are low enough to be acceptable to the United States and yet high enough to compel the soft-type Soviets. Brinkmanship (using a probabilistic threat) can do the job in this situation, whereas a simple dire threat would be too risky.

If p exceeds 0.55, then no value of q satisfies both conditions. If the probability that the Soviets will never give in is greater than 0.55, then any threat large enough to work against the soft-type Soviets ($q \geqslant 0.18$) creates a risk of war too large to be acceptable to the United States. If $p \geqslant 0.55$, therefore, the United States cannot use the brinkmanship strategy to its advantage.

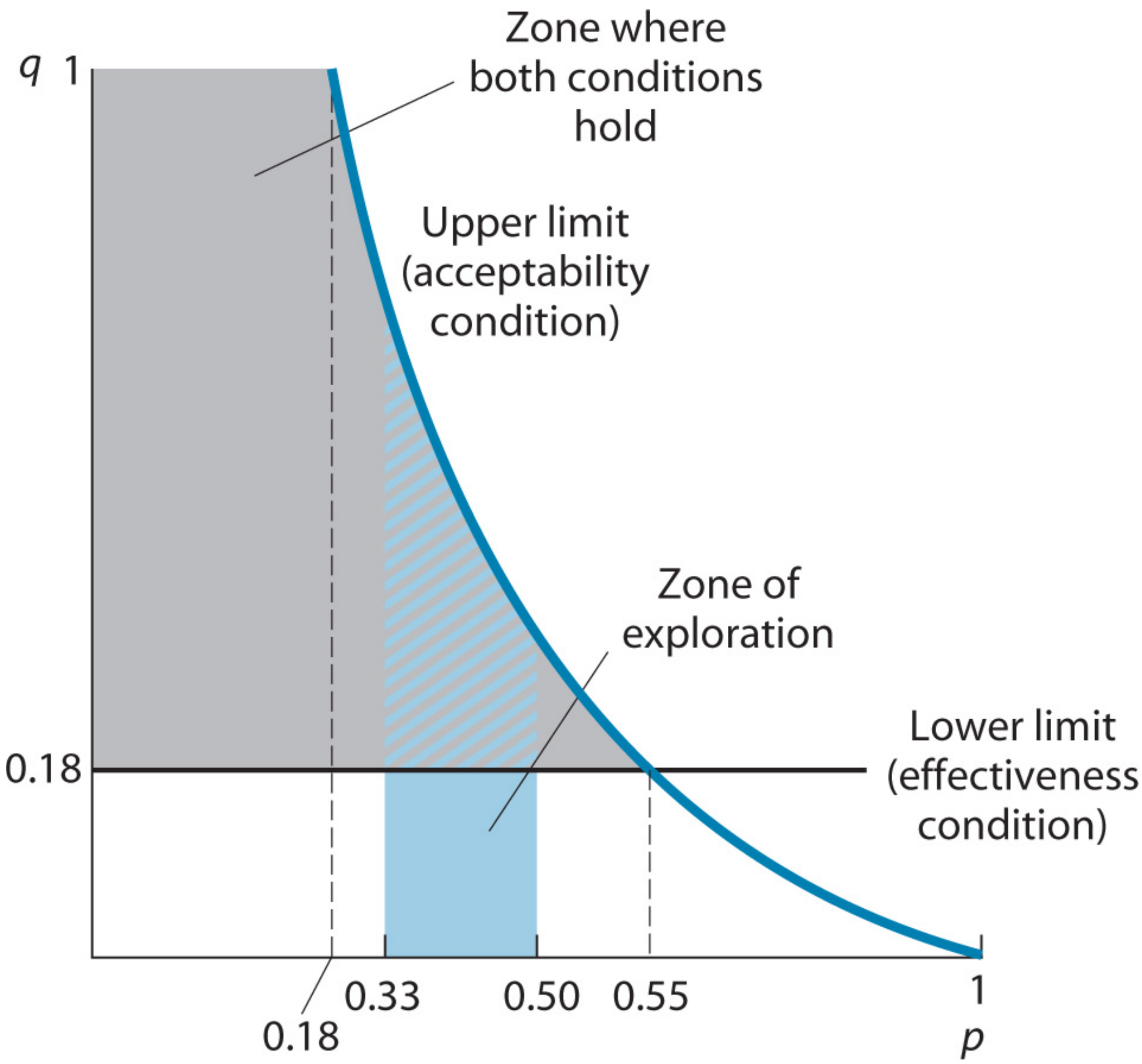

FIGURE 13.5 Conditions of Successful Brinkmanship

If we regard Kennedy's estimate of the probability of war (originally 20% and later increased to "between one-third and one-half") as his estimate that the Soviets were hard-line, it falls squarely within the range of *p* where brinkmanship can be successfully practiced (*p* between 0.18 and 0.55). So, should we regard the episode as a result of Kennedy's brilliant assessment of the situation and exercise of brinkmanship?[6] His calculation would have required assumptions about the payoffs for the two types of Soviets, as well as clarity about his own payoffs. In Exercise U1 we will ask you to calculate the consequences of one change in these assumptions. Now we turn to the practical difficulties of controlling risk.

Endnotes

- Or as vindication of our brilliant choice of numbers? ☺
  Return to reference 6

Glossary

probabilistic threat
: A strategic move in the nature of a threat, but with the added qualification that if the event triggering the threat (the opponent's action in the case of deterrence or inaction in the case of compellence) comes about, a chance mechanism is set in motion, and if its outcome so dictates, the threatened action is carried out. The nature of this mechanism and the probability with which it will call for the threatened action must both constitute prior commitments.

effectiveness condition
: A lower bound on the probability of fulfillment in a brinkmanship threat, expressed as a function of the probability of error, showing the lower limit of risk that will induce the threatened player to comply with the wishes of the threatener.

acceptability condition
: An upper bound on the probability of fulfillment in a brinkmanship threat, expressed as a function of the probability of error, showing the upper limit of risk that the player making the threat is willing to tolerate.

4 BRINKMANSHIP WITH UNCONTROLLED RISK : A GAME OF DYNAMIC CHICKEN

Our account of brinkmanship, while a substantial improvement on the pure threat model of Section 2, remains inadequate. First, it still does not explain why the Soviets installed the missiles in Cuba in the first place. Why didn't they look ahead to the subsequent game of brinkmanship outlined above and conclude that their best strategy was not to play at all? Presumably the Soviets thought it sufficiently likely that the United States would prove to be soft itself and accept the missiles in Cuba, just as the Soviets had learned to live with U.S. missiles in Turkey. In other words, the Soviets themselves were unsure whether the United States was a soft or hard-line type; there was *two-sided* uncertainty about payoffs.

Even more importantly, our discussion in Section 3 assumed the ability to fix q—that is, to achieve a *controlled* loss of control. However, particularly in the last couple of days of the crisis, events were rapidly spinning out of the control of the principals, Kennedy and Khrushchev, into a realm of uncontrolled loss of control!

Daniel Ellsberg, who later became famous for leaking the Pentagon Papers, was a young researcher for the Department of Defense during the crisis. At that time, he and his immediate superiors estimated the probability of war to be very low, between one in a thousand and one in a hundred (*Doomsday*, 189, 199). The United States had overwhelming superiority in nuclear weapons and missiles, so they thought that, given any degree of rationality, Khrushchev simply had to back down, even without a significant concession from the United States. Ellsberg was astounded to hear the higher numbers offered by more senior people in ExComm, but later realized that they were correct. He and other relatively junior participants had not known the extent to which the top people were losing control of the situation. In fact, those leaders themselves came to realize the extent of their loss of control very late in the game (*Doomsday*, 201–22).

At numerous points—for example, when the U.S. Navy was trying to stop and board the freighter *Marucla*—the people involved might have set off an incident with alarming consequences by taking some action in fear of the immediate situation. Most dramatically, a Soviet submarine crew, warned to surface when approaching the quarantine line on October 27, considered firing a nuclear-tipped torpedo that it carried onboard (unknown to the U.S. Navy). The firing-authorization rule required the approval of three officers, only two of whom agreed; the third officer by himself may have prevented all-out nuclear war.[7]

The U.S. Air Force created even greater dangers. A U-2 plane that drifted "accidentally" into Soviet air space almost caused a serious setback. General Curtis LeMay, acting without the president's knowledge or authorization, ordered the Strategic Air Command's nuclear bombers to fly past their "turnaround" points and some distance toward Soviet air space to positions where they would be detected by Soviet radar. Fortunately, the Soviets responded calmly; Khrushchev merely protested to Kennedy.[8]

On Saturday, October 27, Castro ordered his anti-aircraft forces to fire on all U.S. planes overflying Cuba and refused the Soviet ambassador's request to rescind the order (*War*, 544). On the same day, an overflying U.S. U-2 plane was shot down by a Soviet surface-to-air missile; a lower-level local commander had interpreted his orders more broadly than Moscow had intended [*War*, 537; *Tapes*, 682].

With each day that the crisis continued, the probability of nuclear war was increasing, and the top leaders were losing control over it. Combine this rising risk of disaster with each side's uncertainty about the other's payoffs, and we have a game of dynamic chicken, which we introduced in Chapter 9. Instead of two teenagers driving their cars toward each other as the risk of a collision increases, testing each other's bravery (or foolhardiness) in deciding when to swerve, we have the two leaders of the two major powers of the day, testing each other's resolve as the risk of Armageddon increases. Unlike the model in Section 13.3, where Kennedy controlled the level of risk, this model shows the risk increasing steadily through time, and Kennedy and Khrushchev must decide each day whether to continue the confrontation at the new, higher level of risk or to concede the game (Kennedy by accepting the missiles or Khrushchev by withdrawing them).

In Chapter 9, we solved a two-step version of such a game. Here, we apply the method developed in Chapter 9, but we do so for a game with 13 steps, one for each day of the crisis.[9] In our original dynamic chicken example, the value each player placed on winning, W, was that player's private information; in keeping with the context here, we cast each side's perceived cost of catastrophe, C, in that role. We still have two-sided asymmetric information, as in Chapter 9, but the value unknown to the other player has changed. We fix the value of winning at $W = 1$ and the cost of losing at $L = 1$, and take each player's cost of catastrophe C to be uniformly distributed over the interval from 0 to 10.[10] Because you would surely defy a threat made by the other player if your C was below L, our choice of values makes the implied probability that each side is hard-line equal to one-tenth.[11] A player at the soft end (with a high cost of catastrophe, C, close to 10) would concede quickly. Each day, as the probability of disaster rises, only successively tougher players remain; the threshold value of C, beyond which a player concedes, falls.

Figure 13.6 shows the results of our calculations. For easy reference, we include a brief summary of the major events of each day of the crisis in the second column.[12] The third column shows the values we have chosen (our best "guesstimates") for the rising probabilities of disaster. These probabilities start very low, rise slowly over the middle days to conform to Kennedy's estimates cited previously, and then rise sharply in the last two days of the crisis because of the loss of control detailed previously. On the last day of the crisis (Sunday, October 28), the risk rises to 1.000 because if Khrushchev had not conceded then, the U.S. invasion plans would have gone into effect.

The last four columns in Figure 13.6 show the results of the calculations for the dynamic chicken model. In the fourth column, we report the threshold levels of C; a player with a C value above the threshold listed for a particular step of the game would concede at that step. The last three columns give probability calculations: the probability that each player concedes at a particular step, the probability of disaster at each step, and the cumulative probability of disaster at each step. Observe that on the first day of the crisis (October 16), the threshold value of C reported in the fourth column is 9.852, close to the maximum of the range of possible values of C (which is 10). Only the most peace-loving, dovish players (with C values in the top 1.5% of the full range) would concede at that stage.

Our results for the probabilities of concession clarify two features of the crisis at once. First, we see why Khrushchev may have embarked on this adventure at all. He clearly put significant value on the prestige that a victory in this confrontation would yield to the USSR (and to him as its leader) in other communist countries, in the third world, and in many countries of the Western bloc as well. A victory would also strengthen his position in other confrontations with the United States, especially over Berlin. With a C not too close to 10, it would be rational for Khrushchev to launch the game, and not to concede for quite a while. Combine this with the high likelihood that he expected Kennedy to be weak (to have a high C, in the language of our model). Kennedy's performances during their meeting in Vienna in 1961, and during the attempted invasion of Cuba by exiles at the Bay of Pigs the same year, were widely perceived to be weak; this may have led Khrushchev to believe that Kennedy would concede quickly, accepting the missiles in Cuba as a fait accompli. Second, we can infer that both principals, Kennedy and Khrushchev, must have been very hard-line to have lasted as long as they did without conceding. Even if we accept October 22 as the date when the Soviet Presidium decided to concede, that date implies that the Soviets had a *C* value of 4.329, significantly above *L* (=1) and well below the maximum possible in our model. However, it is also possible that both sides underestimated the risk of war for quite a while, and that only the near-confrontations that occurred in the last two days of the crisis brought home to them the extent to which they had lost control of the situation.

Figure 13.6 also shows the probability of concession decreasing in the final few days, dropping to 0.16 on the final Sunday; this probability is quite small because, by then, only the toughest players are left. Our result for the conditional probability of disaster on Saturday (October 27, day 12) is high enough at 46.4% to make McNamara's fear as he left the White House that beautiful fall evening that this "might be the last sunset I saw" (*Doomsday*, 201) quite understandable. And the predicted cumulative probability of disaster over the whole duration of the crisis is high enough (at 58.3%) that we should indeed be thankful that it did not end in a nuclear disaster.

Date (October 1962)	Major events	Probability of disaster at this step if neither concedes	Concession threshold	Probability that each player concedes at this step	Probability of disaster at this step (conditional on it being reached)	Cumulati probabil of disas up to an includin this ste
15	U-2 photos show MRBM missile sites					
16	JFK briefed, convenes ExComm	0.010	9.852	0.015	0.010	0.010
17	New overflight, evidence of IRBMs	0.025	9.120	0.074	0.021	0.030
18	ExComm continues meetings. Differences of opinion; changes of mind; options debated.	0.040	7.971	0.126	0.031	0.055
19		0.050	6.853	0.140	0.037	0.080
20		0.060	5.919	0.136	0.045	0.109
21	ExComm majority for blockage/quarantine	0.075	5.054	0.146	0.055	0.144

You may need to scroll left and right to see the full figure.

Date (October 1962)	Major events	Probability of disaster at this step if neither concedes	Concession threshold	Probability that each player concedes at this step	Probability of disaster at this step (conditional on it being reached)	Cumulati probabil of disas up to an includin this ste
22	JFK addresses nation. Soviets deny, bluster. Soviet Presidium meets in secret and decides USSR will eventually withdraw but will first explore best possible deals in exchange.	0.100	4.329	0.143	0.073	0.186
23	UN Security Council meeting	0.150	3.533	0.184	0.100	0.240
24	Blockade goes into effect. Some Soviet ships approaching quarantine line stop or reverse, others continue.	0.200	2.889	0.182	0.134	0.297
25	U-2 overflights increase. *Marucla* pursued.	0.300	2.330	0.194	0.195	0.371

You may need to scroll left and right to see the full figure.

Date (October 1962)	Major events	Probability of disaster at this step if neither concedes	Concession threshold	Probability that each player concedes at this step	Probability of disaster at this step (conditional on it being reached)	Cumulati probabil of disas up to an includin this ste
26	NSK's conciliatory letter (asking "no invasion" promise). *Marucla* boarded. Castro authorizes anti-aircraft fire on low-flying U.S. reconnaissance flights. Soviet junior officer in Cuba nearly fires missile at overflying U-2 while boss is away from desk. Attitudes harden in ExComm.	0.500	1.801	0.227	0.299	0.456
27	NSK's hard-line letter (Cuba/Turkey swap); JFK independently offers this as secret deal in letter RFK hands over to Dobrynin. Overflying U-2 shot down. Soviet sub crew considers firing nuclear torpedo but fails in getting required unanimity of three officers. ExComm feeling: "We haven't got but one more day." U.S. air strike planned for Monday, October 29.	0.750	1.416	0.214	0.464	0.531

You may need to scroll left and right to see the full figure.

Date (October 1962)	Major events	Probability of disaster at this step if neither concedes	Concession threshold	Probability that each player concedes at this step	Probability of disaster at this step (conditional on it being reached)	Cumulati probabil of disas up to an includin this ste
28	NSK speech withdrawing missiles	1.000	1.190	0.160	0.706	0.583
You may need to scroll left and right to see the full figure.						

FIGURE 13.6 A Dynamic Chicken Model of the Crisis

Endnotes

- This story became public in a conference held in Havana, Cuba, in October 2002, to mark the 40th anniversary of the missile crisis. See Kevin Sullivan, "40 Years After Missile Crisis, Players Swap Stories in Cuba," *Washington Post*, October 13, 2002, p. A28. Vadim Orlov, who was a member of the Soviet submarine crew, identified the officer who refused to fire the torpedo as Vasili Arkhipov, who died in 1999. See also *Doomsday*, pp. 216–17. Return to reference 7
- Richard Rhodes, *Dark Sun: The Making of the Hydrogen Bomb* (New York: Simon & Schuster, 1995), pp. 573–75. LeMay, renowned for his extreme views and his constant chewing of large unlit cigars, is supposed to be the original inspiration for General Jack D. Ripper, in the 1963 movie *Dr. Strangelove*, who orders his bomber wing to launch an unprovoked attack on the Soviet Union. Return to reference 8
- The mathematical details of the model are in our working paper, "We Haven't Got But One More Day: The Cuban Missile Crisis as a Dynamic Chicken Game," (June 2019). Available at SSRN: https://ssrn.com/abstract=3406265. Return to reference 9
- In the formal algebra of the model, we should say the payoffs are $L=-1$ and C between 0 and -10, but the positive values are easier to write in a verbal account. Return to reference 10
- Note that the probabilities in this dynamic chicken interpretation of brinkmanship are not directly comparable to those we calculated in the one-sided version in Section 3.B. Return to reference 11
- JFK is, of course, John F. Kennedy, RFK is his brother Robert, and NSK is Nikita Sergeyevich Khrushchev. Return to reference 12

5 PRACTICING BRINKMANSHIP

The very features of the Cuban missile crisis that make it inaccurate to regard it as a two-person game made it easier for the players to practice brinkmanship. The blockade of Cuba was a relatively small action, unlikely to start a nuclear war at once. But once Kennedy had set the blockade in motion, its operation, escalation, and other features were not totally under his control. So Kennedy was not saying to Khrushchev, “If you defy me (cross a sharp brink), I will coolly and deliberately launch a nuclear war that will destroy both our peoples.” Rather, he was implicitly saying, “The wheels of the blockade have started to turn and are gathering their own momentum. The longer you defy me, the more likely it is that some operating procedure will slip up, the domestic political pressure on me will rise to a point where I must give in to the hawks, or some military guy will run amok. If any of these things come to pass, I may be unable to prevent nuclear war, no matter how much I may regret it at that point. Only you can now defuse the tension by complying with my demand to withdraw the missiles.” And Khrushchev was making similar implicit statements to Kennedy up until the moment when he decided to concede.

We believe that this perspective gives a much better and deeper understanding of the crisis than can most analyses based on simple threats. It tells us why the *risk* of war played such an important role in all discussions. It even makes Allison’s compelling arguments about bureaucratic procedures and internal divisions on both sides an integral part of the picture: These features allowed the top-level players on both sides to lose some control credibly—that is, to practice brinkmanship.

One important condition remains to be discussed. In Chapter 8, we saw that every threat has an associated implicit affirmation—namely, that the bad consequence will not take place if your opponent complies with your wishes. The same is required for brinkmanship. If, as you are increasing the level of risk, your opponent does comply, you must be able to "go into reverse"—begin reducing the risk immediately and quickly remove it from the picture. Otherwise, the opponent would not gain anything by compliance. This might have been a problem in the Cuban missile crisis. If the Soviets had feared that Kennedy could not control hawks such as LeMay ("We ought to just go in there today and knock 'em off"), they would have gained nothing by giving in.

To review and sum up, brinkmanship is the strategy of exposing your rival and yourself to a gradually increasing risk of mutual harm. The actual occurrence of the harmful outcome is not totally within the threatener's control. But the loss of control itself needs to be controlled; the probability of disaster must be kept within certain bounds to ensure that the risk is acceptable to the threatener. This "controlled loss of control" is difficult to achieve, and in the last few days of the Cuban missile crisis, the situation was close to becoming totally uncontrolled by the two principals.

Because the level of risk in a game of brinkmanship is difficult to control, it makes sense to start with a low level of risk and let it rise gradually in its own way. This approach allows you to find out whether the opponent's tolerance for the rising risk runs out before your own does. In other words, this approach allows you to stand eyeball to eyeball with your opponent and to see who blinks first.[13]

A game of brinkmanship can end in one of three ways: with success for one side and loss for the other, or in mutual disaster. Fortunately, the Cuban missile crisis did not end disastrously; if it had, none of us would be here to analyze it as a case study.

Viewed as a strategy that entails increasing the risk of disaster for both sides of an interaction, brinkmanship is everywhere. In most confrontations—for example, between a company and a labor union, a husband and a wife, a parent and a child, or the president and Congress—one player cannot be sure of the other players' objectives and capabilities. Therefore, most threats carry a risk of error, and every threat must contain an element of brinkmanship. We hope that we have given you some understanding of this strategy and that we have impressed on you the risks that it carries. Unsuccessful brinkmanship can lead to a labor strike, the dissolution of a marriage, a shutdown of the U.S. government, or some other disaster that, while small compared with nuclear annihilation, looms large in the context of the game you are playing. You will have to face brinkmanship, or conduct it yourself, on many occasions in your personal and professional lives. Please do so carefully, with a clear understanding of its potentialities and risks.

To help you do so, we now recapitulate the important lessons learned from the handling of the Cuban missile crisis, reinterpreted here in the context of a labor union leader contemplating a strike in pursuit of the union's demand for higher wages, unsure whether this action will result in the whole firm's shutting down:

1. Start small and safe. Your first step should not be an immediate walkout; it should be to schedule a membership meeting at a date a few days or weeks hence, while negotiations continue.
2. Let the risks increase gradually. Your public and private statements, as well as the stirring up of the sentiments of the membership, should induce management to believe that acceptance of its current low-wage offer is becoming less and less likely. If possible, stage small incidents—for example, a few one-day strikes or local walkouts.

3. As this process continues, read and interpret signals in management's actions to figure out whether the firm has enough profit potential to afford the union's high-wage demand.
4. Try to retain enough control over the situation; that is, retain the power to induce your membership to ratify the agreement that you will reach with management; otherwise management will think that the risk of a strike will not decrease even if it concedes to your demands.
5. Remain alert for signs that the situation is getting out of your control, and be ready to reassert control and de-escalate, either when the opponent concedes or by your own concession.

Endnotes

- This image was invoked by Secretary of State Dean Rusk when Soviet ships bound for Cuba appeared to stop or reverse on October 24 (*Midnight*, 88). Return to reference 13

SUMMARY

In some game situations, the risk of error in the presence of a threat may call for the use of as small a threat as possible. When a large threat cannot be reduced in other ways, it can be scaled down by making its fulfillment probabilistic. Strategic use of a *probabilistic threat*, in which you expose your rival and yourself to an increasing risk of harm, is called *brinkmanship*.

Brinkmanship requires a player to relinquish some control over the outcome of the game without completely losing control. You must create a threat with a risk level that is both large enough to be *effective* in compelling or deterring your rival and small enough to be *acceptable* to you. To do so, you must test the limit of your opponent's risk tolerance, while going up to your own limit if necessary, through a *gradual escalation of the risk of mutual harm.*

The Cuban missile crisis of 1962 serves as a case study in the use of brinkmanship. Analyzing the crisis as an example of a simple threat, with the U.S. blockade of Cuba establishing credibility for its threat, is inadequate. A better analysis accounts for the many complexities and uncertainties inherent in the situation and the likelihood that a simple threat was too risky. Because the actual crisis included numerous political and military players, Kennedy could attempt "controlled loss of control" by ordering the blockade and gradually letting incidents and tension escalate, until Khrushchev yielded in the face of the rising risk of nuclear war. A model of dynamic chicken captures such uncertainty and gives better explanations, and even plausible numerical magnitudes, for the unfolding of the crisis.

KEY TERMS

acceptability condition (534)

brinkmanship (518)

effectiveness condition (533)

probabilistic threat (531)

Glossary

brinkmanship
: A threat that creates a risk but not certainty of a mutually bad outcome if the other player defies your specified wish as to how he should act, and then gradually increases this risk until one player gives in or the bad outcome happens.

probabilistic threat
: A strategic move in the nature of a threat, but with the added qualification that if the event triggering the threat (the opponent’s action in the case of deterrence or inaction in the case of compellence) comes about, a chance mechanism is set in motion, and if its outcome so dictates, the threatened action is carried out. The nature of this mechanism and the probability with which it will call for the threatened action must both constitute prior commitments.

effectiveness condition
: A lower bound on the probability of fulfillment in a brinkmanship threat, expressed as a function of the probability of error, showing the lower limit of risk that will induce the threatened player to comply with the wishes of the threatener.

acceptability condition
: An upper bound on the probability of fulfillment in a brinkmanship threat, expressed as a function of the probability of error, showing the upper limit of risk that the player making the threat is willing to tolerate.

SOLVED EXERCISES

1. Consider a game between a union and the company that employs the union membership. The union can threaten to strike (or not) to get the company to meet its wage and benefit demands. When faced with a threatened strike, the company can choose to concede to the demands of the union or to defy its threat of a strike. The union, however, does not know the company's profit position when it decides whether to make its threat; that is, it does not know whether the company is sufficiently profitable to meet its demands—and the company's assertions in this matter cannot be believed. Nature determines whether the company is profitable; the probability that the firm is unprofitable is p.

   The payoff structure is as follows: (i) When the union makes no threat, the union gets a payoff of 0 (regardless of the profitability of the company). The company gets a payoff of 100 if it is profitable, but a payoff of 10 if it is unprofitable. A passive union leaves more profit for the company if there is any profit to be made. (ii) When the union threatens to strike and the company concedes, the union gets 50 (regardless of the profitability of the company) and the company gets 50 if it is profitable but −40 if it is not. (iii) When the union threatens to strike and the company defies the union's threat, the union must strike and gets −100 (regardless of the profitability of the company). The company gets −100 if it is profitable and −10 if it is not. Defiance is very costly for a profitable company but not so costly for an unprofitable one.

   1. What happens when the union uses the pure threat to strike unless the company concedes to the union's

demands?

2. Suppose that the union sets up a situation in which there is some risk, with probability $q < 1$, that it will strike after the company defies its threat. This risk may arise from the union leadership's imperfect ability to keep the membership in line. Draw a game tree similar to Figure 13.4. for this game.
3. What happens when the union uses brinkmanship, threatening to strike with some probability q unless the company accedes to its demands?
4. Derive the effectiveness and acceptability conditions for this game, and determine the values for p and q for which the union can use a pure threat, brinkmanship, or no threat at all.

2. The professor teaching a course has established a rule that any late homework must receive a failing grade, but ultimately a grader decides what grades to assign. Consider the resulting game between the grader and a student. The grader moves first, either threatening to follow through on the harsh official policy or announcing a more lenient approach. The student then decides whether to complete the homework on time or be late. The grader wants homework to be on time, but also does not want to fail the student, since then the student is sure to complain and try to get a grade change from the professor, dragging the grader into an unpleasant process. The grader gets payoff +1 when homework is on time, payoff 0 when homework is late and not failed, or payoff −3 when homework is late and failed. The student prefers not to fail but also prefers to do the work late because he is busy with other activities. With probability p, those other activities are so important that the student will choose to be late even if that leads to failure: This "unwilling type" gets payoff −10 from being on time, 0 from being late and not failing, and −5 from failing. The rest of the time, the

student prefers to be on time rather than fail: This "willing type" gets payoff −1 from being on time, 0 from being late and not failing, and −5 from failing.

1. What happens when the grader uses the pure threat to fail the student unless homework is on time?
2. Suppose that the grader sets up a situation in which there is some risk, with probability $q < 1$, that she will fail the student for late homework. Such risk could arise from uncertainty regarding whether the professor is truly serious about the policy. Draw a game tree similar to Figure 13.4 for this game.
3. What happens when the grader uses brinkmanship, threatening to fail the student with some probability q unless homework is on time?
4. Derive the effectiveness and acceptability conditions for this game, and determine the values for p and q for which the grader can use a pure threat, brinkmanship, or no threat at all.

3. Scenes from many movies illustrate the concept of brinkmanship. Analyze the following descriptions from this perspective. What are the risks the two sides face? How do those risks increase during the course of the execution of the brinkmanship threat?
   1. In the 1980 film *The Gods Must Be Crazy*, the only survivor of a rebel team that tried to assassinate the president of an African country has been captured and is being interrogated. He stands blindfolded with his back to the open door of a helicopter. Above the noise of the helicopter rotors, an officer asks him, "Who is your leader? Where is your hideout?" The man does not answer, and the officer pushes him out the door. In the next scene, we see that although its engine is running, the helicopter is actually on the ground, and the man has fallen 6 feet and landed on his back. The officer appears at the door and says, laughing, "Next time it will be a little higher."

2. In the 1998 film *A Simple Plan*, two brothers remove some of a $4.4 million ransom payment that they find in a crashed airplane. After many intriguing twists of fate, the remaining looter, Hank, finds himself in conference with an FBI agent. The agent, who suspects, but cannot prove, that Hank has some of the missing money, fills Hank in on the story of the money's origins and tells him that the FBI possesses the serial numbers of about 1 of every 10 of the bills in that original ransom payment. The agent's final words to Hank are, "Now it's simply a matter of waiting for the numbers to turn up. You can't go around passing $100 bills without eventually sticking in someone's memory."

4. In this exercise, we provide two examples of the successful use of brinkmanship, where "success" consists of the two parties' reaching a mutually acceptable deal. For each example, (i) identify the interests of the parties; (ii) describe the nature of the uncertainty inherent in the situation; (iii) give the strategies the parties used to escalate the risk of disaster; (iv) discuss whether those strategies were good ones; and (v) (**Optional**) if you can, set up a small mathematical model of the kind presented in this chapter. In each case, we provide a few readings to get you started; you should locate more by using the resources of your library and online resources such as Lexis-Nexis.
   1. The Uruguay Round of international trade negotiations that started in 1986 and led to the formation of the World Trade Organization in 1994. *Reading:* John H. Jackson, *The World Trading System*, 2nd ed. (Cambridge, Mass.: MIT Press, 1997), pp. 44–49 and Chapters 12 and 13.
   2. The Camp David Accords between Israel and Egypt in 1978. *Reading:* William B. Quandt, *Camp David: Peacemaking and Politics* (Washington, D.C.: Brookings Institution, 1986).

5. The following examples illustrate the unsuccessful use of brinkmanship, where brinkmanship is considered "unsuccessful" when the mutually bad outcome (disaster) occurs. Answer the questions listed in Exercise S4 for each example.
    1. The confrontation between the Chinese communist regime and the student prodemocracy demonstrators in Beijing in June 1989. *Readings:* Donald Morrison, ed., *Massacre in Beijing: China's Struggle for Democracy* (New York: Time Magazine Publications, 1989); Suzanne Ogden, Kathleen Hartford, L. Sullivan, and D. Zweig, eds., *China's Search for Democracy: The Student and Mass Movement of 1989* (Armonk, N.Y.: M. E. Sharpe, 1992).
    2. The Caterpillar strike, from 1991 to 1998. *Readings:* "The Caterpillar Strike: Not Over Till It's Over," *Economist*, February 28, 1998; "Caterpillar's Comeback," *Economist*, June 20, 1998; Aaron Bernstein, "Why Workers Still Hold a Weak Hand," *BusinessWeek*, March 2, 1998.
6. Answer the questions listed in Exercise S4 for these potential opportunities for brinkmanship in the future:
    1. A Taiwanese declaration of independence from the People's Republic of China. *Reading:* Ian Williams, "Taiwan's Independence," *Foreign Policy in Focus*, December 20, 2006, available at www.fpif.org/fpiftxt/3815.
    2. The militarization of space; for example, the positioning of weapons in space or the shooting down of satellites. *Reading:* "Disharmony in the Spheres," *Economist*, January 17, 2008, available at www.economist.com/node/10533205.

UNSOLVED EXERCISES

1. In the calculations of Section 3, we assumed that the payoff to the United States is −10 when Soviets of either type (hard-line or soft) defy the U.S. threat, as illustrated in Figure 13.3. Suppose now that this payoff is −25 rather than −10.
   1. Incorporate this change in payoff into a game tree similar to the one in Figure 13.4.
   2. Using the payoffs from your game tree in part (a), find the effectiveness condition for this version of the U.S.–USSR brinkmanship game.
   3. Using the payoffs from part (a), find the acceptability condition for this game.
   4. Draw a diagram similar to that in Figure 13.5, illustrating the effectiveness and acceptability conditions found in parts (b) and (c).
   5. For what values of p, the probability that the Soviets are hard-line, is the pure threat (q = 1) acceptable? For what values of p is the pure threat unacceptable but brinkmanship still possible?
   6. If Kennedy was correct in believing that p lay between one-third and one-half, does your analysis of this version of the game suggest that an effective *and* acceptable probabilistic threat existed? Use this example to explain how a game theorist's assumptions about player payoffs can affect the predictions that arise from the theoretical model.
2. A corrupt Russian policeman is interrogating a mobster who knows where $1 million is hidden. The policeman can threaten to kill the mobster unless he reveals the money's location. The policeman moves first, deciding whether to make the threat, after which the mobster decides whether to reveal the secret location. The policeman wants the money, but also prefers not to murder

the mobster, since then there might be an investigation that would uncover his own crimes. The policeman gets $1,000,000 (normalized to payoff +1) if the mobster reveals the money's location, payoff 0 if the mobster stays quiet and is left alive, and payoff −0.5 if the mobster stays quiet and is killed. With probability $(1 - p)$, the mobster is willing to tell the secret to save his life. This "loose-lipped type" gets payoff 0 when staying quiet and living, payoff −2 when telling the secret and living, and payoff −10 when staying quiet and dying. The rest of the time, with probability p, the mobster is a "tight-lipped type" who would rather die than spill the beans. This type gets payoff 0 when staying quiet and living, payoff −2 when telling the secret and living, and payoff −1 from staying quiet and dying.

1. What happens when the policeman uses the pure threat to kill the mobster unless he tells the secret?
2. The corrupt policeman has a gun that can hold six bullets. By loading the gun with $b = 1, 2, 3, 4,$ or 5 bullets and then spinning the gun's cylinder, the policeman can create a risk $q = b/6 < 1$ that there is a bullet in the chamber that he is about to fire. This allows the policeman to make a probabilistic threat by committing to pull the trigger exactly once—unless the mobster reveals the secret. Draw a game tree similar to Figure 13.4 for this game.
3. What happens when the corrupt policeman uses brinkmanship, playing "Russian roulette" to threaten to kill the mobster with some probability q unless he reveals the money's secret location?
4. Derive the effectiveness and acceptability conditions for this game, and determine the values for p and q for which the corrupt policeman can use a pure threat, brinkmanship, or no threat at all.
5. Suppose that $p = ⅓$. Will the corrupt policeman find it optimal to play Russian roulette with the mobster?

If so, how many bullets will he optimally load into the gun before spinning the cylinder?

3. Answer the questions from Exercise S3 for the following movies:
    1. In the 1941 movie classic *The Maltese Falcon*, the hero, Sam Spade (Humphrey Bogart), is the only person who knows the location of the immensely valuable gem-studded falcon figure, and the villain, Caspar Gutman (Sydney Greenstreet), is threatening to torture him for that information. Spade points out that torture is useless unless the threat of death lies behind it, and Gutman cannot afford to kill Spade, because then the information would die with him. Therefore, he may as well not bother with the threat of torture. Gutman replies, "That is an attitude, sir, that calls for the most delicate judgment on both sides, because, as you know, sir, men are likely to forget in the heat of action where their best interests lie and let their emotions carry them away."
    2. The 1925 Soviet classic *The Battleship Potemkin* (set in the summer of 1905) closes with a squadron of ships from the tsar's Black Sea fleet chasing the mutinous and rebellious crew of the *Potemkin*. The tension mounts as the ships draw ever closer. Men on each side race to their battle stations, load and aim the huge guns, and wait nervously for the order to fire on their countrymen. Neither side wants to attack the other, but neither wants to back down or to die without defending itself. The tsar's ships have orders to take the *Potemkin* by any means necessary, and the crew knows it will be tried for treason if it surrenders.
4. Answer the questions listed in Exercise S4 for these examples of successful brinkmanship:
    1. The negotiations between the South African apartheid regime and the African National Congress to establish a new constitution with majority rule, 1989 to 1994.

Reading: Allister Sparks, *Tomorrow Is Another Country* (New York: Hill and Wang, 1995).
2. Peace in Northern Ireland: disarmament of the IRA in July 2005, the St. Andrews Agreement of October 2006, the elections of March 2007, and the power-sharing government of Ian Paisley and Martin McGuinness. *Reading:* “The Thorny Path to Peace and Power Sharing,” *CBC News*, March 26, 2007, available at www.cbc.ca/news2/background/northern-ireland/timeline.xhtml.

5. Answer the questions listed in Exercise S4 for these examples of unsuccessful brinkmanship:
    1. The U.S. budget confrontation between President Clinton and the Republican-controlled Congress in 1995. *Readings:* Sheldon Wolin, “Democracy and Counterrevolution,” *Nation*, April 22, 1996; David Bowermaster, “Meet the Mavericks,” *U.S. News and World Report*, December 25, 1995–January 1, 1996; “A Flight that Never Seems to End,” *Economist*, December 16, 1995.
    2. The television writers’ strike of 2007–2008. *Readings:* “Writers Guild of America,” online archive of the *New York Times* on the Writers Guild and the strike, available at http://topics.nytimes.com/top/reference/timestopics/organizations/w/writers_guild_of_america/index.xhtml; “Writers Strike: A Punch from the Picket Line,” available at http://writers-strike.blogspot.com.
6. Answer the questions listed in Exercise S4 for these potential opportunities for brinkmanship in the future:
    1. The West Coast states (California, Oregon, and Washington) attempt to secede from the United States. *Readings:* “What If California Attempted to Secede from the U.S.?” available at http://www.bbc.com/future/story/20190221-what-if-california-seceded-from-the-us; “Partition and Secession in California,” available at

https://en.wikipedia.org/wiki/Partition_and_secession_in_California.

2. A nuclear confrontation between India and Pakistan over Kashmir or other issues. *Readings:* "Could the Confrontation between India and Pakistan Lead to Nuclear War?" *Pacific Standard*, https://psmag.com/news/could-the-conflict-between-pakistan-and-india-lead-to-nuclear-war; "Factbox: India and Pakistan—Nuclear Arsenals and Strategies," Reuters, available at https://www.reuters.com/article/us-india-kashmir-pakistan-nuclear-factbo/factbox-india-and-pakistan-nuclear-arsenals-and-strategies-idUSKCN1QI4O5.

14■ Design of Incentives

JAMES MIRRLEES WON THE NOBEL PRIZE in economics in 1996 for his pioneering work on optimal nonlinear income taxation and related policy issues. Many non-economists, and some economists, too, found his work difficult to understand. But the *Economist* magazine gave a brilliant characterization of the broad importance and relevance of the work. It said that Mirrlees showed us "how to deal with someone who knows more than you do."[1]

In Chapter 9, we observed some of the ways in which asymmetries of information affect the analysis of games. But the underlying problem for Mirrlees differed slightly from the problems we considered earlier. In his work, one player (the government) needed to devise a set of rules so that the other players' (the taxpayers') incentives were aligned with the first player's goals. Models with this general framework, in which a less informed player works to create motives for a more informed player to take actions beneficial to the less informed one, now abound, and they are relevant to a wide range of social and economic interactions. Generally, the less informed player is called the *principal* while the more informed one is called the *agent*; hence these models are termed *principal–agent* models.

The principal's goal in a principal–agent problem is to design an *incentive scheme* that will motivate the agent to take actions that benefit the principal, taking into account that the agent knows something (about the world or about herself) that the principal does not. Such incentive schemes are known as *mechanisms*, and the process that the principal uses to devise the best possible incentive scheme is referred to as mechanism design or incentive design. As we will see, mechanism-design ideas apply in many contexts that at first glance might seem to have little in common, including the pricing of goods or services (Section 1), highway procurement (Section 3), employee supervision (Section 5.A), insurance provision (Section 5.B), auctions (Chapter 15), and Mirrlees’s original application: government taxation.

In Mirrlees’s model, the government seeks a balance between efficiency and equity. It wants the more productive members of society to contribute effort to increase its total output; it can then redistribute the proceeds to benefit the poorer members. If the government knew the exact productive potential of every person and could observe the quantity and quality of every person’s effort, it could simply order everyone to contribute according to their ability, and it could distribute the fruits of their effort according to people’s needs. But such detailed information would be costly or even impossible to obtain, and such redistribution schemes can be equally difficult to enforce. Each person has a good idea of his abilities and needs, and chooses his own effort level, but stands to benefit by concealing this information from the government. Pretending to have less ability and more need will enable him to get away with paying less in taxes or getting larger checks from the government; in addition, his incentive to provide effort is reduced if the government takes part of the yield. The government must calculate its tax policy, or design its fiscal mechanism, taking these problems of information and incentives into account. Mirrlees’s contribution was to solve this complex mechanism-design problem within the principal–agent framework.

The economist William Vickrey shared the 1996 Nobel Prize in economics with Mirrlees for his own work in mechanism design in the presence of asymmetric information. Vickrey is best known for designing an auction mechanism to elicit truthful bidding, a topic we will study in greater detail in Chapter 15. But his work extended to other mechanisms, such as congestion pricing on highways, and he and Mirrlees laid the groundwork for extensive research in the field.

Indeed, in the past 30 years, the general theory of mechanism design has made great advances. The 2007 Nobel Prize in economics was awarded to Leonid Hurwicz, Roger Myerson, and Eric Maskin for their contributions to it. Their work, and that of many others, has taken the theory and applied it to numerous specific contexts, including the design of compensation schemes, insurance policies, and of course, tax schedules and auctions. In this chapter, we develop a few prominent applications, using our usual method of numerical examples followed by exercises.

Endnotes

- “Economics Focus: Secrets and the Prize,” *Economist*, October 12, 1996. Return to reference 1

Glossary

mechanism design
: Same as **incentive design**.

incentive design
: The process that a *principal* uses to devise the best possible incentive scheme (or mechanism) in a *principal-agent problem* to motivate the agent to take actions that benefit the principal. By design, such incentive schemes take into account that the agent knows something (about the world or about herself) that the principal does not know. Also called **mechanism design**.

1 PRICE DISCRIMINATION

A firm generally sells to diverse customers with different levels of willingness to pay for its product. Ideally, the firm would like to extract from each customer the maximum that he would be willing to pay. If the firm could charge each customer an individualized price based on that customer's willingness to pay, economists would say that it was practicing perfect (or first-degree) price discrimination.

Such perfect price discrimination may not be possible for many reasons. The most general underlying reason is that even a customer who is willing to pay a lot prefers to pay less. Therefore, the customer will prefer a lower price, and the firm may have to compete with other firms or resellers who undercut its high price. But even if there are no close competitors, the firm usually does not know how much each individual customer is willing to pay, and customers will try to get away with pretending to be unwilling to pay a high price so as to secure a lower price. In some situations, even if the firm could detect a customer's willingness to pay, it would be illegal to practice blatant first-degree price discrimination based on the identity of the buyer. In such situations, the firm must devise a product line and prices so that customers' choices of what they buy (and therefore what they pay) go some way toward the firm's goal of increasing its profit by way of price discrimination.

In our terminology of asymmetric information games developed in Chapter 9, the process by which a firm identifies a customer's willingness to pay from his purchase decisions involves *screening* to achieve *separation of types* (by *self-selection*). The firm does not know each customer's *type* (willingness to pay), so it tries to acquire that information from the customer’s actions. An example that should be familiar to most readers is that of airlines. These firms try to separate business travelers, who are willing to pay more for their tickets, from tourists, who are not willing to pay as much, by offering low prices in return for various restrictions on fares that the business flyers are not willing to accept, such as advance purchase and minimum stay requirements.[2] We develop this particular example in more detail here to make the underlying ideas more precise and quantifiable.

We consider the pricing decisions of a firm called Pie-In-The-Sky (PITS), an airline running a service from Podunk to South Succotash. It carries some business passengers and some tourists; the former type are willing to pay a higher price than the latter type for any particular ticketed seat. To serve the tourists profitably without having to offer the same low price to the business passengers, PITS has to develop a way of creating different versions of the same flight; it then needs to price these options in such a way that each type will choose a different version. As mentioned above, the airline could distinguish between the two types of passengers by offering restricted and unrestricted fares. The practice of offering first-class and economy tickets is another way to distinguish between the two groups; we will use that practice as our example.

Type of service	PITS’s cost	RESERVATION PRICE		PITS’S POTENTIAL PROFIT	
		Tourist	Business	Tourist	Business

You may need to scroll left and right to see the full figure.

Economy	100	140	225	40	125
First	150	175	300	25	150
You may need to scroll left and right to see the full figure.					

FIGURE 14.1 Airline Price Discrimination (in Dollars)

Suppose that 30% of PITS's customers are business travelers and 70% are tourists. The table in Figure 14.1 shows the (maximum) willingness to pay, or the *reservation price* in economics jargon, for each type of customer for each class of service, along with the costs of providing the two types of service and the potential profits available under each option.

We begin by setting up a ticket-pricing scheme that is ideal from PITS's point of view. Suppose it knows the type of each individual customer: its salespeople determine customers' types, for example, by observing their style of dress when they come to make their reservations. Also suppose that there are no legal prohibitions on price discrimination and no possibility that lower-priced tickets can be resold to other passengers. (Actual airlines prevent such resale by requiring positive ID for each ticketed passenger.) Then PITS could practice perfect (first-degree) price discrimination.

How much would PITS charge each type of customer? It could sell a first-class ticket to each business traveler at $300 for a profit of $300 – $150 = $150 per ticket or sell him an economy ticket at $225, for a profit of $225 – $100 = $125 per ticket. The former is better for PITS, so it would want to sell \$300 first-class tickets to business travelers. It could sell a first-class ticket to each tourist at \$175, for a profit of \$175 − \$150 = \$25, or sell him an economy ticket at \$140, for a profit of \$140 − \$100 = \$40. Here, the latter is better for PITS, so it would want to sell \$140 economy tickets to the tourists. Ideally, PITS would like to sell only first-class tickets to business travelers and only economy tickets to tourists, in each case at a price equal to the relevant group's maximum willingness to pay. PITS's total profit per 100 customers from this strategy would be

$$(140 - 100) \times 70 + (300 - 150) \times 30 = 40 \times 70 + 150 \times 30 = 2,800 + 4,500 = 7,300.$$

Thus, PITS's best possible outcome earns it a profit of \$7,300 for every 100 customers it serves.

Now turn to the more realistic scenario in which PITS cannot identify the type of each customer or is not allowed to use such information for purposes of overt price discrimination. How can it use the different ticket versions to screen its customers?

The first thing PITS should realize is that the pricing scheme devised above will not be the most profitable in the absence of identifying information about each customer. Most importantly, it cannot charge the business travelers the full \$300 they are willing to pay for first-class seats while charging only \$140 for an economy seat. Then the business travelers could buy economy seats, for which they are actually willing to pay \$225, for \$140 and get an extra benefit, or *consumer surplus* in the jargon of economics, of \$225 − \$140 = \$85. They might use this surplus, for example, for better food or accommodation on their travels. Paying the maximum \$300 that they are willing to pay for a first-class seat would leave them no consumer surplus. Therefore, they would switch to economy class in this situation, and screening would fail. PITS' s profit per 100 customers would drop to $(140 - 100) \times 100 = \$4,000$.

The maximum that PITS will be able to charge for first-class tickets must give business travelers at least as much extra benefit as the $85 they can get if they buy an economy ticket. Thus, the price of first-class tickets can be at most $300 − $85 = $215. (Perhaps it should be $214 to give business travelers a definite positive reason to choose first class, but we will ignore the trivial difference.) PITS can still charge $140 for an economy ticket to extract as much profit as possible from the tourists, so its total profit in this case (from every 100 customers) will be

$$(140 - 100) \times 70 + (215 - 150) \times 30 = 40 \times 70 + 65 \times 30 = 2,800 + 1,950 = 4,750.$$

This profit is more than the $4,000 that PITS would get if it tried unsuccessfully to implement its perfect price discrimination scheme despite its limited information, but less than the $7,300 it would get if it had full information and successfully practiced perfect price discrimination.

By pricing first-class seats at $215 and economy seats at $140, PITS can successfully screen and separate business travelers from tourists on the basis of their self-selection of its two types of services. But PITS must sacrifice some profit to achieve this indirect discrimination. PITS loses this profit because it must charge the business travelers less than their full willingness to pay. As a result, its profit per 100 passengers drops from the $7,300 it could achieve if it had full and complete information, to the $4,750 it achieves by indirect discrimination based on self-selection. The difference, $2,550, is precisely 85×30, where 85 is the drop in the first-class fare below the business travelers' full willingness to pay for this service, and 30 is the number of these business travelers per 100 passengers served.

Our analysis shows that, in order to achieve separation of types with its ticket-pricing mechanism, PITS has to keep the first-class fare sufficiently low to give the business travelers enough incentive to choose this service. Those travelers have the option of choosing economy class if it provides more benefit (or surplus) to them; PITS has to ensure that they do not "defect" to making the choice that PITS intends for the tourists. Such a requirement, or constraint, on the screener's strategy arises in all problems of mechanism design and is called an *incentive-compatibility constraint.*

The only way PITS could charge business travelers more than \$215 without inducing their defection would be to increase the economy fare. For example, if the first-class fare were \$240 and the economy fare were \$165, then business travelers would get equal consumer surplus from either class; their surplus would be \$300 − \$240 from first class and \$225 − \$165 from economy class, or \$60 from each. At those higher prices, they would still be (only just) willing to buy first-class tickets, and PITS could enjoy higher profits from each first-class ticket sale.

But at \$140, the economy fare is already at the limit of the tourists' willingness to pay. If PITS raised that fare to \$165, it would lose those customers altogether. In order to keep those customers willing to buy, PITS's pricing mechanism must meet an additional requirement, namely, the tourists' *participation constraint.*

PITS's pricing strategy is thus squeezed between the participation constraint of the tourists and the incentive-compatibility constraint of the business travelers. If it charges X for economy and Y for first class, it must keep $X < 140$ to ensure that the tourists still buy tickets, and it must keep $225 - X < 300 - Y$, or $Y < X + 75$, to ensure that the business travelers choose first class and not economy class. Subject to these constraints, PITS wants to charge prices that are as high as possible. Therefore, its profit-maximizing screening strategy is to make *X* as close to 140 and *Y* as close to 215 as possible. Ignoring the small differences that are needed to preserve the less-than signs, let us call the prices 140 and 215. Then charging \$215 for first-class seats and \$140 for economy seats is the solution to PITS's mechanism-design problem.

This pricing strategy being optimal for PITS depends on the specific numbers in our example. If the proportion of business travelers were much higher—say, 50%—PITS would have to revise its optimal ticket prices. If 50% of its customers were business travelers, the sacrifice of \$85 on each business traveler's ticket might be too high to justify keeping the few tourists. PITS might do better not to serve the tourists at all—that is, to violate the tourists' participation constraint and raise the price of first-class service. Indeed, the strategy of screening by self-selection with these percentages of passengers would yield PITS a profit, per 100 customers, of

$$(140 - 100) \times 50 + (215 - 150) \times 50 = 40 \times 50 + 65 \times 50 = 2{,}000 + 3{,}250 = 5{,}250.$$

By contrast, the strategy of serving only business travelers in \$300 first-class seats would yield a profit (per 100 customers) of

$$(300 - 150) \times 50 = 150 \times 50 = 7{,}500,$$

which is higher than its profit with the screening strategy. Thus, if there are only a relatively few customers with low willingness to pay, a seller might find it better not to serve them at all than to offer sufficiently low prices to the mass of high-paying customers to prevent their switching to the lower-priced version of its product or service.

Precisely what proportion of business travelers constitutes the borderline between the two cases? We leave this question as an exercise for you. We will simply point out that an airline's decision to offer low tourist fares may be a profit-maximizing response to the existence of asymmetric information, rather than an indication of some soft spot for vacationers!

Endnotes

- Pricing policies that offer a menu of different versions of a product at different prices are referred to as *second-degree* price discrimination. Some examples include *quantity discounts*, where different quantities of a good are offered at different per-unit prices, and *upgrades* (like the example here) or *damaged goods*, where different qualities of a product are offered. See Raymond Deneckere and Preston McAfee, "Damaged Goods," *Journal of Economics & Management Strategy*, vol. 5, no. 2 (June 1996), pp. 149–174. Readers with an economics background will also be familiar with *third-degree* price discrimination, also referred to as "market segmentation," in which the firm can observe something about the buyer (such as her age, income, or location) and can charge different prices based on that information. Return to reference 2

Glossary

price discrimination
: Perfect, or first-degree, price discrimination occurs when a firm charges each customer an individualized price based on willingness to pay. In general, price discrimination refers to situations in which a firm charges different prices to different customers for the same product.

2 SOME TERMINOLOGY

We have now seen one example of mechanism design in action. There are many others, of course, and we will see additional ones in later sections of this chapter. We pause briefly here, however, to set out the specifics of the terminology used in most models of this type.

Mechanism-design problems are broadly of two kinds. The airline price-discrimination case in Section 1 is an example of the first kind, in which one player is better informed (in the example, the customer knows his own willingness to pay), and his information affects the payoff of the other player (in the example, the airline's pricing and therefore its profits). In the language of Chapter 9, mechanisms in this category are designed to cope with potential *adverse selection*. The less informed player designs a scheme in which the more informed player must make some choice that will reveal the information, albeit at some cost to the less informed player (in the example, the airline's inability to charge the business travelers their full willingness to pay).

In the second kind of mechanism-design problem, one player takes some action that is not observable to others. For example, an employer cannot observe the quality, or sometimes even the quantity, of the effort an employee exerts, and an insurance company cannot observe all the actions that an insured driver or homeowner takes to reduce the risk of an accident or robbery. In the language of Chapter 9, mechanisms for this kind of problem are designed to cope with potential *moral hazard*. The less informed player designs a scheme—for example, offering profit sharing to an employee or imposing deductibles and copayments on the insured—that aligns the other player's incentives to some extent with those of the mechanism designer.[3]

In each case, the less informed player designs the mechanism; she is called the principal in the strategic game. The more informed player is then called the agent; this term is most accurate in the case of the employee and less so in the cases of the customer or the insured, but the jargon has become established and we will adopt it. The game is then called a principal–agent, or agency, problem.

In both kinds of problems, the principal designs the mechanism to maximize her own payoff, subject to two types of constraints. First, the principal knows that the agent will use the mechanism to maximize his own (the agent's) payoff. In other words, the principal's mechanism has to be consistent with the agent's incentives. As we saw in Chapter 9, Section 5.A, this requirement is called the *incentive-compatibility constraint.* Second, given that the agent responds to the mechanism in his own best interest, the agency relationship has to give the agent at least as much expected payoff as he would get elsewhere—for example, by working for a different employer, or by driving instead of flying. In Chapter 9, we termed this requirement the *participation constraint.* We saw specific examples of both constraints in the airline price-discrimination example in the previous section; we will meet many other examples and applications in the rest of this chapter.

Endnotes

- In formal game theory, the two kinds of mechanism-design problems we describe here are often called "hidden information" and "hidden action" problems, respectively. The distinction between the two kinds of problems was emphasized by Oliver Hart and Bengt Holmstrom in their classic paper, "The Theory of Contracts," in Truman Bewley (ed.), *Advances in Economic Theory: Fifth World Congress*, (Cambridge: Cambridge University Press, 1987), pp. 71–156. Return to reference 3

Glossary

principal

The principal is the less-informed player in a principal–agent game of asymmetric information. The principal in such games wants to design a mechanism that creates incentives for the more-informed player (agent) to take actions beneficial to the principal.

agent

The agent is the more-informed player in a principal–agent game of asymmetric information. The principal (less-informed) player in such games attempts to design a mechanism that aligns the agent’s incentives with his own.

principal–agent (agency) problem

A situation in which the less-informed player (principal) wants to design a mechanism that creates incentives for the more-informed player (agent) to take actions beneficial to himself (the principal).

principal–agent (agency) problem

A situation in which the less-informed player (principal) wants to design a mechanism that creates incentives for the more-informed player (agent) to take actions beneficial to himself (the principal).

3 INFORMATION-REVEALING CONTRACTS

When writing procurement contracts for certain services—perhaps highway building or office-space construction—governments and firms face mechanism-design problems of the kind we have been describing. There are two common ways of writing such contracts. In a *cost-plus contract*, the buyer agrees to pay the supplier of the services a sum equal to his cost, plus a small amount of profit. In a *fixed-price contract*, a specific price for the services is agreed upon in advance; the supplier keeps any extra profit if his actual cost turns out to be less than anticipated, and he bears the loss if his actual cost is higher.

Each type of contract has its own good and bad points. The cost-plus contract appears not to give the supplier excessive profit; this characteristic is especially important for public-sector procurement contracts, where the citizens are the ones who ultimately pay for the procured services. But the supplier typically has better information about his cost than does the buyer of his services; therefore, the supplier can be tempted to overstate or pad his costs in order to extract some benefit from the wasteful excess. The fixed-price contract, in contrast, gives the supplier every incentive to keep the cost at a minimum and thus to achieve an efficient use of resources. But with this kind of public-sector contract, the citizens have to pay the set price and give away any excess profit (to the supplier). The optimal contract should balance these two considerations.

A. Highway Construction: Full Information

We first consider the example of a state government designing a procurement mechanism for a highway-construction project. Specifically, suppose that a major highway is to be built by a construction firm to be hired by the state, and that the state government has to decide how many lanes it should have.[4] More lanes yield more social benefit in the form of faster travel and fewer accidents (at least up to a point, beyond which the harm to the countryside will be too great). To be specific, we suppose that the social value V (measured in billions of dollars) of having N lanes on the highway is given by the formula

$$V = 15N - \frac{N^2}{2}.$$

The cost of construction per lane, including an allowance for normal profit, could be either \$3 billion or \$5 billion per lane, depending on the types of soil and minerals located in the construction zone. For now, we assume that the state government can determine the construction cost as well as the firm can. So it chooses N and writes a contract to maximize the benefit to the state (V) net of the fee paid to the firm (call it F); that is, the government's objective is to maximize net benefit, G, where $G = V - F$.[5]

Suppose first that the government knows that the actual cost per lane is 3 (billion dollars per lane of highway). At this cost level, the government has to pay $3N$ to the firm for an N-lane highway. The government then chooses N to maximize G, as above, where the appropriate formula in this situation is

$$G = V - F = 15N - \frac{N^2}{2} - 3N = 12N - \frac{N^2}{2}.$$

Recall that in the appendix to Chapter 5, we gave a formula for finding the correct value to maximize this type of function. Specifically, the solution to the problem of choosing X to maximize

$$Y = A + BX - CX^2$$

is $X = B/(2C)$. Here, Y is G, X is N, and $A = 0$, $B = 12$, and $C = ½$. Applying our solution formula yields the government's optimal choice of $N = 12/(2 \times ½) = 12$. The best highway to choose therefore has 12 lanes, and the cost of that 12-lane highway is \$36 billion. So, the government offers the contract: "Build a 12-lane highway and we will pay you \$36 billion." [6] This price includes normal profit, so the firm is happy to take the contract.

Similarly, if the cost is \$5 billion per lane, the optimal N will be 10. The government will offer a \$50 billion contract for the 10-lane highway. And the firm will accept the contract.

B. Highway Construction: Asymmetric Information

Now suppose that the firm knows how to assess the relevant terrain to determine the actual building cost per lane, but the state government does not. The government can only estimate what the cost will be. We assume that it thinks that there is a two-thirds probability of the cost being 3 (billion dollars per lane) and a one-third probability of the cost being 5.

What if the government tries to go ahead with the full-information optimum we found in Section 3.A and offers a pair of contracts: "Build a 12-lane highway for $36 billion" and "Build a 10-lane highway for $50 billion"? If the actual cost is really only $3 billion per lane, the firm will get more profit by taking the latter contract, even though that one was designed for the situation in which the cost is $5 billion per lane. The true cost of the 10-lane highway will be only $30 billion, and the firm will earn $20 billion in excess profit.[7]

This outcome is not very satisfying for the state government. The contracts offered do not give the firm sufficient incentive to choose between them on the basis of cost; it will always take the $50 billion contract. There must be a better way for the government to design its procurement contract system.

Let us therefore allow the government the freedom to design an optimal screening mechanism to separate the two possible types of projects (low-cost and high-cost). Suppose it offers a pair of contracts: "Contract L: Build N_L lanes and get paid R_L dollars" and "Contract H: Build N_H lanes and get paid R_H dollars." If contracts L and H are designed correctly then, when the firm determines the true cost is low ($3 billion per lane), it will pick contract L (L stands for "low") and, when it determines the true cost is high ($5 billion per lane), it will pick contract H (H stands for "high"). The numbers that the symbols

N_L, R_L, N_H, and R_H represent must satisfy certain conditions for this screening mechanism to work.

First, under each contract, a firm anticipating the relevant cost (low for contract L and high for contract H) must receive enough payment to cover its cost (inclusive of normal profit). Otherwise, it will not agree to the terms; it will not participate in the contract. Thus, the contract must satisfy two *participation constraints*: $3N_L \leqslant R_L$ for the firm when the cost is 3, and $5N_H \leqslant R_H$ for the firm when the cost is 5.

Next, the government needs the two contracts to be such that the firm would not benefit by taking contract H when it knows the true cost is low, and vice versa. That is, the contracts must also satisfy two incentive-compatibility constraints. For example, if the true cost is low, contract L will yield excess profit $R_L - 3N_L$, whereas contract H will yield $R_H - 3N_H$. (Note that in the latter expression, the number of lanes and the payment are as specified in the H contract, but the firm' s cost is still only 3, not 5.) To be incentive-compatible for the low-cost case, the contracts must ensure that when the cost is 3 the firm' s excess profit from contract H is no larger than that from contract L, or that the latter expression no larger than the former. Thus, we need $R_L - 3N_L \geqslant R_H - 3N_H$. Similarly, if the true cost is 5, the firm' s excess profit from the L contract must be no larger than its excess profit from the H contract. To keep the contracts incentive-compatible, we therefore need $R_H - 5N_H \geqslant R_L - 5N_L$.

As before, the government wants to maximize the net benefit of the payment, G. Here, there are two possible outcomes, so the government must actually maximize the net *expected* benefit using the probabilities of the two project types as weights to calculate the expected value of G. Therefore, the government' s objective here is to maximize

$$E[G] = \left(\frac{2}{3}\right)\left[15N_L - \frac{(N_L)^2}{2} - R_L\right] + \left(\frac{1}{3}\right)\left[15N_H - \frac{(N_H)^2}{2} - R_H\right].$$

The problem looks formidable, with four choice variables and four inequality constraints. But it simplifies greatly, because two of the constraints are redundant, and the other two must hold as exact equalities (or, the other two constraints *bind)*, allowing us to solve and substitute for two of the variables.

Note that if the participation constraint when the true cost is high, $5N_H \leqslant R_H$, and the incentive-compatibility constraint when the true cost is low, $R_L - 3N_L \geqslant R_H - 3N_H$, both hold, then we can get the following string of inequalities (using the fact that N_H cannot be negative):

$$R_L - 3N_L \geqslant R_H - 3N_H \geqslant 5N_H - 3N_H \geqslant 2N_H \geqslant 0.$$

The first and last expressions in the inequality string tell us that $R_L - 3N_L \geqslant 0$. Therefore, we need not consider the participation constraint when the true cost is low, $3N_L \leqslant R_L$, separately; it is automatically satisfied when the two other constraints are satisfied.

It is also intuitive that when the true cost is high, the firm will not want to pretend that the cost is low; it would be compensated for the lower cost project while incurring the higher true cost. However, this intuition needs to be verified by the rigorous logic of the analysis. To confirm the intuition, we ignore the second incentive-compatibility constraint, $R_H - 5N_H \geqslant R_L - 5N_L$, and proceed to solve the problem with just the remaining two constraints. Then we return and verify that our solution to the two-constraint problem satisfies the ignored incentive-compatibility constraint anyway. So our solution must also be the solution to the three-constraint problem. (If something better was available, it would also work better for the less constrained problem.)

Thus, we have two constraints to consider: $5N_H \leqslant R_H$ (the participation constraint for the firm when the true project cost is high) and $R_L - 3N_L \geqslant R_H - 3N_H$ (the incentive-compatibility constraint for the firm when the true project cost is low). We write these as $R_H \geqslant 5N_H$ and $R_L \geqslant R_H + 3(N_L - N_H)$. Next, observe that R_L and R_H each figure negatively in $E[G]$, the expected benefit to the government: It wants to make them as small as possible while still satisfying the constraints. This result is achieved by satisfying each constraint with equality; thus, we can set $R_H = 5N_H$ and $R_L = R_H + 3(N_L - N_H) = 3N_L + 2N_H$. These expressions for the contract payments can now be substituted into the expected benefit function, $E[G]$. This substitution yields

$$E[G] = \left(\frac{2}{3}\right)\left[15N_L - \frac{(N_L)^2}{2} - 3N_L - 2N_H\right] + \left(\frac{1}{3}\right)\left[15N_H - \frac{(N_H)^2}{2} - 5N_H\right].$$

$$= 8N_L - \frac{(N_L)^2}{3} + 2N_H - \frac{(N_H)^2}{6}.$$

The expected benefit function now splits cleanly into two parts: One (the first two terms) involves only N_L, and the other (the second two terms) involves only N_H. We can apply our maximization formula (from Section 3.A) separately to each part. In the N_L part, $A = 0$, $B = 8$, and $C = 1/3$, so the optimal $N_L = 8/(2 \times 1/3) = {}^{24}/_{2} = 12$. In the N_H part, $A = 0$ again, $B = 2$, and $C = 1/6$, so the optimal $N_H = 2/(2 \times 1/6) = {}^{12}/_{2} = 6$.

Now we can use the optimal values for N_L and N_H to derive the optimal payment (R) values, using the formulas for R_L and R_H that we derived and substituted into $E[G]$ in the previous paragraph. Substituting $N_L = 12$ and $N_H = 6$ into those formulas gives us $R_H = 5N_H = 5 \times 6 = 30$ and $R_L = R_H + 3(N_L - N_H) = 3 \times 12 + 2 \times 6 = 48$. These calculations give us the optimal values for all of the unknowns in the government’s expected benefit function.

However, remember that we ignored one of the incentive-compatibility constraints. We must ensure that the ignored constraint, $R_H - 5N_H \geq R_L - 5N_L$, holds with our calculated values for the Rs and the Ns. In fact, it does. The left-hand side of the expression is $30 - 5 \times 6 = 0$. And the right-hand side is $48 - 5 \times 12 = -12$, so the ignored constraint is indeed satisfied.

Our solution indicates that the government should offer the following two contracts: "Contract L: Build 12 lanes and get paid \$48 billion" and "Contract H: Build 6 lanes and get paid \$30 billion." How can we interpret this solution so as best to understand the intuition behind it? That intuition is most easily seen when we compare the solution here with the one we found in Section 3.A, when the government had full information about project cost. Figure 14.2 shows these comparisons.

The optimal mechanism with asymmetric information differs in two important respects from the one we found when information was perfect. First, although the contract that the government intends the firm to choose if the project cost is low has the same number of lanes (12) as in the full-information case, its payment to the firm is larger in the asymmetric case (48 instead of 36). Second, the contract that the government intends the firm to choose if the project cost is high has a smaller number of lanes (6 instead of 10), but pays the full cost for that number ($30 = 6 \times 5$) and no more. Both of these differences separate the project types.

With asymmetric information, the firm may be tempted to pretend that the true cost of the project is high when it is in fact low. The optimal procurement mechanism therefore incorporates both a "carrot" to reward the firm for truthfully admitting the true low cost and a "stick" to dissuade it from pretending that the true cost is high. The carrot is the excess profit, $48 - 36 = 12$, that comes from the admission the firm makes implicitly by choosing contract L. The stick is the reduction in excess profit it incurs by choosing contract H, achieved by reducing the number of lanes that will be constructed in that case. The full-information high-cost contract would have the highway be 10 lanes and would pay \$50 billion; if the firm chose this contract while knowing the true cost was low would make excess profit of $50 - 3 \times 10 = \$20$ billion. In the optimal information-constrained high-cost contract, only 6 lanes are constructed, and the firm is paid \$30 billion. If the true cost is low, it makes an excess profit of $30 - 3 \times 6 = \$12$ billion by choosing the high-cost contract. Its benefit from the pretense of high cost (implicit in its choice of contract H even though the true cost is low) is reduced. In fact, the excess profit is reduced exactly to the amount that the firm is guaranteed by the carrot part of the mechanism, thereby exactly offsetting its temptation to pretend that the true cost of the project is high.

	N_L	R_L	N_H	R_L
Perfect information	12	36	10	50
Asymmetric information	12	48	6	30

FIGURE 14.2 Highway-Building Contract Values

Endnotes

- Generally, numerous contractors would be competing for the highway-construction contract. For this example, we restrict ourselves to the case in which there is only one contractor. Return to reference 4
- In reality, the cost per lane would not have only two discrete values, but could take any value along a continuous range of possibilities. The probabilities of each value would then correspondingly form a density function on this range. Our methods will not always yield an integer solution, N, for each possible cost along this range. But we leave these matters to more advanced treatments and confine ourselves to this simple illustrative example. Return to reference 5
- In reality, the contract will contain many clauses specifying quality, timing, inspections, and so forth. We leave out these details to keep the exposition of the basic idea of mechanism design simple. Return to reference 6
- If multiple contractors are competing for the job, the ones not selected might spill the beans about its true cost. But for large highway projects (as for many other large government projects, such as defense contracts), there are often only a few potential contractors, and they do better by colluding among themselves and not revealing their private information. For simplicity, we keep the analysis confined to the case where there is just one contractor. Return to reference 7

4 EVIDENCE CONCERNING INFORMATION-REVELATION MECHANISMS

In the cases considered so far in this chapter, the agent has some private information, which we called that player's *type* in Chapter 9. Further, the principal designs a mechanism that requires the agent to take some action that reveals this information. In the terminology of Chapter 9, these mechanisms are examples of screening for the separation of types by self-selection.

Price-discrimination mechanisms are ubiquitous. All firms have customers who are diverse in their willingness to pay for the firms' products. Ideally, firms would like to discriminate by giving a price break to the less willing customers without giving the same break to the more willing ones. The ability of a firm to practice price discrimination may be limited for reasons other than those of information, including anti-discrimination laws, competition from other firms, or resale by initial buyers. But here we focus on information-based examples of price discrimination, keeping other reasons in the background of the discussion.

Your local coffee shop probably has a "frequent-drinker card"; for every ten cups you buy, for example, you get one free. Why is it in the firm's interest to do this? Frequent drinkers are more likely to be locals, who have the time and incentive to search out the best deals in the neighborhood. To attract those customers away from other competing coffee shops, your coffee shop must offer a sufficiently attractive price. In contrast, infrequent customers are more likely to be strangers in the town, or in a hurry, and have less time and incentive to search for the best deals; when they need a cup of coffee and see a coffee shop, they are willing to pay whatever the price is (within reason). So posting a higher price and giving out frequent-drinker cards enables your coffee shop to give a price break to the price-sensitive regular customers without giving the same price break to the occasional buyers. If you don't have the card, you are revealing yourself as the latter type, willing to pay a higher price.

Many restaurants offer fixed-price three-course menus or blue-plate specials as well as regular à la carte offerings. This strategy enables them to separate diverse customer types with different tastes for soups, salads, main courses, desserts, and so on. Similarly, book publishers start selling new books in a hardcover version and issue a paperback version a year or more later. The price difference between the two versions is generally far greater than the difference in the cost of production of the two kinds of books. The idea behind this pricing scheme is to separate two types of customers, those who need or want to read the book immediately and are willing to pay more for the privilege, and those who are willing to wait until they can get a better price.

The rise of the Internet and our expanding online lives make some sorts of price discrimination more difficult, while also creating new opportunities to target consumers based on extensive, extremely personalized "big data." For products that can be easily found through an online search, like books or consumer electronics, the Internet can make it harder for sellers to use price discrimination.[8] But for highly customized products and services, where there is little, if any, competition, sellers can use everything they have learned about you (including past purchases, browsing history, social networks, physical location, and so on)[9] to offer experiences and set prices that are entirely unique to you. To take an extreme example, consider virtual reality. Everything you say and do within a virtual world (including any virtual money you might earn) is visible to the company running that world, allowing them to know literally everything about your virtual self—and to charge you accordingly.

We invite you to look for other examples of price discrimination and similar screening mechanisms in your own experience. A good source of examples is Tim Harford's *Undercover Economist*.[10]

There is a lot of research literature on the design of procurement mechanisms of the kind we sketched in Section 3.[11] These mechanism-design problems pertain to situations where the buyer confronts just one potential seller, whose cost is private information. This type of interaction accurately describes how contracts for major defense weapon systems or very specialized equipment are designed, as there is usually only one reliable supplier of such products or services. However, in reality, buyers often have the choice of several suppliers, and mechanisms that set the suppliers in competition with one another are beneficial to the buyer. Many such mechanisms take the form of auctions. For example, construction contracts are often awarded by inviting bids and choosing the bidder that offers to do the job for the lowest price (after adjusting for the promised quality of the work, the speed of completion, or other relevant attributes of the bid). We discuss auctions in depth in Chapter 15.

Endnotes

- A 2000 study found substantial price dispersion online for commodity products such as books and CDs, but when its authors restricted attention to large online retailers, there was less variation of prices online than in traditional large brick-and-mortar stores. (This study also found that prices on the Internet were, on average, 9%–16% lower than those in stores.) See Erik Brynjolfsson and Michael D. Smith, "Frictionless Commerce? A Comparison of Internet and Conventional Retailers," *Management Science*, vol. 46, no. 4 (April 2000), pp. 563–85. Return to reference 8
- Geofeedia, a social-media surveillance company launched in 2011, scours social media to determine people's locations and then sells that location-based data to businesses, police departments, and others with an interest in knowing where people are at any given time. See Lee Fang, "The CIA is Investing in Firms that Mine Your Tweets and Instagram Photos," *The Intercept*, April 14, 2016 (available at https://theintercept.com/2016/04/14/in-undisclosed-cia-investments-social-media-mining-looms-large, accessed May 1, 2019) and Colin Lecher and Russell Brandom, "Facebook Caught an Office Intruder Using the Controversial Surveillance Tool It Just Blocked," *The Verge*, October 19, 2016 (available at https://www.theverge.com/2016/10/19/13317890/faceboo-geofeedia-social-media-tracking-tool-mark-zuckerberg-office-intruder, accessed May 1, 2019). Return to reference 9
- Tim Harford, *The Undercover Economist: Exposing Why the Rich Are Rich, the Poor Are Poor—and Why You Can Never Buy a Decent Used Car!* (New York: Oxford University

Press, 2005). The first two chapters give examples of pricing mechanisms. Return to reference 10

- Jean-Jacques Laffont and Jean Tirole, *A Theory of Incentives in Procurement and Regulation* (Cambridge, Mass.: MIT Press, 1993), is the classic of this literature. Return to reference 11

5 INCENTIVES FOR EFFORT: THE SIMPLEST CASE

We now turn from the first type of mechanism-design problem, in which the principal’s goal is to achieve information revelation, to the second type, which deals with moral hazard. The principal’s goal in such situations is to provide an incentive that will induce the best level of effort from the agent, even though that effort level is not observable by the principal.

A. Managerial Supervision

Suppose you are the owner of a company that is undertaking a new project. You have to hire a manager to supervise it. The success of the project is uncertain, but good supervision can increase the probability of success. Managers are only human, though; they will try to get away with as little effort as they can! If your manager’s effort is observable, you can write a contract that compensates the manager for his trouble sufficiently to bring forth good supervisory effort.[12] But if you cannot observe the effort, you have to try to give him incentives based on success of the project—for example, a bonus if the project is successful. Unless good effort absolutely guarantees success, however, such bonuses make the manager’s income uncertain; he gets no bonus if the project fails, even if he has exerted the required effort. And the manager is likely to be averse to this risk or loss, so you have to compensate him for it. You have to design your compensation policy to maximize your own expected profit, recognizing that the manager’s choice of effort depends on the nature and amount of the compensation. The solution to this incentive-design problem is intended to cope with the moral-hazard problem of the manager’s shirking.

Let us consider a numerical example. Suppose that if the project succeeds, it will earn the company a profit of $1 million over material and wage costs. If it fails, the profit will be zero. With good supervision, the probability of success is one-half, but if supervision is poor, the probability of success is only one-quarter. The manager you want to hire is currently in a steady job elsewhere that gets him $100,000; to get him to accept a job with your firm, you must pay him at least as much, but the extra effort costs him (in terms of the extra time diverted from family, friends, or other pursuits) an equivalent of $50,000.

In an ideal world where effort is observable, you can write a contract that states, "If you work for me, and if you exert extra effort, I will pay you $150,000; but if you don't, I will pay you only $100,000." Your expected profit (in millions of dollars) when the manager exerts the extra effort will be

$$0.5 \times 1 + 0.5 \times 0 - 0.15 = 0.35$$

Without the payment for extra effort, the manager will shirk, and your expected profit will be

$$0.25 \times 1 + 0.75 \times 0 - 0.1 = 0.15.$$

So you prefer to include the extra effort clause, and the manager is satisfied, too.

What if the manager's effort cannot be observed? Then the incentive must be based on something that can be observed, and the only possibility in our example is success or failure of the project. Suppose you pay the manager a salary s, plus a bonus b if the project succeeds. Now, extra effort will get the manager $s + 0.5b - 0.05$; without that effort, he will get $s + 0.25b$. Thus, to elicit the extra effort, you must choose the bonus to satisfy $s + 0.5b - 0.05 \geqslant s + 0.25b$, or $0.25b \geqslant 0.05$; that is, $b \geqslant 0.2$. The bonus for success has to be $200,000! That may seem a hefty sum, but observe that the extra effort will increase the probability of the manager's receiving the bonus only from 0.25 to 0.5—that is, by 0.25—and that this extra probability will raise his expected income by just 0.25 times $200,000—that is, by $50,000, exactly enough to compensate him for the cost of the effort.

You don't want to pay the manager any more than you have to. So you want his expected earnings with the extra effort—namely, $s + 0.5b$—to equal his earnings in his current job plus the money-equivalent cost of the extra effort, or 0.10 +

0.05 = 0.15. Using $b = 0.2$, we have $s = 0.15 - 0.5 \times 2 = 0.05$; that is, \$50,000. You are now offering the manager a lower wage, but a large enough bonus for success: He will get \$50,000 if the project fails, but \$250,000 if it succeeds.

However, this situation creates a risk for the manager. He probably dislikes the prospect of facing that risk. Our discussion of insurance in Chapter 9 showed that people will pay to avoid risk, or have to be paid to bear it; here, to attract the manager to this job, you, as the owner, will have to compensate him for accepting the risk. Even more importantly, if the project fails, the manager will get only \$50,000, less than he was earning in his old job. Research in psychology and behavioral economics has shown that people are especially averse to losses measured in relation to the status quo. The manager, recognizing the prospect of such a loss, will demand sufficient compensation to take the job with your company.

Suppose you pay the manager $x > 0.1$ if the project succeeds, but $y < 0.1$ if it fails. The manager values y at less than its monetary value because of his loss aversion. To keep the calculations simple, we suppose that losses get twice as much weight as gains; for example, that the manager views getting \$90,000 after losing \$10,000 as equivalent to getting \$80,000 if loss were not an issue. To reflect this mathematically, suppose the manager regards y as equivalent to z where the loss $0.1 - z = 2 \times (0.1 - y)$, or $z = 2y - 0.1$. Now we can recalculate the contract you would need to offer. The manager expects $0.5 \times x + 0.5 \times z$ if the project succeeds and $0.25 \times x + 0.75 \times z$ if it fails. So, to bring forth his extra effort, you need $0.5x + 0.5z - 0.05 \geqslant 0.25x + 0.75z$, or $x \geqslant z + 0.2$, or $x \geqslant 2y - 0.1 + 0.2 = 2y + 0.1$, or $x - 2y \geqslant 0.1$. This is the incentive-compatibility constraint.

You also need to pay the manager enough, when he exerts the extra effort, to ensure that he receives compensation to cover his previous salary plus the cost of that effort: $0.5x + 0.5z \geqslant 0.1 + 0.05$, or $x + z \geqslant 0.3$, or $x + 2y - 0.1 \geqslant 0.3$, or $x + 2y \geqslant 0.4$. This is the participation constraint.

Adding the two constraints, we have $2x \geqslant 0.5$, or $x \geqslant 0.25$. The cheapest way to satisfy the two constraints is then to set $x = 0.25$ and $y = 0.075$; that is, to pay the manager $75,000 if the project fails and $250,000 if it succeeds.

Note that, compared with the contract that does account for loss aversion, you must pay the manager more after failure ($75,000 rather than $50,000), but the same after success ($250,000). The reason for the difference is that the manager views getting $75,000 *after losing $25,000* as equivalent to getting $50,000 without any loss. In this way, the manager’s loss aversion imposes an extra cost on you.

B. Insurance Provision

Suppose you own a precious pearl necklace worth $100,000. Your parents taught you to be careful with such prized possessions, so you make a habit of storing the necklace in a bank safe-deposit vault when you are not wearing it, and you are constantly on guard against potential thieves in your vicinity whenever you wear it. Given all these precautions, your probability of losing the necklace is only 1% per year, compared with 6% if you took no precautions. So your expected monetary loss is only $1,000, but you are averse to the risk and the loss, and you would be willing to pay much more—say, $3,000—to insure the necklace. Insurance companies pool the independent risks of many customers like you, as we explained in Chapter 9. The premium for this coverage in the market could therefore be much less than your willingness to pay—say, $1,500—which is a 50% margin above what the company expects to pay out for lost necklaces.

Once you have the coverage, however, your incentive to be careful with the necklace is reduced. It costs you time and effort to make all those trips back and forth to the bank, and the anxiety of being constantly on guard when wearing it detracts from your enjoyment of the social occasions when you want to look your best and most relaxed. Suppose that all those inconveniences add up to the equivalent of $500 in extra cost to you. So long as you remain uninsured, you will naturally prefer to be careful, since the extra $500 cost is much less than the $5,000 saved on average by reducing the probability of loss from 6% to 1%. However, once insurance has got you covered, why bother?

Now, think again about the insurance company's perspective on this situation. Once you are no longer careful, the probability of loss increases to 6%, and the insurance company will have to pay out $6,000, on average, giving it an expected loss of $4,500 (before other expenses) if it charges the lower $1,500 premium we calculated earlier. Consequently, the insurance company cannot afford to insure the necklace for less than $6,000, but at that price, you would rather not buy insurance at all.[13]

The insurance contract could offer you coverage conditional on your being careful with the necklace, if your care was observable. Perhaps the insurance company could get documented evidence of your trips to the bank. But in most situations, your caution would be hard to document. Some other solution has to be found.

The usual solution to this problem is to offer partial insurance coverage, which leaves you to bear part of the loss and so gives you sufficient incentive to avoid it. The insurance contract might include a deductible (e.g., stating that you are responsible for the first $40,000 of losses, while the company covers the rest) or co-insurance (e.g., stating that the company covers only 60% of the loss). For a single item of known value, a deductible and a co-insurance payment amount to the same thing. More generally, they can work differently, but we must leave those differences to more advanced treatments of the subject.

Consider a partial insurance contract like the one just described. The premium for 60% coverage will be proportionately smaller than that for full coverage (0.6 × $1,500 = $900), and your willingness to pay for such coverage will also be proportionately smaller (0.6 × $3,000 = $2,400). Now, you have two kinds of choices: (i) whether to buy the partial coverage at a cost of $900 and (ii) whether to be careful. If you buy the partial insurance and choose to be careless, you will bear $40,000 of the loss 6% of the time, for an expected monetary loss of $2,400. Because being careful costs you only $500, you will clearly choose to be careful. Anticipating this, the insurance company can predict that it will only have to pay out $60,000 with probability 1%. Overall, then, both you and the insurance company benefit from agreeing to a partial insurance contract. You get coverage that is worth $2,400 to you at a cost of only $900, while the insurance company gets $900 but has to pay out only $600 on average.

Endnotes

- Most importantly, if a dispute arises, you or the manager must be able to prove to a third party, such as an arbitrator or a court, whether the manager made the stipulated effort or shirked. This requirement, often called *verifiability*, is more stringent than mere observability by the parties to the contract (you and the manager). We intend such public observability or verifiability when we use the more common term *observability*. Return to reference 12
- Presumably, you can lower your loss probability to 1% at a cost of $500 and you are willing to pay $3,000 to insure yourself completely against that remaining 1% risk. So, you cannot be willing to pay more than $3,500 for insurance. Return to reference 13

6 INCENTIVES FOR EFFORT: EVIDENCE AND EXTENSIONS

The theme of the managerial-effort incentive scheme of Section 5.A was the trade-off between giving the manager a more powerful incentive to exert extra effort and requiring him to bear more of the risk of the firm's project. This trade-off is an important consideration in practice, but it must be considered in combination with other features of the relationship between a firm and its employee. The firm has many employees, and the overall outcome for the firm depends on some combination of their actions. Most firms have multiple outputs, and each employee performs multiple tasks. Consequently, it may not be possible to describe the quality and quantity of effort simply as "good" or "bad," or outcomes simply as "success" or "failure." Moreover, the firm and its employees interact over a long time, and they work together on many projects. All of these features require correspondingly more complex incentive schemes. In this section, we outline a few such schemes and refer you to a rich body of literature for further details.[14] The mathematics of these schemes gets correspondingly complex, so we merely give you the intuitions behind them and leave formal rigorous analyses to more advanced courses.

A. Nonlinear Incentive Schemes

Suppose that a project has three possible outcomes—failure, modest success, and huge success—and that an agent's level of effort affects the likelihood of each outcome. What sort of bonus should an employer pay to encourage good effort, and how should that bonus depend on the extent of success? If the size of the bonus is proportional to the size of the success (as measured by sales, profit, or some other metric), the resulting wage schedule is referred to as a linear incentive scheme. For instance, suppose that a salesperson receives base salary s and commission c on each sale, and that we use x to denote the number of sales. The salesperson's total wage will be $w = s + cx$, which increases linearly with x. Of course, there are many other *nonlinear* ways to motivate an agent.

To explore the question of optimal nonlinear incentive design, we return to the managerial supervision example from Section 5.A, but now with three possible outcomes: zero profit (failure), \$500,000 profit (modest success), or \$1 million profit (huge success). If the manager works hard, the probability of failure is one-sixth, the probability of modest success is one-third, and the probability of huge success is one-half. If the manager shirks, these probabilities are reversed to one-half, one-third, and one-sixth, respectively.

Suppose you pay the manager a base salary s, a bonus m for modest success, and a bonus h for huge success. His expected income is $s + m/3 + h/2$ if he exerts good supervisory effort, and $s + m/3 + h/6$ if he does not. Since the money-equivalent cost of his good effort is \$50,000, or 0.05 (million dollars), the incentive-compatibility condition to induce him to exert good effort becomes $s + m/3 + h/2 > s + m/3 + h/6 + 0.05$, or $h > 0.15$. This condition does not help us fix s and m separately, only the term $s + m/3$. The rest of the solution must come from the participation condition and the manager's loss aversion. We omit those details, but you should already be able to see that the incentive scheme focuses on the outcome of huge success; it need not be concerned with modest success, and the bonus need not be proportional to the owner’s profit. What happens here is that the manager’s effort shifts some probability from failure to huge success, leaving the probability of modest success unchanged at ⅓. This example is, of course, a special case, but the point is that the optimal incentive design will depend on how effort changes the probabilities of various outcomes.

Nonlinear incentive schemes like this one are ubiquitous in practice. The most common form incorporates a stipulated, fixed bonus that is paid if a certain performance standard or quota is achieved. Such a *quota-bonus scheme* can create a powerful incentive if the quota can be set at such a level that an increase in the worker’s effort substantially increases the probability of meeting it. As an illustration, consider a firm that wants each salesperson to produce $1 million in sales and is willing to pay up to $100,000 for this level of performance. If it pays a flat 10% commission, each salesperson’s incremental effort in pushing sales from $900,000 to $1 million will bring him $10,000. But if the firm offers a wage of $60,000 and a bonus of $40,000 for meeting the quota of $1 million, then this last bit of effort will earn him $40,000. Thus, the quota gives the salesperson a much stronger incentive to make the incremental effort.

But the quota-bonus scheme is not without its drawbacks. The level at which the quota is set must be judged quite precisely. Suppose the firm misjudges and sets the quota at $1.2 million, and the salesperson knows that the probability of reaching that level of sales, even with superhuman effort, is quite small. The salesperson may then give up, make very little effort, and settle for earning just the base salary. The salesperson’s resulting sales may fall far short of even $1 million. (Conversely, the pure quota-bonus scheme we outlined above gives him no incentive to go beyond the $1 million level.) Finally, the quota must be applied over a specific period, usually a calendar year. This requirement produces even more perverse incentives. A salesperson who has bad luck in the first few months of a year may realize that he has no chance of making his quota and take things easy for the rest of the year. If, in contrast, he has very good luck and meets the quota by July, again, he has no incentive to exert himself for the rest of the year. Or he may be able to manipulate the scheme by conspiring with his customers to shift sales from one year to another to improve his chances of making the quota in both years. A linear scheme is less open to such manipulation.

Therefore, firms usually combine a quota-bonus scheme with a more graduated linear incentive scheme. For example, a salesperson may get a base salary, a low rate of commission for sales between \$500,000 and \$1 million, a higher rate of commission for sales between \$1 million and \$2 million, and so on. Managers of mutual funds, for example, are rewarded for good performance over a calendar year. Their rewards come from their firms in the form of bonuses, but also from the public when they invest more money in those managers' specific funds.

B. Incentives in Teams

Rarely do the employees of a firm work as individuals on separate tasks. Salespeople working in distinct assigned regions come closest to being so separate, although even in their case, the performance of an individual salesperson is affected by the support of others in the office. Usually people work in teams, and the outcome for the team, and for each of its members, depends on the efforts of all. A firm's profit as a whole, for example, depends on the performance of all of its workers and managers. This interaction creates special incentive-design problems.

When one worker's earnings depend on the profit of the firm as a whole, that worker will see only a weak link between his effort and the aggregate profit. Because each worker in a large firm gets only a small fractional share of the aggregate profit, each has a very weak incentive to exert extra effort. Even in a smaller team, each member will be tempted to shirk and become a free rider on the efforts of the others. This outcome mirrors the prisoners' dilemma of collective action that we saw in the street-garden game of Chapters 3 and 4, and throughout Chapter 10. If the team is small and stable over a sufficiently long time, we can expect its members to resolve the dilemma by devising internal and perhaps nonmonetary schemes of rewards and punishments like the ones we saw in Chapter 10, Section 3.

In another context, the existence of many workers on a team can sharpen incentives. Suppose a firm has many workers performing similar tasks, perhaps selling different items from the firm's product line. If there is a common (positively correlated) random component to all of the workers' sales, perhaps based on the strength of the underlying economy, then the sales of one worker relative to those of another worker are a good indicator of their relative effort levels. For example, the efforts of workers 1 and 2, denoted by x_1 and x_2, might be related to their sales, y_1 and y_2, according to the formulas $y_1 = x_1 + r$ and $y_2 = x_2 + r$, where r represents the common random component. In this case, it follows that $y_2 - y_1 = x_2 - x_1$ with no randomness; that is, the difference in observed sales will exactly equal the difference in exerted effort between workers 1 and 2. The firm employing these workers can then reward them according to their relative outcomes. This payment scheme entails no risk for the workers. The trade-off we considered in Section 5, between evoking optimal effort and sharing the profits of the firm, vanishes. Now, if the first worker has a poor sales record and tries to blame it on bad luck, the firm can respond, "Then how come this other worker achieved so much more? Luck was common to the two of you, so you must have made less effort." Of course, if the two workers can collude, they can defeat the firm's purpose, but otherwise the firm can implement a powerful incentive scheme by setting workers in competition with one another. An extreme example of such a scheme is a tournament in which the best performer gets a prize.

Tournaments also help mitigate another potential moral-hazard problem. In reality, whether a worker has met the criteria of success may itself not be easily or publicly observable. Then the owner of the firm may be tempted to claim that no one has performed well enough and that no one should be paid a bonus. A tournament with a prize that must be awarded to someone, or a given aggregate bonus pool that must be distributed among the workers, eliminates this moral hazard on the part of the principal.

C. Multiple Tasks and Outcomes

Employees usually perform several tasks for their employers. These various tasks lead to several measurable outcomes of employee effort. Incentives for providing effort to these different tasks then interact. This interaction makes mechanism design more complex for the firm.

The outcome of each of an agent’s tasks depends partly on the agent’s effort and partly on chance. That is why an outcome-based incentive scheme generally exposes the agent to some risk. If the element of chance is small, then the risk to the agent is small, and the incentive to exert effort can be made more powerful. Of course, the outcomes of different tasks are likely to be affected by chance to different extents. So if the principal considers the tasks one at a time, he will use stronger incentives for effort on the tasks that have smaller elements of chance and weaker incentives for effort on the tasks where outcomes are more uncertain indicators of the agent’s effort. But a powerful incentive for one task will divert the agent’s effort away from the other task, further weakening the agent’s performance on that task. To avoid this diversion of effort toward the task with the stronger incentive, the principal has to weaken the incentive on that task, too.

An example of this problem can be found in our own lives. Professors are supposed to do research as well as teaching. There are many accurate indicators of good research: publications in and appointments to editorial positions for prestigious journals, elections to scientific academies, and so on. By contrast, indicators of good teaching can be observed only imprecisely and with long lags. Students often need years of experience to recognize the value of what they learned in college; in the short term, they may be more impressed by showmanship than by scholarship. If these two tasks required of faculty members were considered in isolation, university administrators would attach powerful incentives to research and weaker incentives to teaching. But if they did so, professors would divert their efforts away from teaching and toward research (even more so than they would otherwise do). Therefore, the imprecise observation of teaching outcomes forces deans and presidents to offer only weak incentives for research as well.

The most cited example of a situation involving multiple tasks and outcomes occurs in school teaching. Some outcomes of teaching, such as test scores, are precisely observable, whereas other valuable indicators of education, such as the ability to work in teams or speak in public, are less accurately measurable. If teachers are rewarded on the basis of their students' test scores, they will "teach to the test," and the other dimensions of their students' education will be ignored. Such "gaming" of an incentive scheme also extends to sports. In baseball, if a hitter is rewarded for hitting home runs, he will neglect other aspects of batting (taking pitches, sacrifice bunts, etc.) that can sometimes contribute more to his team's chances of winning a game. Similarly, salespeople may sacrifice long-term customer relationships in favor of driving home a sale to meet a short-term sales quota.

If the dysfunctional effects of some incentives on other tasks are too severe, other systems of rewarding tasks may be needed. A more holistic but more subjective measure of performance—for example, an overall evaluation by the worker's boss, may be used. This alternative is not without its own problems; workers may then divert their effort into activities that find favor with the boss!

D. Incentives over Time

Many employment relationships last for a long time, and that opens up opportunities for the firm to devise incentive schemes where an employee’s performance at one time is rewarded at a later time. Firms regularly use promotions, seniority-based salaries, and other forms of deferred compensation for this purpose. In effect, workers are underpaid relative to their performance in the earlier stages of their careers with the firm and overpaid in later years. The prospect of future rewards motivates younger workers to exert good effort and also induces them to stay with the firm, thus reducing job turnover. Of course, the firm may be tempted to renege on its implicit promise of overpayment in later years; therefore, such schemes must be credible if they are to be effective. They are more likely to be used effectively in firms that have a long record of stability and a reputation for treating their senior workers well.

A different way that the prospect of future compensation can keep workers motivated is through the use of an efficiency wage. The firm pays a worker more than the going wage, and the excess is a surplus, or economic rent, for the worker. So long as the worker makes good effort, he will go on earning this surplus. But if he shirks, he may be detected, at which point he will be fired and will have to go back to the general labor market, where he can earn only the going wage.

The firm faces a mechanism-design problem when it tries to determine the appropriate efficiency wage level. Suppose the going wage is w_0, and the firm’s efficiency wage is $w > w_0$. Let the monetary equivalent of the worker’s subjective cost of making good effort be e. In each pay period, the worker has the choice of whether to make this effort. If the worker shirks, he saves e. But with probability p, the shirking will be detected. If it is discovered that he has been shirking, the worker will lose the surplus $(w - w_0)$, starting in the next pay period and continuing indefinitely. Let r be the rate of interest from one period to the next. Then, if the worker shirks today, the expected discounted present value of the worker's loss in the next pay period is $p\,(w - w_0)/(1 + r)$. And the worker loses $w - w_0$ with probability p in all future pay periods. A calculation similar to the ones we performed for repeated games in Chapter 10 and its appendix shows that the total expected discounted present value of the future loss to the worker is

$$p\left[\frac{w-w_0}{1+r} + \frac{w-w_0}{(1+r)^2} + \cdots\right] = p(w-w_0)\,\frac{1/(1+r)}{1-1/(1+r)} = \frac{p(w-w_0)}{r}.$$

To deter shirking, the firm needs to make sure that this expected loss is at least as high as the worker's immediate gain from shirking, e. Therefore, the firm must pay an efficiency wage that satisfies

$$\frac{p(w-w_0)}{r} \geq e \quad \text{or} \quad w - w_0 \geq \frac{er}{p} \quad \text{or} \quad w \geq w_0 + \frac{er}{p}.$$

The smallest efficiency wage is the one that makes this expression hold with equality. And the more accurately the firm can detect shirking (that is, the higher is p), the smaller its excess over the going wage needs to be.

A repeated relationship may also enable the firm to design a sharper incentive scheme in another way. In any one period, as we explained above, the worker's observed outcome is a combination of the worker's effort and an element of chance. But if the outcome is poor year after year, the worker cannot credibly blame bad luck year after year. Therefore, the average outcome over a long period can, by the law of large numbers, be used as an accurate measure of the worker's average effort, and the worker can be rewarded or punished accordingly.

Endnotes

- Canice Prendergast, "The Provision of Incentives in Firms," *Journal of Economic Literature*, vol. 37, no. 1 (March 1999), pp. 7–63, is an excellent survey of the theory and practice of incentive mechanisms. Prendergast gives references to the original research literature from which many of the findings and anecdotes mentioned in this section are taken, so we will not repeat the specific citations. James N. Baron and David M. Kreps, *Strategic Human Resources: Frameworks for General Managers* (New York: Wiley, 1999), is a wider-ranging book on personnel management, combining perspectives from economics, sociology, and social psychology; Chapters 8, 11, and 16 and appendixes C and D are closest to the concerns of this chapter and this book. Return to reference 14

Glossary

efficiency wage

A higher-than-market wage paid to a worker as a means of incentivizing him to exert effort. If the worker shirks and is detected, he will be fired and will have to get a lower-wage job in the general labor market.

SUMMARY

The study of *mechanism design* can be summed up as learning “how to deal with someone who knows more than you do.” Such situations occur in numerous contexts, usually in interactions involving a more informed player, called the *agent*, and a less informed player, called the *principal*, who wants to design a mechanism to give the agent the correct incentives to help the principal attain his goal.

Mechanism-design problems are of two types. In the first type, the principal creates a scheme that requires the agent to reveal information. In the second type, which involves moral hazard, the principal creates a scheme to elicit the optimal level of an unobservable action by the agent. In all cases, the principal attempts to maximize his own benefit (or payoff) function subject to the incentive-compatibility and participation constraints imposed by the agent.

Firms use information-revelation schemes in creating pricing structures that separate customers by their willingness to pay for the firm’s product. Procurement contracts are also often designed to separate projects, or contractors, according to various levels of cost. Evidence of both *price discrimination* and screening with procurement contracts can be seen in actual markets.

When facing moral hazard, employers must devise incentives that encourage their employees to provide optimal effort. Similarly, insurance companies must write policies that give their clients the right incentives to take action to reduce the probability of loss. In some simple situations, optimal contracts will be linear schemes, but in the presence of more complex relationships, nonlinear schemes may be more beneficial. Incentive schemes designed for workers in teams, or whose relationships continue over time, are correspondingly more complex than those designed for simpler situations.

KEY TERMS

agent (556)

efficiency wage (573)

incentive design (551)

mechanism design (551)

price discrimination (552)

principal (556)

principal–agent (agency) problem (556)

Glossary

mechanism design

Same as **incentive design**.

incentive design

The process that a *principal* uses to devise the best possible incentive scheme (or mechanism) in a *principal–agent problem* to motivate the agent to take actions that benefit the principal. By design, such incentive schemes take into account that the agent knows something (about the world or about herself) that the principal does not know. Also called **mechanism design**.

price discrimination

Perfect, or first-degree, price discrimination occurs when a firm charges each customer an individualized price based on willingness to pay. In general, price discrimination refers to situations in which a firm charges different prices to different customers for the same product.

principal

The principal is the less-informed player in a principal–agent game of asymmetric information. The principal in such games wants to design a mechanism that creates incentives for the more-informed player (agent) to take actions beneficial to the principal.

agent

The agent is the more-informed player in a principal–agent game of asymmetric information. The principal (less-informed) player in such games attempts to design a mechanism that aligns the agent’s incentives with his own.

principal–agent (agency) problem

A situation in which the less-informed player (principal) wants to design a mechanism that creates incentives for the more-informed player (agent) to take actions beneficial to himself (the principal).

efficiency_wage

A higher-than-market wage paid to a worker as a means of incentivizing him to exert effort. If the worker shirks and is detected, he will be fired and will have to get a lower-wage job in the general labor market.

SOLVED EXERCISES

1. Firms that provide insurance to clients to protect them from the costs associated with theft or accident must necessarily be interested in the behavior of their policyholders. Sketch some ideas for the creation of an incentive scheme that such a firm might use to deter and detect fraud or lack of care on the part of its policyholders.
2. Some firms sell goods and services either singly at a fairly high price or in larger quantities at a discount in order to increase their own profit by separating consumers with different preferences.
    1. List three examples of quantity discounts offered by firms.
    2. How do quantity discounts allow firms to screen consumers by their preferences?
3. Omniscient Wireless Limited (OWL) is planning to roll out a new nationwide broadband wireless telephone service next month. The firm has conducted market research indicating that its 10 million potential customers can be divided into two types, which they call Light users and Regular users. Light users have less demand for wireless phone service, and in particular, they seem unlikely to place any value on more than 300 minutes of calls per month. Regular users have more demand for wireless phone service generally and place a high value on more than 300 minutes of calls per month. OWL analysts have determined that the best service plans to offer to customers entail 300 minutes per month and 600 minutes per month, respectively. They estimate that 50% of customers are Light users and 50% are Regular users, and that the two types have the following willingness to pay for each type of service:

	300 minutes	600 minutes

	300 minutes	600 minutes
Light user (50%)	$20	$30
Regular user (50%)	$25	$70

OWL’ s cost per additional minute of wireless service is negligible, so the cost to the company of providing service is $10 per user, no matter which plan the user chooses.

Each potential customer calculates the net payoff (benefit minus price) that she would get from each of the service plans and buys the plan that would give her the higher net payoff, so long as this payoff is not negative. If both plans give her equal, nonnegative net payoffs, she goes for 600 minutes; if both plans give her negative net payoffs, she does not purchase. OWL wants to maximize its expected profit per potential customer.

1. Suppose the firm were to offer only the 300-minute plan, but not the 600-minute plan. What would be the optimal price to charge, and what would be the average profit per potential customer?
2. Suppose instead that the firm were to offer only the 600-minute plan. What would be the optimal price, and what would be the average profit per potential customer?
3. Suppose the firm wanted to offer both plans. Suppose further that it wanted the Light users to purchase the 300-minute plan and the Regular users to purchase the 600-minute plan. Write down the incentive-compatibility constraint for the Light users.
4. Similarly, write down the incentive-compatibility constraint for the Regular users.
5. Use the results from parts (c) and (d) to calculate the optimal pair of prices to charge for the 300-minute and 600-minute plans, so that each user type

will purchase its intended service plan. What would be the average profit per potential customer?

6. Consider the outcomes described in parts (a), (b), and (e). For each of the three situations, describe whether it is a separating outcome, a pooling outcome, or a semiseparating outcome.

4. Mictel Corporation has a world monopoly on the production of personal computers. It can make two kinds of computers: low-end and high-end. One-fifth of the potential buyers are casual users, and the rest are intensive users.

   The costs of production of the two kinds of machines, as well as the benefits gained from the two by the two types of prospective buyers, are given in the following table (all figures are in thousands of dollars):

		COST	BENEFIT FOR USER TYPE	
			Casual	Intensive
PC TYPE	Low-end	1	4	5
	High-end	3	5	8

You may need to scroll left and right to see the full figure.

   Each type of buyer calculates the net payoff (benefit minus price) that he would get from each kind of machine and buys the kind that would give him the higher net payoff, so long as this payoff is not negative. If both kinds give him equal, nonnegative net payoffs, he goes for the high-end machine; if both kinds give him negative net payoffs, he does not purchase.

   Mictel wants to maximize its expected profit.

1. If Mictel were omniscient, then, when a prospective customer came along, knowing his type, the company could offer to sell him just one kind of machine at a stated price, on a take-it-or-leave-it basis. What kind of machine would Mictel offer, and at what price, to each type of buyer?

   In fact, Mictel does not know the type of any particular buyer. It just makes its catalog available for all buyers to choose from.

2. First, suppose the company produces only the low-end machines and sells them for price x. What value of x will maximize its profit? Why?
3. Next, suppose Mictel produces only the high-end machines and sells them for price y. What value of y will maximize its profit? Why?
4. Finally, suppose the company produces both kinds of machines, selling the low-end machines for price x and the high-end machines for price y. What incentive-compatibility constraints on x and y must the company satisfy if it wants the casual users to buy the low-end machines and the intensive users to buy the high-end machines?
5. What participation constraints must x and y satisfy for the casual users to be willing to buy the low-end machines and for the intensive users to be willing to buy the high-end machines?
6. Given the constraints in parts (d) and (e), what values of x and y will maximize the expected profit when the company sells both kinds of machines? What is the company's expected profit from this policy?
7. Putting it all together, decide what production and pricing policy the company should pursue.

5. Redo Exercise S4, assuming that one-half of Mictel's customers are casual users.

6. Using the insights gained in Exercises S4 and S5, solve Exercise S4 for the general case in which the proportion of casual users is c and the proportion of intensive users is $(1 - c)$. The answers to some parts will depend on the value of c. In these instances, list all relevant cases and how they depend on c.
7. Sticky Shoe, the discount movie theater, sells popcorn and soda at its concession counter. Cameron, Jessica, and Sean are regular patrons of Sticky Shoe, and the valuations of each for popcorn and soda are as follows:

	Popcorn	Soda
Cameron	\$3.50	\$3.00
Jessica	\$4.00	\$2.50
Sean	\$1.50	\$3.50

There are 2,997 other residents of Harkinsville who see movies at Sticky Shoe. One-third of them have valuations identical to Cameron's, one-third to Jessica's, and one-third to Sean's. If a customer is indifferent between buying and not, she buys. It costs Sticky Shoe essentially nothing to produce each additional order of popcorn or soda.

1. If Sticky Shoe sets separate prices for popcorn and soda, what price should it set for each concession to maximize its profit? How much profit does Sticky Shoe make selling concessions separately?
2. What does each type of customer (Cameron, Jessica, Sean) buy when Sticky Shoe sets separate profit-maximizing prices for popcorn and soda?
3. Instead of selling the concessions separately, Sticky Shoe decides always to sell the popcorn and soda together in a combo, charging a single price for both. What single combo price would maximize its

profit? How much profit does Sticky Shoe make selling only combos?

4. What does each type of customer buy when Sticky Shoe sets a single profit-maximizing price for a popcorn and soda combo? How does this answer compare with your answer in part (b)?
5. Which pricing scheme does each customer type prefer? Why?
6. If Sticky Shoe sold the concessions both as a combo and separately, which products (popcorn, soda, or the combo) does it want to sell to each customer type? How can Sticky Shoe make sure that each customer type purchases exactly the product that it intends for them to purchase?
7. What prices—for the popcorn, soda, and the combo—would Sticky Shoe set to maximize its profit? How much profit does Sticky Shoe make selling the concessions at these three prices?
8. How do your answers to parts (a), (c), and (g) differ? Explain why.

8. Section 5.A of this chapter discusses the principal-agent problem in the context of a company deciding whether and how to induce a manager to put in extra effort to increase the chances that the project he will be managing succeeds. The value of a successful project is $1 million; the probability of success given extra effort is 0.5; the probability of success without extra effort is 0.25. The manager's wage in his current job is now $120,000, and the money equivalent of the extra effort is $60,000. The manager is loss averse, and gives money losses relative to $120,000 twice the weight of gains relative to that amount.
   1. What contract does the company offer if it does not want extra effort from the manager?
   2. What is the expected profit to the company when it does not induce extra managerial effort?

3. What contract pair (y, x)—where y is the salary paid for a successful project and x is the salary paid for a failed project—should the company offer the manager to induce extra effort?
4. What is the company's expected profit when it induces extra effort?
5. Which level of effort does the company want to induce from its manager? Why?

9. A company has purchased fire insurance for its main factory. The probability of a fire in the factory without a fire-prevention program is 0.01. The probability of a fire in a factory with a fire-protection program is 0.001. If a fire occurred, the value of the loss would be $300,000. A fire-prevention program would cost $80 to run, but the insurance company cannot observe whether or not the prevention program has been implemented without incurring extra costs.
   1. Why does moral hazard arise in this situation? What is its source?
   2. Can the insurance company eliminate the moral-hazard problem? If so, how? If not, explain why not.
10. Mozart moved from Salzburg to Vienna in 1781, hoping for a position at the Habsburg court. Instead of applying for a position, he waited for the emperor to call him, because "if one makes any move oneself, one receives less pay." Discuss this situation using the theory of games with asymmetric information, including theories of signaling and screening.
11. (**Optional**, requires calculus) You are Oceania's Minister for Peace, and it is your job to purchase war materials for your country. The net benefit, measured in Oceanic dollars, from quantity Q of these materials is $2Q^{1/2} - M$, where M is the amount of money paid for the materials.

    There is just one supplier—Baron Myerson's Armaments (BMA). You do not know BMA's cost of production. Everyone knows that BMA's cost per unit of output is constant, and that it is equal to 0.10 with probability p

= 0.4 and equal to 0.16 with probability 1 − p. Call BMA "low cost" if its per-unit cost is 0.10 and "high cost" if it is 0.16. Only BMA knows its true cost type with certainty.

In the past, your ministry has used two kinds of purchase contracts: cost plus and fixed price. But cost-plus contracts create an incentive for BMA to overstate its cost, and fixed-price contracts may compensate the firm more than is necessary. You decide to offer a menu of two possibilities:

Contract 1: Supply us quantity Q_1, and we will pay you money M_1.

Contract 2: Supply us quantity Q_2, and we will pay you money M_2.

The idea is to set Q_1, M_1, Q_2, and M_2 such that a low-cost BMA will find contract 1 more profitable, and a high-cost BMA will find contract 2 more profitable. If another contract is exactly as profitable, a low-cost BMA will choose contract 1, and a high-cost BMA will choose contract 2. Further, regardless of its cost, BMA will need to receive at least zero economic profit in any contract it accepts.

1. Write expressions for the profit of a low-cost BMA and a high-cost BMA when it supplies quantity Q and is paid M.
2. Write the incentive-compatibility constraints to induce a low-cost BMA to select contract 1 and a high-cost BMA to select contract 2.
3. Give the participation constraints for each type of BMA.
4. Assuming that each of the BMA types chooses the contract designed for it, write the expression for

Oceania's expected net benefit.

Now your problem is to choose Q_1, M_1, Q_2, and M_2 to maximize the expected net benefit found in part (d) subject to the incentive-compatibility (IC) and participation constraints (PC).

5. Assume that $Q_1 > Q_2$, and further assume that constraints IC_1 and PC_2 bind—that is, they will hold with equalities instead of weak inequalities. Use these constraints to derive lower bounds on your feasible choices of M_1 and M_2 in terms of Q_1 and Q_2.
6. Show that when IC_1 and PC_2 bind, IC_2 and PC_1 are automatically satisfied.
7. Substitute out for M_1 and M_2, using the expressions found in part (e) to express your objective function in terms of Q_1 and Q_2.
8. Write the first-order conditions for maximization, and solve them for Q_1 and Q_2.
9. Solve for M_1 and M_2.
10. What is Oceania's expected net benefit from offering this menu of contracts?
11. What general principles of screening are illustrated in the menu of contracts you found?

12. (**Optional**) Revisit Oceania's problem in Exercise S11 to see how the optimal menu of contracts found in that problem compares with some alternative contracts:
    1. If you decided to offer a single fixed-price contract that was intended to attract only the low-cost BMA, what would it be? That is, what single (Q, M) pair would be optimal if you knew BMA had a low cost?
    2. Would a high-cost BMA want to accept the contract offered in part (a)? Why or why not?
    3. Given the probability that BMA has a low cost, what would the expected net benefit to Oceania be from offering the contract in part (a)? How does your

answer compare with the expected net benefit from offering the menu of contracts you found in part (j) of Exercise S11?

4. What single fixed-price contract would you offer to a high-cost BMA?
5. Would a low-cost BMA want to accept the contract offered in part (d)? What would its profit be if it did?
6. Given your answer in part (e), what would be the expected net benefit to Oceania from offering the contract in part (d)? How does your answer compare with the expected net benefit from offering the menu of contracts you found in part (j) of Exercise S11?
7. Consider the case in which an industrial spy within BMA has promised to divulge the true per-unit cost, so that Oceania can offer the optimal single, fixed-price contract geared toward BMA’ s true type. What would Oceania’ s expected net benefit be if it knew that it was going to learn BMA’ s true type? How does your answer compare with those you found in parts (c) and (f) of this exercise and in part (j) of Exercise S11?

UNSOLVED EXERCISES

1. What problems of moral hazard and/or adverse selection arise in your dealings with each of the following? In each case, outline some appropriate incentive schemes and/or signaling and screening strategies to cope with these problems. No mathematical analysis is expected, but you should state clearly the economic reasoning underlying your suggested strategies.
   1. Your financial adviser tells you what stocks to buy or sell.
   2. You consult a real-estate agent when you are selling your house.
   3. You visit your doctor, whether for routine checkups or treatments.
2. MicroStuff is a software company that sells two popular applications, WordStuff and ExcelStuff. It doesn’t cost anything for MicroStuff to make each additional copy of its applications. MicroStuff has three types of potential customers, represented by Ingrid, Javiera, and Kathy. There are 100 million potential customers of each type, whose valuations for each application are as follows:

	WordStuff	ExcelStuff
Ingrid	100	20
Javiera	30	100
Kathy	80	0

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   1. If MicroStuff sets separate prices for WordStuff and ExcelStuff, what price should it set for each application to maximize its profit? How much profit does MicroStuff earn with these prices?

2. What does each type of customer (Ingrid, Javiera, Kathy) buy when MicroStuff sets separate profit-maximizing prices for WordStuff and ExcelStuff?
3. Instead of selling the applications separately, MicroStuff decides always to sell WordStuff and ExcelStuff together in a bundle, charging a single price for both. What single price for the bundle would maximize its profit? How much profit does MicroStuff make selling its software only in bundles?
4. What does each type of customer buy when MicroStuff sets a single profit-maximizing price for a bundle containing WordStuff and ExcelStuff? How does this answer compare with your answer in part (b)?
5. Which pricing scheme does each customer type prefer? Why?
6. If MicroStuff sold the applications both as a bundle and separately, which products (WordStuff, ExcelStuff, or the bundle) would it want to sell to each customer type? How can MicroStuff make sure that each customer type purchases exactly the product that it intends for them to purchase?
7. What prices—for WordStuff, ExcelStuff, and the bundle—would MicroStuff set to maximize its profit? How much profit does MicroStuff make selling the products at these three prices?
8. How do your answers to parts (a), (c), and (g) differ? Explain why.

3. Consider a managerial effort example similar to the one in Section 5. The value of a successful project is \$420,000; the probabilities of success are 0.5 with good supervision and 0.25 without. The manager's expected payoff equals his expected income minus the cost of his effort. His current job pays \$90,000, and his cost for exerting the extra effort for good supervision on your project is \$100,000.
    1. Show that inducing extra effort would require the firm to offer a compensation scheme with a negative

base salary; that is, if the project fails, the manager pays the firm an amount stipulated in the scheme.
2. How might a negative base salary be implemented in reality?
3. Show that if a negative base salary is not feasible, then the firm does better to settle for low pay and no extra effort.

4. Cheapskates is a very minor-league professional hockey team. Its facilities are large enough to accommodate all of the 1,000 fans who might want to watch its home games. It can provide two kinds of seats—ordinary and luxury. There are also two types of fans: 60% of the fans are blue-collar fans, and the rest are white-collar fans. The costs of providing each kind of seat and the fans' willingness to pay for each kind of seat are given in the following table (measured in dollars):

		Cost	WILLINGNESS TO PAY	
			Blue-Collar	White-Collar
SEAT TYPE	Ordinary	4	12	14
	Luxury	8	15	22

You may need to scroll left and right to see the full figure.

Each fan will buy at most one seat, depending on the consumer surplus he would get (maximum willingness to pay minus the actual price paid) from each kind. If the surplus for both kinds is negative, then he won't buy any. If at least one kind gives him a nonnegative surplus, then he will buy the kind that gives him the larger surplus. If the two kinds give him an equal nonnegative surplus, then the blue-collar fan will buy an ordinary seat, and the white-collar fan will buy a luxury seat.

The team owners provide and price their seating to maximize profit, measured in thousands of dollars per game. They set a price for each kind of seat, sell as many tickets as are demanded at these prices, and then provide the numbers of seats of each kind for which the tickets have sold.

1. First, suppose the team owners can identify the type of each individual fan who arrives at the ticket window (presumably by the color of his collar) and can offer him just one kind of seat at a stated price, on a take-it-or-leave-it basis. What is the owners' maximum profit, π^*, under this system?
2. Now, suppose that the owners cannot identify the type of any individual fan, but they still know the proportion of blue-collar fans. Let the price of an ordinary seat be X and the price of a luxury seat be Y. What are the incentive-compatibility constraints that will ensure that the blue-collar fans buy the ordinary seats and the white-collar fans buy the luxury seats? Graph these constraints on an X- Y coordinate plane.
3. What are the participation constraints for the fans' decisions on whether to buy tickets at all? Add these constraints to the graph in part (b).
4. Given the constraints you found in parts (b) and (c), what prices X and Y maximize the owners' profit, π_2, under this price system? What is π_2?
5. The owners are considering whether to set prices so that only the white-collar fans will buy tickets. What is their profit, π_w, if they decide to cater to only the white-collar fans?
6. Comparing π_2 and π_w, determine the pricing policy that the owners will set. How does their profit achieved from this policy compare with the case with full information, where they earn π^*?

   7. What is the "cost of coping with the information asymmetry" in part (f)? Who bears this cost? Why?
5. Redo Exercise U4 above, assuming that 10% of the fans are blue-collar fans.
6. Using the insights you gained in Exercises U4 and U5, solve Exercise U4 for the general case where a fraction B of the fans are blue-collar and a fraction $(1 - B)$ are white-collar. The answers to some parts will depend on the value of B. In these instances, list all relevant cases and how they depend on B.
7. In many situations, agents exert extra effort in order to get promoted to a better-paid position, where the reward for that position is fixed and where agents compete among themselves for those positions. Tournament theory considers a group of agents competing for a fixed set of prizes. In this case, all that matters for winning is one's performance relative to that of the others, rather than one's absolute level of performance.
   1. Discuss the reasons why a firm might wish to employ the tournament scheme described above. Consider the effects on the incentives of both the firm and its employees.
   2. Discuss the reasons why a firm might *not* wish to employ the tournament scheme described above.
   3. State one specific prediction of tournament theory and provide an example of empirical evidence in support of that prediction.
8. Repeat Exercise S8 with the following adjustments: Due to the departure of some of the company's brightest engineers, the probability of the project's success with extra managerial effort is only 0.4, and the probability of its success without extra managerial effort is reduced to 0.24.
9. (**Optional**) A teacher wants to find out how confident his students are about their own abilities. He proposes the following scheme: "After you answer this question, state your estimate of the probability that you are right. I

will then check your answer to the question. Suppose you have given the probability estimate x. If your answer is actually correct, your grade will be $\log(x)$. If your answer is incorrect, it will be $\log(1 - x)$." Show that this scheme will elicit the students' own truthful estimates—that is, if the truth is p, show that a student's stated estimate $x = p$.

10. **(Optional)** Repeat Exercise S11, but assume that the probability that BMA has a low cost is 0.6.
11. **(Optional)** Repeat Exercise S11, but assume that a low-cost BMA has a per-unit cost of 0.2, and a high-cost BMA has a per-unit cost of 0.38. Let the probability that BMA has a low cost be 0.4.
12. **(Optional)** Revisit the situation in which Oceania is procuring arms from BMA (see Exercise S11). Now consider the case in which BMA has three possible cost types: c_1, c_2, and c_3, where $c_3 > c_2 > c_1$. BMA has cost c_1 with probability p_1, cost c_2 with probability p_2, and cost c_3 with probability p_3, where $p_1 + p_2 + p_3 = 1$. In what follows, we will say that BMA is of type i if its cost is c_i, for $i = 1, 2, 3$.

    You offer a menu of three possibilities: "Supply us quantity Q_i, and we will pay you M_i," for $i = 1, 2$, and 3. Assume that more than one contract is equally profitable, so that a BMA of type i will choose contract i. To meet the participation constraint, contract i should give a BMA of type i nonnegative profit.

    1. Write an expression for the profit of a type i BMA when it supplies quantity Q and is paid M.
    2. Give the participation constraints for each BMA type.
    3. Write the six incentive-compatibility constraints. That is, for each type i, give separate expressions that state that the profit that BMA receives under contract i is greater than or equal to the profit it would receive under the other two contracts.

4. Write down the expression for Oceania's expected net benefit, B. This is the objective function (what you want to maximize).

   Now your problem is to choose the three Q_i and the three M_i to maximize expected net benefit, subject to the incentive-compatibility (IC) and participation constraints (PC).

5. Begin with just three constraints: the IC constraint for type 2 to prefer contract 2 over contract 3, the IC constraint for type 1 to prefer contract 1 over contract 2, and the participation constraint for type 3. Assume that $Q_1 > Q_2 > Q_3$. Use these constraints to derive lower bounds on your feasible choices of M_1, M_2, and M_3 in terms of c_1, c_2, and c_3 and Q_1, Q_2, and Q_3. (Note that two or more of the cs and Qs may appear in the expression for the lower bound for each of the Ms.)
6. Prove that these three constraints—the two ICs and one PC in part (e)—will bind at the optimum.
7. Now prove that when the three constraints in part (e) are binding, the other six constraints (the remaining four ICs and two PCs) are automatically satisfied.
8. Substitute out for the M_i to express your objective function in terms of the three Q_i only.
9. Write the first-order conditions for maximization, and solve for each of the Q_i. That is, take the three partial derivatives $\partial Q_i/\partial B$, set them equal to zero, and solve for Q_i.
10. Show that the assumption made above, $Q_1 > Q_2 > Q_3$, will be true at the optimum if

$$\frac{c_3 - c_2}{c_2 - c_1} > \frac{p_1 p_3}{p_2}.$$

15■ Auctions, Bidding Strategy, and Auction Design

THE BIDDING was hot at Christie’s auction house in midtown Manhattan when Andy Warhol’s *Colored Mona Lisa* (1963) came up for sale in May 2015. The painting had been expected to sell “in the region of $35 million,” but the bidding quickly blew past that mark to a final price of $50 million, as at least three bidders outdid one another to secure this iconic masterwork.[1] Elsewhere, around the same time, a different sort of bidding contest was under way, as college students enjoyed what the *Wall Street Journal* called “one of the strongest graduate hiring seasons in recent memory.”[2] More and more graduating students were receiving multiple job offers, which gave them the opportunity to pit potential employers against one another in auctions of their own.

Anything and everything can be (and usually has been) auctioned, even gold medals honoring world-class achievement. In 2014, *USA Today* reported that Olympic gold medals “are hitting the market in record number,” most selling for about $10,000 and a famous few commanding prices well over a million.[3] Even closer to our hearts as game theorists, the gold medal celebrating John Nash’s 1994 Nobel Prize in Economics came up for auction in October 2019; the winner paid $735,000.[4]

In this chapter, we explore auctions from many angles and from the perspective of both bidders, who are competing to secure a scare resource, and auction designers, those who are in control of the scarce resource. We first clarify what is meant by the term *auction* and provide examples that illustrate some of the wide variety of interactions that can be viewed as auctions. We also consider the importance of information in auctions and some interesting auction-related phenomena, such as the *winner's curse*. Finally, we delve into both bidding strategy and auction design, considering optimal behavior across a range of different auction types.

Endnotes

- See Judd Tully, “Christie’s Scores 4th Biggest Art Auction Ever, Pulls in $1.36 Billion in 2 Nights,” Blouin Art Info International, May 14, 2015, available at https://www.blouinartinfo.com/news/story/1157180/christies-scores-4th-biggest-art-auction-ever-pulls-in-136. A video of the live bidding for *Colored Mona Lisa* is available on Christie’s Web site at http://www.christies.com/lotfinder/paintings/andy-warhol-colored-mona-lisa-5896014-details.aspx#features-videos. Return to reference 1
- Lindsay Gellman, “The Workers Who Say ‘Thanks, but No Thanks’ to Jobs: Recruiters Complain College Hires Are Leaving Them in the Lurch,” *Wall Street Journal*, July 14, 2015. Return to reference 2
- According to Ingrid O’Neil, an auctioneer at RR Auction in Boston who auctioned 14 gold medals in September 2014, “[A] bronze medal from the Summer Olympics should bring $5,000–$6,000, a silver $8,000, and a gold $10,000.” On the other hand, one of Jesse Owens’s gold medals from 1936 sold in 2013 for $1.46 million. See Karen Rosen, “Olympic Medals Hit the Market in Record Number,” *USA Today: Sports*, September 16, 2014. Return to reference 3
- The three authors had the privilege of writing a brief essay on Nash’s Nobel-winning contributions for publication in Christie’s catalog for this auction. See “Gold Medals for Games” by Avinash Dixit, David McAdams, and Susan Skeath in Christie’s *Fine Printed Books & Manuscripts including Americana: New York 25*, October 2019 [Auction catalog] (New York: Christie’s Auction House, 2019), pp. 38–39. Return to reference 4

1 WHAT ARE AUCTIONS?

An auction is a game between players competing for a scarce resource. The players in an auction are called bidders, the item they compete for is the *object*, and the action that a bidder takes to try to win the object is his *bid*. The benefit that a bidder gets when winning the object is his valuation (also called *value* or *willingness to pay*) for the object. Each bidder's strategy in the auction game is referred to as his *bidding strategy*. The cost that each bidder incurs within the auction game is referred to as his *payment*. In many, but not all, auctions, there is another player, referred to as the auction designer, who sets the rules of the game that the bidders play. There is also sometimes yet another player, the *auctioneer*, who runs the auction according to the rules laid down by the auction designer.

Real-world auctions come in so many forms and formats that studying them can be a bit bewildering for those new to the subject. We are used to thinking about auctions as mechanisms by which an object is sold, but there are also auctions in which objects are purchased (procurement auctions). Sometimes a single object is being bought or sold (single-object auctions). In other cases (multi-unit auctions), multiple identical objects are bought or sold (e.g., Treasury-bond auctions used to fund U.S. debt), and in still others (combinatorial auctions), combinations of non-identical objects are bought or sold (e.g., the consumer-goods company Procter & Gamble uses a combinatorial procurement auction to meet some of its complex procurement needs[5]).

Even if we restrict our attention to auctions of a single object—as we will do throughout most of this chapter—auctions can take seemingly countless forms in practice. Some are noisy affairs with yelling bidders[6] and fast-talking auctioneers (e.g., classic-car auctions), while others are completely silent, with bidders dropping off their bids in sealed envelopes. Some happen in places where bidders are able to inspect the merchandise (e.g., art auctions) while in others, bids are submitted over the Internet without any need for bidders to congregate. Some last for several days (e.g., eBay auctions) while others are over in seconds (e.g., fish auctions) or even milliseconds (e.g., Google Adwords auctions to determine which ads are placed next to Google search results).

The variety of games that can be interpreted as auctions is enormous, but not every game played to allocate a scarce resource is an auction. The first key feature of any auction is that bidders must be able to do something to increase their likelihood of winning the resource. When scientists compete for federal funding, they submit research proposals explaining the importance of their work, but cannot do anything to influence whether they win funding. The contest among scientists for funding is therefore not an auction—there is *no bidding*. The second key feature of any auction is that bidders must compete directly against one another. In real-estate markets without a lot of buyers, sellers typically interact with potential buyers one at a time, deciding whether to accept the current buyer's terms or wait for the next buyer to come along. Such home sales are therefore not auctions—there is *no direct competition*. As long as there is bidding and direct competition among bidders, the game being played will qualify as an auction. In the remainder of this section, we describe examples of interactions that can be analyzed as auctions, the formats that auctions can take, and the role of asymmetric information in auctions.

A. More Than Just Buying and Selling

The auction concept is extraordinarily broad. It includes situations like the previous examples in which several bidders would each like to buy or sell something—what people typically refer to as "auctions" in everyday language—but also many other sorts of interactions in which there is some conflict or competition over a scarce resource.

Consider the game that firms play when competing to hire a new employee named Anne. Suppose that Anne has been successful in her job search, so much so that three firms plan to make her an offer. Anne tells the hiring manager at each company that she would love to work there, but two other firms with equally attractive jobs are also trying to hire her. Faced with this tough decision, Anne commits to sign with whichever firm offers the highest salary. (Anne also commits not to renegotiate. Each firm's salary offer will have to be its best-and-final offer.)

This hiring contest can be interpreted as an auction in which the object up for bid is Anne herself, a future employee, and she is the auction designer. The bidders are the firms looking to hire her. Because firms make their bids without knowing what others have bid, this auction game has simultaneous moves. Auctions with simultaneous moves are called *sealed-bid auctions*. Several sorts of sealed-bid auctions are commonly used in practice, with different rules for how much the winning bidder needs to pay. The auction format that Anne is using is known as a *first-price auction* because the winning firm pays its own bid. We examine first-price auctions, along with auctions that use different rules for winning payments, in more detail later in this section.

Consider next a sales contest. Adam and Bengt are salespeople in an office that pays the employee with the most sales a bonus at the end of each year. How much each of them sells depends on how much time he spends pursuing new clients. Whoever puts in the most time will take home the bonus, but there's no way for either of them to observe how hard the other is working.

This sales contest can be interpreted as an auction where the bidders are the salespeople, the object is the bonus, and bids correspond to the amount of time spent selling. (The employer is the auction designer here, using the bonus system to encourage each salesperson to pursue new clients.) As in the previous example, bids are unobservable to other bidders, so this is a sealed-bid auction. However, in this case, everyone pays his bid, even the losers. This type of auction is known as an *all-pay auction*. We study all-pay auctions in more detail in Section 3.E.

B. Auction Formats

Auctions are broadly divided into two main categories: sealed-bid auctions, in which each bidder must decide what to bid without being able to observe anything about others' bids, and open-outcry auctions, in which the auction game plays out publicly and bidders can decide what to do at each moment based on what has happened so far. Within the category of sealed-bid auctions, three different formats—based on different rules for how much the winning bidder must pay—are used. We saw the first-price format above in our hiring contest example and the all-pay format in our sales contest example. The third format is a *second-price auction*, in which the winning bidder pays not his own bid, but the highest of the losing bids. Within the category of open-outcry auctions, there are also three different formats: the traditional *ascending-price auction*, in which offered prices rise over time, a similar *descending-price auction*, in which offered prices fall over time, and the *war of attrition*, which is the open-outcry analogue to an all-pay auction. Figure 15.1 summarizes the most notable auction formats in each category, specifying who pays what in each format.

Who pays what?	**Sealed bid**	**Open outcry**
Winner pays own bid. Losers do not pay.	First-price auction [Section 3.C]	Descending-price (“Dutch”) auction [Section 3.B]

Winner pays (exactly or approximately) the highest losing bid. Losers do not pay.	Second-price auction [Section 3.D]	Ascending-price ("English") auction [Section 3.A]
Every bidder pays.	All-pay auction [Section 3.E]	War of attrition [Section 3.F]

FIGURE 15.1 Notable Auction Formats

C. Information in Auctions

Despite the extraordinarily wide variety of auction formats in practice, auction theorists have identified a relatively short list of key features of auction environments that play important roles in determining how sellers should design the auction and how bidders should approach their bidding problem. The most fundamental and important feature of an auction is one that you might not expect: *information*. In particular, what is most important is whether individual bidders possess private information (something they know that others do not know) about the object that is up for auction. Although the levels and types of information available to bidders in auctions can vary widely, three specific cases are of particular interest: (i) the object has an *objective value* known to all bidders; (ii) bidders have *private values* for the object; and (iii) bidders have *common values* for the object.

I. OBJECTIVE VALUEAn item being auctioned has an objective value if each bidder assigns the same value to the object and this value is known to all bidders. For instance, imagine that a game-theory professor holds an auction in class, with students as the bidders, but with a strange twist—she is going to auction off \$20. She announces that she will be conducting an auction to sell a single \$20 bill (definitely not counterfeit). Anyone can yell out a bid at any time, but each bid must be in whole dollars, must be at least \$1, and must be at least \$1 more than the previous high bid.

What would happen? There would be confusion at first, probably, as the students slowly adjusted to the strange notion of bidding money to win money. However, eventually the auction would begin with some smart aleck in the back yelling out, "I'll bid \$1!" But it wouldn't end there. Paying just $1 for a $20 bill is such a good deal that someone else would inevitably cry out a higher bid. And on and on, until eventually the price would get up to $19 or $20—or, if there’s another smart aleck out there looking for a laugh, perhaps $21.

No matter what, we would expect the auction to fetch approximately $20 for the $20 bill being sold. This may seem obvious, but there’s a deeper point here. Imagine that you have a stamp or coin collection—something that you know must be valuable but that you cannot precisely value on your own. The beauty of holding an auction in such a situation is that, as a seller, you don’t need to know the object’s worth! As long as bidders know how much it’s worth (and don’t collude to keep the price low), bidding competition among them will tend to drive the price up to a level reflecting its true value.

II. PRIVATE VALUESBidders in an auction have private values if (1) each bidder has private information—something he knows that others do not know—about how much the object is worth to him and if (2) knowing other bidders’ private information would not change any bidder’s willingness to pay (*valuation*) for the object. For example, imagine that our game-theory professor holds a picnic at her house after the final exam, grilling steak and shrimp for all her students to enjoy. However, to make things interesting, she holds an auction to determine who will get the shrimp (and donates all proceeds to charity). Each student’s willingness to pay depends on how much they want shrimp that day (rather than steak)—this is their private information—and the auction has private values because each student’s value depends *only* on his own hunger for shrimp.

When bidders have private values, they would still like to know others’ private information. For instance, in a first-price auction for an object that you really love (and which is worth $100 to you), you would like to know how others feel about it. If others also like the object, you might need to submit a high bid (say $80 or $90) in order to have a decent chance of winning it. On the other hand, if no one else likes it, you may be able to win the object with a very low bid (say, $20). Knowing how much others want the object has no effect on how much you want it—your value is $100 no matter what they think—only on the bid you believe you need to make to win. This is a defining feature of auctions with private values: bidders' value assessments do not depend on others' private information.

III. COMMON VALUESBidders in an auction have common values when the object to be auctioned has the same value to all of them, but each bidder has only an imprecise estimate of that value. In an auction with common values, bidders would all agree about the object's value *if* they all had the same information. However, bidders in such auctions do not have the same information, since each bidder is privy only to his own estimate of the object's value.

In an auction with common values, the winner is typically the bidder who has most *overestimated* the value of the object. For instance, suppose that a jar full of pennies is being auctioned. Each bidder is allowed to inspect the jar to make his own best guess of its value. Even if the bidders' guesses are correct *on average*, the highest estimate across all the bidders is often much higher than the true value. Unless bidders submit bids that are much lower than their own estimates, the winner therefore tends to pay more for the jar than it is actually worth. This tendency of bidders to overpay in common-value auctions, which we study in detail in the next section, is known as the *winner's curse*.

Endnotes

- In 2006, Procter & Gamble vice president for global purchases Rick Hughes and coauthors reported that P&G was spending about $3 billion each year in combinatorial procurement auctions and saving an estimated $300 million per year relative to what it had previously been paying for the same goods and services. See Tuomas Sandholm et. al., "Changing the Game in Strategic Sourcing at Procter & Gamble: Expressive Competition Enabled by Optimization," *Interfaces*, vol. 36, no. 1, pp. 55–68. Return to reference 5
- Mark Hinog, "Car Auction Has the Loudest '69' We've Ever Heard Someone Scream," *sbnation.com*, May 20, 2017, available at https://www.sbnation.com/lookit/2017/5/20/15669582/mecam-auto-auctions-69-yell-nice-nice-niiiiiiiiice. Return to reference 6

Glossary

auction
: A game in which multiple players (called bidders) compete for a scarce resource.

bidder
: A player in an auction game.

valuation
: The benefit that a bidder gets from winning the object in an auction.

auction designer
: A player who sets the rules of an auction game.

procurement auction
: An auction in which multiple bidders compete to supply an item. Bids in a procurement auction are prices that bidders are willing to receive to supply the good. The lowest bidder wins and is paid her bid.

multi-unit auction
: An auction in which multiple identical objects are sold.

combinatorial auction
: An auction of multiple dissimilar objects in which bidders are able to bid on and win combinations of objects.

sealed-bid auction
: An auction mechanism in which bids are submitted privately in advance of a specified deadline, sometimes in sealed envelopes.

open-outcry auction
: An auction mechanism in which bids are made openly for all to hear or see.

private information
: Information known by only one player.

objective value
: An auction is called an objective-value auction when the object up for sale has the same value to all bidders and

each bidder knows that value.

private value

An auction is called a private-value auction when each bidder has private information about their own valuation of the object up for sale, but knowing others’ private information would not change any bidder’s own willingness to pay for the object. An important special case is when each bidder knows their own valuation but others do not.

common value

An auction is called a common-value auction when the object up for sale has the same value to all bidders, but each bidder knows only an imprecise estimate of that value.

single-object auction

An auction in which a single indivisible object is sold.

2 THE WINNER’ S CURSE

The winner’ s curse is a phenomenon often experienced in common-value auctions whereby the winner of the auction pays more (on average) than the object is worth. The winner’ s curse arises when a bidder fails to take account of the fact that, when she wins, she is likely to have made an overly optimistic estimate of the object’ s value. Fortunately, the name is not entirely apt, because the “curse” can be avoided once you understand it. The goal of this section is to help you understand the winner’ s curse well enough to avoid it.

In the early 1980s, two economics professors, Max Bazerman and William Samuelson, ran an auction experiment with MBA students at Boston University.[7] In each of 12 class sessions, students (from 34 to 54 per class) participated in a first-price auction for a jar full of pennies, nickels, or other small items such as paper clips with an assigned value. Students would first write down their best guesses about the value of the items in the jar, then submit their bids. Results were revealed at the end of the term, and the winner of each auction was paid the value of the items in the jar (in cash, not pennies or paper clips) minus their winning bid. As it turned out, students systematically underestimated the value of the jars being auctioned, guessing, on average, that the jars had a value of \$5.13, whereas in fact each jar was worth exactly \$8. Nonetheless, the average winning bid was \$10.01—\$2.01 *more* than the actual value. So, at the end of the term, the winning students wound up paying the professors: a winner’ s curse!

What happened? Even though students’ guesses about the value of the jar were, on average, on the low side, each class was big enough that there were always a few students who overestimated the jar’ s value by a substantial amount.

Because these students had so overestimated the jar's value, they were at risk of bidding, and hence paying, more than its true value. Thus, an auction's winner typically is not an average bidder, but rather one who has *overestimated* the true value of the object at auction. This explains why winners of auctions with common values routinely overpay—they don't account for the fact that, when they win the auction, they are likely to have overestimated the object's value.

Consider another example from the world of business. In May 2018, PayPal paid $2.2 billion to acquire iZettle, a financial-technology company whose smartphone-connected payment systems make it easy for small businesses to securely process credit-card and debit-card payments. There's no way for us to know whether PayPal got a good deal for its $2.2 billion, but let's consider how the winner's curse *could* have been at play in this transaction. (All numbers in this example other than the $2.2 billion purchase price are made up. We have no inside information about this transaction.)

iZettle made its name with innovative devices like its 2015 credit-card reader, which connects to iPhones through the audio jack, and other technologies with the potential to transform commerce—but someone could come out tomorrow with an even cleverer approach and render iZettle's technology obsolete. The company's value could therefore be extremely high, or close to nothing at all. In the face of such uncertainty, how should PayPal think about the acquisition and what price to offer?

To keep things simple, suppose that iZettle's technology is either Strong (will continue to dominate the marketplace) or Weak (will be made obsolete in the near future). If the technology is Strong, iZettle's "stand-alone value" if it doesn't get acquired by PayPal is $6 billion, but if the technology is Weak, iZettle's stand-alone value is $0. Moreover, due to synergies with PayPal's existing businesses, iZettle's value to PayPal is 150% of its stand-alone value ($9 billion if the technology is Strong or $0 if it is Weak).

Suppose for a moment that PayPal and iZettle are equally uncertain about the technology's potential, each believing that it has a 50% chance of being Strong. iZettle's expected stand-alone value would then be 50% × $6 billion = $3 billion, while PayPal's expected value for the company would be 150% × $3 billion = $4.5 billion. This leaves plenty of room for negotiation (see Chapter 17 for more on negotiation games), as any price between $3 billion and $4.5 billion would make both sides better off.

Our analysis so far hinges on the assumption that iZettle and PayPal share the same belief about the technology's potential—that neither side has private information. But what if iZettle's founders (Jacob de Geer and Magnus Nilsson) have a better sense of their own technology than PayPal's negotiators? In particular, suppose that de Geer and Nilsson have a *feeling* about their technology: a "good feeling" half of the time that the technology's likelihood of being Strong is 80%, and a "bad feeling" the other half of the time that the technology's likelihood of being Strong is only 20%. (The overall likelihood of its being Strong is still 50% = ½ × 80% + ½ × 20%.)

To see how private information changes the game here and creates the potential for a winner's curse, imagine a scenario in which PayPal enters the negotiation with a lowball bid equal to just half of its estimate of iZettle's value. As discussed above, PayPal's estimate (based on its belief that the technology has a 50% chance of being Strong) is $4.5 billion, so its lowball offer is $2.25 billion. PayPal is expecting this initial offer to be rejected, but then iZettle accepts!

What this tells us is that *de Geer and Nilsson must have a bad feeling* about the technology. Why? If they had had a good feeling—if they had thought the technology had an 80% chance of being Strong—their estimate of the company's stand-alone value would have been 80% × $6 billion = $4.8 billion, and they would have laughed off an offer of $2.25 billion. But if they have a bad feeling, their estimate of the company's stand-alone value would be only 20% × $6 billion = $1.2 billion, and $2.25 billion would look pretty good to them! Of course, once PayPal owned the company and came to understand why de Geer and Nilsson had a bad feeling, they would wish that they hadn't acquired the company at all, since its expected value to them would then be only 150% × $1.2 billion = $1.8 billion. They would have made a lowball offer but wound up overpaying by $445 million—a winner's curse.

The mathematics associated with calculating optimal (equilibrium) bidding strategies in auctions with common values is complex, not least because other bidders are simultaneously making similar calculations. Although the math behind bidding equilibria is beyond our scope here, we can provide you with some general advice.[8] If you imagine yourself in the position of PayPal bidding for iZettle, you can see that the question, "Would I be willing to purchase this company for $2.2 billion, given what I know before submitting my bid?" is very different from the question, "Would I still be willing to purchase this company for $2.2 billion, given what I know before submitting my bid *and* given the knowledge that $2.2 billion is good enough to win?" Whether you find yourself in a one-on-one negotiation or in an auction with several competing bidders, it is the second question that reveals correct strategic thinking, because you win with any given bid only when all others bid less—and that happens only if all others have lower estimates of the value of the object than you do.

If you fail to use your game-theoretic training and do not take the winner's curse into account, you should expect to consistently lose substantial amounts. But at least you won't be alone in this misfortune, as the winner's curse is all around us. Winners of auctions for oil- and gas-drilling rights took substantial losses on their leases for many years.[9] Baseball players who as free agents went to new teams were found to have been overpaid in comparison with those who re-signed with their old teams.[10] On the other hand, if you understand the winner's curse and take it into account, you can avoid it. True, you may win less often, but at least you will truly "win" when you do.

Endnotes

- Max H. Bazerman and William F. Samuelson, "I Won the Auction but Don't Want the Prize," *Journal of Conflict Resolution*, vol. 27, no. 4 (December 1983), pp. 618–34. Return to reference 7
- See Steven Landsburg, *The Armchair Economist* (New York: Free Press, 1993), p. 175. Return to reference 8
- A comprehensive study estimated that in 1,000 lease auctions in the 1950s and 1960s, 12 of the 18 major bidders consistently overbid; the median firm in this group overbid by more than 30%, and one firm (Texaco) bid *seven times* what it ought to have bid. See Kenneth Hendricks, Robert H. Porter, and Bryan Boudreau, "Information, Returns, and Bidding Behavior in OCS Auctions: 1954–1969," *Journal of Industrial Economics*, vol. 35, no. 4 (June 1987), pp. 517–54. Return to reference 9
- James Cassing and Richard W. Douglas, "Implications of the Auction Mechanism in Baseball's Free Agent Draft," *Southern Economic Journal*, vol. 47, no. 1 (July 1980), pp. 110–21. Return to reference 10

Glossary

winner’s curse

A situation in a common-value auction where the winner fails to take account of the fact that when she wins, she is likely to have made an overly optimistic estimate of the object’s value. Bidders who correctly anticipate this possibility can avoid the winner’s curse by lowering their bids appropriately.

3 BIDDING IN AUCTIONS

In May 2016, 10 bottles of Chateau Mouton Rothschild 1945 Bordeaux sold at auction for $343,000 at Sotheby's New York, a record amount for a wine considered by many to be one of the finest ever bottled. Such eye-popping prices might turn some off to wine auctions, but according to *Bloomberg* magazine, "auctions can be a good way to find hard-to-get vintages and even to score bargains."[11] Better still, bidders in wine auctions these days can participate remotely over the Internet. So, you can bid to win that lot of Chateau Margaux 1952 Bordeaux you've been eyeing recently without leaving the comfort of your living-room couch.

Let's suppose that you actually do decide to bid for a lot of wine and that your valuation for the wine is V. How should you bid? The answer depends on what auction format is being used to sell the wine.

A. Ascending-Price (“English”) Auction

For most people, the word *auction* evokes an image of an ascending-price auction—at a traditional auction house or perhaps at an online auction site like eBay—in which bidders “call out” ever-higher prices until finally a price is reached that no one is willing to top, at which point the object is sold to the final bidder. Real-world ascending-price auctions each have their own particular rules about exactly what happens during the auction and what bidders are allowed to do. We don’t want to get bogged down in such details, and so will focus on an idealized auction game that auction theorists refer to as the “English auction”—named in honor of the English auction houses that have long dominated the world of book and art auctions.[12]

In an English auction, the price is increased continuously over time, each bidder decides when to “drop out,” and the winner is whoever stays in the auction longest. More precisely, the auction begins as the seller announces a minimal acceptable price r, known as the reserve price, and potential bidders decide whether to participate. If no one participates, the auction ends, and the object is returned unsold to the seller. If only one bidder participates, that bidder wins and pays the reserve price. Finally, in the most interesting case, when multiple bidders participate, the price increases continuously over time, and each bidder decides when to drop out of the auction. (The price at which a bidder drops out is his drop-out price.) The price stops increasing when only one bidder remains active in the auction, at which point that bidder wins and pays the final price.

Should you find yourself in an English auction having value V for the object up for sale, your optimal bidding strategy is simple: Don't participate at all if the reserve price r exceeds your value V; otherwise, enter the auction and stay in the bidding until the price reaches V. Why is this your best strategy? First, consider the question of whether you should enter the auction. If the reserve price exceeds your value and you enter, you might wind up winning, which is bad news, since you will then have to pay more than the object is worth to you. On the other hand, so long as the reserve price is less than your value, there's a chance that you will be able to win the object at a price less than your value. So, you should enter the bidding only if $r < V$. Moreover, so long as the current price p is less than your value V, you stand to gain by staying in the bidding, since there's a chance that all other bidders will drop out in the near future, leaving you to win the object at a price less than your value. If, however, $p \geqslant V$ and you stay in the auction, you run the risk of winning at a price that exceeds your value. Thus, your optimal strategy is to wait until the price reaches V and then drop out. In fact, the argument here shows that *dropping out at price $p = V$ is a weakly dominant strategy*; this choice remains optimal no matter what bidding strategies others use.

Given that all bidders remain in the bidding until the price reaches their value, what is the final price in an English auction? For concreteness, suppose that Anna has the highest value, V_A, Ben has the second-highest value, V_B, and the *bid increment*, the minimum amount that any new bid must increase the current price, is $D > 0$. So as long as the current price p is less than $V_B - D$, both Anna and Ben are willing to increase the bid by D in order to become the current high bidder. Thus, the auction cannot possibly end at any price less than $V_B - D$. On the other hand, once the current price p exceeds $V_B - D$, Ben is no longer willing to increase the bid. Therefore, the highest that the auction price can possibly go is $V_B + D$, a possibility that could arise, for instance, if Anna was the current high bidder at price $V_B - D$, Ben raised the bid to V_B, his maximum willingness to pay, and then Anna topped that with a bid of $V_B + D$, at which point the auction would end. Note that, no matter what, the final price must be between $V_B - D$ and $V_B + D$; it will be *approximately* equal to the second-highest bidder value when the bid increment is small.

The English auction presents bidders with a single, simple decision regarding when to drop out of the bidding. However, most real-world ascending-price auctions are more complex. For instance, in a Sotheby's-style art auction, bidders can engage in jump bidding (submitting a bid that is significantly higher than the previous bid, and well beyond whatever minimum bid increment exists), as a way of signaling their interest in the object and perhaps deterring others from competing further. Bidders can also mask their participation in the auction by submitting secret instructions ahead of time to the auctioneer. And the auctioneer can employ a shill bidder (someone employed to continue to outbid a legitimate bidder) to drive up the price even after only one serious bidder remains. Similarly, in eBay auctions, bidders can wait until the last moment to submit their bids, a tactic known as sniping, or they can use eBay's "proxy bidder" to continue to bid on their behalf when they submit a maximum price that exceeds the minimum bid increment. For more on these additional features of real-world ascending-price auctions and the appropriate ways to bid in such auctions, see the references provided in Section 5.

B. Descending-Price ("Dutch") Auction

The Aalsmeer Flower Auction building in the Netherlands, known as the "Wall Street of flowers," is the second-largest building in the world (second only to the Pentagon) and consists of 128 acres filled with flowers from Ecuador, Ethiopia, and everywhere in between. About 20 million flowers are sold daily at Aalsmeer. With so many flowers to sell, the auction has to be fast, and it is; each lot goes in about four seconds.[13] Bidders sit in stadium seating facing a light display that looks like a big clock, with a button (actually three buttons: red, yellow, and green) next to their seats that they use to bid. Each point on the "clock" corresponds to a price. The light starts at a high price that no one is willing to pay, then rapidly moves around the clock to lower and lower prices until someone pushes his button. The light then stops moving, the auction ends, and the bidder who pushed his button first gets the lot at that final price.

Should you find yourself bidding at Aalsmeer, or in some other descending-price (or Dutch) auction with value V for the object up for sale, how should you bid? Figuring out the optimal price to bid in a Dutch auction involves some challenging mathematics, but we can gain some reasonable insight by working through a simple numerical example.

Imagine that you are willing to pay \$100 for a lot of flowers. You've been observing the auctions at Aalsmeer for some time now, and you've noticed that the price at which bidders jump in for similar flowers ranges from \$50 to \$150 (with every price in this range being equally likely). Now the auction has begun, at a starting price of \$200, and the price is quickly dropping. When should you jump in to win? Obviously not at any price above \$100, because then you would have to pay more than the flowers are worth to you. So you wait and, lo and behold, the price falls to $100 and no one has bid yet. You could push your button to win at $100, but since you would be paying your full value for the flowers, you would walk away from the auction with zero profit! So, as the price hits $100, you should not hit your button at that moment. You should wait at least a little longer. But how long?

Let's step back for a moment and consider this bidding problem from your perspective *before the auction begins*. Even though the auction has not yet begun, you can think ahead to how the auction might proceed and decide, for each price that might be reached, whether you will want to jump in and win the object or wait and let the price fall further. Let P denote the first price at which you will want to jump into the bidding. If someone else jumps in before price P is reached, you will lose the auction and get zero profit; because others' jump-in prices range uniformly from 50 to 150, the probability that this happens is

$$\frac{150-P}{150-50}=\frac{150-P}{100}.$$

Alternatively, if no one jumps in before price P is reached, you will win the object, pay P, and get profit equal to your value $100 minus the price P; this happens with probability $\frac{P-50}{100}$. Overall, your expected profit [denoted by $E\Pi(P)$] equals the probability that you win multiplied by your profit when you win:

$$E\Pi(P)=\frac{P-50}{100}\times(100-P)=\frac{-P^2+150P-5{,}000}{100}.$$

Using the methods introduced in Chapter 5 to maximize a quadratic equation, we find that $E\Pi(P)$ is maximized at $P^* = 75$;[14] so, when you are willing to pay \$100, you should wait until the price reaches \$75 to jump in.

The price at which a bidder should optimally jump in to win the object depends on his belief about others' bidding strategies. To see the point, reconsider the previous numerical example and continue to suppose that you are willing to pay \$100, but now suppose that you believe that the price at which someone else will jump in is uniformly distributed from \$0 to \$100 (rather than from \$50 to \$150). The probability that no one else jumps in before price P is now $\frac{P}{100-0}=\frac{P}{100}$, higher than before. Your expected profit from jumping in at price P is now

$$E\Pi(P)=\frac{P}{100}\times(100-P)=\frac{-P^2+100P}{100},$$

which is maximized at $P^* = 50$.

Such calculations can be difficult to perform in practice, especially if you do not know others' bidding strategies. Even so, we can glean some basic insights from the logic behind these calculations. First, you should always wait until the price has fallen below your valuation before jumping in to win the object. Second, how far you should allow the price to drop below your value depends on how you assess the strength of the bidding competition—you should wait longer to jump in when the competition is weaker.

C. First-Price Auction

In a first-price auction, all bidders submit their bids simultaneously, and the highest bidder wins and pays his bid. First-price auctions are widely used for everything from selling stamps to buying fighter jets. Figuring out the optimal price to bid in a first-price auction involves some challenging mathematics, but as we did with the Dutch auction, we can gain some insight by working through a simple numerical example.

Imagine that you are willing to pay \$100 for a batch of stamps for sale in a first-price auction. You've been observing similar auctions for some time now,[15] and you've noticed that the winning price for similar batches of stamps ranges from \$50 to \$150 (with every price in this range being equally likely). What should you bid? Obviously not anything above \$100, because then you would have to pay more than the stamps are worth to you. And obviously not \$100, since then you would pay your full value when you win and have no profit. So, you should bid less than \$100. But how much less?

Suppose that you shade your bid (that is, bid below your value), bidding $P < 100$. Your chance of winning the auction depends on how likely it is that your P exceeds the next highest bid, which we know is at least \$50. The probability that you win when the range of winning prices is between \$50 and \$150 is then $\frac{P-50}{150-50}=\frac{P-50}{100}$, and when you win, your profit will be $100 - P$. Overall, then, your expected profit when bidding P equals

$$E\Pi(P)=\frac{P-50}{100}\times(100-P),$$

which is maximized at P = 75 (as we showed in footnote 14). So, if your value is \$100, you should shade your bid and submit a bid of \$75. As in the Dutch auction, your optimal bid will be at some price less than your true valuation that depends on how likely you are to win at that price along with the profit you would earn.

You probably noticed that what we did here in determining your optimal bid in the first-price stamp auction generated a calculation identical to the one we solved to find your optimal bid in the descending-price flower auction above. We were able to do that because the first-price auction and descending-price auction are strategically equivalent! Any game-theoretic analysis of one of these auction formats applies word for word to the other format, as long as you replace the phrase “jump-in price” with “sealed bid,” or vice versa.

To see why the descending-price auction (DPA) and first-price auction (FPA) are strategically equivalent, consider how the outcome of each game depends on players’ chosen strategies: The bidder with the highest jump-in price wins and pays their jump-in price in the DPA, while the bidder with the highest sealed bid wins and pays their sealed bid in the FPA. So, if we were to create a game table for each auction, they would look exactly the same—the only difference being that strategies are called “jump-in prices” in one auction format and “sealed bids” in the other.

Of course, there may be other factors outside of the game itself that cause bidders to behave differently in descending-price auctions than they do in first-price auctions. For instance, perhaps seeing the other bidders sitting nearby causes some bidders to bid more aggressively at Aalsmeer than they would in an otherwise equivalent sealed-bid auction. These factors are obviously important practical considerations for real-world auctioneers, but outside the scope of our analysis here.

D. Second-Price Auction

In a second-price auction, all bidders submit their bids simultaneously, and the highest bidder wins and pays the second-highest bid. The second-price auction is not widely used in practice,[16] but is famous nonetheless as a special case of the Vickrey auction, which won William Vickrey the Nobel Prize in economics in 1996. (The Vickrey auction in its most general form is difficult to explain in words, but when there is only a single object for sale, it reduces to the second-price auction.)

What makes the second-price (Vickrey) auction fascinating is that *bidders have a weakly dominant strategy to submit bids equal to their true valuations* (that is, to employ truthful bidding). To see why, let's work through a simple example. Suppose you are a collector of antique china, and you have discovered that a local estate auction will be selling off a nineteenth-century Meissen Blue Onion tea set in a second-price auction. As someone experienced with vintage china but lacking this particular set for your collection, you are willing to pay up to $3,000 for it. However, you have no idea what other bidders' values for the set will be. If they are inexperienced, they may not realize the considerable value of the set. On the other hand, if they have sentimental attachments to Meissen or the Blue Onion pattern, they may value it more highly than you.

According to Vickrey's famous result, bidding truthfully (for you, bidding $3,000) is your weakly dominant strategy. To see why, suppose that you were to bid some amount $B <$ \$3,000 and let P denote the highest bid submitted by any other bidder in the auction. (We use notation P here to remind you that P is the *price* that you will pay if you wind up winning the auction.) If $P >$ \$3,000, it doesn't matter whether you bid B or \$3,000, because you will lose the auction either way. Similarly, if $P < B$, it doesn't matter whether you bid B or \$3,000, because you will win the auction and pay P either way. However, if $B < P <$ \$3,000, you win and pay P if you bid \$3,000 (for positive profit \$3,000 − P) but lose with bid B (for zero profit). So, shading your bid always leaves you worse off than if you bid truthfully. (A similar argument shows that bidding more than your value is also weakly dominated by truthful bidding.)

Vickrey's remarkable finding that truthful bidding is a dominant strategy in second-price auctions has many other applications. For example, if each member of a group is asked what she would be willing to pay for a public project that will benefit the whole group, each has an incentive to understate her own contribution—to become a free rider on the contributions of the rest. We have already seen examples of such effects in the collective-action games of Chapter 11. A variant of the Vickrey scheme can elicit the truth in such games as well.

There is an interesting connection between the second-price auction and the English (ascending-price) auction considered in Section 3.A. In both auction formats, each bidder has an incentive to bid or stay in the bidding up to her private value, with the end result that the bidder with the highest value wins and pays the value of the highest losing bidder (or the minimum bid increment above that losing value). Consequently, when bidders have private values, second-price and English auctions have identical equilibrium outcomes. Overall, then, the four auction formats we have considered thus far fall into two basic groups. In the first group are the first-price auction and the Dutch auction. These formats are strategically equivalent and hence always generate identical equilibrium outcomes. In the second group are the second-price auction and the English auction, which generate identical equilibrium outcomes when bidders have private values. The second-price and English auctions are not strategically equivalent, however, and can generate different equilibrium outcomes when bidders have common values.

E. All-Pay Auction

The contest among firms and other special interests to influence policy in the United States and other countries can be viewed as an auction in which each firm invests (by giving money to politicians and by attempting to sway public opinion) to increase the likelihood of getting its own way. Because all participants must pay their bids, whether they win or lose, such sealed-bid auctions are called all-pay auctions.

Lobbying and bribing of public officials is the most famous example of the all-pay auction in action, but other examples can be found in any context where people compete by exerting effort. Consider the Olympic Games. As you read this, thousands of elite athletes around the world have put their lives on hold to train full-time for the Olympics. Many will fail to qualify for their national teams, and all but one in each sport will fail to win a gold medal. But whether they win or lose, they all pay the price of lost income and lost opportunities during training. The workplace tournaments we discussed in Chapter 14, Section 6.B, are similar. Once you start thinking along these lines, you will realize that all-pay auctions are, if anything, more common in real life than situations resembling the standard formal auctions where only the winner pays!

How should you bid (that is, what should your strategy be for expenditure of time, effort, and money) in an all-pay auction? Once you decide to participate, your bid is wasted unless you win, so you have a strong incentive to bid very aggressively. In experiments using this auction format, the sum of all the bids often exceeds the value of the prize by a large amount, and the auctioneer makes a handsome profit.[17] In that case, everyone submitting extremely aggressive bids cannot be the equilibrium outcome; it seems wiser to stay out of such destructive competition altogether. But if everyone else did that, then one bidder could walk away with the prize for next to nothing; thus, not bidding cannot be an equilibrium strategy either. This analysis suggests that the equilibrium lies in mixed strategies.

Consider a specific all-pay auction with n bidders. To keep the notation simple, we choose units of measurement so that the prize has objective value equal to 1. Bidding more than 1 is sure to bring a loss, so we restrict bids to those between 0 and 1. It is easier to let the bid be a continuous variable x, where x can take on any (real) value in the interval [0, 1]. Because the equilibrium will be in mixed strategies, each person's bid, x, will be a continuous random variable. Because you win the object only if all other bidders submit bids below yours, we can express your equilibrium mixing strategy as $Prob(x)$, the probability that your bid takes on a value less than x; for example, $Prob(½) = 0.25$ would mean that your equilibrium strategy entailed bids below ½ one-quarter of the time (and bids above ½ three-quarters of the time).[18]

As usual, we can find the mixed-strategy equilibrium by using an indifference condition. Each bidder must be indifferent about the choice of any particular value of x, given that the others are playing their equilibrium mixes. Suppose you, as one of the n bidders, bid x. You win if all of the remaining $(n - 1)$ bidders are bidding less than x. The probability of anyone else bidding less than x is $Prob(x)$; the probability of two others bidding less than x is $Prob(x) \times Prob(x)$, or $[Prob(x)]^2$; the probability of all $(n - 1)$ of them bidding less than x is $Prob(x) \times Prob(x) \times Prob(x)$ . . . multiplied $(n - 1)$ times, or $[Prob(x)]^{n-1}$. Thus, with a probability of $[Prob(x)]^{n-1}$, you win 1. Remember that you pay x no matter what happens. Therefore, your net expected payoff for any bid of x is $[Prob(x)]^{n-1} - x$. But you could get 0 for sure by bidding 0. Thus, because you must be indifferent about the choice of any particular x, including 0, the condition that defines the equilibrium is $[Prob(x)]^{n-1} - x = 0$. In a full mixed-strategy equilibrium, this condition must be true for all x. Therefore, the equilibrium mixed-strategy bid is $Prob(x) = x^{1/(n-1)}$.

A couple of sample calculations will illustrate what is implied here. First, consider the case in which $n = 2$; then $Prob(x) = x$ for all x. Therefore, the probability of bidding a number between two given levels x_1 and x_2 is $Prob(x_2) - Prob(x_1) = x_2 - x_1$. Because the probability that the bid lies in any range is simply the length of that range, any one bid must be just as likely as any other bid. That is, your equilibrium mixed-strategy bid should be random and uniformly distributed over the whole range from 0 to 1.

Next, let $n = 3$. Then $Prob(x) = \sqrt{x}$. For $x = ¼$, $Prob(x) = ½$; so the probability of bidding ¼ or less is ½. The bids are no longer uniformly distributed over the range from 0 to 1; they are more likely to be in the lower end of the range.

Further increases in n reinforce this tendency. For example, if $n = 10$, then $Prob(x) = x^{1/9}$, and $Prob(x)$ equals ½ when $x = (½)^9 = 1/512 = 0.00195$. In this situation, your bid is as likely to be smaller than 0.00195 as it is to be anywhere within the whole range from 0.00195 to 1. Thus, your bids are likely to be very close to 0.

Your average bid should correspondingly be smaller the larger the number n. In fact, a more precise mathematical calculation shows that if everyone bids according to this strategy, the average or expected bid of any one player will be just $(1/n)$.[19] With n players bidding, on average, $1/n$ each, the total expected bid is 1, and the auctioneer makes zero expected profit. This calculation provides more precise confirmation that the equilibrium strategy eliminates overbidding.

The idea that you should bid closer to 0 when the total number of bidders is larger makes excellent intuitive sense, and the finding that equilibrium bidding eliminates overbidding lends further confidence to the theoretical analysis. Unfortunately, many people in actual all-pay auctions either do not know or forget this theory and bid to excess.

Interestingly, philanthropists have figured out how to take this tendency to overbid and harness it for social benefit. Building on the historical lessons learned from prizes offered in 1919 by a New York hotelier for the first nonstop transatlantic flight (won by Charles Lindbergh in 1927) and even earlier, in 1714, by the British government for a method to precisely measure longitude for sea navigation (eventually awarded to John Harrison in the 1770s), several U.S. and international foundations have begun offering incentive prizes for various socially worthwhile innovations. One foundation in particular, the XPRIZE Foundation, has as its sole purpose the provision of incentive prizes; its first prize was awarded in 2004 for the first private space flight. More recently, the $20 million Carbon XPRIZE seeks to promote new technologies that can convert CO_2 emissions into valuable products, such as building materials or alternative fuels. Ten finalists were selected in September 2018, each with a proven new technology to convert CO_2 into everything from carbon nanotubes to methanol and concrete; winners will be announced in March 2020, after this book has gone to print. Some foundation experts estimate that as much as 40 times the amount of money that would otherwise be devoted to a particular innovation gets spent when incentive prizes are available. Thus, the tendency to overbid in all-pay auctions can actually have a beneficial effect on society (if not on the individual pursuing the prize).[20]

F. War of Attrition

A war of attrition is a contest between multiple players in which each player decides when to retreat from the contest, the victor is whoever remains the longest, and choosing to remain longer in the contest is costly for each player. A war of attrition can be viewed as an auction, with the *bidders* being the contestants, the *object* up for auction being victory, each player's *bid* being how long he remains in the contest before retreating, and each player's *payment* corresponding to the costs[21] he incurs by remaining in the contest as long as he did.

The war of attrition is an open-outcry analogue to the all-pay auction, but the connection between the war of attrition and the all-pay auction is not as close as the connection between the Dutch and first-price auction formats (which are strategically equivalent) or the connection between the English and second-price auction formats (which lead to identical equilibrium outcomes when bidders have private values). In the war of attrition, each loser pays his full bid, but the winner only has to pay the highest losing bid. By contrast, in the all-pay auction, all bidders pay their full bids. For example, suppose that two firms are seeking to bribe a corrupt official. If the firms submit their bribes all at once in sealed envelopes, then this game would be an all-pay auction, with the corrupt official pocketing all of the cash in both envelopes. On the other hand, if the firms paid the official over time in small installments, then the game would be a war of attrition, and the winning firm would only need to pay the amount paid by the losing firm. For example, suppose that Firm A "bids" \$200 and Firm B "bids" \$150 in both games. The official would pocket \$350 = \$200 + \$150 in the all-pay auction, but only \$300 = \$150 + \$150 in the war of attrition. Because of this difference in the *payment rules* of these two auctions, equilibrium bidding strategies in the all-pay auction will not be the same as those in the war of attrition.

We examined the war of attrition (also called dynamic chicken and brinkmanship with two-sided asymmetric information) in great detail earlier in the book, in Chapter 9, Section 7, and in Chapter 13. While we will not repeat those analyses here, it is worth repeating some of the main insights that emerged from them. First, as a war of attrition continues with neither side retreating, each side should infer that the other player is more likely to be Tough (with a lot to gain from victory and/or with low costs associated with continuing the contest). Second, because a war of attrition continues only if both sides are relatively tough, a war of attrition that has already lasted a long time should be expected to last a long time more.

Endnotes

- Elin McCoy, “How to Buy Wine at Auction—and Why You Should,” *Bloomberg*, September 2, 2016, available at https://www.bloomberg.com/news/articles/2016-09-02/how-to-buy-wine-at-auction-and-why-you-should. Return to reference 11
- For more on the history and famous ambience of English auctions, see Marion Laffey Fox, “Inside the Great English Auction Houses,” *Christian Science Monitor*, April 17, 1984. Return to reference 12
- See Martha Stewart’s video *Aalsmeer Flower Auction In Amsterdam*, available at https://www.marthastewart.com/918460/aalsmeer-flower-auction-amsterdam, for an excellent overview of the auction process at Aalsmeer. Return to reference 13
- Completing the square, we can rewrite $-P^2 + 150P - 5{,}000$ as $-(P - 75)^2 + (75)^2 - 5{,}000$, which is maximized when $P - 75 = 0$, or when $P = 75$. Return to reference 14
- A simplifying assumption here is that other bidders do not change their behavior once you decide to participate in the auction. Of course, once there is another bidder (you) in the auction, their bidding incentives will naturally change. We account for this in the appendix to this chapter when deriving equilibrium bidding strategies. Return to reference 15
- Lawrence M. Ausubel and Paul Milgrom, “The Lovely but Lonely Vickrey Auction,” in Peter C. Cramton, Yoav Shoham, and Richard Steinberg, *Combinatorial Auctions* (Cambridge, Mass.: MIT Press, 2006), pp.17–40. Return to reference 16
- One of us (Dixit) has auctioned $10 bills to his Games of Strategy class and made a profit of as much as $60 from a 20-student section. At Princeton, there is a tradition of giving the professor a polite round of applause at the

end of a semester. Once Dixit offered \$20 to the student who kept applauding continuously the longest. This is an example of an unusual open-outcry, all-pay auction with payments in kind (applause). Although most students dropped out after 5 to 20 minutes, three went on applauding for 4½ hours! Return to reference 17

- *Prob*(x) is called the *cumulative probability distribution function* for the random variable x. The more familiar probability density function for x is its derivative, $Prob'(x) = prob(x)$. Then $prob(x)\,dx$ denotes the probability that the variable takes on a value in a small interval from x to $x + dx$. Return to reference 18
- The expected bid of any one player is calculated as the expected value of x, by using the probability density function, $prob(x)$. In this case, $prob(x) = Prob'(x) = [1/(n - 1)]x^{(2-n)/(n-1)}$, and the expected value of x is the sum, or integral, of this from 0 to 1, namely, $\int x\, prob(x)\, dx = 1/n$. Return to reference 19
- For more on incentive prizes, see Matthew Leerberg, "Incentivizing Prizes: How Foundations Can Utilize Prizes to Generate Solutions to the Most Intractable Social Problems," Duke University Center for the Study of Philanthropy and Voluntarism Working Paper, Spring 2006. Information on the X Prize Foundation is available at www.xprize.org. Return to reference 20
- In many applications, the cost of remaining in the contest arises from the continued *risk* of some bad outcome, such as the risk of nuclear war in the U.S.-USSR confrontation studied in Chapter 13. Even though nuclear war did not occur, the United States and the Soviet Union each bore the risk that nuclear war could have erupted—a risk either side could have reduced by retreating earlier in the confrontation. Return to reference 21

Glossary

ascending-price auction
: An open-outcry auction in which prices are announced in increasing order either by an auctioneer (in the case of an English auction) or by bidders themselves (in the case of jump bidding). The last person to bid or accept the announced price wins the auction and pays that price.

English auction
: A type of *ascending-price auction* in which the auctioneer calls out a sequence of increasing prices, bidders decide when to drop out of the bidding, and the last bidder remaining pays the last announced price.

reserve price
: The minimum price set by the seller of an item up for auction; if no bids exceed the reserve, the item is not sold.

drop-out price
: In an English auction, the price at which a bidder drops out of the bidding.

jump bidding
: Submitting a bid that is significantly higher than the previous bid and well beyond whatever minimum bid increment exists.

shill bidder
: A fake bidder created by sellers at an auction to place fictitious bids for an object they are selling.

sniping
: Waiting until the last moment to make a bid.

descending-price auction
: An open-outcry auction in which the auctioneer announces possible prices in descending order. The first person to accept the announced price wins the auction and pays that price. Also called **Dutch auction**.

Dutch auction

Same as a **descending-price auction**.

auction
: A game in which multiple players (called bidders) compete for a scarce resource.

first-price auction
: A sealed-bid auction in which the highest bidder wins and pays the amount of her bid.

shading
: A strategy in which bidders bid slightly below their true valuation of an object.

second-price auction
: A sealed-bid auction in which the highest bidder wins the auction but pays a price equal to the value of the second-highest bid; a special case of the *Vickrey auction*.

Vickrey auction
: An auction design proposed by William Vickrey in which truthful bidding is a weakly dominant strategy for each bidder. When a single object is sold, the Vickrey auction is the same as the *second-price auction*.

truthful bidding
: A practice by which bidders in an auction bid their true valuation of an object.

all-pay auction
: An auction in which each person who submits a bid must pay her highest bid amount at the end of the auction, even if she does not win the auction.

war of attrition
: A contest between multiple players in which each player decides when to retreat, the victor is whoever remains the longest, and choosing to remain longer is costly for each player.

4 AUCTION DESIGN

Suppose now that, instead of being one of the many bidders at a particular auction, you are designing an auction to sell an object you already own, with the goal of maximizing your expected revenue. What auction format should you use? In this section, we focus on comparing the two most commonly used auction formats: the English auction and the first-price/Dutch auction. [We write *first-price/Dutch* to remind you that the first-price (sealed-bid) auction and the Dutch (descending-price) auction are strategically equivalent.] To provide you with some insight into the problem, we begin by exploring a simple numerical example.

Suppose there are two bidders (denoted by $i = 1, 2$) who will take part in your auction. These bidders have private values, V_i, that are uncorrelated with each other and uniformly distributed over the interval [0, 1]. (*Uniformly distributed* means that each bidder’s private value is equally likely to be any number in the specified interval—here, from 0 to 1. *Uncorrelated* means that knowing one bidder’s value tells you nothing about the other bidder’s likely value; thus, each bidder has no idea what the other bidder’s value is, and hence views it as being uniformly distributed over [0, 1].)

If you use an English auction, each bidder will stay in the bidding until the price reaches his value; the auction will end once the price hits the lower bidder’s value. Because each bidder’s value is uniformly distributed over [0, 1], we can show (as in the appendix to this chapter) that the lower bidder’s value, on average, equals ⅓.[22] We conclude that, if you choose to use an English auction, your expected revenue will be ⅓.

If you decide to use a first-price auction (FPA), the bidders have an incentive to shade their bids below their true values, but the winner will pay his full bid. In the appendix to this chapter, we provide a complete mathematical derivation of the equilibrium bidding strategies for this case, showing that each of the two bidders submits a bid equal to exactly half of his value; that is, each bidder's bidding strategy takes the form $b_i(V_i) = V_i/2$. (A bidder with value 1 bids ½, while a bidder with value ½ bids ¼, etc.) Because the higher bidder's value is, on average, ⅔, and because each bidder submits a bid equal to half of his value, the winning bid will, on average, equal ⅓. At least in our example, then, it doesn't matter which of these auction formats you use to sell your item—you'll get the same expected revenue!

This result may seem like a coincidence, but in fact, the "revenue equivalence" that we have discovered here is the consequence of a deeper result, the so-called revenue equivalence theorem, one of the most famous results in auction theory.[23] The revenue equivalence theorem (abbreviated hereafter as RET) tells us that two different auction formats with the same set of bidders and the same reserve prices will generate the same average revenue for the seller under three conditions: (1) that bidders have uncorrelated private values drawn from the same distribution, (2) that bidders care only for expected monetary payoffs, without any aversion to loss or risk, and (3) that in each auction, bidders play a symmetric Nash equilibrium[24] in which any bidder with the lowest possible value gets zero payoff.

RET is an interesting theoretical result, but it also serves the highly practical purpose of focusing our attention on the aspects of the auction environment that matter for auction design. *By focusing on which RET assumptions fail* in a given auction environment, we can gain insight into how to design an auction for that environment. What, then, are the key RET assumptions? There are a number of them. We identify six of them below, then devote the remainder of this section to deeper analysis of the effects of these assumptions on an auction's expected revenue. (Our discussion doesn't mention every RET assumption, just the ones that are most important for you to understand. Those who want to learn more can consult the additional references provided in Section 5.)

First, RET assumes that the *same bidders* participate in the two auctions that are being compared. An auction that is more attractive to bidders, so that more participate, will (all else being equal) obviously generate greater expected revenue. Second, RET assumes that the *reserve price* is the same in the two auctions. As we discuss in Section 4.A below, changing the reserve price can affect revenue. Third, RET assumes that the bidders are *risk neutral*. If bidders are risk averse, RET does not hold, and the auction format can affect how much revenue is generated, as we discuss in Section 4.B. Fourth, RET assumes that bidders have *private values*. If bidders have common values instead, the auction format you choose can make a difference to your revenue; we discuss the case of common values in Section 4.C. Fifth, RET assumes that bidders' values are *uncorrelated*. If bidders' values are correlated, the auction format, again, can affect the revenue you earn; we address this possibility in Section 4.C. Sixth and finally, RET assumes that bidders play a *Nash equilibrium*. We consider two reasons why bidders might fail to play a Nash equilibrium in practice: incorrect beliefs about the game (due to inexperience or overconfidence, for example) in Section 4.D and collusion in Section 4.E. In both cases, the auction format can affect your expected revenue.

A. Reserve Prices

In 1997, Overture (later acquired by Yahoo!) became the first search engine to use an auction to sell the advertisements that appear next to its search results. Such "sponsored-search auctions" eventually became big money makers,[25] but in those early years, the auction format that Overture (and later Yahoo!) used had not yet been optimized to maximize revenue. Enter Michael Schwarz, an auction-theory expert who joined Yahoo!'s newly-formed economics research unit in 2006. Schwarz had previously been an economics professor at Harvard University, but now he was tasked with improving the auction design that Yahoo! relied on for its revenue.

Schwarz and his collaborator Michael Ostrovsky (a Stanford economics professor) identified reserve prices as an aspect of the auction design that could be improved, showing that Yahoo! could substantially raise its revenue by increasing its reserve prices on many keywords. The idea was that, by refusing to place ads at bargain-basement rates, Yahoo! could induce bidders to make higher bids in the first place. The results were felt quickly, as Yahoo! experienced an impressive boost in revenues in the quarters immediately following its adoption of the new reserve pricing system.

How would you figure out the optimal reserve price to set for a particular object? In our earlier numerical example, we assumed that there were just two bidders having values drawn uniformly from the interval [0, 1]. Here we suppose that there may be *any number*, $n \geqslant 1$, of bidders,[26] but we continue to assume that the bidders have uncorrelated values drawn uniformly from the interval [0, 1]. We also suppose that you can set any reserve price, $r \geqslant 0$. According to the revenue equivalence theorem, your expected revenue if you use a first-price auction (FPA) is equal to your expected revenue in a second-price auction (SPA) with the same number of bidders and the same reserve price. So let's focus on the SPA, which is easier to analyze because truthful bidding is a weakly dominant strategy.

As we show in the appendix to this chapter, the expected revenue (denoted by *REV*) that you get from choosing the SPA with n bidders and reserve price r is given by the following formula:

$$REV(n,r)=\frac{n-1}{n+1}+r^n\left(1-\frac{2nr}{n+1}\right).$$

Let $r*(n)$ denote the optimal reserve price when there are n bidders; $r*(n)$ is the level of r that maximizes $REV(n, r)$. Remarkably, $r*(n)$ = ½, no matter how many bidders there are. So, your expected revenue when setting an optimal reserve price [call it $REV*(n)$] takes the form

$$REV^*(n)=REV\left(n,\frac{1}{2}\right)=\frac{n-1}{n+1}+\frac{(1/2)^n}{n+1}.$$

Note that if you didn't use a reserve price, your expected revenue would be $$REV(n,0)=\frac{n-1}{n+1};$$ so the extra expected revenue you get from using an optimal reserve price (rather than no reserve price at all) is $$\frac{(1/2)^n}{n+1}.$$ When n = 2, this extra revenue, (, is an increase of 25% relative to what you would have earned without any reserve price [$REV(2, 0)$ = ⅓]. On the other hand, if there are 10 bidders, the extra revenue from using an optimal reserve price is only $^{1}/_{11,264}$, an increase of 0.027% relative to not having any reserve price.

In summary, the key observation regarding reserve prices for anyone designing an auction is that *setting a reserve price can raise your expected revenue by a substantial amount, but only when the number of bidders is small*.

B. Bidder Attitudes Toward Risk

If you choose to sell your object using an English auction, each bidder has an incentive to remain in the bidding until the price reaches his value, no matter what his attitude toward risk is. So your expected revenue in the English auction is unaffected by risk aversion. On the other hand, bidders in the Dutch auction have an incentive to jump in at higher prices when they are risk averse. To understand why, consider again the logic of bidding in the Dutch auction (see Section 3.B). To earn a profit, bidders need to wait until the price has dropped below their value and then continue to wait a little longer, taking the risk that someone else will jump in and grab the object. Risk-averse bidders will be willing to pay a higher price in order to avoid that risk, causing revenues in the Dutch auction to be higher when bidders are risk averse than when they are risk neutral. So, if you use a Dutch auction (or the strategically equivalent first-price auction) when your bidders are risk averse, you will have greater expected revenue than you would have from an English auction.

In summary, the key observation regarding bidder attitudes toward risk for anyone designing an auction is that *when bidders are risk averse, the first-price/Dutch auction generates greater expected revenue than the English auction*.

C. Common Values and Correlated Estimates

Suppose that the object you plan to auction has a common value and that each bidder has an estimate, $\hat{V}_i,$ of that common value. For instance, before a drilling-rights auction, oil companies interested in bidding on a given tract will each run their own seismic tests and make an estimate of how much oil is under the ground. These estimates are naturally correlated with one another because if there is actually a large amount of oil under the ground, each oil company is likely to make a high estimate, and vice versa.

Each oil company in this example would love to know what the others know, because this would allow it to improve its own estimate and to bid more successfully in the auction. But consider for a moment what would happen if all of them somehow managed to learn everything that any of them knew about the oil tract up for auction. Having the same information, all of them would then have the same estimate, $\hat{V},$ of the common value. With all bidders having the same valuation, $\hat{V},$ the auction would then be one with *objective values*—like an auction of a $20 bill. In such an auction, the auction designer could extract the bidders' full willingness to pay, $\hat{V},$ and leave them with zero profit.

Notice that this kind of "private-information leakage" is good for you as the seller here; it increases the intensity of bidder competition, leading to higher expected revenue. As we saw in Chapter 9, players in a game who possess private information must be given an incentive to act on (and hence reveal) that information, and such incentives must always leave money on the table for the player in question. In the auction context, the "money on the table" corresponds to bidder surplus (or profit), the difference between the winning price and the bidder's true value. Anything you as the seller can do to reduce bidders' private information will therefore tend to have the effect of reducing their surplus—and hence increasing your revenue.

In a first-price auction, nothing is revealed about bidders' private information during the auction because bidders submit sealed bids. On the other hand, in an English auction, some private information is revealed every time a bidder drops out of the auction, Others can make inferences about what that bidder's value estimate must have been for him to drop out at that particular price.

The overall effect of this private-information revelation on expected revenue in an English auction is not intuitively obvious, because seeing others' bids may cause bidders to bid more or less aggressively. If surprisingly many bidders drop out early at low prices, others will adjust their estimates downward and will also drop out earlier. On the other hand, if surprisingly few bidders drop out at low prices, others will adjust their estimates upward and stay in the bidding even longer. Auction theorists have determined that private-information revelation generally increases expected revenue, so your expected revenue with an English auction will be larger than with a first-price auction—the intuition again being that the English auction induces more of bidders' private information to be revealed.

In summary, the key observation regarding common values and correlated estimates for anyone designing an auction is that *when bidders have common values and positively correlated estimates of the object's value, the English auction tends to generate greater expected revenue than the first-price/Dutch auction.*[27]

D. Incorrect Beliefs about the Game

DealDash.com sells flat-screen TVs and other big-ticket items using a penny auction (also known as a *bid-fee auction*). In a DealDash auction, the price starts at zero and the auction ends as soon as 10 seconds go by without any bidding. During each 10-second window, bidders can jump in to restart the clock by paying a “bidding fee” of 60 cents. The name *penny auction* comes from the fact that when someone jumps in to bid, the price increases by a penny. In a video on DealDash.com that explains how its auction works,[28] a 52″ HDTV Sony flat-screen TV that retails for about \$600 ultimately sells at a price of \$35.63. This may seem like a great deal, until you realize that—because DealDash collected 60 cents for every penny of the sale price—bidders overall paid 61 × \$35.63 = \$2,173.43, well over three times the TV’s actual value. And this isn’t an anomaly. Studies of penny auctions have found that penny-auction sites enjoy remarkable profit margins of 50% or more.

It’s possible that bidders enjoy a thrill from participating, and occasionally winning, in penny auctions, so much so that they don’t mind losing money on average—much like gamblers in a casino or people playing the lottery. However, it’s equally plausible that many people who participate in these auctions have *incorrect beliefs* about the game and hence are prone to make choices that are actually not in their best interest. For instance, some bidders may overestimate their own strategic prowess, thinking that they can outbid others even though, in reality, their skills and ability to win are just average. Many studies have documented this tendency toward overconfidence, known as *illusory superiority*.[29] For instance, 88% of University of Oregon students thought that they were better-than-average drivers (relative to other students who participated in the same study), 87% of Stanford MBA students thought that their academic performance was above average, and 90% of professors at the University of Nebraska–Lincoln believed that they were better-than-average teachers.

People who *think* that they can outwit others to win money consistently in penny auctions will be the ones who wind up participating in these auctions. But, of course, only some people can be better than average, and most will end up consistently losing money. The issue is not that these people are dumb or incompetent, but that they have incorrect beliefs about the game and hence are unable to play their best response—which, with penny auctions, would be not to play at all!

The key observation to take away from this discussion is that *in reality, some bidders fail to play a best response and, hence, cannot be playing a Nash equilibrium. Auctions designed to take advantage of such mistakes can generate greater expected revenue*.

E. Collusion

Each summer, school districts across the country solicit bids on annual supply contracts for milk. Dairies submit bids in these procurement auctions, and the low bidder is selected to supply the milk that will be served to students during the following school year. Auctions could in principle be a great way for school districts to secure a good price for milk, but there's a problem: *pervasive collusion* among the dairies bidding in these auctions.

In the mid-1980s, Florida Attorney General Robert A. Butterworth began uncovering milk-auction collusion, using bid-analysis techniques developed by auction theorists who had been studying collusion in highway-construction procurement auctions. In 1993, the *New York Times* reported that "federal and state investigators have found evidence in at least 20 states that executives at the nation's largest national and regional dairy companies have conspired—sometimes for decades—to rig bids on milk products sold to schools and military bases."[30]

As the auctioneer Mike Brandly explains on *Auctioneer's Blog*,[31] "Bidders collude by agreeing with each other to not bid against each other." One way that dairies did that was to divide the school districts in a given state into *territories*, and then promise not to bid on contracts in another dairy's territory. Of course, each dairy would have had the temptation to break that promise and try to steal some extra business. However, because the winner of any public contract is a matter of public record, any such betrayal would have been easily detectable, ending the collusive arrangement and hence returning the dairies to a less profitable competitive mode.

The longevity of the dairy-auction cartels relied in large part on the fact that the dairies were playing a repeated game (like those we analyzed in Chapter 10), with the ability to observe others' actions and to retaliate against anyone who violated the collusive agreement. But the same principle also applies in auctions that are not repeated. If an auction format allows bidders to observe others' actions and to retaliate, then colluding will be easier than if the auction were designed to obscure bidders' actions until it is too late to retaliate.

For instance, suppose that two bidders in a first-price auction have made a "handshake agreement" to collude. Bidder 1 has value \$100, bidder 2 has value \$80, and both are fairly certain that no one else is willing to pay more than \$50. They agree that bidder 2 will bid \$50 (thereby allowing bidder 1 to win the item at a slightly higher price) and that after the auction, bidder 1 will pay \$10 to bidder 2. If bidder 2 goes along with this plan, she will walk away with \$10. However, if she cheats and bids \$60, she can win the item for a profit of \$20 = \$80 − \$60. She can get away with such cheating in a first-price auction because there is no way for bidder 1 to see what bidder 2 has bid until it's too late. On the other hand, in an English auction, bidder 2 has no hope of winning by cheating. If she stays in the bidding past \$50, bidder 1 will just stay in the bidding himself. Bidder 2 therefore has no temptation to cheat, making collusion easier to sustain in an English auction.

The key observation to take away from this analysis of bidder collusion is that *the English auction is more prone to collusion than the Dutch/first-price auction*.

Endnotes

- We provide an appendix to this chapter for those interested in more mathematical details. For now, please just take our word for it that with two bidders, the lower bidder's value, on average, equals ⅓ and the higher bidder's value, on average, equals ⅔. Return to reference 22
- See Vijay Krishna, *Auction Theory*, 2nd ed. (Burlington, Mass.: Academic Press, 2010), Chapter 3 for details. Return to reference 23
- A *strategy* in an auction specifies the bid $b_i(V_i)$ that bidder i will make for any given value V_i. Bidding strategies are "symmetric" if $b_i(V_i) = b(V_i)$ for all i or, in words, if all bidders with the same value submit the same bid. Return to reference 24
- In 2012, Google hosted over 5.1 billion searches per day; the corresponding sponsored-search auctions generated $43.7 billion for Google that year, 95% of its total revenue. For more on the auction formats used by different search engines, see Hal Varian, "Online Ad Auctions," *American Economic Review*, vol. 99, no. 2 (May 2009), pp. 430–34. Return to reference 25
- The case with one bidder ($n = 1$) corresponds to having a single buyer, also known as "monopoly pricing." So, when you learned about monopoly pricing in your introductory microeconomics class, you were actually learning about auction theory! Return to reference 26
- In a landmark 1982 paper, Paul Milgrom and Robert Weber established the revenue ranking theorem: so long as bidders' value estimates are positively correlated (in a particular technical sense known as "affiliation"), expected revenue in the English auction is greater than in the second-price auction, which is itself greater than in the first-price/Dutch auction. See Paul Milgrom and

Robert Weber, “A Theory of Auctions and Competitive Bidding,” *Econometrica*, vol. 50, no. 5 (September 1982), pp. 1089–1122. Return to reference 27

- See https://www.dealdash.com/help/how-it-works(accessed July 23, 2018). Return to reference 28
- Illusory superiority is also known as “superiority bias” and “the Lake Wobegon effect” (after a fictional Minnesota town where “all the children are above average”). See Ola Svenson, “Are We All Less Risky and More Skillful Than Our Fellow Drivers?” *Acta Psychologica*, vol. 47, no. 2 (February 1981), pp. 143–48; “It’s Academic,” *Stanford GSB Reporter*, April 24, 2000, pp. 14–15; and K. Patricia Cross, “Not Can, But *Will* College Teaching Be Improved?” *New Directions for Higher Education*, vol. 1977, no. 17 (Spring), pp. 1–15. Return to reference 29
- Diana B. Henriques and Dean Baquet, “Evidence Mounts of Rigged Bidding in Milk Industry,” *New York Times*, May 23, 1993. Return to reference 30
- Mike Brandly, “What is Collusion at an Auction?” May 14, 2000, available at https://mikebrandlyauctioneer.wordpress.com/2010/05/14/what-is-collusion-at-an-auction/(accessed May 27, 2019). Return to reference 31

Glossary

revenue equivalence theorem (RET)
: A famous result in auction theory specifying conditions under which two auctions will generate the same expected revenue for the seller.

penny auction
: An auction format in which each bidder may pay a bidding fee (say, 60 cents) to advance the price by one cent. The auction continues until no one is willing to advance the price any longer, at which point the last bidder wins and pays the final price.

5 FURTHER READING

This chapter only scratches the surface of auction theory, auction design, and the broader field known as *market design*. In this section, we provide reading recommendations for those who would like to dig in deeper and learn more about the subject.

Books with compelling stories about auctions and other markets:

John McMillan, *Reinventing the Bazaar: A Natural History of Markets* (New York: W. W. Norton, 2003).

Alvin Roth, *Who Gets What—and Why: The New Economics of Matchmaking and Market Design* (Wilmington, Mass.: Mariner Books, 2016).

Paul Milgrom, *Putting Auction Theory to Work* (New York: Cambridge University Press, 2004).

Scholarly books and articles on auction-theory topics:

Multi-unit auctions (Section 1): Ali Hortaçsu and David McAdams, "Empirical Work on Auctions of Multiple Objects," *Journal of Economic Literature*, vol. 56, no. 1 (March 2018), pp. 157–84.

Combinatorial auctions (Section 1): Peter Cramton, Yoav Shoham, and Richard Steinberg (eds.), *Combinatorial Auctions* (Cambridge, Mass: MIT Press, 2005).

Internet-enabled auctions (Section 1): David Lucking-Reiley, "Auctions on the Internet: What's Being Auctioned, and How?" *Journal of Industrial Economics*, vol. 48, no. 3 (September 2000), pp. 227–52; and on the issue of participation and winner's curse, Patrick Bajari and Ali Hortaçsu, "The Winner's Curse, Reserve Prices, and Endogenous Entry: Empirical Insights from eBay Auctions," *RAND Journal of Economics*, vol. 34, no. 2 (Summer 2003), pp. 329–55.

Experimental evidence on bidder behavior, including winner's curse (Section 2): John H. Kagel and Dan Levin, "Auctions: A Survey of Experimental Research," in *The Handbook of Experimental Economics*, vol. 2, ed. John H. Kagel and Alvin E. Roth (Princeton: Princeton University Press, 2016), pp. 563–637.

Jump bidding (Section 3.A): Christopher Avery, "Strategic Jump Bidding in English Auctions," *Review of Economic Studies*, vol. 65, no. 2 (April 1998), pp. 185–210; Robert F. Easley and Rafael Tenorio, "Bidding Strategies in Internet Yankee Auctions," SSRN Working Paper 170028 (June 1999).

Shill bidding (Section 3.A): David McAdams and Michael Schwarz, "Who Pays when Auction Rules are Bent?" *International Journal of Industrial Organization*, vol. 25, no. 5 (October 2007), pp. 1144–57; Georgia Kosmopoulou and Dakshina G. De Silva, "The Effect of Shill Bidding upon Prices: Experimental Evidence," *International Journal of Industrial Organization*, vol. 25, no. 2 (April 2007), pp. 291–313.

Sniping (Section 3.A): Axel Ockenfels and Alvin Roth, "Late and Multiple Bidding in Second-Price Internet Auctions: Theory and Evidence Concerning Different Rules for Ending an Auction," *Games and Economic Behavior*, vol. 55, no. 2 (May 2006), pp. 297–320.

Vickrey auctions (Section 3.D): Vickrey's original article is "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, vol. 16, no. 1 (March 1961), pp. 8–37. See also David Lucking-Reiley, "Vickrey Auctions in Practice: From Nineteenth-Century Philately to Twenty-First-Century e-Commerce," *Journal of Economic Perspectives*, vol. 14, no. 3 (Summer 2000), pp. 183–92.

All-pay auctions (Section 3.E): Many real-world contests can be interpreted as all-pay auctions.

- **In lobbying and political corruption:** Michael R. Baye, Dan Kovenock, and Casper G. de Vries, "Rigging the Lobbying Process: An Application of the All-Pay Auction," *American Economic Review*, vol. 83, no. 1 (March 1993), pp. 289–94.
- **In crowdsourcing over the Internet:** Dominic DiPalantino and Milan Vojnovic, "Crowdsourcing and All-Pay Auctions," *Proceedings of the 10th ACM Conference on Electronic Commerce* (July 2009), pp. 119–28.
- **In patent races:** Michael R. Baye and Heidrun C. Hoppe, "The Strategic Equivalence of Rent-Seeking, Innovation, and Patent-Race Games," *Games and Economic Behavior*, vol. 44, no. 2 (August 2003), pp. 217–26.

War of attrition (Section 3.F): Many real-world contests can be interpreted as wars of attrition.

- **Animal contests over territory and mates:** Steven N. Austad, "A Game Theoretical Interpretation of Male Combat in the Bowl and Doily Spider (*Frontinella pyramitela*)," *Animal Behaviour*, vol. 31, no. 1 (February 1983), pp. 59–73; James H. Marden and Jonathan K. Waaget, "Escalated Damselfly Territorial Contests Are Energetic Wars of Attrition," *Animal Behaviour*, vol. 39, no. 5 (May 1990), pp. 954–59.
- **Business exit from a declining industry:** Yuya Takahashi, "Estimating a War of Attrition: The Case of the US Movie Theater Industry," *American Economic Review*, vol. 105, no. 7 (July 2015), pp. 2204–41.
- **Political delay before stabilizing out-of-control deficits:** Alberto Alesina and Allan Drazen, "Why Are

Stabilizations Delayed?" *American Economic Review*, Vol. 81, no. 5 (December 1991), pp. 1170–88.

Auctions with risk-averse bidders (Section 4.B): Eric Maskin and John Riley, "Optimal Auctions with Risk Averse Buyers," *Econometrica*, vol. 52, no. 6 (November 1984), pp. 1473–1518; Steven Matthews, "Comparing Auctions for Risk Averse Buyers: A Buyer's Point of View," *Econometrica*, vol. 55, no. 3 (May 1987), pp. 633–46.

Auctions with collusion (Section 4.E): Robert C. Marshall and Leslie Marx, *The Economics of Collusion: Cartels and Bidding Rings* (Cambridge, Mass.: MIT Press, 2012).

SUMMARY

Auctions arise whenever players compete over a scarce resource and the game they play involves bidding and direct competition among *bidders*. Auctions take many forms, from art lovers raising their paddles in an art-house auction and firms competing to hire an employee to lobbyists raising campaign cash in an all-pay contest over legislative policy. Auction theory identifies the key features of an auction environment that determine how much you should bid (if you are a bidder) and how you design the auction (if you are the seller).

The most important feature of an auction environment is whether bidders have private values or common values for the object to be auctioned. In an auction with *common values*, how much the object is worth to you depends on what others know about it, and you are more likely to win the object when they do not want it. Bidders who do not account for this effect are likely to suffer the *winner's curse*, paying more for the object than it is worth. Fortunately, the winner's curse can be avoided once you understand it.

In an auction with *private values*, how you should bid depends on the details of the auction format. In an *English auction* or a *second-price auction*, *truthful bidding* is your best strategy. In a *first-price auction* or a *Dutch auction*, you should always bid less than your true value. And in an *all-pay auction*, your best strategy may involve submitting a low bid even when your value is high.

Sellers can often increase their expected revenue by committing to keep the object unless a *reserve price* is met. Many other factors also affect how you should bid in or design an auction, including risk aversion, correlation of bidders’ private information, bidders’ incorrect beliefs about the game, and collusion among bidders.

KEY TERMS

all-pay auction (601)

ascending-price auction (595)

auction (587)

auction designer (587)

bidder (587)

combinatorial auction (587)

common value (591)

descending-price auction (597)

drop-out price (595)

Dutch auction (597)

English auction (595)

first-price auction (598)

jump bidding (596)

multi-unit auction (587)

objective value (590)

open-outcry auction (589)

penny auction (609)

private information (590)

private value (591)

procurement auction (587)

reserve price (595)

revenue equivalence theorem (RET) (605)

sealed-bid auction (589)

second-price auction (599)

shading (599)

shill bidder (596)

single-object auction (587)

sniping (596)

truthful bidding (600)

valuation (587)

Vickrey auction (599)

war of attrition (603)

winner’s curse (592)

Glossary

auction
: A game in which multiple players (called bidders) compete for a scarce resource.

bidder
: A player in an auction game.

valuation
: The benefit that a bidder gets from winning the object in an auction.

auction designer
: A player who sets the rules of an auction game.

procurement auction
: An auction in which multiple bidders compete to supply an item. Bids in a procurement auction are prices that bidders are willing to receive to supply the good. The lowest bidder wins and is paid her bid.

multi-unit auction
: An auction in which multiple identical objects are sold.

combinatorial auction
: An auction of multiple dissimilar objects in which bidders are able to bid on and win combinations of objects.

sealed-bid auction
: An auction mechanism in which bids are submitted privately in advance of a specified deadline, sometimes in sealed envelopes.

open-outcry auction
: An auction mechanism in which bids are made openly for all to hear or see.

private information
: Information known by only one player.

objective value
: An auction is called an objective-value auction when the object up for sale has the same value to all bidders and

each bidder knows that value.

private value

An auction is called a private-value auction when each bidder has private information about their own valuation of the object up for sale, but knowing others’ private information would not change any bidder’s own willingness to pay for the object. An important special case is when each bidder knows their own valuation but others do not.

common value

An auction is called a common-value auction when the object up for sale has the same value to all bidders, but each bidder knows only an imprecise estimate of that value.

winner’s curse

A situation in a common-value auction where the winner fails to take account of the fact that when she wins, she is likely to have made an overly optimistic estimate of the object’s value. Bidders who correctly anticipate this possibility can avoid the winner’s curse by lowering their bids appropriately.

ascending-price auction

An open-outcry auction in which prices are announced in increasing order either by an auctioneer (in the case of an English auction) or by bidders themselves (in the case of jump bidding). The last person to bid or accept the announced price wins the auction and pays that price.

English auction

A type of *ascending-price auction* in which the auctioneer calls out a sequence of increasing prices, bidders decide when to drop out of the bidding, and the last bidder remaining pays the last announced price.

reserve price

The minimum price set by the seller of an item up for auction; if no bids exceed the reserve, the item is not sold.

drop-out price

In an English auction, the price at which a bidder drops out of the bidding.

jump bidding

Submitting a bid that is significantly higher than the previous bid and well beyond whatever minimum bid increment exists.

shill bidder

A fake bidder created by sellers at an auction to place fictitious bids for an object they are selling.

sniping

Waiting until the last moment to make a bid.

descending-price auction

An open-outcry auction in which the auctioneer announces possible prices in descending order. The first person to accept the announced price wins the auction and pays that price. Also called **Dutch auction**.

Dutch auction

Same as a **descending-price auction**.

first-price auction

A sealed-bid auction in which the highest bidder wins and pays the amount of her bid.

shading

A strategy in which bidders bid slightly below their true valuation of an object.

second-price auction

A sealed-bid auction in which the highest bidder wins the auction but pays a price equal to the value of the second-highest bid; a special case of the *Vickrey auction*.

Vickrey auction

An auction design proposed by William Vickrey in which truthful bidding is a weakly dominant strategy for each bidder. When a single object is sold, the Vickrey auction is the same as the *second-price auction*.

truthful bidding

A practice by which bidders in an auction bid their true valuation of an object.

all-pay auction
: An auction in which each person who submits a bid must pay her highest bid amount at the end of the auction, even if she does not win the auction.

war of attrition
: A contest between multiple players in which each player decides when to retreat, the victor is whoever remains the longest, and choosing to remain longer is costly for each player.

revenue equivalence theorem (RET)
: A famous result in auction theory specifying conditions under which two auctions will generate the same expected revenue for the seller.

penny auction
: An auction format in which each bidder may pay a bidding fee (say, 60 cents) to advance the price by one cent. The auction continues until no one is willing to advance the price any longer, at which point the last bidder wins and pays the final price.

single-object auction
: An auction in which a single indivisible object is sold.

SOLVED EXERCISES

1. In each of the following examples, does the object being auctioned have an objective value or not? Why or why not? In each case, imagine that the object in question is being auctioned by your game-theory professor in your game-theory class:
   1. \$20 Amazon gift card
   2. Lunch with the professor
   3. A bottle of water
2. A house painter has a regular contract to work for a builder. On these jobs, her cost estimates are generally right: sometimes a little high, sometimes a little low, but correct on average. When her regular work is slack, she bids competitively for other jobs. “Those are different,” she says. “They almost always end up costing more than I estimate.” If we assume that her estimating skills do not differ between the two types of jobs, what can explain the difference?
3. Consider an auction where n identical objects are offered, and there are $(n + 1)$ bidders. The actual value of an object is the same for all bidders and equal for all objects, but each bidder has only an independent estimate, subject to error, of this common value. The bidders submit sealed bids. The top n bidders get one object each, and each pays what she has bid. What considerations will affect your bidding strategy? How?
4. You are in the market for a used car and see an ad for the model that you like. The owner has not set a price, but invites potential buyers to make offers. Your prepurchase inspection gives you only a very rough idea of the value of the car; you think it is equally likely to be anywhere in the range of \$1,000 to \$5,000 (so your calculation of the average value is \$3,000). The current owner knows the exact value and will accept your offer if it exceeds that value. If your offer is accepted and you get the car, then you will find out the truth. But you have some special repair skills and know that when you own the car, you will be able to work on it and increase its value by a third (33.3 . . . %) of whatever it is worth.
   1. What is your expected profit if you offer \$3,000? Should you make such an offer?
   2. What is the highest offer that you can make without losing money on the deal?

5. It’ s your birthday, and there’ s one cupcake left. Your two best friends Bob (B) and Apple (A), each want it. Each friend i = A, B is willing to pay V_i for the entire cupcake or $V_i/3$ for half of the cupcake (they don’ t like sharing), where V_i are uncorrelated private values drawn uniformly from \$0 to \$12.
   1. Suppose that you hold a second-price auction with no reserve price. Compute Bob’ s expected surplus when his realized value is V_B = \$12, V_B = \$9, V_B = \$6, and V_B = \$3. Hint: Expected surplus = (probability that Bob wins) × (Bob’ s surplus when winning). Bob wins whenever he has the highest value, which happens with probability $V_B/2$, and pays Apple’ s valuation, which, when Bob wins, is uniformly distributed from \$0 to V_B.
   2. If you instead give Bob half of the cupcake, Bob’ s surplus is $v_B/3$. Compare this *gift surplus* with the *auction surplus* that you computed in part (a). For each of the values V_B = \$12, V_B = \$9, V_B = \$6, and V_B = \$3, does Bob prefer to get half of the cupcake for free or to compete in an auction for the whole cupcake?
6. In this problem, we consider a special case of the first-price, sealed-bid auction and show what the equilibrium amount of bid shading should be. Consider a first-price, sealed-bid auction with n risk-neutral bidders. Each bidder has a private value independently drawn from a uniform distribution over the interval [0, 1]. That is, for each bidder, all values between 0 and 1 are equally likely. The complete strategy of each bidder is a “bid function” that will tell us, for any value v, what amount $b(V)$ that bidder will choose to bid. Deriving the equilibrium bid functions requires solving a differential equation, but instead of asking you to derive the equilibrium using a differential equation, this problem proposes a candidate equilibrium and asks you to confirm that it is indeed a Nash equilibrium.

   It is proposed that the equilibrium bid function for n = 2 is $b(V) = V/2$ for each of the two bidders. That is, if we have two bidders, each should bid half her value, which represents considerable shading.

   1. Suppose you’ re bidding against just one opponent whose value is uniformly distributed over the interval [0, 1], and who always bids half her value. What is the probability that you

will win if you bid $b = 0.1$? If you bid $b = 0.4$? If you bid $b = 0.6$?

2. Put together the answers to part (a). What is the correct mathematical expression for *Prob*(win), the probability that you win, as a function of your bid b?
3. Find an expression for the expected profit you make when your value is v and your bid is b, given that your opponent is bidding half her value. Remember that there are two cases: either you win the auction, or you lose the auction. You need to average the profit between these two cases.
4. What is the value of b that maximizes your expected profit? The answer should be a function of your value V.
5. Use your results to argue that it is a Nash equilibrium for both bidders to follow the same bid function $b(V) = V/2$.

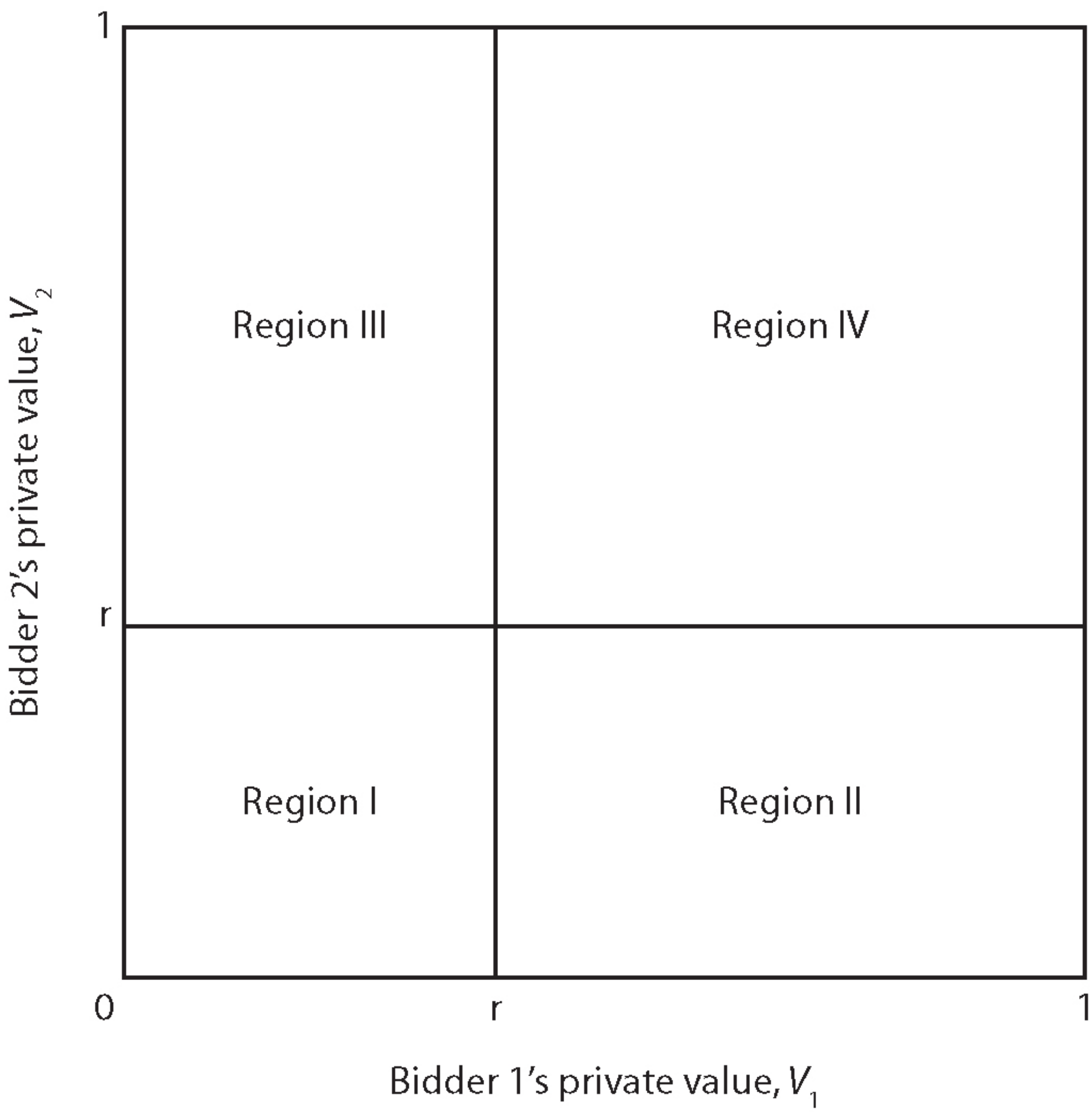
7. **(Optional)** This question looks at equilibrium bidding strategies in all-pay auctions in which bidders have private values for the good, as opposed to the scenario in Section 4, where the all-pay auction is for a good with a publicly known value. Consider an all-pay auction where each player's private value is distributed uniformly between 0 and 1, and those of different players are independent of each other.
   1. Verify that the Nash equilibrium bid function is $b(V) = [(n - 1)/n]V^n$. Use an approach similar to that of Exercise S6. Remember that in an all-pay auction, you pay your bid even when you lose, so your payoff is $V - b$ when you win and $-b$ when you lose.
   2. Now consider the effect of changing the number of bidders n. For clarity, call the bid function $b_n(V)$. Show that $b_3(V) > b_2(V)$ if and only if $V > ¾$, and that $b_4(V) > b_3(V)$ if and only if $V > 8/9$. Using the results for those cases, what can you say about the general case comparing $b_{n+1}(V) > b_n(V)$?

UNSOLVED EXERCISES

1. In each of the following examples, do bidders have private values or not? Why or why not? In each case, imagine that the object in question is being auctioned to a group of Wall Street investors.
    1. Shares of stock in Mystery Enterprises Agricultural Technologies (MEAT), a (fictional) biotech startup that sells “Mystery Meat” under the tagline “There’s no mystery—it tastes like beef!!”
    2. A lifetime supply of Mystery Meat.
    3. A hamburger (made from traditional beef).
2. “In the presence of very risk-averse bidders, a person selling her house in an auction will have a high expected profit by using a first-price, sealed-bid auction.” True or false? Explain your answer.
3. Suppose that three risk-neutral bidders are interested in purchasing a Princess Beanie Baby. The bidders (numbered 1 through 3) have valuations of $12, $14, and $16, respectively. The bidders will compete in auctions as described in parts (a) through (d); in each case, bids can be made in $1 increments at any value from $5 to $25.
    1. Which bidder wins an open-outcry English auction? What is the final price paid, and what is the profit to the winning bidder?
    2. Which bidder wins a second-price, sealed-bid auction? What is the final price paid, and what is the profit to the winning bidder? Contrast your answer here with that for part (a). What is the cause of the difference in profits in these two cases?
    3. In a sealed-bid, first-price auction, all the bidders will bid a positive amount (at least $1) less than their true valuations. What is the likely outcome in this auction? Contrast your answer with those for parts (a) and (b). Does the seller of the Princess Beanie Baby have any clear reason to choose one of these auction mechanisms over the other?
    4. Risk-averse bidders would reduce the shading of their bids in part (c); assume, for the purposes of this question, that they do not shade at all. If that were the case, what would be the winning price (and profit for the bidder) in part (c)?

Would the seller care about which type of auction she chooses? Why?

4. In this exercise, you will explore how much setting a reserve price can increase the seller's expected revenue. Two bidders $i = 1, 2$ have (uncorrelated) private values that are uniformly distributed over the interval from 0 to 1. The figure below illustrates the four basic possible outcomes. First, if $V_1 < r$ and $V_2 < r$ (Region I), then no sale occurs; this happens with probability $Prob(V_1 < r) \times Prob(V_2 < r) = r \times r = r^2$. Second, if $V_1 > r$ and $V_2 < r$ (Region II), then bidder 1 wins and pays the reserve price r; this happens with probability $Prob(V_1 > r) \times Prob(V_2 < r) = (1 - r) \times r = r - r^2$. Third, if $V_1 < r$ and $V_2 > r$ (Region III), then bidder 2 wins and pays r; this also happens with probability $r - r^2$. Finally, if $V_1 > r$ and $V_2 > r$ (Region IV), then the bidder with the higher value wins and pays the lower value; this happens with probability $(1 - r)^2$, and when it happens, the price on average equals $\frac{2r+1}{3}$.[32]



1. What is the seller' s expected revenue without any reserve price?
2. Provide a formula expressing the seller' s expected revenue as a function of *r*. Hint: Determine the probability that bidder values fall in each of the four possible regions shown in the figure and the expected price (with price = 0 if no sale is made) when values fall in each region.
3. Does the seller earn more expected revenue, less expected revenue, or the same amount when setting a reserve price *r* = ½ than when setting no reserve price? How much expected revenue is gained (or lost) by setting reserve price *r* = ½? How often is the object left unsold?

4. Does the seller earn more expected revenue, less expected revenue, or the same amount when setting a reserve price $r = \frac{3}{4}$ than when setting no reserve price? How much expected revenue is gained (or lost) by setting reserve price $r = \frac{3}{4}$? How often is the object left unsold?

5. You are a turnaround artist, specializing in identifying underperforming companies, buying them, improving their performance and stock price, and then selling them. You have found such a prospect, Sicco. This company’s marketing department is mediocre; you believe that if you take over the company, you will increase its value by 75% of whatever it was before. But its accounting department is very good; it can conceal assets, liabilities, and transactions to a point where the company’s true value is hard for outsiders to identify. (But insiders know the truth perfectly.) You think that the company’s value in the hands of its current management is somewhere between $10 million and $110 million, uniformly distributed over this range. The current management will sell the company to you if, and only if, your bid exceeds the true value known to them.
   1. If you bid $110 million for the company, your bid will surely succeed. Is your expected profit positive?
   2. If you bid $50 million for the company, what is the probability that your bid succeeds? What is your expected profit if you do succeed in buying the company? Therefore, at the time when you make your bid of $50 million, what is your expected profit? (Warning: In calculating this expectation, don’t forget the probability of your getting the company.)
   3. What should you bid if you want to maximize your expected profit? (Hint: Assume your bid is $\$X$ million. Carry out the same analysis as in part (b) above, and find an algebraic expression for your expected profit as seen from the time when you are making your bid. Then choose X to maximize this expression.)
6. The idea of the winner’s curse can be expressed slightly differently from its usage in the chapter: “The only time your bid matters is when you win, which happens when your estimate is higher than the estimates of all the other bidders. Therefore you should focus on this case. That is, you should always act as if all the others have received estimates lower than yours, and use this ‘information’ to revise your own estimate.” Here we ask you to apply this idea to a very different situation.

A jury consists of 12 people who hear and see evidence presented at a trial and collectively reach a verdict of guilt or innocence. To simplify the process somewhat, assume that the jurors hold a single simultaneous vote to determine the verdict. Each juror is asked to vote Guilty or Not Guilty. The accused is convicted if all 12 vote Guilty and is acquitted if one or more vote Not Guilty; this is known as the unanimity rule. Each juror's objective is to arrive at a verdict that is the most accurate verdict in light of the evidence, but each juror interprets the evidence in accord with her own thinking and experience. Thus, she arrives at an estimate of the guilt or the innocence of the accused that is individual and private.

1. If jurors vote truthfully—that is, in accordance with their individual private estimates of the guilt of the accused—will the verdict be Not Guilty more often under a unanimity rule or under a majority rule, where the accused is convicted if 7 jurors vote Guilty? Explain. What might we call the "juror's curse" in this situation?
2. Now consider the case in which each juror votes strategically, taking into account the potential problems of the juror's curse and using all the devices of information inference that we have studied. Are individual jurors more likely to vote Guilty under a unanimity rule when voting truthfully or when voting strategically? Explain.
3. Do you think strategic voting to account for the juror's curse would produce too many Guilty verdicts? Why or why not?

7. (**Optional**) This exercise is a continuation of Exercise S6; it looks at the general case where n is any positive integer. It is proposed that the equilibrium bid function with n bidders is $b(V) = V(n - 1)/n$. For $n = 2$, we have the case explored in Exercise S6: Each of the bidders bids half of her value. If there are nine bidders ($n = 9$), then each should bid $^{9}/_{10}$ of her value, and so on.
   1. Now there are $n - 1$ other bidders bidding against you, each using the bid function $b(V) = V(n - 1)/n$. For the moment, let's focus on just one of your rival bidders. What is the probability that she will submit a bid less than 0.1? Less than 0.4? Less than 0.6?
   2. Using the above results, find an expression for the probability that the other bidder has a bid less than your

bid amount b.

3. Recall that there are $n-1$ other bidders, all using the same bid function. What is the probability that your bid b is larger than *all* of the other bids? That is, find an expression for $Prob(\text{win})$, the probability that you win, as a function of your bid b.
4. Use this result to find an expression for your expected profit when your value is V and your bid is b.
5. What is the value of b that maximizes your expected profit?
6. Use your results to argue that it is a Nash equilibrium for all n bidders to follow the same bid function $b(V) = V(n-1)/n$.

Endnotes

- When two bidders have private values uniformly drawn from an interval $[A, B]$, the higher value, on average, equals $(A + 2B)/3$ while the lower value, on average, equals $(2A + B)/3$. When bidder values are in Region IV, they are each drawn from the interval $[r, 1]$. Return to reference 32

■ Appendix: Computing Bidding Equilibria

In this appendix, we derive equilibrium bidding strategies for the second-price auction, first-price auction, and all-pay auction in an extended version of the numerical example in the main text, allowing for any number of bidders and for any reserve price. In particular, there are any number, $n \geqslant 1$, of risk-neutral bidders, each with uncorrelated (independent) private values, V_i, drawn uniformly from the interval [0, 1] (uniformly distributed).

A. Math Facts

Our analysis in this appendix takes advantage of several basic facts about the uniform distribution and leverages mathematical notation to express complex formulas as simply as possible. The following six facts are used throughout our bidding strategy calculations below.

I. MATH FACT ONE (MF1) The first math fact addresses the average highest and average second-highest bidder value. Because bidders' values are uniformly distributed over the interval [0, 1], each bidder's value is, on average, equal to ½. But how high do bidders' values tend to be *when they win* and *when they come in second*? Given bidder values V_1, V_2, . . . , V_n, let $V^{(1)} = \max\{V_1, V_2, . . . , V_n\}$ denote the highest bidder's value, and let $V^{(2)}$ denote the second-highest bidder's value. $V^{(1)}$ is the value of the bidder who wins; its average value is denoted by $E[V^{(1)}]$.[33] $V^{(2)}$ is the value of the bidder who comes in second; its average value is denoted by $E[V^{(2)}]$. We will make extensive use of the fact that $E[V^{(1)}]=\frac{n}{n+1}$ and $E[V^{(2)}]=\frac{n-1}{n+1}$.[34] For example, in the special case with two bidders considered in the main text, $E[V^{(1)}]=\frac{2}{3}$ and $E[V^{(2)}]=\frac{1}{3}$.

II. MATH FACT TWO (MF2)The second math fact addresses the average highest bidder value when the reserve price is not met. Suppose that $V^{(1)} < r$, so that all bidders' values are less than the reserve price. Conditional on that being true, $V^{(2)}$, on average, equals $\frac{n-1}{n+1}r.$ This can be written more succinctly, using mathematical notation, as

$$E[V^{(2)} | V^{(1)} < r] = \frac{n-1}{n+1}r.$$

III. MATH FACT THREE (MF3)The third math fact relates to the likelihood that the reserve price is not met. Each bidder, I, is unwilling to pay the reserve price so long as $V_i < r$. Since bidder i's value is uniformly distributed over [0,1], this happens with probability r, or, using mathematical notation, $Prob[V_i < r] = r$. Of course, the highest bidder's value, $V^{(1)}$, can be less than the reserve price, r, only if all n bidders' values are less than r; so, $Prob[V^{(1)} < r] = (Prob[V_i < r])^n = r^n$.

IV. MATH FACT FOUR (MF4)The fourth math fact relates to the average second-highest bidder value when only one bidder meets the reserve price. Suppose that $V^{(2)} < r < V^{(1)}$, so that exactly one bidder's value exceeds the reserve price. Conditional on that being true, the second-highest bidder's value is, on average, equal to $\frac{n-1}{n}r,$ or, in mathematical notation,

$$E[V^{(2)} | V^{(2)} < r < V^{(1)}] = \frac{n-1}{n}.$$

V. MATH FACT FIVE (MF5)The fifth math fact relates to the likelihood that only one bidder meets the reserve price. It happens that exactly one bidder's value exceeds the reserve price with probability $nr^{n-1}(1 - r)$ or, more succinctly, $Prob[V^{(2)} < r < V^{(1)}] = nr^{n-1}(1 - r)$.

VI. MATH FACT SIX (MF6)The sixth math fact determines the average second-highest bidder value, conditional on the highest bidder value. Suppose that bidder i has the highest value. The second-highest value must, on average, be equal to $\frac{n-1}{n}V_i$; that is,

$$E[V^{(2)} | V_i = V^{(1)}] = \frac{n-1}{n}V_i.$$

B. Second-Price Auction Revenue with Any Reserve

Consider a second-price auction (SPA) in which the reserve price is set at some value r, $r \geqslant 0$. Let $REV(n, r)$ denote the expected revenue in this SPA with n bidders and reserve price r. In the SPA, each bidder has a dominant strategy to stay out of the auction if $V_i < r$ and, otherwise, to enter the auction and bid truthfully. If the reserve price were 0, all bidders would participate, and expected revenue would be

$$REV(n,0)=E[V^{(2)}]=\frac{n-1}{n+1}$$ by MF1.

Depending on the highest bidder value, $V^{(1)}$, and the second-highest value, $V^{(2)}$, adding a reserve price may or may not change the auction outcome. There are three possibilities.[35]

Case 1: $V^{(1)} < r$. In this case, setting the reserve price causes the object not to sell, hurting the seller because revenue $V^{(2)}$ is lost. This case occurs with probability $\text{Prob}(V^{(1)} < r) = r^n$ (by MF3), and when it does, revenue decreases from an average of

$$E[V^{(2)}|V^{(1)}<r]=\frac{n-1}{n+1}r$$ to 0 (by MF2).

The overall expected harm to the seller in this case is therefore

$$r^n \frac{n-1}{n+1}r=\frac{n-1}{n+1}r^{n+1}.$$

Case 2: $V^{(2)} < r < V^{(1)}$. In this case, the object sells at price r rather than at price $V^{(2)}$, benefiting the seller. This case occurs with probability Prob$(V^{(2)} < r < V^{(1)}) = nr^{n-1}(1-r)$ (by MF5), and when it does,

$$E[V^{(2)} | V^{(2)} < r < V^{(1)}] = \frac{n-1}{n} r$$

(by MF4). The overall expected benefit to the seller in this case is therefore

$$nr^{n-1}(1-r)\left(r - \frac{n-1}{n} r \right) = r^n(1-r).$$

Case 3: $V^{(2)} > r$. In this case, the reserve price has no effect since the object still sells at price $V^{(2)}$. Adding up all these effects, we conclude that expected revenue for the SPA would be

$$REV(n,r) = REV(n,0) + r^n(1-r) - \frac{n-1}{n+1} r^{n+1},$$

which, after plugging in $REV(n,0) = \frac{n-1}{n+1}$ and collecting terms, yields the formula

$$REV(n,r) = \frac{n-1}{n+1} + r^n - \frac{2n}{n+1} r^{n+1}.$$

C. Optimal Reserve Price

We can show that a seller's optimal reserve price, $r^*(n)$, the reserve price that maximizes expected revenue, is equal to ½ for all n. (This derivation uses calculus.) To calculate $r^*(c)$, we first take the derivative of $REV(n, r)$, derived in Section B , with respect to r:

$$r: \frac{dREV(n,r)}{dr} = nr^{n-1} - 2nr^n.$$

Since $r^*(n)$ maximizes $REV(n, r)$ (and $REV(n, r)$ is a smooth function), it must be true that

$$\frac{dREV(n,r^*(n))}{dr} = 0.$$

We conclude that $nr^*(n)^{n-1}(1 - 2r^*(n)) = 0$, which is possible only if $r^*(n) = 0$

$$\text{if } r^*(n) = 0 \text{ or } r^*(n) = \frac{1}{2}.$$

or Comparing these two possibilities, note that

$$REV(n,0) = \frac{n-1}{n+1}$$

and that

$$REV\left(n, \frac{1}{2}\right) = \frac{n-1}{n+1} + \frac{(1/2)^n}{n+1} > REV(n,0).$$

So, it must be true that the optimal reserve price

$$r^*(n) = \frac{1}{2}$$

for all n.

The argument thus far applies to the second-price auction (SPA), but, according to the revenue equivalence theorem (RET), it also applies to the first-price auction (FPA). To see why, let $REV^{\mathrm{FPA}}(n, r)$ and $REV^{\mathrm{SPA}}(n, r)$ denote expected revenue in the FPA and SPA, respectively, given n bidders and reserve price r. Because all RET assumptions hold in this example, RET implies that $REV^{\mathrm{FPA}}(n, r) = REV^{\mathrm{SPA}}(n, r)$ for all (n, r). The fact that $REV^{\mathrm{SPA}}(n, r)$ is maximized at

$$r^*(n) = \frac{1}{2}$$

for all n therefore implies that $REV^{\mathrm{FPA}}(n, r)$ is maximized at $r^*(n) = ½$. Indeed, the same argument shows that $r^*(n) = ½$ is the optimal reserve price in any of the standard auction formats; it is true not just for the SPA and the FPA/Dutch auction, but also for the English auction, the all-pay auction, and the war of attrition.

D. Equilibrium Bidding Strategies in the First-Price Auction

There's another way that the revenue equivalence theorem (RET) can come in handy when analyzing auctions. It provides a shortcut when computing equilibrium bidding strategies in tough-to-analyze auctions. (In what follows, we focus for simplicity on the case with no reserve price.)

Bidder i wins the object in the SPA when his value is the highest ($V_i = V^{(1)}$) and pays a price equal to the second-highest value, which, on average, equals

$$E[V^{(2)} \mid V_i = V^{(1)}] = \frac{n-1}{n} V_i$$

(by MF6). By RET, we know that the FPA must generate expected revenue equal to that of the SPA, and that for this to happen, bidders in the FPA must pay the same amount (when they win) as they do in the SPA. Since bidders pay their full bid when they win in the FPA, but pay only $\frac{n-1}{n} V_i$ in the SPA, this tells us that the equilibrium bidding strategy in the FPA must be $b^{\text{FPA}}(V_i) = \frac{n-1}{n} V_i.$ That's it!

To check that this solution is indeed an equilibrium, suppose that all other bidders use this bidding strategy and that bidder i has value V_i. Let $S(b_i, V_i)$ denote bidder i's expected surplus when bidding b_i given value V_i, and let $b_i^*(V_i)$ be the bid that maximizes $S(b_i, V_i)$. Bid b_i is high enough to win the auction so long as

$$b_i > \max_{j \neq i} \frac{n-1}{n} V_j;$$

this happens with probability

$$Prob\left[\max_{j \neq i} V_j < b\frac{n}{n-1}\right]$$

$$= \left(b\frac{n}{n-1}\right)^{n-1}$$

(by the same logic behind MF3), and when it happens, generates bidder surplus $V_i - b_i$. So,

$$V_i - b_i. \text{ So, } S(b_i, V_i) = \left(b\frac{n}{n-1}\right)^{n-1} \times (V_i - b_i),$$

which, after rearranging, becomes

$$S(b_i, V_i) = \left(\frac{n}{n-1}\right)^{n-1} (b_i^{n-1} V_i - b_i^n).$$

Taking the derivative with respect to b,

$$\frac{dS(b_i, V_i)}{db} = \left(\frac{n}{n-1}\right)^{n-1} ((n-1)(b_i)^{n-2} V_i - n(b_i)^{n-1}),$$

which equals zero only at $b_i = 0$ or $$b_i = \frac{n-1}{n} V_i.$$

Checking these two bid levels, note that $S(0, V_i) = 0$ while

$$S\left(\frac{n-1}{n} V_i, V_i\right) = \frac{(V_i)^n}{n} > 0.$$

We conclude that the optimal bid $$b_i^*(V_i) = \frac{n-1}{n} V_i,$$ as desired. In particular, in the special case considered in the text with $n = 2$, $$n = 2,\ b_i^*(V_i) = \frac{V_i}{2}.$$

E. Equilibrium Bidding Strategies in the All-Pay Auction

By RET, we know that the all-pay auction must generate expected revenue equal to that of the SPA, and that for this to happen, bidders in the all-pay auction must pay the same amount (on average) as they do in the SPA. Since bidders always pay their full bid in the all-pay auction, but pay only $\frac{n-1}{n}V_i$ with probability $Prob\left[\max_{j \neq i} V_j < V_i\right] = (V_i)^{n-1}$ in the SPA, this tells us that the equilibrium bidding strategy in the all-pay auction must be

$$b^{\mathrm{APA}}(V_i) = \frac{n-1}{n} V_i \times (V_i)^{n-1} = \frac{n-1}{n}(V_i)^n.$$

That's it!

Endnotes

- Given any random variable X, the notation $E[X]$ means "the average (or 'expected') value of X." Similarly, given random variables X and Y, $E[X \mid Y < y]$ means "the average value of X, conditional on Y being less than y." Return to reference 33
- Interested readers can verify these facts for themselves—or you can just trust us. To dive deeper into the fascinating mathematics of these so-called 'order statistics,' consult Barry C. Arnold, N. Balakrishnan, and H. N. Nagaraja, *A First Course in Order Statistics*, Classics in Applied Mathematics (Philadelphia, Pa.: Society for Industrial and Applied Mathematics, 2008). Return to reference 34
- We ignore the zero-probability events in which $V^{(1)} = r$ or $V^{(2)} = r$. Return to reference 35

16■ Strategy and Voting

WHEN MANY OF YOU think about voting, you probably imagine first a national presidential election, then perhaps a local mayoral election, and maybe even an election for class president at your school. But some of you may also be reminded of last year's Heisman Trophy–winning college football player, the latest Academy Award–winning film, or the most recent Supreme Court decision. *All* of these situations involve voting, although they differ in the number of voters involved, the ballot length or number of choices available to those voters, and the procedures used to tally the votes and determine the final winner. In each case, strategic thinking may play a role in how ballots are marked. And strategic considerations can be critical in choosing the method by which votes are taken and then counted.

Voting procedures vary widely not because some votes elect Oscar winners and others elect presidents, but because certain procedures have attributes that make them better (or worse) for specific voting situations. In the past decade, for example, concerns about how elections based on the plurality rule (the candidate with the most votes wins) encourage the existence of a two-party political system have led to changes in voting rules in more than a dozen U.S. cities.[1] These changes have led in some cases to election outcomes that differed from those that would have arisen under the old plurality-rule system. Jean Quan, the mayor of Oakland, California, for example, won her post in November 2010 under that city's new ranked-choice voting system despite being the first choice of only 24% of the voters, while the eventual runner-up had 35% of the first-place votes. In the last round of the ranked-choice vote, Quan won 51% of the votes, with 49% going to the runner-up. We investigate such seemingly paradoxical outcomes in Section 2 of this chapter.

Given the fact that different voting procedures can produce different outcomes, you should immediately see the scope for strategic behavior in choosing a procedure that can generate an outcome you prefer. Perhaps, then, you can also imagine a situation in which voters might find it beneficial to vote for someone, or something, that is not their top choice in order to avoid having their absolute last choice option be the winner. This type of strategic behavior is common when the voting procedure allows it. As a voter, you should be aware of the benefits associated with such *strategic misrepresentation of preferences* and of the possibility that others may use such tactics against you.

In this chapter, we first introduce you to the range of voting procedures available and to some of the paradoxical outcomes that can arise when specific procedures are used. We then consider how one might judge the performance of those procedures before addressing the impact of strategic voting. Finally, we present two different versions of a well-known result known as the *median voter theorem*—as a two-person zero-sum game with discrete strategies and with continuous ones.

Endnotes

- This result is known in political science as Duverger’s law. We discuss it in greater detail in Section 4.A. Return to reference 1

1 VOTING RULES AND PROCEDURES

Numerous voting procedures are available to help voters choose from a slate of alternatives (that is, candidates or issues). With as few as three available alternatives, election design becomes interestingly complex. We describe in this section a variety of procedures from three broad classes of voting, or vote-aggregation, methods. We have not attempted to provide an exhaustive survey—the number of possible voting procedures is enormous, and the simple taxonomy that we provide here can be broadened extensively by using combinations of these procedures—but rather to give you a flavor of the broader literatures in economics and political science on the subject.[2]

A. Binary Methods

Vote-aggregation methods can be classified according to the number of options or candidates considered by the voters at any given time. Binary methods require voters to choose between only two alternatives at a time. In elections in which there are exactly two candidates, votes can be aggregated by using the well-known principle of majority rule, which simply requires that the alternative with a majority of votes wins. When dealing with a slate of more than two alternatives, pairwise voting—a method consisting of a repetition of binary votes—can be used. Pairwise voting is a multistage procedure; it entails voting on pairs of alternatives in a series of majority votes to determine which alternative is most preferred.

One pairwise voting procedure is called the Condorcet method, after the eighteenth-century French theorist Antoine Nicholas Caritat, marquis de Condorcet. This procedure requires a separate majority vote for each possible pair of alternatives (a "round robin"); if there are n alternatives, there are $n(n-1)/2$ such pairs. If one alternative wins all of its $(n-1)$ binary elections, it wins the entire election and is termed a Condorcet winner. There may be no Condorcet winner, but there cannot be two or more (except in the very unusual circumstances that there are ties in all of the binary elections between the two, or more, alternatives that win all of their other elections). Other pairwise procedures produce "scores" such as the Copeland index, which measures an alternative's win-loss record in a round robin of contests. The first round of the World Cup soccer tournament uses a type of Copeland index to determine which teams from each group move on to the second round of play.[3]

Another well-known pairwise procedure, used when there are three possible alternatives, is the amendment procedure, required by the parliamentary rules of the U.S. Congress when legislation is brought to a vote. When a bill is brought before Congress, any amended version of the bill must first win a majority vote against the original version of the bill. The winner of that vote is then paired against the status quo, and members vote on whether to adopt the version of the bill that won the first round. The amendment procedure can be used to consider any three alternatives by pairing two in a first-round election and then putting the third up against the winner in a second-round vote.

B. Plurative Methods

Plurative methods allow voters to consider three or more alternatives simultaneously. One group of plurative voting methods applies information on the positions of alternatives on a voter’s ballot to assign points used when tallying ballots; these voting methods are known as positional methods. The familiar plurality rule is a special-case positional method in which each voter casts a single vote for her most preferred alternative. That alternative is assigned a single point when votes are tallied; the alternative with the most votes (or points) wins. Note that a plurality winner need *not* gain a majority, or 51%, of the vote. Thus, for instance, in the 2012 presidential election in Mexico, Enrique Peña Nieto captured the presidency with only 38.2% of the vote; his opponents gained 31.6%, 25.4%, and 2.3% of the vote. Such narrow margins of victory have led to concerns about the legitimacy of past Mexican presidential elections, especially in 2006, when the margin of victory was a mere 0.58 percentage points. Another special-case positional method, the antiplurality method, asks voters to vote against one of the available alternatives or, equivalently, to vote for all but one. For counting purposes, the alternative voted against is allocated −1 point, or else all alternatives except that one receive 1 point while the alternative voted against receives 0.

One of the best-known positional methods is the Borda count, named after Jean-Charles de Borda, a fellow countryman and contemporary of Condorcet. Borda described the new procedure as an improvement on the plurality rule. The Borda count requires voters to rank all of the possible alternatives in an election and to indicate their rankings on their ballot cards. Points are assigned to each alternative on the basis of its position on each voter’s ballot. In a three-person election, for example, the candidate at the top of a ballot gets 3 points, the next candidate 2 points, and the bottom candidate 1 point. After the ballots are collected, each candidate's points are summed, and the one with the most points wins the election. A Borda count procedure is used in a number of sports-related elections, including professional baseball's Cy Young Award and college football's championship elections.

Many variations on the Borda count can be devised simply by altering the rule used for the allocation of points to alternatives based on their positions on a voter's ballot. One system might allocate points in such a way as to give the top-ranked alternative relatively more than the others—for example, 5 points for the most preferred alternative in a three-way election, but only 2 and 1 for the second- and third-ranked options. In elections with larger numbers of candidates—say, eight—the top two choices on a voter's ballot might receive preferred treatment, gaining 10 and 9 points, respectively, while the others receive 6 or fewer.

An alternative to these positional plurative methods is the relatively recently invented approval voting method, which allows voters to cast a single vote for each alternative of which they "approve."[4] Unlike positional methods, approval voting does not distinguish between alternatives on the basis of their positions on the ballot. Rather, all approval votes are treated equally, and the alternative that receives the most approvals wins. In elections in which more than one winner can be selected (in electing a school board, for instance), a threshold level of approvals is set in advance, and alternatives with more than the required minimum of approvals are elected. Proponents of this method argue that it favors relatively moderate alternatives over those at either end of a spectrum; opponents claim that unwary voters could elect an unwanted novice candidate by indicating too many "encouragement" approvals on their ballots. Despite these disagreements, several professional societies and the United Nations have adopted approval voting to elect their officers, and some states have used or are considering using this method for public elections.

C. Mixed Methods

Some multistage voting procedures combine plurative and binary voting in mixed methods. The majority runoff procedure, for instance, is a two-stage method used to reduce a large group of possibilities to a binary decision. In a first-stage election, voters indicate their most preferred alternative, and these votes are tallied. If one candidate receives a majority of votes in the first stage, she wins. However, if there is no majority choice, a second-stage election pits the two most preferred alternatives against each other. The winner is chosen by majority rule in the second stage. French presidential elections use the majority runoff procedure, which can yield unexpected results if three or four strong candidates split the vote in the first round. In the spring of 2002, for example, the far-right candidate Jean-Marie Le Pen came in second, ahead of France’s socialist Prime Minister Lionel Jospin, in the first round of the presidential election. This result aroused surprise and consternation among French citizens, 30% of whom hadn’t even bothered to vote in the first round, and some of whom had taken it as an opportunity to express their preference for various candidates of the far and fringe left. Le Pen’s advance to the runoff election led to considerable political upheaval, although he lost in the end to the incumbent president, Jacques Chirac.

Another mixed procedure consists of voting in successive rounds. Voters consider a number of alternatives in each round of voting. The worst-performing alternative is eliminated after each round, and voters then consider the remaining alternatives in the next round. The elimination continues until only two alternatives remain; at that stage, the method becomes binary, and a final majority runoff determines a winner. A procedure with rounds is used to choose sites for the Olympic Games.

One can eliminate the need for successive rounds of voting by having each voter indicate her preference ordering by ranking all the candidates on a single ballot. Then a single transferable vote method can be used to tally votes. If no alternative receives a majority of all first-place votes, the bottom-ranked alternative is eliminated, and all first-place votes for that candidate are "transferred" to the candidate ranked second on those ballots; similar reallocation occurs in later rounds as additional alternatives are eliminated until a majority winner emerges. This voting method, also referred to as instant runoff voting (IRV) or ranked-choice voting, has been slowly gaining traction. In 2010, IRV was used in a dozen U.S. cities. In 2016, Maine became the first state to use IRV for both primary and general elections for both statewide and national offices (governor, state legislature, U.S. House, and U.S. Senate), and Maine voters reaffirmed IRV in a 2018 referendum. However, some who experimented with IRV have abandoned it. For example, North Carolina used IRV for statewide judicial elections in 2010, but then returned to plurality rule, and Aspen, Colorado used IRV for city council elections in 2008, but then returned to a more traditional runoff system.[5]

In 2015, Democrat Ethan Strimling was elected mayor of Portland, Maine, in a three-way race using instant runoff voting, defeating incumbent mayor Michael Brennan (also a Democrat) and Green Independent Party leader Tom MacMillan. Strimling led the voting from the first to the final round, but some outside observers commented that Strimling might have lost if the election had used the plurality rule—the idea being that Democratic voters might have rallied around the incumbent Brennan to avoid splitting the Democratic vote and allowing MacMillan to win.[6] (You are asked to consider this example in more detail in Exercise S3.)

The single transferable vote method is sometimes combined with other procedures meant to promote proportional representation in an election. Proportional representation means that a state electorate consisting of 55% Republicans, 25% Democrats, and 20% Independents, for example, would yield a body of representatives mirroring the party affiliations of that electorate—in other words, that 55% of the U.S. representatives from such a state would be Republican, and so on. Such methods contrast starkly with the plurality rule, which would elect *all* Republicans (assuming that the voter mix in each district exactly mirrors the overall voter mix in the state). Under proportional representation rules, candidates who attain a certain quota of votes are elected, and others who fall below a certain quota are eliminated, depending on the exact specifications of the voting procedure. Votes for those candidates who are eliminated are again transferred by using the voters' preference orderings. This procedure continues until an appropriate number of candidates from each party is elected. Versions of this type of procedure are used in parliamentary elections in both Australia and New Zealand.

Clearly, there is room for considerable strategic thinking in the choice of a vote-aggregation method, and strategy is important even after the rule has been chosen. We examine some of the issues related to rule making and agenda setting in Section 2. Furthermore, strategic behavior on the part of voters, often called strategic voting or strategic misrepresentation of preferences, can also alter election outcomes under any set of rules, as we will see later in this chapter.

Endnotes

- The classic textbook on this subject, which was instrumental in making game theory popular in political science, is William Riker, *Liberalism against Populism* (San Francisco: W. H. Freeman, 1982). A general survey is "Symposium: Economics of Voting," *Journal of Economic Perspectives*, vol. 9, no. 1 (Winter 1995). An important early research contribution is Michael Dummett, *Voting Procedures* (Oxford: Clarendon Press, 1984). Donald Saari, *Chaotic Elections* (Providence, R.I.: American Mathematical Society, 2000), develops some new ideas that we use later in this chapter. Return to reference 2
- Note that such indexes, or scores, must have precise mechanisms in place to deal with ties; World Cup soccer uses a system that undervalues a tie to encourage more aggressive play. See Barry Nalebuff and Jonathan Levin, "An Introduction to Vote Counting Schemes," *Journal of Economic Perspectives*, vol. 9, no. 1 (Winter 1995), pp. 3–26. Return to reference 3
- Unlike the many voting methods that have histories going back several centuries, the approval voting method was designed and named by then-graduate student Robert Weber in 1971. Weber is now a professor of managerial economics and decision sciences at Northwestern University, specializing in game theory. Return to reference 4
- Some Aspen residents complained that they did not know all of the candidates well enough to rank them, and that more time was needed for them to make informed choices in the runoff. See Carolyn Sackariason, "Aspen Voters to Vote on How They Vote-Again," *Aspen Times*, July 22, 2009. Return to reference 5
- Drew Spencer Penrose, "Seven Ways Ranked Choice Voting is Empowering Voters in 2015," FairVote.org, November 4, 2015, available at https://www.fairvote.org/seven-ways-

ranked-choice-voting-is-empowering-voters-in-2015(accessed May 29, 2019). Return to reference 6

Glossary

binary method
: A class of voting methods in which voters choose between only two alternatives at a time.

majority rule
: A voting method in which the winning alternative is the one that garners a majority (more than 50%) of the votes.

pairwise voting
: A voting method in which only two alternatives are considered at the same time.

multistage procedure
: A voting procedure in which there are multiple rounds of voting.

Condorcet method
: A voting method in which the winning alternative must beat each of the other alternatives in a round-robin of pairwise contests.

Condorcet winner
: The alternative that wins an election run using the *Condorcet method*.

Copeland index
: An index measuring an alternative's record in a round-robin of contests where different numbers of points are allocated for wins, ties, and losses.

amendment procedure
: A procedure in which any amended version of a proposal must win a vote against the original version before the winning version is put to a vote against the status quo.

plurative method
: Any voting method that allows voters to consider a slate of three or more alternatives simultaneously.

positional method
: A voting method that determines the identity of the winning alternative using information on the position of

alternatives on a voter's ballot to assign points used when tallying ballots.

plurality rule
: A voting method in which two or more alternatives are considered simultaneously and the winning alternative is the one that garners the largest number of votes; the winner needs only gain more votes than each of the other alternatives and does not need 50% of the vote as would be true in *majority rule*.

antiplurality method
: A positional voting method in which the electorate is asked to vote against one item on the slate (or to vote for all but one).

Borda count
: A positional voting method in which the electorate indicates its order of preference over a slate of alternatives. The winning alternative is determined by allocating points based on an alternative's position on each ballot.

approval voting
: A voting method in which voters cast votes for all alternatives of which they approve.

mixed method
: A multistage voting method that uses plurative and binary votes in different rounds.

majority runoff
: A two-stage voting method in which a second round of voting ensues if no alternative receives a majority in the first round. The top two vote-getters are paired in the second round of voting to determine a winner.

round
: A single vote within a larger *multistage procedure* that consists of multiple sequentially held votes.

single transferable vote
: A voting method in which each voter indicates her preference ordering over all candidates on a single initial ballot. If no alternative receives a majority of

all first-place votes, the bottom-ranked alternative is eliminated and all first-place votes for that candidate are "transferred" to the candidate listed second on those ballots; this process continues until a majority winner emerges. Also called **instant-runoff voting (IRV)** or **ranked-choice voting**.

instant-runoff voting (IRV)
: Same as **single transferable vote**.

ranked-choice voting
: Another name for **single transferable vote**.

proportional representation
: This voting system requires that the number of seats in a legislature be allocated in proportion to each party's share of the popular vote.

strategic voting
: Voting in conformity with your optimal rational strategy found by doing rollback analysis on the game tree of the voting procedure.

strategic misrepresentation of preferences
: Refers to strategic behavior of voters when they use rollback to determine that they can achieve a better outcome for themselves by not voting strictly according to their preference orderings.

2 VOTING PARADOXES

Election outcomes can depend critically on the type of procedure used to aggregate votes. Even when people vote according to their true preferences, certain voter preferences or voting procedures can give rise to counterintuitive outcomes. This section describes some of the most famous of those outcomes—the so-called voting paradoxes—as well as some examples of how election results can change under different vote-aggregation methods with no change in voter preferences and no strategic voting.

A. The Condorcet Paradox

The Condorcet paradox is one of the most famous and important of the voting paradoxes.[7] As mentioned earlier, the Condorcet method calls for the winner to be the candidate who gains a majority of votes in each round of a round robin of pairwise comparisons. The paradox arises when no Condorcet winner emerges from this process.

To illustrate the paradox, we construct an example in which three people vote on three alternatives using the Condorcet method. Consider three city councillors (Left, Center, and Right) who are asked to rank their preferences for three alternative welfare policies: one that extends the welfare benefits currently available (call this one Generous, or G), another that decreases available benefits (Decreased, or D), and yet another that maintains the status quo (Average, or A). They are then asked to vote on each pair of policies to establish a *group ranking*, or social ranking. This ranking is meant to describe how the council as a whole judges the merits of the alternative welfare systems.

Suppose Councillor Left prefers to keep benefits as high as possible, whereas Councillor Center is most willing to maintain the status quo, but is concerned about the state of the city budget, and so is least willing to extend welfare benefits. Finally, Councillor Right most prefers reducing benefits, but prefers an increase in benefits to the status quo; she expects that extending benefits will soon cause a serious budget crisis and turn public opinion so much against benefits that a more permanent state of low benefits will result, whereas the status quo could go on indefinitely. We illustrate these preference orderings in Figure 16.1, where the “curly” greater-than symbol, $\succ$, is used to indicate that one alternative is preferred to another. (Technically, $\succ$ is referred to as a *binary ordering relation*.)

With these preferences, if Generous is paired against Average, Generous wins. In the next pairing, of Average against Decreased, Average wins. And in the final pairing, of Generous against Decreased, the vote is again 2 to 1, this time in favor of Decreased. Therefore, if the council votes on alternative pairs of policies, a majority prefer Generous over Average, Average over Decreased, *and* Decreased over Generous. No one policy has a majority over both of the others. The group's preferences are cyclical: G $\succ$ A $\succ$ D $\succ$ G.

Left	Center	Right
G > A > D	A > D > G	D > G > A

FIGURE 16.1 Councillor Preference Orderings for Welfare Policies

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This cycle of preferences is an example of an intransitive ordering of preferences. The concept of rationality is usually taken to mean that individual preference orderings are transitive (the opposite of intransitive). If someone is given choices A, B, and C, and you know that she prefers A to B and B to C, then transitivity implies that she also prefers A to C. (The terminology comes from the transitivity of numbers in mathematics; for instance, if 3 > 2 and 2 > 1, then we know that 3 > 1.) A transitive preference ordering does not cycle as the social ranking derived in our city council example does; hence, we say that such an ordering is intransitive.

Notice that all of the *councillors* have transitive preference orderings of the three welfare policy alternatives, but the *council* does not. This is the Condorcet paradox: Even if all individual preference orderings are transitive, there is no guarantee that the social preference ordering induced by Condorcet's voting procedure will also be transitive. This result has far-reaching implications for public servants as well as for the general public. It calls into question the basic notion of the "public interest" because such interests may not be easily defined, or may not even exist. Our city council does not have any well-defined set of group preferences among the welfare policies. The lesson is that societies, institutions, or other large groups of people should not always be analyzed as if they acted like individuals.

The Condorcet paradox can even arise more generally. There is no guarantee that the social ranking induced by *any* formal group voting process will be transitive just because individual preferences are. However, some estimates have shown that the paradox is most likely to arise when large groups of people are considering large numbers of alternatives. Smaller groups considering smaller numbers of alternatives are more likely to have similar preferences among those alternatives; in such situations, the paradox is much less likely to appear.[8] In fact, the paradox arose in our example because the council completely disagreed not only about which alternative was best, but also about which was worst. The smaller the group, the less likely such outcomes are to occur.

B. The Agenda Paradox

The second paradox that we consider also entails a binary voting procedure, but this example considers the ordering of alternatives in that procedure. In a parliamentary setting with a committee chair who determines the specific order of voting for a three-alternative election, the chair has substantial power over the final outcome. In fact, the chair can take advantage of the intransitive social preference ordering that arises from some sets of individual preferences and, by selecting an appropriate agenda, manipulate the outcome of the election in any manner she desires.

Consider again the city councillors Left, Center, and Right, who must decide among Generous, Average, and Decreased welfare policies. The councillors' preference rankings of the alternatives were shown in Figure 16.1. Let us now suppose that one of the councillors has been appointed chair of the council by the mayor, and the chair is given the right to decide which two welfare policies get voted on first and which goes up against the winner of that initial vote. With the given set of councillor preferences, as long as the chair knows all of the preference orderings, she can get any outcome that she wants. If Left were chosen as chair, for example, she could orchestrate a win for Generous by setting Average against Decreased in the first round, with the winner to go up against Generous in round two. The fact that any final ordering can be obtained by choosing an appropriate procedure under these circumstances is known as the agenda paradox.

The only determinant of the outcome in our city council example is the ordering of the agenda. Thus, setting the agenda is the real game here, and because the chair sets the agenda, the appointment or election of the chair is the true outlet for strategic behavior. Here, as in many other strategic situations, what appears to be the game (in this case, choice of a welfare policy) is not the true game at all; rather, those participating in the game engage in strategic play at an earlier point (deciding the identity of the chair) and vote according to set preferences in the eventual election.

There is a subtlety here worth noting: The preceding demonstration of the agenda setter's power assumes that in the first round, voters choose between the two alternatives (Average and Decreased) only on the basis of their rankings of these two alternatives, with no regard for the eventual outcome of the procedure. Such behavior is called sincere voting or truthful voting; actually, myopic or nonstrategic voting would be a better name. If Center is a strategic game player, she should realize that if she votes for Decreased in the first round (even though she prefers Average between the pair presented at that stage), then Decreased will win the first round and will also win against Generous in the second round with support from Right. Center prefers Decreased over Generous as the eventual outcome. Therefore, she should do this rollback analysis and vote strategically in the first round. But should she do so if everyone else is also voting strategically? We examine the game of strategic voting and find its equilibrium in Section 4.

C. The Reversal Paradox

Positional voting methods can also lead to paradoxical results. The Borda count, for example, can yield the reversal paradox when the slate of candidates presented to voters changes. This paradox arises in an election with at least four alternatives when one of them is removed from consideration after votes have been submitted, making recalculation necessary.

Suppose there are four candidates for a (hypothetical) special commemorative Cy Young Award to be given to a retired major-league baseball pitcher. The candidates are Steve Carlton, Sandy Koufax, Robin Roberts, and Tom Seaver. Seven prominent sportswriters are asked to rank these pitchers on their ballot cards. The top-ranked candidate on each card will get 4 points; decreasing numbers of points will be allotted to candidates ranked second, third, and fourth.

Across the seven voting sportswriters, there are three different preference orderings of the candidate pitchers; these preference orderings, with the number of writers having each ordering, are shown in Figure 16.2. When the votes are tallied, Seaver gets $(2 \times 3) + (3 \times 2) + (2 \times 4) = 20$ points; Koufax gets $(2 \times 4) + (3 \times 3) + (2 \times 1) = 19$ points; Carlton gets $(2 \times 1) + (3 \times 4) + (2 \times 2) = 18$ points; and Roberts gets $(2 \times 2) + (3 \times 1) + (2 \times 3) = 13$ points. Seaver wins the election, followed by Koufax, Carlton, and Roberts in last place.

Now suppose it is discovered that Roberts is not really eligible for the commemorative award, because he never actually won a Cy Young Award, having reached the pinnacle of his career in the years just before the institution of the award in 1956. This discovery requires points to be recalculated, ignoring Roberts on the ballots. The top spot on each card now gets 3 points, while the second and third spots receive 2 and 1, respectively. Ballots from sportswriters with preference ordering 1, for example, now give Koufax and Seaver 3 and 2 points, respectively, rather than the 4 and 3 from the first calculation; those ballots also give Carlton a single point for last place.

Adding votes with the revised point system shows that Carlton receives 15 points, Koufax receives 14 points, and Seaver receives 13 points. Winner has turned loser as the new results reverse the standings in the original election. No change in preference orderings accompanies this result. The only difference between the two elections is the number of candidates being considered. In Section 3, we identify the key vote-aggregation principle violated by the Borda count that leads to the reversal paradox.

Ordering 1 (2 voters)	Ordering 2 (3 voters)	Ordering 3 (2 voters)
Koufax > Seaver > Roberts > Carlton	Carlton > Koufax > Seaver > Roberts	Seaver > Roberts > Carlton > Koufax

FIGURE 16.2 Sportswriter Preference Orderings for Pitchers

D. Change the Voting Method, Change the Outcome

As should be clear from the preceding discussion, election outcomes are likely to differ under different sets of voting rules. As an example, consider 100 voters who can be broken down into three groups on the basis of their preference rankings of three candidates (A, B, and C), as shown in Figure 16.3. With these preferences as shown, and depending on the vote-aggregation method used, any of the three candidates could win the election.

With a simple plurality rule, candidate A wins with 40% of the vote, even though 60% of the voters rank her lowest of the three candidates. Supporters of candidate A would obviously prefer this type of election. If they had the power to choose the voting method, then the plurality rule, a seemingly fair procedure, would win the election for A in spite of the majority’s strong dislike for that candidate.

The Borda count would produce a different outcome. In a Borda system with 3 points going to the most preferred candidate, 2 points to the middle candidate, and 1 to the least preferred candidate, A would get 40 first-place votes and 60 third-place votes, for a total of $(40 \times 3) + (60 \times 1) = 180$ points. Candidate B would get 25 first-place votes and 75 second-place votes, for a total of $(25 \times 3) + (75 \times 2) = 225$ points; and C would get 35 first-place votes, 25 second-place votes, and 40 third-place votes, for a total of $(35 \times 3) + (25 \times 2) + (40 \times 1) = 195$ points. With this procedure, B wins, with C in second place and A last. Candidate B would also win with the antiplurality method, in which electors cast votes for all but their least preferred candidate.

And what about candidate C? She could win the election if a majority or an instant runoff system were used. In either method, A and C, with 40 and 35 votes in the first round, would survive to face each other in the runoff. The majority runoff system would call voters back to the polls to consider A and C; the instant runoff system would eliminate B and reallocate B’s votes from Group 2 voters to their next preferred alternative, candidate C. Then, because A is the least preferred alternative for 60 of the 100 voters, candidate C would win the runoff 60 to 40.

Another example of how different procedures can lead to different outcomes can be seen in the case of the 2010 Oakland mayoral election described in the introduction to this chapter. Olympics site selection voting is now done using instant runoff instead of several rounds of plurality-rule voting with elimination. The change was made after similar unusual results in voting for the 1996 and 2000 host cities. In both cases, the plurality winner in all but the final round lost to the one remaining rival city in the last round: Athens lost out to Atlanta for the 1996 Games, and Beijing lost out to Sydney for the 2000 Games.

Group 1 (40 voters)	Group 2 (25 voters)	Group 3 (35 voters)
A > B > C	B > C > A	C > B > A

FIGURE 16.3 Group Preference Orderings for Candidates

Endnotes

- It is so famous that economists have been known to refer to it as *the* voting paradox. Political scientists appear to know better, in that they are far more likely to use its formal name. As we will see, there are any number of possible voting paradoxes, not just the one named for Condorcet. Return to reference 7
- See Peter Ordeshook, *Game Theory and Political Theory* (Cambridge: Cambridge University Press, 1986), p. 58. Return to reference 8

Glossary

Condorcet paradox

Even if all individual voter preference orderings are transitive, there is no guarantee that the social preference ordering generated by Condorcet’s voting method will also be transitive.

social ranking

The preference ordering of a group of voters that arises from aggregating the preferences of each member of the group.

intransitive ordering

A preference ordering that cycles and is not *transitive*. For example, a preference ordering over three alternatives A, B, and C is intransitive if A is preferred to B and B is preferred to C but it is not true that A is preferred to C.

transitive ordering

A preference ordering for which it is true that if option A is preferred to B and B is preferred to C, then A is also preferred to C.

agenda paradox

A voting situation where the order in which alternatives are paired when voting in multiple rounds determines the final outcome.

sincere voting

Voting at each point for the alternative that you like best among the ones available at that point, regardless of the eventual outcome. Also called **truthful voting**.

truthful voting

Same as **sincere voting**.

reversal paradox

This paradox arises in an election with at least four alternatives when one of these is removed from consideration after votes have been submitted and the removal changes the identity of the winning alternative.

3 EVALUATING VOTING SYSTEMS

The discussion of the various voting paradoxes in Section 2 suggests that voting methods can suffer from a number of faults that lead to unusual, unexpected, or even unfair outcomes. This suggestion leads us to ask, Is there one voting system with properties that most people would regard as most desirable, including being “fair”—that is, most accurately capturing the preferences of the electorate? Kenneth Arrow listed some such properties and proved that no vote-aggregation system can satisfy them all; the result is his celebrated impossibility theorem.[9]

The technical content of Arrow’s theorem makes it beyond our scope to prove completely. But the sense of the theorem is straightforward. Arrow argued that no vote-aggregation method could conform to all six of the critical principles that he identified:

1. The social or group ranking must rank all alternatives (be complete).
2. It must be transitive.
3. It must satisfy a condition known as *positive responsiveness*, or the Pareto property: Given two alternatives, A and B, if the electorate unanimously prefers A to B, then the aggregate ranking should place A above B.
4. The ranking must not be imposed by external considerations (such as customs) independent of the preferences of individual members of the society.
5. It must not be dictatorial—no single voter should determine the group ranking.
6. And it must be independent of irrelevant alternatives; that is, no change in (i.e., addition to or subtraction from) the set of alternatives should change the rankings of the unaffected alternatives.

Often, the theorem is abbreviated by imposing the first four conditions and focusing on the difficulty of simultaneously obtaining the last two; this simplified form states that we cannot have independence of irrelevant alternatives (IIA) without dictatorship.[10]

You should be able to see immediately that some of the voting methods considered earlier do not conform to all of Arrow's principles. The requirement of IIA, for example, is violated by the single transferable vote procedure as well as by the Borda count, as we saw in Section 2.C. The Borda count is, however, nondictatorial and satisfies the Pareto property. All the other systems that we have considered satisfy IIA but break down on one of the other principles.

Arrow's theorem has provoked extensive research into the robustness of his conclusion to changes in the underlying assumptions. Political scientists, economists, and mathematicians have searched for a way to reduce the number of criteria or relax Arrow's principles minimally to find a procedure that satisfies the criteria without sacrificing the core principles; their efforts have been largely unsuccessful. Most economic and political theorists now accept the idea that some form of compromise is necessary when choosing a vote- or preference-aggregation method. Here are a few prominent examples of such compromises, each representing the approach of a particular field—political science, economics, or mathematics.

A. Black’s Condition

As the discussion in Section 2.A showed, the pairwise voting procedure does not satisfy Arrow’s requirement for transitivity of the social ranking, even when all individual rankings are transitive. One way to surmount this obstacle to meeting Arrow’s conditions, as well as to prevent the Condorcet paradox, is to place restrictions on the preference orderings held by individual voters. Such a restriction, known as the requirement of single-peaked preferences, was put forth by the political scientist Duncan Black in the late 1940s.[11] Black’s seminal paper on group decision making actually pre-dates Arrow’s impossibility theorem and was formulated with the Condorcet paradox in mind, but voting theorists have since shown its relevance to Arrow’s work; in fact, the requirement of single-peaked preferences is sometimes referred to as Black’s condition.

For a preference ordering to be single peaked, it must be possible to order the alternatives being considered along some specific dimension (for example, by the expenditure level associated with each of several alternative policies). To illustrate this requirement, we draw a graph in Figure 16.4, with the specified dimension on the horizontal axis and a voter’s preference ranking (or payoffs) on the vertical axis. For the single-peaked requirement to be met, each voter must have a single ideal or most preferred alternative, and alternatives farther away from the most preferred point must provide steadily lower payoffs. The two voters in Figure 16.4, Mr. Left and Ms. Right, have different ideal points along the policy dimension, but for each, the payoff falls steadily as the policy moves away from his or her ideal point.

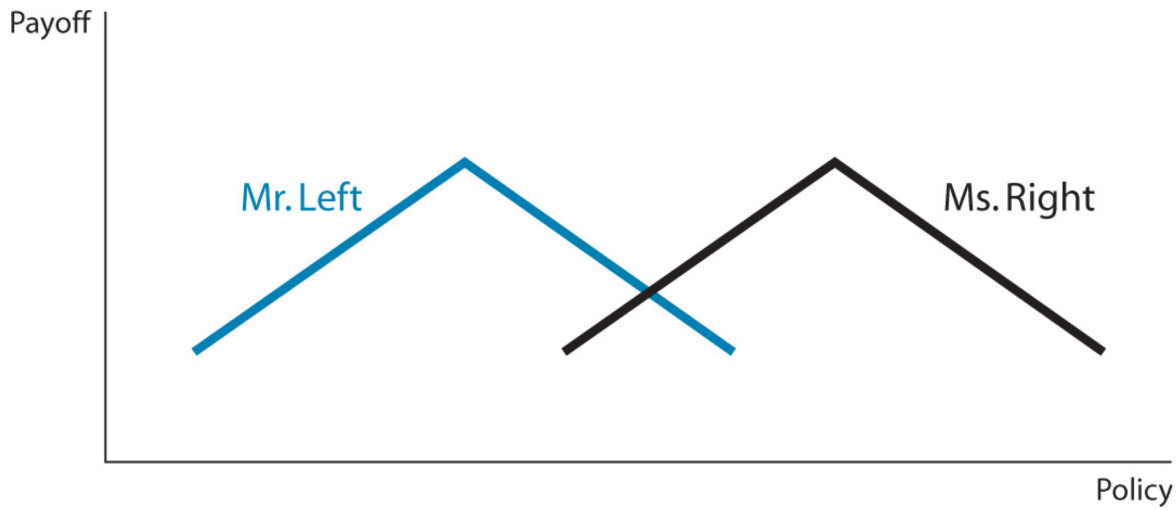


FIGURE 16.4 Single-Peaked Preferences

Black shows that if the preferences of each voter are single peaked, then the pairwise (majority) voting procedure produces a transitive social ranking. The Condorcet paradox is prevented, and pairwise voting satisfies Arrow’s transitivity condition.

B. Robustness

An alternative, more recent method of compromise with Arrow's principles comes from the economic theorists Partha Dasgupta and Eric Maskin.[12] They suggest a new criterion, called robustness, by which to judge voting methods. Robustness is measured by considering how often a voting procedure that is nondictatorial and that satisfies IIA as well as the Pareto property also satisfies the requirement of transitivity: For how many sets of voter preference orderings does such a procedure satisfy transitivity?

With the use of the robustness criterion, simple majority rule can be shown to be *maximally robust*—that is, it is nondictatorial, satisfies IIA and the Pareto property, and provides transitive social rankings for the largest possible set of voter preference orderings. Behind majority rule on the robustness scale lie other voting procedures, including the Borda count and the plurality rule. The robustness criterion is appealing in its ability to establish one of the most commonly used voting procedures—the one most often associated with the democratic process—as a candidate for the best vote-aggregation procedure.

C. Intensity Ranking

Another class of attempts to escape from Arrow's impossibility theorem focuses on the difficulty of satisfying Arrow's IIA requirement. A recent theory of this kind comes from the mathematician Donald Saari.[13] He suggests that a vote-aggregation method might use more information about voters' preferences than is contained in their mere ordering of any pair of alternatives, X and Y; rather, it could take into account each individual voter's *intensity* of preference between that pair of alternatives. This intensity can be measured by counting the number of other alternatives, Z, W, V, . . . , that a voter places between X and Y. Saari therefore replaces the IIA condition, number 6 of Arrow's principles, with a different condition, which he labels IBI (intensity of binary independence) and which we will number 6′:

6′. Society's relative ranking of any two alternatives should be determined only by (1) each voter's relative ranking of the pair and (2) the intensity of this ranking.

This condition is weaker than IIA because it effectively applies IIA only to such additions or deletions of "irrelevant" alternatives that do not change the intensity of people's preferences between the "relevant" ones. With this revision, the Borda count satisfies the Arrow theorem. It is the only positional voting method that does so.

Saari also hails the Borda count as the only procedure that appropriately observes ties within collections of ballots, a criterion that he argues is essential for a good vote-aggregation system to satisfy. Ties can occur two ways: through Condorcet terms or through reversal terms within voter preference orderings. In a three-candidate election among alternatives A, B, and C, the Condorcet terms are the preference orderings A ≻ B ≻ C, B ≻ C ≻ A, and C ≻ A ≻ B. In a set of three ballots with these preferences appearing on one ballot apiece, the ballots should logically offset one another, or constitute a tie.

Reversal terms are preference orderings that contain a reversal in the location of a *pair* of alternatives. In the same election, two ballots with preference orderings of A ≻ B ≻ C and B ≻ A ≻ C should logically lead to a tie in a pairwise contest between A and B. Only the Borda procedure treats collections of ballots with Condorcet terms or reversal terms as tied. Although the Borda count can lead to the reversal paradox, as shown in Section 2.C, it retains many proponents. The *only* time that the Borda procedure produces paradoxical results is when alternatives are dropped from consideration after ballots have been collected. Because such results can be prevented by using only ballots for the complete set of final candidates, the Borda procedure has gained favor in some circles as one of the best vote-aggregation methods.

Other researchers have made different suggestions regarding criteria that a good aggregation system should satisfy. Some of them include the *Condorcet criterion* (that a Condorcet winner should be selected by a voting system, if such a winner exists), the *consistency criterion* (that an election including all voters should elect the same alternative as would two elections held for two arbitrary divisions of the entire set of voters), and lack of manipulability (that a voting system should not encourage strategic voting on the part of voters). We cannot consider each of these suggestions at length, but we do address strategic voting in the following section.

Endnotes

- A full description of this theorem, often called "Arrow's general possibility theorem," can be found in Kenneth Arrow, *Social Choice and Individual Values*, 2nd ed. (New York: Wiley, 1963). Return to reference 9
- See Nicholson and Snyder's treatment of Arrow's impossibility theorem in their *Microeconomic Theory*, 11th ed. (New York: Cengage Learning, 2012), Chapter 19 , for more detail at a level appropriate for intermediate-level economics students. Return to reference 10
- Duncan Black, "On the Rationale of Group Decision-Making," *Journal of Political Economy*, vol. 56, no. 1 (February 1948), pp. 23–34. Return to reference 11
- See Partha Dasgupta and Eric Maskin, "On the Robustness of Majority Rule," *Journal of the European Economic Association*, vol. 6 (2008), pp. 949–73. Return to reference 12
- For more precise information about Saari's work on Arrow's theorem, see D. Saari, "Mathematical Structure of Voting Paradoxes I: Pairwise Vote," *Economic Theory*, vol. 15 (2000), pp. 1–53. Additional information on this result and on the robustness of the Borda count can be found in D. Saari, *Chaotic Elections* (Providence, R.I.: American Mathematical Society, 2000). Return to reference 13

Glossary

impossibility theorem

A theorem that indicates that no preference aggregation method can satisfy the six critical principles identified by Kenneth Arrow.

single-peaked preferences

A preference ordering in which alternatives under consideration can be ordered along some specific dimension and each voter has a single ideal or most-preferred alternative with alternatives farther away from the most-preferred point providing steadily lower payoffs. Also called **Black's condition**.

Black's condition

Same as the condition of **single-peaked preferences**.

robustness

A measure of the number of sets of voter preference orderings that are nondictatorial, satisfy independence of irrelevant alternatives and the Pareto property, and also produce a transitive *social ranking*.

Condorcet terms

A set of ballots that would generate the Condorcet paradox and that should together logically produce a tied vote among three possible alternatives. In a three-candidate election among A, B, and C, the Condorcet terms are three ballots that show A preferred to B preferred to C; B preferred to C preferred to A; C preferred to A preferred to B.

reversal terms

A set of ballots that would generate the *reversal paradox* and that should together logically produce a tied vote between a pair of alternatives. In a three-candidate election among A, B, and C, the reversal terms are two ballots that show a reversal in the location of a pair of alternatives. For example, one ballot with A preferred to B preferred to C and another with B preferred to A preferred to C should produce a tie between A and B.

4 STRATEGIC VOTING

Several of the voting systems that we have considered yield considerable scope for strategic misrepresentation of preferences by voters. In Section 2.B, we showed how the power of an agenda-setting chair could be countered by a councillor voting in the first round against her true preference, so as to knock out her least preferred alternative and send a more preferred one into the second round. More generally, voters can choose to vote for candidates, issues, or policies that are not actually their most preferred among the alternatives presented in an early round of voting if such behavior can alter the final election results in their favor. In this section, we consider a number of ways in which strategic voting behavior can affect elections.

A. Plurality Rule

Plurality-rule elections, often perceived as the fairest by many voters, still provide opportunities for strategic behavior. In U.S. presidential elections, for instance, there are generally two candidates, representing the two major political parties, in contention. When such a race is relatively close, there is potential for a third candidate to enter the race and divert votes away from the leading candidate; if the entry of this third player truly threatens the chances of the leader winning the election, the third player is called a spoiler.

In the 2016 U.S. presidential election, Libertarian candidate Gary Johnson and Green Party candidate Jill Stein got, respectively, 3.3% and 1.1% of the popular vote, earning zero electoral votes but potentially still swinging the election. Johnson had a centrist appeal and so probably drew voters from both Hillary Clinton, the Democratic candidate, and Donald Trump, the Republican. Jill Stein, on the other hand, got her support from the far left and hence probably drew voters mainly from Hillary Clinton. If those Stein voters had instead cast their ballots for Clinton, they would have flipped Michigan, Pennsylvania, and Wisconsin in Clinton's favor, which would have been enough to put Clinton in the White House.[14] Stein also harshly criticized Clinton during the campaign, potentially causing some voters to stay at home who would have otherwise voted for Clinton, even as she praised Trump (saying at one point that "maybe he's the peace president"[15]) and denied that Russia was in any way interfering in the election. However, as later revealed by the special counsel investigation led by Robert Mueller, Russian agents aggressively promoted Stein's candidacy (and anti-Clinton views) on social media as part of their broader effort to elect Trump.[16]

In plurality-rule elections with a spoiler candidate, those who prefer the spoiler to the leading major candidate but least prefer the trailing major candidate may do best to strategically misrepresent their preferences to prevent the election of their least favorite candidate. That is, you should vote for the leader in such a case, even though you would prefer the spoiler, because the spoiler is unlikely to garner a plurality; voting for the leader then prevents the trailing candidate, your least favorite, from winning.[17]

Of course, sometimes a third-party candidate who appears to be a spoiler might actually have a shot at winning an election. If voters who prefer the third-party candidate strategically misrepresent their preferences, voting instead for a major-party candidate, the third-party candidate could lose even if she has more support than anyone else. For instance, in the 1992 U.S. presidential election, businessman Ross Perot ran as a third-party candidate against Republican George H. W. Bush and Democrat Bill Clinton. In a survey conducted after the election, *Newsweek* found that a plurality of 40% of voters surveyed said they would have voted for Perot if they had thought he could have won. However, because they thought he was likely to lose, they cast their ballots for their second-favorite candidate (Bush or Clinton), causing Perot to lose the election with only 18.9% of the popular vote and zero electoral votes.[18]

Because of strategic voting, third parties in the American political system face a "chicken-and-egg problem": They cannot successfully challenge the two major parties until people believe that they can make a successful challenge. One way to crack this problem would be to switch from plurality-rule elections to another system, such as instant runoff voting, in which voters have an incentive to vote according to their true preferences. For example, consider again the 1992 election between Ross Perot (P), Bill Clinton (C), and George H. W. Bush (B), and suppose that voters' preferences among the three candidates were as follows: $P \succ B \succ C$ (20%); $P \succ C$

$\succ$ B (20%); C $\succ$ P $\succ$ B (30%); C $\succ$ B $\succ$ P (5%); B $\succ$ P $\succ$ C (20%); and B $\succ$ C $\succ$ P (5%). In the actual plurality-rule election, some voters who preferred Perot voted instead for their second-favorite candidate, fearing that Perot was a spoiler with no chance of winning. Using our (imagined) numbers, if half of Perot supporters voted strategically in this way, then Perot would get 20%, Clinton would get 45%, and Bush would get 35%; this is roughly what transpired in the actual election.

But what if, instead, the election had been held using instant runoff voting? Voters who preferred Perot would have had an incentive to vote according to their true preferences, listing him as their favorite candidate and then either Bush or Clinton as their second favorite. (If Perot had turned out to have the least support, their second-favorite votes would then have been tallied in the instant runoff between Bush and Clinton.) With all votes reflecting voters' true preferences, the first-round votes would have gone 40% to Perot, 35% to Clinton, and 25% to Bush, at which point Bush would have lost, and his supporters' votes would have been redistributed to their favorite remaining candidate. The instant runoff would have added 20% to Perot's support and 5% to Clinton's, leaving Perot the winner with 60% in the final tally.

In elections for legislatures, where many candidates are chosen, the performance of third parties is very different under a system of proportional representation of the whole population in the whole legislature than under a system of plurality in separate constituencies. Britain uses the constituency and plurality system. In the past 50 years, the Labor and Conservative Parties have shared power. The Liberal Party, despite sizable third-place support in the electorate, has suffered from strategic voting and therefore has had disproportionately few seats in Parliament. Italy uses the nationwide candidate list and proportional representation system; there is no need to vote strategically in such a system, and even small parties can have a significant presence in the legislature. Often, no party has a clear majority of seats, and small parties can affect policy through bargaining for alliances.

A party cannot flourish if it is largely ineffective in influencing a country's political choices. Therefore, we tend to see just two major parties in countries with the plurality system and several parties in those with the proportional representation system. Political scientists call this observation *Duverger's law*. The plurality system tends to produce only two major parties—often one of them with a clear majority of seats in the legislature—and therefore more decisive government. But it runs the risk that the minority's interests will be overlooked—that is, of producing a "tyranny of the majority." A proportional representation system gives more of a voice to minority views. But it can produce inconclusive bargaining for power and legislative gridlock. Interestingly, each country seems to believe that its system performs worse than the other and considers switching; in Britain, there are strong voices calling for proportional representation, and Italy has been seriously considering a constituency system.

B. Pairwise Voting

When you know that you are bound by a pairwise voting method such as the amendment procedure, you can use your prediction of the second-round outcome to determine your optimal voting strategy in the first round. It may be in your interest to appear committed to a particular candidate or policy in the first round, even if it is not your most preferred alternative, so that your least favorite alternative cannot win the entire election in the second round.

We return here to our example of the city council with an agenda-setting chair from Section 2.B. Here, we add the assumption that, because they have worked together closely for so long, all three councillors' preference orderings (shown in Figure 16.1) are known to the entire council. Suppose Councillor Left, who most prefers the Generous (G) welfare package, is appointed chair and sets the Average (A) and Decreased (D) policies against each other in a first vote, with the winner facing off against G in the second round. If the three councillors vote strictly according to their preferences, A will beat D in the first vote and G will then beat A in the second vote; the chair's preferred outcome will be chosen. The city councillors are likely to be well-trained strategists, however, who can look ahead to the final round of voting and use rollback to determine which way to vote in the opening round.

In the scenario just described, Councillor Center's least preferred policy will be chosen in the election. Therefore, rollback analysis says that she should vote strategically in the first round to alter the election's outcome. If Center votes for her most preferred policy in the first round, she will vote for A, which will then beat D in that round and lose to G in round two. However, she could instead vote strategically for D in the first round, which would lift D over A on the first vote. Then, when D is set up against G in the second round, G will lose to D. Councillor Center's misrepresentation of her preference ordering with respect to A and D helps her to change the winner of the election from G to D. Although D is not her most preferred outcome, it is better than G from her perspective.

This strategy works well for Center if she can be sure that no other strategic votes will be cast in the election. Thus, we need to analyze both rounds of voting fully to verify the Nash equilibrium strategies for the three councillors. We do so by using rollback on the two simultaneous-vote rounds of the election, starting with the two possible second-round contests, A versus G and D versus G.

Figure 16.5 illustrates the outcomes that arise in each of the possible second-round elections. The two tables in Figure 16.5a show the winning policy (not payoffs to the players) when A has won the first round and is pitted against G; the tables in Figure 16.5b show the winning policy when D has won the first round. In both cases, Councillor Left chooses the row of the final outcome, Center chooses the column, and Right chooses the actual table (left or right).

You should be able to establish that each councillor has a dominant strategy in each second-round election. In the A versus G election, Left's dominant strategy is to vote for G, Center's dominant strategy is to vote for A, and Right's dominant strategy is to vote for G; G will win this election. If the councillors consider D versus G, Left's dominant strategy is still to vote for G, and Right and Center both have a dominant strategy to vote for D; in this vote, D wins. A quick check shows that all the councillors vote according to their true preferences in this round. Thus, these dominant strategies are all the same: "Vote for the alternative that I prefer." Because there is no future to consider in the second-round vote, the councillors simply vote for whichever policy ranks higher in their preference ordering.[19]

(a) A versus G election

RIGHT votes:

A

		CENTER	
		A	**G**
LEFT	A	A	A
	G	A	G

G

		CENTER	
		A	**G**
LEFT	A	A	G
	G	G	G

(b) D versus G election

RIGHT votes:

D

		CENTER	
		D	**G**
LEFT	D	D	D
	G	D	G

G

		CENTER	
		D	**G**
LEFT	D	D	G

	CENTER		
	D		G
	G	G	G

FIGURE 16.5 Election Outcomes in Two Possible Second-Round Votes

We can now use the results from our analysis of Figure 16.5 to consider optimal strategies in the first round of voting, in which the councillors choose between policies A and D. Because we know how the councillors will vote in the next round given each winner here, we can show the outcome of the entire election in each case (Figure 16.6).

As an example of how we arrived at these outcomes, consider the G in the upper-left cell of the right-hand table in Figure 16.6. The outcome in that cell is obtained when Left and Center both vote for A in the first round while Right votes for D. Thus, A and G are paired in the second round, and as we saw in Figure 16.5, G wins. The other outcomes are derived in similar fashion.

Given the outcomes in Figure 16.6, Councillor Left (who is the chair and has set the agenda) has a dominant strategy to vote for A in this first round. Similarly, Councillor Right has a dominant strategy to vote for D. Neither of these councillors misrepresents her preferences or votes strategically in either round. Councillor Center, however, has a dominant strategy to vote for D here even though she strictly prefers A to D. As the preceding discussion suggested, she has a strong incentive to misrepresent her preferences in the first round of voting, and she is the only one who votes strategically. Center’s behavior changes the winner of the election from G (the winner without strategic voting) to D.

RIGHT votes:

A

		CENTER	
		A	D
LEFT	A	G	G
	D	G	D

D

		CENTER	
		A	D
LEFT	A	G	D
	D	D	D

FIGURE 16.6 Election Outcomes Based on First-Round Votes

Remember that the chair, Councillor Left, set the agenda in the hope of having her most preferred alternative chosen. Instead, her *least* preferred alternative has prevailed. So it might appear that the power to set the agenda may not be so beneficial after all. But Councillor Left should anticipate the other councillors' strategic behavior. Then she can choose the agenda so as to take advantage of her understanding of games of strategy. In fact, if she sets D against G in the first round and then the winner against A, the Nash equilibrium outcome is G, the chair's most preferred outcome. With that agenda, Right misrepresents her preferences in the first round to vote for G over D to prevent A, her least preferred outcome, from winning. You should verify that setting this agenda is Councillor Left's optimal strategy. In the full voting game, where setting the agenda is considered an initial, pre-voting round, we should expect to see the Generous welfare policy adopted when Councillor Left is chair.

We can also see an interesting pattern emerge when we look more closely at voting behavior in the strategic version of the election. There are pairs of councillors who vote "together" (i.e., for the same policy) in both rounds. Under the original agenda (with A versus D in the first round), Right and Center vote together in both rounds, and in the suggested alternative agenda (with D versus G in the first round), Right and Left vote together in both rounds. In other words, a sort of long-lasting coalition has formed between two councillors in each case.

Strategic voting of this type appears to have taken place in the U.S. Congress on more than one occasion. One example was a federal school construction funding bill considered in 1956.[20] Before being brought to a vote against the status quo of no funding, the bill was amended in the House of Representatives to require that the funding be offered only to states with no racially segregated schools. Under the parliamentary voting rules of Congress (described in Section 1.A), a vote on whether to accept this so-called Powell Amendment was taken first, with the winning version of the bill considered afterward. Political scientists who have studied the history of this bill argue that opponents of school funding strategically misrepresented their preferences regarding the amendment in an effort to defeat the full bill. A key group of representatives voted for the amendment, but then joined opponents of racial integration in voting against the full bill in the final vote; the bill was defeated. The voting records of this group indicate that many of them had voted against racial integration in other circumstances, implying that their vote for integration in the case of this amendment was merely an instance of strategic voting and not an indication of their true feelings regarding school integration.

C. Strategic Voting with Incomplete Information

The analysis in Section 4.B showed that sometimes group members have incentives to vote strategically to prevent their least preferred alternative from winning an election. Our example assumed that the council members knew the preference orderings that were possible and how many other councillors had those preference orderings. Now suppose their information is incomplete: Each council member knows the possible preference orderings, her own actual ordering, and the probability that each of the others has each particular ordering, but not the actual distribution of the different preference orderings among the other councillors. In this situation, each councillor's strategy needs to be conditioned on her beliefs about that distribution and on her beliefs about how sincere the other councillors' votes will be.[21]

For example, suppose we still have a three-member council considering the three alternative welfare policies described earlier, following the (original) agenda set by Councillor Left; that is, the council considers policies A and D in the first round of voting, with the winner facing G in the second round. We assume that there are still three different possible preference orderings, as illustrated in Figure 16.1, and that the councillors know that these orderings are the only possibilities. The difference is that no councillor knows for sure exactly how many other councillors have each preference ordering. Rather, each councillor knows her own type, and she knows that there is some positive probability of observing each type of voter (Left, Center, or Right), with the probabilities p_L, p_C, and p_R summing to 1.

We saw earlier that all three councillors vote truthfully in the last round of balloting. We also saw that Left-type and

Right-type councillors vote truthfully in the first round as well. This result remains true in the incomplete information case. Right-type voters prefer to see D win the first-round election; given this preference, Right always does at least as well by voting for D over A (if both other councillors have voted the same way) as she would by voting otherwise, and she sometimes does better by voting this way (if the other two votes split between D and A). Similarly, Left-type voters prefer to see A survive to vie against G in round two; these voters always do at least as well as otherwise—and sometimes do better—by voting for A over D.

At issue, then, is only the behavior of the Center-type voters. Because they do not know the types of the other councillors, and because they have an incentive to vote strategically with some preference distributions—specifically, the case in which it is known for certain that there is one voter of each type—their behavior will depend on the probabilities that the various voter types occur within the council. We consider here one of two polar cases, in which a Center-type voter believes that other Center types will vote truthfully, and we look for a symmetric, pure-strategy Nash equilibrium. The other case, in which she believes that other Center types will vote strategically, is taken up in Exercise U9.

To make outcome comparisons possible, we specify the payoffs for the Center-type voters associated with the possible winning policies. Center-type preferences are A $\succ$ D $\succ$ G. Suppose that if A wins, Center types receive a payoff of 1, and if G wins, Center types receive a payoff of 0. If D wins, Center types receive some intermediate payoff, call it u, where $0 < u < 1$.

Now suppose our Center-type councillor must decide how to vote in the first round (A versus D) in an election in which she believes that both other councillors will vote truthfully, regardless of their type. If both other councillors choose either A or D, then Center's vote is immaterial to the final outcome; she is indifferent between A and D. If the other two councillors split their votes, however, then Center can influence the election outcome. Her problem is that she needs to decide whether to vote truthfully herself.

If the other two councillors split between A and D, and if both are voting truthfully, then the vote for D must have come from a Right-type voter. But the vote for A could have come from *either* a Left type *or* a (truthful) Center type. If the A vote came from a Left-type voter, then Center knows that there is one voter of each type. If she votes truthfully for A in this situation, A will win the first round but lose to G in the end; Center's payoff will be 0. If Center votes strategically for D, D beats A and G, and Center's payoff is u. In contrast, if the A vote came from a Center-type voter, then Center knows there are two Center types and a Right type, but no Left type, on the council. In this case, a truthful vote for A helps A win the first round, and then A also beats G by a vote of 2 to 1 in round two; Center gets her highest payoff of 1. If Center were to vote strategically for D, D would win both rounds again, and Center would get u.

To determine Center's optimal strategy, we need to compare her expected payoff from truthful voting with her expected payoff from strategic voting. When she votes truthfully for A in the first round, Center's payoff depends on how likely it is that the other A vote comes from a Left type versus a Center type. Those probabilities are straightforward to calculate. The probability that the other A vote comes from a Left type is just the probability of a Left type being one of the remaining voters, or $p_L/(p_L + p_C)$; similarly, the probability that the other A vote comes from a Center type is $p_C/(p_L + p_C)$. Then Center's payoffs from truthful voting are 0 with probability $p_L/(p_L + p_C)$ and 1 with probability $p_C/(p_L + p_C)$, so her expected payoff is $p_C/(p_L + p_C)$. With a strategic vote for D, D wins regardless of the identity of the third voter—D wins with certainty—so Center's expected payoff is just u. Center's final decision is to vote truthfully as long as $p_C/(p_L + p_C) > u$.

Note that Center's decision-making condition is an intuitively reasonable one. If the probability of there being other Center-type voters is large, or relatively larger than the probability of there being a Left-type voter, then the Center types vote truthfully. Voting strategically is useful to Center only when she is the only voter of her type on the council.

We add two additional comments on the existence of imperfect information and its implications for strategic behavior. First, if the number of councillors, n, is larger than three but odd, then the expected payoff to a Center type from voting strategically remains equal to u, and $[p_C/(p_L + p_C)]^{(n - 1)/2}$ is her expected payoff from voting truthfully.[22] Thus, a Center type should vote truthfully only when $[p_C/(p_L + p_C)]^{(n - 1)/2} > u$. Because $p_C/(p_L + p_C) < 1$ and $u > 0$, this inequality will *never* hold for large enough values of n. This result tells us that a truthful-voting equilibrium can never persist in a large enough council! Second, imperfect information about the preferences of other voters yields additional scope for strategic behavior. With agendas that include more than two rounds of voting, voters can use their early-round votes to signal their types. The extra rounds give other voters the opportunity to update their prior beliefs about the probabilities p_C, p_L, and p_R and a chance to act on that information. With only two rounds of pairwise votes, there is no time to use any information gained during round one, because truthful voting is a dominant strategy for all voters in the final round.

D. Scope for Strategic Voting

The extent to which a voting procedure is susceptible to strategic misrepresentation of preferences—that is, to strategic voting of the types illustrated in this section—is another topic that has generated considerable interest among voting theorists. Arrow's principles do not require invulnerability to strategic behavior, but the literature has considered how such a requirement would relate to his conditions. Similarly, theorists have considered the scope for strategic voting in various procedures, producing rankings of voting methods.

In an election with a large number of voters, any individual's vote, and likewise any individual's strategic voting, is unlikely to have any effect; strategic voting requires collective action by a group of like-minded voters, which is quite difficult. Strategic voting is more likely to have an effect in smaller electorates, for example, in committees rather than in nationwide elections.

The economist William Vickrey, perhaps better known for his Nobel Prize–winning work on auctions (see Chapter 15), did some of the earliest work considering strategic behavior of voters. He pointed out that procedures satisfying Arrow's IIA condition were most immune to strategic behavior. He also set out several conditions under which strategic voting is more likely to be attempted and to be successful. In particular, he noted that situations with smaller numbers of informed voters and smaller sets of available alternatives may be most susceptible to strategic voting, given a voting method that is itself susceptible to such behavior. This result means, however, that weakening the IIA requirement to help voting procedures satisfy Arrow's conditions makes way for procedures that are more prone to strategic voting. In particular, Saari's intensity ranking version of IIA (called

IBI), mentioned in Section 3.C, may allow more procedures to satisfy this modified version of Arrow’s theorem but may simultaneously allow more of those procedures that are susceptible to strategic voting to do so.

Like Arrow’s impossibility theorem on the limits of vote aggregation, the most general result on invulnerability to strategic voting is a negative one. Specifically, the Gibbard–Satterthwaite theorem shows that if there are three or more alternatives to consider, the only voting procedure that prevents strategic voting is dictatorship: One voter is assigned the role of dictator, and her preferences determine the election outcome.[23] Combining the Gibbard–Satterthwaite outcome with Vickrey’s discussion of IIA may help the reader understand why Arrow’s theorem is often reduced to a consideration of which procedures can simultaneously satisfy nondictatorship and IIA.

Finally, some theorists have argued that voting procedures should be evaluated not on their ability to satisfy Arrow’s conditions, but on their vulnerability to strategic voting. The relative vulnerability of a voting system can be determined by the amount of information about the preferences of other voters that is required by voters to vote strategically and alter an election successfully. Some research based on this criterion suggests that of the procedures so far discussed, the plurality rule is the most vulnerable to strategic voting (that is, requires the least information). In decreasing order of vulnerability are approval voting, the Borda count, the amendment procedure, majority rule, and the single transferable vote method (IRV).[24]

It is important to note that this ranking of voting procedures by level of vulnerability to strategic voting depends only on the amount of information necessary to alter the outcome; it is not based on the ease of putting such information to good use or on whether strategic voting is most easily achieved by individual voters or by groups. In practice, strategic voting by *individual* voters to alter the outcome in a plurality-rule election is quite difficult.

Endnotes

- "Jill Stein: Democratic Spoiler Or Scapegoat?" FiveThirtyEight Chat, December 7, 2016, available at https://fivethirtyeight.com/features/jill-stein-democratic-spoiler-or-scapegoat/(accessed May 29, 2019). Return to reference 14
- Mike Pesca, "Jill Stein Thinks Nuclear War Is Less Likely under Trump," *Slate*, October 19, 2016, available at https://slate.com/news-and-politics/2016/10/jill-stein-thinks-nuclear-war-is-less-likely-under-trump.xhtml(accessed May 29, 2019). Return to reference 15
- According to Andrew Weiss, a Russian expert at the Carnegie Endowment for International Peace, "The Russian embrace of fringe voices like Stein goes back more than a decade to the earlier days of RT [Russia Today]," an English-language television network funded by the Russian government on which Stein appeared as a frequent guest. See Robert Windrem, "Russians Launched Pro–Jill Stein Social Media Blitz to Help Trump Win Election, Reports Say," *NBC News*, Dec. 22, 2018. Return to reference 16
- Note that an approval voting method would not suffer from this same problem. Return to reference 17
- "Ross Reruns," *Newsweek*, Special Election Recap Issue, November 18, 1996, p. 104. Return to reference 18
- It is a general result in the voting literature that voters faced with pairs of alternatives will always vote truthfully at the last round of voting. Return to reference 19
- A more complete analysis of the case can be found in Riker, *Liberalism against Populism*, pp. 152–57. Return to reference 20
- This result can be found in P. Ordeshook and T. Palfrey, "Agendas, Strategic Voting, and Signaling with Incomplete Information," *American Journal of Political Science*, vol. 32, no. 2 (May 1988), pp. 441–66. The structure of the

example to follow is based on Ordeshook and Palfrey’s analysis. Return to reference 21

- A Center type can affect the election outcome only if all other votes are split evenly between A and D. Thus, there must be exactly $(n - 1)/2$ Right-type voters choosing D in the first round and $(n - 1)/2$ other voters choosing A. If those A voters are Left types, then A won’t win the second-round election, and Center will get 0 payoff. For Center to get a payoff of 1, it must be true that all of the other A voters are Center types. The probability of this occurring is $[p_C/(p_L+p_C)]^{(n-1)/2}$; then Center’s expected payoff from voting truthfully is as stated. See Ordeshook and Palfrey, p. 455. Return to reference 22
- For the theoretical details on this result, see A. Gibbard, “Manipulation of Voting Schemes: A General Result,” *Econometrica*, vol. 41, no. 4 (July 1973), pp. 587–601, and M. A. Satterthwaite, “Strategy-Proofness and Arrow’s Conditions,” *Journal of Economic Theory*, vol. 10 (1975), pp. 187–217. The theorem carries both their names because each proved the result independently of the other. Return to reference 23
- H. Nurmi’s classification can be found in his *Comparing Voting Systems* (Norwell, Mass.: D. Reidel, 1987). Return to reference 24

Glossary

spoiler
: Refers to a third candidate who enters a two-candidate race and reduces the chances that the leading candidate actually wins the election.

Gibbard–Satterthwaite theorem
: With three or more alternatives to consider, the only voting method that prevents strategic voting is dictatorship; one person is identified as the dictator and her preferences determine the outcome.

5 THE MEDIAN VOTER THEOREM

All of the preceding sections have focused on the behavior, strategic and otherwise, of voters in elections. However, strategic analysis can also be applied to *candidate* behavior in such elections. Given a particular distribution of voters and voter preferences, candidates will, for instance, need to determine their optimal strategies for building their political platforms. The median voter theorem tells us that when there are just two candidates in an election, when voters are distributed in a "reasonable" way along the political spectrum, and when each voter has "reasonably" consistent (meaning singled-peaked) preferences, both candidates will position themselves on the political spectrum at the same place as the median voter. The median voter is the "middle" voter in that distribution—more precisely, the one at the 50th percentile.

The full game here has two stages. In the first stage, candidates choose their locations on the political spectrum. In the second stage, voters elect one of the candidates. The second-stage game is open to all the varieties of strategic misrepresentation of preferences discussed earlier; hence, we have reduced the choice of candidates to two for our analysis to prevent such behavior from arising in equilibrium. With only two candidates, second-stage votes directly correspond to voter preferences, and the first-stage location decisions of the candidates remain the interesting part of the larger game. It is in that first stage that the median voter theorem defines Nash equilibrium behavior.

A. Discrete Political Spectrum

Let us first consider a population of 90 million voters, each of whom has a preferred position on a five-point political spectrum: Far Left (FL), Left (L), Center (C), Right (R), or Far Right (FR). We suppose that these voters are spread across the political spectrum. The discrete distribution of their locations is shown by a histogram, or bar chart, in Figure 16.7. The height of each bar indicates the number of voters located at that position. In this example, we have supposed that, of the 90 million voters, 40 million are Left, 20 million are Far Right, and 10 million each are Far Left, Center, and Right.

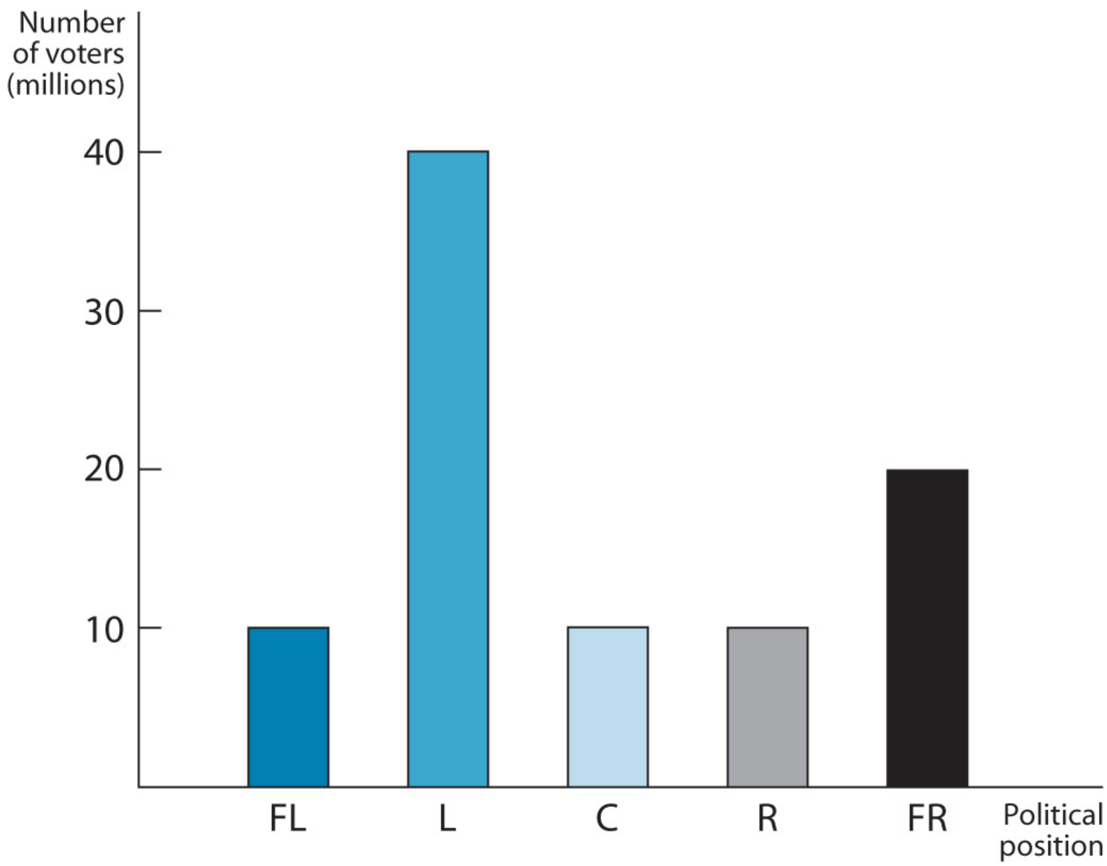


FIGURE 16.7 Discrete Distribution of Voters

In an election, each voter will vote for the candidate who publicly identifies herself as being closer to her own position on the spectrum. If both candidates are politically equidistant from a group of like-minded voters, each voter will flip a coin to decide which candidate to choose; this process will give each candidate one-half of the voters in that group.

Now suppose there is an upcoming presidential election between a former first lady (Claudia) and a former first lady hopeful (Dolores), each now running for office on her own.[25] Given the distribution of voters illustrated in Figure 16.7, we can construct a payoff table for the two candidates showing the number of votes that each can expect to receive under all of the possible combinations of political platform choices. This five-by-five table is shown in Figure 16.8, with totals denoted in millions of votes. The candidates will choose their location strategies to maximize the number of votes that they receive (and thus to increase their chances of winning).[26]

Here is how the votes are allocated. When both candidates choose the *same* position (the five cells along the top-left to bottom-right diagonal of the table), each candidate gets exactly one-half of the votes; because all voters are equidistant from each candidate, all of them flip coins to decide their choices, and each candidate garners 45 million votes. When the two candidates choose *different* positions, the more left-leaning candidate gets all the votes at or to the left of her position, while the more right-leaning candidate gets all the votes at or to the right of her position. In addition, each candidate gets the votes in central positions closer to her than to her rival, and the two of them split the votes from any voters in a position equidistant between them. Thus, if Claudia locates herself at L while Dolores locates herself at FR, Claudia gets the 40 million votes at L, the 10 million at FL, *and* the 10 million at C (because C is closer to L than to FR). Dolores gets the 20 million votes at FR and the 10 million at R (because R is closer to FR than to L). The payoffs are (60, 30). Similar calculations determine the outcomes in the rest of the table.

The table in Figure 16.8 is large, but the game can be solved very quickly. We begin with the now familiar search for dominant, or dominated, strategies for the two players. Immediately we see that for Claudia, FL is dominated by L and FR is dominated by R. For Dolores, too, FL is dominated by L and FR by R. With these extreme strategies eliminated, R is dominated by C for each candidate. With the two R strategies gone, C is dominated by L for each candidate.

The only remaining cell in the table is (L, L); this is the Nash equilibrium.

		DOLORES				
		FL	L	C	R	FR
CLAUDIA	FL	45, 45	10, 80	30, 60	50, 40	55, 35
	L	80, 10	45, 45	50, 40	55, 35	60, 30
	C	60, 30	40, 50	45, 45	60, 30	65, 25
	R	40, 50	35, 55	30, 60	45, 45	70, 20
	FR	35, 55	30, 60	25, 65	20, 70	45, 45

You may need to scroll left and right to see the full figure.

FIGURE 16.8 Payoff Table for Candidates' Positioning Game

We now note three important characteristics of the equilibrium in the candidate-location game. First, both candidates locate at the *same* position in equilibrium. This outcome illustrates the principle of minimum differentiation, a general result in all two-player games of locational competition, whether it be political platform choices by presidential candidates, hot-dog-cart location choices by street vendors, or product feature choices by electronics manufacturing firms.[27] When the persons who vote for or buy from you can be arranged on a well-defined spectrum of preferences, you do best by looking as much like your rival as possible. This principle explains a diverse collection of behaviors on the part of political candidates and businesses. It may help you understand, for example, why there is never just one gas station at a heavily traveled intersection or why all brands of four-door sedans (or minivans, or sport utility vehicles) seem to look the same even though every brand claims to be coming out continually with a "new" look.

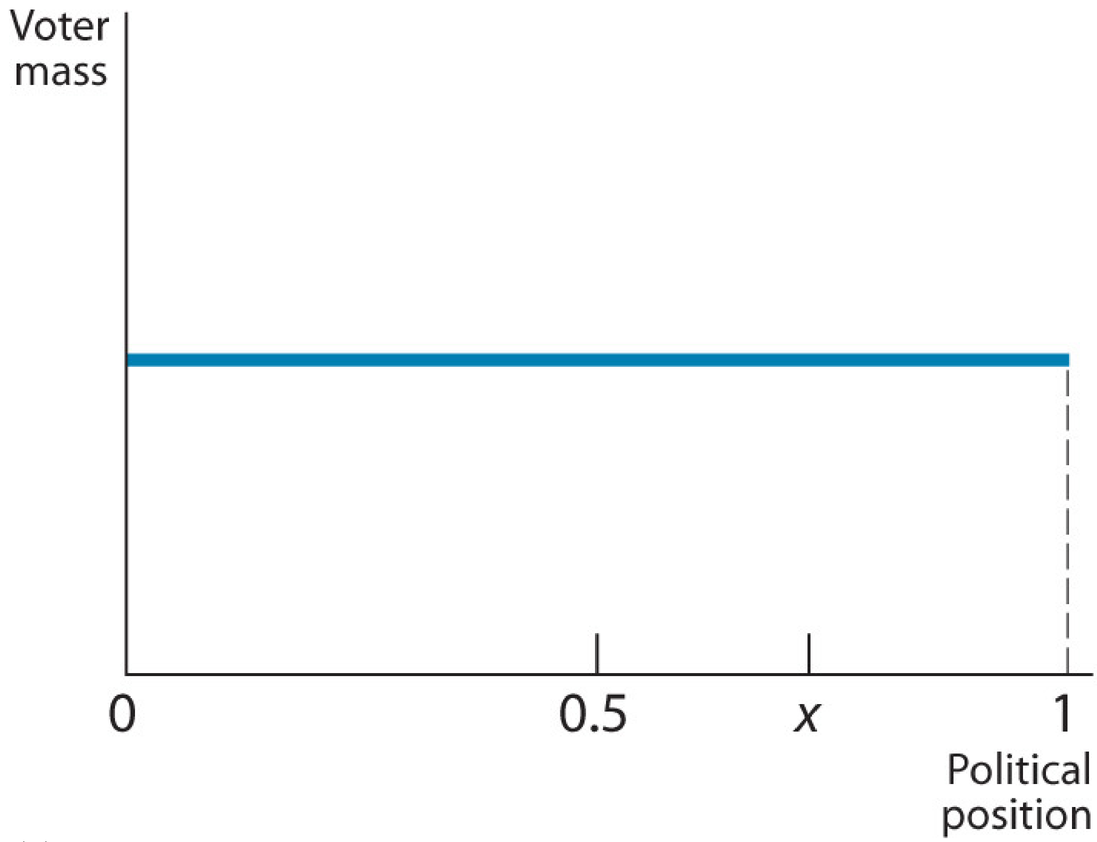
Second, and perhaps most crucially, both candidates locate at the position of the median voter in the population. In our example, with a total of 90 million voters, the median voter is number 45 million from each end. The numbers within one location can be assigned arbitrarily, but the location of the median voter is clear; here, the median voter is located at the L position on the political spectrum. So that is where both candidates locate themselves, which is the result predicted by the median voter theorem.

Third, observe that the location of the median voter need not coincide with the geometric center of the spectrum. The two will coincide if the distribution of voters is symmetric, but the median voter can be to the left of the geometric center if the distribution is skewed to the left (as is true in Figure 16.7) and to the right if the distribution is skewed to the right. This observation helps explain why state political candidates in Massachusetts, for example, *all* tend to be more liberal than candidates for similar positions in Texas or South Carolina.

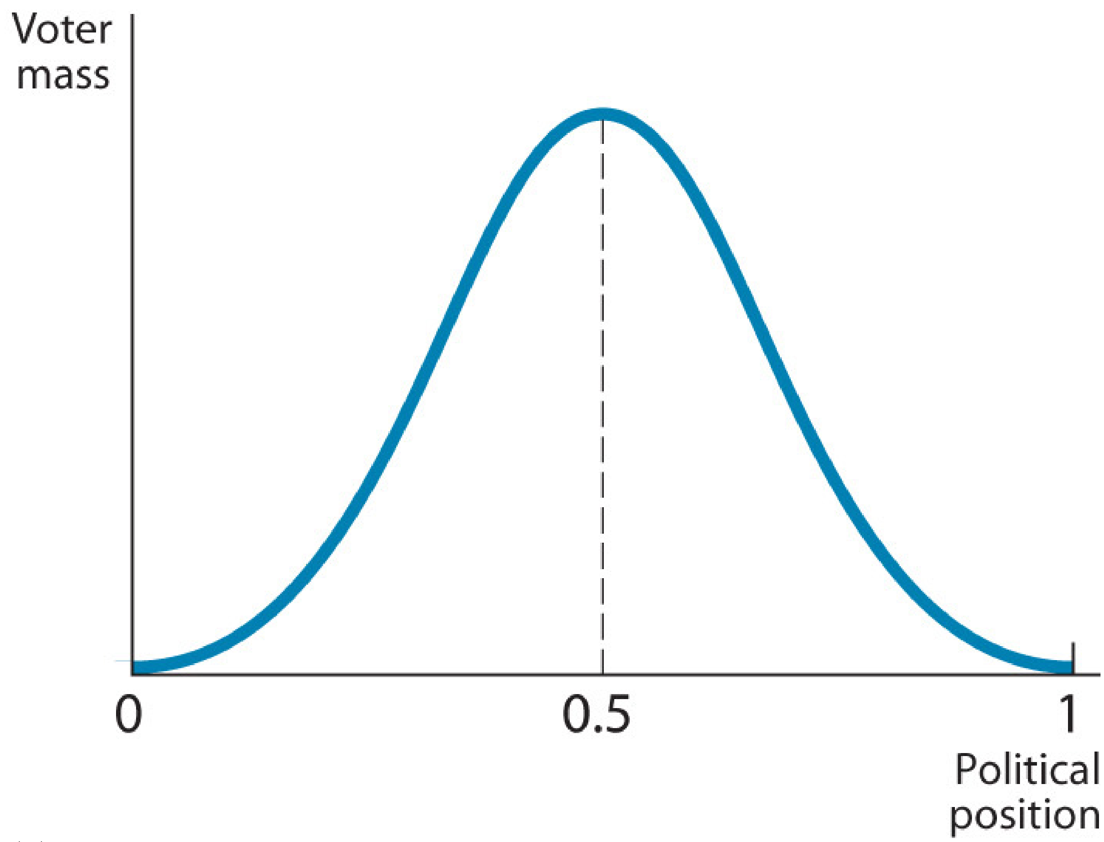
The median voter theorem can be expressed in different ways. One version states simply that the position of the median voter is the equilibrium position of the candidates in a two-candidate election. Another version says that the position that the median voter most prefers will be the Condorcet winner; this position will defeat every other position in a pairwise contest. For example, if M is this median position and L is any position to the left of M, then a candidate positioned at M will get all the votes of people who most prefer a position at or to the right of M, plus some to the left of M but closer to M than to L. Thus, M will get more than 50% of the votes. The two versions amount to the same thing because, in a two-candidate election, both candidates seeking to win a majority of votes will adopt the Condorcet-winner position. In addition, to guarantee that the result holds for a particular population of voters, the theorem (in either form) requires that each voter's preferences be "reasonable." *Reasonable* here means single peaked, as in Black's condition, described in Section 3.A and Figure 16.4. Each voter has a unique, most preferred position on the political spectrum, and her payoff decreases away from that position in either direction.[28] In actual U.S. presidential elections, the theorem is borne out by the tendency for the candidates from the two major parties to make very similar promises to the electorate.

B. Continuous Political Spectrum

The median voter theorem can also be proved for a continuous distribution of political positions. Rather than having five, three, or any finite number of positions from which to choose, a continuous distribution assumes there are effectively an infinite number of political positions. These political positions are then associated with locations along the real number line between 0 and 1.[29] Voters are still distributed along the political spectrum as before, but because the distribution is now continuous rather than discrete, we use a voter distribution function rather than a histogram to illustrate voter locations. Two common distribution functions—the uniform distribution and the (symmetric) normal distribution—are illustrated in Figure 16.9.[30] The area under each curve represents the total number of votes available; for any given point along the interval from 0 to 1, such as x in Figure 16.9a, the number of votes up to that point is determined by finding the area under the distribution function from 0 to x. It should be clear that the median voter in each of these distributions is located at the center of the spectrum, at position 0.5.



(a) Uniform distribution



(b) Normal distribution

FIGURE 16.9Continuous Distribution of Voters

It is not feasible to construct a payoff table for our two candidates in the continuous-spectrum case; such tables must necessarily be finitely dimensioned and thus cannot accommodate an infinite number of possible strategies for players. We can, however, solve the game by applying the same strategic logic that we used for the discrete-spectrum case discussed in Section 5.A.

Consider the options of Claudia and Dolores as they contemplate the possible political positions open to them. Each knows that she must find her Nash equilibrium strategy—her best response to the equilibrium strategy of her rival. We can define a set of strategies that are best responses quite easily in this game, even though the complete set of possible strategies is impossible to delineate.

Suppose Dolores locates at a random position on the political spectrum, such as x in Figure 16.9a. Claudia can then calculate how the votes will be split for all possible positions that she might choose. If she chooses a position to the left of x, she gets all the votes to her left and half of the votes lying between her position and Dolores' s. If she locates to the right of x, she gets all the votes to her right and half of the votes lying between her position and x. Finally, if she, too, locates at x, she and Dolores split the votes 50–50. These three possibilities effectively summarize all of Claudia' s location choices, given that Dolores has chosen to locate at x.

But which of the response strategies just outlined is Claudia' s *best* response? The answer depends on the location of x relative to the median voter. If x is to the right of the median, then Claudia knows that her best response will be to maximize the number of votes that she gains, which she can do by locating an infinitely small bit to the left of x.[31] In that case, she effectively gets all the votes from 0 to x, and Dolores gets those from x to 1. When x is to the right of the median, as in Figure 16.9a, then the number of voters represented by the area under the distribution curve from 0 to x is by definition larger than the number of voters from x to 1, so Claudia would win the election. Similarly, if x is to the left of the median, Claudia' s best response will be to locate an infinitely small bit to the right of x and thus gain all the votes from x to 1. When x is exactly at the median, Claudia does best by also choosing to locate at x. The best-response strategies for Dolores are constructed exactly the same way and, given the location of her rival, are exactly the same as those described for Claudia. Our analysis shows that Claudia' s best response to Dolores' s location is to locate slightly closer to the median voter and, when Dolores locates at the position of the median voter, to locate in the same place. The same is true in reverse for Dolores' s best response to Claudia' s location. Therefore, the only equilibrium is when both locate at the median voter' s position.

More complex mathematics is needed to prove the continuous version of the median voter theorem to the satisfaction of a true mathematician. For our purposes, however, the discussion here should convince you of the validity of the theorem in both its discrete and continuous forms. The most important limitation of the median voter theorem is that it applies when there is just one issue, or a one-dimensional spectrum of political differences. If there are two or more dimensions—for example, if being conservative versus liberal on social issues does not coincide with being conservative versus liberal on economic issues—then the voter population is spread out in a two-dimensional "issue space," and the median voter theorem no longer holds. The preferences of every individual voter can still be single peaked, in the sense that the individual voter has a most preferred point and her payoff value drops away from this point in all directions, like the height going away from the peak of a hill. But we cannot identify a median voter in two dimensions, such that exactly the same number of voters have their most preferred point to one side of the median voter position as to the other side. In two dimensions, there is no unique sense of "side," and the numbers of voters to the two sides can vary, depending on just how we define *side*.

Endnotes

- Any resemblance between our hypothetical candidates and actual past or possible future candidates in the United States is not meant to imply an analysis or prediction of their performances relative to the Nash equilibrium. Nor is our distribution of voters meant to typify U.S. voter preferences. Return to reference 25
- To keep the analysis simple, we ignore the complications created by the Electoral College and suppose that only the popular vote matters. Return to reference 26
- Economists learn this result within the context of Hotelling's model of spatial location. See Harold Hotelling, "Stability in Competition," *Economic Journal*, vol. 39, no. 1 (March 1929), pp. 41–57. Return to reference 27
- However, the distribution of voters' positions along the political spectrum does not have to be single peaked, as indeed the histogram in Figure 15.7 is not—there are two peaks, at L and FR. Return to reference 28
- This construction is the same one used in Chapters 11 and 12 for analyzing large populations of individual members. Return to reference 29
- We do not delve deeply into the mechanics underlying distribution theory or the integral calculus required to calculate the exact proportion of the voting population lying to the left or right of any particular position on the continuous political spectrum. Here, we present only enough information to convince you that the median voter theorem continues to hold in the continuous case. Return to reference 30
- Such a location, infinitesimally removed from x to the left, is feasible in the continuous case. In our discrete example, candidates had to locate at exactly the same position. Return to reference 31

Glossary

median voter theorem
: If the political spectrum is one-dimensional and every voter has single-peaked preferences, then [1] the policy most preferred by the median voter will be the Condorcet winner, and [2] power-seeking politicians in a two-candidate election will choose platforms that converge to the position most preferred by the median voter. (This is also known as the **principle of minimum differentiation.**)

median voter
: The voter in the middle—at the 50th percentile—of a distribution.

discrete distribution
: A probability distribution in which the random variables may take on only a discrete set of values such as integers.

histogram
: A bar chart; data are illustrated by way of bars of a given height (or length).

principle of minimum differentiation
: Same as part [2] of the *median voter theorem*.

continuous distribution
: A probability distribution in which the random variables may take on a continuous range of values.

distribution function
: A function that indicates the probability that a variable takes on a value less than or equal to some number.

uniform distribution
: A common statistical distribution in which the *distribution function* is horizontal; data are distributed uniformly at each location along the range of possible values.

normal distribution
: A commonly used statistical distribution for which the *distribution function* looks like a bell-shaped curve.

SUMMARY

Elections can be held using a variety of different voting procedures that differ in the order in which issues are considered or the manner in which votes are tallied. Voting procedures are classified as *binary*, *plurative*, or *mixed methods*. Binary methods include *majority rule* as well as *pairwise* procedures such as the *Condorcet method* and the *amendment procedure*. *Positional methods* such as the *plurality rule* and the *Borda count*, as well as *approval voting*, are plurative methods. And *majority runoffs*, *instant runoffs*, and *proportional representation* are mixed methods.

Voting paradoxes (such as the *Condorcet*, the *agenda*, and the *reversal paradox*) show that counterintuitive election results can arise owing to difficulties associated with aggregating voter preferences or to small changes in the list of candidates or issues being considered. Outcomes in any given election under a given set of voter preferences can differ depending on the voting procedure used. Certain principles for evaluating voting methods can be described, although Arrow's *impossibility theorem* shows that no one system satisfies all of these criteria at the same time. Researchers in a broad range of fields have considered alternatives to the principles that Arrow identified.

Voters have scope for strategic behavior in the game that chooses the voting procedure or in an election itself through the *misrepresentation of their own preferences*. Voters may strategically misrepresent their preferences to achieve their most preferred or to avoid their least preferred outcome. In the presence of imperfect information, voters may decide whether to vote strategically on the basis of their beliefs about others' behavior and their knowledge of the distribution of preferences.

Candidates may also behave strategically in building a political platform. A general result known as the *median voter theorem* shows that in elections with only two candidates, both locate at the preference position of the *median voter*. This result holds when voters are distributed along the preference spectrum either *discretely* or *continuously*.

KEY TERMS

agenda paradox (635)

amendment procedure (629)

antiplurality method (630)

approval voting (630)

binary method (628)

Black's condition (639)

Borda count (630)

Condorcet method (629)

Condorcet paradox (633)

Condorcet terms (641)

Condorcet winner (629)

continuous distribution (655)

Copeland index (629)

discrete distribution (652)

distribution function (656)

Gibbard–Satterthwaite theorem (651)

histogram (652)

impossibility theorem (638)

instant runoff voting (IRV) (631)

intransitive ordering (634)

majority rule (629)

majority runoff (631)

median voter (652)

median voter theorem (652)

mixed method (631)

multistage procedure (629)

normal distribution (656)

pairwise voting (629)

plurality rule (629)

plurative method (629)

positional method (629)

principle of minimum differentiation (654)

proportional representation (632)

ranked-choice voting (631)

reversal paradox (636)

reversal terms (641)

robustness (640)

round (631)

sincere voting (635)

single-peaked preferences (639)

single transferable vote (631)

social ranking (633)

spoiler (642)

strategic misrepresentation of preferences (632)

strategic voting (632)

transitive ordering (634)

truthful voting (635)

uniform distribution (656)

Glossary

binary method
: A class of voting methods in which voters choose between only two alternatives at a time.

majority rule
: A voting method in which the winning alternative is the one that garners a majority (more than 50%) of the votes.

pairwise voting
: A voting method in which only two alternatives are considered at the same time.

multistage procedure
: A voting procedure in which there are multiple rounds of voting.

Condorcet method
: A voting method in which the winning alternative must beat each of the other alternatives in a round-robin of pairwise contests.

Condorcet winner
: The alternative that wins an election run using the *Condorcet method*.

Copeland index
: An index measuring an alternative's record in a round-robin of contests where different numbers of points are allocated for wins, ties, and losses.

amendment procedure
: A procedure in which any amended version of a proposal must win a vote against the original version before the winning version is put to a vote against the status quo.

plurative method
: Any voting method that allows voters to consider a slate of three or more alternatives simultaneously.

positional method
: A voting method that determines the identity of the winning alternative using information on the position of

alternatives on a voter's ballot to assign points used when tallying ballots.

plurality rule
: A voting method in which two or more alternatives are considered simultaneously and the winning alternative is the one that garners the largest number of votes; the winner needs only gain more votes than each of the other alternatives and does not need 50% of the vote as would be true in *majority rule*.

antiplurality method
: A positional voting method in which the electorate is asked to vote against one item on the slate (or to vote for all but one).

Borda count
: A positional voting method in which the electorate indicates its order of preference over a slate of alternatives. The winning alternative is determined by allocating points based on an alternative's position on each ballot.

approval voting
: A voting method in which voters cast votes for all alternatives of which they approve.

mixed method
: A multistage voting method that uses plurative and binary votes in different rounds.

majority runoff
: A two-stage voting method in which a second round of voting ensues if no alternative receives a majority in the first round. The top two vote-getters are paired in the second round of voting to determine a winner.

round
: A single vote within a larger *multistage procedure* that consists of multiple sequentially held votes.

single transferable vote
: A voting method in which each voter indicates her preference ordering over all candidates on a single initial ballot. If no alternative receives a majority of

all first-place votes, the bottom-ranked alternative is eliminated and all first-place votes for that candidate are "transferred" to the candidate listed second on those ballots; this process continues until a majority winner emerges. Also called **instant-runoff voting (IRV)** or **ranked-choice voting**.

instant-runoff voting (IRV)
: Same as **single transferable vote**.

ranked-choice voting
: Another name for **single transferable vote**.

proportional representation
: This voting system requires that the number of seats in a legislature be allocated in proportion to each party's share of the popular vote.

strategic voting
: Voting in conformity with your optimal rational strategy found by doing rollback analysis on the game tree of the voting procedure.

Condorcet paradox
: Even if all individual voter preference orderings are transitive, there is no guarantee that the social preference ordering generated by Condorcet's voting method will also be transitive.

social ranking
: The preference ordering of a group of voters that arises from aggregating the preferences of each member of the group.

intransitive ordering
: A preference ordering that cycles and is not *transitive*. For example, a preference ordering over three alternatives A, B, and C is intransitive if A is preferred to B and B is preferred to C but it is not true that A is preferred to C.

transitive ordering
: A preference ordering for which it is true that if option A is preferred to B and B is preferred to C, then A is also preferred to C.

agenda paradox

A voting situation where the order in which alternatives are paired when voting in multiple rounds determines the final outcome.

sincere voting

Voting at each point for the alternative that you like best among the ones available at that point, regardless of the eventual outcome. Also called **truthful voting**.

truthful voting

Same as **sincere voting**.

reversal paradox

This paradox arises in an election with at least four alternatives when one of these is removed from consideration after votes have been submitted and the removal changes the identity of the winning alternative.

impossibility theorem

A theorem that indicates that no preference aggregation method can satisfy the six critical principles identified by Kenneth Arrow.

single-peaked preferences

A preference ordering in which alternatives under consideration can be ordered along some specific dimension and each voter has a single ideal or most-preferred alternative with alternatives farther away from the most-preferred point providing steadily lower payoffs. Also called **Black's condition**.

Black's condition

Same as the condition of **single-peaked preferences**.

robustness

A measure of the number of sets of voter preference orderings that are nondictatorial, satisfy independence of irrelevant alternatives and the Pareto property, and also produce a transitive *social ranking*.

Condorcet terms

A set of ballots that would generate the Condorcet paradox and that should together logically produce a tied vote among three possible alternatives. In a three-candidate election among A, B, and C, the Condorcet terms are three ballots that show A preferred to B preferred to C; B preferred to C preferred to A; C preferred to A preferred to B.

reversal terms

A set of ballots that would generate the *reversal paradox* and that should together logically produce a tied vote between a pair of alternatives. In a three-candidate election among A, B, and C, the reversal terms are two ballots that show a reversal in the location of a pair of alternatives. For example, one ballot with A preferred to B preferred to C and another with B preferred to A preferred to C should produce a tie between A and B.

spoiler

Refers to a third candidate who enters a two-candidate race and reduces the chances that the leading candidate actually wins the election.

Gibbard–Satterthwaite theorem

With three or more alternatives to consider, the only voting method that prevents strategic voting is dictatorship; one person is identified as the dictator and her preferences determine the outcome.

median voter theorem

If the political spectrum is one-dimensional and every voter has single-peaked preferences, then [1] the policy most preferred by the median voter will be the Condorcet winner, and [2] power-seeking politicians in a two-candidate election will choose platforms that converge to the position most preferred by the median voter. (This is also known as the **principle of minimum differentiation.**)

median voter

The voter in the middle—at the 50th percentile—of a distribution.

discrete distribution

A probability distribution in which the random variables may take on only a discrete set of values such as integers.

histogram

A bar chart; data are illustrated by way of bars of a given height (or length).

principle of minimum differentiation

Same as part [2] of the *median voter theorem*.

continuous distribution

A probability distribution in which the random variables may take on a continuous range of values.

distribution function

A function that indicates the probability that a variable takes on a value less than or equal to some number.

uniform distribution

A common statistical distribution in which the *distribution function* is horizontal; data are distributed uniformly at each location along the range of possible values.

normal distribution

A commonly used statistical distribution for which the *distribution function* looks like a bell-shaped curve.

strategic misrepresentation of preferences

Refers to strategic behavior of voters when they use rollback to determine that they can achieve a better outcome for themselves by not voting strictly according to their preference orderings.

SOLVED EXERCISES

1. Consider a vote being taken by three roommates, A, B, and C, who share a triple dorm room. They are trying to decide which of three elective courses to take together this term. (Each roommate has a different major and is taking required courses in her major for the rest of her courses.) Their choices are Philosophy, Geology, and Sociology, and their preferences for the three courses are as shown here:

A	B	C
Philosophy	Sociology	Geology
Geology	Philosophy	Sociology
Sociology	Geology	Philosophy

The roommates have decided to have a two-round vote and will draw straws to determine who sets the agenda. Suppose A wins the right to set the agenda and wants the Philosophy course to be chosen. How should she set the agenda to achieve this outcome if she knows that everyone will vote truthfully in all rounds? What agenda should she set if she knows that they will all vote strategically?

2. Suppose that voters 1 through 4 are being asked to consider three different candidates—A, B, and C—in a positional (Borda-count) election. Their preference orderings are as shown here:

1	2	3	4
A	A	B	C
B	B	C	B
C	C	A	A

Assume that the voters will cast their votes truthfully (no strategic voting). Find a variation on the Borda system—a number of points to be allotted to a voter’ s first, second, and third preferences—in which candidate A wins.

3. In 2015, Democrat Ethan Strimling (S) was elected mayor of Portland, Maine, in a three-way race using instant runoff voting, over incumbent mayor Michael Brennan (B, also a Democrat) and Green Independent Party leader Tom MacMillan (M). Suppose instead that this election is run

under the plurality rule and that there are three types of voters: Green Party supporters, who are 45% of all voters and whose preference ranking is M ≻ S ≻ B; Brennan Democrats, who are 15% of all voters and whose preference ranking is B ≻ S ≻ M; and Strimling Democrats, who are 40% of all voters and whose preference ranking is S ≻ B ≻ M. For simplicity, suppose that the voters in each of these three groups can coordinate their actions and vote together as a bloc.

1. Suppose that M voters are nonstrategic, always casting their votes for MacMillan, but that B and S voters are strategic. Show that the game B and S voters play has ordinal payoffs identical to those in the battle-of-the-sexes game shown in Figure 4.15. Explain why, in the mixed-strategy equilibrium of this game, all three candidates have a chance of winning the election.
2. Suppose now that B voters are nonstrategic, always casting their votes for Brennan, but that M and S voters are strategic. Construct the ordinal payoff matrix for the game played by M and S voters. Show that, in any Nash equilibrium of this game, Brennan always wins the election.

4. Nashville has been the capital of Tennessee since 1826, but before that, the capital had moved from Knoxville to Kingston and then to Murfreesboro. Might the capital move again someday? To explore the likelihood of that happening, consider a voting game played by residents of Tennessee' s four biggest cites: Nashville (N), population 660,000; Memphis (M), population 650,000; Knoxville (K), population 190,000; and Chattanooga (C), population 180,000. The figure below shows the location of each city within the state. Residents of these four cities vote to choose the state' s new capital.

Residents prefer the capital to be as close to them as possible. Nashville residents have the preference ordering N ≻ C ≻ K ≻ M, Memphis residents have the preference ordering M ≻ N ≻ C ≻ K, Chattanooga residents have the preference ordering C ≻ K ≻ N ≻ M, and Knoxville residents have the preference ordering K ≻ C ≻ N ≻ M.

1. Would there a Condorcet winner in this voting game?
2. Which city would win under instant runoff voting (IRV), assuming nonstrategic voting?

3. What if Nashville were smaller and Chattanooga were bigger? In particular, suppose that Nashville shrunk to Chattanooga’s current size (population 180,000) and Chattanooga grew to Nashville’s current size (population 660,000). Would there be a Condorcet winner under these circumstances? Which city would win under IRV, again assuming nonstrategic voting?

5. Consider a group of 50 residents attending a town meeting in Massachusetts. They must choose one of three proposals for dealing with town garbage. Proposal 1 asks the town to provide garbage collection as one of its services; Proposal 2 calls for the town to hire a private garbage collector to provide collection services; and Proposal 3 calls for residents to be responsible for their own garbage. There are three types of voters. The first type prefers Proposal 1 to Proposal 2 and Proposal 2 to Proposal 3; there are 20 of these voters. The second type prefers Proposal 2 to Proposal 3 and Proposal 3 to Proposal 1; there are 15 of these voters. The third type prefers Proposal 3 to Proposal 1 and Proposal 1 to Proposal 2; there are 15 of them.
   1. Under the plurality rule, which proposal wins?
   2. Suppose voting proceeds with the use of a Borda count in which voters list the proposals, in order of preference, on their ballots. The proposal listed first (or at the top) on a ballot gets three points; the proposal listed second gets two points; and the proposal listed last gets one point. In this situation, with no strategic voting, how many points does each proposal gain? Which proposal wins?
   3. What strategy can the second and third types of voters use to alter the outcome of the Borda count vote in part (b) to one that both types prefer? If they use this strategy, how many points does each proposal get, and which one wins?
6. During the Cuban missile crisis (described more fully in Chapter 13), serious differences of opinion arose within ExComm, the group advising President John F. Kennedy, which we summarize here. There were three policy options: Soft (a blockade of Cuba), Medium (a limited air strike), and Hard (a massive air strike or invasion of Cuba). There were also three groups in ExComm. The civilian doves ranked the alternative Soft best, Medium next, and Hard last. The civilian hawks ranked Medium best, Hard next, and Soft last. The military ranked Hard best, but they felt “so strongly about the dangers inherent in the limited strike that they would prefer taking no military action rather than to take that limited strike.”[32] In other words, they ranked Soft second and Medium last. Each group constituted about one-third of ExComm, and so any two of the groups would form a majority.
   1. If the policy were to be decided by a majority vote in ExComm and the members voted sincerely, which alternative, if any, would win?

2. What outcome would arise if members voted strategically? What outcome would arise if one group had agenda-setting power? (Model your discussion in these two cases on the analyses found in Sections 2.B and 4.B.)

7. In his book *A Mathematician Reads the Newspaper*, John Allen Paulos gives the following caricature based on the 1992 Democratic presidential primary caucuses.[33] There are five candidates: Jerry Brown, Bill Clinton, Tom Harkin, Bob Kerrey, and Paul Tsongas. There are 55 voters, with different preference orderings for these candidates. There are six different orderings, which we label I through VI. The preference orderings (1 for best to 5 for worst), along with the numbers of voters with each ordering, are shown in the following table (the candidates are identified by the first letters of their last names):

		GROUPS AND THEIR SIZES					
		I 18	II 12	III 10	IV 9	V 4	VI 2
RANKING	1	T	C	B	K	H	H
	2	K	H	C	B	C	B
	3	H	K	H	H	K	K
	4	B	B	K	C	B	C
	5	C	T	T	T	T	T

You may need to scroll left and right to see the full figure.

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1. First, suppose that all voters vote sincerely. Consider the outcomes of each of several different election rules. Show each of the following outcomes:
   1. Under the plurality method (the one with the most first preferences), Tsongas wins.
   2. Under the runoff method (the top two first preferences go into a second round), Clinton wins.
   3. Under the elimination method (at each round, the one with the fewest first preferences in that round is eliminated, and the rest go into the next round), Brown wins.
   4. Under the Borda count method (5 points for first preference, 4 for second, and so on; the candidate with the most points wins), Kerrey wins.
   5. Under the Condorcet method (pairwise comparisons), Harkin wins.
2. Suppose that you are a Brown, Kerrey, or Harkin supporter. Under the plurality method, you would get your worst outcome. Can you benefit by voting strategically? If so, how?

3. Are there opportunities for strategic voting under each of the other methods as well? If so, explain who benefits from voting strategically and how they can do so.

8. As mentioned in the chapter, some localities have replaced runoff elections and even primaries with instant runoff voting to save time and money. Most jurisdictions, however, still use a two-stage system in which, if a candidate fails to receive a majority of votes in the first round, a second runoff election is held weeks later between the two candidates who earned the most votes.

   For instance, France employs a two-stage system for its presidential elections. No primaries are held. Instead, all candidates from all parties are on the ballot in the first round, which usually guarantees a second round, since it is very difficult for a single candidate to earn a majority of votes among such a large field. Although a runoff in the French presidential election is always expected, it doesn't mean that French elections are without the occasional surprise, as we saw in Section 1.C.

   Instant runoff voting can be explained in five steps:

   1. Voters rank all candidates according to their preferences.
   2. The votes are counted.
   3. If a candidate has earned a majority of the votes, that candidate is the winner. If not, go to step 4.
   4. Eliminate candidate(s) with the fewest votes. (Eliminate more than one candidate at the same time only if they tie for the fewest votes.)
   5. Redistribute the votes for eliminated candidates to the next-ranked choices on those ballots. Once this is done, return to step 2.

   1. Instant runoff voting is slowly gaining traction. Given its potential for saving money and time, it might be surprising that the institution isn't more widely adopted. Why might some oppose instant runoff voting? (Hint: Which candidates, parties, and interests benefit from the two-stage systems that are currently in place?)
   2. What other concerns or criticisms might be raised about instant runoff voting?

9. An election has a slate of three candidates and takes place under the plurality rule. There are numerous voters, spread along a political spectrum from left to right. Represent this spread by a horizontal line whose extreme points are 0 (left) and 1 (right). Voters are uniformly distributed along this spectrum, so the number of voters in any segment of the line is proportional to the length of that segment. Thus, a third of the voters are in the segment from 0 to ⅓, a quarter in the

segment from ½ to ¾, and so on. Each voter votes for the candidate whose declared political position is closest to the voter's own position. The candidates have no ideological attachment and take up any position along the line, each seeking only to maximize her share of votes.

1. Suppose you are one of the three candidates. The leftmost of the other two positions herself at point x, and the rightmost at point $(1 - y)$, where $x + y < 1$ (so that the rightmost candidate is a distance y from 1). Show that your best response is to take up the following positions under the given conditions:
   1. Just slightly to the left of x if $x > y$ and $3x + y > 1$
   2. Just slightly to the right of $(1 - y)$ if $y > x$ and $x + 3y > 1$
   3. Exactly halfway between the other candidates if $3x + y < 1$ and $x + 3y < 1$
2. In a graph with x and y along the axes, show the areas (the combinations of x and y values) where each of the response rules (i) - (iii) in part (a) is best for you.
3. From your analysis, what can you conclude about the Nash equilibrium of the game where the three candidates each choose positions?

UNSOLVED EXERCISES

1. Repeat Exercise S1 for the situation in which B sets the agenda and wants to ensure that Sociology wins.
2. Repeat Exercise S2 to find a variation on the Borda weighting system in which candidate B wins.
3. In the small town of Embarrass, Minnesota, famous for its funny name, the Miss Embarrass contest is no laughing matter. Each August, several young ladies from around the Embarrass River Valley vie for the honor of being crowned Queen at the Embarrass Region Fair. (Everything so far is true; the rest is made up.) The Miss Embarrass contest has traditionally used the plurality method to determine the winner. However, one of the organizers recently read about a FairVote.org study that found that instant runoff voting (IRV) promotes more civil campaigning, and that format is now being considered.

   All 1,000 Embarrass area residents who attend the fair are eligible to vote. This year, Ashley (A) and Becky (B) are the front-runners, trailed by Olivia (O). At the start of fair season, 420 people are "Ashley fans," with the preference ordering A ≻ B ≻ O; 400 people are "Becky fans," with the preference ordering B ≻ A ≻ O; and 180 are "Olivia fans," with the preference ordering O ≻ A = B. However, fairgoers' preferences may change before the vote, depending on whether Ashley and/or Becky engages in "negative campaigning" by spreading a nasty rumor during the fair about her opponent. (For simplicity, we assume that Olivia is sure to run a positive campaign.)

   As you consider the questions below, assume (i) that Ashley and Becky each prefer to remain civil and each prefer not to have a nasty rumor spread about them, but most of all want to win the election; (ii) that negative campaigning suppresses turnout, causing 40 fans of the targeted candidate to forgo voting in the election; and (iii) that if only one candidate stays civil, all Olivia fans will prefer the civil candidate as their second favorite. (If Ashley and Becky are both civil or both negative, Olivia fans will flip a coin if forced to decide between them.)

   1. Suppose that the plurality rule continues to be used for the election. Draw the ordinal payoff matrix for the "civility game" played by Ashley and Becky when they choose between Civil and Negative. Show that this game does not have a pure-strategy Nash equilibrium. Explain in words why, in the unique mixed-strategy Nash equilibrium, Ashley and Becky are each sometimes crowned Miss Embarrass.

2. Suppose that the organizers decide to switch to instant runoff voting for the election. Show that being civil is now a superdominant strategy for both Ashley and Becky. Hint: Draw the ordinal payoff matrix for the civility game with Ashley as the row player. Civil is superdominant for Ashley if the best two payoffs (4 and 3) appear in the Civil row. Similarly, Civil is superdominant for Becky if the two best payoffs appear in the Civil column.

4. Consider a version of recent political developments in Britain related to the issue of leaving the European Union (Brexit), only slightly caricatured here. There are three alternatives: Remain in the EU (labeled R), a negotiated Soft Brexit (S), and crashing out of the EU without a negotiated agreement in a Hard Brexit (H). There are three types of voters: 45% of voters are Remainers, with the preference ranking $R \succ S \succ H$; 25% of voters are Moderate leavers, with the preference ranking $S \succ H \succ R$; and 30% of voters are Extremist leavers, with the preference ranking $H \succ R \succ S$ (because they regard a negotiated Soft Brexit as the worst compromise and irreversible, whereas with R they hope to continue their fight and get H in the future).

   For each of the possible voting methods described below, calculate the outcome, first assuming that all types of voters vote sincerely, and then assuming that they all vote strategically (which requires collective action for each group).

   1. Plurality rule: All alternatives are on the ballot at the same time, and the one with the most votes wins.
   2. Two-round ballot: All alternatives are on the first ballot. Each voter votes for just one. If one gets over 50% of the votes, it wins; if not, a second round of voting pits the top two vote getters against each other. The option with the most votes wins.
   3. Exhaustive ballot: All alternatives are on the first ballot. Each voter votes for just one option. If one gets over 50% of the votes, it wins; if not, the one with the fewest votes is eliminated, and another vote is run among all the remaining options. This process continues until an option gains 50% of the votes in a round.
   4. Ranked choice (instant runoff): All alternatives are on the ballot, and each voter ranks them all from best (1) to worst (3). If one option gets more than 50% first-choice votes, it wins. If not, the one with the fewest first-choice votes is eliminated, and all first-choice votes for that alternative are transferred to the alternative ranked second on those ballots. The one that has the majority in this recalculation is the winner.
   5. Borda count: All alternatives are on the ballot, and each voter ranks them all from best (1) to worst (3). All these numbers are

added over all the voters, and the alternative that gets the smallest total is the winner.

6. Pairwise voting: A ballot for each pair of alternatives is prepared and voters vote on each pairing (R versus S, S versus H, and H versus R). The alternative that wins all of its contests is the winner.
7. Sequential decisions: The first round of voting is Remain versus Leave. If Remain wins, that ends the process. If Leave wins, there a second round of voting between Soft and Hard, and the winning alternative is implemented.

5. Every year, college football's Heisman Trophy is awarded by means of a Borda count procedure. Each voter submits first-, second-, and third-place votes, worth 3 points, 2 points, and 1 point, respectively. Thus, the Borda count point scheme used may be called (3-2-1), where the first digit is the point value of a first-place vote, the second digit the point value of a second-place vote, and the third digit the point value of a third-place vote. In 2004, the vote totals for the top five candidates under the Borda system were as follows:

Player	1st Place	2st Place	3st Place
Leinhart (USC)	267	211	102
Peterson (Oklahoma)	154	180	175
White (Oklahoma)	171	149	146
Smith (Utah)	98	112	117
Bush (USC)	118	80	83

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1. Compare the Borda count scores of Leinhart and Peterson. By what margin of Borda points did Leinhart win?
2. It seems only fair that a point scheme should give a first-place vote at least as much weight as a second-place vote and a second-place vote at least as much weight as a third-place vote. That is, for a point scheme (x-y-z), we should have $x \geqslant y \geqslant z$. Given this "fairness" restriction, is there any point scheme under which Leinhart would have lost? If so, provide such a scheme. If not, explain why not.
3. Even though White had more first-place votes than Peterson, Peterson had a higher Borda count total. If first-place votes were weighted enough, White's edge in first-place votes could have given him a higher Borda count. Assume that second-place votes are worth 2 points and third-place votes are worth 1 point, so that the point scheme is (x–2-1). What is the lowest integer value of x such that White would get a higher Borda count than Peterson?

4. Suppose that the data in the table represent truthful voting. For simplicity, let's suppose that the election was a simple plurality vote instead of a Borda count. Note that Leinhart and Bush are both from USC, whereas Peterson and White are both from Oklahoma. Suppose that, due to Oklahoma loyalty, those voters who prefer White all have Peterson as their second choice. If those voters were to vote strategically in a plurality-rule election, could they change the outcome of the election? Explain.
5. Similarly, suppose that due to USC loyalty, those voters who prefer Bush all have Leinhart as their second choice. If all four voting groups (those who most prefer Leinhart, then Bush; those who most prefer Peterson, then White; those who most prefer White, then Peterson; and those who most prefer Bush, then Leinhart) were to vote strategically in a plurality-rule election, who would be the winner of the Heisman Trophy?
6. In 2004, there were 923 Heisman voters. Under the actual (3-2-1) system, what is the minimum integer number of first-place votes that it would have taken to guarantee victory (that is, without the help of any second-or third-place votes)? Note that a player's name may appear on a ballot only once.

6. Olympic figure skaters complete two programs in their competition, one short and one long. In each program, the skaters are scored and then ranked by a panel of nine judges. The skaters' positions in the rankings are used to determine their final scores. A skater's ranking depends on the number of judges placing her first (or second or third); the skater judged to be best by the most judges is ranked 1, and so on. In the calculation of a skater's final score, the short program gets half the weight of the long program. That is, Final score = 0.5(Rank in short program) + Rank in long program. The skater with the lowest final score wins the gold medal. In the event of a tie, the skater judged best in the long program by the most judges takes the gold.

   In the women's individual figure-skating competition at the 2002 Olympics in Salt Lake City, Michelle Kwan was in first place after the short program. She was followed by Irina Slutskaya, Sasha Cohen, and Sarah Hughes, who were in second, third, and fourth places, respectively. In the long program, the judges' cards for these four skaters were as follows:

		JUDGE NUMBER								
		1	2	3	4	5	6	7	8	9
KWAN	Points	11.3	11.5	11.7	11.5	11.4	11.5	11.4	11.5	11.4

You may need to scroll left and right to see the full figure.

		JUDGE NUMBER								
		1	2	3	4	5	6	7	8	9
	Rank	2	3	2	2	2	3	3	2	3
SLUTSKAYA	Points	11.3	11.7	11.8	11.6	11.4	11.7	11.5	11.4	11.5
	Rank	3	1	1	1	4	1	2	3	2
COHEN	Points	11.0	11.6	11.5	11.4	11.4	11.4	11.3	11.3	11.3
	Rank	4	2	4	3	3	4	4	4	4
HUGHES	Points	11.4	11.5	11.6	11.4	11.6	11.6	11.3	11.6	11.6
	Rank	1	4	3	4	1	2	1	1	1

You may need to scroll left and right to see the full figure.

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1. In the long program, Slutskaya skated last of the top skaters. Use the information from the judges' cards to determine the judges' long-program ranks for Kwan, Cohen, and Hughes *before* Slutskaya skated. Then, using the standings already given for the short program in conjunction with your calculated ranks for the long program, determine the final scores and standings among these three skaters *before* Slutskaya skated. (Note that Kwan's rank in the short program was 1, so her partial score after the short program is 0.5.)
2. Given your answer to part (a), what would have been the final outcome of the competition if the judges had ranked Slutskaya's long program above all three of the others?
3. Use the judges' cards to determine the actual final scores for all four skaters *after* Slutskaya skated. Who won each medal?
4. Which of Arrow's principles does the Olympic figure-skating scoring system violate? Explain.

7. The 2008 presidential nomination season saw 21 Republican primaries and caucuses on Super Tuesday—February 5, 2008. By that day—only a month after the Iowa caucuses that began the process—more than half of the Republican contenders had dropped out of the race, leaving only four: John McCain, Mitt Romney, Mike Huckabee, and Ron Paul. McCain, Romney, and Huckabee had each previously won at least one state. McCain had beaten Romney in Florida the week before Super Tuesday, and at that point it looked like only the two of them stood a realistic chance of wining the nomination. In this primary season, as is typical for the Republican party, nearly every GOP contest (whether primary or caucus) was winner-take-all, so winning a given state would earn a candidate all of the convention delegates allotted to that state by the Republican National Committee.

The West Virginia caucus was the first contest to reach a conclusion on Super Tuesday because that caucus took place in the afternoon, it was brief, and the state is in the eastern time zone. News of the result was available hours before the close of polls in many of the other states voting that day.

The following problem is based on the results of that West Virginia caucus. As we might expect, the caucusers did not all share the same preference orderings of the candidates. Some favored McCain, whereas others liked Romney or Huckabee. The caucusers also had varied preferences about whom they wanted to win if their favorite candidate did not. Simplifying substantially from reality (but based on the actual voting), assume that there were seven types of West Virginia caucusers that day, whose prevalence and preferences were as follows:

	I (16%)	II (28%)	III (13%)	IV (21%)	V (12%)	VI (6%)	VII (4%)
1st	McCain	Romney	Romney	Huckabee	Huckabee	Paul	Paul
2nd	Romney	McCain	Huckabee	Romney	McCain	Romney	Huckabee
3rd	Huckabee	Huckabee	McCain	McCain	Romney	Huckabee	Romney
4th	Paul	Paul	Paul	Paul	Paul	McCain	McCain
You may need to scroll left and right to see the full figure.							

At first, no one knew the distribution of preferences of those in attendance at the caucus, so everyone voted truthfully. Thus, Romney won a plurality of the votes in the first round, with 41%.

After each round of this caucus, if no candidate wins a majority, the candidate with the smallest number of votes is dropped from consideration, and his or her supporters vote for one of the remaining candidates in the following rounds.

1. What would the results of the second round have been under truthful (nonstrategic) voting for the remaining three candidates?
2. If West Virginia had held pairwise votes among the four candidates, which one would have been the Condorcet winner with truthful voting?
3. In reality, the results of the second round of the caucus were as follows:

   Huckabee 52%

   Romney 47%

McCain 1%

Given the preferences of the McCain voters, why might this have happened? (Hint: How would the outcome have been different if West Virginia had voted last on Super Tuesday?)

4. After the fact, Romney's campaign cried foul and accused the McCain and Huckabee supporters of making a backroom deal.[34] Should Romney's campaign have suspected collusion between the McCain and Huckabee camps in this case? Explain why or why not.

8. Return to the discussion of instant runoff voting (IRV) in Exercise S8.
   1. Consider the following IRV ballots of five voters:

	Ana	Bernard	Cindy	Desmond	Elizabeth
1st	Jack	Jack	Kate	Locke	Locke
2nd	Kate	Kate	Locke	Kate	Jack
3rd	Locke	Locke	Jack	Jack	Kate
You may need to scroll left and right to see the full figure.					

Which—if any—of the five voters have an incentive to vote strategically? If so, who and why? If not, explain why not.

   2. Consider the following table, which gives the IRV ballots of a small town of seven citizens voting on five policy proposals put forward by the mayor:

	Anderson	Brown	Clark	Davis	Evans	Foster	García
1st	V	V	W	W	X	Y	Z
2nd	W	X	V	X	Y	X	Y
3rd	X	W	Y	V	Z	Z	X
4th	Y	Y	X	Y	V	W	W
5th	Z	Z	Z	Z	W	V	V
You may need to scroll left and right to see the full figure.							

Assuming that all candidates (or policies) that tie for the fewest votes are eliminated at the same time, under what conditions is an eventual majority winner guaranteed? Put another way, under what conditions might there not be an unambiguous majority winner? (Hint: How important is it for Evans, Foster, and García to fill out their ballots completely?) How will these conditions change if a new citizen, Harris, moves into town and votes?

9. Recall the three-member council considering three alternative welfare policies in Section 4.C. There, three councillors (Left, Center, and Right) considered policies A and D in a first-round vote, with the winner facing policy G in a second-round election. But no one knows for sure exactly how many councillors have each set of possible preferences. The possible preference orderings are shown in Figure 16.1. Each councillor knows her own type, and she knows the probabilities of observing each type of voter, p_L, p_C, and p_R (with $p_L + p_C + p_R = 1$). The behavior of the Center-type voters in the first-round election is the only unknown in this situation and will depend on the probabilities that the various preference types occur. Suppose here that a Center-type voter believes (in contrast to the case considered in the chapter) that other Center types will vote strategically; suppose further that the Center type's payoffs are as in Section 4.C: 1 if A wins, 0 if G wins, and $0 < u < 1$ if D wins.
   1. Under what configuration of the other two votes does the Center-type voter's first-round vote matter to the outcome of the election? Given her assumption about the behavior of other Center-type voters, how would she identify the source of the first-round votes?
   2. Following the analysis in Section 4.C, determine the expected payoff to the Center type when she votes truthfully. Compare this with her expected payoff when she votes strategically. What is the condition under which the Center type votes strategically?

Endnotes

- Ernest R. May and Philip D. Zelikow, eds., *The Kennedy Tapes: Inside the White House during the Cuban Missile Crisis* (Cambridge, Mass.: Harvard University Press, 1997), p. 97. Return to reference 32
- John Allen Paulos, *A Mathematician Reads the Newspaper* (New York: Basic Books, 1995), pp. 104–6. Return to reference 33
- See Susan Davis, "Romney Cries Foul in W. Va. Loss," *Wall Street Journal*, February 5, 2008, available at http://blogs.wsj.com/washwire/2008/02/05/huckabee-wins-first-super-tuesday-contest/?mod=WSJBlog. Return to reference 34

17 ■ Bargaining

PEOPLE ENGAGE IN BARGAINING throughout their lives. Children start by negotiating to share toys and to play games with other children. Couples bargain about matters of housing, child rearing, and the adjustments that each must make for the other's career. Buyers and sellers bargain over price, workers and bosses over wages. Countries bargain over policies of mutual trade liberalization; superpowers negotiate mutual arms reduction. And the authors of this book had to bargain among themselves—generally very amicably—about what to include or exclude, how to structure the exposition, and so forth. To get a good result from such bargaining, the participants must devise good strategies. In this chapter, we explain some of the basic ideas of bargaining and some strategies derived from them.

All bargaining situations have two things in common. First, the total payoff that the parties to the negotiation are capable of creating and enjoying as a result of reaching an agreement should be greater than the sum of the individual payoffs that they could achieve separately—the whole must be greater than the sum of the parts. Without the possibility of this excess value, or *surplus*, the negotiation would be pointless. If two children considering whether to play together cannot see a net gain from having access to a larger total stock of toys or to one another's company, then it is better for each to "take his toys and play by himself." The world is full of uncertainty, and the expected benefits of an agreement may not materialize. But when engaged in bargaining, the parties must at least perceive some gain therefrom: Even Faust, when he agreed to sell his soul to the Devil, thought the benefits of knowledge and power that he would gain were worth the price that he would eventually have to pay.

The second important general point about bargaining follows from the first: Bargaining is not a zero-sum game. When a surplus exists, the negotiation is about how to divide it up. Each bargainer tries to get more for himself and leave less for the others. This may appear to be a zero-sum game, but behind it lies the danger that if an agreement is not reached, no one will get any surplus at all. This mutually harmful alternative, as well as *both* parties' desire to avoid it, is what creates the potential for the threats—explicit and implicit—that make bargaining such a strategic matter.

Before the advent of game theory, one-on-one bargaining was generally thought not to have any determinate equilibrium solution. Observations of widely different outcomes in otherwise similar-looking situations lent support to this view. Theorists were not able to achieve any systematic understanding of why one party got more than another and attributed these results to vague and inexplicable differences in "bargaining power."

Even the simple theory of Nash equilibrium does not take us any further. Suppose two people are to split $1. Let us construct a game in which each is asked to announce how much of the money he wants. The moves are simultaneous. If the players' announced amounts x and y add up to 1 or less, each gets the amount he announced. If they add up to more than 1, neither gets anything. Then *any* pair (x, y) adding up to 1 constitutes a Nash equilibrium in this game; *given* the announcement of the other, each player cannot do better than to stick to his own announcement.[1]

Further advances in game theory have brought progress along two quite different lines, each using a distinct mode of game-theoretic reasoning. In Chapter 2, we distinguished between *cooperative games*, in which the players decide and implement their actions jointly, and *noncooperative games*, in which the players decide and take their actions separately. Each of the two lines of advance in bargaining theory uses one of these two concepts. One approach views bargaining as a cooperative game, in which the parties find and implement a solution jointly, perhaps using a neutral third party such as an arbitrator for enforcement. The other approach views bargaining as a noncooperative game, in which the parties choose strategies separately, and looks for an equilibrium. In contrast to our earlier simple game of simultaneous announcements, whose equilibrium was indeterminate, this approach imposes more structure and specifies a sequential-move game of offers and counteroffers, which leads to a determinate equilibrium. As in Chapter 2, we emphasize that the labels *cooperative* and *noncooperative* refer to joint versus separate actions, not to nice versus nasty behavior or to compromise versus breakdown. The equilibria of noncooperative bargaining games can entail a lot of compromise.

Endnotes

- As we saw in Chapter 5, Section 2.B, this type of game can be used as an example to bolster the critique that the Nash equilibrium concept is too imprecise. In the bargaining context, we might say that the multiplicity of equilibria is just a formal way of showing the indeterminacy that previous analysts had claimed. Return to reference 1

1 THE NASH COOPERATIVE SOLUTION

In this section, we present John Nash's cooperative-game approach to bargaining.[2] First we present the idea in a simple numerical example; then we develop the more general algebra.

A. Numerical Example

Imagine two Silicon Valley entrepreneurs, Andy and Bill. Andy produces a microchip set that he can sell to any computer manufacturer for $900. Bill has developed a software package that can retail for $100. The two meet and realize that their products are ideally suited to each other and that, with a bit of trivial tinkering, they can produce a combined system of hardware and software worth $3,000 in each computer. Thus, together they can produce an extra value of $2,000 per unit, and they can expect to sell millions of these units each year. The only obstacle that remains on this path to fortune is to agree to a division of the spoils. Of the $3,000 revenue from each unit, how much should go to Andy and how much to Bill?

Bill’s starting position is that without his software, Andy’s chip set is just so much metal and sand, so Andy should get only the $900 he could charge for the software and Bill himself should get $2,100. Andy counters that without his hardware, Bill’s programs are just symbols on paper or magnetic signals on a diskette, so Bill should get only $100, and $2,900 should go to Andy.

Watching them argue, you might suggest they “split the difference.” But that is not an unambiguous recipe for agreement. Bill might offer to split the profit on each unit equally with Andy. Under this scheme, each would get a profit of $1,000, meaning that $1,100 of the revenue would go to Bill and $1,900 to Andy. Andy’s response might be that each of them should get a percentage of the profit reflecting his contribution to the joint enterprise. Thus, Andy should get $2,700 and Bill $300.

The final agreement depends on their stubbornness or patience if they negotiate directly with each other. If they try to have the dispute arbitrated by a third party, the arbitrator’s decision depends on her sense of the relative value of hardware and software and on the rhetorical skills of the two principals as they present their arguments to her. For the sake of definiteness, suppose the arbitrator decides that the division of the profit should be 4:1 in favor of Andy; that is, Andy should get four-fifths of the surplus while Bill gets one-fifth, or Andy should get four times as much as Bill. What is the actual division of revenue under this scheme? Suppose Andy gets a total of x and Bill gets a total of y; thus Andy’s profit is $(x - 900)$ and Bill’s is $(y - 100)$. The arbitrator’s decision implies that Andy’s profit should be four times as large as Bill’s; so $x - 900 = 4(y - 100)$, or $x = 4y + 500$. The total revenue available to both is \$3,000; so it must also be true that $x + y = 3{,}000$, or $x = 3{,}000 - y$. Then $x = 4y + 500 = 3{,}000 - y$, or $5y = 2{,}500$, or $y = 500$, and thus $x = 2{,}500$. This division mechanism leaves Andy with a profit of $2{,}500 - 900 = \$1{,}600$ and Bill with $500 - 100 = \$400$, which is the 4:1 split in favor of Andy that the arbitrator wants.

Next we develop these simple data into a general algebraic formula that you will find useful in many practical applications. Then we go on to examine more specifics of what determines the ratio between the players’ shares of the surplus in a bargaining game.

B. General Theory

Suppose two bargainers, A and B, seek to split a total value v, which they can achieve if and only if they agree on a specific division. If no agreement is reached, A will get a and B will get b, each by acting alone or in some other way acting outside of their relationship. Call these their *backstop* payoffs or, in the jargon of the Harvard Negotiation Project, their BATNAs (best alternatives to a negotiated agreement).[3] Often a and b are both zero, but more generally, we need only assume that $a + b < v$, so that the agreement yields a positive surplus $(v - a - b)$; if this were not the case, the whole bargaining process would be moot because each side would just take up its outside opportunity and get its BATNA.

Consider the following rule: Each player is to be given his BATNA plus a share of the surplus, a fraction h of the surplus for A and a fraction k for B, such that $h + k = 1$. Writing x for the amount that A finally ends up with, and similarly y for B, we translate these statements as

$$x = a + h(v - a - b) = a(1 - h) + h(v - b)$$

$$x - a = h(v - a - b)$$

and

$$y = b + k(v - a - b) = b(1 - k) + k(v - a)$$

$$y - b = k(v - a - b).$$

We call this set of expressions the Nash formula. Another way of looking at them is to say that the surplus $(v - a - b)$ gets divided between the two bargainers in the proportion $h{:}k$, or

$$\frac{y-b}{x-a} = \frac{k}{h}$$

or, in slope-intercept form,

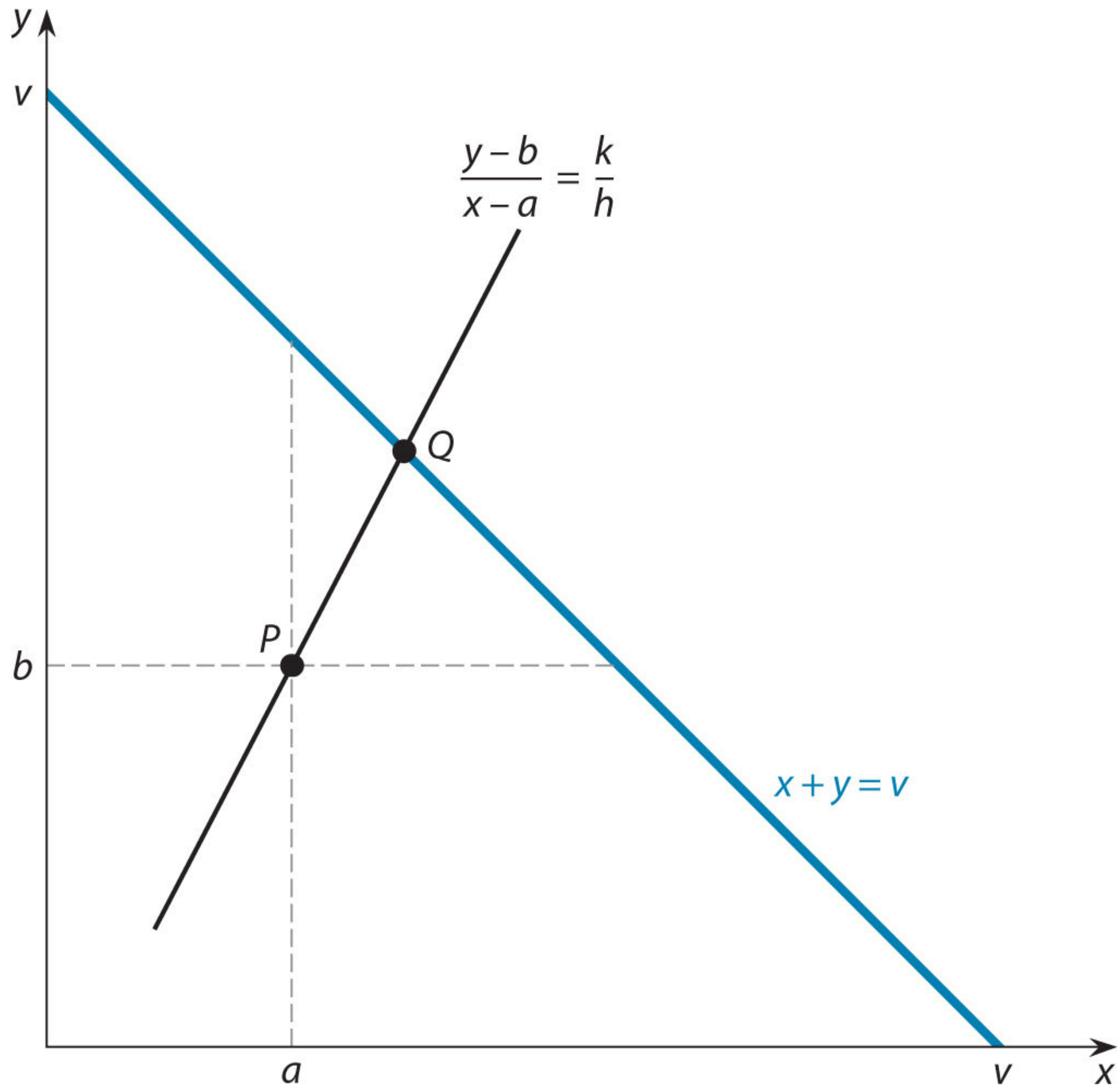

FIGURE 17.1 The Nash Cooperative Solution in the Simplest Case

$$y = b + \frac{k}{h}(x - a) = \left(b - \frac{ak}{h}\right) + \frac{k}{h}x.$$

To use up the whole surplus, x and y must also satisfy $x + y = v$. The Nash formula for x and y actually gives the solutions to these last two simultaneous equations.

The outcome of this formula is the Nash cooperative solution to the bargaining problem.[4] Its geometric representation is shown in Figure 17.1. The backstop, or BATNA, is the point P, with coordinates (a, b). All points (x, y) that divide the gains in proportions $h{:}k$ between the two players lie along the straight line passing through P and having slope k/h; this slope is just the line $y = b + (k/h)(x - a)$ that we derived earlier. All points (x, y) that use up the whole surplus lie along the straight line joining $(v, 0)$ and $(0, v)$; this line is the second equation that we derived—namely, $x + y = v$. Nash's cooperative solution is at the intersection of the lines, at the point Q. The coordinates of this point are the parties' payoffs after the agreement.

The Nash formula says nothing about how or why such a solution might come about. And this vagueness is its merit—it can be used to encapsulate the results of many different theories taking many different perspectives.

At the simplest, you might think of the Nash formula as a shorthand description of the outcome of a bargaining process that we have not specified in detail. Then h and k can stand for the two parties' relative bargaining strengths. This shorthand description is a cop-out; a more complete theory should explain where these bargaining strengths come from and why one party might have more bargaining strength than the other. We will offer such explanations in a particular context later in this chapter. In the meantime, by summarizing any and all of the sources of bargaining strength in these numbers h and k, the formula has given us a good tool.

Nash's own interpretation of his cooperative solution differed from the view that it was a "shorthand description of some underlying game" as described in the previous paragraph. Indeed, Nash's interpretation differed from the whole approach to game theory that we have taken thus far in this book and deserves more careful explanation. In all the games that we have studied so far, the players chose and played their strategies separately from one another. We have looked for equilibria in which each player's strategy was in his own best interest, given the strategies of the others. Some such outcomes were very bad for some or even all of the players, the prisoners' dilemma being the most prominent example. In such situations, there was scope for the players to get together and agree that all would follow some particular strategy. But in our framework, there was no way in which they could be sure that the agreement would hold. After reaching an agreement, the players would disperse, and when it was each player's turn to act, he would actually take the action that served his own best interest. The agreement on joint action would unravel in the face of such separate temptations. In considering repeated games in Chapter 10, we found that the implicit threat of the collapse of an ongoing relationship might sustain an agreement, and in Chapter 9, we allowed for communication by signals. But individual action was of the essence, and any mutual benefit could be achieved only if it did not fall prey to the selfishness of separate individual actions. In Chapter 2, we called this approach to game theory *noncooperative*, emphasizing that the term signifies how actions are taken, not whether outcomes are jointly good. The important point, again, is that any joint good outcome has to be an equilibrium of separate actions in such games.

But what if joint action *is* possible? For example, the players might take all their actions immediately after the agreement is reached, in one another's presence. Or they might delegate the implementation of their joint agreement to a neutral third party, or to an arbitrator. In other words, the game might be *cooperative* (again, in the sense of joint action). Nash modeled bargaining as a cooperative game.

The thinking of a group that is going to implement a joint agreement by joint action can be quite different from that of a set of individual people who know that they are *interacting* strategically but are *acting* noncooperatively. Whereas the latter set will think in terms of an equilibrium and then delight or grieve depending on whether they like the results, the former can think first of what is a good outcome and then see how to implement it. In other words, the theory defines the outcome of a cooperative game in terms of some general principles or properties that seem reasonable to the theorist.

Nash formulated a set of such principles for bargaining and proved that they imply a unique outcome. His principles are roughly as follows: (1) The outcome should not change if the scale by which the payoffs are measured changes linearly; (2) the outcome should be efficient; and (3) if the set of possibilities is reduced by removing some that are irrelevant, in the sense that they would not be chosen anyway, then the outcome should not be affected.

The first of these principles conforms to the intuition that a change of scale merely changes the units of measurement for the entities being bargained over—for example, inches to centimeters or dollars to euros—and should not affect the result. A nonlinear rescaling would be different; as we mentioned in Chapter 9, that would correspond to a change in the players' attitudes toward risk, but we will leave that topic to more advanced treatments.

The second principle, that the outcome should be efficient, simply means that no available mutual gain should go unexploited. In our simple example of A and B splitting a total value of v, it would mean that x and y must sum to v, and not to any smaller amount; in other words, the solution has to lie on the $x + y = v$ line in Figure 17.1. More generally, the complete set of logically conceivable agreements in a bargaining game, when plotted on a graph as in Figure 17.1, will be bounded above and to the right by the subset of agreements that leave no mutual gain unexploited. This subset need not lie along a straight line such as $x + y = v$ (or $y = v - x$); it could lie along any curve of the form $y = f(x)$.

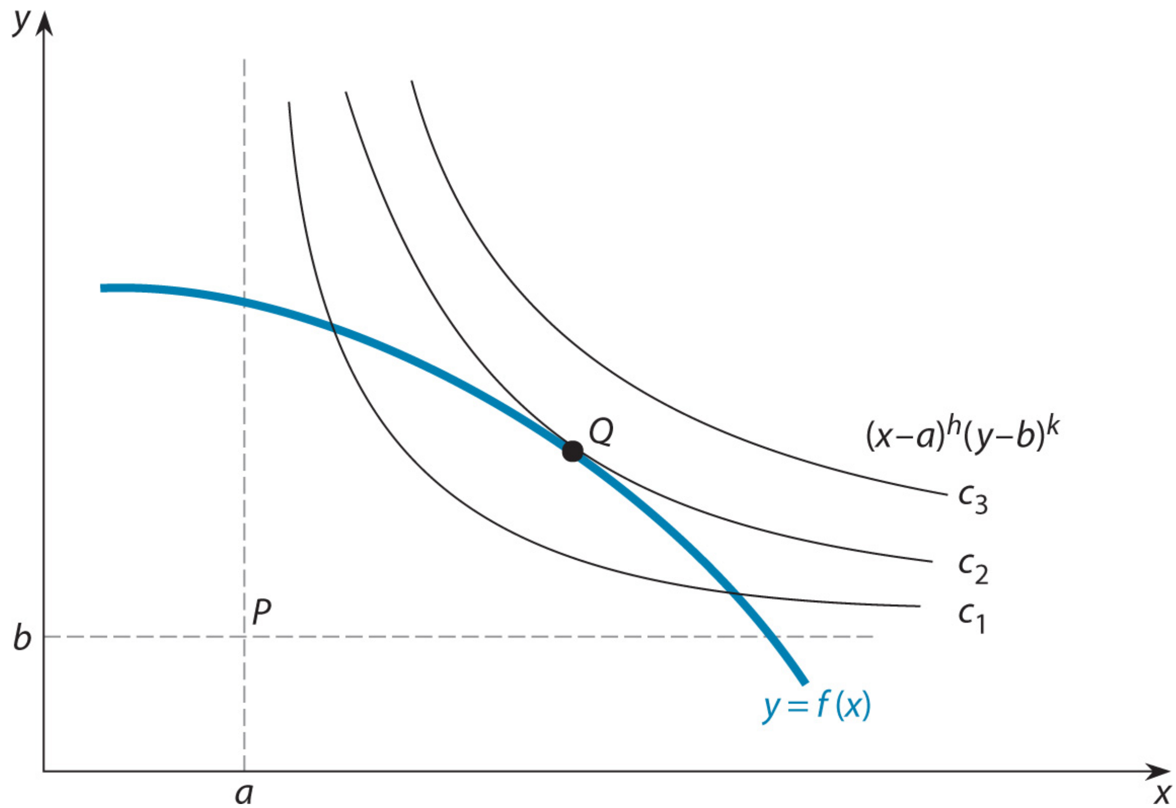

FIGURE 17.2 The General Form of the Nash Cooperative Solution

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In Figure 17.2, all the points on and below (that is, "southwest" of) the thick blue curve labeled $y = f(x)$ constitute the complete set of conceivable outcomes. The curve itself consists of the efficient outcomes; there are no conceivable outcomes that include more of both x and y than the outcomes on $y = f(x)$, so there are no unexploited mutual gains left. Therefore, we call the curve $y = f(x)$ the efficient frontier of the bargaining problem.

To illustrate a curved efficient frontier, we consider two friends who jointly own a lottery ticket that has won a jackpot of \$200,000 and are bargaining over its division. Each wants more money for himself, but the added pleasure each gets from an extra dollar diminishes as he gets more and more money. To be precise, we suppose that the pleasure payoff increases as the square root of the amount of money, so increasing the dollar amount fourfold increases the pleasure only by a factor of 2. If the first friend gets \$$z$ of the money and the other gets the remaining \$(200,000 − z), their respective payoffs x and y will be

$$x = \sqrt{z} \text{ and } y = \sqrt{200{,}000 - z}.$$

Therefore, we can describe the set of possible payoff outcomes for the two friends by the equation

$x^2 + y^2 = z + (200{,}000 - z) = 200{,}000.$

This equation defines a quarter-circle in the positive quadrant and represents the efficient frontier of the friends' bargaining problem.

Now suppose that, if the friends fail to agree on a division of the jackpot, organizers of the lottery will submit their dispute to an arbitrator. This arbitrator would charge a 10% fee and divide the remaining \$180,000 equally between the two friends. So each would get \$90,000 and a payoff of

$\sqrt{90{,}000} = 300.$ That then is the BATNA of each, and the BATNA point (300, 300) lies inside the quarter-circle efficient frontier.

The third principle also seems appealing. If an outcome that a bargainer wouldn't have chosen anyway drops out of the picture, what should it matter? This assumption is closely connected to the "independence of irrelevant alternatives" assumption of Arrow's impossibility theorem, which we met in Chapter 16, Section 3, but we must leave the development of this connection to more advanced treatments of the subject.

Nash proved that the cooperative outcome that satisfied all three of these assumptions could be characterized by a mathematical maximization problem: Choose x and y to maximize $(x - a)^h(y - b)^k$ subject to $y = f(x)$. Here, x and y are the outcomes, a and b the

BATNAs, and h and k two positive numbers summing to 1, which are like the bargaining strengths of the Nash formula. The values for h and k cannot be determined by Nash's three assumptions alone; thus they leave a degree of freedom in the theory and in the outcome. Nash actually imposed a fourth assumption on the problem—that of symmetry between the two players; this additional assumption led to the outcome $h = k = ½$ and fixed a unique solution. We have given the more general formulation that subsequently became common in game theory and economics.

Figure 17.2 also gives a geometric representation of the objective of the maximization. The black curves labeled c_1, c_2, and c_3 are the level curves, or contours, of the function being maximized; along each such curve, $(x - a)^h(y - b)^k$ is constant and equals c_1, c_2, or c_3 (with $c_1 < c_2 < c_3$) as indicated. The whole space could be filled with such curves, each with its own value of the constant, and curves farther to the northeast would have higher values of the constant.

It is immediately apparent that the highest possible value of the function is at that point of tangency, Q, between the efficient frontier and one of the level curves.[5] The location of Q is defined by the property that the contour passing through Q is tangent to the efficient frontier. This tangency is the usual way to illustrate Nash's cooperative solution geometrically.[6]

In our example of Figure 17.1, we can also derive the Nash cooperative solution mathematically; to do so requires calculus, but the ends here are more important—at least to the study of games of strategy—than the means. For the solution, it helps to write $X = x - a$ and $Y = y - b$. Thus, X is the amount of the surplus that goes to A, and Y is the amount of the surplus that goes to B. The efficiency of the outcome guarantees that $X + Y = x + y - a - b = v - a - b$, which is just the total surplus and which we will write as S. Then $Y = S - X$, and

$$(x - a)^h(y - b)^k = X^hY^k = X^h(S - X)^k.$$

In the Nash cooperative solution, X takes on the value that maximizes this function. Elementary calculus tells us that the way to find X is to take the derivative of this expression with respect to X and set it equal to zero. Using the rules of calculus for taking the derivatives of powers of X and of the product of two functions of X, we get

$$hX^{h-1}(S - X)^{k} - X^{h}k(S - X)^{k-1} = 0.$$

When we cancel the common factor $X^{h-1}(S - X)^{k-1}$, this equation becomes

$$h(S-X) - kX = 0$$

$$hY - kX = 0$$

$$kX = hY$$

$$\frac{X}{h} = \frac{Y}{k}.$$

Finally, expressing the equation in terms of the original variables x and y, we have $(x - a)/h = (y - b)/k$, which is just a rearranged version of the Nash formula we presented above. The punch line: Nash's three conditions lead to the formula we originally stated as a simple way of splitting the bargaining surplus.

The three principles, or desired properties, that determine the Nash cooperative solution are simple and even appealing. But in the absence of a good mechanism to make sure that the parties take the actions stipulated by the agreement, these principles may come to nothing. A player who can do better by strategizing on his own than by using the Nash cooperative solution may simply reject the principles. If an arbitrator can enforce a solution, the player may simply refuse to go to arbitration. Therefore the

Nash cooperative solution will seem more compelling if it can be given an alternative interpretation—namely, as the Nash equilibrium of a noncooperative game played by the bargainers. This can indeed be done, and we will develop an important special case of it in Section 4.

Endnotes

- John F. Nash Jr., “The Bargaining Problem,” *Econometrica*, vol. 18, no. 2 (1950), pp. 155–62. Return to reference 2
- See Roger Fisher and William Ury, *Getting to Yes*, 2nd ed. (New York: Houghton Mifflin, 1991). Return to reference 3
- This outcome is also commonly known as the *Nash bargaining solution*. Our terminology more clearly distinguishes the cooperative solution from the alternative noncooperative solutions to bargaining games (based on Nash equilibrium) that are analyzed later in this chapter (Sections 3 through 7). Return to reference 4
- One and only one of the (convex) level curves can be tangential to the (concave) efficient frontier; in Figure 17.2, this level curve is labeled c_2. All lower-level curves (such as c_1) cut the frontier at two points; all higher-level curves (such as c_3) do not meet the frontier at all. Return to reference 5
- If you have taken an elementary microeconomics course, you will have encountered the concept of social optimality, illustrated graphically by the tangent point between the production possibility frontier of an economy and a social indifference curve. Our Figure 17.2 is similar in spirit; the efficient frontier in bargaining is like the production possibility frontier, and the level curves of the objective in cooperative bargaining are like social indifference curves. Return to reference 6

Glossary

best alternative to a negotiated agreement (BATNA)
: In a bargaining game, this is the payoff a player would get from his other opportunities if the bargaining in question failed to reach an agreement.

surplus
: A player's surplus in a bargaining game is the excess of his payoff over his BATNA.

Nash cooperative solution
: This outcome splits the bargainers' surpluses in proportion to their bargaining powers.

efficient
: An outcome of a bargaining game is called efficient if there is no feasible alternative that would leave one bargainer with a higher payoff without reducing the payoff of the other.

efficient frontier
: This is the northeast boundary of the set of feasible payoffs of the players, such that in a bargaining game it is not possible to increase the payoff of one person without lowering that of another.

2 VARIABLE-THREAT BARGAINING

The outcome of bargaining depends on the players' backstops (BATNAs). The case of equal bargaining strengths suffices to illustrate this result: With $h = k = ½$, the outcomes for players A and B become $x = (v + a - b)/2$ and $y = (v + b - a)/2$. So far, we have taken the BATNAs to be fixed by considerations outside of this bargaining game. But what if one or both players can change the BATNAs? If such manipulation is possible, the bargaining game acquires a pregame stage not unlike the pregame we described in our discussion of strategic moves in Chapter 8. In the pregame to the bargaining game, players can manipulate BATNAs, within some limitations set by the rules of the pregame, to get themselves a better outcome in the upcoming game. This two-stage game (manipulation of BATNAs followed by bargaining) is called variable-threat bargaining. The difference between this situation and the ones described in Chapter 8 is that here, the upcoming game is played cooperatively while the pregame is played noncooperatively, whereas both stages are noncooperative in our strategic move analysis. The principle of rollback applies here just the same, however. When choosing their noncooperative actions to change BATNAs, the players look ahead to the effect of those actions on the outcome of the upcoming cooperative game.

We can see from the formula for outcomes in the case of equal bargaining strengths that Player A stands to gain by increasing $(a - b)$, while Player B stands to gain by decreasing the same difference. It should be intuitive that each player can gain by increasing his own BATNA and by decreasing the other's. But, perhaps surprisingly, it is also true that A can gain even if both a and b *decrease*, so long as $(a - b)$ *increases*; that is, if B's BATNA decreases *more* than A's. On reflection, this should make sense: If a breakdown of negotiations would hurt B more than it hurts A, that should lead to a bargaining outcome that is more favorable to A.

This type of result is nicely illustrated in a scene from the movie *Ransom*. Sean, the son of multimillionaire Tom Mullen (played by Mel Gibson), has been kidnapped. The men holding him are demanding a ransom of $2 million. Mullen goes on live TV and, in a sequence of shots, we see the people involved, including the lead kidnapper, Jimmy Shaker (played by Gary Sinise), watching from various locations. With the full $2 million spread out on a table before him, Mullen announces that he is instead offering it as a reward on the kidnapper’s head. But if the kidnapper returns the boy alive and unharmed, Mullen will withdraw the bounty.

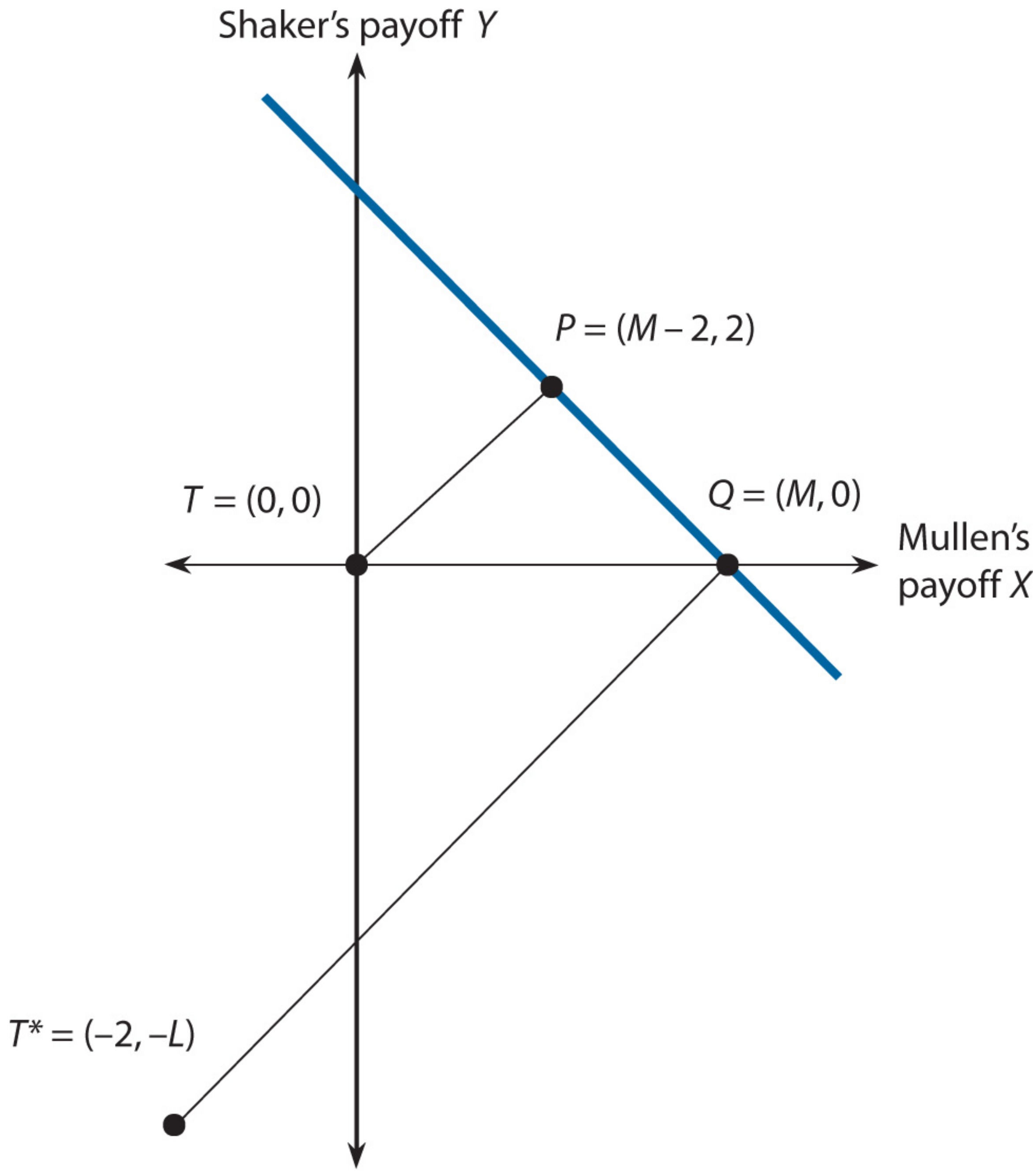

FIGURE 17.3 The Variable-Threat Game of Ransom

Let us represent this interaction in game-theoretic language. Figure 17.3 shows Mullen's payoff on the horizontal axis and Shaker's on the vertical axis. The origin is at the point of their initial wealths. Before the kidnapping, Mullen also has his son; we let M denote his son's value to him in money-equivalent terms (millions of dollars). So the payoff point in the status quo is $(M, 0)$, or the point Q in the figure. The line through Q with slope -1 shows all payoff points that can be achieved by transferring money from one player to another with Q representing the initial, no-transfers, point; this line is therefore the bargaining frontier.

In the original scenario, after the kidnapping but prior to any additional actions, if negotiation fails, then Mullen loses his son, but Shaker gets no ransom, so the BATNA is the origin, labeled T in Figure 17.3. Shaker has asked for a \$2 million ransom. If the ransom is paid and Mullen gets his son back, the payoffs (in millions of dollars) will be $M - 2$ to Mullen and 2 to Shaker. That outcome is shown as point P in the figure.

Mullen's strategy moves the BATNA point. Now, if the negotiation fails, Mullen will lose his son and end up paying the \$2 million bounty to the person who kills Shaker (probably one of Shaker's confederates), while Shaker will lose his life. Let Shaker's valuation of his own life be denoted by L (in millions of dollars). The payoffs at the new BATNA are -2 to Mullen and $-L$ to Shaker, and the new BATNA is point T^* in the figure.

With this new BATNA in place, Mullen offers Shaker a solution to the bargaining problem—namely, going back to the status quo point Q. That point is attained if Shaker brings Sean back unharmed and Mullen withdraws the bounty on Shaker's head. The figure is drawn in a way that suggests that this outcome conforms to the Nash cooperative solution with equal bargaining strengths. It is easy to verify that this solution implicitly sets $M = 4$ and $L = 6$. For other values of M and L, the Nash cooperative solution for Mullen's counterproposal need not be exactly at Q.

Under what circumstances will Mullen's strategy give him an outcome better than the P he would get by acceding to Shaker's original demand? The Nash cooperative solution for the threat point T^* would have to be located to the southeast of that for T along the bargaining frontier. This situation arises if the line T^*Q lies below the line TP (as is true in Figure 17.3). That configuration of the two lines occurs when $L > 2$, or when Shaker values his own life more than the ransom money.

Because T^* is to the southwest of T, Mullen's change in the threat point (from T to T^*) worsens both players' BATNAs. His strategy aims to encourage Shaker's acceptance of his alternative proposal, because that proposal carries the threat of an even bigger loss for Shaker than for Mullen if the negotiation fails. In other words, Mullen is implicitly saying to Shaker, "This will hurt you more than it will hurt me." We often hear such statements made in arguments and disputes; now we see the strategic role they play in negotiations.[7]

For scenarios in which Shaker values his life very highly, with $L > 6$ in our figure, the line from T^* would meet the bargaining frontier at a point southeast of the status quo point, Q. The Nash cooperative solution in that case would correspond to a negative payoff for Shaker, and he would end up with wealth below his original level. We could interpret such an outcome as equivalent to a situation in which Shaker is paying Mullen to take his son back! This kind of outcome may seem impossible in practice. If Mullen's threat, T^*, is that severe, the bargaining outcome may actually be at Q. However, it is not outside the realm of possibility that kidnappers would agree to pay to be rid of their hostage, at least in some other branches of fiction.[8]

The variable-threat strategy of stating, "This will hurt you more than it will hurt me" has been used in real-world bargaining situations, too. For example, smart labor unions threaten or launch strikes at times when they will deliver the biggest hit to firms' profits. The British coal miners' union did this consistently in the 1970s, although, in that case, the hit was not so much to the nationalized coal industry's profits as to the political popularity of the government affected by the disruption of power supplies. Conversely, Prime Minister Margaret Thatcher's strategy of provoking the union's leader, Arthur Scargill, into striking in the spring and summer of 1984, when the demand for coal was lower and the disruption caused by a strike correspondingly smaller, was instrumental in the collapse of the strike and led to a collapse of the union itself.

The same strategy was used in the Major League Baseball strike of 1980.[9] The strike started during the exhibition games of the preseason. The players returned to work (actually, to play) at the start of the regular season, but resumed the strike after Memorial Day. This curious discontinuous strike can be better understood when we examine the time-varying costs of the strike to the two sides. During the exhibition games, the players are not paid salaries, but the team owners earn substantial revenues from fans who combine a vacation in a warmer clime with following their favorite team's stars and prospects. Once the regular season starts, the players get salaries, but attendance at games, and therefore the owners' revenues, are low; those numbers grow substantially only after Memorial Day. Therefore the discontinuous strike was the players' correct strategy to maximize the owners' losses relative to their own.

Endnotes

- Movies have their own requirements of dramatic tension and denouement that override game-theoretic logic. To conform to those demands, *Ransom* does not have an efficient resolution on the bargaining frontier where one of the players accedes to the other’s demand, but rather twists that end in chases and gunfights. That spin on the story is not material to the basic bargaining game we want to illustrate. Return to reference 7
- In O. Henry’s short story “The Ransom of Red Chief,” two small-time crooks kidnap a banker’s 10-year-old son. He turns out to be a brat who makes their lives so impossible that they pay the father to take him back. The text is available at http://fiction.eserver.org/short/ransom_of_red_chief.xhtml (accessed February 17, 2016). Return to reference 8
- Lawrence M. DeBrock and Alvin E. Roth, “Strike Two: Labor-Management Negotiations in Major League Baseball,” *Bell Journal of Economics*, vol. 12, no. 2 (Autumn, 1981), pp. 413–25. Return to reference 9

Glossary

variable-threat bargaining
: A two-stage game where at the first stage you can take an action that will alter the BATNAs of both bargainers (within certain limits), and at the second stage bargaining results in the Nash solution on the basis of these BATNAs.

3 ALTERNATING-OFFERS MODEL I: TOTAL VALUE DECAYS

In this section, we move back to the more realistic noncooperative-game approach and think about the process of individual strategizing that may produce an equilibrium in a bargaining game. Our standard picture of this process is one of alternating offers. One player—say, A—makes an offer. The other—say, B—either accepts it or makes a counteroffer. If he does the latter, then A can either accept it or come back with another offer of his own. And so on. Thus, we have a sequential-move game, and we can look for its rollback equilibrium.

To find a rollback equilibrium, we must start at the end of the game and work backward. But where is the end? Why should the process of offers and counteroffers ever terminate? Perhaps more drastic is the question, Why would it ever start? Why would the two bargainers not stick to their original positions and refuse to budge? It is costly to both if they fail to agree at all, but the benefit of an agreement is likely to be smaller to the one who makes the first, or the larger, concession. The reason why anyone concedes must be that continuing to stand firm would cause an even greater loss of benefit. This loss takes one of two broad forms. The available surplus may decay (shrink) with each offer, a possibility that we consider in this section. The alternative possibility is that time has value and impatience is important, and so a delayed agreement is worth less; we examine this possibility in Section 5.

Consider the following story of bargaining over a shrinking surplus. A fan arrives at a professional football (or basketball) game without a ticket. He is willing to pay as much as $25 to watch each quarter of the game. He finds a scalper, who states a price for a ticket. If the fan is not willing to pay this price, he goes to a nearby bar to watch the first quarter on its big-screen TV. At the end of the quarter, he comes out, finds the scalper still there, and makes a counteroffer for the ticket. If the scalper does not agree, the fan goes back to the bar. He comes out again at the end of the second quarter, when the scalper makes him yet another offer. If that offer is not acceptable to the fan, he goes back into the bar, emerging at the end of the third quarter to make yet another counteroffer. The value of watching the rest of the game is declining as the quarters go by.[10]

Rollback analysis enables us to predict the outcome of this alternating-offers bargaining process. At the end of the third quarter, the fan knows that if he does not buy the ticket then, the scalper will be left with a small piece of paper of no value. So the fan will be able to make a very small offer that, for the scalper, will still be better than nothing. Thus, on his last offer, the fan can get the ticket almost for free. Backing up one period, we see that at the end of the second quarter, the scalper has the initiative in making an offer. But he must look ahead and recognize that he cannot hope to extract the whole of the remaining two quarters' value from the fan. If the scalper offers the ticket for more than $25—the value of the *third* quarter to the fan—the fan will turn down the offer because he knows that he can get the fourth quarter later for almost nothing, so the scalper can ask for $25 at most. Now consider the situation at the end of the first quarter. The fan knows that if he does not buy the ticket now, the scalper can expect to get only $25 later, and so $25 is all that the fan needs to offer now to secure the ticket. Finally, before the game even begins, the scalper can look ahead and ask for $50; this $50 includes the $25 value of the *first* quarter to the fan plus the $25 for which the fan can get the remaining three quarters' worth. Thus, the two will strike an immediate agreement, and the ticket will change hands for $50, but the price is determined by the full forward-looking rollback reasoning process.[11]

This story can be easily turned into a more general theory of negotiation between two bargainers, A and B. Suppose A makes the first offer to split the total surplus, which we call v (measured in some currency—say, dollars). If B refuses the offer, the total surplus available drops by x_1 to $(v - x_1)$. B offers a split of this amount. If A refuses B's offer, the total drops by a further amount x_2 to $(v - x_1 - x_2)$. A offers a split of this amount. This offer and counteroffer process continues until finally, say, after 10 rounds, $v - x_1 - x_2 - . . . - x_{10} = 0$, so the game ends. As usual with sequential-move games, we begin our analysis at the end.

If the game has gone to the point where only x_{10} is left, B can make a final offer whereby he gets to keep almost all of the surplus, leaving a measly cent or so to A. Left with the choice of that or absolutely nothing, A should accept the offer. To avoid the finicky complexity of keeping track of tiny cents, let us call this outcome "x_{10} to B, 0 to A." We will do the same in the other (earlier) rounds.

Knowing what is going to happen in round 10, we turn to round 9. Here, A will make the offer, and $(x_9 + x_{10})$ is left. A knows that he must offer at least x_{10} to B, or else B will refuse the offer and take the game to round 10, where he can get that much. Bargainer A does not want to offer any more to B. So, on round 9, A will offer a split where he keeps x_9 and leaves x_{10} to B.

Then, on the round before that, when $(x_8 + x_9 + x_{10})$ is left, B will offer a split where he gives x_9 to A and keeps $(x_8 + x_{10})$. Working backward to the very first round, A will offer a split where he keeps (x_1 + x_3 + x_5 + x_7 + x_9) and gives (x_2 + x_4 + x_6 + x_8 + x_{10}) to B. This offer will be accepted.

You can remember these formulas by means of a simple trick: *Hypothesize* a sequence in which all offers are refused. (This sequence is *not* what actually happens.) Then add up the amounts of the surplus that would be destroyed by the refusals of one player. This total is what the other player gets in the actual equilibrium. For example, if B refused A’s first offer, the total available surplus would drop by x_1, so x_1 becomes part of what goes to A in the equilibrium of the game.

If each player has a positive BATNA, the analysis must be modified somewhat to take that into account. In the last round, B must offer A at least the BATNA a. If x_{10} is greater than a, B is left with ($x_{10} - a$); if not, the game must terminate before this round is reached. Now, at round 9, A must offer B the larger of two amounts: the ($x_{10} - a$) that B can get in round 10, or the BATNA b that B can get outside the agreement. The analysis can proceed all the way back to round 1 in this way; we leave it to you to complete the rollback reasoning for this case.

We have found the rollback equilibrium of the alternating-offers bargaining game, and in the process of deriving the outcome, we have also described the full strategies (complete contingent plans of action) behind the equilibrium—namely, what each player *would* do if the game reached some later stage. In fact, actual agreement is immediate on the very first offer. The later stages are not reached; they are off-equilibrium nodes and paths of play. But, as is usual with rollback reasoning, the foresight about what rational players would do at those nodes if they were reached is what informs the initial action.

The other important point to note is that *gradual decay* (with several potential rounds of offers) leads to a more even or fairer split of the surplus than does *sudden decay* (with only one round of bargaining permitted). In the latter case, no agreement would result if B turned down A’s very first offer; then, in a rollback equilibrium, A would get to keep (almost) the whole surplus, giving B an “ultimatum” to accept a measly cent or else get nothing at all. The subsequent rounds give B the credible ability to refuse a very uneven first offer.

Endnotes

- Just to keep the argument simple, we imagine this process as one-on-one bargaining. Actually, there may be several fans and several scalpers, turning the situation into a *market*. You can access our supplemental chapter on interactions in markets at digital.wwnorton.com/gamesofstrategy5. Return to reference 10
- To keep the analysis simple, we have omitted the possibility that the game might get exciting, and so the value of the ticket might actually increase as the quarters go by. This uncertainty would make the problem much more complex, but also more interesting. The ability of game theory to deal with such possibilities should inspire you to go beyond this book or course to study more advanced game theory. Return to reference 11

Glossary

alternating offers
: A sequential move procedure of bargaining in which, if the offer made by one player is refused by the other, then the refuser gets the next turn to make an offer, and so on.

decay
: Shrinkage over time of the total surplus available to be split between the bargainers, if they fail to reach an agreement for some length of time during the process of their bargaining.

impatience
: Preference for receiving payoffs earlier rather than later. Quantitatively measured by the discount factor.

4 ALTERNATING-OFFERS MODEL II: IMPATIENCE

In this section, we consider a different kind of cost of delay in reaching an agreement. Suppose the actual monetary value of the total surplus available for splitting does not decay, but players prefer to receive money earlier rather than later, and will accept somewhat less now rather than having to wait until later to get a bit more. For concreteness, we will say that both bargainers believe that having only 95 cents right now is as good as having $1 one round later. They make alternate offers as in Section 3.

A player who prefers having something right away to having the same thing later is *impatient*; he attaches less importance to the future than to the present. We came across this idea in Chapter 10, Section 2, and saw two reasons for it. First, Player A may be able to invest his money—say, $1—now and get his principal back along with interest and capital gains at a rate of return r, for a total gain of $(1 + r)$ in the next period (tomorrow, next week, next year, or whatever is the length of the period). Second, there may be some risk that the game will end between now and the next offer (as in the sudden end after 3 to 5 minutes in the classroom game described earlier). If p is the probability that the game continues, then the chance of getting a dollar next period has an expected value of only p now.

Suppose we consider a bargaining process between two players with zero BATNAs. Let us start the process with one of the two bargainers—say, A—making an offer to split $1. If the other player, B, rejects A's offer, then B will have an opportunity to make his own offer one round later. The two bargainers are in identical situations when each makes his offer, because the amount to be split is always $1. Thus, in equilibrium, the amount that goes to the player currently in charge of making the offer (call it x) is the same, regardless of whether that person is A or B. We can use rollback reasoning to find an equation that can be solved for x.

Suppose A starts the alternating-offers process. He knows that B can get x in the next round when it is B's turn to make the offer. Therefore, A must give B at least an amount that is equivalent, in B's eyes, to getting x in the next round; A must give B at least $0.95x$ now. (Remember that, for B, 95 cents received now is equivalent to $1 received in the next round; so $0.95x$ now is as good as x in the next round.) Player A will not give B any more than is required to induce B's acceptance. Thus, A offers B exactly $0.95x$ and is left with $(1 - 0.95x)$. But the amount that A gets when making the offer is just what we called x. Therefore, $x = 1 - 0.95x$, or $(1 + 0.95)x = 1$, or $x = 1/1.95 = 0.512$.

Two things about this calculation should be noted. First, even though the process allows for an unlimited sequence of alternating offers and counteroffers, in equilibrium, the very first offer A makes gets accepted, and the bargaining ends. Because time has value, this outcome is efficient. The cost of delay governs how much A must offer B to induce acceptance; it thus affects A's rollback reasoning. Second, the player who makes the first offer gets more than half of the surplus—namely, 0.512 rather than 0.488. Thus, each player gets more when he makes the first offer than when the other player makes the first offer. But this advantage is far smaller than that in an ultimatum game with no future rounds of counteroffers.

Now suppose the two players are not equally patient (or impatient, as the case may be). Player B still regards $1 in the next round as equivalent to 95 cents now, but A regards it as equivalent to only 90 cents now. Thus, A is willing to accept a smaller amount of money to get his money sooner; in other words, A is more impatient. This inequality in rates of impatience can translate into unequal payoffs from the bargaining process in equilibrium. To find the equilibrium for this example, we write x for the amount that A gets when he starts the process and y for what B gets when he starts the process.

Player A knows that he must give B at least $0.95y$; otherwise B will reject the offer in favor of the y that he knows he can get when it becomes his turn to make the offer. Thus, the amount that A gets, x, must be $1 - 0.95y$; $x = 1 - 0.95y$. Similarly, when B starts the process, he knows that he must offer A at least $0.90x$, and then $y = 1 - 0.90x$. These two equations can be solved for x and y:

$x = 1 - 0.95(1 - 0.9x)$		$y = 1 - 0.9(1 - 0.95y)$
$[1 - 0.95(0.9)]x = 1 - 0.95$	and	$[1 - 0.9(0.95)]y = 1 - 0.9$
$0.145x = 0.05$		$0.145y = 0.10$
$x = 0.345$		$y = 0.690$

Note that x and y do not add up to 1, because each of these amounts is the payoff to a given player when he makes the first offer. Thus, when A makes the first offer, A gets 0.345 and B gets 0.655; when B makes the first offer, B gets 0.69 and A gets 0.31. Once again, each player does better when he makes the first offer than when the other player makes the first offer, and once again the advantage from being able to make the first offer is small.

The outcome of this case with unequal rates of impatience differs from that of the preceding case, with equal rates of impatience, in a major way. With unequal rates of impatience, the more impatient player, A, gets a lot less than B even when he is able to make the first offer. We expect that the person who is willing to accept less to get it sooner ends up getting less, but the difference is very dramatic. The proportion of A's share to B's share is almost 1:2.

As usual, we can now build on these examples to develop the more general algebra. Suppose A regards \$1 immediately as being equivalent to \$$(1 + r)$ one round later or, equivalently, A regards \$$1/(1 + r)$ immediately as being equivalent to \$1 one round later. For brevity, we substitute a for $1/(1 + r)$ in the calculations that follow. Likewise, suppose Player B regards \$1 today as being equivalent to \$$(1 + s)$ one round later; we use b for $1/(1 + s)$. If r is high (or equivalently, if a is low), then Player A is very impatient. Similarly, B is impatient if s is high (or if b is low).

Here we look at bargaining that takes place in alternating rounds, with a total of \$1 to be divided between two players, both of whom have zero BATNAs. (You can solve the even more general case easily once you understand this one.) What is the rollback equilibrium?

We can find the payoffs in equilibrium by extending the simple argument used earlier. Suppose A's payoff in the rollback equilibrium is x when he makes the first offer; B's payoff in the rollback equilibrium is y when he makes the first offer. We look for a pair of equations linking the values x and y and then solve these equations to determine the equilibrium payoffs.[12]

When A is making the offer, he knows that he must give B an amount that B regards as being equivalent to y one period later. This amount is $by = y/(1 + s)$. Then, after making the offer to B, A can keep only what is left: $x = 1 - by$.

Similarly, when B is making the offer, he must give A the equivalent of x one period later—namely, ax. Therefore, $y = 1 - ax$. Solving these two equations is now a simple matter.

We have $x = 1 - b(1 - ax)$, or $(1 - ab)x = 1 - b$. Expressed in terms of r and s, this equation becomes

$$x = \frac{1-b}{1-ab} = \frac{s+rs}{r+s+rs}.$$

Similarly, $y = 1 - a(1 - by)$, or $(1 - ab)y = 1 - a$. This equation becomes

$$y = \frac{1-a}{1-ab} = \frac{r+rs}{r+s+rs}.$$

Although this quick solution might seem like sleight of hand, it follows the same steps used earlier, and we will soon give a different method of reasoning yielding exactly the same answer. First, let us examine some features of this answer.

First, note that, as in our simple unequal-impatience example, the two magnitudes x and y add up to more than 1:

$$x+y = \frac{r+s+2rs}{r+s+rs} > 1.$$

Remember that x is what A gets when he has the right to make the first offer, and y is what B gets when he has the right to make the first offer. When A makes the first offer, B gets $(1 - x)$, which is less than y; this just shows A’s advantage from being the first proposer. Similarly, when B makes the first offer, B gets y and A gets $(1 - y)$, which is less than x.

However, usually r and s are small numbers. When offers can be made at short intervals, such as a week or a day or an hour, the interest that your money can earn from one offer to the next, or the probability that the game ends precisely within the next interval, is quite small. For example, if r is 1% (0.01) and s is 2% (0.02), then the formulas yield x = 0.668 and y = 0.337; so the advantage of making the first offer is only 0.005. (A gets 0.668 when making the first offer, but 1 − 0.337 = 0.663 when B makes the first offer; the difference is only 0.005.) More formally, when r and s are both small compared with 1, their product rs is very small indeed; thus, we can ignore rs to write an approximate solution for the split that does not depend on which player makes the first offer:

$$x = \frac{s}{r+s} \quad \text{and} \quad y = \frac{r}{r+s}.$$

Now $x + y$ is approximately equal to 1.

Most importantly, x and y in the approximate solution are the shares of the surplus that go to the two players, and $y/x = r/s$; that is, the shares of the players are inversely proportional to their rates of impatience as measured by r and s. If B is twice as impatient as A, then B gets half as much as A; so the shares are ⅓ and ⅔, or 0.333 and 0.667, respectively. Thus, we see that patience is an important advantage in bargaining. Our formal analysis supports the intuition that if you are very impatient, the other player can offer you a quick but poor deal, knowing that you will accept it.

This effect of impatience hurts the United States in numerous negotiations that its government agencies and diplomats conduct with other countries. The American political process puts a great premium on speed. The media, interest groups, and rival politicians all demand results and are quick to criticize the administration or the diplomats for any delay.

Under this pressure to deliver, the negotiators are always tempted to come back with results of any kind. Such results are often poor from the long-term U.S. perspective; the other countries' concessions often have loopholes, and their promises are less than credible. The U.S. administration hails the deals as great victories, but they usually unravel after a few years. The financial crisis of 2008 offers another, even more dramatic, example. When the housing boom collapsed, some major financial institutions that held mortgage-backed assets faced bankruptcy. That led them to curtail credit, which in turn threatened to throw the U.S. economy into a severe recession. The crisis exploded in September, in the midst of a presidential election campaign. The Treasury, the Federal Reserve, and political leaders in Congress all wanted to act fast. This impatience led them to offer generous terms of rescue to many financial institutions, when a slower process would have yielded an outcome that cost the taxpayers much less and offered them much better prospects of sharing in future gains on the assets being rescued.

Similarly, individuals who have suffered losses are in a much weaker position when they negotiate with insurance companies on coverage. The companies often make lowball offers of settlement to people who have suffered a major loss, knowing that their clients urgently want to make a fresh start and are therefore very impatient.

As a conceptual matter, the formula $y/x = r/s$ ties our noncooperative-game approach to bargaining to Nash's cooperative solution discussed in Section 1. The formula for shares of the available surplus that we derived there becomes, with zero BATNAs, $y/x = k/h$. In the cooperative approach, the shares of the two players stood in the same proportions as their bargaining strengths, but those strengths were assumed to be imposed somehow from the outside. Now we have an explanation for the bargaining strengths in terms of some more basic characteristics of the players: h and k are inversely proportional to the players' rates of impatience r and s. In other words, Nash's cooperative solution can also be given an alternative, and perhaps more satisfactory, interpretation as the rollback equilibrium of a noncooperative game of offers and counteroffers, if we interpret the abstract bargaining-strength parameters in the cooperative solution correctly in terms of the players' characteristics, such as impatience.

Finally, note that agreement is once again immediate—the very first offer is accepted. As usual, the rollback analysis imposes discipline by making the first proposer recognize that the other would credibly reject a less adequate offer.

To conclude this section, we offer an alternative derivation of the same (precise) formula for the equilibrium offers that we derived earlier. Suppose this time that there are 100 rounds of bargaining; A is the first proposer and B the last. Start the backward induction in the 100th round; B will keep the whole dollar. Therefore, in the 99th round, A will have to offer B the equivalent of $1 in the 100th round—namely, b, and A will keep $(1 - b)$. Then proceed backward:

In round 98, B offers $a(1 - b)$ to A and keeps

$$1 - a(1 - b) = 1 - a + ab.$$

In round 97, A offers $b(1 - a + ab)$ to B and keeps

$$1 - b(1 - a + ab) = 1 - b + ab - ab^2.$$

In round 96, B offers $a(1 - b + ab - ab^2)$ to A and keeps

$$1 - a + ab - a^2b + a^2b^2.$$

In round 95, A offers $b(1 - a + ab - a^2b + a^2b^2)$ to B and keeps

$$1 - b + ab - ab^2 + a^2b^2 - a^2b^3.$$

Proceeding in this way and following the established pattern, we see that in round 1, A gets to keep

$$1 - b + ab - ab^2 + a^2b^2 - a^2b^3 + . . . + a^{49}b^{49} - a^{49}b^{50}$$

$$= (1 - b)[1 + ab + (ab)^2 + . . . + (ab)^{49}].$$

The consequence of allowing more and more rounds is now clear: We just get more and more of these terms, growing geometrically by the factor *ab* for every two offers. To find A's payoff when he is the first proposer in an infinitely long sequence of offers and counteroffers, we have to find the limit of the infinite geometric sum. In the appendix to Chapter 10, we saw how to sum such series. Using the formula obtained there, we get the answer

$$(1 - b)[1 + ab + (ab) + (ab)^2 + \cdots + (ab)^{49} + \cdots] = \frac{1 - b}{1 - ab}.$$

This is exactly the solution for *x* that we obtained before. By a similar argument, you can find B's payoff when he is the proposer and, in doing so, improve your understanding and technical skills at the same time.

Endnotes

- We are taking a shortcut; we have simply assumed that such an equilibrium exists and that the payoffs are uniquely determined. More rigorous theory proves these conditions. For a step in this direction, see John Sutton, “Non-Cooperative Bargaining: An Introduction,” *Review of Economic Studies*, vol. 53, no. 5 (October 1986), pp. 709–24. The fully rigorous (and quite difficult) theory is given in Ariel Rubinstein, “Perfect Equilibrium in a Bargaining Model,” *Econometrica*, vol. 50, no. 1 (January 1982), pp. 97–109. Return to reference 12

5 EXPERIMENTAL EVIDENCE

The theory of noncooperative bargaining is fairly simple, and many people have staged laboratory or classroom experiments that create such conditions to observe what the experimental subjects actually do. We mentioned some of these experiments briefly in Chapter 3, when considering the validity of rollback reasoning. Here we examine them in more detail in the specific context of bargaining.[13]

The simplest bargaining experiment is the ultimatum game that we introduced in Chapter 3, Section 6. In that game, there is only one round: Player A makes an offer, and if B does not accept it, the bargaining ends and both get nothing. The general structure of these experiments is as follows: A pool of players is brought together, either in the same room or via an online network. The players are paired; one person in each pair is designated the *proposer* (A, the one who makes an offer) and the other is designated the *chooser* (B, the one who accepts or refuses the offer). The pair is given a fixed surplus, usually $1 or some other sum of money, to split.

Rollback reasoning suggests that A should offer B the minimal unit—say, 1 cent out of a dollar—and that B should accept such an offer. Actual results are dramatically different. In the case in which the subjects are together in a room and the assignment of the role of proposer is made randomly, the most common offer is a 50:50 split. Very few offers worse than 75:25 are made (with the proposer to keep 75% and the chooser to get 25%), and if made, they are often rejected.

This finding can be interpreted in two ways: Either the players cannot or do not perform the calculation required for rollback, or the payoffs of the players include something other than the money they get out of this round of bargaining. Surely the calculation in the ultimatum game is simple enough that anyone should be able to do it, and the subjects in most of these experiments are college students. A more likely explanation is the one that we put forth in Chapter 3, Section 6, and Chapter 5, Section 4—that the theory, which assumed payoffs to consist only of the sum earned in the one round of bargaining, is too simplistic.

Participants can have payoffs that include things other than money. First, they may have self-esteem or pride that prevents them from accepting a very unequal split. Even if A does not include this consideration in his own payoff, if he thinks that B might, then it is a good strategy for A to offer enough to make it likely that B will accept. A balances his higher payoff with a smaller offer to B against the risk of getting nothing if B rejects an offer deemed too unequal.

A second possibility is that when the participants in the experiment are gathered in a room, the anonymity of pairing cannot be guaranteed. If the participants come from a group of people, such as college classmates or villagers, who have ongoing relationships outside this game, they may value those relationships. Then the proposers fear that if they offer too unequal a split in this game, those relationships may suffer. Therefore, they will be more generous in their offers than the simplistic theory suggests. If this is the explanation for the results, then ensuring greater anonymity should enable the proposers to make more unequal offers, and experiments do find this to be the case.

Finally, people may have a sense of fairness drilled into them during their nurture and education. This sense of fairness may have evolutionary value for society as a whole and may therefore have become a social norm. Whatever its origin, it may lead the proposers to be relatively generous in their offers, quite irrespective of the fear of their rejection. One of us (Skeath) has conducted classroom experiments with several different ultimatum games. Students who had bargaining partners previously known to them were noticeably “fairer” in their splits. In addition, several students cited specific cultural backgrounds as explanations for behavior that was inconsistent with theoretical predictions.

Experimenters have tried variants of the basic ultimatum game to differentiate between these explanations. The question of ongoing relationships can be addressed by stricter procedures that visibly guarantee anonymity. Doing so by itself has some effect on the outcomes, but still does not produce offers as extreme as those predicted by the purely selfish rollback argument of the theory. The other two explanations—namely, fear of the offer’s rejection and an ingrained sense of fairness—remain to be sorted out.

The fear of rejection can be removed by considering a variant called the *dictator game.* Again, the participants are matched in pairs. One person (say, A) is designated to determine the split, and the other (say, B) is simply a passive recipient of what A decides. Now the split becomes decidedly more uneven, but even here a majority of As choose to keep no more than 70%. This result suggests a role for an ingrained sense of fairness.

But such a sense has its limits. In some experiments, a sense of fairness was created when the experimenter randomly assigned the roles of proposer and chooser. In one variant, the participants were given a simple quiz, and those who performed best were made proposers. This procedure created a sense that the role of proposer had been earned, and the outcomes showed more unequal splits. When the dictator game was played with earned dictator roles and with stricter anonymity conditions, most As kept everything, but some (about 5%) still offered a 50:50 split.

One of us (Dixit) carried out a classroom experiment in which students in groups of 20 were gathered together. They all logged into the game system and were matched randomly and anonymously in pairs, and each pair tried to agree on how to split 100 points. Roles of proposer and chooser were not assigned; either could make an offer or accept the other’s offer. Offers could be made and changed at any time. The pairs could exchange messages instantly on their screens. The bargaining round ended at a random time after 3 to 5 minutes; if an agreement was not reached in that time by a pair, both got zero. There were 10 such rounds, with different random pairs of opponents each time. Thus, the game itself offered no scope for cooperation through repetition. In a classroom context, the students had ongoing relationships outside the game, but they did not generally know or guess with whom they were playing in any round, even though no great attempt was made to enforce anonymity. Each student’s score for the whole game was the sum of his point score for the 10 rounds. The stakes were quite high, because the score accounted for 5% of the course grade!

The highest total of points achieved was 515. Those who quickly agreed on 50:50 splits did the best, and those who tried to hold out for very uneven scores or who refused to split a difference of 10 points or so between the offers and ran the risk of time running out on them did poorly.[14] It seems that moderation and fairness do get rewarded, even as measured in terms of one’s own payoff.

Endnotes

- For more details, see Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton, N.J.: Princeton University Press, 1993), pp. 263–69, and *The Handbook of Experimental Economics*, ed. John H. Kagel and Alvin E. Roth (Princeton, N.J.: Princeton University Press, 1995), pp. 255–74. Return to reference 13
- Those who were best at the mathematical aspects of game theory, such as problem sets, did a little worse than the average, probably because they tried too hard to eke out an extra advantage and met resistance. And women did slightly better than men. Return to reference 14

Glossary

ultimatum game
: A form of bargaining where one player makes an offer of a particular split of the total available surplus, and the other has only the all-or-nothing choice of accepting the offer or letting the game end without agreement, when both get zero surplus.

6 MANIPULATING INFORMATION IN BARGAINING

We have seen that the outcomes of bargaining depend crucially on various characteristics of the parties to the bargain, most importantly their BATNAs and their impatience. We have proceeded thus far by assuming that the players know each other's characteristics as well as their own. In fact, we have assumed that each player knows that the other knows, and so on; that is, that their characteristics are common knowledge. In reality, we often engage in bargaining without knowing the other side's BATNA or degree of impatience; sometimes we do not even know our own BATNA very precisely.

As we saw in Chapter 9, a game with such uncertainty or asymmetry of information has associated with it an important game of signaling and screening. Bargaining is replete with opportunities for manipulating information using such strategies. A player with a good BATNA or a high degree of patience wants to signal this fact to the other player. However, because someone without these good attributes will want to imitate them, the other player will be skeptical and will examine the signals critically for their credibility. And each player will also try screening, using strategies that induce the other to take actions that will reveal his characteristics truthfully.

In this section, we look at some information-manipulation strategies used by buyers and sellers in the housing market. Most Americans are active in the housing market several times in their lives, and professional real-estate agents or brokers have even more extensive experience in that market. Moreover, housing is one of the few markets in the United States where haggling or bargaining over price is accepted and even expected. Therefore, considerable experience with strategies is available. We draw on this experience for many of our examples and interpret it in the light of our game-theoretic ideas and insights.[15]

When you contemplate buying a house in a new neighborhood, you are unlikely to know the range of prices for the particular type of house in which you are interested. Your first step should be to find out what this range is so that you can then determine your BATNA. And that does not mean looking at newspaper ads or realtors' listings, which indicate only asking prices. Local newspapers and some Internet sites list recent actual transactions and the actual prices; you should check them against the asking prices of the same houses to get an idea of the state of the market and the range of bargaining that might be possible.

Next comes screening: finding out the other side's BATNA and level of impatience. If you are a buyer, you can find out why the house is being sold and how long it has been on the market. If it is empty, why? And how long has it been that way? If the owners are getting divorced, or have moved elsewhere and are financing another house with an expensive bridge loan, it is likely that they have a low BATNA or are rather impatient.

You should also find out other relevant things about the other side's preferences, even though those preferences may seem irrational to you. For example, some people consider an offer too far below the asking price an insult and will not sell at any price to someone who makes such an offer. Norms of this kind vary across regions and times. It pays to find out what the common practices are.

Most importantly, the *acceptance* of an offer more accurately reveals a player's true willingness to pay than anything else and therefore is open to exploitation by the other player. A brilliant game-theorist friend of ours tried just such a ploy. He was bargaining for a floor lamp. Starting with the seller’ s asking price of $100, the negotiation proceeded to a point where our friend made an offer to buy the lamp for $60. The seller said yes, at which point our friend thought, “This guy is willing to sell it for $60, so his true rock-bottom price must be even lower. Let me try to find out whether it is.” So our friend said, “How about $55?” The seller got very upset, refused to sell for any price, and asked our friend to leave the store and never come back.

The seller’ s behavior conformed to the norm that considers it the utmost in bad faith to renege on an offer once it is accepted. This norm makes good sense in the context of all bargaining games that take place in society. If an offer on the table cannot be accepted in good faith by the other player without fear of the kind of exploitation attempted by our friend, then each bargainer will wait to get the other to accept an offer, thereby revealing his true rock-bottom acceptance level, and the whole process of bargaining will grind to a halt. Therefore, such behavior has to be disallowed. Establishing a social norm to which people adhere instinctively, as the seller in the example did, is a good way for society to achieve this aim.

An offer may explicitly state that it is open for only a specified and limited time; this stipulation can be part of the offer itself. Job offers usually specify a deadline for acceptance; stores have sales for limited periods. But in that case, the offer is truly a *package* of price and time, and reneging on either dimension violates the norms of bargaining. For example, customers get quite angry if they arrive at a store in the sale period and find an advertised item unavailable. The store must offer a rain check, which allows the customer to buy the item at its sale price when it is next available at the regular price, but even this offer causes some inconvenience to the customer and risks some loss of goodwill. The store can specify "limited quantities, no rain checks" very clearly in its advertising of the sale; even then, many customers instinctively get upset if they find that the store has run out of the item.

Next on our list of strategies to use in one-on-one bargaining, as in the housing market, comes signaling your own high BATNA or patience. The best way to signal patience is to *be* patient. Do not come back with counteroffers too quickly, "let the sellers think they've lost you." This signal is credible because someone not genuinely patient would find it too costly to mimic the leisurely approach. Similarly, you can signal a high BATNA by starting to walk away, a tactic that is common in negotiations at bazaars in other countries and some flea markets and tag sales in the United States.

Even if your BATNA is low, you may commit yourself to not accepting an offer below a certain level. This constraint acts just like a high BATNA, because the other party cannot hope to get you to accept anything less. In the housing context, you can claim your inability to concede any further by inventing (or creating) a tightwad parent who is providing the down payment or a spouse who does not really like the house and will not let you offer any more. Sellers can try similar tactics. A parallel in wage negotiations is the *mandate.* A meeting is convened at which the workers pass a resolution—the mandate—authorizing the union leaders to represent them in the negotiations, but with the constraint that the negotiators must not accept an offer below a certain level specified in the resolution. Then, at the meeting with the management, the union leaders can say that their hands are tied; there is no time to go back to the membership to get their approval for any lower offer.

Most of these strategies entail some risk. While you are signaling patience by waiting, the seller of the house you want may find another willing buyer. As employer and union wait for one another to concede, tensions may mount so high that a strike that is costly to both sides cannot be prevented. In other words, many strategies of information manipulation are instances of brinkmanship. We saw in Chapter 13 how such games can have an outcome that is bad for both parties. The same is true in bargaining. *Threats* of breakdown of negotiations or of strikes are strategic moves intended to achieve quicker agreement or a better deal for the player making the move; however, an *actual* breakdown or strike is an instance of a threat gone wrong. The player making the threat—initiating the brinkmanship—must assess the risk and the potential rewards when deciding whether and how far to proceed down this path.

Endnotes

- We have taken the insights of practitioners from Andrée Brooks, “Honing Haggling Skills,” *New York Times*, December 5, 1993. Return to reference 15

7 BARGAINING WITH MANY PARTIES AND ISSUES

Our discussion thus far has been confined to the classic situation where two parties are bargaining about the split of a given total surplus. But many real-life negotiations include several parties or several issues simultaneously. Although the games get more complicated in these cases, often the enlargement of the group or the set of issues actually makes it easier to arrive at a mutually satisfactory agreement. In this section, we take a brief look at such matters.[16]

A. Multi-Issue Bargaining

In a sense, we have already considered multi-issue bargaining. The negotiation over price between a seller and a buyer always involves *two* things: (1) the object offered for sale or considered for purchase and (2) money. The potential for mutual benefit arises when the buyer values the object more than the seller does—that is, when the buyer is willing to give up more money in return for getting the object than the seller is willing to accept in return for giving up the object. Both players can be better off as a result of their bargaining agreement.

The same principle applies more generally. International trade is the classic example. Consider two hypothetical countries, Freedonia and Ilyria. If Freedonia can convert 1 loaf of bread into 2 bottles of wine (by using less of its resources, such as labor and land, in the production of bread and using those resources to produce more wine instead), and if Ilyria can convert 1 bottle of wine into 1 loaf of bread (by switching its resources in the opposite direction), then between them, they can create more goods "out of nothing." For example, suppose that Freedonia can produce 200 more bottles of wine if it produces 100 fewer loaves of bread, and that Ilyria can produce 150 more loaves of bread if it produces 150 fewer bottles of wine. These switches in resource use create an extra 50 loaves of bread and an extra 50 bottles of wine relative to what the two countries produced originally. This extra bread and wine is the surplus that they can create if they can agree on how to divide it between them. For example, suppose Freedonia gives 175 bottles of wine to Ilyria and gets 125 loaves of bread. Then each country will have 25 more loaves of bread and 25 more bottles of wine than it did before. But there is a whole range of possible exchanges corresponding to different divisions of the gain. At one extreme, Freedonia could give up all the 200 extra bottles of wine that it has produced in exchange for 101 loaves of bread from Ilyria, in which case Ilyria would get almost all the gain from trade. At the other extreme, Freedonia could give up only 151 bottles of wine in exchange for 150 loaves of bread from Ilyria, so that Freedonia would get almost all the gain from trade.[17] Between these limits lies the frontier where the two can bargain over the division of the gains from trade.

The general principle should now be clear: When two or more issues are on the bargaining table at the same time and the two parties are willing to trade more of one thing against less of the other at different rates, then a mutually beneficial deal exists. The mutual benefit can be realized by trading at a rate somewhere between the two parties' different rates of willingness to trade. The division of gains depends on the choice of the rate of trade. The closer it is to one side's willingness ratio, the less that side gains from the deal.

Now you can also see how the possibilities for mutually beneficial deals can be expanded by bringing more issues to the table at the same time. With more issues, you are more likely to find divergences in the ratios of valuation between the two parties and are thereby more likely to locate possibilities for mutual gain. In regard to a house, for example, many of the fittings or furnishings may be of little use to the seller in the new house to which he is moving, but they may be of sufficiently good fit and taste that the buyer values having them. Then, if the seller cannot be induced to lower the price of the house, he may be amenable to including these items in the original price to close the deal.

However, the expansion of issues is not an unmixed blessing. If you value something greatly, you may fear putting it on the bargaining table; you may worry that the other side will extract big concessions from you, knowing that you want to protect that one item of great value. At the worst, a new issue on the table may make it possible for one side to deploy threats that lower the other side’s BATNA. For example, a country engaged in diplomatic negotiations may be vulnerable to an economic embargo; then it would much prefer to keep the political and economic issues distinct.

B. Multiparty Bargaining

Having many parties simultaneously engaged in bargaining may also facilitate agreement, because instead of having to look for pairwise deals, the parties can seek a circle of concessions. International trade is again the prime example. Suppose the United States can produce wheat very efficiently, but is less productive in cars; Japan is very good at producing cars, but has no oil; and Saudi Arabia has a lot of oil, but cannot grow wheat. In pairs, they can achieve little, but the three together have the potential for a mutually beneficial deal.

As with multiple issues, expanding the bargaining to multiple parties is not simple. In our example, the deal would be that the United States would send an agreed amount of wheat to Saudi Arabia, which would send an agreed amount of oil to Japan, which would in turn ship an agreed number of cars to the United States. But suppose that Japan reneges on its part of the deal. Saudi Arabia cannot retaliate against the United States because, in this scenario, it is not offering anything to the United States that it can potentially withhold. Saudi Arabia can only break its deal to send oil to Japan. Thus, enforcement of multilateral agreements may be problematic. The General Agreement on Tariffs and Trade (GATT) between 1946 and 1994, as well as the World Trade Organization (WTO) since then, have indeed found it difficult to enforce their members' agreements and to levy punishments on countries that violate the rules.

Endnotes

- Mutually-beneficial agreements are possible whenever there are “gains from trade,” a topic typically covered in microeconomics courses; see, for example, Robert S. Pindyck and Daniel L. Rubenfeld, *Microeconomics*, 9th ed. (Upper Saddle River, NJ: Pearson, 2017), Chapter 16. For a more thorough treatment, see Howard Raiffa, *The Art and Science of Negotiation* (Cambridge, MA: Harvard University Press, 1982), Parts III and IV. Return to reference 16
- Economics uses the concept *ratio of exchange*, or price, which here is expressed as the number of bottles of wine that trade for each loaf of bread. The crucial point is that the possibility of gain for both countries exists with any ratio that lies between the 2:1 rate at which Freedonia can convert bread into wine and the 1:1 rate at which Ilyria can do so. At a ratio of exchange close to 2:1, Freedonia gives up almost all of its 200 extra bottles of wine and gets little more than the 100 loaves of bread that it sacrificed to produce the extra wine; thus Ilyria has almost all of the gain. Conversely, at a ratio close to 1:1, Freedonia has almost all of the gain. The issue in the bargaining is the division of gain, and therefore the ratio or the price at which the two should trade. Return to reference 17

SUMMARY

Bargaining negotiations attempt to divide the *surplus* (excess value) that is available to the parties if an agreement can be reached. Bargaining can be analyzed as a *cooperative* game in which parties find and implement a solution jointly or as a (structured) *noncooperative* game in which parties choose strategies separately and attempt to reach an equilibrium.

Nash's cooperative solution is based on three principles: the outcome's invariance with linear changes in the payoff scale, its *efficiency*, and its invariance with removal of irrelevant outcomes. The solution is a rule that states the proportions of division of surplus, beyond the backstop payoff levels (also called *BATNAs* or *best alternatives to a negotiated agreement*) available to each party, based on the parties' relative bargaining strengths. Strategic manipulation of the backstops can be used to increase a party's payoff.

In a noncooperative setting of *alternating offers* and counteroffers, rollback reasoning is used to find an equilibrium; this reasoning generally includes a first-round offer that is immediately accepted. If the surplus value *decays* with refusals, the sum of the (hypothetical) amounts destroyed owing to the refusals of a single player is the payoff to the other player in equilibrium. If delay in agreement is costly owing to *impatience*, the equilibrium agreement shares the surplus roughly in inverse proportion to the parties' rates of impatience. Experimental evidence indicates that players often offer more than is necessary to reach an agreement in such games; this behavior is thought to be related to lack of player anonymity as well as beliefs about fairness.

The presence of information asymmetries in bargaining games makes signaling and screening important. Some parties will wish to signal their high BATNAs or extreme patience; others will want to screen to obtain truthful revelation of such characteristics. When more issues are on the table or more parties are participating, agreements may be easier to reach, but bargaining may be riskier or the agreements more difficult to enforce.

KEY TERMS

alternating offers (684)

best alternative to a negotiated agreement (BATNA) (675)

decay (685)

efficient (678)

efficient frontier (679)

impatience (685)

Nash cooperative solution (676)

surplus (675)

ultimatum game (692)

variable-threat bargaining (681)

Glossary

alternating offers
: A sequential move procedure of bargaining in which, if the offer made by one player is refused by the other, then the refuser gets the next turn to make an offer, and so on.

best alternative to a negotiated agreement (BATNA)
: In a bargaining game, this is the payoff a player would get from his other opportunities if the bargaining in question failed to reach an agreement.

decay
: Shrinkage over time of the total surplus available to be split between the bargainers, if they fail to reach an agreement for some length of time during the process of their bargaining.

efficient
: An outcome of a bargaining game is called efficient if there is no feasible alternative that would leave one bargainer with a higher payoff without reducing the payoff of the other.

efficient frontier
: This is the northeast boundary of the set of feasible payoffs of the players, such that in a bargaining game it is not possible to increase the payoff of one person without lowering that of another.

impatience
: Preference for receiving payoffs earlier rather than later. Quantitatively measured by the discount factor.

Nash cooperative solution
: This outcome splits the bargainers' surpluses in proportion to their bargaining powers.

ultimatum game
: A form of bargaining where one player makes an offer of a particular split of the total available surplus, and the

other has only the all-or-nothing choice of accepting the offer or letting the game end without agreement, when both get zero surplus.

variable-threat bargaining
: A two-stage game where at the first stage you can take an action that will alter the BATNAs of both bargainers (within certain limits), and at the second stage bargaining results in the Nash solution on the basis of these BATNAs.

surplus
: A player's surplus in a bargaining game is the excess of his payoff over his BATNA.

SOLVED EXERCISES

1. Ali and Baba are bargaining to split a total that starts out at $100. Ali makes the first offer, stating how the $100 will be divided between them. If Baba accepts this offer, the game is over. If Baba rejects it, a dollar is withdrawn from the total, so it is now only $99. Then Baba gets the second turn to make an offer of a division. The turns alternate in this way, a dollar being removed from the total after each rejection. Ali’s BATNA is $2.25 and Baba’s BATNA is $3.50. What is the rollback equilibrium outcome of the game?
2. William, a worker at Acme Enterprises, is responsible for managing Acme’s relationship with Must Buy, an important supplier. William has learned how to work well with Must Buy, allowing Acme to get a benefit of $280,000 per year from the relationship (not counting William’s salary), whereas Acme would only get only $100,000 per year from the relationship if William left and someone else had to take his role. Currently, everyone at Acme (including William) earns a salary of $100,000 per year, but William knows that he’s worth a lot more to the company. On the other hand, William also knows that his boss has three times as much bargaining power as he does.
    1. One day, William drops by his boss’s office and demands to renegotiate his salary. His boss counters, “That’s fine. But you should understand that if we can’t reach an agreement, you’re gone—and I’ll make sure that you never work again!” Should William back down (and keep his current salary) or press forward with the negotiation? If he presses forward, what does Nash’s theory of bargaining predict will happen? Assume that the boss’s threat “You’ll never work again” is credible.

    2. What if the boss’s threat isn’t credible? Yes, William will lose his job if the negotiation fails, but if so, William will be able to find another job. However, this new job will have a lower salary of $80,000 per year. What is the Nash cooperative solution now?
    3. Before going into the boss’s office, William realizes that he has even more to offer the company. If his boss agrees to give him a raise, William can teach his coworker, Jill, how to increase the profitability of her supplier relationship by $60,000 per year. However, if the negotiation fails, William will still get fired and have to take that $80,000 per year job. What is the Nash cooperative solution now?
3. Two hypothetical countries, Euphoria and Militia, are holding negotiations to settle a dispute. They meet once a month, starting in January, and take turns making offers. Suppose the total at stake is 100 points. The government of Euphoria is facing reelection in November. Unless that government produces an agreement at the October meeting, it will lose the election, an outcome it regards as being just as bad as getting zero points from an agreement. The government of Militia does not really care about reaching an agreement; it is just as happy to prolong the negotiations, or even to fight, rather than settle for anything significantly less than 100.
    1. What will be the outcome of the negotiations? What difference will the identity of the first mover make?
    2. In light of your answer to part (a), discuss why actual negotiations often continue right down to the deadline.
4. In 1974, the U.S. Congress passed the Budget Act, creating the House Budget Committee and requiring that all budget resolutions pass through the Committee before coming to the House floor for a vote. This act gives the chair of the Budget Committee great negotiating power,

since the chair controls what the committee can consider and the committee controls what Congress can consider. Actual budget negotiations are very complex, but for our purposes here, suppose that the committee is considering a budget with total expenditures of $4 trillion, of which $3.88 trillion are for “must-fund programs” that will be funded even if Congress fails to reach an agreement. This leaves $120 billion per year that can be allocated to “pork” projects that, while not essential, benefit voters in the places where those projects are located. From this perspective, the substance of budget negotiations is to determine where the pork flows.

Suppose that each round of this negotiation takes one month, with the chair deciding how much pork to give to his own district (or to allies who will now owe him a favor) and how much to give to others. If the first offer is rejected, one month’s worth of pork ($120/12 = $10 billion) will be lost and the whole process will start over, with the chair making another offer of how to divide the remaining $110 billion, and so on until $10 billion remains in the final month. What is the rollback equilibrium outcome of the game?

UNSOLVED EXERCISES

1. Recall the variant of the pizza pricing game in Exercise U3, part (b), in Chapter 10, in which one store (Donna's Deep Dish) was much larger than the other (Pierce's Pizza Pies). The payoff table for that version of the game is

		PIERCE'S PIZZA PIES	
		High	Low
DONNA'S DEEP DISH	High	156, 60	132, 70
	Low	150, 36	130, 50

   The noncooperative Nash equilibrium is (High, Low), yielding profits of 132 to Donna's and 70 to Pierce's, for a total of 202. If the two could achieve (High, High), their total profit would be 156 + 60 = 216, but Pierce's would not agree to this pricing.

   Suppose the two stores can reach an enforceable agreement whereby both charge High and Donna's pays Pierce's a sum of money. The alternative to this agreement is simply the noncooperative Nash equilibrium. Donna's has 2.5 times as much bargaining power as Pierce's. In the resulting agreement, what sum will Donna's pay to Pierce's?

2. This exercise considers a variation of William's negotiation game (from Exercise S2). As before, William's current salary is $100,000 per year, he creates $280,000 per year value for his boss at Acme Enterprises, and his boss has three times as much bargaining strength as William. Also, as in part (a) of

Exercise S2, William initially believes that he has no hope of ever getting another job. But now William gets a call from another company, Wily Industries, inquiring whether William would like to work for them. William won't be quite as productive at Wily, creating $240,000 of value there rather than his $280,000 at Acme; also, like his current boss at Acme, the boss at Wily has three times as much bargaining strength as William. The question then arises: Does it matter who William negotiates with first?

1. Suppose that William first negotiates with Wily Industries and that the Wily negotiation fails. William then has the option to negotiate with his boss at Acme. Verify that, in this scenario, William will choose not to negotiate for a higher wage with Acme and hence will continue earning $100,000 per year. [Feel free to consult the solution to Exercise S2, part (a).]
2. Given your finding in (a), what is the predicted Nash cooperative solution of this sequential-move bargaining game if William negotiates first with Wily Industries? Will William switch jobs, and what will his new wage be?
3. What if William first negotiates with his boss at Acme, explaining that he has received interest from Wily but has not yet entered negotiations with them? As in Exercise S2, assume that William loses his job if the negotiation with Acme fails. The difference now is that if William is tossed out of Acme, he can then negotiate with Wily. Verify that in this scenario, the predicted Nash cooperative outcome of negotiation between William and Wily will be for William to accept an offer at Wily at a salary of $60,000 per year.
4. Given your finding in (c), what is the predicted Nash cooperative outcome of this sequential-move bargaining game if William negotiates first with Acme

    Enterprises, his current employer? Will William switch jobs, and what will his new wage be?
  5. Which is better for William: to negotiate first with Acme or to negotiate first with Wily?
3. Consider two players who bargain over a surplus initially equal to a whole-number amount V, using alternating offers. That is, Player 1 makes an offer in round 1; if Player 2 rejects this offer, she makes an offer in round 2; if Player 1 rejects this offer, she makes an offer in round 3; and so on. Suppose that the available surplus decays by a constant value of $c = 1$ each period. For example, if the players reach agreement in round 2, they divide a surplus of $V - 1$; if they reach agreement in round 5, they divide a surplus of $V - 4$. This means that the game will be over after V rounds, because at that point there will be nothing left to bargain over. (For comparison, remember the football ticket example in Section 3, in which the value of the ticket to the fan started at \$100 and declined by \$25 per quarter over the four quarters of the game.) In this problem, we first solve for the rollback equilibrium of this game, and then solve for the equilibrium of a generalized version of this game in which the two players can have BATNAs.
   1. Let's start with a simple version. What is the rollback equilibrium when $V = 4$? In which period will the players reach agreement? What payoff x will Player 1 receive, and what payoff y will Player 2 receive?
   2. What is the rollback equilibrium when $V = 5$?
   3. What is the rollback equilibrium when $V = 10$?
   4. What is the rollback equilibrium when $V = 11$?
   5. Now we're ready to generalize. What is the rollback equilibrium for any whole-number value of V? (Hint: You may want to consider even values of V separately from odd values.)

Now consider BATNAs. Suppose that if no agreement is reached by the end of round V, Player A gets a payoff of a and Player B gets a payoff of b. Assume that a and b are whole numbers satisfying the inequality $a + b < V$, so that the players can get higher payoffs by reaching agreement than they can by not reaching agreement.

1. (f) Suppose that $V = 4$. What is the rollback equilibrium for any possible values of a and b? [Hint: You may need to write down more than one formula, just as you did in part (e). If you get stuck, try assuming specific values for a and b, and then change those values to see what happens. In order to roll back, you'll need to figure out the turn at which the value of V has declined to the point where a negotiated agreement would no longer be profitable for the two bargainers.]
2. (g) Suppose that $V = 5$. What is the rollback equilibrium for any possible values of a and b?
3. (h) For any whole-number values of a, b, and V, what is the rollback equilibrium?
4. (i) Relax the assumption that a, b, and V are whole numbers: Let them be any nonnegative numbers such that $a + b < V$. Also relax the assumption that the value of V decays by exactly 1 each period: Let the value decay each period by some constant amount $c > 0$. What is the rollback equilibrium to this general problem?

4. Let x be the amount that Player A asks for, and let y be the amount that Player B asks for, when making the first offer in an alternating-offers bargaining game with impatience. Their rates of impatience are r and s, respectively.
   1. If we use the approximate formulas $x = s/(r + s)$ for x and $y = r/(r + s)$ for y, and if B is twice as impatient as A, then A gets two-thirds of the surplus

and B gets one-third. Verify that this result is correct.

2. Let $r = 0.01$ and $s = 0.02$, and compare the x and y values found by using the approximation method with the more exact solutions for x and y found by using the formulas $x = (s + rs)/(r + s + rs)$ and $y = (r + rs)/(r + s + rs)$ derived in the chapter.

Glossary

■

Here we define the key terms that appear in the text. We aim for verbal definitions that are logically precise but not mathematical or detailed like those found in more advanced textbooks.

acceptability condition An upper bound on the probability of fulfillment in a brinkmanship threat, expressed as a function of the probability of error, showing the upper limit of risk that the player making the threat is willing to tolerate.

action node A node at which one player chooses an action from two or more that are available.

addition rule If the occurrence of *X* requires the occurrence of *any one* of several disjoint *Y*, *Z*, . . . , then the probability of *X* is the sum of the separate probabilities of *Y*, *Z*, . . .

adverse selection A form of information asymmetry where a player's type (available strategies, payoffs . . .) is his private information, not directly known to others.

affirmation A response to the *favored action* that is a best response, as specified within a strategic move. Only the second mover in a game can feasibly make an affirmation. However, credibility is not required since the specified action is already the player's best response. The strategic move referred to as "making a threat" entails declaring both a threat and an affirmation.

agenda paradox A voting situation where the order in which alternatives are paired when voting in multiple rounds determines the final outcome.

agent The agent is the more-informed player in a principal–agent game of asymmetric information. The principal (less-informed) player in such games attempts to design a mechanism that aligns the agent’s incentives with his own.

all-pay auction An auction in which each person who submits a bid must pay her highest bid amount at the end of the auction, even if she does not win the auction.

alternating offers A sequential move procedure of bargaining in which, if the offer made by one player is refused by the other, then the refuser gets the next turn to make an offer, and so on.

amendment procedure A procedure in which any amended version of a proposal must win a vote against the original version before the winning version is put to a vote against the status quo.

antiplurality method A positional voting method in which the electorate is asked to vote against one item on the slate (or to vote for all but one).

approval voting A voting method in which voters cast votes for all alternatives of which they approve.

ascending-price auction An open-outcry auction in which prices are announced in increasing order either by an auctioneer (in the case of an English auction) or by bidders themselves (in the case of jump bidding). The last person to bid or accept the announced price wins the auction and pays that price.

assurance game A game where each player has two strategies, say, Cooperate and Not, such that the best response of each is to Cooperate if the other cooperates, Not if not, and the outcome from (Cooperate, Cooperate) is better for both than the outcome of (Not, Not).

asymmetric information Information is said to be asymmetric in a game if some aspects of it—rules about what actions are permitted and the order of moves if any, payoffs as functions of the players strategies, outcomes of random choices by “nature,” and of previous actions by the actual players in the game—are known to some of the players but are not common knowledge among all players.

auction A game in which multiple players (called bidders) compete for a scarce resource.

auction designer A player who sets the rules of an auction game.

babbling equilibrium In a game where communication among players (which does not affect their payoffs directly) is followed by their choices of actual strategies, a babbling equilibrium is one where the strategies are chosen ignoring the communication, and the communication at the first stage can be arbitrary.

backward induction Same as **rollback**.

battle of the sexes A game where each player has two strategies, say, Hard and Soft, such that [1] (Hard, Soft) and (Soft, Hard) are both Nash equilibria, [2] of the two Nash equilibria, each player prefers the outcome where he is Hard and the other is Soft, and [3] both prefer the Nash equilibria to the other two possibilities, (Hard, Hard) and (Soft, Soft).

Bayesian Nash equilibrium A Nash equilibrium in an asymmetric information game where players use Bayes’ theorem and draw correct inferences from their observations of other players’ actions.

Bayes’ theorem An algebraic formula for estimating the probabilities of some underlying event by using knowledge of some consequences of it that are observed.

belief The notion held by one player about the strategy choices of the other players and used when choosing his own optimal strategy.

best alternative to a negotiated agreement (BATNA) In a bargaining game, this is the payoff a player would get from his other opportunities if the bargaining in question failed to reach an agreement.

best response The strategy that is optimal for one player, given the strategies actually played by the other players, or the belief of this player about the other players' strategy choices.

best-response analysis Finding the Nash equilibria of a game by calculating the best-response functions or curves of each player and solving them simultaneously for the strategies of all the players.

best-response curve A graph showing the best strategy of one player as a function of the strategies of the other player(s) over the entire range of those strategies.

best-response rule A function expressing the strategy that is optimal for one player, for each of the strategy combinations actually played by the other players, or the belief of this player about the other players' strategy choices.

bidder A player in an auction game.

binary method A class of voting methods in which voters choose between only two alternatives at a time.

Black's condition Same as the condition of **single-peaked preferences**.

Borda count A positional voting method in which the electorate indicates its order of preference over a slate of alternatives. The winning alternative is determined by allocating points based on an alternative’s position on each ballot.

branch Each branch emerging from a node in a game tree represents one action that can be taken at that node.

brinkmanship A threat that creates a risk but not certainty of a mutually bad outcome if the other player defies your specified wish as to how he should act, and then gradually increases this risk until one player gives in or the bad outcome happens.

cheap talk equilibrium In a game where communication among players (which does not affect their payoffs directly) is followed by their choices of actual strategies, a cheap talk equilibrium is one where the strategies are chosen optimally given the players’ interpretation of the communication, and the communication at the first stage is optimally chosen by calculating the actions that will ensue.

chicken A game where each player has two strategies, say Tough and Weak, such that [1] both (Tough, Weak) and (Weak, Tough) are Nash equilibria, [2] of the two, each prefers the outcome where she plays Tough and the other plays Weak, and [3] the outcome (Tough, Tough) is worst for both.

coercion In this context, forcing a player to accept a lower payoff in an asymmetric equilibrium in a collective action game, while other favored players are enjoying higher payoffs. Also called **oppression** in this context.

collective action A problem of achieving an outcome that is best for society as a whole, when the interests of some or all individuals will lead them to a different outcome as the equilibrium of a noncooperative game.

combinatorial auction An auction of multiple dissimilar objects in which bidders are able to bid on and win combinations of objects.

commitment An action taken at a pregame stage, stating what action you would take unconditionally in the game to follow.

common value An auction is called a common-value auction when the object up for sale has the same value to all bidders, but each bidder knows only an imprecise estimate of that value.

compellence An attempt to induce the other player(s) to act to change the status quo in a specified manner.

compound interest When an investment goes on for more than one period, compound interest entails calculating interest in any one period on the whole accumulation up to that point, including not only the principal initially invested but also the interest earned in all previous periods, which itself involves compounding over the period previous to that.

Condorcet method A voting method in which the winning alternative must beat each of the other alternatives in a round-robin of pairwise contests.

Condorcet paradox Even if all individual voter preference orderings are transitive, there is no guarantee that the social preference ordering generated by Condorcet's voting method will also be transitive.

Condorcet terms A set of ballots that would generate the Condorcet paradox and that should together logically produce a tied vote among three possible alternatives. In a three-candidate election among A, B, and C, the Condorcet terms are three ballots that show A preferred to B preferred to C; B preferred to C preferred to A; C preferred to A preferred to B.

Condorcet winner The alternative that wins an election run using the *Condorcet method*.

constant-sum game A game in which the sum of all players' payoffs is a constant, the same for all their strategy combinations. Thus, there is a strict conflict of interests among the players—a higher payoff to one must mean a lower payoff to the collectivity of all the other players. If the payoff scales can be adjusted to make this constant equal to zero, then we have a *zero-sum game*.

contingent strategy In repeated play, a plan of action that depends on other players' actions in previous plays. (This is implicit in the definition of a strategy; the adjective "contingent" merely reminds and emphasizes.)

continuation The continuation of a strategy from a (noninitial) node is the remaining part of the plan of action of that strategy, applicable to the subgame that starts at this node.

continuous distribution A probability distribution in which the random variables may take on a continuous range of values.

continuous strategy A choice over a continuous range of real numbers available to a player.

contract In this context, a way of achieving credibility for one's strategic move by entering into a legal obligation to perform the committed, threatened, or promised action in the specified contingency.

convention A mode of behavior that finds automatic acceptance as a focal point, because it is in each individual's interest to follow it when others are expected to follow it too (so the game is of the assurance type). Also called **custom**.

convergence of expectations A situation where the players in a noncooperative game can develop a common understanding of the strategies they expect will be chosen.

cooperative game A game in which the players decide and implement their strategy choices jointly, or where joint-action agreements are directly and collectively enforced.

coordination game A game with multiple Nash equilibria, where the players are unanimous about the relative merits of the equilibria, and prefer any equilibrium to any of the nonequilibrium possibilities. Their actions must somehow be coordinated to achieve the preferred equilibrium as the outcome.

Copeland index An index measuring an alternative’s record in a round-robin of contests where different numbers of points are allocated for wins, ties, and losses.

credibility A strategy is credible if its continuation at all nodes, on or off the equilibrium path, is optimal for the subgame that starts at that node.

credibility device A means by which a player acquires credibility, for instance, when declaring a promise or threat as part of a strategic move.

custom Same as **convention**.

decay Shrinkage over time of the total surplus available to be split between the bargainers, if they fail to reach an agreement for some length of time during the process of their bargaining.

decision node A decision node in a decision or game tree represents a point in a game where an action is taken.

decision tree Representation of a sequential decision problem facing one person, shown using nodes, branches, terminal nodes, and their associated payoffs.

default action In the context of strategic moves, the action that the other player (the player not making a strategic move) will take in the absence of a strategic move, as opposed to the *favored action*.

descending-price auction An open-outcry auction in which the auctioneer announces possible prices in descending order. The first person to accept the announced price wins the auction and pays that price. Also called **Dutch auction**.

deterrence An attempt to induce the other player(s) to act to maintain the status quo.

diffusion of responsibility A situation where action by one or a few members of a large group would suffice to bring about an outcome that all regard as desirable, but each thinks it is someone else's responsibility to take this action.

discount factor In a repeated game, the fraction by which the next period's payoffs are multiplied to make them comparable with this period's payoffs.

discrete distribution A probability distribution in which the random variables may take on only a discrete set of values such as integers.

disjoint Events are said to be disjoint if two or more of them cannot occur simultaneously.

distribution function A function that indicates the probability that a variable takes on a value less than or equal to some number.

dominance solvable A game where iterated elimination of dominated strategies leaves a unique outcome, or just one strategy for each player.

dominant strategy A strategy X is dominant for a player if the outcome when playing X is always better than the outcome when playing any other strategy, no matter what strategies other players adopt.

dominated strategy A strategy X is dominated by another strategy Y for a player if the outcome when playing X is always worse than the outcome when playing Y, no matter what strategies other players adopt.

doomsday device An automaton that will under specified circumstances generate an outcome that is very bad for all players. Used for giving credibility to a severe threat.

drop-out price In an English auction, the price at which a bidder drops out of the bidding.

Dutch auction Same as a **descending-price auction**.

dynamic chicken A game of chicken in which the choice to play Weak may be made at any time, the game ends as soon as either player chooses Weak, and the risk of the mutually worst outcome increases gradually over time if neither player has played Weak; a special case of the *war of attrition*.

effectiveness condition A lower bound on the probability of fulfillment in a brinkmanship threat, expressed as a function of the probability of error, showing the lower limit of risk that will induce the threatened player to comply with the wishes of the threatener.

effective rate of return Rate of return corrected for the probability of noncontinuation of an investment to the next period.

efficiency wage A higher-than-market wage paid to a worker as a means of incentivizing him to exert effort. If the worker shirks and is detected, he will be fired and will have to get a lower-wage job in the general labor market.

efficient An outcome of a bargaining game is called efficient if there is no feasible alternative that would leave one bargainer with a higher payoff without reducing the payoff of the other.

efficient frontier This is the northeast boundary of the set of feasible payoffs of the players, such that in a bargaining game it is not possible to increase the payoff of one person without lowering that of another.

English auction A type of *ascending-price auction* in which the auctioneer calls out a sequence of increasing prices, bidders decide when to drop out of the bidding, and the last bidder remaining pays the last announced price.

equilibrium A configuration of strategies where each player's strategy is his best response to the strategies of all the other players.

equilibrium path of play The *path of play* actually followed when players choose their rollback equilibrium strategies in a sequential game.

evolutionary game A situation where the strategy of each player in a population is fixed genetically, and strategies that yield higher payoffs in random matches with others from the same population reproduce faster than those with lower payoffs.

evolutionary stability A population is evolutionarily stable if it cannot be successfully invaded by a new mutant phenotype.

evolutionarily stable strategy (ESS) A phenotype or strategy that can persist in a population, in the sense that all the members of a population or species are of that type; the population is evolutionarily stable (static criterion). Or, starting from an arbitrary distribution of phenotypes in the population, the process of selection will converge to this strategy (dynamic criterion).

expected payoff The probability-weighted average (statistical mean or expectation) of the payoffs of one player in a game, corresponding to all possible realizations of a random choice of nature or mixed strategies of the players.

expected utility The probability-weighted average (statistical mean or expectation) of the utility of a person, corresponding to all possible realizations of a random choice of nature or mixed strategies of the players in a game.

expected value The probability-weighted average of the outcomes of a random variable, that is, its statistical mean or expectation.

extensive form Representation of a game by a game tree.

external effect When one person's action alters the payoff of another person or persons. The effect or spillover is *positive* if one's action raises others' payoffs (for example, network effects) and *negative* if it lowers others' payoffs (for example, pollution or congestion). Also called **externality** or **spillover effect**.

externality Same as **external effect**.

external uncertainty A player's uncertainty about external circumstances such as the weather or product quality.

favored action The action that a player making a *strategic move* wants the other player to take, as opposed to the *default action*.

feasibility Possibility within the (unchangeable) physical and/or procedural restrictions that apply in a given game.

first-mover advantage This exists in a game if, considering a hypothetical choice between moving first and moving second, a player would choose the former.

first-price auction A sealed-bid auction in which the highest bidder wins and pays the amount of her bid.

fitness The expected payoff of a phenotype in its games against randomly chosen opponents from the population.

focal point A configuration of strategies for the players in a game, which emerges as the outcome because of the convergence of the players' expectations on it.

free rider A player in a collective-action game who intends to benefit from the positive externality generated by others' efforts without contributing any effort of his own.

game (game of strategy) An action situation where there are two or more mutually aware players, and the outcome for each depends on the actions of all.

game table A spreadsheetlike table whose dimension equals the number of players in the game; the strategies available to each player are arrayed along one of the dimensions (row, column, page, . . .); and each cell shows the payoffs of all the players in a specified order, corresponding to the configuration of strategies that yield that cell. Also called **payoff table**.

game tree Representation of a game in the form of nodes, branches, and terminal nodes and their associated payoffs.

genotype A gene or a complex of genes, which give rise to a phenotype and which can breed true from one generation to another. (In social or economic games, the process of breeding can be interpreted in the more general sense of teaching or learning.)

Gibbard–Satterthwaite theorem With three or more alternatives to consider, the only voting method that prevents strategic voting is dictatorship; one person is identified as the dictator and her preferences determine the outcome.

grim strategy A strategy of noncooperation forever in the future, if the opponent is found to have cheated even once. Used as a threat of punishment in an attempt to sustain cooperation.

hawk–dove game An evolutionary game where members of the same species or population can breed to follow one of two strategies, Hawk and Dove, and depending on the payoffs, the game between a pair of randomly chosen members can be either a prisoners' dilemma or chicken.

histogram A bar chart; data are illustrated by way of bars of a given height (or length).

impatience Preference for receiving payoffs earlier rather than later. Quantitatively measured by the discount factor.

imperfect information A game is said to have perfect information if each player, at each point where it is his turn to act, knows the full history of the game up to that point, including the results of any random actions taken by nature or previous actions of other players in the game, including pure actions as well as the actual outcomes of any mixed strategies they may play. Otherwise, the game is said to have imperfect information.

impossibility theorem A theorem that indicates that no preference aggregation method can satisfy the six critical principles identified by Kenneth Arrow.

incentive-compatibility condition (constraint) A constraint on an incentive scheme or screening device that makes it optimal for the agent (more-informed player) of each type to reveal his true type through his actions.

incentive design The process that a *principal* uses to devise the best possible incentive scheme (or mechanism) in a *principal–agent problem* to motivate the agent to take actions that benefit the principal. By design, such incentive schemes take into account that the agent knows something (about the world or about herself) that the principal does not know. Also called **mechanism design**.

independent events Events Y and Z are independent if the actual occurrence of one does not change the probability of the other occurring. That is, the conditional probability of Y occurring given that Z has occurred is the same as the ordinary or unconditional probability of Y.

infinite horizon A repeated decision or game situation that has no definite end at a fixed finite time.

information set A set of nodes among which a player is unable to distinguish when taking an action. Thus, his strategies are restricted by the condition that he should choose the same action at all points of an information set. For this, it is essential that all the nodes in an information set have the same player designated to act, with the same number and similarly labeled branches emanating from each of these nodes.

initial node The starting point of a sequential-move game. (Also called the **root** of the tree.)

instant-runoff voting (IRV) Same as **single transferable vote**.

intermediate valuation function A rule assigning payoffs to nonterminal nodes in a game. In many complex games, this must be based on knowledge or experience of playing similar games, instead of explicit rollback analysis.

internalize the externality To offer an individual a reward for the external benefits he conveys on the rest of society, or to inflict a penalty for the external costs he imposes on the rest, so as to bring his private incentives in line with social optimality.

intransitive ordering A preference ordering that cycles and is not *transitive*. For example, a preference ordering over three alternatives A, B, and C is intransitive if A is preferred to B and B is preferred to C but it is not true that A is preferred to C.

invasion The appearance of a small proportion of mutants in the population.

irreversible Cannot be undone by a later action. In a sequential-move game, the first mover's action must be irreversible and *observable* before the second mover's action is irreversible.

iterated elimination of dominated strategies Considering the players in turns and repeating the process in rotation, eliminating all strategies that are dominated for one at a time, and continuing doing so until no such further elimination is possible. Also called **successive elimination of dominated strategies**.

jump bidding Submitting a bid that is significantly higher than the previous bid and well beyond whatever minimum bid increment exists.

leadership In a prisoners' dilemma with asymmetric players, this is a situation where a large player chooses to cooperate even though he knows that the smaller players will cheat.

locked in A situation where the players persist in a Nash equilibrium that is worse for everyone than another Nash equilibrium.

majority rule A voting method in which the winning alternative is the one that garners a majority (more than 50%) of the votes.

majority runoff A two-stage voting method in which a second round of voting ensues if no alternative receives a majority in the first round. The top two vote-getters are paired in the second round of voting to determine a winner.

marginal private gain The change in an individual's own payoff as a result of a small change in a continuous-strategy variable that is at his disposal.

marginal social gain The change in the aggregate social payoff as a result of a small change in a continuous-strategy variable chosen by one player.

mechanism design Same as **incentive design**.

median voter The voter in the middle—at the 50th percentile—of a distribution.

median voter theorem If the political spectrum is one-dimensional and every voter has single-peaked preferences, then [1] the policy most preferred by the median voter will be the Condorcet winner, and [2] power-seeking politicians in a two-candidate election will choose platforms that converge to the position most preferred by the median voter. (This is also known as the **principle of minimum differentiation.**)

mixed method A multistage voting method that uses plurative and binary votes in different rounds.

mixed strategy A mixed strategy for a player consists of a random choice, to be made with specified probabilities, from his originally specified pure strategies.

monomorphism All members of a given species or population exhibit the same behavior pattern.

moral hazard A situation of information asymmetry where one player's actions are not directly observable to others.

move An action at one node of a game tree.

multiplication rule If the occurrence of *X* requires the simultaneous occurrence of *all* the several independent *Y, Z,* . . . , then the probability of *X* is the *product* of the separate probabilities of *Y, Z,* . . .

multistage procedure A voting procedure in which there are multiple rounds of voting.

multi-unit auction An auction in which multiple identical objects are sold.

mutation Emergence of a new genotype.

Nash cooperative solution This outcome splits the bargainers' surpluses in proportion to their bargaining powers.

Nash equilibrium A configuration of strategies (one for each player) such that each player's strategy is best for him, given those of the other players. (Can be in pure or mixed strategies.)

negatively correlated Two random variables are said to be negatively correlated if, as a matter of probabilistic average, when one is above its expected value, the other is below its expected value.

never a best response A strategy is never a best response for a player if, for each list of strategies that the other players choose (or for each list of strategies that this player believes the others are choosing), some other strategy is this player’s best response. (The other strategy can be different for different lists of strategies of the other players.)

node This is a point from which branches emerge, or where a branch terminates, in a decision or game tree.

noncooperative game A game where each player chooses and implements his action individually, without any joint-action agreements directly enforced by other players.

nonexcludable Benefits that are available to each individual, regardless of whether he has paid the costs that are necessary to secure the benefits.

nonrival Benefits whose enjoyment by one person does not detract anything from another person’s enjoyment of the same benefits.

norm A pattern of behavior that is established in society by a process of education or culture, to the point that a person who behaves differently experiences a negative psychic payoff.

normal distribution A commonly used statistical distribution for which the *distribution function* looks like a bell-shaped curve.

normal form Representation of a game in a game matrix, showing the strategies (which may be numerous and complicated if the game has several moves) available to each player along a separate dimension (row, column, etc.) of the matrix and the outcomes and payoffs in the multidimensional cells. Also called **strategic form**.

objective value An auction is called an objective-value auction when the object up for sale has the same value to all bidders and each bidder knows that value.

observable Known to other players before they make their responding actions. Together with irreversibility, this is an important condition for a game to be sequential-move.

off-equilibrium path A path of play that does not result from the players' choices of strategies in a subgame-perfect equilibrium.

off-equilibrium subgame A subgame starting at a node that does not lie on the equilibrium path of play.

open-outcry auction An auction mechanism in which bids are made openly for all to hear or see.

opponent's indifference property An equilibrium mixed strategy of one player in a two-person game has to be such that the other player is indifferent among all the pure strategies that are actually used in her mixture.

oppression In this context, same as **coercion**.

ordinal payoffs Each player's ranking of the possible outcomes in a game.

pairwise voting A voting method in which only two alternatives are considered at the same time.

participation condition (constraint) A constraint on an incentive scheme or a screening device that should give the more-informed player an expected payoff at least as high as he can get outside this relationship.

path of play A route through the game tree (linking a succession of nodes and branches) that results from a configuration of strategies for the players that are within the rules of the game. (See also *equilibrium path of play*.)

payoff The objective, usually numerical, that a player in a game aims to maximize.

payoff matrix Same as **payoff table** and **game table**.

payoff table Same as **game table**.

penalty We reserve this term for one-time costs (such as fines) introduced into a game to induce the players to take actions that are in their joint interests.

penny auction An auction format in which each bidder may pay a bidding fee (say, 60 cents) to advance the price by one cent. The auction continues until no one is willing to advance the price any longer, at which point the last bidder wins and pays the final price.

perfect Bayesian equilibrium (PBE) An equilibrium where each player’s strategy is optimal at all nodes given his beliefs, and beliefs at each node are updated using Bayes’ rule in the light of the information available at that point including other players’ past actions.

perfect information A game is said to have perfect information if players face neither strategic nor external uncertainty.

phenotype A specific behavior or strategy, determined by one or more genes. (In social or economic games, this can be interpreted more generally as a customary strategy or a rule of thumb.)

playing the field A many-player evolutionary game where all animals in the group are playing simultaneously, instead of being matched in pairs for two-player games.

pluralistic ignorance A situation of collective action where no individual knows for sure what action is needed, so everyone takes the cue from other people's actions or inaction, possibly resulting in persistence of wrong choices.

plurality rule A voting method in which two or more alternatives are considered simultaneously and the winning alternative is the one that garners the largest number of votes; the winner needs only gain more votes than each of the other alternatives and does not need 50% of the vote as would be true in *majority rule*.

plurative method Any voting method that allows voters to consider a slate of three or more alternatives simultaneously.

polymorphism An evolutionarily stable equilibrium in which different behavior forms or phenotypes are exhibited by subsets of members of an otherwise identical population.

pooling Same as **pooling of types**.

pooling equilibrium A perfect Bayesian equilibrium in a game of asymmetric information, where the actions in the equilibrium cannot be used to distinguish type.

pooling of types An outcome of a signaling or screening game in which different types follow the same strategy and get the same payoffs, so types cannot be distinguished by observing actions.

positional method A voting method that determines the identity of the winning alternative using information on the position of alternatives on a voter's ballot to assign points used when tallying ballots.

positive feedback When one person's action increases the payoff of another person or persons taking the same action, thus increasing their incentive to take that action too.

positively correlated Two random variables are said to be positively correlated if, as a matter of probabilistic average, when one is above its expected value, the other is also above its expected value, and vice versa.

present value (PV) The total payoff over time, calculated by summing the payoffs at different periods each multiplied by the appropriate discount factor to make them all comparable with the initial period's payoffs.

price discrimination Perfect, or first-degree, price discrimination occurs when a firm charges each customer an individualized price based on willingness to pay. In general, price discrimination refers to situations in which a firm charges different prices to different customers for the same product.

primary criterion Comparison of the fitness of a mutant with that of a member of the dominant population, when each plays against a member of the dominant population.

principal The principal is the less-informed player in a principal–agent game of asymmetric information. The principal in such games wants to design a mechanism that creates incentives for the more-informed player (agent) to take actions beneficial to the principal.

principal–agent (agency) problem A situation in which the less-informed player (principal) wants to design a mechanism that creates incentives for the more-informed player (agent) to take actions beneficial to himself (the principal).

principle of minimum differentiation Same as part [2] of the *median voter theorem*.

prisoners' dilemma A game where each player has two strategies, say Cooperate and Defect, such that [1] for each player, Defect dominates Cooperate, and [2] the outcome (Defect, Defect) is worse for both than the outcome (Cooperate, Cooperate).

private information Information known by only one player.

private value An auction is called a private-value auction when each bidder has private information about their own valuation of the object up for sale, but knowing others' private information would not change any bidder's own willingness to pay for the object. An important special case is when each bidder knows their own valuation but others do not.

probabilistic threat A strategic move in the nature of a threat, but with the added qualification that if the event triggering the threat (the opponent's action in the case of deterrence or inaction in the case of compellence) comes about, a chance mechanism is set in motion, and if its outcome so dictates, the threatened action is carried out. The nature of this mechanism and the probability with which it will call for the threatened action must both constitute prior commitments.

probability The probability of a random event is a quantitative measure of the likelihood of its occurrence. For events that can be observed in repeated trials, it is the long-run frequency with which it occurs. For unique events or other situations where uncertainty may be in the mind of a person, other measures are constructed, such as subjective probability.

procurement auction An auction in which multiple bidders compete to supply an item. Bids in a procurement auction are prices that bidders are willing to receive to supply the good. The lowest bidder wins and is paid her bid.

promise A response to the *favored action* that benefits the other player and that is not a best response, as specified within a *strategic move*. It is only feasibly made by the second mover in a game. Credibility is required because the specified action is not a best response. The strategic move referred to as "making a promise" entails declaring both a promise and a warning, whereas "making a combined threat and promise" entails declaring both a promise and a threat.

proportional representation This voting system requires that the number of seats in a legislature be allocated in proportion to each party's share of the popular vote.

pruning Using rollback analysis to identify and eliminate from a game tree those branches that will not be chosen when the game is rationally played.

punishment We reserve this term for costs that can be inflicted on a player in the context of a repeated relationship (often involving termination of the relationship) to induce him to take actions that are in the joint interests of all players.

pure coordination game A coordination game where the payoffs of each player are the same in all the Nash equilibria. Thus, all players are indifferent among all the Nash equilibria, and coordination is needed only to ensure avoidance of a nonequilibrium outcome.

pure public good A good or facility that benefits all members of a group, when these benefits cannot be excluded from a member who has not contributed efforts or money to the provision of the good, and the enjoyment of the benefits by one person does not significantly detract from their simultaneous enjoyment by others.

pure strategy A rule or plan of action for a player that specifies without any ambiguity or randomness the action to take in each contingency or at each node where it is that player’s turn to act.

quantal-response equilibrium (QRE) Solution concept that allows for the possibility that players make errors, with the probability of a given error smaller for more costly mistakes.

ranked-choice voting Another name for **single transferable vote**.

rational behavior Perfectly calculating pursuit of a complete and internally consistent objective (payoff) function.

rational irrationality Adopting a strategy that is not optimal after the fact, but serves a rational strategic purpose of lending credibility to a threat or a promise.

rationalizability A solution concept for a game. A list of strategies, one for each player, is a rationalizable outcome of the game if each strategy in the list is rationalizable for the player choosing it.

rationalizable A strategy is called rationalizable for a player if it is his optimal choice given some belief about what (pure or mixed strategy) the other player(s) would choose, provided this belief is formed recognizing that the other players are making similar calculations and forming beliefs in the same way. (This concept is more general than that of the Nash equilibrium and yields outcomes that can be justified on the basis only of the players' common knowledge of rationality.)

refinement A restriction that narrows down possible outcomes when multiple Nash equilibria exist.

repeated play A situation where a one-time game is played repeatedly in successive periods. Thus, the complete game is mixed, with a sequence of simultaneous-move games.

reputation Relying on the effect on payoffs in future or related games to make threats or promises credible, when they would not have been credible in a one-off or isolated game.

reserve price The minimum price set by the seller of an item up for auction; if no bids exceed the reserve, the item is not sold.

response rule A rule that specifies how you will act in response to various actions of other players.

revenue equivalence theorem (RET) A famous result in auction theory specifying conditions under which two auctions will generate the same expected revenue for the seller.

reversal paradox This paradox arises in an election with at least four alternatives when one of these is removed from consideration after votes have been submitted and the removal changes the identity of the winning alternative.

reversal terms A set of ballots that would generate the *reversal paradox* and that should together logically produce a tied vote between a pair of alternatives. In a three-candidate election among A, B, and C, the reversal terms are two ballots that show a reversal in the location of a pair of alternatives. For example, one ballot with A preferred to B preferred to C and another with B preferred to A preferred to C should produce a tie between A and B.

robustness A measure of the number of sets of voter preference orderings that are nondictatorial, satisfy independence of irrelevant alternatives and the Pareto property, and also produce a transitive *social ranking*.

rollback Analyzing the choices that rational players will make at all nodes of a game, starting at the terminal nodes and working backward to the initial node. Also called **backward induction**.

rollback equilibrium The strategies (complete plans of action) for each player that remain after rollback analysis has been used to prune all the branches that can be pruned.

root Same as **initial node**.

round A single vote within a larger *multistage procedure* that consists of multiple sequentially held votes.

salami tactics A method of defusing threats by taking a succession of actions, each sufficiently small to make it nonoptimal for the other player to carry out his threat.

sanction Punishment approved by society and inflicted by others on a member who violates an accepted pattern of behavior.

screening Strategy of a less-informed player to elicit information credibly from a more-informed player.

screening devices Methods used for screening.

sealed-bid auction An auction mechanism in which bids are submitted privately in advance of a specified deadline, sometimes in sealed envelopes.

secondary criterion Comparison of the fitness of a mutant with that of a member of the dominant population, when each plays against a mutant.

second-mover advantage A game has this if, considering a hypothetical choice between moving first and moving second, a player would choose the latter.

second-price auction A sealed-bid auction in which the highest bidder wins the auction but pays a price equal to the value of the second-highest bid; a special case of the *Vickrey auction*.

selection The dynamic process by which the proportion of fitter phenotypes in a population increases from one generation to the next.

self-selection Where different types respond differently to a screening device, thereby revealing their type through their own action.

semiseparating equilibrium A perfect Bayesian equilibrium in a game of asymmetric information, where the actions in the equilibrium convey some additional information about the players' types, but some ambiguity about these types remains.

separating equilibrium A perfect Bayesian equilibrium in a game of asymmetric information, where the actions in the equilibrium reveal player type.

separation of types An outcome of a signaling or screening game in which different types follow different strategies and get the different payoffs, so types can be identified by observing actions.

sequential moves The moves in a game are sequential if the rules of the game specify a strict order such that at each action node only one player takes an action, with knowledge of the actions taken (by others or himself) at previous nodes.

shading A strategy in which bidders bid slightly below their true valuation of an object.

shill bidder A fake bidder created by sellers at an auction to place fictitious bids for an object they are selling.

signaling Strategy of a more-informed player to convey his "good" information credibly to a less-informed player.

signals Devices used for signaling.

simultaneous moves The moves in a game are simultaneous if each player must take his action without knowledge of the choices of others.

sincere voting Voting at each point for the alternative that you like best among the ones available at that point, regardless of the eventual outcome. Also called **truthful voting**.

single-object auction An auction in which a single indivisible object is sold.

single-peaked preferences A preference ordering in which alternatives under consideration can be ordered along some specific dimension and each voter has a single ideal or most-preferred alternative with alternatives farther away from the most-preferred point providing steadily lower payoffs. Also called **Black's condition**.

single transferable vote A voting method in which each voter indicates her preference ordering over all candidates on a single initial ballot. If no alternative receives a majority of all first-place votes, the bottom-ranked alternative is eliminated and all first-place votes for that candidate are "transferred" to the candidate listed second on those ballots; this process continues until a majority winner emerges. Also called **instant-runoff voting (IRV)** or **ranked-choice voting**.

sniping Waiting until the last moment to make a bid.

social optimum In a collective-action game where payoffs of different players can be meaningfully added together, the social optimum is achieved when the sum total of the players' payoffs is maximized.

social ranking The preference ordering of a group of voters that arises from aggregating the preferences of each member of the group.

spillover effect Same as **external effect**.

spoiler Refers to a third candidate who enters a two-candidate race and reduces the chances that the leading candidate actually wins the election.

strategic form Same as **normal form**.

strategic misrepresentation of preferences Refers to strategic behavior of voters when they use rollback to determine that they can achieve a better outcome for themselves by not voting strictly according to their preference orderings.

strategic move Action taken at a pregame stage that changes the strategies or the payoffs of the subsequent game (thereby changing its outcome in favor of the player making the move).

strategic order The order of moves from a game-theoretic point of view, determined by considerations of observability and irreversibility. It may differ from the chronological order of actions and, in turn, determine whether the game has sequential or simultaneous moves.

strategic uncertainty A player’s uncertainty about an opponent’s moves made in the past or made at the same time as her own.

strategic voting Voting in conformity with your optimal rational strategy found by doing rollback analysis on the game tree of the voting procedure.

strategy A complete plan of action for a player in a game, specifying the action he would take at all nodes where it is his turn to act according to the rules of the game (whether these nodes are on or off the equilibrium path of play). If two or more nodes are grouped into one information set, then the specified action must be the same at all these nodes.

subgame A game comprising a portion or remnant of a larger game, starting at a noninitial node of the larger game.

subgame-perfect equilibrium (SPE) A configuration of strategies (complete plans of action) such that their continuation in any subgame remains optimal (part of a rollback equilibrium), whether that subgame is on- or off-equilibrium. This ensures credibility of all the strategies.

successive elimination of dominated strategies Same as **iterated elimination of dominated strategies**.

superdominant A strategy is superdominant for a player if the worst possible outcome when playing that strategy is better than the best possible outcome when playing any other strategy.

surplus A player’ s surplus in a bargaining game is the excess of his payoff over his BATNA.

terminal node This represents an end point in a game tree, where the rules of the game allow no further moves, and payoffs for each player are realized.

threat A response to the default action that harms the other player and that is not a best response, as specified within a strategic move. It can only feasibly be made by the second mover in a game. Credibility is required because the specified action is not a best response. The strategic move referred to as “making a threat” entails declaring both a threat and an affirmation, whereas “making a combined threat and promise” entails declaring both a threat and a promise.

tit-for-tat (TFT) In a repeated prisoners’ dilemma, this is the strategy of [1] cooperating on the first play and [2] thereafter doing each period what the other player did the previous period.

transitive ordering A preference ordering for which it is true that if option A is preferred to B and B is preferred to C, then A is also preferred to C.

trigger strategy In a repeated game, this strategy cooperates until and unless a rival chooses to defect, and then switches to noncooperation for a specified period.

truthful bidding A practice by which bidders in an auction bid their true valuation of an object.

truthful voting Same as **sincere voting.**

type Players who possess different private information in a game of asymmetric information are said to be of different types.

ultimatum game A form of bargaining where one player makes an offer of a particular split of the total available surplus, and the other has only the all-or-nothing choice of accepting the offer or letting the game end without agreement, when both get zero surplus.

uniform distribution A common statistical distribution in which the *distribution function* is horizontal; data are distributed uniformly at each location along the range of possible values.

valuation The benefit that a bidder gets from winning the object in an auction.

variable-threat bargaining A two-stage game where at the first stage you can take an action that will alter the BATNAs of both bargainers (within certain limits), and at the second stage bargaining results in the Nash solution on the basis of these BATNAs.

Vickrey auction An auction design proposed by William Vickrey in which truthful bidding is a weakly dominant strategy for each bidder. When a single object is sold, the Vickrey auction is the same as the *second-price auction*.

warning A response to the default action that is a best response, as specified within a strategic move. Only the second mover in a game can feasibly make a warning. However, credibility is not required since the specified action is already the player’s best response. The strategic move referred to as “making a promise” entails declaring both a promise and a warning.

war of attrition A contest between multiple players in which each player decides when to retreat, the victor is whoever remains the longest, and choosing to remain longer is costly for each player.

weakly dominant A strategy is weakly dominant for a player if the outcome when playing that strategy is never worse than the outcome when playing any other strategy, no matter what strategies other players adopt.

winner’s curse A situation in a common-value auction where the winner fails to take account of the fact that when she wins, she is likely to have made an overly optimistic estimate of the object’s value. Bidders who correctly anticipate this possibility can avoid the winner’s curse by lowering their bids appropriately.

zero-sum game A game where the sum of the payoffs of all players equals zero for every configuration of their strategy choices. (This is a special case of a *constant-sum game*, but in practice no different because adding a constant to all the payoff numbers of any one player makes no difference to his choices.)

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