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DATA RECONSTRUCTION BY INTERLEAVED SAMPLING

H.Makitani, S.Kawamura
Yokogawa Electric Corporation
2-9-32 Nakacho, Musashino-shi
Tokyo, 180, Japan

Abstract

According to the Sampling Theorem, a given signal must be bandlimited to $f_0 / 2$ (f_0 is the sampling frequency) to be recoverable from its sampling data. In this paper, however, it is shown that by using several trains of interleaved sampling data, this limit can be expanded.

Introduction

When measuring waveforms using an A/D converter, the aliasing effect can be a problem. To avoid this , the random sampling technique is commonly used. But digital signal processing, such as DFT or digtal filtering, cannot be performed to these random(interleaved) sampled data, because they are not equally spaced. So, it may be required to interpolate and obtain equally spaced data from interleaved trains of data. Linden[1] and Bracewell[2] showed the analytical method to reconstruct an original signal which is bandlimited to twice the Nyquist frequency, from its 2 trains of sampling data. In this paper, we will introduce a more generalized method using N trains of data to reconstruct an original signal.

Sampling Theorem

It is well known that a bandlimited signal $x(t)$, with $X(f)=0$ for $|f| \geqq f_0/2$, is uniquely determined by its samples $x(nT_0)$, $n=0,\pm1,\pm2,..(T_0=1/f_0)$.

Let $p(t)$ denote the impulse train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_0) \tag{1}$$

Its Fourier transform $P(f)$ is given as follows.

$$P(f) = f_0 \sum_{k=-\infty}^{\infty} \delta(f-kf_0) \quad \text{where } f_0=1/T_0 \tag{2}$$

Introducing the sample train by

$$x_p(t) = x(t)\,p(t) = \sum_{n=-\infty}^{\infty} x(nT_0)\,\delta(t-nT_0) \tag{3}$$

we can obtain $X_p(f)$ according to the convolution theorem

$$X_p(f) = f_0 \sum_{k=-\infty}^{\infty} X(f-kf_0) \tag{4}$$

That is, Xp(f) is a periodic function of frequency, consisting of a sum of shifted replicas of X(f). So if X(f) is bandlimited (no overlap between the shifted replicas), it is possible to reproduce X(f) by lowpass filtering.

$$X(f) = W(f)\,X_p(f) \tag{5}$$

where

$$\begin{aligned} W(f) &= 1/f_0 \quad (|f| < f_0/2) \\ &= 0 \quad\quad (|f| \geqq f_0/2) \end{aligned} \tag{6}$$

and in the time domain

$$\begin{aligned} x(t) &= w(t) * x_p(t) \\ &= \int_{-\infty}^{\infty} w(\tau)\,x_p(t-\tau)d\tau \\ &= \int_{-f_0/2}^{f_0/2} w(\tau) \sum_{n=-\infty}^{\infty} x(nT_0)\,\delta(t-\tau-nT_0)d\tau \\ &= \sum_{n=-\infty}^{\infty} x(nT_0)\,w(t-nT_0) \end{aligned} \tag{7}$$

where

$$w(t) = T_0 \sin(\pi t/T_0)/\pi t \tag{8}$$

This means obtaining a window function $w(t)$, which reproduces $X(f)$ from $X_p(f)$, $x(t)$ can be calculated from its sampled data.

Interleaved Sampling

Let $p_a(t)$ be a shifted impulse train

$$p_a(t) = p(t-a) = \sum_{n=-\infty}^{\infty} \delta(t-nT_0-a) \tag{9}$$

From the shifting theorem, its transform may be given by

$$P_a(f) = f_0 \sum_{k=-\infty}^{\infty} e^{-i2\pi kf_0 a}\,\delta(f-kf_0) \tag{10}$$

Accordingly, the shifted sample train and its transform are given as follows

$$x_{pa}(t) = x(t)\,p_a(t) = \sum_{n=-\infty}^{\infty} x(nT_0+a)\,\delta(t-nT_0-a) \tag{11}$$

$$X_{pa}(f) = f_0 \sum_{k=-\infty}^{\infty} e^{-i2\pi kf_0 a}\,X(f-kf_0) \tag{12}$$

Reconstruction procedure

When aliasing occurs, it is impossible to reproduce $X(f)$ from (5). But it can be expressed in terms of $X_{pa_j}(f)$ as follows.

For simplicity, let N be even. It is supposed that $x(t)$ is bandlimited to $Nf_0/2$, that is

$$X(f) = 0 \quad \text{for } |f| \geqq Nf_0/2$$

We divide the interval $[-Nf_0/2, Nf_0/2]$ into N segments in the frequency domain . In each segment, (12) becomes

$$X_{pa}(f) = f_0 \sum_{k=m-N/2}^{m+N/2-1} e^{-i2\pi kf_0 a}\,X(f-kf_0) \tag{13}$$

$$(m-1)f_0 < f < mf_0$$
$$m = -N/2+1, \dots, N/2$$

which means $X_{pa}(f)$ can be expressed as a sum of N functions of $X(f)$ including a factor of $e^{-i2\pi kf_0 a}$.

We will explain the procedure for the case $N = 4$. $X(f)$ may be bandlimited to $2f_0$. The interval is divided into 4 segments $[-2f_0,-f_0]$, $[-f_0,0]$, $[0,f_0]$, $[f_0,2f_0]$. Let a_j ($j = 1,2,3,4$) be the delay time of j-th train of sampling data. It follows in the interval $[-2f_0,-f_0]$ ($m=-1$) from (13)

$$\begin{bmatrix} X_{pa_1}(f) \\ X_{pa_2}(f) \\ X_{pa_3}(f) \\ X_{pa_4}(f) \end{bmatrix} = f_0 \begin{bmatrix} b_1{}^{-3} & b_1{}^{-2} & b_1{}^{-1} & b_1{}^{0} \\ b_2{}^{-3} & b_2{}^{-2} & b_2{}^{-1} & b_2{}^{0} \\ b_3{}^{-3} & b_3{}^{-2} & b_3{}^{-1} & b_3{}^{0} \\ b_4{}^{-3} & b_4{}^{-2} & b_4{}^{-1} & b_4{}^{0} \end{bmatrix} \begin{bmatrix} X(f+3f_0) \\ X(f+2f_0) \\ X(f+f_0) \\ X(f) \end{bmatrix}$$

$$\text{where } b_j = e^{-i2\pi f_0 a_j} \tag{14}$$

If all a_j are given, we can obtain the inverse matrix and express $X(f)$ in terms of $X_{pa_j}(f)$.

$$X(f) = A_{-1,1}X_{pa_1}(f) + \dots + A_{-1,4}X_{pa_4}(f) \tag{15}$$

Note that $A_{m,j}$ is constant.

In the same manner, $X(f)$ in other segment can be expressed similarly.Considering the sum of all intervals in the frequency domain

$$X(f) = W_1(f)\,X_{pa_1}(f) + \dots + W_4(f)\,X_{pa_4}(f) \tag{16}$$

where

$$W_j(f) = 0 \; (|f| \geqq 2f_0)$$
$$W_j(f) = A_{m,j}$$
$$((m-1)f_0 < f < mf_0,\; m = -1,0,1,2)$$

and in the time domain

$$x(t) = w_1(t) * x_{pa_1}(t) + \dots + w_4(t) * x_{pa_4}(t) \tag{17}$$

Each convolution becomes

$$\begin{aligned} w_j(t) * x_{pa_j}(t) &= \int_{-2f_0}^{2f_0} w_j(\tau) \sum x(nT_0+a_j)\,\delta(t-\tau-nT_0-a_j)d\tau \\ &= \sum_{n=-\infty}^{\infty} x(nT_0+a_j)\,w_j(t-nT_0-a_j) \end{aligned} \tag{18}$$

where $x(nT_0+a_j)$ are the interleaved sampling data. $w_j(t)$ is obtained from

$$
\begin{aligned}
w_j(t) &= \int_{-\infty}^{\infty} W_j(f)\, e^{i2\pi ft} df \\
&= A_{1,j} \int_{-2f_0}^{-f_0} e^{i2\pi ft} df + \ldots + A_{2,j} \int_{f_0}^{2f_0} e^{i2\pi ft}\, df
\end{aligned}
\tag{19}
$$

In this way, $x(t)$, for any t, can be calculated.

Conclusion

We evaluated this metod using 16 trains of data, and it worked well. Since it is numerically necessary to truncate summation, when infinite summation is required, this error must be analyzed. The effect of distribution of delay times(a_j) also must be considered.

References

[1] D.A. Linden, "A Discussion of Sampling Theorems", Proc. IRE., vol. 47, p.1219, July 1959

[2] Ronald N. Bracewell, The Fourier Transform and Its Applications. MacGraw-Hill, 1986, pp.201-202.