1. **Background:**

Centrality is the measure of how central a particular node in a graph or network is (Freeman). An example demonstrating how central a node is would be if Tim knows Alex and Alex knows Jaxson, Alex would be more central than Tim and Jaxson because he knows both of them. There are many forms of centrality measure including simplistic measure such as degree, closeness, betweenness, and load. There are also more complicated measures that have been developed throughout history for other applications such as the Page Rank centrality measure.

In order to know if certain centrality measures are able to predict other centrality measures using different machine learning models, the centrality measures will have to be calculated on graphs that are generated a specified amount of times. These graphs feature scale-free, small-world, random, and scale-free small-world networks.

Scale-free graphs are graphs whose degree distribution follows the power law. Small-world graphs are graphs which have high clustering coefficients which means that nodes are typically in cliques but individual cliques don’t have a large amount of interactions. Scale-free small-world graphs are graphs which have all the properties of both scale-free and small-world graphs. Random graphs are graphs that randomly generate nodes and edges connecting the nodes with now regard to degree distribution or clustering coefficients.

An important thing to note is that the particular values of centrality measures are not necessarily important when comparing vertexes. The differences between the centrality values are not important so the vertices will be compared by examining how the centrality values rank against each other.

Below will be a discussion on the different centrality measures that are used in this research and an examination of related work.

**IIA. Degree Centrality:**

In network science, degree centrality has traditionally been considered to be the simplest measure of centrality and first item to look at when examining centrality (Opsahl et al.). Degree centrality can be defined as the ability for a node to receive information that is flowing through a network. This is measured by the number of links that node has to other nodes (Opsahl et al.).

The degree centrality for a vertex , for a given graph , where the graph can be defined as for vertices and edges is defined as where N is the is the total number of vertices. The definition of , the degree of vertex v. The degree of a vertex is the number of edges that are connected to the vertex (Opsahl et al.)

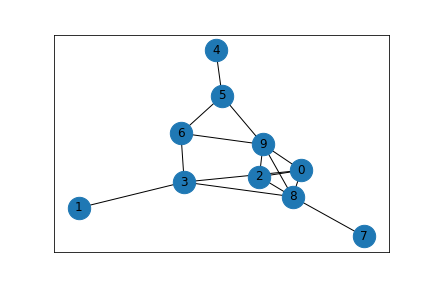


Fig. 1. A randomly generated graph with 10 vertices and 15 edges to demonstrate degree centrality.

In Figure 1, the vertices with the highest degree centrality would be vertices nine and eight with a degree centrality value of 0.5556. This value was calculated examining how many edges are connected to the vertex. By summing up those edges, then dividing by the one less than the total amount of nodes, the degree centrality for that vertex is obtained. The vertices with the lowest degree centrality are vertices one, seven, and four with a degree centrality of 0.1111.

**IIB. Closeness Centrality:**

Within graphs, a node is considered to have a high value of ‘closeness’ if it has a relatively low average of shortest path distance to all other nodes or equivalently, the inverse of the total distance (Brandes et al.). This is insinuating that a node with a high value is generally closer to all other nodes in the graph. To calculate this value, take one less than the number of nodes in a graph and divide it by the sum of the shortest path between a node and all other nodes in a graph.

The closeness centrality can be defined mathematically as follows. The closeness centrality for a vertex , for a given graph , where the graph can be defined as for vertices and edges is defined as where is the number of vertices in a graph and is the shortest path function between vertex y and x (Rochat).

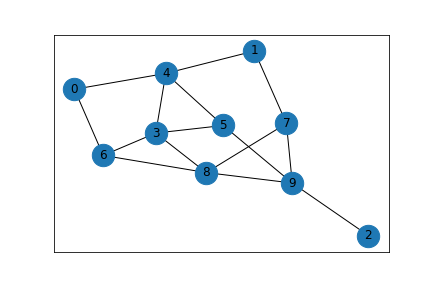


Fig. 2. A randomly generated graph with 10 vertices and 15 edges to demonstrate closeness centrality.

In Figure 2, the vertex with the highest closeness centrality is vertex eight with a value of 0.6429. This value was calculated using the formula above by dividing one less than the number of nodes by the sum of the shortest distance from that node and all other nodes in the graph. The vertex with the lowest closeness centrality is vertex two with a value of 0.3913.

**IIC. Betweenness Centrality:**

The betweenness centrality measure is the measure of how many times that a particular node occurs on the shortest path between two other nodes. This centrality measure is similar to the closeness centrality because both of them involve the calculation of the shortest path between nodes. To calculate this value, take the sum of the shortest paths between nodes s and t that pass-through node v then divide the number of shortest paths between nodes s and t. (Brandes, "Maintaining the duality of closeness and betweenness centrality.").

The betweenness centrality can be defined mathematically as follows. The closeness centrality for a vertex , for a given graph , where the graph can be defined as for vertices and edges is defined as where is the number of shortest paths between s and t given they contain vertex v and is the number of shortest paths between s and t (Brandes, "Maintaining the duality of closeness and betweenness centrality.").

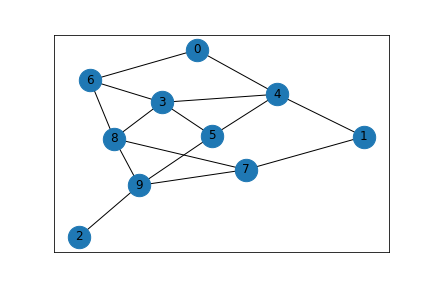


Fig. 3. A randomly generated graph with 10 vertices and 15 edges to demonstrate betweenness centrality.

Based on the graph depicted in Figure 3, the vertex with the highest betweenness centrality value is vertex nine with a value of 0.2639. This value was found by summing the shortest path between two nodes that involve a node then divide it by all shortest paths between two nodes. The vertex with the lowest centrality rating is vertex two with a value of 0.0000.

**IID. Load Centrality:**

The load centrality measure is similar to the betweenness centrality in that it measures the amount of flow that goes through a particular node; however, the load centrality measures the unit amount of information that get split between other nodes. Information is continually split between adjacent nodes until the target is reached. The total amount of information that passes through the node is defined as its load (Brandes, "On variants of shortest-path betweenness centrality and their generic computation.").

The betweenness centrality can be defined mathematically as follows. The closeness centrality for a vertex , for a given graph , where the graph can be defined as for vertices and edges is defined as where N is the number of vertices, is the vertex, and is defined to be 0.80. (Goh).

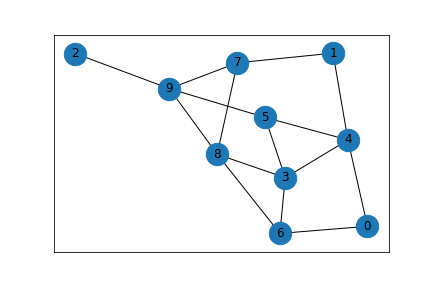


Fig. 4. A randomly generated graph with 10 vertices and 15 edges to demonstrate load centrality.

Based on the mathematical formula defined above, the graph depicted in Figure 4, and the implementation of this centrality calculation in the network python library, the node with the highest level of load is node nine with a value of 0.2639 and the node with the lowest level of load is node two with a load centrality value of 0.

**IIE. Local Reaching Centrality:**

The local reaching centrality is the measure for a node and its proportion of all other nodes that are reachable in a graph of that particular node. This gives a fundamental assumption that all nodes that are reachable for a node are located in some finite distance away (Mones et al.).

The betweenness centrality can be defined mathematically as follows. The closeness centrality for a vertex , for a given graph , where the graph can be defined as for vertices and edges is defined as where is the distance formula and is the number of vertices in a graph (Mones et al.).

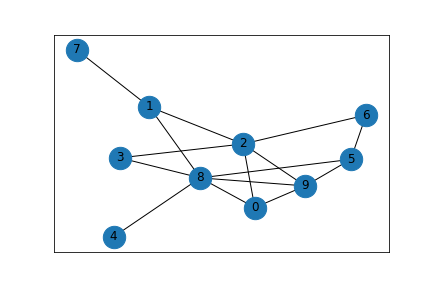


Fig. 5. A randomly generated graph with 10 vertices and 15 edges to demonstrate load centrality.

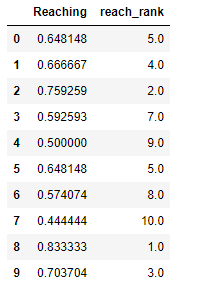


Table 1. A displaying showing the local reaching calculation and rank for all nodes depicted in figure 5.

As you can see in the results of the calculation of the graph in Figures 5 and Table 1, the node with the highest reaching is node 8 and the node with the lowest reaching is node 7. It is interesting to note that node 7 and node 4 are very removed from the graph as they have a degree of one and they both have the lowest reach rank.

**IIF. Harmonic Centrality:**

The harmonic centrality measure is similar to the closeness centrality however it addresses the issues of unreachable nodes. The harmonic difference will correct the issues with the average shortest path measure because disconnected nodes can have a potentially misleading value because the average distance could be low if the graph is almost entirely disconnected (Boldi and Vigna).

The harmonic centrality can be defined mathematically as follows. The closeness centrality for a vertex , for a given graph , where the graph can be defined as for vertices and edges is defined as where is the shortest average distance function (Boldi and Vigna).

Below is an example of a randomly generated network and the betweenness centrality associated with some of the nodes will be demonstrated.

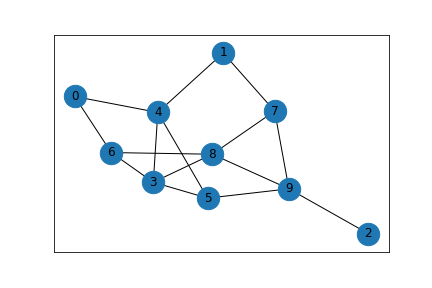


Fig. 6. A randomly generated graph with 10 vertices and 15 edges to demonstrate harmonic centrality.



Table 3. Correlation table for closeness rank and harmonic rank for the graph depicted in figure 6.

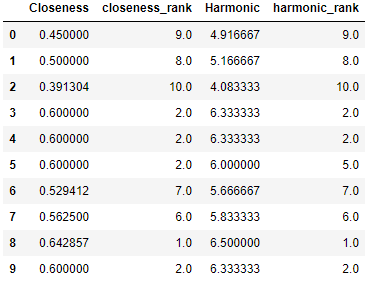


Table 2. Pandas DataFrame with closeness centrality data and harmonic centrality data for the graph depicted in figure 6.

The graph depicted in Figure 6 is a simple graph with all of the nodes connected to one another. So, it should be expected that the results of calculating the harmonic centrality to be similarly ranked as the closeness centrality ranking. In fact, there is an expectation for harmonic centrality to be closely correlated to closeness centrality (Boldi and Vigna). As you can see in Table 3, the closeness ranks, and the harmonic ranks are very close together and the nodes typically have the same rank and show a strong correlation as shown in Table 3.

**IIG. Page Rank:**

The page rank centrality measure has many applications including being the basis for how Google designed its search function (Page et al.). Page rank will rank websites based on the quality of websites that reference that particular website. In terms of networks, page rank will work in a very similar sense as it works in populating a search result. Page rank can be calculated iteratively and will return a probability that a node is accessed via another link. The page rank value at any particular time can be shown as where is the node that is accessed. The page rank for a vertex , for a given graph , where the graph can be defined as for vertices and edges is defined as where is the number of links from a node and c is a factor for normalization. (Page). Although this is a simplified definition of the page rank calculation, it should suffice for our simplified network application rather than ranking web pages for a search engine (Page et al.).

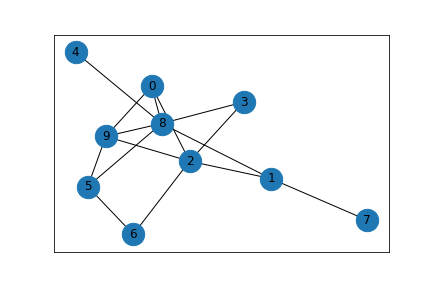


Fig. 7. A randomly generated graph with 10 vertices and 15 edges to demonstrate betweenness centrality.

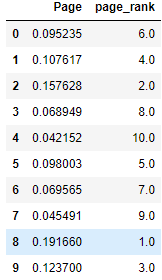


Table 4. A randomly generated graph with 10 vertices and 15 edges to demonstrate betweenness centrality.

The networkx python library provides a page rank function that will be calculated on a specific graph. For the graph in Figure 7, the table in Table 4 shows the rank and page rank value for all of the nodes. The node with the highest page rank value is node 8 and the node with the lowest page rank value is node 4.

1. **References:**

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