Homework 0

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Question 1 (Calculus Review)

(a). $\frac{\partial f}{\partial x} = 2a_1xy^2 + a_4y + a_5$ $\qquad \frac{\partial^2 f}{\partial y^2} = 2a_1x^2$

$\frac{\partial f}{\partial y} = 2a_1x^2y + a_4x$ $\qquad \frac{\partial^2 f}{\partial x \partial y} = 4a_1xy + a_4$

$\frac{\partial^2 f}{\partial x^2} = 2a_1y^2$

(b) $\frac{\partial f}{\partial x} = 2a_1xy^2 + 2a_2xy + a_3y^2 + a_4y + a_5$

$\frac{\partial f}{\partial y} = 2a_1x^2y + a_2x^2 + 2a_3xy + a_4x + a_6$

$\frac{\partial^2 f}{\partial x^2} = 2a_1y^2 + 2a_2y$

$\frac{\partial^2 f}{\partial y^2} = 2a_1x^2 + 2a_3x$

$\frac{\partial^2 f}{\partial x \partial y} = 4a_1xy + 2a_2x + 2a_3y + a_4$

(c) $\sigma'(x) = \frac{\partial \sigma}{\partial x} = \frac{e^{-x}}{(1+e^{-x})^2}$

$\sigma(x)(1-\sigma(x)) = \frac{1}{1+e^{-x}} \cdot (1 - \frac{1}{1+e^{-x}}) = \frac{1}{1+e^{-x}} \cdot \frac{e^{-x}}{1+e^{-x}} = \frac{e^{-x}}{(1+e^{-x})^2}$

$\sigma'(x) = \frac{\partial \sigma}{\partial x} = \sigma(x)(1-\sigma(x))$

(d) $\frac{\partial y_1}{\partial x} = 8x - 3$ $\qquad \frac{\partial y_2}{\partial x} = 12x^3 - 6x^2$

$8x - 3 = 0$ $\qquad \cancel{12x}$ $12x^3 - 6x^2 = 0$

$x = \frac{3}{8}$ $\qquad x_1 = 0 \quad x_2 = \frac{1}{2}$

$\frac{\partial^2 y_1}{\partial x^2} = 8 > 0$ $\qquad \frac{\partial^2 y_3}{\partial x^2} = 36x^2 - 12x$

so no local maximum point $\qquad y''_{(x=0)} = 0$

local minimum point is $(\frac{3}{8}, \frac{39}{16})$ $\qquad y''_{(x=\frac{1}{2})} = 3 > 0$

$4 \cdot (\frac{3}{8})^2 - 3 \cdot \frac{3}{8} + 3 = \frac{39}{16}$ $\qquad$ so no local maximum point

local minimum point is $(\frac{1}{2}, -\frac{1}{16})$

$$\frac{\partial y_3}{\partial x} = 4 - \frac{1}{2\sqrt{1-x}}$$

$$4 - \frac{1}{2\sqrt{1-x}} = 0$$

$$x_1 = \frac{63}{64} \quad x_2 = 1$$

$$\frac{\partial y_3}{\partial x} > 0 \quad x < \frac{63}{64}$$

$$\frac{\partial y_3}{\partial x} < 0 \quad \frac{63}{64} < x < 1$$

so local maximum point is $\left(\frac{63}{64}, \frac{65}{16}\right)$

local minimum point is $(1, 4)$

$$\frac{\partial y_4}{\partial x} = 1 - \frac{1}{x^2}$$

$$1 - \frac{1}{x^2} = 0$$

$$x = \pm 1$$

$$\frac{\partial^2 y_4}{\partial x^2} = \frac{1}{x^3}$$

if $x = 1$ $\frac{\partial^2 y_4}{\partial x^2} = 1 > 0$

$x = -1$ $\frac{\partial^2 y_4}{\partial x^2} = -1 < 0$

so local maximum point is $(-1, -2)$

local minimum point is $(1, 2)$

Question 2 (Probabiity Review)

(a) $P(A) = 20\% + 10\% = 30\%$

$P(B) = 1 - 20\% - 10\% - 40\% + 10\% = 40\%$

$P(A \cup B) = 1 - 40\% = 60\%$

$P(\bar{A}\bar{B}) = 1 - P(A \cup B) = 1 - 60\% = 40\%$

(b) (i) $r = 1 - \frac{1}{6} - \frac{1}{6} - \frac{1}{12} - \frac{1}{12} - \frac{1}{6} = \frac{1}{3}$

(ii) $P(X=2, Y=3) = \frac{1}{6}$

(iii) $P(X=3) = r = \frac{1}{3}$ $\quad P(X=3 \mid Y=2) = \frac{\frac{1}{3}}{\frac{1}{12} + \frac{1}{3}} = \frac{4}{5}$

(iv) $E(X) = 1 \times \left(\frac{1}{6} + \frac{1}{12} + \frac{1}{12}\right) + 2 \times \left(\frac{1}{6} + \frac{1}{6}\right) + 3 \times \frac{1}{3} = 2$

$E(Y) = 1 \times \left(\frac{1}{6} + \frac{1}{6}\right) + 2 \times \left(\frac{1}{12} + \frac{1}{3}\right) + 3 \times \left(\frac{1}{12} + \frac{1}{6}\right) = \frac{23}{12}$

$E(XY) = 1 \times \frac{1}{6} + 2 \times \left(\frac{1}{12} + \frac{1}{6}\right) + 3 \times \frac{1}{12} + 6 \times \left(\frac{1}{6} + \frac{1}{3}\right) = \frac{47}{12}$

(v) $E(X^2) = 1 \times \frac{1}{3} + 4 \times \frac{1}{3} + 9 \times \frac{1}{3} = \frac{14}{3}$

$E(Y^2) = 1 \times \frac{1}{3} + 4 \times \frac{5}{12} + 9 \times \frac{1}{4} = \frac{17}{4}$

(vi) $Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{47}{12} - 2 \times \frac{23}{12} = \frac{1}{12}$

(vii) $Var(X) = E(X^2) - (E(X))^2 = \frac{14}{3} - 4 = \frac{2}{3}$

$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{17}{4} - (\frac{23}{12})^2 = \frac{83}{144}$

(viii) $Corr(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{\frac{1}{12}}{\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{83}{144}}} = \sqrt{\frac{3}{166}}$

(ix) $E(X+Y) = E(X)+E(Y) = 2 + \frac{23}{12} = \frac{47}{12}$

$E(X+Y^2) = E(X) + E(Y^2) = 2 + \frac{17}{4} = \frac{25}{4}$

$Var(X+Y) = Var(X) + Var(Y) + 2cov(X,Y) = \frac{2}{3} + \frac{83}{144} + 2 \cdot \sqrt{\frac{3}{166}}$

Question 3 (Linear Algebra Review)

(a) A 3×5 B 6×1 A^T 5×3

(b) (i) A 3×3 B 2×2 can not compute AB and BA

(ii) $AC = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} 7 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 21 & 14 & 14 \\ 20 & 10 & 10 \\ 56 & 28 & 28 \end{bmatrix}$

$CA = \begin{bmatrix} 7 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 31 & 39 & 40 \\ 10 & 12 & 12 \\ 18 & 18 & 16 \end{bmatrix}$

(iii) $AD = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 32 \\ 17 & 19 \\ 43 & 45 \end{bmatrix}$

D 3×2 A 3×3 can not compute DA

(iv) D 3×2 C 3×3 can not compute DC

$CD = \begin{bmatrix} 7 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 41 \\ 13 & 13 \\ 18 & 22 \end{bmatrix}$

$$P^T C = \begin{bmatrix} 4 & 4 & 1 \\ 2 & 6 & 3 \end{bmatrix} \begin{bmatrix} 7 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 38 & 18 & 18 \\ 32 & 18 & 18 \end{bmatrix}$$

(v) $Bu = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \end{bmatrix}$

u 2x1 B 2x2 can not compute uB

(vi) A 3x3 u 2x1 can not compute Au

(vii) $AV = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 31 \end{bmatrix}$

v 3x1 A 3x3 can not comput VA

(viii) B 2X2 V 3x1 can not compute AV+BV

(c) (i) $\|u\|_1 = 1+3 = 4$ $\quad \|u\|_\infty = \sqrt[\infty]{1^\infty + 3^\infty}$

$\|u\|_2 = \sqrt{1+3^2} = \sqrt{10}$

$\|u\|_2^2 = 10$

(ii) $\|v\|_1 = 2+4+1 = 7$ $\quad \|v\|_\infty = \sqrt[\infty]{2^\infty + 4^\infty + 1^\infty}$

$\|v\|_2 = \sqrt{2^2+4^2+1} = \sqrt{21}$

$\|v\|_2^2 = 21$

(iii) $\|v+w\|_1 = 3+2+3 = 8$

$\|v+w\|_2 = \sqrt{3^2+3^2+2^2} = \sqrt{22}$

$\|v+w\|_\infty = \sqrt[\infty]{3^\infty + 3^\infty + 2^\infty}$

(iv) $\|AV\|_2 = \sqrt{18^2+13^2+31^2} = \sqrt{1454}$

(d) $\langle u, v\rangle = 1+2 = 3$ $\quad \cos\langle u, v\rangle = \frac{3}{\sqrt{5}\cdot\sqrt{2}} = \frac{3}{\sqrt{10}}$ the angle is 18.4°

$\langle u, w\rangle = 0$ $\quad \cos\langle u, w\rangle = \frac{0}{\sqrt{5}\cdot\sqrt{2}} = 0$ the angle is 90°

$\langle v, w\rangle = -\frac{1}{2}$ $\quad \cos\langle v, w\rangle = \frac{-\frac{1}{4}}{\sqrt{2}\cdot\frac{5}{\sqrt{4}}}$ the angle is 108.4°

(e) If the dot product is positive, that means the angle is $0 \sim 90°$

If the dot product is zero, that means the angle is $90°$

If the dot product is negative, that means the angle is $90° \sim 180°$

(f) the inverse of A is $\begin{bmatrix} -\frac{1}{11} & \frac{3}{11} \\ \frac{4}{11} & -\frac{1}{11} \end{bmatrix}$

(g) the inverse of A^{-1} is $\begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$

(h) if $B = A^T A$

$$B^T = (A^T A)^T = A^T (A^T)^T = A^T A = B$$

so $A^T A$ is symmetric