Ex 8.1

Apparently $f(z) = e^{iz^2}$ is holomorphic.

$$\int_{\Gamma_R} f(z)\,dz = \int_0^R e^{it^2}\cdot 1\,dt + \int_0^{\frac{\pi}{4}} e^{i(Re^{it})^2}\cdot iRe^{it}\,dt$$

$$+ \int_R^0 e^{i(te^{\frac{\pi}{4}i})^2}\cdot e^{i\frac{\pi}{4}}\,dt$$

$= 0$ since Γ_R is a closed toy contour.

$$\int_0^R e^{it^2}\cdot 1\,dt = \int_0^R \cos t^2\,dt + \int_0^R i\cdot\sin t^2\,dt$$

$$\int_0^{\frac{\pi}{4}} e^{i(Re^{it})^2}\cdot iRe^{it}\,dt = \int_0^{\frac{\pi}{4}} e^{iR^2\cos 2t - R^2\sin 2t + it}\cdot iR$$

$$= \int_0^{\frac{\pi}{4}} \frac{iR\cos[R^2\cos 2t + t] - R\sin[R^2\cos 2t + t]}{e^{R^2\sin 2t}}\,dt$$

$$\left|\frac{iR\cos[R^2\cos 2t + t] - R\sin[\cos 2t R^2 + t]}{e^{R^2\sin 2t}}\right| = 0 \text{ when } R \to \infty$$

$$\int_R^0 e^{i(te^{\frac{\pi}{4}i})^2}\, e^{i\frac{\pi}{4}}\,dt$$

$$= \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\int_R^0 e^{-t^2}\,dt \xrightarrow{R\to\infty} \left(\frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4}\right)$$

so when $R \to \infty$,

$$\int_{\Gamma_R} f(z)\,dz = \left(\int_0^\infty \cos t^2\,dt - \frac{\sqrt{2\pi}}{4}\right) + \left(\int_0^\infty \sin t^2\,dt - \frac{\sqrt{2\pi}}{4}\right) i$$

$$= 0$$

so $\int_0^\infty \cos t^2\,dt = \int_0^\infty \sin t^2\,dt = \frac{\sqrt{2\pi}}{4}$

Ex 8.2

Apparently $f(z) = \frac{e^{iz}-1}{2iz}$ is holomorphic.

so $\int_\varepsilon^R \frac{e^{it}-1}{2it}\,dt + \int_{-R}^{-\varepsilon} \frac{e^{it}-1}{2it}\,dt + \int_{C_R} f(z)\,dz + \int_{-C_\varepsilon} f(z)\,dz$

$= 0$, as it's a closed toy contour

$$\int_\varepsilon^R \frac{e^{it}-1}{2it}\,dt + \int_{-R}^{-\varepsilon} \frac{e^{it}-1}{2it}\,dt$$

$$= \int_\varepsilon^R \left[\frac{1}{2i}\cdot\frac{\cos t}{t} + \frac{1}{2}\frac{\sin t}{t} - \frac{1}{2it}\right] dt - \int_{-\varepsilon}^{-R} \left[\frac{1}{2i}\frac{\cos t}{t} + \frac{1}{2}\frac{\sin t}{t} - \frac{1}{2it}\right] dt$$

$= \int_\varepsilon^R \frac{\sin t}{t}\,dt = \int_0^\infty \frac{\sin t}{t}\,dt$ as $\varepsilon \to 0$, $R \to \infty$

$$\int_{C_R} f(z)\,dz = \int_0^{\pi} \frac{e^{i(Re^{it})}-1}{2iRe^{it}} \cdot iRe^{it}\,dt$$

$$= \int_0^{\pi} \frac{e^{i\cos t}}{2e^{R\sin t}} - \frac{1}{2}\,dt$$

$$\left|\frac{e^{i\cos t}}{2e^{R\sin t}}\right| = \frac{1}{2e^{R\sin t}} \xrightarrow{R\to\infty} 0$$

so as $R \to \infty$, $\int_{C_R} f(z)\,dz = -\frac{\pi}{2}$

$$\int_{C_\varepsilon} f(z)\,dz = \int_{-\pi}^{0} \frac{e^{i\cos t}}{2e^{\varepsilon\sin t}} - \frac{1}{2}\,dt$$

$$\xrightarrow{\varepsilon\to 0} 0$$

so $\int_0^{\infty} \frac{\sin t}{t}\,dt - \frac{\pi}{2} + 0 + 0 = 0$

$$\int_0^{\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2}$$