DIVISION ALGEBRAS AND THE PICARD NUMBER OF A RAMIFIED CYCLIC COVERING

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1. Introduction and Statement of Problem

Let k denote a field in which n is invertible, and assume k contains ζ, a primitive nth root of unity. Let $A = k[x_1,\dots,x_m]$ be the affine coordinate ring of $\mathbb{A}_k^m$ and $K = k(x_1,\dots,x_m)$ the field of rational functions. Given an irreducible polynomial f in A we consider the affine variety in $\mathbb{A}_k^{m+1} = \operatorname{Spec} k[x_1,\dots,x_m,x]$ defined by the equation $z^n = f$. Let $T = A[z]/(z^n - f)$, $R = A[f^{-1}]$, and $S = T[z^{-1}]$. Then T is a ramified cyclic extension of A, and S is a Galois extension of R. Identifying z with $\sqrt[n]{f}$, the quotient field of T (and S) is $L = K(z)$ and L/K is a Kummer extension with cyclic Galois group. Let σ denote the K-algebra automorphism of $L = K(\sqrt[n]{f})$ defined by $z =\mapsto -z$. Let $G = \{1,\sigma\}$ be the cyclic group generated by σ. Then G is a group of A-automorphisms of T, a group of R-automorphisms of S, and a group of K-automorphisms of $K(z)$. The rings together with their quotient fields appear in the following commutative diagram.

$$\begin{tikzcd}
T = A[\sqrt[n]{f}] \arrow[r] & S = R[\sqrt[n]{f}] \arrow[r] & L = K(\sqrt[n]{f}) \\
A \arrow[u] \arrow[r] & R = A[f^{-1}] \arrow[u] \arrow[r] & K \arrow[u]
\end{tikzcd} \tag{1}$$

This article studies connections between K-division algebras and divisor classes on the affine varieties $\operatorname{Spec} T$ and $\operatorname{Spec} S$. Arithmetic in the Brauer group of K is exploited to study the Picard group $\operatorname{Pic} S$ and the class group $\operatorname{Cl}(T)$. We give sufficient conditions on f such that the Picard group $\operatorname{Pic} S$ is nontrivial. For many examples, the Picard numbers are computed. Associated to the Galois extension S/R is the so-called seven term exact sequence of Chase, Harrison and Rosenberg:

$$\begin{aligned}(2)\quad 1 \to \mathrm{H}^1(G,S^*) \xrightarrow{\alpha_1} \operatorname{Pic}(R) \xrightarrow{\alpha_2} (\operatorname{Pic} S)^G \xrightarrow{\alpha_3} \\ \mathrm{H}^2(G,S^*) \xrightarrow{\alpha_4} \mathrm{B}(S/R) \xrightarrow{\alpha_5} \mathrm{H}^1(G,\operatorname{Pic} S) \xrightarrow{\alpha_6} \mathrm{H}^3(G,S^*)\end{aligned}$$

[1, Corollary 5.5] or [9, Theorem 13.3.1]. Since A and $R = A[1/f]$ are factorial, $\operatorname{Pic} A = \operatorname{Pic} R = 0$. Since G is cyclic, [10, Theorem 8.5.20] and the exact sequence (2) imply that $\mathrm{H}^i(G,S^*) = \langle 1\rangle$ for $i = 1,3,\dots$. In our context, (2) reduces to the exact sequence

$$\langle 1\rangle \to (\operatorname{Pic} S)^G \xrightarrow{\alpha_3} \mathrm{H}^2(G,S^*) \xrightarrow{\alpha_4} \mathrm{B}(S/R) \xrightarrow{\alpha_5} \mathrm{H}^1(G,\operatorname{Pic} S) \to \langle 1\rangle \tag{3}$$

In Section 2 below, the goal is to derive sufficient conditions on n and f such that there exist nontrivial elements in the image of α_5. In Section 3, we derive sufficient conditions

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on n and f such that there exists a homomorphism

$$\mathrm{B}(S/R) \xrightarrow{\gamma_5} \mathrm{H}^1(G, \mathrm{Cl}(T)) \tag{4}$$

and for any $N > 0$, the image of γ_5 contains a subgroup of order N or greater.

2. Double Covers

In this section we continue to use the notation established above, with some modifications. The cyclic covering T/A is assumed to be quadratic. Thus, $n = 2$, and $L = K(\sqrt{f})$. The varieties are surfaces, thus $m = 2$, and we write $A = k[x,y]$. The polynomial f is always square-free, but not necessarily irreducible. Let $f = f_1 \cdots f_n$ be the factorization of f into irreducibles in the factorial ring A. The group of units of R is equal to $k^* \times \langle f_1 \rangle \times \cdots \langle f_n \rangle$, which is isomorphic to $k^* \times \mathbb{Z}^{(n)}$. By the Kummer sequence, $\mathrm{H}^1(R, \mu_2) \cong (\mathbb{Z}/2)^{(n)}$. Since $\mathrm{H}^1(R, \mu_2)$ classifies the étale double covers of R, we view S as a representative of the class $[S]$ in $\mathrm{H}^1(R, \mu_2)$ corresponding to $f = f_1 \cdots f_n$. Fixing $[S]$ in one factor of the cup product $\smile$: $\mathrm{H}^1(R, \mu_2) \times \mathrm{H}^1(R, \mu_2) \to \mathrm{H}^2(R, \mu_2)$ [18, p. 172] and following with the Kummer theory map $\mathrm{H}^2(R, \mu_2) \to {}_2\mathrm{B}(R)$, we have a homomorphism $(\cdot) \smile [S] : \mathrm{H}^1(R, \mu_2) \to {}_2\mathrm{B}(R)$. The image of $(\cdot) \smile [S]$ is denoted by $\mathrm{B}^{\smile}(S/R)$. If we pass to the quotient fields, $K \to K(\sqrt{f})$, every element of the Brauer group $\mathrm{B}(K)$ split by $K(\sqrt{f})$ is a cyclic crossed product, hence is in the image of the cup product map. In this sense, the classes of Azumaya algebras in $\mathrm{B}^{\smile}(S/R)$ represent the obvious elements in $\mathrm{B}(S/R)$. The short exact sequence of Theorem 2.1(a) is a special case of (2).

Theorem 2.1. *In the notation established above, the following are true.*

(a) There is an exact sequence of abelian groups

$$0 \to \mathrm{B}^{\smile}(S/R) \to \mathrm{B}(S/R) \xrightarrow{\alpha_5} \mathrm{Pic}\, S \otimes \mathbb{Z}/2 \to 0.$$

(b) The restriction-corestriction sequence

$$0 \to \mathrm{B}(S/R) \to {}_2\mathrm{B}(R) \xrightarrow{\mathrm{res}^2} {}_2\mathrm{B}(S) \xrightarrow{\mathrm{cor}^2} {}_2\mathrm{B}(R) \to 0$$

is exact.

(c) The $\mathbb{Z}/2$-rank of $\mathrm{Pic}\, S \otimes \mathbb{Z}/2$ is less than or equal to the $\mathbb{Z}/2$-rank of ${}_2\mathrm{B}(R)$.

Proof. [6, Theorem 2.1] and its proof. □

Theorem 2.2. *In the notation established above, assume f is irreducible. The following are true.*

(a) $\mathrm{B}^{\smile}(S/R) = \langle 0 \rangle$.

(b) $\alpha_5 : \mathrm{B}(S/R) \cong \mathrm{Pic}\, S \otimes \mathbb{Z}/2$.

(c) $\dim_{\mathbb{Z}/2} \mathrm{H}^1(S, \mu_2) = \dim_{\mathbb{Z}/2} \mathrm{H}^1(R, \mu_2) = 1$.

(d) $\dim_{\mathbb{Z}/2} \mathrm{H}^2(S, \mu_2) = 2 \dim_{\mathbb{Z}/2} \mathrm{H}^2(R, \mu_2)$.

(e) For all $i > 0$, $\mathrm{H}^i(G, S^*) = \langle 1 \rangle$.

(f) $(\mathrm{Pic}\, S)^G = \langle 0 \rangle$.

Proof. [6, Theorem 2.8] □

Proposition 2.3. *If I is a prime ideal of S of height one, then I is a free R-module of rank two. There exist elements a, b in I such that $I = aS + bS$.*

Proof. Let I be a height one prime ideal in S. Then I is a rank one reflexive module and because S is non-singular, I is a rank one projective S-module (for example, [10, Theorem 12.6.9] or [13, Corollary II.6.16]). Since S is a free R-module of rank two, it follows that I is a projective R-module of rank two. By [19], the R-module I decomposes into a direct sum of two rank one projective modules. Since $\operatorname{Pic} R = 0$, it follows that I is a free R-module. $\square$

2.1. **Motivational Examples.**

Example 2.4. Let $f = f_1 f_2 f_3 f_4 \in k[x,y]$, where f_1, f_2, f_3, f_4 are four linear polynomials in general position. Let $R = k[x,y][f^{-1}]$, $S = R[\sqrt{f}]$. Using [4, Theorem 4], we see that ${}_2\mathrm{B}(R) = (\mathbb{Z}/2)^{(6)}$ and a basis consists of the symbol algebras $\{(f_i, f_j)_2 \mid i < j\}$. The group $B^{\smile}(S/R)$ is the subgroup of ${}_2\mathrm{B}(R)$ generated by $\{(f, f_i)_2 \mid 1 \le i \le 4\}$. One computes that $\mathrm{B}^{\smile}(R)$ is a group of order 2^3. Let $F_i = Z(f_i)$ be the line defined by $f_i = 0$. Let $P_{12} = F_1 \cap F_2$ and $P_{34} = F_3 \cap F_4$. Let ℓ be the linear equation of the line L through P_{12} and P_{34}. Let $\Lambda = (f, \ell)_2$. As in [4, Theorem 4], one computes

$$
\begin{aligned}
(f,\ell)_2 &\sim (f_1 f_2, \ell)_2 (f_3 f_4, \ell)_2 \\
&\sim (f_1, f_2)_2 (f_3, f_4)_2
\end{aligned}
\tag{5}
$$

is in $\mathrm{B}(S/R)$ and not in $\mathrm{B}^{\smile}(S/R)$. By Theorem 2.1, $\alpha_5(\Lambda)$ represents a non-trivial element of $\operatorname{Pic}(S) \otimes \mathbb{Z}/2$.

Example 2.5. As in Example 2.4, let f_1, f_2, f_3, f_4 be four linear polynomials in general position. Let $F_i = Z(f_i)$ be the line defined by $f_i = 0$. Let $P_{12} = F_1 \cap F_2$, $P_{34} = F_3 \cap F_4$, and let ℓ be the linear equation of the line L through P_{12} and P_{34}. Let F_0 be the line at infinity and let P_{05} be the point $F_0 \cap L$. Let F_5 be a line through P_{05} which is in general position with respect to F_1, F_2, F_3, F_4, L. Let $f = f_1 f_2 f_3 f_4 f_5$, $R = k[x,y][f^{-1}]$, and $S = R[\sqrt{f}]$. Then ${}_2\mathrm{B}(R) = (\mathbb{Z}/2)^{(10)}$ and a basis consists of the symbol algebras $\{(f_i, f_j)_2 \mid i < j\}$. The group $B^{\smile}(S/R)$ is the subgroup of ${}_2\mathrm{B}(R)$ generated by $\{(f, f_i)_2 \mid 1 \le i \le 5\}$. One computes that $\mathrm{B}^{\smile}(S/R)$ is a group of order 2^4. Let $\Lambda = (f, \ell)_2$. One computes

$$
\begin{aligned}
(f,\ell)_2 &\sim (f_1 f_2, \ell)_2 (f_3 f_4, \ell)_2 \\
&\sim (f_1, f_2)_2 (f_3, f_4)_2
\end{aligned}
\tag{6}
$$

is in $\mathrm{B}(S/R)$ and not in $\mathrm{B}^{\smile}(S/R)$. By Theorem 2.1, $\alpha_5(\Lambda)$ represents a non-trivial element of $\operatorname{Pic}(S) \otimes \mathbb{Z}/2$.

Example 2.6. Pick a linear polynomial $\ell \in k[x,y]$, and let $L = Z(\ell)$ be the line in $\mathbb{A}^2$ defined by ℓ. Generalizing Example 2.5, a large class of f are presented such that $G = Z(\ell)$ is split by $R[\sqrt{f}]$. Let $m \ge 2$ and pick distinct points $P_1, \dots, P_m$ on L. Let $F_1, \dots, F_{2m}$ be general lines in $\mathbb{A}^2$ satisfying $P_i \in F_{2i-1} \cap F_{2i}$. Let $f_j = 0$ be the linear equation for F_j and set $f = f_1 f_2 \cdots f_{2m}$. Let $R = k[x,y][f^{-1}]$ and $S = R[\sqrt{f}]$. Then ${}_2\mathrm{B}(R) = (\mathbb{Z}/2)^{(r)}$ where $r = 1 + 2 + \cdots + (2m-1)$ and a basis consists of the symbol algebras $\{(f_i, f_j)_2 \mid i < j\}$. The group $B^{\smile}(S/R)$ is the subgroup of ${}_2\mathrm{B}(R)$ generated by $\{(f, f_j)_2 \mid 1 \le j \le 2m-1\}$. One computes that $\mathrm{B}^{\smile}(S/R)$ is a $\mathbb{Z}/2$-module of rank $2m-1$. Let $\Lambda = (f, \ell)_2$. One computes

$$
\begin{aligned}
(f,\ell)_2 &\sim (f_1 f_2, \ell)_2 (f_3 f_4, \ell)_2 \cdots (f_{2m-1} f_{2m}, \ell)_2 \\
&\sim (f_1, f_2)_2 (f_3, f_4)_2 \cdots (f_{2m-1}, f_{2m})_2
\end{aligned}
\tag{7}
$$

is in $\mathrm{B}(S/R)$ and not in $\mathrm{B}^{\smile}(S/R)$. By Theorem 2.1, $\alpha_5(\Lambda)$ represents a non-trivial element of $\operatorname{Pic}(S) \otimes \mathbb{Z}/2$.

2.2. **Division Algebras over K and Primes of S.** As in diagram (1), $A = k[x,y]$, f is square-free, $T = A[z]/(z^2 - f)$, $R = A[f^{-1}]$ and $S = T[z^{-1}]$. Let $\pi : \operatorname{Spec} T \to \operatorname{Spec} A$ be the corresponding morphism of surfaces. Since R and S are regular surfaces, $\mathrm{B}(S/R) \to \mathrm{B}(L/K)$ is one-to-one. An element of $\mathrm{B}(S/R)$ is represented by a central K-division algebra $\Lambda \in \mathrm{B}(L/K)$ and the ramification divisor of Λ is contained in $F = Z(f)$. By the crossed product theorem, the division algebra Λ is a symbol $(f,h)_2$ for some h in K^* (for instance, see [20, Corollary 7.11]). Since h is unique up to norms from L^*, we can assume h is a square-free element of A. Factoring h into irreducibles, the Brauer class of Λ is a product of classes of the form $(f,g)_2$, where g is an irreducible element of A. Denote by $C = Z(g)$ the irreducible curve on $\operatorname{Spec} A$ defined by g. Consider the divisor $\tilde{C} = \pi^{-1}(C)$ on $\operatorname{Spec} T$. The diagrams

$$\begin{tikzcd} \tilde{C} \arrow[r, "\subseteq"] \arrow[d, "\pi"'] & \operatorname{Spec} T \arrow[d, "\pi"] \\ C \arrow[r, "\subseteq"] & \operatorname{Spec} A \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r] & T/Tg \\ A \arrow[r] \arrow[u] & A/Ag \arrow[u] \end{tikzcd} \tag{8}$$

commute, where (8) shows the morphisms of varieties on the left, and the coordinate rings on the right.

Proposition 2.7. *As above, $\pi : \operatorname{Spec} T \to \operatorname{Spec} A$ is the affine double plane defined by $z^2 = f$, where $A = k[x,y]$ and $K = k(x,y)$. Assume g is irreducible in A and the K-symbol algebra $(f,g)_2$ ramifies only along $F = Z(f)$. If $C = Z(g)$, then $\tilde{C} = \pi^{-1}(C)$ is not irreducible. The curve $\tilde{C}$ is reducible with only one irreducible component if and only if g divides f. Otherwise $\tilde{C} = \tilde{C}_1 + \tilde{C}_2$ is reducible and has two irreducible components.*

Proof. If g divides f, then any prime of T containing g also contains z. In this case, g has a unique minimal prime in T, namely $P = (g,z)$. In the local ring T_P, the element g has valuation 2. This shows $\operatorname{Div}(g) = 2P$. So $\tilde{C}$ is reducible with only one irreducible component. Note that in this case, $(f,g)_2$ is in $\mathrm{B}^{\smile}(S/R)$.

Now assume g does not divide f. Then g is irreducible in $R = A[f^{-1}]$. Let Q denote the prime ideal Rg in R. The field $K(C) = R_Q/QR_Q$ is the function field of C. Because $S = T \otimes_A R$ is Galois over R, $S \otimes_R K(C)$ is separable of degree two over $K(C)$. Either $S \otimes_R K(C)$ is a field, or a direct sum of two copies of $K(C)$ (for example, see [10, Corollary 5.5.9] or [15, Proposition III.4.1]). If $S \otimes_R K(C)$ is a field, then Sg is a prime ideal in S, so $\tilde{C}$ is irreducible. In this case, the ramification of $(f,g)_2$ along the divisor C is the non-zero class of $S \otimes_R K(C)$ in $\mathrm{H}^1(K(C), \mu_2)$. This case does not arise because we are assuming $(f,g)_2$ is unramified along C.

The last possibility is that $S \otimes_R K(C)$ is a direct sum of two copies of $K(C)$. In this case there are two minimal primes of Sg. Let P be one of them. The other is necessarily $\sigma(P)$ (for example [10, Theorem 6.3.6] or [17, (5.E), Theorem 5]). Because the residue fields of R_Q and S_P are equal, the image of QR_Q generates the maximal ideal of S_P. This means g is a local parameter for S_P. The divisor of g on $\operatorname{Spec} S$ is $P + \sigma(P)$. □

In Proposition 2.8 we prove a partial converse to Proposition 2.7. If $C = Z(g)$ splits over S into $\tilde{C} = \tilde{C}_1 + \tilde{C}_2$ where $\tilde{C}_1$ and $\tilde{C}_2$ are disjoint, then the K-symbol algebra $(f,g)_2$ is shown to represent a Brauer class in the image of $\mathrm{B}(R) \to \mathrm{B}(K)$.

Proposition 2.8. *In the context of Proposition 2.7, suppose $g \in R$ is irreducible and that $S/(g)$ is isomorphic to the direct sum of two copies of $R/(g)$.*

	$(1,0)$	$(z,0)$	$(0,g)$	$(0,z-h)$
$(1,0)$	$(1,0)$	$(z,0)$	$(0,g)$	$(0,z-h)$
$(z,0)$	$(z,0)$	$(f,0)$	$(0,zg)$	$(0,z(z-h))$
$(0,g)$	$(0,g)$	$(0,-zg)$	$(g,0)$	$(-z-h,0)$
$(0,z-h)$	$(0,z-h)$	$(0,-z(z-h))$	$(z-h,0)$	$(-u,0)$

TABLE 1. Multiplication table for $\Delta(I)$ in Proposition 2.8.

(a) *There is an element h in $R-(0)$ such that the minimal primes of g in S are $I=(g,z-h)$ and $\sigma(I)=(g,z+h)$.*
(b) *The symbol algebra $(f,g)_2$ over K represents a class ξ in $\mathrm{B}(S/R)$.*
(c) *The coset $\alpha_5(\xi)$ in $\operatorname{Pic} S\otimes\mathbb{Z}/2$ is represented by the ideal I.*

Proof. We are given that
$$\frac{S}{(g)}=\frac{(R/(g))\,[z]}{(z^2-f)}$$
is the trivial quadratic extension of $R/(g)$. This means f is a non-zero square in $R/(g)$. There exist u, h in $R-(0)$ such that $f=ug+h^2$. Look at the ideal $I=(g,z-h)$ in S. Since
$$S/I=\frac{k[x,y,z][f^{-1}]}{(g,z^2-f,z-h)}\cong\frac{k[x,y][f^{-1}]}{(g)}$$
we see that I is prime of height one. A typical element of S can be written in the form $a+b(z-h)$, for $a,b\in R$. If a,b,c,d are from R, then a typical element of I is
$$\begin{aligned}(a+b(z-h))g+(c+d(z-h))(z-h)&=ag+b(z-h)g+c(z-h)+d(z-h)^2\\&=ag+b(z-h)g+c(z-h)+d(z^2-h^2-2zh+2h^2)\\&=(a+du)g+(bg+c-2dh)(z-h)\end{aligned}$$
so $I=Rg+R(z-h)$. By Proposition 2.3, g, $z-h$ is a free R-basis for I. Since z is invertible in S, $I\sigma(I)=(g^2,g(z+h),g(z-h),ug)=Sg$. Let $\Delta(I)$ be the generalized cross product algebra, as defined in [6, §2.2]. Then $\Delta(I)$ is an Azumaya R-algebra which is split by S. As an R-module $\Delta(I)$ is generated by $(1,0)$, $(z,0)$, $(0,g)$, and $(0,z-h)$. Using equation [6, (16)], the multiplication table for $\Delta(I)$ is constructed in Table 1. Upon extending the ring of scalars to K, it is clear that $\Delta(I)\otimes_R K$ is isomorphic to the symbol algebra $(f,g)_2$. Therefore $(f,g)_2$ is unramified on $Z(g)$, represents a class ξ in $\mathrm{B}(S/R)$, and $\alpha_5(\xi)$ is represented by the divisor class of the ideal $I=(g,z-h)$. □

Suppose f and g are as in Proposition 2.7 and g does not divide f. If $C=Z(g)$ is rational and simply connected, then $\tilde{C}=\tilde{C}_1+\tilde{C}_2$ is reducible if and only if the local intersection multiplicity of C and F at each point is even [13, Corollary IV.2.4].

Proposition 2.9. *As always, $A=k[x,y]$ and $K=k(x,y)$. Suppose f and g are in A, f is square-free, g is irreducible, g does not divide f, and the K-symbol algebra $(f,g)_2$ is unramified along each prime divisor of $R=A[f^{-1}]$. If $C=Z(g)$ on $\operatorname{Spec} R$ is either nonsingular, or has only unibranched singularities, then $S/(g)$ is isomorphic to a direct sum of two copies of $R/(g)$.*

Proof. We are in the context of the paragraph preceding Proposition 2.7. Let $\Lambda=(f,g)_2$. The ramification $a_C(\Lambda)$ along C is given by the tame symbol. But R is factorial and g is irreducible. Therefore $a_C(\Lambda)$ is the quadratic extension $K(C)[z]/(z^2-f)$, which by assumption represents the zero class in $\mathrm{H}^1(K(C),\mathbb{Z}/2)$. Let $\bar{C}$ denote the normalization of C. Because C has at most unibranched singularities, the natural map $\mathrm{H}^1(C,\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(\bar{C},\mathbb{Q}/\mathbb{Z})$ is an isomorphism. For any closed point $p \in \bar{C}$, the natural map

$$\mathrm{H}^1(\bar{C},\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(\bar{C}-p,\mathbb{Q}/\mathbb{Z})$$

is one-to-one by cohomological purity [18, Theorem VI.5.1]. By a direct limit argument, the natural map

$$\mathrm{H}^1(\bar{C},\mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(K(C),\mathbb{Q}/\mathbb{Z})$$

is one-to-one. Therefore, the unramified quadratic extension $S/(g)$ represents the zero class in $\mathrm{H}^1(C,\mathbb{Q}/\mathbb{Z})$. So $S/(g)$ is isomorphic to a direct sum of two copies of $R/(g)$. □

Example 2.10. This example shows that if the curve $R/(g)$ has a nodal singularity, the conclusion of Proposition 2.9 can fail. Let $f = x+1$, $T = k[x,y,z]/(z^2-f)$. Let $g = y^2-x^2(x+1)$. In T the element g factors into $(y-xz)(y+xz)$. Each factor is irreducible because the map $x \mapsto z^2-1$, $y \mapsto xz$ induces $T/(y-xz) \cong k[z]$. Since

$$\frac{T}{(y-xz,y+xz)} \cong \frac{k[z]}{(z(z^2-1))}$$

the elements $y-xz$ and $y+xz$ are not relatively prime, even in $S = T[z^{-1}]$. The conclusion of Proposition 2.9 is not satisfied. Now look at the symbol algebra $\Lambda = (f,g)_2$ over $K = k(x,y)$. Since $1 \sim (x+1,x)_2$, we have

$$\begin{aligned}\Lambda &\sim (x+1,x^{-2})_2(x+1,y^2-x^2(x+1))_2\\ &\sim (x+1,(y/x)^2-(x+1))_2\\ &\sim 1\end{aligned}$$

Therefore, $(f,g)_2$ is split, hence unramified over R.

2.3. **A Construction.** Suppose our goal is to construct a double plane $\operatorname{Spec} T \to \mathbb{A}^2$ with the property that the class group on the unramified set $\operatorname{Spec} S \subseteq \operatorname{Spec} T$ is non-trivial and easy to compute. An approach based on Theorem 2.1 is to find f such that we can compute elements that are in $\mathrm{B}(S/R)$ but not in $\mathrm{B}^{\smile}(S/R)$. The preceding examples provide some insight on how to pick elements f and g in A such that $(f,g)_2$ is in $\mathrm{B}(S/R)$ and not in $\mathrm{B}^{\smile}(S/R)$. Start with a sequence of distinct irreducible polynomials $f_1,\dots,f_N$ in $A = k[x,y]$, where $N \geq 3$. Put $f = f_1 f_2 \cdots f_j + (f_{j+1}\cdots f_N)^2$, for some j such that $2 \leq j < N$. If f is square-free, then z^2-f is irreducible and $T = A[z]/(z^2-f)$ is integrally closed. Let g be any one of $f_1,\dots,f_j$ and $h = f_{j+1}\cdots f_N$. By construction, g does not divide f. Let $R = A[f^{-1}]$. The map

$$\frac{(R/(g))[z]}{(z^2-h^2)} \xrightarrow{\beta} \frac{R}{(g)} \oplus \frac{R}{(g)} \tag{9}$$

is an isomorphism, where β maps $z \mapsto (h,-h)$. If $S = T[z^{-1}]$, then $S/(g)$ is isomorphic to the ring on the left hand side of (9). By Proposition 2.8, the symbol algebra $\Lambda = (f,g)_2$ ramifies only along the zeros of f. Also, the homomorphic image of $[\Lambda]$ under α_5 is the divisor class of the ideal $I = (g,z-h)$. Upon restriction to the quotient field $K = k(x,y)$, the symbol algebra $(f,g)_2$ is a division algebra if the ideal $I = (g,z-h)$ represents a non-trivial class in $\operatorname{Pic} S \otimes \mathbb{Z}/2$. The converse of this last statement is false, as shown in Example 2.6.

Example 2.11. This example is based on the construction of Section 2.3. Let ℓ_1, ℓ_2, ℓ_3 be three general linear polynomials in $k[x,y]$. Let $f = \ell_1\ell_2 - \ell_3^2$. We can assume f is irreducible. Let $F = Z(f)$, $L_i = Z(\ell_i)$, and F_0 the line at infinity. Let $L_1 \cdot L_3 = P_1$ and $L_3 \cdot F_0 = P_{03}$. We see that $F \cdot L_1 = 2P_1$. By a general position argument, we can assume $F_0 \cdot F = P_{01} + P_{02}$. For the symbol algebra $(f, \ell_1)_2$, the weighted path in the graph $\Gamma = \Gamma(F + L_1 + F_0)$ is shown in Figure 1. The cycle $F \to P_{01} \to F_0 \to P_{02} \to F$ is non-trivial.

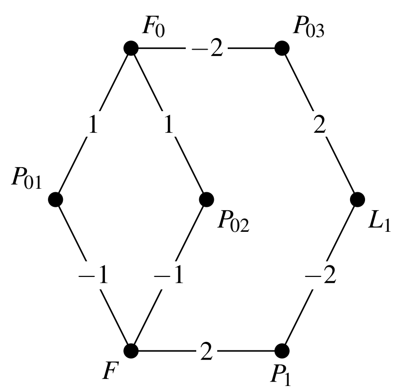

FIGURE 1. The weighted path of $(f,g)_2$ in Example 2.11.

The fact that the cycle in Figure 1 represents a non-trivial element in $\mathrm{H}_1(\Gamma, \mathbb{Z}/2)$ proves that $(f, \ell_1)_2$ is a non-trivial element of $\mathrm{B}(S/R)$. Since f is irreducible, Theorem 2.2 says $\mathrm{B}^{\smile}(S/R) = (0)$. Therefore, the ideal $I = (\ell_1, z - \ell_3)$ is a non-trivial element of $\operatorname{Pic} S \otimes \mathbb{Z}/2$.

Example 2.12. This example is based on the construction of Section 2.3. Start with a sequence of distinct irreducible polynomials $f_1, \dots, f_n$ in $A = k[x,y]$, where $n \geq 2$. Set $f = f_1 f_2 \cdots f_n + h^2$, for some $h \in A$ such that f is irreducible. Let $R = A[f^{-1}]$, and $S = R[z]/(z^2 - f)$. Theorem 2.2 says $\mathrm{B}(S/R) \cong \operatorname{Pic}(S) \otimes \mathbb{Z}/2$. Let g be any one of $f_1, \dots, f_n$. Let $F = Z(f)$, F_0 the line at infinity, $G = Z(g)$, and $H = Z(h)$. At a finite point P, the local intersection multiplicity $(F \cdot G)_P$ is divisible by 2. Assume there exists P_0 in $F_0 \cap F$ such that P_0 is not a point of G and the local intersection multiplicity $(F_0 \cdot F)_{P_0}$ is odd. If we assume $\deg G$ is odd, then the weighted path in the graph $\Gamma(F + G + F_0)$ of the symbol algebra $(f,g)_2$, has loops of the type $F \to P_0 \to F_0 \to P_{0,j} \to F$. Therefore, $(f,g)_2$, is a division algebra and the ideal $I = (g, z - h)$ is a non-trivial element of $\operatorname{Pic} S \otimes \mathbb{Z}/2$.

Example 2.13. This example is based on Example 2.12. This example shows that it is not necessary to assume the degree of p_1 is odd. Let ℓ_1, ℓ_2 be linear polynomials in $k[x,y]$ and c an irreducible conic such that $f = \ell_1 c + \ell_2^2$ is an irreducible cubic. Assume ℓ_1, ℓ_2, c, and the line at infinity F_0 are in general position. In this example we prove that $(f,c)_2$ is a division algebra. Let $C = Z(c)$, $F = Z(F)$, $L_i = Z(\ell_i)$, and F_0 the line at infinity. Let

$$\begin{aligned} C \cdot L_1 &= P_1 + P_2 \\ C \cdot L_2 &= P_3 + P_4 \\ L_1 \cdot L_2 &= P_5 \\ L_1 \cdot F_0 &= P_6 \\ L_2 \cdot F_0 &= P_7 \\ C \cdot F_0 &= P_8 + P_9 \end{aligned} \tag{10}$$

Then

$$\begin{aligned} F \cdot C &= 2F \cdot L_2 + F \cdot F_0 \\ &= 2L_2 \cdot C + C \cdot F_0 \\ &= 2P_3 + 2P_4 + P_8 + P_9 \\ F \cdot F_0 &= L_1 \cdot F_0 + C \cdot F_0 \\ &= P_6 + P_8 + P_9 \end{aligned} \tag{11}$$

From this we compute the weighted path in the graph $\Gamma(F + C + F_0)$ for the symbol algebra $(f, c)_2$, with coefficients in $\mathbb{Z}/2$. The graph and edge weights are shown in Figure 2. There

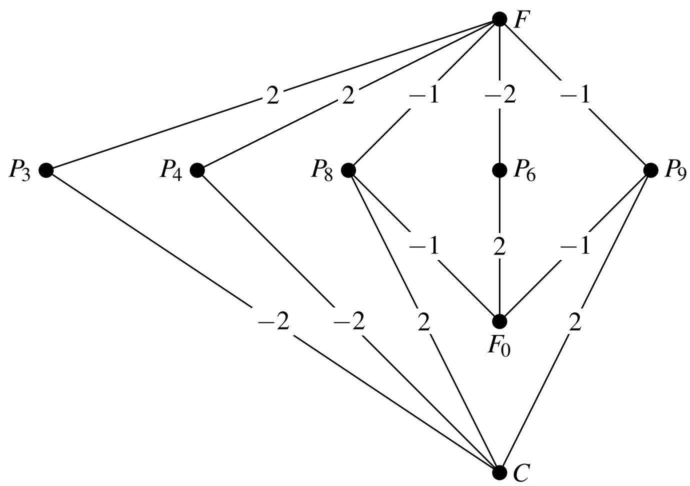

FIGURE 2. The weighted path of $(f, c)_2$ in Example 2.13

is one non-trivial loop, $F \to P_8 \to F_0 \to P_9 \to F$. Therefore $(f, c)_2$ is a division algebra and corresponds to a non-trivial element in $\mathrm{B}(S/R)$.

Notice that $F \cdot L_1 = 2P_5 + P_7$. Compute the weighted path in the graph $\Gamma(F + L_1 + F_0)$ for the symbol algebra $(f, \ell_1)_2$, with coefficients in $\mathbb{Z}/2$. There is a non-trivial cycle, $F \to P_6 \to F_0 \to P_7 \to F \to P_8 \to F_0 \to P_9 \to F$. This proves $(f, \ell_1)_2$ is a division algebra and corresponds to a non-trivial element in $\mathrm{B}(S/R)$. It follows that the order of $\operatorname{Pic} S \otimes \mathbb{Z}/2$ is at least 4. The ideals $(c, z - \ell_2)$ and $(\ell_1, z - \ell_2)$ are independent in $\operatorname{Pic} S \otimes \mathbb{Z}/2$.

Example 2.14. Say ℓ and c are in $k[x, y]$, where ℓ is a line and c is an irreducible conic. Let $C = Z(c)$, $L = Z(\ell)$. Let F_0 denote the line at infinity. Assume L, C and F_0 are in general position. Let $f = \ell c - 1$ and assume f is irreducible. An argument similar to that used in Example 2.13 shows that over $k(x,y)$, both $(f,\ell)_2$ and $(f,c)_2$ are division algebras.

3. A Cyclic Covering of a Polynomial Ring

Let k be a field in which n is invertible, and assume k contains ζ, a primitive nth root of unity. Let $\bar{k}$ be an algebraic closure of k. The example we consider is a cyclic covering of $\mathbb{A}^m = \operatorname{Spec} k[x_1,\dots,x_m]$ in $\mathbb{A}^{m+1} = \operatorname{Spec} k[x_1,\dots,x_m,z]$ defined by a single equation of the form $z^n = f$, where f is an irreducible polynomial in $k[x_1,\dots,x_m]$. Our notation in this section will agree with that of Section 1. To define f, start with a sequence of irreducible polynomials $f_1,\dots,f_\nu$ in $A = k[x_1,\dots,x_m]$ such that the polynomial $f = f_1 f_2 \cdots f_\nu + 1$ is irreducible in $\bar{k}[x_1,\dots,x_m]$. Let $T = A[z]/(z^n - f)$, $R = A[f^{-1}]$ and $S = R[z]/(z^n - f)$. Since f is irreducible, an application of Eisenstein's Criterion (for example, [11, Theorem 3.7.6]) shows T is an integral domain. The quotient field of A is $K = k(x_1,\dots,x_m)$ and that of T is $L = K[z]/(z^n - f)$. From the Jacobian and Serre's Criteria (for example, [10, Theorems 11.6.5 and 11.4.8] or [13, Theorem I.5.1 and Proposition II.8.23]), we know that $\bar{T} = T \otimes_k \bar{k}$ is normal. Since $T \to T \otimes_k \bar{k}$ is faithfully flat, T is integrally closed in L (see, for example, [10, Example 11.6.7]). Let σ be the A-algebra automorphism of T defined by $\sigma(z) = \zeta z$. Then we will also view σ as an R-automorphism of S and K-automorphism of L. Since f and n are invertible in R, by Kummer Theory S/R is Galois with group $G = \langle \sigma \rangle$ (see for example [9, Example 12.9.5] or [18, § III.4, pp. 125-126]). Since R is regular, so is S (see for example [10, Corollary 11.5.4]). The map $\pi : \operatorname{Spec} T \to \operatorname{Spec} A$ ramifies only over the hypersurface $F = Z(f)$. Lying above F is the irreducible hypersurface defined by $z = 0$ and the ramification index is n. If we set $U = \operatorname{Spec} R$ and $V = \operatorname{Spec} S$, then we are in the context of [7, Section 1.1]. In particular, [7, Theorem 1.1] applies and there is a homomorphism

$$\gamma_5 : \mathrm{B}(S/R) \to \mathrm{H}^1(G, \mathrm{Cl}(T)) \tag{12}$$

of abelian groups. The goal of Section 3 is to derive sufficient conditions on $f_1,\dots,f_\nu$ such that there exists a subgroup of $\mathrm{B}(S/R)$ of order $n^{\nu-1}$ which embeds in $\mathrm{H}^1(G,\mathrm{Cl}(T))$. This result appears below in Proposition 3.3. To compute the subgroup, and its image under γ_5, the proof applies the results of [7, Sections 3 and 4].

Lemma 3.1. *Let f_1, f_2 be polynomials in $k[x_1,\dots,x_m]$ such that f_1 is irreducible in $k[x_1,\dots,x_m]$ and $f = f_1 f_2 + 1$ is irreducible in $\bar{k}[x_1,\dots,x_m]$. For any $0 \le j < n$, consider the ideal $I = (\zeta^j z - 1, f_1)$ in $T = k[x_1,\dots,x_m]/(z^n - f)$. In the context of the previous paragraph the following are true.*

(a) I is a height one prime ideal in T.

(b) I is an invertible fractional ideal of T in L, the quotient field of T, hence I represents a class in $\operatorname{Pic}(T) \subseteq \operatorname{Cl}(T)$.

(c) Under the action of $G = \langle \sigma \rangle$ on $\operatorname{Pic}(T)$, the norm of I is the principal ideal Tf_1. That is, $Tf_1 = I\sigma(I)\cdots\sigma^{n-1}(I)$.

Proof. [7, Lemma 5.1]. □

In the notation of the first paragraph of Section 3, consider the ideals $P_1 = (z-1, f_1)$, $\dots$, $P_{n-1} = (z-1, f_{\nu-1})$ in the ring T. For each i, Lemma 3.1 shows the norm of P_i is equal to Tf_i. Therefore we can construct the A-algebra $\Lambda_i = \Delta(T/A, P_i, f_i)$ as in [7, Definition 3.2]. By [7, Corollary 3.10], the generic stalk of Λ_i is $\Lambda_i \otimes_A K = (L/K, \sigma, f_i^{-1})$, which we identify with the symbol algebra $(f, f_i^{-1})_n$ over K. By [7, Corollary 3.12], $\Lambda_i \otimes_A R$ is an Azumaya R-algebra that is split by S. By [7, Theorem 4.17], the homomorphism

(12) maps the Brauer class $[\Lambda_i \otimes_A R]$ to the 1-cocycle in $\mathrm{H}^1(G, \mathrm{Cl}(T))$ represented by the class of P_i. We have shown

Proposition 3.2. *Assume f_1, f_2, $\dots f_\nu$, are irreducible polynomials in $k[x_1,\dots,x_m]$, and the polynomial $f = f_1 f_2 \cdots f_\nu + 1$ is irreducible in $\bar{k}[x_1,\dots,x_m]$. Then in the above context, the following are true.*

(a) For each i, $\Lambda_i \otimes_A R = \Delta(T/A, P_i, f_i) \otimes_A R$ is an Azumaya R-algebra split by S.

(b) Under the homomorphism γ_5 of (12), the Brauer class $[\Lambda_i \otimes_A R]$ in $\mathrm{B}(S/R)$ is mapped by γ_5 to the 1-cocycle in $\mathrm{H}^1(G, \mathrm{Cl}(T))$ represented by the class of P_i.

Now we apply Proposition 3.2 to algebraic surfaces. For the following, the polynomial ring A is $k[x,y]$. We derive sufficient conditions on the polynomials $f_1,\dots,f_\nu$ in A such that the Brauer classes represented by $\Lambda_1, \dots, \Lambda_{\nu-1}$ are $\mathbb{Z}/n$-independent in the group $\mathrm{B}(S/R)$. In the usual way embed $\mathbb{A}^2_{\bar{k}}$ as an open subset of the projective plane $\mathbb{P}^2_{\bar{k}}$ and let F_∞ denote the line at infinity. For $i = 1,\dots,\nu$, let $F_i = Z(f_i)$ be the projective plane curve in $\mathbb{P}^2_{\bar{k}}$ defined by f_i. Let $d_i = \deg f_i$ be the degree of f_i. The degree of $f = f_1 f_2 \cdots f_\nu + 1$ is $d = d_1 + \cdots + d_\nu$. Proposition 3.3 is a variation of [7, Proposition 5.3].

Proposition 3.3. *In the above context, assume $\nu \geq 2$ and f_1, f_2, $\dots f_\nu$ are irreducible polynomials in $k[x,y]$ satisfying the following.*

(A) In $\mathbb{P}^2_{\bar{k}}$ the curve $Z(f_1 f_2 \cdots f_\nu)$ intersects F_∞ in $d = d_1 + \cdots + d_\nu$ distinct points.

(B) $f = f_1 f_2 \cdots f_\nu + 1$ is irreducible in $\bar{k}[x,y]$.

(C) One of the following sets of conditions is satisfied:

  (i) $1 = \gcd(d,n) = \gcd(d_1,n) = \cdots = \gcd(d_{\nu-1},n)$.

  (ii) $\gcd(d,n) = 1$ and $0 \equiv d_1 \equiv \cdots d_{\nu-1} \pmod{n}$.

Then the following are true.

(a) The classes represented by the symbol algebras $(f,f_1)_n,\dots,(f,f_{\nu-1})_n$ generate a subgroup of $\mathrm{B}(L/K)$ of order $n^{\nu-1}$.

(b) The classes represented by $\Lambda_1 \otimes_A R, \dots, \Lambda_{\nu-1} \otimes_A R$ generate a subgroup of $\mathrm{B}(S/R)$ of order $n^{\nu-1}$.

(c) The classes represented by the ideals $P_1, \dots, P_{\nu-1}$ generate a subgroup of $\mathrm{H}^1(G, \mathrm{Cl}\, T)$ of order $n^{\nu-1}$.

Proposition 3.3 is proved utilizing the cycle space of the graph associated to a plane curve. Before the proof, we review the definition.

Definition 3.4. Let Y be a reduced curve in $\mathbb{P}^2_{\bar{k}}$ and write $Y = Y_1 \cup \cdots \cup Y_m$, where the Y_i are the distinct irreducible components of Y. For each i let $\tilde{Y}_i \to Y_i$ be the normalization and define $\tilde{Y}$ to be the disjoint union $\tilde{Y}_1 \cup \cdots \cup \tilde{Y}_m$. There is a natural map $\pi : \tilde{Y} \to Y$. Let $P = \{p_1,\dots,p_s\}$ be the singular set of Y, and $\tilde{P} = \pi^{-1}(P) = \{q_1 \dots, q_e\}$. The diagram

$$\begin{array}{ccc} \tilde{P} = \{q_1 \dots, q_e\} & \xrightarrow{\subseteq} & \tilde{Y} = \tilde{Y}_1 \cup \cdots \cup \tilde{Y}_m \\ \pi \big\downarrow & & \big\downarrow \pi \\ P = \{p_1,\dots,p_s\} & \xrightarrow{\subseteq} & Y = Y_1 \cup \cdots \cup Y_m \end{array}$$

commutes. To the curve Y is associated a bipartite graph $\Gamma(Y)$ with vertex set $\{\tilde{Y}_1,\dots,\tilde{Y}_m\} \cup \{p_1,\dots,p_s\}$ and edge set $\tilde{P}$. The edge $q \in \tilde{P}$ connects the vertex $\tilde{Y}_i \in \tilde{Y}$ to the vertex $p_j \in P$ if and only if $q \in \tilde{Y}_i$ and $\pi(q) = p_j$. By [5, Corollary 1.3] there is an isomorphism of abelian groups ${}_n\mathrm{B}(\mathbb{P}^2 - Y) \to \mathrm{H}^1(\tilde{Y}, \mathbb{Q}/\mathbb{Z}) \oplus \mathrm{H}_1(\Gamma(Y), \mathbb{Z}/n)$ (modulo torsion divisible by $\mathrm{char}\, k$). The element in the cycle space of $\Gamma(Y)$ associated to a symbol algebra can be computed using local intersection multiplicities [5, Theorem 2.1]. Suppose Y_i and Y_j intersect at the point p with local intersection multiplicity $\mu = (F_i, F_j)_p$. The definition simplifies if we assume both Y_i and Y_j are nonsingular at p. This is true in the application below. Let f_i and f_j be local equations for the two curves and consider the symbol algebra $(f_i, f_j)_n$ over the field of rational functions on $\mathbb{P}^2$. Then near the vertex p, the cycle in Γ corresponding to $(f_i, f_j)_n$ looks like $F_i \xrightarrow{\mu} p \xrightarrow{-\mu} F_j$.

Proof of Proposition 3.3. The diagram

$$\begin{array}{ccc} \mathrm{B}(R) & \longrightarrow & \mathrm{B}(R \otimes_k \bar{k}) \\ \downarrow & & \downarrow \\ \mathrm{B}(k(x,y)) & \longrightarrow & \mathrm{B}(\bar{k}(x,y)) \end{array} \tag{13}$$

commutes. The ring R is a localization of $k[x,y]$ in $K = k(x,y)$, so the vertical arrows in (13) are one-to-one. Part (b) follows from Proposition 3.2 and Part (a). The diagram

$$\begin{array}{ccc} \mathrm{B}(S/R) & \xrightarrow{\gamma_5} & \mathrm{H}^1(G, \mathrm{Cl}(T)) \\ \downarrow & & \downarrow \\ \mathrm{B}(\bar{S}/\bar{R}) & \longrightarrow & \mathrm{H}^1(G, \mathrm{Cl}(T \otimes_k \bar{k})) \end{array} \tag{14}$$

commutes. By [12, Proposition 2.1], $\mathrm{H}^2(G, (T \otimes_k \bar{k})^*) = \langle 1 \rangle$, hence by [7, Theorem 1.1, Eq. (4)] the arrow in the bottom row of (14) is one-to-one. Therefore (c) follows from Proposition 3.2 and (b). To prove (a), by (13) it is enough to show the symbol algebras $(f, f_1)_n, \dots, (f, f_{\nu-1})_n$ generate a subgroup of order $n^{\nu-1}$ in $\mathrm{B}(\bar{k}(x,y))$. For the remainder of the proof, we assume k is algebraically closed. If we write

$$F_i \cdot F_\infty = Q_{i1} + \cdots + Q_{id_i}, \tag{15}$$

for $1 \le i \le \nu$, then the set $\{Q_{ij}\}$ contains d distinct points. Then

$$\begin{aligned} F \cdot F_\infty &= \sum_{i=1}^{n} \sum_{j=1}^{d_i} Q_{ij}, \text{ and} \\ F \cdot F_i &= dQ_{i1} + \cdots + dQ_{id_i} \text{ for } 1 \le i \le \nu. \end{aligned} \tag{16}$$

For each i, the symbol algebra $(f, f_i)_n$ represents a Brauer class on the open complement of the curve $F + F_1 + \cdots + F_\nu + F_\infty$ in $\mathbb{P}^2$. We use [5, Theorem 2.1] to associate to (f, f_i) a cycle in the edge space of the graph Γ associated to the plane curve $F + F_1 + \cdots + F_\nu + F_\infty$. The edge weights are computed from the local intersection multiplicities. From (15) and (16) we compute the weighted path in the graph $\Gamma(F + F_1 + F_\infty)$ for the symbol algebra $(f, f_1)_n$. The homology is computed with coefficients in $\mathbb{Z}/n$. The graph and edge weights are shown in Figure 3. For each $i = 1, \dots, \nu$ the graph for (f, f_i) is similar. It suffices to show that the cycles in the graph Γ corresponding to $(f, f_1)_n, \dots, (f, f_{\nu-1})_n$ generate a subgroup of order $n^{\nu-1}$ in $\mathrm{H}_1(\Gamma, \mathbb{Z}/n)$. We sketch a proof of this assuming condition (C)(i) is satisfied. The proof when (C)(ii) is satisfied is left to the reader. Find $u_1, \dots, u_{\nu-1}$ such that $d_i u_i \equiv 1 \pmod{n}$. The cycle in the graph for the symbol algebra $(f, f_1^{u_1})_n$ is shown in Figure 4. Figure 5 shows the cycle in the graph for $(f, f_1^{u_1} f_j^{-u_j})_n$, when $1 < j < \nu$. It is not hard to see that in the edge space of the graph Γ over $\mathbb{Z}/n$ the cycles for $(f, f_1^{u_1})_n, (f, f_1^{u_1} f_2^{-u_2})_n, \dots, (f, f_1^{u_1} f_{\nu-1}^{-u_{\nu-1}})_n$ are independent. This proves the symbol algebras $(f, f_1)_n, \dots, (f, f_{\nu-1})_n$ generate a subgroup of order $n^{\nu-1}$ in $\mathrm{B}(L/K)$. $\square$

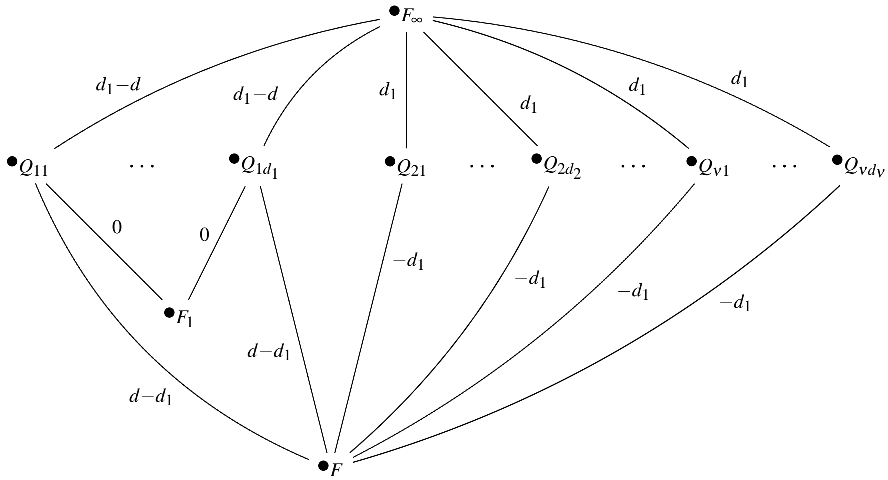

FIGURE 3. The weighted path of $(f, f_1)_n$ in Proposition 3.3

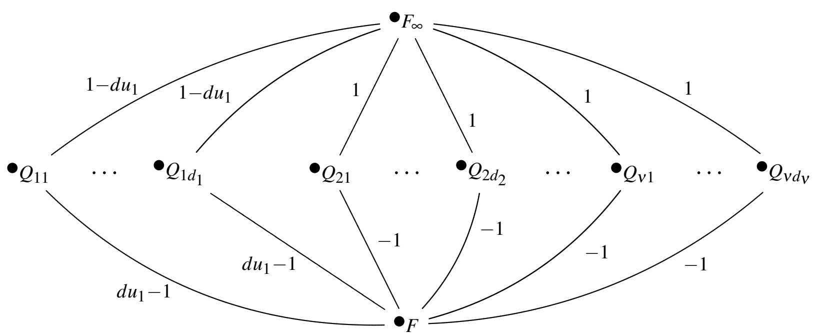

FIGURE 4. The weighted path of $(f, f_1^{u_1})_n$ in Proposition 3.3

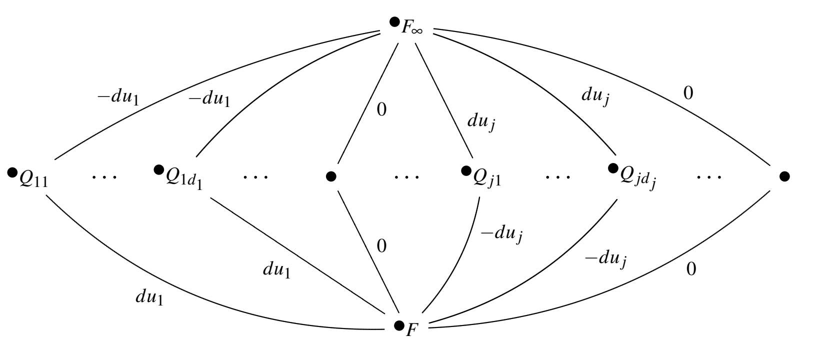

FIGURE 5. The weighted path of $(f, f_1^{u_1} f_j^{-u_j})_n$ in Proposition 3.3

3.1. **More examples.**

Example 3.5. As in Example 2.4, we consider a double plane ramified over four lines. We consider the case where two of the four lines are parallel. Start with a linear polynomial $\ell \in k[x,y]$ which defines the line $L = Z(\ell)$ in $\mathbb{A}^2$. Pick a point P on L. Let F_1 and F_2 be general lines which are parallel to L. Let F_3 and F_4 be general lines that intersect L at P. Let f_i be the equation for F_i. Let $f = f_1 f_2 f_3 f_4$, $R = k[x,y][f^{-1}]$, and $S = R[\sqrt{f}]$. Then ${}_2\mathrm{B}(R)$ is isomorphic to $(\mathbb{Z}/2)^{(5)}$. A basis consists of the symbol algebras

$$\{(f_1,f_3)_2,(f_1,f_4)_2,(f_2,f_3)_2,(f_2,f_4)_2,(f_3,f_4)_2\}.$$

The group $B^{\smile}(S/R)$ is the subgroup of ${}_2\mathrm{B}(R)$ generated by $\{(f,f_1)_2,\dots,(f,f_4)_2\}$. One computes that $\mathrm{B}^{\smile}(S/R)$ is a $\mathbb{Z}/2$-module of rank 3, with a basis being

$$\{(f_1,f_3)_2(f_1,f_4)_2,(f_2,f_3)_2(f_2,f_4)_2,(f_1,f_3)_2(f_2,f_3)_2(f_3,f_4)_2\}.$$

Let $\Lambda = (f,\ell)_2$. One computes $(f,\ell)_2 \sim (f_3 f_4,\ell)_2 \sim (f_3,f_4)_2$ which is in $\mathrm{B}(S/R)$, but not $\mathrm{B}^{\smile}(S/R)$. Theorem 2.1 says $\alpha_5(\Lambda)$ represents a non-trivial element of $\mathrm{Pic}(S)\otimes\mathbb{Z}/2$.

The case where f_1 and f_2 are parallel, and f_3 and f_4 are parallel is the subject of Example 3.6, where it is shown that α_5 is zero. The double plane ramified over four lines passing through a common point is studied in [8], where it is shown that α_5 is zero.

Example 3.6. Let $f = (x^2-1)(y^2-1) \in k[x,y]$. Set $R = k[x,y][f^{-1}]$ and $S = R[\sqrt{f}]$. Let $T = k[x,y,z]/(z^2 = f)$. As computed in [16], $\mathrm{Cl}(T) \cong (\mathbb{Z}/2)^{(3)}$. By [6, Theorem 2.4], $\mathrm{H}^1(G,\mathrm{Cl}(T)) \cong \mathrm{Cl}(T)\otimes\mathbb{Z}/2 \cong (\mathbb{Z}/2)^{(3)}$. As shown in [6, Theorem 2.5], $\mathrm{H}^1(G,\mathrm{Cl}(T)) \to \mathrm{B}(S/R)$ is onto. Using [4], one can check that ${}_2\mathrm{B}(R) \cong (\mathbb{Z}/2)^{(4)}$ and $\mathrm{B}^{\smile}(S/R) \cong (\mathbb{Z}/2)^{(3)}$. This proves $\mathrm{B}^{\smile}(S/R) = \mathrm{B}(S/R)$ and $\mathrm{Pic}\,S\otimes\mathbb{Z}/2 = (0)$. Consider the symbol algebra $(f,y-x)_2$. Check that

$$\begin{aligned}(f,y-x)_2 &\sim ((x-1)(y-1),y-x)_2((x+1)(y+1),y-x)_2\\ &\sim (x-1,y-1)_2(x+1,y+1)_2\\ &\sim (f,(x-1)(y+1))_2\end{aligned} \tag{17}$$

Upon restriction to the field $K = k(x,y)$, $(f,y-x)_2$ is a division algebra. The ideal $S(y-x)$ has two minimal primes, namely $(y-x,z-x^2+1)$ and $(y-x,z+x^2-1)$ and they are comaximal. The ring $S/(y-x)$ is a direct sum of two copies of $R/(y-x)$. The ring in Example 2.4 was a double plane ramified over four lines in general position. The ring in Example 3.5 was a double plane ramified over four lines, three of which are in general position. In both of these examples, it was shown that $\mathrm{Pic}\,S\otimes\mathbb{Z}/2$ was non-trivial. By comparison, in this example we find that $\mathrm{Pic}\,S\otimes\mathbb{Z}/2$ is trivial because the four lines are not sufficiently general.

One can check that the K-symbol algebra $\Lambda = (x-1,y-1)_2$ represents a class in $\mathrm{B}(R)$ that is not in $\mathrm{B}^{\smile}(S/R)$. If L is the quotient field of T, then $\Lambda\otimes_K L$ is a division algebra over L. Moreover, $\Lambda\otimes_K L$ is unramified at every height one prime of T. By [3, Corollary 3] the sequence

$$0 \to \mathrm{B}(L/T) \to \mathrm{B}(T) \to \mathrm{B}(V) \to 0 \tag{18}$$

is exact, where V is the set of regular points of $\mathrm{Spec}\,T$. We know that a maximal $\mathscr{O}_V$-order in $\Lambda\otimes L$ is Azumaya. It would be informative to have a description of an Azumaya T-algebra whose generic stalk is Brauer equivalent to $\Lambda\otimes L$.

Example 3.7. Now we give an example where, in the context of Section 2.3, the symbol algebra $(f,g)_2$ is split, the ideal $I = (g,z-h)$ represents a non-trivial class in $\mathrm{Pic}\,S$, and I is in $2\,\mathrm{Pic}\,S$. Start with the special case $n = 3$ of [6, §3.3]. Let $f_1 = 2xy-1$, $\ell_1 = x-1$,

$\ell_2 = x+1$, $f = f_1\ell_1\ell_2$, $R = k[x,y][f^{-1}]$, and $S = R[z]/(z^2 - f)$. By [6, Proposition 3.4] we know that $\operatorname{Pic} S$ is infinite cyclic and is generated by the class of $I_1 = (z-1,x)$. The divisor of x is $\operatorname{Div}(x) = I_1 + I_2$, where $I_2 = (z+1,x)$. Take $g = x^2+y^2-1$. Check that

$$\begin{aligned} f_1 &= g-(x-y)^2 \\ f_2f_3 &= g-y^2 \end{aligned} \tag{19}$$

so the symbol algebras $(f_1,g)_2$ and $(f_2f_3,g)_2$ are split. It follows that $(f_1,g)_2(f_2f_3,g)_2 \sim (f,g)_2$ is also split. Multiply on both sides of (19),

$$\begin{aligned} f = f_1f_2f_3 &= g\left(g-(x-y)^2-y^2\right)+(x-y)^2y^2 \\ &= g\left(2xy-y^2-1\right)+(x-y)^2y^2 \end{aligned} \tag{20}$$

In the notation of Proposition 2.8, $u = 2xy-y^2-1$ and $h = (x-y)y$. Let $I = (g, z-(x-y)y)$. Then I is a height one prime of S, $\sigma(I)+I = S$, and $I\sigma(I) = Sg$. The divisor of g is $\operatorname{Div}(g) = I+\sigma(I)$. Let

$$\begin{aligned} m &= g-(z-xy+y^2) \\ &= x^2+xy-1-z = x(x+y)-(z+1) \end{aligned} \tag{21}$$

Note that m is in I and $m(z+xy-y^2) = g(1+z-xy)$. Since $z+xy-y^2$ and $1+z-xy$ are not in I, the valuation of m at I is one. Any prime ideal that contains m must contain g or $1+z-xy$. The ideal I is generated by m and g. Since $m+(1+z-xy) = x^2$, if a prime contains m and not g, then it contains x. Any ideal that has both m and x also has $z+1 = xy+x^2-m$. Therefore, the only minimal primes of m are I and I_2. One checks that $(z+1)^2 = x^2(2xy+1)-2\left(x(x+y)-(z+1)\right)$, from which it follows $m \in I_2^2$. Lastly, it is straightforward to check that $(z+1)^2 - x^2(2xy+1)$ is not in I_2^3, so the divisor of m is $\operatorname{Div}(m) = 2I_2 + I$. Therefore, I is a non-trivial element in $2\operatorname{Pic}(S)$.

Example 3.8. Let $f \in A = k[x,y]$ be a general polynomial of degree six. Let $R = A[f^{-1}]$, $S = R[\sqrt{f}]$. As observed in [14, §7] and [2, Example 1.2], S is an open subset of a $K3$ surface. Because f is general, it follows that $\operatorname{Pic} S = 0$. We are in the context of Theorem 2.2. Therefore, $\mathrm{B}(S/R) = 0$ and the sequence

$$0 \to {}_2\mathrm{B}(R) \xrightarrow{\text{res}} {}_2\mathrm{B}(S) \xrightarrow{\text{cor}} {}_2\mathrm{B}(R) \to 0$$

is exact.

Example 3.9. Let $a_1,\dots,a_n$ be distinct elements of k, and set $\ell_i = x-a_i$ for $i = 1,\dots,n$. Let $f = y^2-\ell_1\cdots\ell_n$, $R = A[f^{-1}]$, and $S = R[\sqrt{f}]$. We are in the context of Theorem 2.2. As observed in [8], S is a nonsingular affine rational surface, $\mathrm{B}(S/R) = {}_2\mathrm{B}(R) \cong (\mathbb{Z}/2)^{n-1}$, and the sequence

$$0 \to {}_2\mathrm{B}(S/R) \to \mathrm{B}(R) \to \mathrm{B}(S) \to 0$$

is exact.

Example 3.10. The surface $z^2 = y(y-p(x))$.

The notation in this example agrees with that of Section 2.

In this section we study the divisor classes and algebra classes on the surface defined by $z^2 = y(y-p(x))$, where $p(x) \in k[x]$ is a monic polynomial of degree $d > 1$. Let $f_1 = y$, $f_2 = y-p(x)$, and $f = f_1f_2$. Let $A = k[x,y]$, $R = A[f^{-1}]$, $T = A[z]/(z^2-f)$ and $S = T[z^{-1}]$. In A let $p(x) = \ell_1^{e_1}\cdots\ell_v^{e_v}$ be the unique factorization into irreducibles. Let $\alpha_1,\dots,\alpha_v$ be the distinct roots of $p(x)$. Let $F_i = Z(f_i)$ which we embed into $\mathbb{P}^2$ in the usual way. Let F_0 be the line at infinity. Then $F_1\cdot F_2 = e_1P_1+\cdots+e_vP_v$, $F_1\cdot F_0 = P_{01}$ and

$F_2 \cdot F_0 = dP_{02}$. The graph Γ of the curve $F = F_0 + F_1 + F_2$ is seen in Figure 6. If $i > 1$, the node P_i and its edges exist only if $v \geq i$. This explains why the edges to $P_2, \dots, P_v$ are dashed. We will compute the following, where $D = \gcd(e_1, \dots, e_v)$.

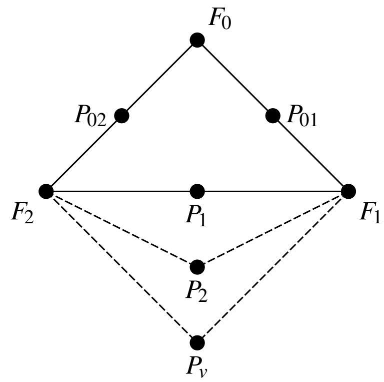

FIGURE 6. The graph in Example 3.10.

(a) $\mathrm{B}(R) \cong (\mathbb{Q}/\mathbb{Z})^{(v)}$
(b) $\mathrm{B}^{\smile}(S/R) \cong 0$ if $2 \mid D$, otherwise $\mathbb{Z}/2$
(c) $\mathrm{B}(S/R) = {}_2\mathrm{B}(R) \cong (\mathbb{Z}/2)^{(v)}$
(d) $\mathrm{Cl}(T) \cong \mathbb{Z}/(2D) \oplus \mathbb{Z}^{(v-1)}$ and $\mathrm{H}^1(G, \mathrm{Cl}(T)) \cong (\mathbb{Z}/2)^{(v)}$
(e) $\mathrm{Pic}(S) \cong \mathbb{Z}/D \oplus \mathbb{Z}^{(v-1)}$ and $\mathrm{H}^1(G, \mathrm{Cl}(S)) \cong (\mathbb{Z}/2)^{(v)}$ if $2 \mid D$, otherwise $(\mathbb{Z}/2)^{(v-1)}$
(f) $\mathrm{B}(S) \cong (\mathbb{Q}/\mathbb{Z})^{(v)}$

Using [4, Theorem 4], $\mathrm{B}(R) \cong (\mathbb{Q}/\mathbb{Z})^{(v)}$, which is (a). The group $\mathrm{B}^{\smile}(S/R)$ is generated by the symbol $(f, f_1)_2 \sim (f_2, f_1)_2 \sim (f, f_2)_2$, hence is cyclic. The weighted element of the edge space is computed as in [5, § 2] and is shown in Figure 7. Therefore we get (b):

$$\mathrm{B}^{\smile}(S/R) = \begin{cases} 0 & \text{if } 2 \mid e_i \text{ for all } i \\ \mathbb{Z}/2 & \text{otherwise.} \end{cases} \tag{22}$$

This and [6, sequence (9)] imply

$$\mathrm{H}^1(S, \mu_2) \cong \begin{cases} (\mathbb{Z}/2)^{(3)} & \text{if } 2 \mid e_i \text{ for all } i \\ (\mathbb{Z}/2)^{(2)} & \text{otherwise.} \end{cases} \tag{23}$$

Let $L_i = Z(\ell_i)$. Then $L_i \cdot F_0 = P_{02}$, $L_i \cdot F_1 = P_i$, and $L_i \cdot F_2 = P_i + (d-1)P_{02}$. Using the method of [4, Theorem 4], the weighted path associated to the symbol algebra $(fy^{-2}, \ell_i)_m$ is computed to be $F_2 \to P_{02} \to F_0 \to P_{01} \to F_1 \to P_i \to F_2$. For $i = 1, \dots, v$, these cycles make up a basis for $\mathrm{H}_1(\Gamma, \mathbb{Z}/m)$. Therefore, a basis for ${}_m\mathrm{B}(R)$ consists of the classes of the algebras

$$(fy^{-2}, \ell_1)_m, \dots, (fy^{-2}, \ell_v)_m. \tag{24}$$

This shows that a basis of ${}_2\mathrm{B}(R)$ consists of $(f, \ell_1)_2, \dots, (f, \ell_v)_2$, all of which are in $\mathrm{B}(S/R)$, which proves (c). By [6, Theorem 2.1], this shows

$$\mathrm{Pic}\, S \otimes \mathbb{Z}/2 \cong \begin{cases} (\mathbb{Z}/2)^{(v)} & \text{if } 2 \mid e_i \text{ for all } i \\ (\mathbb{Z}/2)^{(v-1)} & \text{otherwise.} \end{cases} \tag{25}$$

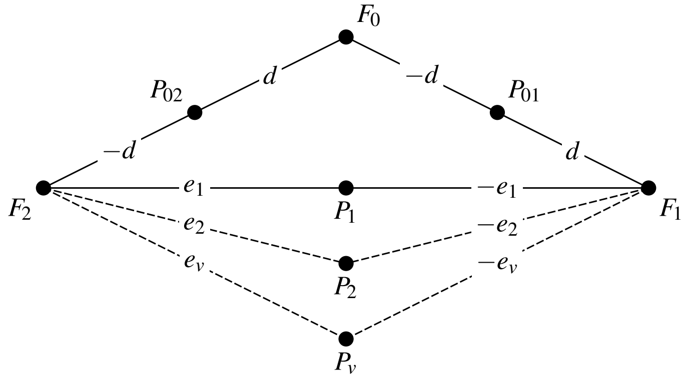

FIGURE 7. The weighted path of $(f_2, f_1)_2$ in Example 3.10.

Consider the homomorphism of k-algebras

$$T = \frac{k[x,y,z]}{(z^2 - y(y-p(x))} \xrightarrow{\beta} U = k[x,w][(1-w^2)^{-1}] \tag{26}$$

defined by $x \mapsto x$, $y \mapsto p(x)/(1-w^2)$, $z \mapsto wp(x)/(1-w^2)$. One checks that β is well defined and becomes an isomorphism upon adjoining $1/p(x), 1/z$ to T and $1/p(x), 1/w$ to U. Both rings in (26) are integral domains of Krull dimension two, hence β is one-to-one. Since U is rational, so is T. We have

$$k[x,w][p(x)^{-1}, w^{-1}, (1-w^2)^{-1}]^* = k^* \times \langle w \rangle \times \langle 1-w \rangle \times \langle 1+w \rangle \times \prod_{i=1}^{v} \langle \ell_i \rangle. \tag{27}$$

Using β we find $zy^{-1} \mapsto w$, $(z-y)y^{-1} \mapsto w-1$, $(z+y)y^{-1} \mapsto w+1$, hence

$$T[p(x)^{-1}, z^{-1}]^* = k^* \times \langle zy^{-1} \rangle \times \langle (z-y)y^{-1} \rangle \times \langle (z+y)y^{-1} \rangle \times \prod_{i=1}^{v} \langle \ell_i \rangle. \tag{28}$$

Since U is factorial, Nagata's Theorem says the class group of T is generated by the minimal primes of z, z^2-y^2, y, and $\ell_1, \dots, \ell_v$. It is routine to verify that

$$\begin{aligned} \operatorname{Div}(z) &= (z,y) + (z, y-p(x)) \\ \operatorname{Div}(z-y) &= (z,y) + e_1(z-y, \ell_1) + \cdots + e_v(z-y, \ell_v) \\ \operatorname{Div}(z+y) &= (z,y) + e_1(z+y, \ell_1) + \cdots + e_v(z+y, \ell_v) \\ \operatorname{Div}(y) &= 2(z,y) \\ \operatorname{Div}(\ell_i) &= (z-y, \ell_i) + (z+y, \ell_i) \\ \operatorname{Div}(y-p(x)) &= 2(z, y-p(x)). \end{aligned} \tag{29}$$

The class group $\operatorname{Cl}(T)$ is generated by the $2v+2$ prime divisors

$$(z,y), (z, y-p(x)), (z-y, \ell_1), \dots, (z-y, \ell_v), (z+y, \ell_1), \dots, (z+y, \ell_v). \tag{30}$$

Using the principal divisors $\operatorname{Div}(\ell_i) = (z-y, \ell_i) + (z+y, \ell_i) \sim 0$ and $\operatorname{Div}(z) = (z,y) + (z, y-p(x)) \sim 0$, we can eliminate half of the generators and all but two of the relations. The group $\operatorname{Cl}(T)$ is generated by the $v+1$ divisors $(z,y), (z-y, \ell_1), \dots, (z-y, \ell_v)$

modulo the two principal divisors $2(z,y)$, $(z,y)+e_1(z-y,\ell_1)+\cdots+e_v(z-y,\ell_v)$. If $D=\gcd(e_1,\dots,e_v)$, then

$$\begin{aligned}\mathrm{Cl}(T)&\cong\mathbb{Z}/(2D)\oplus\mathbb{Z}^{(v-1)}\\ \mathrm{H}^1(G,\mathrm{Cl}(T))&\cong(\mathbb{Z}/2)^{(v)}\end{aligned}\tag{31}$$

which is (d). Note that (31) together with part (c) show that $\mathrm{B}(S/R)$ is isomorphic to $\mathrm{H}^1(G,\mathrm{Cl}(T))$, which agrees with the conclusion of [6, Theorem 2.7]. We will see below that $T^*=k^*$. The kernel of $\mathrm{Cl}(T)\to\mathrm{Cl}(S)$ is generated by the divisor (z,y). So $\mathrm{Cl}(S)$ is generated by the v divisors $(z-y,\ell_1),\dots,(z-y,\ell_v)$ modulo the principal divisor $e_1(z-y,\ell_1)+\cdots+e_v(z-y,\ell_v)$. If $D=\gcd(e_1,\dots,e_v)$, then

$$\begin{aligned}\mathrm{Cl}(S)&\cong\mathbb{Z}/D\oplus\mathbb{Z}^{(v-1)}\\ \mathrm{H}^1(G,\mathrm{Cl}(S))&\cong\begin{cases}(\mathbb{Z}/2)^{(v)} & \text{if } 2\mid e_i \text{ for all } i\\ (\mathbb{Z}/2)^{(v-1)} & \text{otherwise.}\end{cases}\end{aligned}\tag{32}$$

proving (d). Note that (32) agrees with (25). Using (28), we see that z, y, $z-y$, $\ell_1,\dots,\ell_v$ make up a basis for $S[p(x)^{-1}]]^*/k^*$. The elements z and y are units of S. The minimal primes of $z-y$, $\ell_1,\dots,\ell_v$ in S are $(z-y,\ell_1),\dots,(z-y,\ell_v),(z+y,\ell_1),\dots,(z+y,\ell_v)$. In the Nagata sequence

$$1\to S^*\to S[p(x)^{-1}]^*\xrightarrow{\mathrm{Div}}\bigoplus_{i=1}^{v}\big(\mathbb{Z}(z-y,\ell_i)\oplus\mathbb{Z}(z+y,\ell_i)\big)\tag{33}$$

the elements $\mathrm{Div}(z-y),\mathrm{Div}(\ell_1),\dots,\mathrm{Div}(\ell_v)$ generate a free $\mathbb{Z}$-module of rank $v+1$. This proves $S^*=k^*\times\langle z\rangle\times\langle y\rangle$. Using [6, Theorem 2.2] and [10, Theorem 8.5.20],

$$\mathrm{H}^i(G,S^*)=\begin{cases}R^*=k^*\times\langle y\rangle\times\langle y-p(x)\rangle & \text{if } i=0\\ \langle 1\rangle & \text{if } i=1,3,\dots\\ \langle y\rangle/\langle y^2\rangle & \text{if } i=2,4,\dots\end{cases}$$

From the Nagata sequence

$$1\to T^*\to S^*\xrightarrow{\mathrm{Div}}\mathbb{Z}F_1\oplus\mathbb{Z}F_2$$

we know that S^*/T^* is free of rank two. It follows that $T^*=k^*$. Consider the isomomorphism

$$\mathrm{B}(S[p(x)^{-1}])\xrightarrow{\beta}\mathrm{B}(k[x,w][p(x)^{-1},w^{-1},(1-w^2)^{-1}])\tag{34}$$

induced by the map β of (26). Using [4, Theorem 4], compute the Brauer group on the right hand side of (34). It is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{(3v)}$, and a basis for the subgroup annihilated by m is made up of the $3v$ symbol algebras $(w,\ell_i)_m$, $(w-1,\ell_i)_m$, $(w+1,\ell_i)_m$ for $i=1,\dots,v$. Using β, it follows that the symbol algebras

$$(zy^{-1},\ell_i)_m,((z-y)y^{-1},\ell_i)_m,((z+y)y^{-1},\ell_i)_m\tag{35}$$

for $i=1,\dots,v$, make up a basis for the subgroup ${}_m\mathrm{B}(S[p(x)^{-1}])$. There is an exact sequence [5, Corollary 1.4]

$$0\to\mathrm{B}(S)\to\mathrm{B}(S[p(x)^{-1}])\xrightarrow{a}\mathrm{H}^1(S/(p(x)),\mu)\to 0.\tag{36}$$

The ring $S/(p(x))$ is the disjoint union of $2v$ copies of the algebraic torus $k[z,z^{-1}]$. Therefore, $\mathrm{H}^1(S/(p(x)),\mu)\cong(\mathbb{Q}/\mathbb{Z})^{(2v)}$. Look at the component of $S/(p(x))$ corresponding to the minimal prime $I_i=(z-y,\ell_i)$. The residue field at I_i is isomorphic to the quotient field of $S/I_i \cong k[y,y^{-1}]$ which we identify with $k(y)$. In the local ring S_{I_i}, the valuations are $\nu(z)=0$, $\nu(y)=0$, $\nu(\ell_i)=1$, $\nu(z-y)=e_i$. For the algebras in (35), the ramification map a in (36) agrees with the tame symbol. For $((z-y)y^{-1},\ell_i)_m$, the tame symbol is y^{-1}, which gives rise to an element of order m in $\mathrm{H}^1(k(y),\mathbb{Z}/m)$. This is the only algebra in the list (35) which is ramified at I_i. Similarly, the only algebra in the list (35) which ramifies at the prime $(z+y,\ell_i)$ is $((z+y)y^{-1},\ell_i)_m$. It follows that in sequence (36), the group ${}_m\mathrm{B}(S[p(x)^{-1}])$ maps onto $\mathrm{H}^1((S/(p(x)),\mu_m)$ and a basis for ${}_m\mathrm{B}(S)$ consists of $(zy^{-1},\ell_i)_m$ for $i=1,\dots,v$. This shows $\mathrm{B}(S)\cong(\mathbb{Q}/\mathbb{Z})^{(v)}$, which is (f). By (24)

$$(zy^{-1},\ell_i)_m \sim (zy^{-1},\ell_i)^2_{2m} \sim (z^2y^{-2},\ell_i)_{2m} \sim (fy^{-2},\ell_i)_{2m}$$

is in the image of $\mathrm{B}(R)$. Therefore, the sequence

$$0 \to {}_2\mathrm{B}(R) \to \mathrm{B}(R) \to \mathrm{B}(S) \to 0 \tag{37}$$

is exact. As a homomorphism of abstract groups, the natural map $\mathrm{B}(R)\to\mathrm{B}(S)$ is "multiplication by 2".

3.2. **A Nonnormal Surface.** The surface $z^n=y^{n-1}(y-p(x))$. Let k be an algebraically closed field and $n\geq 2$ an integer which is invertible in k. In this section we study the divisor classes and algebra classes on the surface defined by $z^n=y^{n-1}(y-p(x))$, where $p(x)\in k[x]$ is a monic polynomial of degree $d>1$. Let $f_1=y$, $f_2=y-p(x)$, and $f=f_1^{n-1}f_2$. Let $A=k[x,y]$, $R=A[f^{-1}]$, $T=A[z]/(z^n-f)$ and $S=T[z^{-1}]$. The quotient field of A and R is $K=k(x,y)$. The quotient field of T and S is $L=K[z]/(z^n-f)$. In A let $p(x)=\ell_1^{e_1}\cdots\ell_v^{e_v}$ be the unique factorization into irreducibles. Let $\alpha_1,\dots,\alpha_v$ be the distinct roots of $p(x)$. Let $F_i=Z(f_i)$ which we embed into $\mathbb{P}^2$ in the usual way. Let F_0 be the line at infinity. Then $F_1\cdot F_2=e_1P_1+\cdots+e_vP_v$, $F_1\cdot F_0=P_{01}$ and $F_2\cdot F_0=dP_{02}$. The graph Γ of the curve $F=F_0+F_1+F_2$ is seen in Figure 8. If $i>1$, the node P_i and its edges exist only if $v\geq i$. This explains why the edges to $P_2,\dots,P_v$ are dashed. As in [9, Example 12.9.5], if σ is the A-algebra automorphism of T defined by $z\mapsto\zeta z$, then S/R is a cyclic Galois extension with group $\langle\sigma\rangle$. In our setting, (3) is an exact sequence. If $n=2$, then f is square-free, and as in Section 2, T is a normal surface. If $n\geq 3$, then T is not integrally closed. One way to see this is to compute the singular locus of the surface $z^n=y^{n-1}(y-p(x))$ in $\mathbb{A}^2$ using the jacobian criterion (for example [10, Theorem 11.6.5]). Alternatively, if P is the prime ideal of height one generated by y and z in T, then the local ring T_P is not integrally closed. For instance, yz^{-1} is in L, yz^{-1} is not in T_P, but $(yz^{-1})^{n-1}$ is in T_P.

Proposition 3.11. *In the above context,*

1. *(1)* $\mathrm{B}(R)\cong(\mathbb{Q}/\mathbb{Z})^{(v)}$.
2. *(2) The image of* $\alpha_4:\mathrm{H}^2(G,S^*)\to\mathrm{B}(S/R)$ *is a cyclic* $\mathbb{Z}/n$*-module.*
3. *(3)* $\mathrm{B}(S/R)={}_n\mathrm{B}(R)\cong(\mathbb{Z}/n)^{(v)}$.
4. *(4)* $\mathrm{H}^1(G,\operatorname{Pic}S)$ *contains a subgroup that has order* $n^{\nu-1}$.
5. *(5)* $\mathrm{B}(S)\cong(\mathbb{Q}/\mathbb{Z})^{(v)}$.
6. *(6) The sequence* $0\to{}_n\mathrm{B}(R)\to\mathrm{B}(R)\to\mathrm{B}(S)\to 0$ *is exact.*

Proof. (1): Using [4, Theorem 5], $\mathrm{B}(R)\cong(\mathbb{Q}/\mathbb{Z})^{(v)}$.

(2): By [9, Section 13.4], the image of α_4 is generated by cyclic crossed products. For the Kummer extension S/R, cyclic crossed products can be identified as symbol algebras $(f,u)_n$, for $u\in R^*$. The group of units in R is $k^*\times\langle y\rangle\times\langle y-p(x)\rangle$, and $(f,y)_n\sim(y-p(x),y)_n\sim(y^{-1},y-p(x))_n\sim(y^{n-1},y-p(x))_n\sim(f,y-p(x))_n$. Hence the image of α_4 is cyclic. The weighted element of the edge space corresponding to the symbol algebra $(y-p(x),y)_n$ is computed as in [5, § 2] and is shown in Figure 9. Therefore, as a function

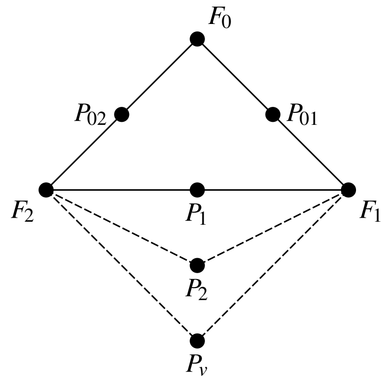

FIGURE 8. The graph in Section 3.2.

of the multiplicities $e_1,\dots,e_v$, the order of the image of α_4 can be any positive divisor of n.

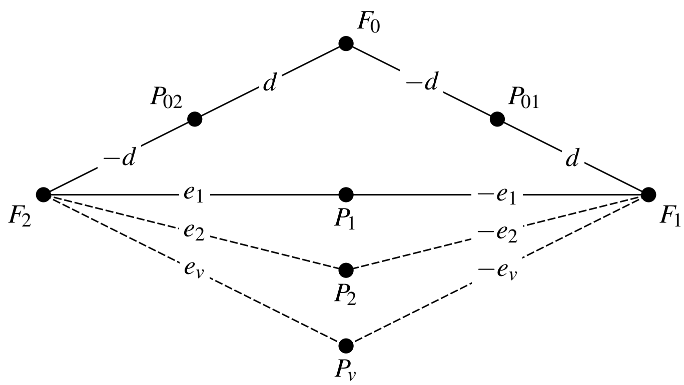

FIGURE 9. The weighted path of $(y-p(x),y)_n$ in Proposition 3.11.

(3): Let $L_i = Z(\ell_i)$. Then $L_i \cdot F_0 = P_{02}$, $L_i \cdot F_1 = P_i$, and $L_i \cdot F_2 = P_i + (d-1)P_{02}$. Let $m > 1$ be an integer that is invertible in k. Using the method of [5, § 2], the weighted path associated to the symbol algebra $(fy^{-n},\ell_i)_m$ is computed to be $F_2 \to P_{02} \to F_0 \to P_{01} \to F_1 \to P_i \to F_2$. For $i = 1,\dots,v$, these cycles make up a basis for $\mathrm{H}_1(\Gamma,\mathbb{Z}/m)$. Therefore, a basis for ${}_m\mathrm{B}(R)$ consists of the classes of the algebras

$$(fy^{-n},\ell_1)_m,\dots,(fy^{-n},\ell_v)_m. \tag{38}$$

This shows that a basis of ${}_n\mathrm{B}(R)$ consists of $(f,\ell_1)_n,\dots,(f,\ell_v)_n$, all of which are in $\mathrm{B}(S/R)$, which proves (3).

(4): This follows from (2), (3) and the exact sequence (3).

(5): Consider the homomorphism of k-algebras

$$T = \frac{k[x,y,z]}{(z^n - y^{n-1}(y-p(x))} \xrightarrow{\beta} U = k[x,w][(1-w^n)^{-1}] \tag{39}$$

defined by $x \mapsto x$, $y \mapsto p(x)/(1-w^n)$, $z \mapsto wp(x)/(1-w^n)$. One checks that β is well defined and becomes an isomorphism upon adjoining $1/p(x), 1/z$ to T and $1/p(x), 1/w$ to U. Both rings in (39) are integral domains of Krull dimension two, hence β is one-to-one. Since U is rational, so is T. There is an isomomorphism

$$\mathrm{B}(S[p(x)^{-1}]) \xrightarrow{\beta} \mathrm{B}(k[x,w][p(x)^{-1}, w^{-1}, (1-w^n)^{-1}]) \tag{40}$$

which is induced by the map β of (39). Using [4, Theorem 4], compute the Brauer group on the right hand side of (40). It is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{(n+1)\nu}$, and a basis for the subgroup annihilated by m is made up of the symbol algebras

$$\{(w,\ell_i)_m \mid i=1,\dots,\nu\} \cup \{(w-\zeta^j,\ell_i)_m \mid i=1,\dots,\nu,\ j=0,\dots,n-1\}.$$

Using β, it follows that the symbol algebras

$$\{(zy^{-1},\ell_i)_m \mid i=1,\dots,\nu\} \cup \{((z-y\zeta^j)y^{-1},\ell_i)_m \mid i=1,\dots,\nu,\ j=0,\dots,n-1\} \tag{41}$$

make up a basis for the subgroup ${}_m\mathrm{B}(S[p(x)^{-1}])$. There is an exact sequence [5, Corollary 1.4]

$$0 \to \mathrm{B}(S) \to \mathrm{B}(S[p(x)^{-1}]) \xrightarrow{a} \mathrm{H}^1(S/(p(x)),\mu) \to 0. \tag{42}$$

The ring $S/(p(x))$ is the disjoint union of $n\nu$ copies of the algebraic torus $k[z,z^{-1}]$. Therefore, $\mathrm{H}^1(S/(p(x)),\mu) \cong (\mathbb{Q}/\mathbb{Z})^{(n\nu)}$. Look at the component of $S/(p(x))$ corresponding to the minimal prime $I_{ij} = (z-y\zeta^j,\ell_i)$. The symbol algebra $((z-y\zeta^j)y^{-1},\ell_i)_m$ is mapped by the ramification map a in (42) to the Kummer extension $(S/I_{ij})[y^{-1/m}]$, which represents an element of order m in $\mathrm{H}^1(S/(p(x)),\mathbb{Z}/m)$. It follows that in sequence (42), the group ${}_m\mathrm{B}(S[p(x)^{-1}])$ maps onto $\mathrm{H}^1(S/(p(x)),\mu_m)$ and a basis for ${}_m\mathrm{B}(S)$ consists of $(zy^{-1},\ell_i)_m$ for $i=1,\dots,\nu$. This shows $\mathrm{B}(S) \cong (\mathbb{Q}/\mathbb{Z})^{(\nu)}$, which is (5).

(6): By (38), the symbol algebra

$$(zy^{-1},\ell_i)_m \sim (zy^{-1},\ell_i)_{nm}^n \sim (z^n y^{-n},\ell_i)_{nm} \sim (fy^{-n},\ell_i)_{nm}$$

is in the image of $\mathrm{B}(R) \to \mathrm{B}(S)$. Therefore, the sequence

$$0 \to {}_n\mathrm{B}(R) \to \mathrm{B}(R) \to \mathrm{B}(S) \to 0 \tag{43}$$

is exact. As a homomorphism of abstract groups, the natural map $\mathrm{B}(R) \to \mathrm{B}(S)$ is "multiplication by n". □

Proposition 3.12. *In the context of Section 3.2, if* $D = \gcd(e_1,\dots,e_\nu)$, *then*

(1) $S^* = k^* \times \langle z\rangle \times \langle y\rangle$.
(2) $\operatorname{Pic} S = \mathrm{Cl}(S) \cong (\mathbb{Z}/D)^{n-1} \oplus \mathbb{Z}^{(n-1)(\nu-1)}$.

Proof. We have

$$k[x,w][p(x)^{-1},w^{-1},(1-w^n)^{-1}]^* = k^* \times \langle w\rangle \times \prod_{j=0}^{n-1}\langle w-\zeta^j\rangle \times \prod_{i=1}^{\nu}\langle \ell_i\rangle. \tag{44}$$

Using the map β of (39) we find $zy^{-1} \mapsto w$, $(z-y\zeta^j)y^{-1} \mapsto w-\zeta^j$, hence

$$S[p(x)^{-1}]^* = T[p(x)^{-1},z^{-1}]^* = k^* \times \langle zy^{-1}\rangle \times \prod_{j=0}^{n-1}\left\langle \frac{z-y\zeta^j}{y}\right\rangle \times \prod_{i=1}^{\nu}\langle \ell_i\rangle. \tag{45}$$

Using (45), we see that $z, y, z-y\zeta, \dots, z-y\zeta^{n-1}, \ell_1,\dots,\ell_\nu$ make up a basis for the abelian group $S[p(x)^{-1}]^*/k^*$. The elements z and y are units of S. In S the minimal primes of

ℓ_i are $(z-y\zeta^j,\ell_i)$, $j=0,\dots,n-1$ and the minimal primes of $z-y\zeta^j$ are $(z-y\zeta^j,\ell_i)$, $i=1,\dots,\nu$. It is routine to verify that

$$\begin{aligned} \operatorname{Div}(\ell_i) &= \sum_{j=0}^{n-1}(z-y\zeta^j,\ell_i) \\ \operatorname{Div}(z-\zeta^j y) &= \sum_{i=1}^{\nu} e_i(z-y\zeta^j,\ell_i). \end{aligned} \tag{46}$$

Since $S[p(x)^{-1}]$ is factorial, the Nagata sequence ([10, Theorem 11.4.14])

$$1 \to S^* \to S[p(x)^{-1}]^* \xrightarrow{\operatorname{Div}} \bigoplus_{i=1}^{\nu}\bigoplus_{j=0}^{n-1}\mathbb{Z}(z-\zeta^j y,\ell_i) \to \operatorname{Cl}(S) \to 0 \tag{47}$$

is exact. In (47), the image of Div is a free $\mathbb{Z}$-module of rank $\nu+n-1$. This proves $S^* = k^* \times \langle z\rangle \times \langle y\rangle$. Using (46), one checks that the nonzero elementary divisors of the map Div are 1 with multiplicity ν and D with multiplicity $n-1$. □

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