A First Course in

# LINEAR ALGEBRA

# Lecture Notes

for Math 1503

# Spectral Theory: Eigenvalues and Eigenvectors





## A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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## Motivation: Calculating Powers of a Matrix

### Example

Let $A = \left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array}\right]$. Find $A^{100}$.

How can we do this efficiently?

Consider the matrix $P = \left[\begin{array}{rr} 1 & -2 \\ 1 & 1 \end{array}\right]$. Observe that $P$ is invertible (why?), and that

$$P^{-1} = \frac{1}{3}\left[\begin{array}{rr} 1 & 2 \\ -1 & 1 \end{array}\right].$$

Furthermore,

$$P^{-1}AP = \frac{1}{3}\left[\begin{array}{rr} 1 & 2 \\ -1 & 1 \end{array}\right]\left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array}\right]\left[\begin{array}{rr} 1 & -2 \\ 1 & 1 \end{array}\right] = \left[\begin{array}{rr} 2 & 0 \\ 0 & 5 \end{array}\right] = D,$$

where $D$ is a diagonal matrix.



### Example (continued)

This is significant, because

$$\begin{array}{rcl} P^{-1}AP & = & D \\ P(P^{-1}AP)P^{-1} & = & PDP^{-1} \\ (PP^{-1})A(PP^{-1}) & = & PDP^{-1} \\ IAI & = & PDP^{-1} \\ A & = & PDP^{-1}, \end{array}$$

and so

$$\begin{array}{rcl} A^{100} & = & (PDP^{-1})^{100} \\ & = & (PDP^{-1})(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) \\ & = & PD(P^{-1}P)D(P^{-1}P)D(P^{-1}\cdots P)DP^{-1} \\ & = & PDIDIDI\cdots IDP^{-1} \\ & = & PD^{100}P^{-1}. \end{array}$$

## Example (continued)

Now,

$$D^{100} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} A^{100} &= PD^{100}P^{-1} \\ &= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix} \left(\frac{1}{3}\right) \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2^{100} + 2 \cdot 5^{100} & 2^{100} - 2 \cdot 5^{100} \\ 2^{100} - 5^{100} & 2 \cdot 2^{100} + 5^{100} \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2^{100} + 2 \cdot 5^{100} & 2^{100} - 2 \cdot 5^{100} \\ 2^{100} - 5^{100} & 2^{101} + 5^{100} \end{bmatrix} \end{aligned}$$



## Theorem

If $A$ is an $n \times n$ matrix and $P$ is an invertible $n \times n$ matrix such that $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$ for each $k = 1, 2, 3, \ldots$

The process of finding an invertible matrix $P$ and a diagonal matrix $D$ so that $A = PDP^{-1}$ is referred to as diagonalizing the matrix $A$, and $P$ is called the diagonalizing matrix for $A$.

## Questions

- When is it possible to diagonalize a matrix?
- How do we find a diagonalizing matrix?

## Answer

Eigenvalues and eigenvectors.

# Eigenvalues and Eigenvectors

## Definition

Let $A$ be an $n \times n$ matrix, $\lambda$ a real number, and $X \neq 0$ an $n$-vector. If $AX = \lambda X$, then $\lambda$ is an eigenvalue of $A$, and $X$ is an eigenvector of $A$ corresponding to $\lambda$, or a $\lambda$-eigenvector.

## Example

Let $A = \left[ \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right]$ and $X = \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]$. Then

$$AX = \left[ \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right] \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] = \left[ \begin{array}{c} 3 \\ 3 \end{array} \right] = 3 \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] = 3X.$$

This means that 3 is an eigenvalue of $A$, and $\left[ \begin{array}{c} 1 \\ 1 \end{array} \right]$ is an eigenvector of $A$ corresponding to 3 (or a 3-eigenvector of $A$).



## What an eigenvalue and eigenvector tell us about a matrix

Suppose that $A$ is an $n \times n$ matrix, with eigenvalue $\lambda$ and corresponding eigenvector $X$. Then $X \neq 0$ is an $n$-vector, $\lambda \in \mathbb{R}$, and $AX = \lambda X$.
It follows that

$$\begin{aligned} \lambda X - AX &= 0 \\ \lambda I X - AX &= 0 \\ (\lambda I - A)X &= 0 \end{aligned}$$

Since $X \neq 0$, $X$ is a nontrivial solution to the linear system with coefficient matrix $\lambda I - A$, and therefore the matrix $\lambda I - A$ is not invertible. Since a matrix is invertible if and only if its determinant is not equal to zero, it follows that

$$\det(\lambda I - A) = 0.$$

## The Characteristic Polynomial

### Definition

The characteristic polynomial of an $n \times n$ matrix $A$ is defined to be

$$c_A(x) = \det(xI - A).$$

### Example

The characteristic polynomial of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ is

$$\begin{aligned} c_A(x) &= \det(xI - A) \\ &= \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\right) \\ &= \det\begin{bmatrix} x-4 & 2 \\ 1 & x-3 \end{bmatrix} \\ &= (x-4)(x-3) - 2 \\ &= x^2 - 7x + 10. \end{aligned}$$



## Finding Eigenvalues and Eigenvectors

### Theorem

Let $A$ be an $n \times n$ matrix.

1. The eigenvalues of $A$ are the roots of $c_A(x)$.
2. The $\lambda$-eigenvectors $X$ are the nontrivial solutions to $(\lambda I - A)X = 0$.

### Procedure:

Let $A$ be an $n \times n$ matrix.

- **Eigenvalues:** Find $\lambda$ by solving the equation

$$c_A(x) = \det(xI - A) = 0$$

- **Eigenvectors:** For each $\lambda$, find $X \neq 0$ by finding the basic solutions to

$$(A - \lambda I)X = 0$$

- **Check:** For each pair of $\lambda, X$ check that $AX = \lambda X$.

## Example (continued)

For $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$, we've found

$$c_A(x) = x^2 - 7x + 10 = (x-2)(x-5),$$

so $A$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$.
The 2-eigenvectors of $A$ (meaning the eigenvectors of $A$ corresponding to $\lambda_1 = 2$) are found by solving the homogeneous system $(2I - A)X = 0$.
This is the homogeneous system with coefficient matrix:

$$2I - A = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}.$$



## Example (continued)

Solve the system in the standard way, by putting the augmented matrix of the system in reduced row-echelon form.

$$\left[\begin{array}{rr|r} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The general solution is

$$X = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R}.$$

However, since eigenvectors are nonzero, the 2-eigenvectors of $A$ are all vectors

$$X = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R} \text{ and } t \neq 0.$$

## Example (continued)

To find the 5-eigenvectors of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ solve the homogeneous system $(5I - A)X = 0$, with coefficient matrix

$$5I - A = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Therefore the 5-eigenvectors of $A$ are the vectors

$$X = \begin{bmatrix} -2s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ where } s \in \mathbb{R} \text{ and } s \neq 0.$$



# Basic Eigenvectors

## Definition

A basic eigenvector of an $n \times n$ matrix $A$ is any nonzero multiple of a basic solution to $(\lambda I - A)X = 0$, where $\lambda$ is an eigenvalue of $A$.

## Basic eigenvectors of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$

$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is is a basic eigenvector of $A$ corresponding to the eigenvalue 2.

$X = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basic eigenvector of $A$ corresponding to the eigenvalue 5.

## Eigenvalues with multiplicity greater than one

### Problem

Find the characteristic polynomial and eigenvalues of the matrix

$$A = \left[\begin{array}{rrr} 4 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{array}\right].$$

### Solution

$$\begin{aligned} c_A(x) = \det(xI - A) &= \det \left[\begin{array}{ccc} x-4 & -1 & -2 \\ 0 & x-3 & 2 \\ 0 & 1 & x-2 \end{array}\right] \\ &= (x-4)[(x-3)(x-2)-2] \\ &= (x-4)(x^2-5x+4) \\ &= (x-4)(x-4)(x-1) \\ &= (x-4)^2(x-1). \end{aligned}$$

Therefore, $A$ has eigenvalues 1 and 4, with 4 being an eigenvalue of multiplicity two.



### Definition

The multiplicity of an eigenvalue $\lambda$ of $A$ is the number of times $\lambda$ occurs as a root of $c_A(x)$.

### Example

We have seen that $A = \left[\begin{array}{rrr} 4 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{array}\right]$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$ of multiplicity two. To find an eigenvector of $A$ corresponding to $\lambda_1 = 1$, solve the homogeneous system $(I - A)X = 0$:

$$\left[\begin{array}{rrr|r} -3 & -1 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is $X = \left[\begin{array}{r} -s \\ s \\ s \end{array}\right]$ where $s \in \mathbb{R}$. We get a basic eigenvector by choosing $s = 1$ (in fact, any nonzero value of $s$ gives us a basic eigenvector).

## Example (continued)

Therefore, $X = \left[\begin{array}{r} -1 \\ 1 \\ 1 \end{array}\right]$ is a (basic) eigenvector of $A$ corresponding to $\lambda_1 = 1$.

To find an eigenvector of $A = \left[\begin{array}{rrr} 4 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{array}\right]$ corresponding the $\lambda_2 = 4$, solve the system $(4I - A)X = 0$:

$$\left[\begin{array}{rrr|r} 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is $X = \left[\begin{array}{r} s \\ -2t \\ t \end{array}\right]$ where $s, t \in \mathbb{R}$.



## Example (continued)

In this case, the general solution has two parameters, which leads to two basic eigenvectors that are not scalar multiples of each other, i.e., since

$$X = \left[\begin{array}{r} s \\ -2t \\ t \end{array}\right] = s\left[\begin{array}{r} 1 \\ 0 \\ 0 \end{array}\right] + t\left[\begin{array}{r} 0 \\ -2 \\ 1 \end{array}\right] \text{ where } s, t \in \mathbb{R},$$

we obtain basic eigenvectors

$$X_1 = \left[\begin{array}{r} 1 \\ 0 \\ 0 \end{array}\right] \text{ and } X_2 = \left[\begin{array}{r} 0 \\ -2 \\ 1 \end{array}\right].$$

We can obtain other pairs of basic 4-eigenvectors for $A$ by taking any nonzero scalar multiple of $X_1$, and any nonzero scalar multiple of $X_2$.

Notice that every 4-eigenvector of $A$ is a nonzero linear combination of basic 4-eigenvectors.

## Problem

For

$$A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix},$$

find $c_A(x)$, the eigenvalues of $A$, and basic eigenvector(s) for each eigenvalue.

## Solution

$$\begin{aligned} \det(xI - A) &= \begin{vmatrix} x-3 & 4 & -2 \\ -1 & x+2 & -2 \\ -1 & 5 & x-5 \end{vmatrix} = \begin{vmatrix} x-3 & 4 & -2 \\ 0 & x-3 & -x+3 \\ -1 & 5 & x-5 \end{vmatrix} \\ &= \begin{vmatrix} x-3 & 4 & 2 \\ 0 & x-3 & 0 \\ -1 & 5 & x \end{vmatrix} = (x-3)\begin{vmatrix} x-3 & 2 \\ -1 & x \end{vmatrix} \end{aligned}$$

Therefore, $c_A(x) = (x-3)(x^2 - 3x + 2) = (x-3)(x-2)(x-1)$.



## Solution (continued)

Since $c_A(x) = (x-3)(x-2)(x-1)$, the eigenvalues of $A$ are $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$. Notice that each of these eigenvalues has multiplicity one.

To find a basic eigenvector corresponding to $\lambda_1 = 3$, solve $(3I - A)X = 0$.

$$\left[\begin{array}{rrr|r} 0 & 4 & -2 & 0 \\ -1 & 5 & -2 & 0 \\ -1 & 5 & -2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $X = \begin{bmatrix} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$. Choosing $t = 2$ gives us

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

as a basic eigenvector corresponding to $\lambda_1 = 3$.

## Solution (continued)

To find a basic eigenvector corresponding to $\lambda_2 = 2$, solve $(2I - A)X = 0$.

$$\left[\begin{array}{rrr|r} -1 & 4 & -2 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & 5 & -3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $X = \left[\begin{array}{r} 2s \\ s \\ s \end{array}\right] = s\left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right]$, $s \in \mathbb{R}$. Choosing $s = 1$ gives us

$$X_2 = \left[\begin{array}{r} 2 \\ 1 \\ 1 \end{array}\right]$$

as an eigenvector corresponding to $\lambda_2 = 2$.



## Solution (continued)

Finally, to find a basic eigenvector corresponding to $\lambda_3 = 1$, solve $(I - A)X = 0$.

$$\left[\begin{array}{rrr|r} -2 & 4 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $X = \left[\begin{array}{r} r \\ r \\ r \end{array}\right] = r\left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right]$, $r \in \mathbb{R}$. Choosing $r = 1$ gives us

$$X_3 = \left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right]$$

is an eigenvector corresponding to $\lambda_3 = 1$.

## Solution (continued)

Summarizing, for $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$, we have found three eigenvalues, and a corresponding eigenvector for each as follows.

$$\lambda_1 = 3 \text{ and } X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}; \lambda_2 = 2 \text{ and } X_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}; \lambda_3 = 1 \text{ and } X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

An easy way to check your work: compute $AX_1$ and see if you get $3X_1$.

$$AX_1 = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 3X_1.$$

You should check that $AX_2 = 2X_2$ and that $AX_3 = 1X_3 = X_3$,



# Eigenvalues and eigenvectors (review)

## Let $A$ be an $n \times n$ matrix

1. Compute the charasteristic polynomial of $A$,
$$c_A(x) = \det(xI - A).$$
2. Factorize $c_A(x)$ and find its roots.
3. For each root $\lambda$ of $c_A(x)$ solve the homogeneous system
$$(\lambda I - A)X = 0.$$
(It always has a nontrivial solution.)
4. $\lambda$-eigenvectors are the (nontrivial) solutions to this system.

## Similar Matrices

### Definition

Let $A$, and $B$ be $n \times n$ matrices. Suppose there exists an invertible matrix $P$ such that

$$A = P^{-1}BP$$

Then $A$ and $B$ are called similar matrices.

**How do similar matrices help us in spectral theory?**

### Theorem

Let $A$ and $B$ be similar matrices, so that $A = P^{-1}BP$ where $A, B$ are $n \times n$ matrices and $P$ is invertible. Then $A$ and $B$ have the same eigenvalues.

### Proof

Assume $BX = \lambda X$. Let $Y = P^{-1}X$. Then

$$AY = (P^{-1}BP)P^{-1}X = P^{-1}BX = P^{-1}\lambda X = \lambda Y.$$



## Using Similar and Elementary Matrices

### Problem

Find the eigenvalues for the matrix

$$A = \left[\begin{array}{rrr} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{array}\right]$$

### Solution

We will use elementary matrices to simplify $A$ before finding the eigenvalues. Left multiply $A$ by $E(2,2)$, and right multiply by the inverse of $E(2,2)$.

$$\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{array}\right]\left[\begin{array}{rrr} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{array}\right]\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{array}\right] = \left[\begin{array}{rrr} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{array}\right]$$

Notice that the resulting matrix and $A$ are similar matrices (with $E(2,2)$ playing the role of $P$) so they have the same eigenvalues.

## Solution (continued)

We do this step again, on the resulting matrix above.

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix} = B$$

Again by properties of similar matrices, the resulting matrix here (labeled $B$) has the same eigenvalues as our original matrix $A$. The advantage is that it is much simpler to find the eigenvalues of $B$ than $A$.
Finding these eigenvalues follows the usual procedure and is left as an exercise.



## Example (Triangular Matrices)

Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 5 & 1 & -2 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

The characteristic polynomial of $A$ is

$$c_A(x) = \det(xI - A) = \det \begin{bmatrix} x-2 & 1 & 0 & -3 \\ 0 & x-5 & -1 & 2 \\ 0 & 0 & x & -7 \\ 0 & 0 & 0 & x+4 \end{bmatrix} = (x-2)(x-5)x(x+4).$$

Therefore the eigenvalues of $A$ are $2, 5, 0$ and $-4$, exactly the entries on the main diagonal of $A$.

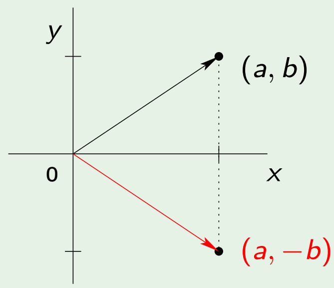
## Eigenvalues of Triangular Matrices

If $A$ is an $n \times n$ upper triangular (or lower triangular) matrix, then the eigenvalues of $A$ are the entries on the main diagonal of $A$.

# Geometric Interpretation of Eigenvalues and Eigenvectors

## Example

Recall that in $\mathbb{R}^2$, reflection in the $x$-axis is a linear transformation that transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$.



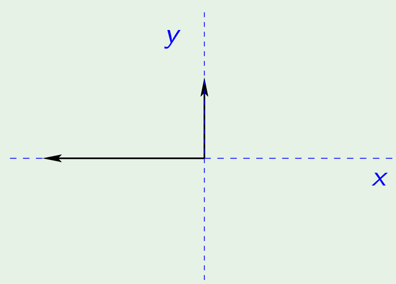

Let $A$ be the matrix that induces reflection in the $x$-axis. If $\lambda$ were an eigenvalue of $A$ and $X$ a corresponding eigenvector, then $AX = \lambda X$ implies that, geometrically, reflecting $X$ in the $x$-axis is the same as changing $X$ to a vector parallel to $X$.

How could this be possible?

Can you picture what an eigenvector of $A$ would look like?



## Example (continued)




- The reflection of $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ in the $x$-axis is $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ is an eigenvector of $A$ that corresponds to the eigenvalue $\lambda = 1$.
- The reflection of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in the $x$-axis is $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ that corresponds to the eigenvalue $\lambda = -1$.

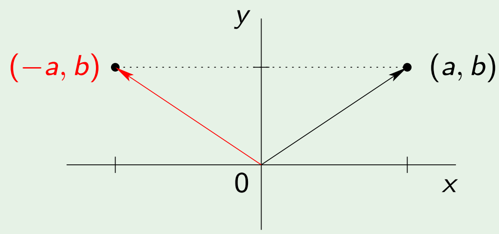
This makes sense, since we know that reflection in the $x$-axis is induced by the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which has eigenvalues 1 and $-1$.

## Example

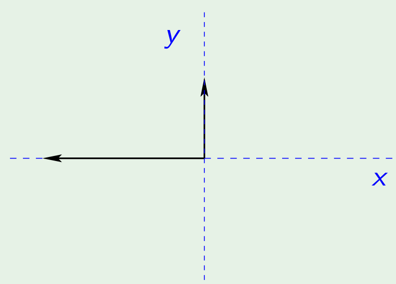
In $\mathbb{R}^2$, reflection in the $y$-axis is a linear transformation that transforms $\left[\begin{array}{c} a \\ b \end{array}\right]$ to $\left[\begin{array}{r} -a \\ b \end{array}\right]$.




Let $A$ be the matrix that induces reflection in the $y$-axis. If $\lambda$ were an eigenvalue of $A$ and $X$ a corresponding eigenvector, then $AX = \lambda X$ implies that, geometrically, reflecting $X$ in the $y$-axis is the same as changing $X$ to a vector parallel to $X$.



## Example (continued)




- The reflection of $\left[\begin{array}{r} -2 \\ 0 \end{array}\right]$ in the $y$-axis is $\left[\begin{array}{c} 2 \\ 0 \end{array}\right]$, so $\left[\begin{array}{r} -2 \\ 0 \end{array}\right]$ is an eigenvector of $A$ that corresponds to the eigenvalue $\lambda = -1$.
- The reflection of $\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ in the $y$-axis is $\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$, so $\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ is an eigenvector of $A$ that corresponds to the eigenvalue $\lambda = 1$.

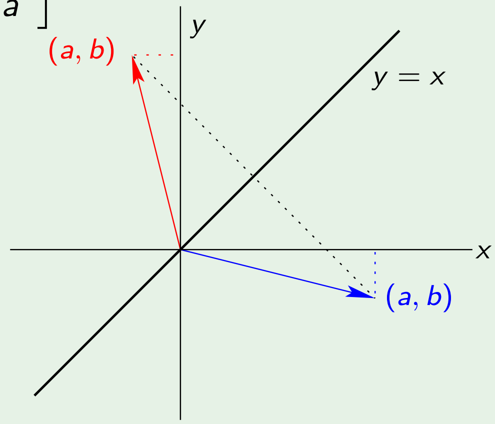
This makes sense, since we know that reflection in the $y$-axis is induced by the matrix

$$A = \left[\begin{array}{rr} -1 & 0 \\ 0 & 1 \end{array}\right],$$

which has eigenvalues 1 and $-1$.

## Example

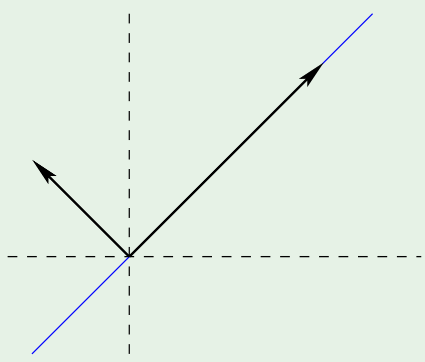
Reflection in the line $y = x$ is a linear transformation that transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.




Let $A$ be the matrix that induces reflection in the line $y = x$. If $\lambda$ were an eigenvalue of $A$ and $X$ a corresponding eigenvector, then $AX = \lambda X$ implies that, geometrically, reflecting $X$ in the $y$-axis is the same as changing $X$ to a vector parallel to $X$.



## Example (continued)



- The reflection of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ in the line $y = x$ is $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is an eigenvector of $A$ that corresponds to the eigenvalue $\lambda = 1$.
- The reflection of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in the line $y = x$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ that corresponds to the eigenvalue $\lambda = -1$.

Therefore, 1 and $-1$ are eigenvalues of $A$.

## $\mathbb{R}^2$: Reflections in lines through the origin

In $\mathbb{R}^2$, if $y = mx$ is a line through the origin, then reflection in the line $y = mx$ is a linear transformation. Let $X$ be a vector in $\mathbb{R}^2$ with tail at the origin.

If $A$ is the matrix that induces reflection in the line $y = mx$, then

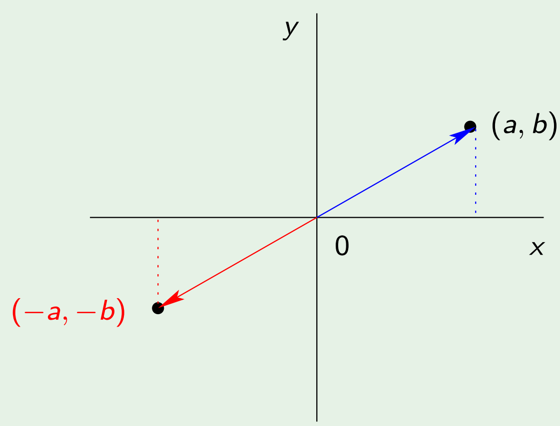
- the reflection of a vector $X$ that is parallel to $y = mx$ is simply $X$;
- the reflection of a vector $X$ that is perpendicular to $y = mx$ is $-X$.

Therefore, 1 and $-1$ are eigenvalues of $A$; in fact, these are the only two eigenvalues of $A$ and each has multiplicity one. This follows from the fact that $A$ is a $2 \times 2$ matrix, so it's characteristic polynomial has degree two.



## Example (Rotation through $\pi$)

We denote by $R_\pi : \mathbb{R}^2 \to \mathbb{R}^2$ counterclockwise rotation about the origin through an angle of $\pi$.

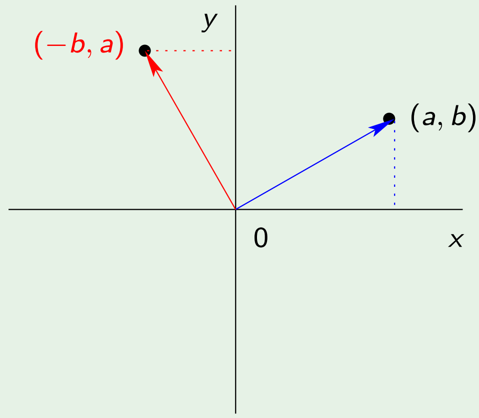



$R_\pi$ is a linear transformation that transforms $X$ to $-X$.

Let $A$ denote the matrix that induces rotation through $\pi$. Then $AX = -X$ for every nonzero vector $X$, meaning that every nonzero vector of $\mathbb{R}^2$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda = -1$.

## Example (Rotation through $\pi/2$)

We denote by $R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$ counterclockwise rotation about the origin through an angle of $\pi/2$.




$R_\pi$ is a linear transformation that transforms $\left[\begin{array}{c} a \\ b \end{array}\right]$ to $\left[\begin{array}{r} -b \\ a \end{array}\right]$.

Notice that there is no nonzero vector $X$ that can be rotated through an angle of $\pi/2$ to produce a vector parallel to $X$. Therefore, $A$ has no real eigenvalues.



## Example (continued)

Let $A$ denote the matrix that induces rotation through $\pi/2$. Then

$$A = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right],$$

and $c_A(x) = x^2 + 1$. Therefore, $A$ has complex eigenvalues $i$ and $-i$.