A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Section 4.10

$\mathbb{R}^n$: An Overview of Spanning, Linear Independence, and Basis

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Definition (Recall: Linear Combination)

Let $\vec{u}_1, \cdots, \vec{u}_n, \vec{v}$ be vectors. Then $\vec{v}$ is said to be a **linear combination** of the vectors $\vec{u}_1, \cdots, \vec{u}_n$ if there exist scalars, $a_1, \cdots, a_n$ such that

$$\vec{v} = a_1 \vec{u}_1 + \cdots + a_n \vec{u}_n$$

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Definition (Span of a Set of Vectors)

The collection of all linear combinations of a set of vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is known as the span of these vectors and is written as $\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

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Additional Terminology. If $U = \text{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$, then

- U is spanned by the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$.
- the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ span U.
- the set of vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a spanning set for U.

Problem

Let $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ *and* $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T \in \mathbb{R}^3$*. Show that* $\vec{w} = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix}^T$ *is in span* $\{\vec{u}, \vec{v}\}$*.*

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Solution

For a vector to be in span $\{\vec{u}, \vec{v}\}$, it must be a linear combination of these vectors. If $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$, we must be able to find scalars a, b such that

$$\vec{w} = a\vec{u} + b\vec{v}$$

$$\left[\begin{array}{c} 4 \\ 5 \\ 0 \end{array}\right] = a \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right] + b \left[\begin{array}{c} 3 \\ 2 \\ 0 \end{array}\right]$$

This is equivalent to the following system of equations

$$\begin{aligned} a + 3b &= 4 \\ a + 2b &= 5 \end{aligned}$$

Solution (continued)

We solving this system the usual way, constructing the augmented matrix and row reducing to find the reduced row-echelon form .

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 1 & 2 & 5 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|r} 1 & 0 & 7 \\ 0 & 1 & -1 \end{array}\right]$$

The solution is $a = 7, b = -1$. This means that

$$\vec{w} = 7\vec{u} - \vec{v}$$

Therefore we can say that $\vec{w}$ is in span $\{\vec{u}, \vec{v}\}$.

Span of a Set of Vectors

Example

Let $\vec{x} \in \mathbb{R}^3$ be a nonzero vector. Then $\text{span}\{\vec{x}\} = \{k\vec{x} \mid k \in \mathbb{R}\}$ is a line through the origin having direction vector $\vec{x}$.

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Problem

Describe the span of the vectors $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$.

Solution

Notice that any linear combination of the vectors $\vec{u}$ and $\vec{v}$ yields a vector $\begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$ in the YZ-plane.

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Suppose we take an arbitrary vector $\left[\begin{array}{c} 0 \\ y \\ z \end{array}\right]$ in the YZ-plane. It turns out we can write any such vector as a linear combination of $\vec{u}$ and $\vec{v}$.

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$$\begin{bmatrix} 0 \\ y \\ z \end{bmatrix} = (-3y + 2z) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (2y - z) \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

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Hence, $\text{span}\{\vec{u}, \vec{v}\}$ is the YZ-plane.

Span of a Set of Vectors

Consider the previous example where the span of $\vec{u}$ and $\vec{v}$ was the YZ-plane. Suppose we add another vector $\vec{w}$, and consider the span of $\vec{u}, \vec{v}$, and $\vec{w}$. What would happen to the span?

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Scenario 1 Suppose $\vec{w}$ is a vector in the YZ-plane. For example, $\vec{w} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$. Then $\vec{w}$ is in the span of $\vec{u}, \vec{v}$. Adding $\vec{w}$ to the set doesn't change the span at all.

$$\mathrm{span}\{\vec{u}, \vec{v}, \vec{w}\} = \mathrm{span}\{\vec{u}, \vec{v}\}$$

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Notice that now, the three vectors span $\mathbb{R}^3$. Any vector in $\mathbb{R}^3$ can be written as a linear combination of $\vec{u}, \vec{v}, \vec{w}$ as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (-4x + 5y + 2z) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (2x + 2y - z) \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + (x) \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

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You can see that the span of these three vectors depended on whether $\vec{w}$ was in $\text{span}\{\vec{u}, \vec{v}\}$ or not. In the next section, we will examine the distinction between these two scenarios using the concept of linear independence.

Linearly Independent Set of Vectors

Definition

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If a set of vectors is not linearly independent, we call it linearly dependent.

A Linearly Dependent Set

Problem

Consider the vectors $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$. Is the set $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

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Solution

Notice that we can write $\vec{w}$ as a linear combination of $\vec{u}, \vec{v}$ as follows:

$$\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = (-10) \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (7) \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $\vec{w}$ is in $\text{span}\{\vec{u}, \vec{v}\}$. By the definition, this set is not linearly independent (it is linearly dependent).

A Linearly Independent Set

Problem

Consider the vectors

$$\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

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Is the set $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent?

Solution

We cannot write any of the three vectors as a linear combination of the other two. (We will see how to show this soon.) Therefore the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

Linear Independence as a Linear Combination

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Theorem

The collection of vectors, $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is linearly independent if and only if whenever

$$\sum_{i=1}^{n} a_i \vec{u}_i = \vec{0}$$

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it follows that each $a_i = 0$.
Thus $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is linearly independent exactly when the system of linear equations $AX = 0$ has only the trivial solution, where A is the $n \times k$ matrix having these vectors as columns.

Linear Independence

We can state the conclusion of this theorem in another way: The set of vectors $\{\vec{u}_1, ..., \vec{u}_k\}$ is linearly independent if and only if there is no nontrivial linear combination which equals zero. If a linear combination of the vectors equals zero, then all the coefficients of the combination are zero.

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If the set is linearly independent, then

$$a_1\vec{u}_1 + ... + a_k\vec{u}_k = 0$$

implies that

$$a_1 = a_2 = ... = a_k = 0$$

Linear Independence

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Problem

Determine whether the following set of vectors are linearly independent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution

Construct the 3 x 3 matrix A having these vectors as columns:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since all columns are pivot columns (and the rank of A is 3), the vectors are linearly independent.

Problem

Determine whether the following vectors are linearly independent. If they are linearly dependent, write one of the vectors as a linear combination of the others.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 17 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 11 \\ 11 \end{bmatrix} \right\}$$

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$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 17 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 11 \\ 11 \end{bmatrix} \right\}$$

Solution

Construct the matrix A using these vectors as columns.

$$A = \begin{bmatrix} 1 & 2 & 0 & 8 \\ 2 & 0 & 1 & 5 \\ 4 & 0 & 3 & 11 \\ 1 & 3 & 0 & 11 \end{bmatrix}$$

Solution (continued)

The reduced row-echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Since the rank of A is $3 < 4$, the vectors are linearly dependent. Therefore, there are infinitely many solutions to $AX = 0$, one of which is

$$\begin{bmatrix} -2 \\ -1 \\ -3 \\ 1 \end{bmatrix}$$

Solution (continued)

Therefore we can write:

$$-2\begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 8 \\ 5 \\ 11 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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We can rewrite this as:

$$2\begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 11 \\ 11 \end{bmatrix}$$

This shows that one of the vectors can be written as a linear combination of the other three vectors. While here we chose the fourth vector, we could have chosen any of the vectors to isolate.

Subspaces

Definition

Let V be a nonempty collection of vectors in $\mathbb{R}^n$. Then V is a subspace if whenever a and b are scalars and $\vec{u}$ and $\vec{v}$ are vectors in V, $a\vec{u} + b\vec{v}$ is also in V.

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Subspaces are closely related to the span of a set of vectors which we discussed earlier.

Theorem

Let V be a nonempty collection of vectors in $\mathbb{R}^n$. Then V is a subspace of $\mathbb{R}^n$ if and only if there exist vectors $\{\vec{u}_1, ..., \vec{u}_k\}$ in V such that

$$V = \operatorname{span}\{\vec{u}_1, ..., \vec{u}_k\}$$

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If V is a subspace of $\mathbb{R}^n$, then there exist linearly independent vectors $\{\vec{u}_1, \ldots, \vec{u}_k\}$ of V such that

$$V = \operatorname{span}\{\vec{u}_1, \ldots, \vec{u}_k\}$$

In other words, subspaces of $\mathbb{R}^n$ consist of spans of finite, linearly independent collections of vectors in $\mathbb{R}^n$.

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The following theorem claims that any two bases of a subspace must be of the same size.

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The following theorem claims that any two bases of a subspace must be of the same size.

Theorem

Let V be a subspace of $\mathbb{R}^n$ and suppose $\{\vec{u}_1, \ldots, \vec{u}_k\}$ and $\{\vec{v}_1, \ldots, \vec{v}_m\}$ are two bases for V. Then $k = m$.

Dimension

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Definition

Let V be a subspace of $\mathbb{R}^n$. Then the dimension of V is the number of a vectors in a basis of V.

Properties of $\mathbb{R}^n$

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- If $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is a linearly independent set of a vectors in $\mathbb{R}^n$, then $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is a basis for $\mathbb{R}^n$.
- Suppose $\{\vec{u}_1, \ldots, \vec{u}_m\}$ spans $\mathbb{R}^n$. Then $m \geq n$.

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- Suppose $\{\vec{u}_1, ..., \vec{u}_m\}$ spans $\mathbb{R}^n$. Then $m \geq n$.
- If $\{\vec{u}_1, ..., \vec{u}_n\}$ spans $\mathbb{R}^n$, then $\{\vec{u}_1, ..., \vec{u}_n\}$ is linearly independent.
- If $\{\vec{u}_1, \vec{u}_2, ..., \vec{u}_k\}$ is a set of vectors in $\mathbb{R}^n$ with $k > n$, then the set is linearly dependent.

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- If $\{\vec{u}_1, \ldots, \vec{u}_n\}$ spans $\mathbb{R}^n$, then $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is linearly independent.
- If $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a set of vectors in $\mathbb{R}^n$ with $k > n$, then the set is linearly dependent.

It follows then that a basis is a minimal spanning set. If a subspace has dimension d, then any spanning set has size at least d, and any spanning set of size d must be a basis (and is therefore independent).

Row and Column Space

Definition

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Problem

Find the rank of the matrix A and describe the column and row spaces efficiently.

$$A = \left[\begin{array}{rrrrr} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{array} \right]$$

Example: Column Space

Solution

To find the column space, we first find the reduced row-echelon form of A:

$$\begin{bmatrix} 1 & 0 & -9 & 9 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example: Column Space

Solution

To find the column space, we first find the reduced row-echelon form of A:

$$\begin{bmatrix} 1 & 0 & -9 & 9 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $\text{rank}(A) = 2$.

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Therefore $\text{rank}(A) = 2$.

Note the first two columns are the pivot columns. All columns of the above reduced row-echelon matrix are in

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Example: Column Space

Solution (continued)

To construct the column space, we use the pivot columns of the original matrix - in this case, the first and second columns. Therefore the column space of A is

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}$$

Example: Row Space

Solution (continued)

To find the row space of A we again look at the reduced row-echelon form of the matrix.

$$\left[\begin{array}{rrrrr} 1 & 0 & -9 & 9 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Row Space

Solution (continued)

To find the row space of A we again look at the reduced row-echelon form of the matrix.

$$\left[\begin{array}{rrrrr} 1 & 0 & -9 & 9 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The row space of A is the span of the non-zero rows of the above matrix:

$$\text{span}\left\{\left[\begin{array}{ccccc} 1 & 0 & -9 & 9 & 2 \end{array}\right], \left[\begin{array}{ccccc} 0 & 1 & 5 & -3 & 0 \end{array}\right]\right\}$$

Example: Row Space

Solution (continued)

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Notice that the vectors used in the description of the column space are from the original matrix, while those in the row space are from the reduced row-echelon form of the original matrix.

Null Space

Definition

The null space of A, or kernel of A is defined as:

$$\ker(A) = \{X : AX = 0\}$$

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To find $\ker(A)$, we solve the system of equations $AX = 0$.

Problem

Find $\ker(A)$ for the matrix A:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$

Null Space

Solution

The first step is to set up the augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 3 & 3 & 0 \end{array}\right]$$

Null Space

Solution

The first step is to set up the augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 3 & 3 & 0 \end{array}\right]$$

Place this matrix in reduced row-echelon form:

$$\left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Null Space

Solution (continued)

The solution to this system of equations is

$$\left\{ \begin{bmatrix} 3t \\ t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

Null Space

Solution (continued)

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Therefore the null space of A is the span of this vector:

$$\ker(A) = \operatorname{span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Nullity

Definition

The dimension of the null space of a matrix is called the nullity, denoted $\text{null}(A)$.

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Theorem

Let A be an $m \times n$ matrix. Then,

$$\mathrm{rank}(A) + \mathrm{null}(A) = n$$

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Theorem

Let A be an $m \times n$ matrix. Then,

$$\text{rank}(A) + \text{null}(A) = n$$

For instance, in the last example, A was a 3×3 matrix. The rank was 2 and the nullity was 1 (since the null space had dimension 1).

$$\text{rank}(A) + \text{null}(A) = 2 + 1 = 3 = n$$