A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Spectral Theory: 7.2 Diagonalization

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Diagonal Matrices

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Notation

An $n \times n$ diagonal matrix

$$D = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

is written $D = \mathrm{diag}(a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$.

Diagonalizability

Definition

An $n \times n$ matrix A is said to be diagonalizable if there exists an invertible $n \times n$ matrix P such that $A = PDP^{-1}$.
Equivalently: A is diagonalizable if $A \sim D$ for some diagonal matrix D.

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Let A be an $n \times n$ matrix. The process of finding an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$ is referred to as diagonalizing the matrix A, and P is called the diagonalizing matrix for A.

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The key to diagonalizing a matrix (finding the matrices P and D) lies in the eigenvectors and eigenvalues of the matrix A.

Reminder

Let A be an $n \times n$ matrix and λ a real number. If λ is an eigenvalue of A, then

$$AX = \lambda X$$

for some **nonzero** vector X in $\mathbb{R}^n$. Such a vector X is called a λ-eigenvector of A or an eigenvector of A corresponding to λ.

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Let A be an $n \times n$ matrix.

1. *A is diagonalizable if and only if it has eigenvectors $X_1, X_2, \ldots, X_n$ so that $P = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$ is invertible.*

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1. *A is diagonalizable if and only if it has eigenvectors $X_1, X_2, \ldots, X_n$ so that $P = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$ is invertible.*
2. *If P is invertible, then*

$$P^{-1}AP = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where λ_i is the eigenvalue of A corresponding to the eigenvector X_i, i.e., $AX_i = \lambda_i X_i$.

Example

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

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Solution

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-1 & 0 \\ 0 & 0 & x+3 \end{vmatrix} = (x-1)^2(x+3).$$

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Eigenvectors for $\lambda_1 = 1$: solve $(I - A)X = 0$.

$$\left[\begin{array}{rrr|r} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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Solution (continued)

Therefore, basic eigenvectors corresponding to $\lambda_1 = 1$ are $\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$ and $\left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]$.

Eigenvectors for $\lambda_2 = -3$: solve $(-3I - A)X = 0$.

$$\left[\begin{array}{rrr|r} -4 & 0 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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Therefore, a basic eigenvector corresponding to $\lambda_2 = -3$ is $\left[\begin{array}{r} -1 \\ 0 \\ 4 \end{array}\right]$

Let

$$P = \left[\begin{array}{rrr} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{array}\right].$$

Then P is invertible (easily checked by computing $\det P$).

Solution (continued)

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Furthermore,

$$P^{-1}AP = D = \mathrm{diag}(-3, 1, 1) = \left[\begin{array}{rrr} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

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Furthermore,

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The eigenvalues of A in D (from left to right) occur in the same order as their corresponding eigenvectors as columns of P.

Example

Diagonalize the matrix $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$

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Solution

You can check that A has eigenvalues and corresponding basic eigenvectors:

$$\lambda_1 = 3 \text{ and } X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}; \lambda_2 = 2 \text{ and } X_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}; \lambda_3 = 1 \text{ and } X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

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Let $P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

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Let $P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. Then P is invertible (check this!), so by the previous theorem,

$$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Eigenvalues, Eigenvectors, and Diagonalization

Theorem

Let A be an $n \times n$ matrix, and suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. For each i, let X_i be a λ_i-eigenvector of A. Then $\{X_1, X_2, \ldots, X_m\}$ is linearly independent.

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Diagonalizability

Determining whether or not a square matrix A is diagonalizable can be done using eigenvalues and eigenvectors of the matrix A.

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Proof.

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ denote the n (distinct) eigenvalues of A, and let X_i be an eigenvector of A corresponding to λ_i, $1 \leq i \leq n$. Then $\{X_1, X_2, \ldots, X_n\}$ is a linearly independent set in $\mathbb{R}^n$ (i.e. a basis).

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It follows that $P = [X_1 \; X_2 \cdots X_n]$ is invertible, and therefore A is diagonalizable. $\square$

Example

Show that the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 8 & 6 & -2 \\ 0 & 0 & -3 \end{bmatrix}$$

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is diagonalizable.

Solution

A has characteristic polynomial

$$c_A(x) = (x+3)(x-2)(x-4),$$

and thus A has distinct eigenvalues $-3, 2$ and 4.

Since A is 3×3 and has three distinct eigenvalues, A is diagonalizable.

Definition

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In other words, the eigenspace $E_\lambda(A)$ is all X (including $X = \vec{0}$) such that $AX = \lambda X$. In other words, $E_\lambda(A) = \text{null}(\lambda I - A)$.

Definition (recall)

Let A be an $n \times n$ matrix with characteristic polynomial given by $\det(\lambda I - A)$. Then, the multiplicity of an eigenvalue λ of A is the number of times λ occurs as a root of that characteristic polynomial.

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Lemma

If A is an $n \times n$ matrix, then

$$\dim(E_\lambda(A)) \leq m$$

where λ is an eigenvalue of A of multiplicity m.

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Lemma

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$$\dim(E_\lambda(A)) \leq m$$

where λ is an eigenvalue of A of multiplicity m.

This result tells us that if λ is an eigenvalue of A, then the number of linearly independent λ-eigenvectors is never more than the multiplicity of λ.

The crucial consequence of the above Lemma is the characterization of matrices that are diagonalizable.

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Theorem

Let A be an $n \times n$ matrix A. Then A is diagonalizable if and only if for each eigenvalue λ of A, $\dim(E_\lambda(A))$ is equal to the multiplicity of λ.

Example

Let

$$A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]$$

Determine if A is diagonalizable

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Solution

Through the usual procedure, we find that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$. Solving as usual, we find that the eigenvectors are given by

$$t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and the basic eigenvector is } X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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This means that $\dim(E_\lambda(A)) = 1$, but the multiplicity of $\lambda = 1$ is 2. Therefore this matrix is not diagonalizable.

Complex Eigenvalues

If a matrix has eigenvalues that have imaginary parts (and aren't simply real numbers), we can still find eigenvectors and possibly diagonalize the matrix.

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Problem

Diagonalize, if possible, the matrix $A = \left[\begin{array}{rr} 1 & 1 \\ -1 & 1 \end{array} \right]$.

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Problem

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Solution

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & -1 \\ 1 & x-1 \end{vmatrix} = x^2 - 2x + 2.$$

The roots of $c_A(x)$ are distinct complex numbers: $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$, so A is diagonalizable. Corresponding eigenvectors are

$$X_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} i \\ 1 \end{bmatrix},$$

respectively.

Solution (continued)

A diagonalizing matrix for A is

$$P = \left[\begin{array}{cc} -i & i \\ 1 & 1 \end{array} \right],$$

and

$$P^{-1}AP = \left[\begin{array}{cc} 1+i & 0 \\ 0 & 1-i \end{array} \right].$$

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Notice that A is a real matrix, but has complex eigenvalues (and eigenvectors).

Eigenvalues of a Real Symmetric Matrix

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The eigenvalues of any real symmetric matrix are real.

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Proof.

Let A be an $n \times n$ real symmetric matrix, and let λ be an eigenvalue of A. To prove that λ is real, it is enough to prove that $\overline{\lambda} = \lambda$, i.e., λ is equal to its (complex) conjugate.

We use $\overline{A}$ to denote the matrix obtained from A by replacing each entry by its conjugate. Since A is real, $\overline{A} = A$.

Suppose

$$X = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

is a λ-eigenvector of A. Then $AX = \lambda X$.

Proof (continued).

Let $y = X^T\overline{X} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{bmatrix}$.

Proof (continued).

Let $y = X^T \overline{X} = \left[\begin{array}{cccc} z_1 & z_2 & \cdots & z_n \end{array} \right] \left[\begin{array}{c} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{array} \right]$.

Then $y = z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$;

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Then $y = z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$; since $X \neq 0$, y is a positive real number. So

$$\lambda y \quad = \quad \lambda(X^T\overline{X}) = (\lambda X^T)\overline{X} = (\lambda X)^T\overline{X}$$

Proof (continued).

Let $y = X^T\overline{X} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{bmatrix}$.

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$$\begin{aligned} \lambda y &= \lambda(X^T\overline{X}) = (\lambda X^T)\overline{X} = (\lambda X)^T\overline{X} \\ &= (AX)^T\overline{X} = X^T A^T \overline{X} \end{aligned}$$

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Proof (continued).

Let $y = X^T\overline{X} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{bmatrix}$.

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$$\begin{aligned} \lambda y &= \lambda(X^T\overline{X}) = (\lambda X^T)\overline{X} = (\lambda X)^T\overline{X} \\ &= (AX)^T\overline{X} = X^TA^T\overline{X} \\ &= X^TA\overline{X} \quad \text{(since } A \text{ is symmetric)} \\ &= X^T\,\overline{AX} \quad \text{(since } A \text{ is real)} \end{aligned}$$

Proof (continued).

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\lambda y &= \lambda(X^T\overline{X}) = (\lambda X^T)\overline{X} = (\lambda X)^T\overline{X} \\
&= (AX)^T\overline{X} = X^T A^T \overline{X} \\
&= X^T A \overline{X} \quad \text{(since } A \text{ is symmetric)} \\
&= X^T\ \overline{A}\overline{X} \quad \text{(since } A \text{ is real)} \\
&= X^T(\overline{AX}) = X^T(\overline{\lambda X}) = X^T\ \overline{\lambda}\overline{X}
\end{aligned}
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Proof (continued).

Let $y = X^T\overline{X} = \left[\begin{array}{cccc} z_1 & z_2 & \cdots & z_n \end{array}\right]\left[\begin{array}{c} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{array}\right]$.

Then $y = z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$; since $X \neq 0$, y is a positive real number. So

$$\begin{aligned} \lambda y &= \lambda(X^T\overline{X}) = (\lambda X^T)\overline{X} = (\lambda X)^T\overline{X} \\ &= (AX)^T\overline{X} = X^TA^T\overline{X} \\ &= X^TA\overline{X} \quad \text{(since } A \text{ is symmetric)} \\ &= X^T\ \overline{AX} \quad \text{(since } A \text{ is real)} \\ &= X^T(\overline{AX}) = X^T(\overline{\lambda X}) = X^T\ \overline{\lambda X} \\ &= \overline{\lambda}(X^T\overline{X}) \end{aligned}$$

Proof (continued).

Let $y = X^T\overline{X} = \left[\begin{array}{cccc} z_1 & z_2 & \cdots & z_n \end{array}\right] \left[\begin{array}{c} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{array}\right]$.

Then $y = z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$; since $X \neq 0$, y is a positive real number. So

$$\begin{aligned}
\lambda y &= \lambda(X^T\overline{X}) = (\lambda X^T)\overline{X} = (\lambda X)^T\overline{X} \\
&= (AX)^T\overline{X} = X^T A^T \overline{X} \\
&= X^T A \overline{X} \quad \text{(since } A \text{ is symmetric)} \\
&= X^T\ \overline{A}\overline{X} \quad \text{(since } A \text{ is real)} \\
&= X^T(\overline{AX}) = X^T(\overline{\lambda X}) = X^T\ \overline{\lambda}\overline{X} \\
&= \overline{\lambda}(X^T\overline{X}) \\
&= \overline{\lambda} y.
\end{aligned}$$

Proof (continued).

Let $y = X^T\overline{X} = \left[\begin{array}{cccc} z_1 & z_2 & \cdots & z_n \end{array}\right] \left[\begin{array}{c} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{array}\right]$.

Then $y = z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$; since $X \neq 0$, y is a positive real number. So

$$\begin{aligned} \lambda y &= \lambda(X^T\overline{X}) = (\lambda X^T)\overline{X} = (\lambda X)^T\overline{X} \\ &= (AX)^T\overline{X} = X^TA^T\overline{X} \\ &= X^TA\overline{X} \quad \text{(since } A \text{ is symmetric)} \\ &= X^T\ \overline{AX} \quad \text{(since } A \text{ is real)} \\ &= X^T(\overline{AX}) = X^T(\overline{\lambda X}) = X^T\ \overline{\lambda X} \\ &= \overline{\lambda}(X^T\overline{X}) \\ &= \overline{\lambda}y. \end{aligned}$$

Thus, $\lambda y = \overline{\lambda}y$. Since $y \neq 0$, it follows that $\lambda = \overline{\lambda}$, and therefore λ is real. □