A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

6.3: Complex Numbers; Roots of Complex Numbers

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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If θ is any angle and n is a positive integer, $\left(e^{i\theta}\right)^n = e^{in\theta}$. This implies that for any real number $r > 0$ and any positive integer n,

$$(re^{i\theta})^n = r^n e^{in\theta}.$$

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This leads to the following result.

Corollary

Let q be a nonzero complex number and n a positive integer. Then $z^n = q$ has exactly n complex solutions, i.e., q has exactly n complex n^{th} roots.

Example

For any positive real number a, $z^2 = a$ has two complex (in this case, real) solution, $z = \sqrt{a}$ and $z = -\sqrt{a}$. This is equivalent to the statement that a has two complex (in this case, real) square roots.

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- One particular example: 25 has two square roots, 5 and -5, and these are the two solutions to $z^2 = 25$.
- But we all knew that. A more interesting example is that -1 has no real square roots, but suddenly it has two (complex) square roots, i and $-i$. These are the two (complex) solutions to $z^2 = 1$.

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  ($\mathbb{Z}$ denotes the set of integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$).

Example (continued)

Dividing both sides of $3\theta = \frac{\pi}{2} + 2\pi\ell$ by 3:

$$\theta = \frac{\pi}{6} + \frac{2}{3}\pi\ell,$$

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We now convert these to Cartesian form.

Example (continued)

$$e^{\pi i/6} \quad =$$

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This process is summarized in the following procedure.

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Let w be a complex number. We wish to find the n^{th} roots of w, that is all z such that $z^n = w$.

There are n distinct n^{th} roots and they can be found as follows:.

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1. Express both z and w in polar form $z = re^{i\theta}, w = se^{i\phi}$. Then $z^n = w$ becomes:

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2. Solve the following two equations:

$$\begin{aligned} r^n &= s \\ e^{in\theta} &= e^{i\phi} \end{aligned} \qquad (1)$$

Continued

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3. The solutions to $r^n = s$ are given by $r = \sqrt[n]{s}$.
4. The solutions to $e^{in\theta} = e^{i\phi}$ are given by:

$$n\theta = \phi + 2\pi\ell, \text{ for } \ell = 0, 1, 2, \cdots, n-1$$

or

$$\theta = \frac{\phi}{n} + \frac{2}{n}\pi\ell, \text{ for } \ell = 0, 1, 2, \cdots, n-1$$

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5. Using the solutions r, θ to the equations given in (1) construct the n^{th} roots of the form $z = re^{i\theta}$.

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2. The equation becomes $r^4 e^{i4\theta} = 4e^{2\pi i/3}$, so we need to solve

$$\begin{aligned} r^4 &= 4 \\ e^{i4\theta} &= e^{2\pi i/3} \end{aligned}$$

Solution (continued)

3. Since $r^4 = 4$, $r^2 = \pm 2$. But r is real, and so $r^2 = 2$, implying $r = \pm\sqrt{2}$. However $r \geq 0$, and therefore $r = \sqrt{2}$.

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Therefore,

$$\theta = \frac{2\pi}{12} + \frac{2\pi\ell}{4} = \frac{\pi}{6} + \frac{\pi\ell}{2} = \frac{\pi(3\ell+1)}{6}, \text{ for } \ell = 0, 1, 2, 3.$$

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$$\ell = 0: \quad z = \sqrt{2}e^{\pi i/6} \quad = \sqrt{2}\left(\frac{(\sqrt{3}}{2} + \frac{1}{2}i\right) \quad = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$$

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\ell = 1: \quad z = \sqrt{2}e^{2\pi i/3} \quad &= \sqrt{2}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \quad &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i
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\ell = 1: \quad z &= \sqrt{2}e^{2\pi i/3} &&= \sqrt{2}(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i) &&= -\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{6}}{2}i \\
\ell = 2: \quad z &= \sqrt{2}e^{7\pi i/6} &&= \sqrt{2}(-\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i) &&= -\tfrac{\sqrt{6}}{2} - \tfrac{\sqrt{2}}{2}i
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Solution (continued)

5. Thus $r = \sqrt{2}$ and $\theta = \left(\frac{3\ell+1}{6}\right)\pi$, $\ell = 0, 1, 2, 3$. Converting to Cartesian form:

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Therefore, the fourth roots of $2(\sqrt{3}i - 1)$ are:

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A complex number z is a root of unity if there exists a positive integer n so that $z^n = 1$.

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$$\theta = \frac{2\pi\ell}{6} = \frac{\pi\ell}{3}, \ \ell = 0, 1, 2, 3, 4, 5.$$

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If you graph these six point in the complex plane, you'll see that they result in six equally spaced points on the unit circle, one of them being $(1, 0)$.

Roots of Unity

For any integer $n \geq 1$, the (complex) solutions to $z^n = 1$ are

$$z = e^{2\pi \ell i/n} \text{ for } \ell = 0, 1, 2, \ldots, n-1.$$

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For any integer $n \geq 1$, the (complex) solutions to $z^n = 1$ are

$$z = e^{2\pi \ell i/n} \text{ for } \ell = 0, 1, 2, \ldots, n-1.$$

Furthermore, the n^{th} roots of unity correspond to n equally spaced points on the unit circle, one of them being $(1, 0)$.