A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Linear Transformations: Rotation Transformations in $\mathbb{R}^2$

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Recall: Matrix Transformation

Definition

Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\vec{x}) = A\vec{x} \text{ for each } \vec{x} \in \mathbb{R}^n$$

is called the matrix transformation induced by A.

Rotations in $\mathbb{R}^2$

Definition

The transformation

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denotes counterclockwise rotation about the origin through an angle of θ.

Rotation through an angle of θ preserves scalar multiplication.

Rotation through an angle of θ preserves vector addition.

R_θ is a linear transformation

Since R_θ preserves addition and scalar multiplication, R_θ is a linear transformation, and hence a matrix transformation.

The matrix that induces R_θ can be found by computing $R_\theta(E_1)$ and $R_\theta(E_2)$, where

$$E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$R_\theta(E_1) = R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

and

$$R_\theta(E_2) = R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

The Matrix for R_θ

The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and is induced by the matrix

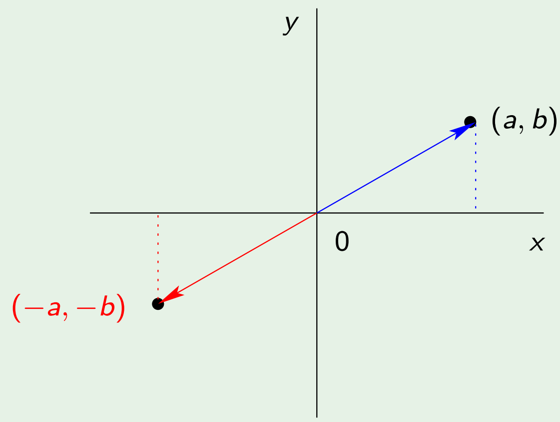
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Example (Rotation through π)

We denote by

$$R_\pi : \mathbb{R}^2 \to \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of π.



We see that $R_\pi \left[\begin{array}{c} a \\ b \end{array}\right] = \left[\begin{array}{c} -a \\ -b \end{array}\right] = \left[\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right] \left[\begin{array}{c} a \\ b \end{array}\right]$, so R_π is a matrix transformation.

Rotation

Problem

The transformation $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes a counterclockwise rotation about the origin through an angle of $\frac{\pi}{2}$ radians. Find the matrix of $R_{\frac{\pi}{2}}$.

Solution

First,

$$R_{\frac{\pi}{2}} \left[\begin{array}{c} a \\ b \end{array}\right] = \left[\begin{array}{c} -b \\ a \end{array}\right]$$

Furthermore $R_{\frac{\pi}{2}}$ is a matrix transformation, and the matrix it is induced by is

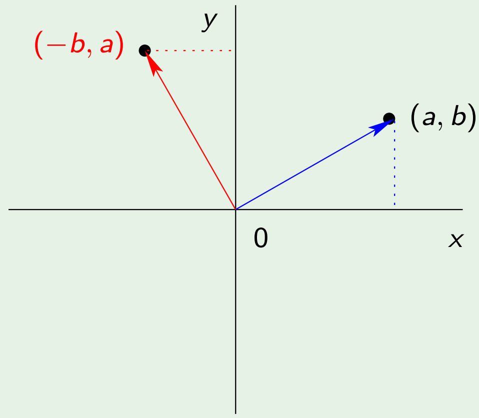
$$\left[\begin{array}{c} -b \\ a \end{array}\right] = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{c} a \\ b \end{array}\right].$$

Example (Rotation through $\pi/2$)

We denote by

$$R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of $\pi/2$.



We see that $R_{\pi/2} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so $R_{\pi/2}$ is a matrix transformation.

Reflection in $\mathbb{R}^2$

Example

In $\mathbb{R}^2$, reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

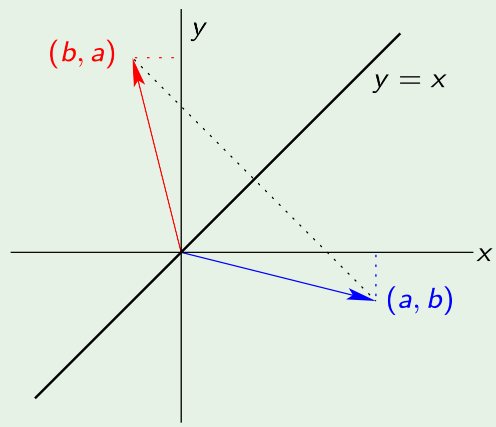
Example

In $\mathbb{R}^2$, reflection in the y-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} -a \\ b \end{bmatrix}$. This is a matrix transformation because

$$\begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example

Reflection in the line $y = x$ transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.



This is a matrix transformation because

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$