A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Complex Numbers: The Quadratic Formula

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Real Quadratics

Definition

A real quadratic is an expression of the form $ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

To find the roots of a real quadratic, we can either factor by inspection, or use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ in the quadratic formula is called the discriminant, and

- if $b^2 - 4ac \geq 0$, then the roots of the quadratic are real;
- if $b^2 - 4ac < 0$, then the quadratic has no real roots.

Definition

A real quadratic $ax^2 + bx + c$ is called irreducible if the discriminant is less than zero, i.e., $b^2 - 4ac < 0$.

Notice that if $b^2 - 4ac < 0$, then

$$\sqrt{b^2 - 4ac} = \sqrt{(-1)(4ac - b^2)} = (\pm)i\sqrt{4ac - b^2}.$$

It follows that the roots of an irreducible quadratic are

$$\frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \begin{cases} -\frac{b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}i \\ -\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}i \end{cases},$$

and we see that the two roots are complex conjugates of each other. We denote the two roots by

$$u = -\frac{b}{2a} + \frac{\sqrt{4ac - b^2}}{2a}i \text{ and } \overline{u} = -\frac{b}{2a} - \frac{\sqrt{4ac - b^2}}{2a}i.$$

Real Quadratics with Complex Roots

Example

The quadratic $x^2 - 14x + 58$ has roots

$$\begin{aligned} x &= \frac{14 \pm \sqrt{196 - 4 \times 58}}{2} \\ &= \frac{14 \pm \sqrt{196 - 232}}{2} \\ &= \frac{14 \pm \sqrt{-36}}{2} \\ &= \frac{14 \pm 6i}{2} \\ &= 7 \pm 3i, \end{aligned}$$

so the roots are $7 + 3i$ and $7 - 3i$.

Conversely, given $u = a + bi$ with $b \neq 0$, there is an irreducible quadratic having roots u and $\overline{u}$.

Problem

Find an irreducible quadratic with $u = 5 - 2i$ as a root. What is the other root?

Solution

$$\begin{aligned} (x - u)(x - \overline{u}) &= (x - (5 - 2i))(x - (5 + 2i)) \\ &= x^2 - (5 - 2i)x - (5 + 2i)x + (5 - 2i)(5 + 2i) \\ &= x^2 - 10x + 29. \end{aligned}$$

Therefore, $x^2 - 10x + 29$ is an irreducible quadratic with roots $5 - 2i$ and $5 + 2i$.
Notice that $-10 = -(u + \overline{u})$ and $29 = u\overline{u} = |u|^2$.

Exercise

Find an irreducible quadratic with root $u = -3 + 4i$, and find the other root.

Answer

$x^2 + 6x + 25$ has roots $u = -3 + 4i$ and $\overline{u} = -3 - 4i$.

Quadratics with Complex Coefficients

Problem

Find the roots of the quadratic $x^2 - (3 - 2i)x + (5 - i) = 0$.

Solution

Using the quadratic formula

$$x = \frac{3 - 2i \pm \sqrt{(-(3 - 2i))^2 - 4(5 - i)}}{2}.$$

Now,

$$(-(3 - 2i))^2 - 4(5 - i) = 5 - 12i - 20 + 4i = -15 - 8i,$$

so

$$x = \frac{3 - 2i \pm \sqrt{-15 - 8i}}{2}.$$

To find $\pm\sqrt{-15 - 8i}$, solve $z^2 = -15 - 8i$ for z.

Solution (continued)

Let $z = a + bi$ and $z^2 = -15 - 8i$. Then

$$(a^2 - b^2) + 2abi = -15 - 8i,$$

so $a^2 - b^2 = -15$ and $2ab = -8$.

Solving for a and b gives us $z = 1 - 4i, -1 + 4i$, i.e., $z = \pm(1 - 4i)$. Therefore,

$$x = \frac{3 - 2i \pm (1 - 4i)}{2},$$

and

$$\tfrac{3-2i+(1-4i)}{2} = \tfrac{4-6i}{2} = 2 - 3i,$$

$$\tfrac{3-2i-(1-4i)}{2} = \tfrac{2+2i}{2} = 1 + i.$$

Thus the roots of $x^2 - (3 - 2i)x + (5 - i)$ are $2 - 3i$ and $1 + i$.

Exercise

Find the roots of $x^2 - 3ix + (-3 + i)$.

Answer

$1 + i$ and $-1 + 2i$.

Problem

Verify that $u_1 = (4 - i)$ is a root of

$$x^2 - (2 - 3i)x - (10 + 6i)$$

and find the other root, u_2.

Solution

First,

$$\begin{aligned} u_1^2 - (2-3i)u_1 - (10+6i) &= (4-i)^2 - (2-3i)(4-i) - (10+6i) \\ &= (15-8i) - (5-14i) - (10+6i) \\ &= 0, \end{aligned}$$

so $u_1 = (4 - i)$ is a root.

Solution (continued)

Recall that if u_1 and u_2 are the roots of the quadratic, then

$$u_1 + u_2 = (2 - 3i) \text{ and } u_1 u_2 = -(10 + 6i).$$

Solve for u_2 using either one of these equations.
Since $u_1 = 4 - i$ and $u_1 + u_2 = 2 - 3i$,

$$u_2 = 2 - 3i - u_1 = 2 - 3i - (4 - i) = -2 - 2i.$$

Therefore, the other root is $u_2 = -2 - 2i$.
You can easily verify your answer by computing $u_1 u_2$:

$$u_1 u_2 = (4 - i)(-2 - 2i) = -10 - 6i = -(10 + 6i).$$