A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Linear Transformations: Matrix of a Linear Transformation

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Recall: Linear Transformations

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (preservation of addition)
2. $T(a\vec{x}) = aT(\vec{x})$ (preservation of scalar multiplication)

Matrix Transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then we can find an $n \times m$ matrix A such that

$$T(\vec{x}) = A\vec{x}$$

In this case, we say that T is induced, or determined, by A and we write

$$T_A(\vec{x}) = A\vec{x}$$

Problem

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$ for each $\vec{x} \in \mathbb{R}^3$ is another matrix transformation, that is, $T(\vec{x}) = A\vec{x}$ for some matrix A. Can you find a matrix A that works?

Solution

First, since $T : \mathbb{R}^3 \to \mathbb{R}^4$, we know that A must have size 4×3. Now consider the product

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix},$$

and try to fill in the values of the matrix.

Solution (continued)

We can deduce from the product that T is induced by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Is there an easier way to find the matrix of T?

For some transformations guess and check will work, but this is not an efficient method. The next theorem gives a method for finding the matrix of T.

Definition

The set of columns $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of I_n is called the standard basis of $\mathbb{R}^n$.

Matrix and Linear Transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation. Furthermore, T is induced by the unique matrix

$$A = \left[\begin{array}{cccc} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{array}\right],$$

where $\vec{e}_j$ is the j^{th} column of I_n, and $T(\vec{e}_j)$ is the j^{th} column of A.

Corollary

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} x+2y \\ x-y \end{array}\right]$$

for each $\vec{x} \in \mathbb{R}^2$. Find the matrix, A, of T.

Solution

To find A, we must find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$.

$$T\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} 1+2(0) \\ 1-0 \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \text{ and } T\left[\begin{array}{c} 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} 0+2(1) \\ 0-1 \end{array}\right] = \left[\begin{array}{r} 2 \\ -1 \end{array}\right]$$

The columns $T(\vec{e}_1)$ and $T(\vec{e}_2)$ become the columns of A, so

$$A = \left[\begin{array}{rr} 1 & 2 \\ 1 & -1 \end{array}\right],$$

and $T(\vec{x}) = A\vec{x}$ for every $\vec{x} \in \mathbb{R}^2$. Therefore A is the matrix for T.

Find the Matrix of T

Problem

Sometimes T is not defined so nicely for us. Suppose T is given as

$$T\left[\begin{array}{c} 1 \\ 1 \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \end{array}\right],\ T\left[\begin{array}{r} 0 \\ -1 \end{array}\right] = \left[\begin{array}{c} 3 \\ 2 \end{array}\right]$$

Find the matrix A of T.

Solution (continued)

We need to write $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of the vectors provided. First, find x and y such that

$$\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = x\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + y\left[\begin{array}{c} 0 \\ -1 \end{array}\right]$$

Once we find x and y we can compute

$$\begin{aligned} T\left[\begin{array}{c} 1 \\ 0 \end{array}\right] &= xT\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + yT\left[\begin{array}{c} 0 \\ -1 \end{array}\right] \\ &= x\left[\begin{array}{c} 1 \\ 2 \end{array}\right] + y\left[\begin{array}{c} 3 \\ 2 \end{array}\right] \end{aligned}$$

Solution (continued)

Finding x and y involves solving the following system of equations.

$$\begin{aligned} x &= 1 \\ x - y &= 0 \end{aligned}$$

The solution is $x = 1, y = 1$.
Hence, we can find $T(\vec{e}_1)$ as follows.

$$T\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = 1\left[\begin{array}{c} 1 \\ 2 \end{array}\right] + 1\left[\begin{array}{c} 3 \\ 2 \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \end{array}\right] + \left[\begin{array}{c} 3 \\ 2 \end{array}\right] = \left[\begin{array}{c} 4 \\ 4 \end{array}\right]$$

This is the first column of the matrix A. Similarly, we can find $T(\vec{e}_2)$ which will be the second column of A. The resulting matrix is

$$A = \left[\begin{array}{cc} 4 & -3 \\ 4 & -2 \end{array}\right]$$

Determining if a Transformation is Linear

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a transformation defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x+2y \end{bmatrix}$.

One way to show that T is a linear transformation is to show that it preserves addition and scalar multiplication. However, now that we know that linear transformations are matrix transformations, we can use this to our advantage.

If T were a linear transformation, then T would be induced by the matrix

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}.$$

Since

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x+2y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix},$$

T is a matrix transformation, and therefore a linear transformation.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x+y \end{bmatrix}$.

If T were a linear transformation, then T would be induced by the matrix

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

However,

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x+y \end{bmatrix}.$$

We see from this that if $x = 0$ or $y = 0$, then $xy = 0$, so $A\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix}$.

But if we take $x = y = 1$, then

$$A\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \text{ while } T\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

i.e., $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq T\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore, T in not a linear transformation.