A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

6.1: Complex Numbers

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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- The set of real numbers, $\mathbb{R}$, consists of all rational and irrational numbers (note that integers are rational numbers). However, we still can't solve
$$x^2 + 1 = 0$$
because this requires $x^2 = -1$, but any real number x has the property that $x^2 \geq 0$.

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  - If $b = 0$, then z is a real number.

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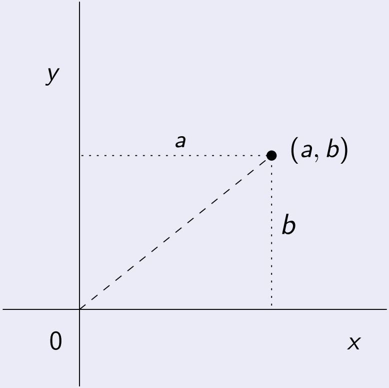

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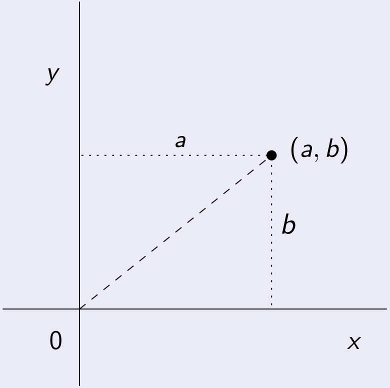

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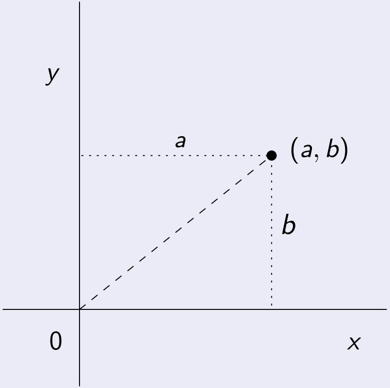

- Real numbers: $a + 0i$ lie on the x-axis.
- Pure imaginary numbers: $0 + bi$ $(b \neq 0)$ lie on the y-axis.

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4. For every $z = a + bi$ there exists a complex number $-z = -a - bi$ such that $z + (-z) = 0$. (existence of an additive inverse)

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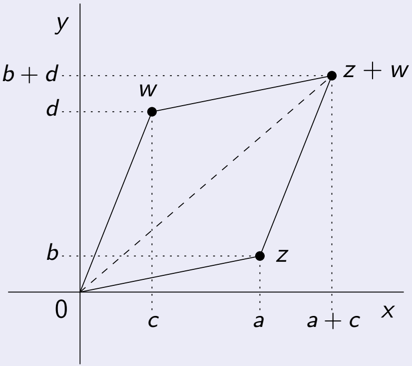

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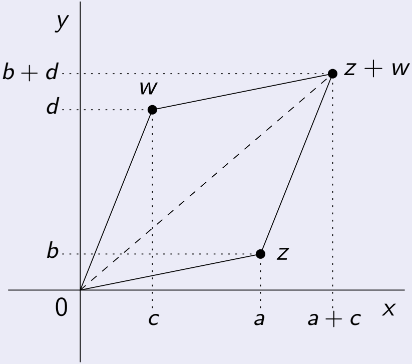

0, z, w, and $z + w$ are the vertices of a parallelogram.

Multiplication of Complex Numbers

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Example

$$\begin{aligned}(2 - 3i)(-3 + 4i) = \quad &= ((2)(-3) - (-3)(4)) + ((2)(4) + (-3)(-3))i \\ &= (-6 + 12) + (8 + 9)i \\ &= 6 + 17i\end{aligned}$$

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Therefore, if $z^2 = -3 + 4i$*, then* $z = 1 + 2i$ *or* $z = -1 - 2i$*.*

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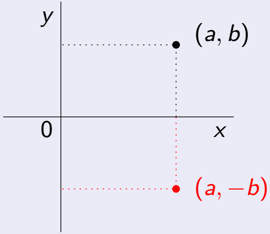

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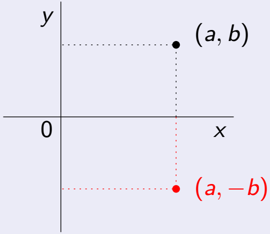

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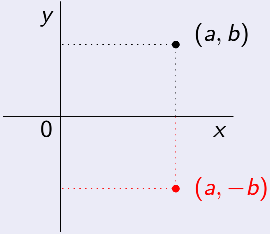

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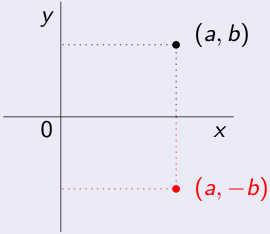

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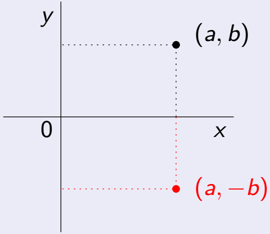

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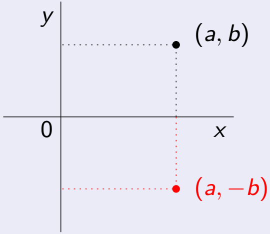

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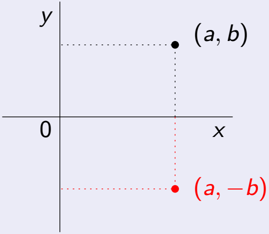

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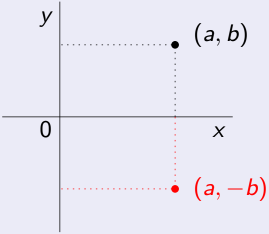

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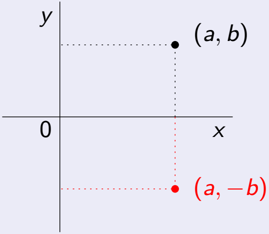

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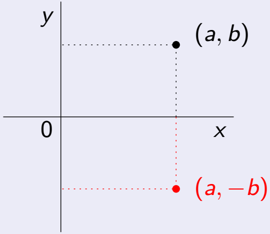

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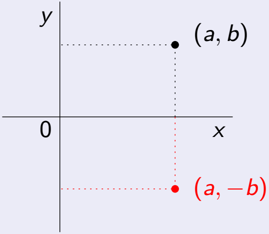

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Lecture 2

Division of Complex Numbers

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$$
\begin{aligned}
\frac{z}{w} = \frac{a+bi}{c+di} &= \frac{a+bi}{c+di} \times \frac{c-di}{c-di} \\
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The quotient $\frac{z}{w}$ is obtained by multiplying both top and bottom of $\frac{z}{w}$ by $\overline{w}$ and then simplifying the expression.

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You can always check that $zz^{-1} = 1$.

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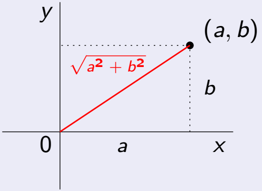
Geometrically, $|z| = \sqrt{a^2 + b^2}$ is the distance from z to the origin.



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7. The Triangle Inequality
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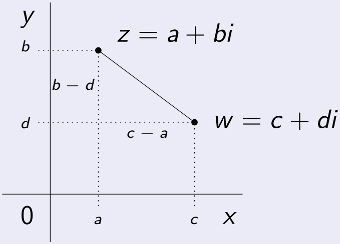

This shows that the distance between z and w in the complex plane is just the absolute value of their difference.

The Triangle Inequality: Geometrically

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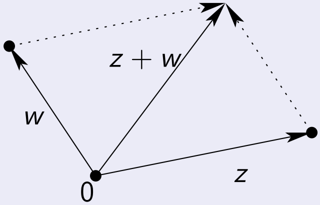

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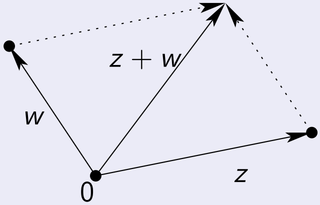

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Since the length of any side of a triangle is at most the sum of the lengths of the other two sides, we get $|z + w| \leq |z| + |w|$.