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4. λ-eigenvectors are the (nontrivial) solutions to this system.

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Let A, and B be $n \times n$ matrices. Suppose there exists an invertible matrix P such that

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Assume $BX = \lambda X$. Let $Y = P^{-1}X$. Then

$$AY = (P^{-1}BP)P^{-1}X = P^{-1}BX = P^{-1}\lambda X = \lambda Y.$$

Using Similar and Elementary Matrices

Problem

Find the eigenvalues for the matrix

$$A = \left[\begin{array}{rrr} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{array} \right]$$

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Solution

We will use elementary matrices to simplify A before finding the eigenvalues. Left multiply A by $E(2 \times 2 + 3)$, and right multiply by the inverse of $E(2 \times 2 + 3)$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

Notice that the resulting matrix and A are similar matrices (with $E(2 \times 2 + 3)$ playing the role of P) so they have the same eigenvalues.

Solution (continued)

We do this step again, on the resulting matrix above.

$$\left[\begin{array}{rrr} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\left[\begin{array}{rrr} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{array}\right]\left[\begin{array}{rrr} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{rrr} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{array}\right] = B$$

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Again by properties of similar matrices, the resulting matrix here (labeled B) has the same eigenvalues as our original matrix A. The advantage is that it is much simpler to find the eigenvalues of B than A.

Finding these eigenvalues follows the usual procedure and is left as an exercise.

Example (Triangular Matrices)

Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 5 & 1 & -2 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

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Eigenvalues of Triangular Matrices

If A is an $n \times n$ upper triangular (or lower triangular) matrix, then the eigenvalues of A are the entries on the main diagonal of A.

Geometric Interpretation of Eigenvalues and Eigenvectors

Example

Recall that in $\mathbb{R}^2$, reflection in the x-axis is a linear transformation that transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$.

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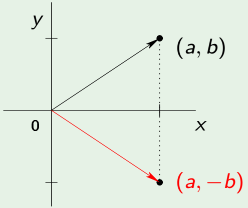

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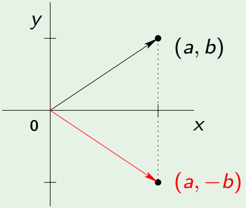

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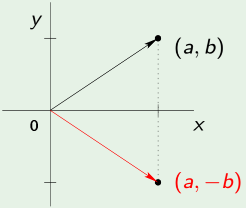

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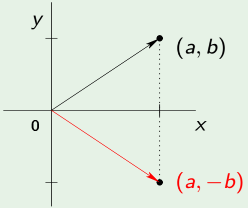

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How could this be possible?

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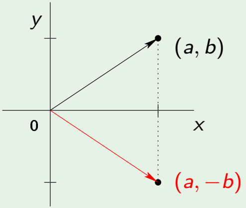

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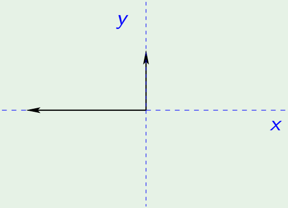
Can you picture what an eigenvector of A would look like?

Example (continued)



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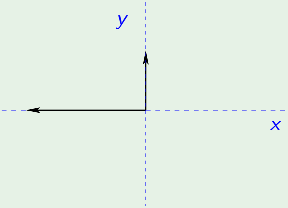

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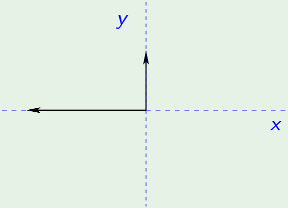

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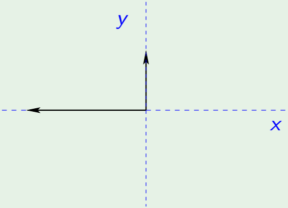

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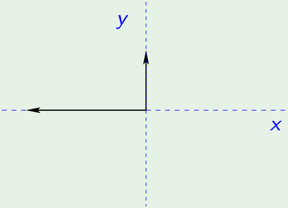

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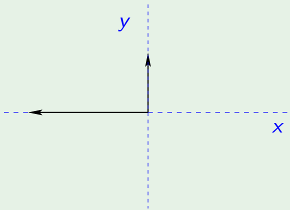

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This makes sense, since we know that reflection in the x-axis is induced by the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which has eigenvalues 1 and -1.

Example

In $\mathbb{R}^2$, reflection in the y-axis is a linear transformation that transforms $\left[\begin{array}{c} a \\ b \end{array}\right]$ to $\left[\begin{array}{r} -a \\ b \end{array}\right]$.

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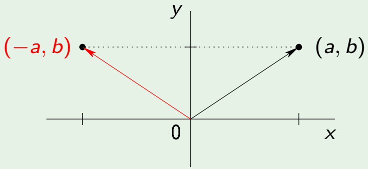

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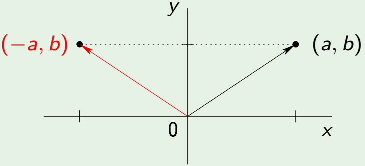

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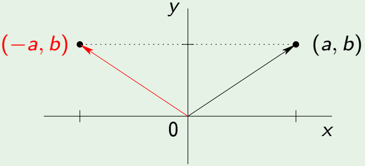

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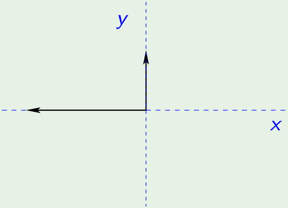

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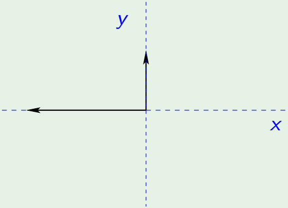

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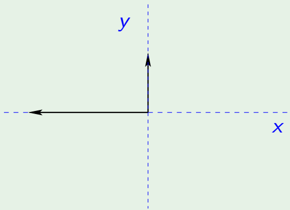

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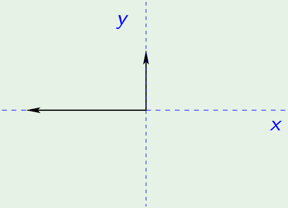

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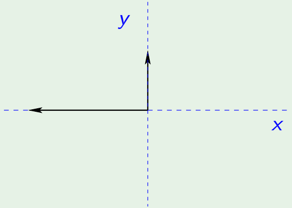

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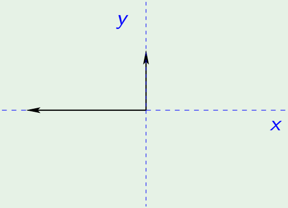

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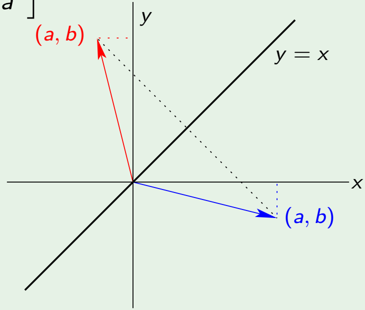

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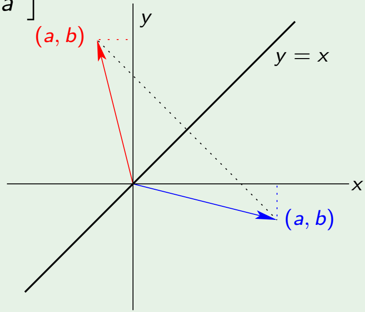

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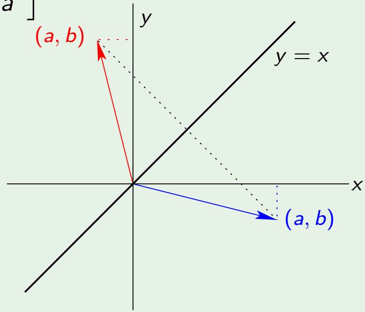

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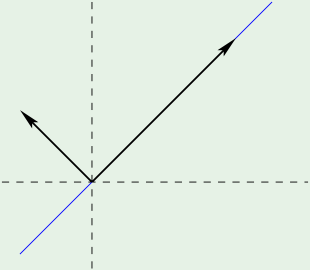

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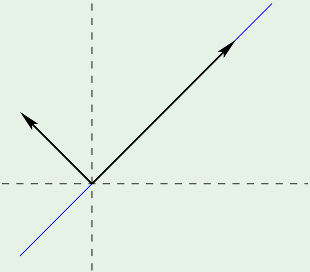

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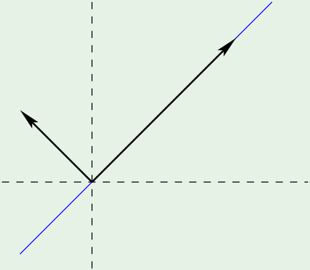

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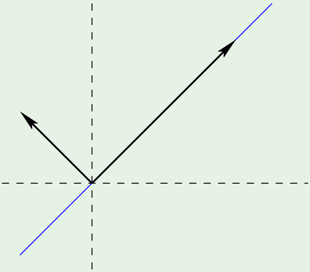

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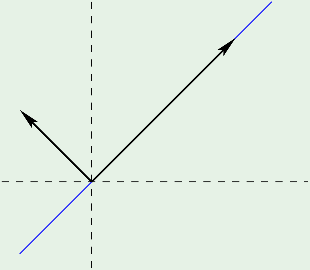

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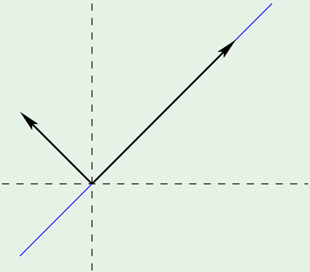
Example (continued)



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Therefore, 1 and -1 are eigenvalues of A.

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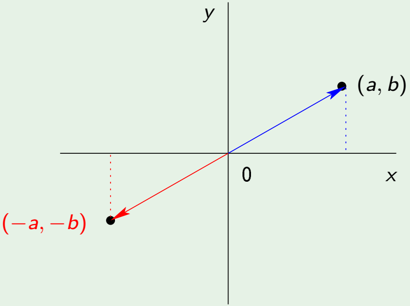

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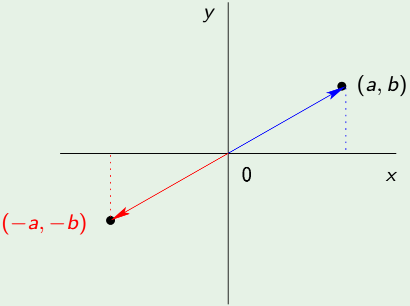

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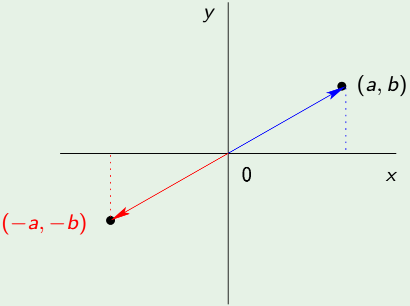

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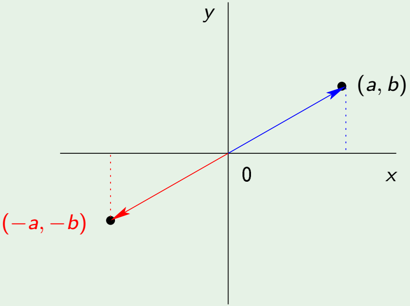

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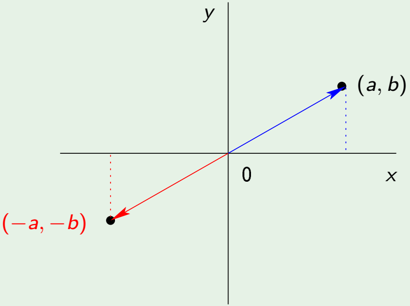

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Let A denote the matrix that induces rotation through π. Then $AX = -X$ for every nonzero vector X, meaning that every nonzero vector of $\mathbb{R}^2$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -1$.

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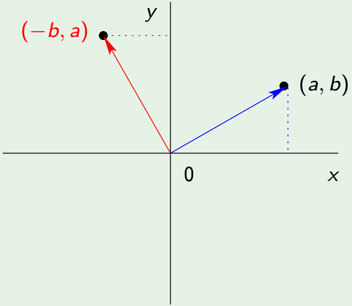

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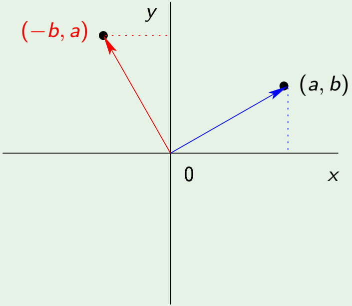

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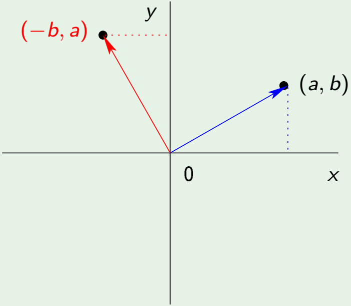

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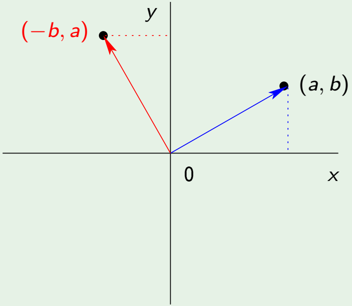

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Notice that there is no nonzero vector X that can be rotated through an angle of $\pi/2$ to produce a vector parallel to X. Therefore, A has no real eigenvalues.

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A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Spectral Theory: 7.2 Diagonalization

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

In addition we recognize the following contributors. All new content contributed is released under the same license as noted below.

- **Tim Alderson, University of New Brunswick**
- **Ilijas Farah, York University**
- **Ken Kuttler, Brigham Young University**
- **Asia Weiss, York University**

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Notation

An $n \times n$ diagonal matrix

$$D = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

is written $D = \operatorname{diag}(a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$.

Diagonalizability

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An $n \times n$ matrix A is said to be diagonalizable if there exists an invertible $n \times n$ matrix P such that $A = PDP^{-1}$.
Equivalently: A is diagonalizable if $A \sim D$ for some diagonal matrix D.

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The key to diagonalizing a matrix (finding the matrices P and D) lies in the eigenvectors and eigenvalues of the matrix A.

Reminder

Let A be an $n \times n$ matrix and λ a real number. If λ is an eigenvalue of A, then

$$AX = \lambda X$$

for some **nonzero** vector X in $\mathbb{R}^n$. Such a vector X is called a λ-eigenvector of A or an eigenvector of A corresponding to λ.

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Let A be an $n \times n$ matrix.

1. *A is diagonalizable if and only if it has eigenvectors $X_1, X_2, \ldots, X_n$ so that $P = \left[\begin{array}{cccc} X_1 & X_2 & \cdots & X_n \end{array} \right]$ is invertible.*

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2. *If P is invertible, then*

$$P^{-1}AP = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where λ_i is the eigenvalue of A corresponding to the eigenvector X_i, i.e., $AX_i = \lambda_i X_i$.

Example

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

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Solution

$$c_A(x) = \det(xI - A) = \left|\begin{array}{ccc} x-1 & 0 & -1 \\ 0 & x-1 & 0 \\ 0 & 0 & x+3 \end{array}\right| = (x-1)^2(x+3).$$

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Eigenvectors for $\lambda_1 = 1$: solve $(I - A)X = 0$.

$$\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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Solution (continued)

Therefore, basic eigenvectors corresponding to $\lambda_1 = 1$ are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda_2 = -3$: solve $(-3I - A)X = 0$.

$$\left[\begin{array}{rrr|r} -4 & 0 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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$$P = \left[\begin{array}{rrr} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{array}\right].$$

Then P is invertible (easily checked by computing $\det P$).

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The eigenvalues of A in D (from left to right) occur in the same order as their corresponding eigenvectors as columns of P.

Example

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Solution

You can check that A has eigenvalues and corresponding basic eigenvectors:

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$$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Eigenvalues, Eigenvectors, and Diagonalization

Theorem

Let A be an $n \times n$ matrix, and suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. For each i, let X_i be a λ_i-eigenvector of A. Then $\{X_1, X_2, \ldots, X_m\}$ is linearly independent.

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Diagonalizability

Determining whether or not a square matrix A is diagonalizable can be done using eigenvalues and eigenvectors of the matrix A.

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Proof.

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ denote the n (distinct) eigenvalues of A, and let X_i be an eigenvector of A corresponding to λ_i, $1 \leq i \leq n$. Then $\{X_1, X_2, \ldots, X_n\}$ is a linearly independent set in $\mathbb{R}^n$ (i.e. a basis).

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Proof.

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ denote the n (distinct) eigenvalues of A, and let X_i be an eigenvector of A corresponding to λ_i, $1 \leq i \leq n$. Then $\{X_1, X_2, \ldots, X_n\}$ is a linearly independent set in $\mathbb{R}^n$ (i.e. a basis).
It follows that $P = [X_1 \ X_2 \cdots X_n]$ is invertible, and therefore A is diagonalizable. $\square$

Example

Show that the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 8 & 6 & -2 \\ 0 & 0 & -3 \end{bmatrix}$$

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Solution

A has characteristic polynomial

$$c_A(x) = (x+3)(x-2)(x-4),$$

and thus A has distinct eigenvalues $-3, 2$ and 4.

Since A is 3×3 and has three distinct eigenvalues, A is diagonalizable.

Definition

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$. The eigenspace of A corresponding to λ, written $E_\lambda(A)$ is the subspace spanned by the set of all eigenvectors corresponding to λ.

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In other words, the eigenspace $E_\lambda(A)$ is all X (including $X = \vec{0}$) such that $AX = \lambda X$. In other words, $E_\lambda(A) = \text{null}(\lambda I - A)$.

Definition (recall)

Let A be an $n \times n$ matrix with characteristic polynomial given by $\det(\lambda I - A)$. Then, the multiplicity of an eigenvalue λ of A is the number of times λ occurs as a root of that characteristic polynomial.

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Lemma

If A is an $n \times n$ matrix, then

$$\dim(E_\lambda(A)) \leq m$$

where λ is an eigenvalue of A of multiplicity m.

Lecture 2

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Lemma

If A is an $n \times n$ matrix, then

$$\dim(E_\lambda(A)) \leq m$$

where λ is an eigenvalue of A of multiplicity m.

This result tells us that if λ is an eigenvalue of A, then the number of linearly independent λ-eigenvectors is never more than the multiplicity of λ.

The crucial consequence of the above Lemma is the characterization of matrices that are diagonalizable.

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Theorem

Let A be an $n \times n$ matrix A. Then A is diagonalizable if and only if for each eigenvalue λ of A, $\dim(E_\lambda(A))$ is equal to the multiplicity of λ.

Example

Let

$$A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]$$

Determine if A is diagonalizable

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Through the usual procedure, we find that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$. Solving as usual, we find that the eigenvectors are given by

$$t \left[\begin{array}{c} 1 \\ 0 \end{array} \right] \text{ and the basic eigenvector is } X_1 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$$

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This means that $\dim(E_\lambda(A)) = 1$, but the multiplicity of $\lambda = 1$ is 2. Therefore this matrix is not diagonalizable.

Complex Eigenvalues

If a matrix has eigenvalues that have imaginary parts (and aren't simply real numbers), we can still find eigenvectors and possibly diagonalize the matrix.

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Complex Eigenvalues

If a matrix has eigenvalues that have imaginary parts (and aren't simply real numbers), we can still find eigenvectors and possibly diagonalize the matrix.

Problem

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Solution

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & -1 \\ 1 & x-1 \end{vmatrix} = x^2 - 2x + 2.$$

The roots of $c_A(x)$ are distinct complex numbers: $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$, so A is diagonalizable. Corresponding eigenvectors are

$$X_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} i \\ 1 \end{bmatrix},$$

respectively.

Solution (continued)

A diagonalizing matrix for A is

$$P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix},$$

and

$$P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

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Notice that A is a real matrix, but has complex eigenvalues (and eigenvectors).

Eigenvalues of a Real Symmetric Matrix

Theorem

The eigenvalues of any real symmetric matrix are real.

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Proof.

Let A be an $n \times n$ real symmetric matrix, and let λ be an eigenvalue of A. To prove that λ is real, it is enough to prove that $\overline{\lambda} = \lambda$, i.e., λ is equal to its (complex) conjugate.

We use $\overline{A}$ to denote the matrix obtained from A by replacing each entry by its conjugate. Since A is real, $\overline{A} = A$.

Suppose

$$X = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

is a λ-eigenvector of A. Then $AX = \lambda X$.

Proof (continued).

Let $y = X^T\overline{X} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{bmatrix}$.

Proof (continued).

Let $y = X^T\overline{X} = \left[\begin{array}{cccc} z_1 & z_2 & \cdots & z_n \end{array}\right] \left[\begin{array}{c} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{array}\right]$.

Then $y = z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$;

Proof (continued).

Let $y = X^T\overline{X} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{bmatrix}$.

Then $y = z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$; since $X \neq 0$, y is a positive real number. So

$$\lambda y \quad = \quad \lambda(X^T\overline{X}) = (\lambda X^T)\overline{X} = (\lambda X)^T\overline{X}$$

Proof (continued).

Let $y = X^T\overline{X} = \left[\begin{array}{cccc} z_1 & z_2 & \cdots & z_n \end{array}\right]\left[\begin{array}{c} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{array}\right]$.

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$$\begin{aligned} \lambda y &= \lambda(X^T\overline{X}) = (\lambda X^T)\overline{X} = (\lambda X)^T\overline{X} \\ &= (AX)^T\overline{X} = X^TA^T\overline{X} \end{aligned}$$

Proof (continued).

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\end{aligned}$$

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Thus, $\lambda y = \overline{\lambda}y$. Since $y \neq 0$, it follows that $\lambda = \overline{\lambda}$, and therefore λ is real. $\square$