A First Course in

LINEAR ALGEBRA

Lecture Notes

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$\mathbb{R}^n$: Spanning Sets of Vectors

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A First Course in Linear Algebra

Lecture Notes

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These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Notation

Definition

$\mathbb{R}$ denotes the set of real numbers. $\mathbb{R}^n$ is the set of all n-tuples of real numbers, i.e.,

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n\}.$$

Vectors are denoted as follows: $\vec{u}, \vec{v}, \vec{x}$, etc.

Example

$$\vec{u} = \left[\begin{array}{r} -2 \\ 3 \\ 0.7 \\ 5 \\ \pi \end{array} \right]$$ is a vector in $\mathbb{R}^5$, written $\vec{u} \in \mathbb{R}^5$.

To save space on the page, the same vector $\vec{u}$ may be written instead as a row matrix by taking the transpose of the column:

$$\vec{u} = \left[\begin{array}{ccccc} -2, & 3, & 0.7, & 5, & \pi \end{array} \right]^T.$$

Definition (Recall: Linear Combination)

Let $\vec{u}_1, \cdots, \vec{u}_n, \vec{v}$ be vectors. Then $\vec{v}$ is said to be a **linear combination** of the vectors $\vec{u}_1, \cdots, \vec{u}_n$ if there exist scalars, $a_1, \cdots, a_n$ such that

$$\vec{v} = a_1\vec{u}_1 + \cdots + a_n\vec{u}_n$$

Definition (Span of a Set of Vectors)

The collection of all linear combinations of a set of vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is known as the span of these vectors and is written as $\mathrm{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

Additional Terminology. If $U = \mathrm{span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$, then

- U is spanned by the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$.
- the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$ span U.
- the set of vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is a spanning set for U.

Example

Let $\vec{x} \in \mathbb{R}^3$ be a nonzero vector. Then $\text{span}\{\vec{x}\} = \{k\vec{x} \mid k \in \mathbb{R}\}$ is a line through the origin having direction vector $\vec{x}$.

Example

Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ be nonzero vectors that are not parallel. Then

$$\text{span}\{\vec{x}, \vec{y}\} = \{k\vec{x} + t\vec{y} \mid k, t \in \mathbb{R}\}$$

is a plane through the origin containing $\vec{x}$ and $\vec{y}$.

How would you find the equation of this plane?

Problem

Let $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ *and* $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T \in \mathbb{R}^3$*. Show that* $\vec{w} = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix}^T$ *is in span* $\{\vec{u}, \vec{v}\}$*.*

Solution

For a vector to be in span $\{\vec{u}, \vec{v}\}$, it must be a linear combination of these vectors. If $\vec{w} \in \text{span}\,\{\vec{u}, \vec{v}\}$, we must be able to find scalars a, b such that

$$\vec{w} = a\vec{u} + b\vec{v}$$

We proceed as follows.

$$\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

This is equivalent to the following system of equations

$$\begin{aligned} a + 3b &= 4 \\ a + 2b &= 5 \end{aligned}$$

Solution (continued)

We solving this system the usual way, constructing the augmented matrix and row reducing to find the reduced row-echelon form .

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 1 & 2 & 5 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|r} 1 & 0 & 7 \\ 0 & 1 & -1 \end{array}\right]$$

The solution is $a = 7, b = -1$. This means that

$$\vec{w} = 7\vec{u} - \vec{v}$$

Therefore we can say that $\vec{w}$ is in span $\{\vec{u}, \vec{v}\}$.

Problem

Let $\vec{u} = \left[\begin{array}{ccc} 1 & 1 & 1 \end{array} \right]^T$ and $\vec{v} = \left[\begin{array}{ccc} 3 & 2 & 0 \end{array} \right]^T \in \mathbb{R}^3$. Show that $\vec{w} = \left[\begin{array}{ccc} 4 & 5 & 0 \end{array} \right]^T$ is in span $\{\vec{u}, \vec{v}\}$.

This is almost identical to the previous, except that $\vec{u}$ (above) has one entry that is different.

Solution

In this case, the system of linear equations is inconsistent which you can verify. Therefore $\vec{w} \notin$ span $\{\vec{u}, \vec{v}\}$.

Problem

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, $U_1 = \text{span}\{\vec{x}, \vec{y}\}$, and $U_2 = \text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\}$. Prove that $U_1 = U_2$.

Solution

To show that $U_1 = U_2$, prove that $U_1 \subseteq U_2$, and $U_2 \subseteq U_1$.

Since $2\vec{x} - \vec{y}, 2\vec{y} + \vec{x} \in U_1$, it follows that $\text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\} \subseteq U_1$, i.e., $U_2 \subseteq U_1$.

Also, since

$$\begin{aligned} \vec{x} &= \frac{2}{5}(2\vec{x} - \vec{y}) + \frac{1}{5}(2\vec{y} + \vec{x}), \\ \vec{y} &= -\frac{1}{5}(2\vec{x} - \vec{y}) + \frac{2}{5}(2\vec{y} + \vec{x}), \end{aligned}$$

$\vec{x}, \vec{y} \in U_2$. Therefore, $\text{span}\{\vec{x}, \vec{y}\} \subseteq U_2$, i.e., $U_1 \subseteq U_2$. The result now follows.

Definition

Let $\vec{e}_j$ denote the j^{th} column of I_n, the $n \times n$ identity matrix; $\vec{e}_j$ is called the j^{th} coordinate vector of $\mathbb{R}^n$.

Claim

$\mathbb{R}^n = \text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.

Proof.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Then $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$, where $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Therefore, $\vec{x} \in \text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, and thus $\mathbb{R}^n \subseteq \text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.

Conversely, since $\vec{e}_i \in \mathbb{R}^n$ for each i, $1 \leq i \leq n$ (and $\mathbb{R}^n$ is a vector space), it follows that $\text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} \subseteq \mathbb{R}^n$. The equality now follows. $\square$

Problem

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$

Show that $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Solution

If you check, you'll find that $\vec{e}_2$ can not be written as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$, and $\vec{u}_4$.