A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Linear Transformations: Properties

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A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Recall: Linear Transformations

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (preservation of addition)
2. $T(a\vec{x}) = aT(\vec{x})$ (preservation of scalar multiplication)

Composition of Linear Transformations

Definition

Suppose $T : \mathbb{R}^k \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations. The composite (or composition) of S and T is

$$S \circ T : \mathbb{R}^k \to \mathbb{R}^m,$$

is defined by

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) \text{ for all } \vec{x} \in \mathbb{R}^k.$$

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, bend left, "T"] \arrow[rr, bend right, "S \circ T"'] & \mathbb{R}^n \arrow[r, bend left, "S"] & \mathbb{R}^m \end{tikzcd}$$

Be careful with the order of the transformations! We write $S \circ T$, but it is the transformation T that is applied first, followed by the transformation S.

Theorem

Let $\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$ be linear transformations, and suppose that S is induced by matrix A, and T is induced by matrix B. Then $S \circ T$ is a linear transformation, and $S \circ T$ is induced by the matrix AB.

Example

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations defined by

$$S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \text{ and } T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \text{ for all } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$

Example (continued)

Then S and T are induced by matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

respectively. The composite of S and T is the transformation $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(S \circ T) \begin{bmatrix} x \\ y \end{bmatrix} = S\left(T \begin{bmatrix} x \\ y \end{bmatrix} \right),$$

and has matrix (or is induced by the matrix)

$$AB = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Example (continued)

Therefore the composite of S and T is the linear transformation

$$(S \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = AB\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix},$$

for each $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Compare this with the composite of T and S which is the linear transformation

$$(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix},$$

for each $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Inverse of a Linear Transformations

Definition

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations such that for each $\vec{x} \in \mathbb{R}^n$,

$$(S \circ T)(\vec{x}) = \vec{x} \text{ and } (T \circ S)(\vec{x}) = \vec{x}.$$

Then T and S are invertible transformations; S is called an inverse of T, and T is called an inverse of S. (Geometrically, S reverses the action of T, and T reverses the action of S.)

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix transformation induced by matrix A. Then A is invertible if and only if T has an inverse. In the case where T has an inverse, the inverse is unique and is denoted T^{-1}. Furthermore, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is induced by the matrix A^{-1}.

Inverse of a Linear Transformations

Fundamental Identities relating T and T^{-1}

1. $T^{-1} \circ T = 1_{\mathbb{R}^n}$
2. $T \circ T^{-1} = 1_{\mathbb{R}^n}$

Example

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a transformation given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

Then T is a linear transformation induced by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Notice that the matrix A is invertible. Therefore the transformation T has an inverse, T^{-1}, induced by

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Example (continued)

Consider the action of T and T^{-1}.

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

$$T^{-1}\begin{bmatrix} x+y \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x+y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore

$$T^{-1}\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$$