A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

$\mathbb{R}^n$: Vectors

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

In addition we recognize the following contributors. All new content contributed is released under the same license as noted below.

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- **Asia Weiss, York University**

What is $\mathbb{R}^n$?

Notation and Terminology

- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{R}^2$ denotes the set of all column vectors with two entries.
- $\mathbb{R}^3$ denotes the set of all column vectors with three entries.
- In general, $\mathbb{R}^n$ denotes the set of all column vectors with n entries.

Scalar quantities versus vector quantities

- A scalar quantity has only magnitude; e.g. time, temperature.
- A (non-zero) vector quantity has both magnitude and direction; e.g. displacement, force, wind velocity.

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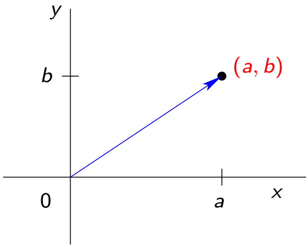
- A (non-zero) vector quantity has both magnitude and direction; e.g. displacement, force, wind velocity.

Whereas two scalar quantities are equal if they are represented by the same value, two vector quantities are equal if and only if they have the same magnitude and direction.

$\mathbb{R}^2$ and $\mathbb{R}^3$

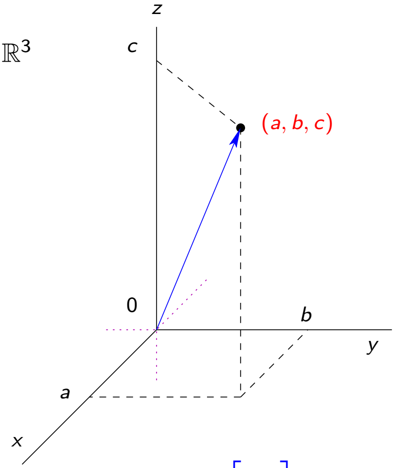
Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have convenient geometric representations as **position vectors** of points in the 2-dimensional (Cartesian) plane and in 3-dimensional space, respectively.

$\mathbb{R}^2$



The vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

$\mathbb{R}^3$



The vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Notation

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  is often used to denote the position vector of the point.
- Instead of using a capital letter to denote the vector (as we generally do with matrices), we emphasize the importance of the geometry and the direction with an arrow over the name of the vector.

Notation and Terminology

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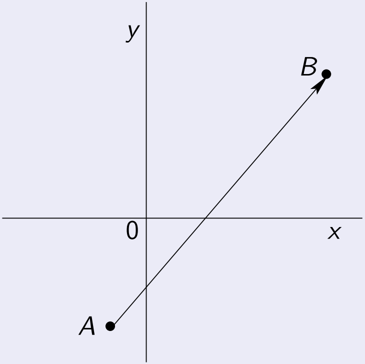
- Any vector
$$\vec{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] \text{ in } \mathbb{R}^n$$
is associated with the point $(x_1, x_2, \ldots, x_n)$.
- Often, there is no distinction made between the vector $\vec{x}$ and the point $(x_1, x_2, \ldots, x_n)$, and we say that both $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and
$$\vec{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] \in \mathbb{R}^n.$$

Geometric Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Let A and B be two points in $\mathbb{R}^2$ or $\mathbb{R}^3$.



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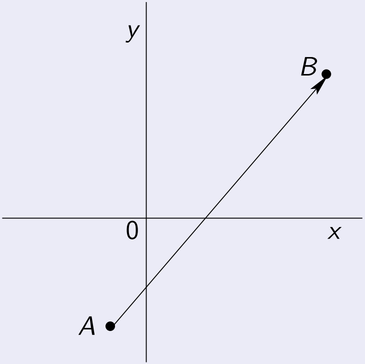

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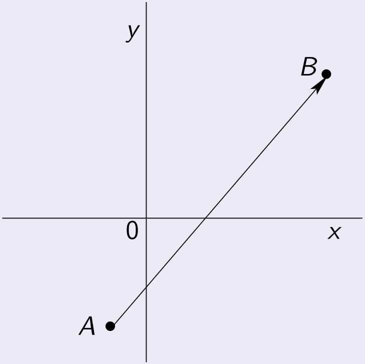
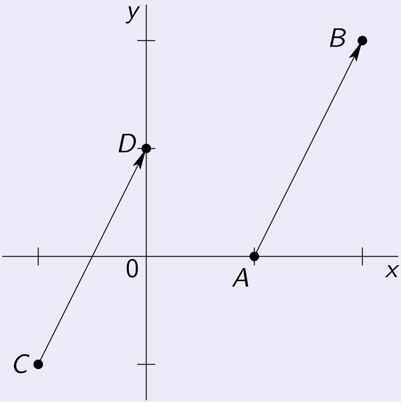


- $\overrightarrow{AB}$ is the geometric vector from A to B.
- A is the tail of $\overrightarrow{AB}$.
- B is the tip of $\overrightarrow{AB}$.
- the magnitude of $\overrightarrow{AB}$ is its length, and is denoted $||\overrightarrow{AB}||$.

Equality of geometric vectors



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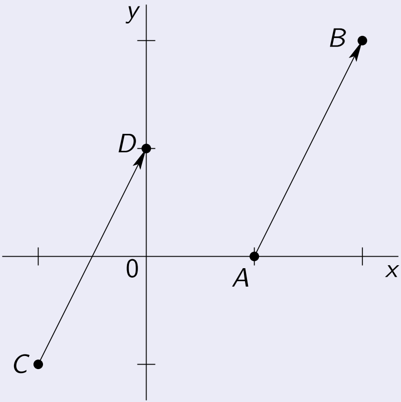

- $\overrightarrow{AB}$ is the vector from $A = (1, 0)$. to $B = (2, 2)$.
- $\overrightarrow{CD}$ is the vector from $C = (-1, -1)$ to $D = (0, 1)$.

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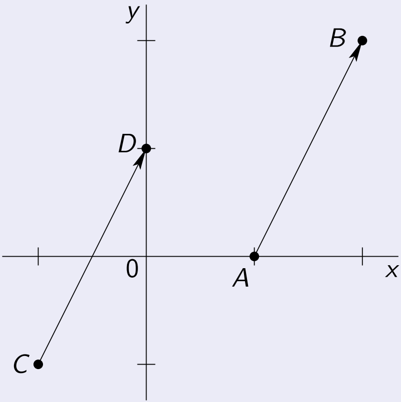

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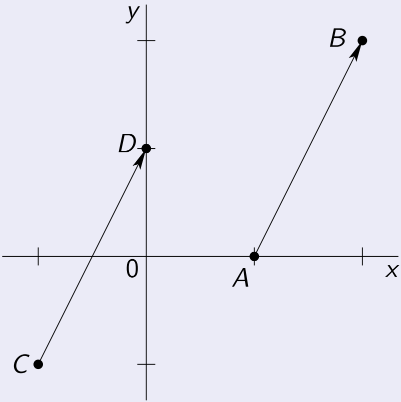

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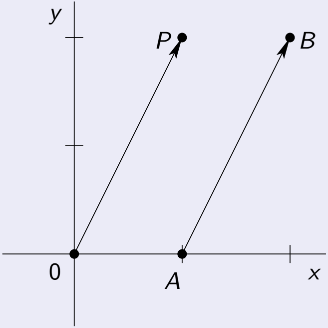
The fact that the points A and B are different from the points C and D is not important. For geometric vectors, the location of the vector in the plane (or in 3-dimensional space) is not important; the important properties are its length and direction.

Coordinatizing Vectors – Part 1



$\overrightarrow{0P}$ is the position vector for $P = (1, 2)$,

and $\overrightarrow{0P} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

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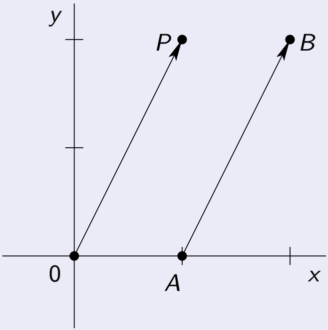

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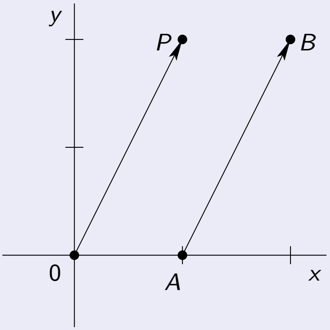

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A geometric vector is coordinatized by putting it in standard position, meaning with its tail at the origin, and then identifying the vector with its tip.

Algebra in $\mathbb{R}^n$

Addition in $\mathbb{R}^n$

Since vectors in $\mathbb{R}^n$ are $n \times 1$ matrices, addition in $\mathbb{R}^n$ is precisely matrix addition using column matrices, i.e.,

- If $\vec{u}$ and $\vec{v}$ are in $\mathbb{R}^n$, then $\vec{u} + \vec{v}$ is obtained by adding together corresponding entries of the vectors.
- The zero vector in $\mathbb{R}^n$ is the $n \times 1$ zero matrix, and is denoted $\vec{0}$.

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Example

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. Then,

$$\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

Properties of Vector Addition

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3. $\vec{u} + \vec{0} = \vec{u}$ (existence of an additive identity).
4. $\vec{u} + (-\vec{u}) = \vec{0}$ (existence of an additive inverse).

Scalar Multiplication

Since vectors in $\mathbb{R}^n$ are $n \times 1$ matrices, scalar multiplication in $\mathbb{R}^n$ is precisely matrix scalar multiplication using column matrices, i.e., If $\vec{u}$ is a vector in $\mathbb{R}^n$ and $k \in \mathbb{R}$ is a scalar, then $k\vec{u}$ is obtained by multiplying every entry of $\vec{u}$ by k.

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Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $k = 4$. Then,

$$k\vec{u} = 4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$

Properties of Scalar Multiplication

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3. $k(p\vec{u}) = (kp)\vec{u}$ (scalar multiplication is associative).
4. $1\vec{u} = \vec{u}$ (existence of a multiplicative identity).

Some notation you may encounter

Often, in $\mathbb{R}^2$ the notation $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is used. Whereas in $\mathbb{R}^3$ the notation is $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So we have

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\vec{i} + b\vec{j}$$

and

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\vec{i} + b\vec{j} + c\vec{k}$$

The Geometry of Vector Addition

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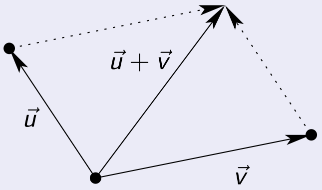
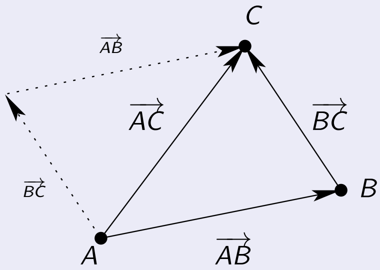
2. The zero vector, $\vec{0}$ has length zero and no direction.
3. Addition. Let $\vec{u}, \vec{v}$ be vectors. Then $\vec{u}+\vec{v}$ is the diagonal of the parallelogram defined by $\vec{u}$ and $\vec{v}$, and having the same tail as $\vec{u}$ and $\vec{v}$.



Tip-to-Tail Method for Vector Addition

For points A, B and C,

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$



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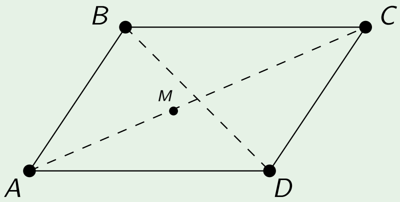
To see this, denote the parallelogram by its vertices, $ABCD$.



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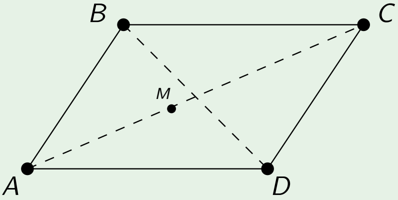

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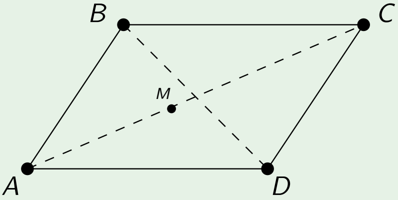

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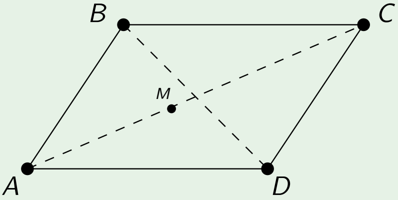

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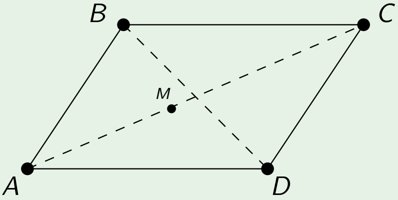

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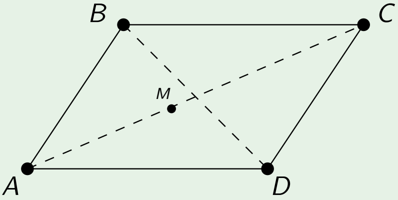

- Let M denote the midpoint of $\overrightarrow{AC}$.
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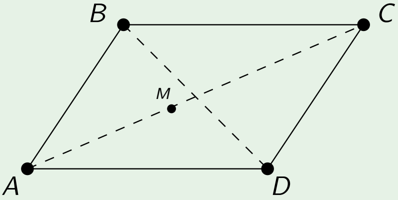

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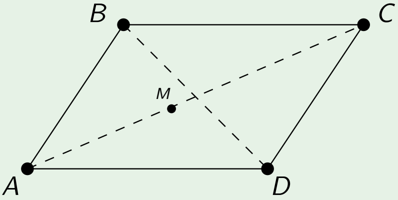

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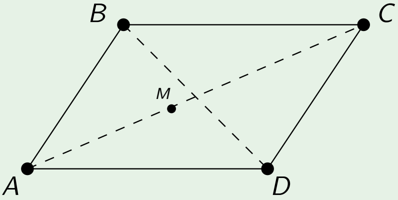

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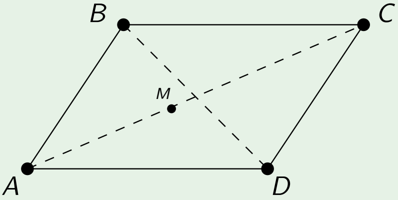

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$$\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM} = \overrightarrow{CD} + \overrightarrow{MC} = \overrightarrow{MC} + \overrightarrow{CD} = \overrightarrow{MD}.$$

Since $\overrightarrow{BM} = \overrightarrow{MD}$, these vectors have the same magnitude and direction,

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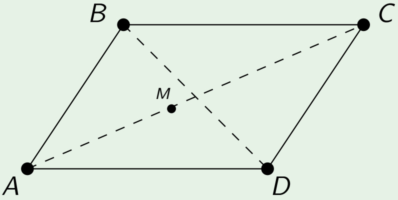

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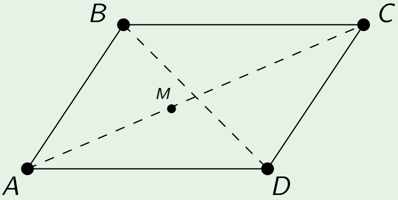
$$\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM} = \overrightarrow{CD} + \overrightarrow{MC} = \overrightarrow{MC} + \overrightarrow{CD} = \overrightarrow{MD}.$$

Since $\overrightarrow{BM} = \overrightarrow{MD}$, these vectors have the same magnitude and direction, implying that M is the midpoint of $\overrightarrow{BD}$.

Example

The diagonals of any parallelogram bisect each other.

To see this, denote the parallelogram by its vertices, $ABCD$.



- Let M denote the midpoint of $\overrightarrow{AC}$. Then $\overrightarrow{AM} = \overrightarrow{MC}$.
- It now suffices to show that $\overrightarrow{BM} = \overrightarrow{MD}$.

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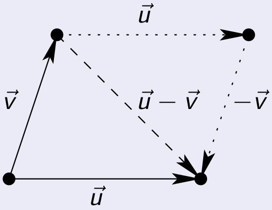
Since $\overrightarrow{BM} = \overrightarrow{MD}$, these vectors have the same magnitude and direction, implying that M is the midpoint of $\overrightarrow{BD}$.

Therefore, the diagonals of $ABCD$ bisect each other.

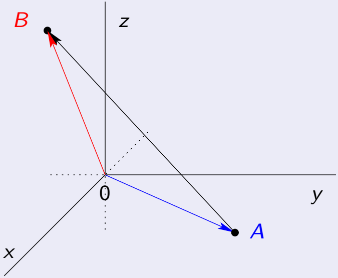
The Geometry of Vector Subtraction

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. The vector $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ is obtained from the parallelogram defined by $\vec{u}$ and $\vec{v}$ by taking the vector from the tip of $\vec{v}$ to the tip of $\vec{u}$, i.e., the diagonal of the parallelogram, directed towards the tip of $\vec{u}$.



Coordinatizing Vectors – Part 2

Let $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$.



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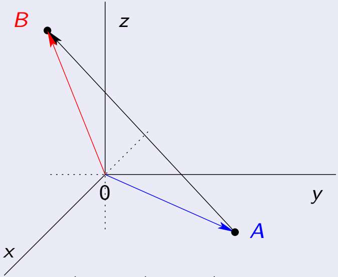
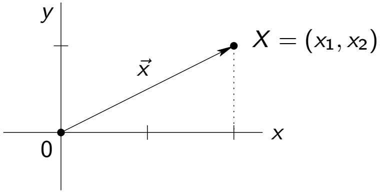


We see from the figure that $\overrightarrow{0A} + \overrightarrow{AB} = \overrightarrow{0B}$, and hence

$$\overrightarrow{AB} = \overrightarrow{0B} - \overrightarrow{0A} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}.$$

Length of a Vector, $\mathbb{R}^2$

If $\vec{x} = \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] \in \mathbb{R}^2$,



then the length of the vector $\vec{x}$ is the distance from the origin 0 to the point $X = (x_1, x_2)$ given by $d(0, X)$.

The length of $\vec{x}$, denoted $\|\vec{x}\|$, is given by:

$$d(0, X) = \|\vec{x}\| = \sqrt{x_1^2 + x_2^2}.$$

Length of a Vector, $\mathbb{R}^3$

This extends clearly to $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$.

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Suppose we want to find the distance between points other than the origin?

Length of a Vector, $\mathbb{R}^3$

Consider two arbitrary points in $\mathbb{R}^3$, $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$. Then the distance between them is written $d(A, B)$ and is given by the distance formula.

Distance Formula

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

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Distance Formula

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Now let $P = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$, and

$$\vec{p} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$$

Length of a Vector, $\mathbb{R}^3$

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$$\vec{p} = \left[\begin{array}{c} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{array} \right]$$

Then the length of $\vec{p}$ is equal to the distance between the origin and P, which are both equal to the distance between points A and B

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Length of a Vector, $\mathbb{R}^n$

More generally, if $P = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$ are points in $\mathbb{R}^n$, then the distance between P and Q is the length of the vector $\overrightarrow{PQ}$, written $\|\overrightarrow{PQ}\|$.

$$d(P,Q) = \|\overrightarrow{PQ}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + \cdots + (q_n - p_n)^2}.$$

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The formula for calculating the length of a vector generalizes to $\mathbb{R}^n$: if

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n,$$

then

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

which represents the distance from the origin to the point $(x_1, x_2, \ldots, x_n)$.

Properties of Distance

Let P and Q be two points in $\mathbb{R}^n$, and $d(P, Q)$ the distance between them. Then the following properties hold.

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Example

For $P = (1, -1, 3)$ and $Q = (3, 1, 0)$, the distance between P and Q is $d(P, Q) = \sqrt{2^2 + 2^2 + (-3)^2} = \sqrt{17}$.

Example

Let $\vec{p} = \left[\begin{array}{c} -3 \\ 4 \end{array}\right]$ and $\vec{q} = \left[\begin{array}{c} 3 \\ -1 \\ -2 \end{array}\right]$. Then $-2\vec{q} = (-2)\vec{q} = \left[\begin{array}{c} -6 \\ 2 \\ 4 \end{array}\right]$.

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and

$$\begin{aligned} \|-2\vec{q}\| &= \sqrt{(-6)^2 + 2^2 + 4^2} \\ &= \sqrt{36+4+16} \\ &= \sqrt{56} = \sqrt{4 \times 14} \\ &= 2\sqrt{14} = 2\|\vec{q}\|. \end{aligned}$$

The Geometry of Scalar Multiplication

- Scalar Multiplication. If $\vec{v} \neq \vec{0}$ and $a \in \mathbb{R}$, $a \neq 0$, then $a\vec{v}$ has length $\|a\vec{v}\| = |a| \cdot \|\vec{v}\|$, and

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- Parallel Vectors. Two nonzero vectors are called parallel if they have the same direction or opposite directions. It follows that nonzero vectors $\vec{v}$ and $\vec{w}$ are parallel if and only if one is a scalar multiple of the other.

Problem

Let $P = (1, -2, 1), Q = (-3, 0, 5)$, $X = (2, -1, 5)$ and $Y = (4, -2, 3)$ be points in $\mathbb{R}^3$. Is $\overrightarrow{PQ}$ parallel to $\overrightarrow{XY}$? Is $\overrightarrow{PX}$ parallel to $\overrightarrow{QY}$?

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$\overrightarrow{PQ} = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$, $\overrightarrow{XY} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$, and these vectors are parallel if $\overrightarrow{PQ} = k\overrightarrow{XY}$ for some scalar k,

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Solution

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Problem

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Solution

$\overrightarrow{PQ} = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$, $\overrightarrow{XY} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$, and these vectors are parallel if $\overrightarrow{PQ} = k\overrightarrow{XY}$ for some scalar k, i.e.,

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$\overrightarrow{PX} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $\overrightarrow{QY} = \begin{bmatrix} 7 \\ -2 \\ -2 \end{bmatrix}$,

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Solution

$\overrightarrow{PQ} = \left[\begin{array}{r} -4 \\ 2 \\ 4 \end{array}\right]$, $\overrightarrow{XY} = \left[\begin{array}{r} 2 \\ -1 \\ -2 \end{array}\right]$, and these vectors are parallel if $\overrightarrow{PQ} = k\overrightarrow{XY}$ for some scalar k, i.e.,

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Problem

Let $P=(1,-2,1), Q=(-3,0,5)$, $X=(2,-1,5)$ and $Y=(4,-2,3)$ be points in $\mathbb{R}^3$. Is $\overrightarrow{PQ}$ parallel to $\overrightarrow{XY}$? Is $\overrightarrow{PX}$ parallel to $\overrightarrow{QY}$?

Solution

$\overrightarrow{PQ}=\left[\begin{array}{r}-4\\2\\4\end{array}\right]$, $\overrightarrow{XY}=\left[\begin{array}{r}2\\-1\\-2\end{array}\right]$, and these vectors are parallel if $\overrightarrow{PQ}=k\overrightarrow{XY}$ for some scalar k, i.e.,

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This gives a system of three equations in one variable, which is consistent, and has unique solution $k=-2$. Therefore, $\overrightarrow{PQ}$ is parallel to $\overrightarrow{XY}$.

$\overrightarrow{PX}=\left[\begin{array}{r}1\\1\\4\end{array}\right]$, $\overrightarrow{QY}=\left[\begin{array}{r}7\\-2\\-2\end{array}\right]$, and these vectors are parallel if $\overrightarrow{PX}=\ell\overrightarrow{QY}$ for some scalar ℓ. You will find that no such ℓ exists, so $\overrightarrow{PX}$ is not parallel to $\overrightarrow{QY}$.

Unit Vectors

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A unit vector is a vector of length one.

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Example

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix},$$ are examples of unit vectors.

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A unit vector is a vector of length one.

Example

$$\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{c} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{array}\right],$$ are examples of unit vectors.

Example

If $\vec{v} \neq \vec{0}$, then

$$\frac{1}{\|\vec{v}\|}\vec{v}$$

is a unit vector in the same direction as $\vec{v}$.

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$\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ is not a unit vector, since $\|\vec{v}\| = \sqrt{14}$.

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If $\vec{v}$ and $\vec{w}$ are nonzero that have

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- opposite directions, then $\vec{v} = -\frac{\|\vec{v}\|}{\|\vec{w}\|}\vec{w}$.

Vector problems and examples

Problem

Find the point, M, that is midway between $P_1 = (-1, -4, 3)$ and $P_2 = (5, 0, -3)$.

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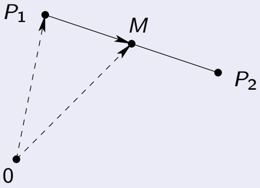

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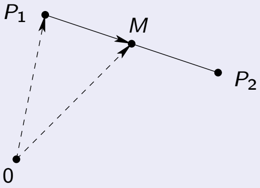

$\overrightarrow{0M}$

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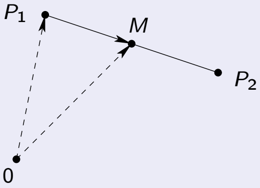

$$\overrightarrow{0M} = \overrightarrow{0P_1} + \overrightarrow{P_1M}$$

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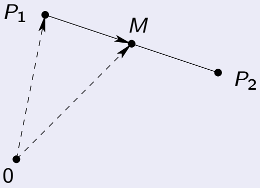

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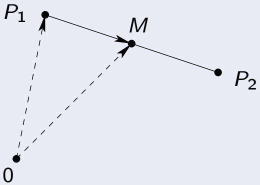

$$\overrightarrow{0M} = \overrightarrow{0P_1} + \overrightarrow{P_1M} = \overrightarrow{0P_1} + \frac{1}{2}\overrightarrow{P_1P_2} \quad = \quad \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6 \\ 4 \\ -6 \end{bmatrix}$$

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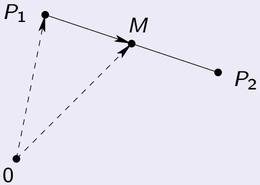

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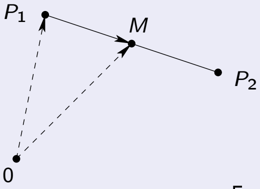

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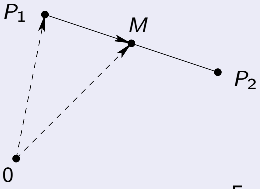

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Therefore $M = (2, -2, 0)$.

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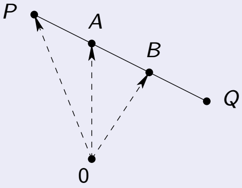

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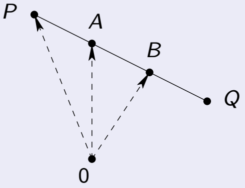

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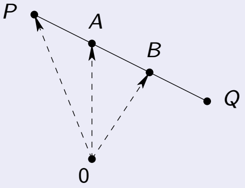

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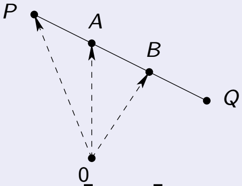

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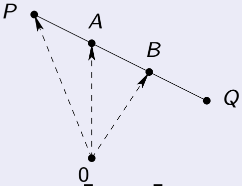

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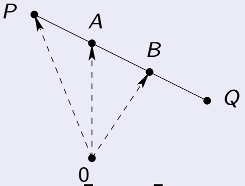

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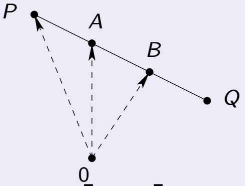

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Therefore, the two points are $A = (4, 0, 4)$ and $B = (6, -3, 3)$.

Example

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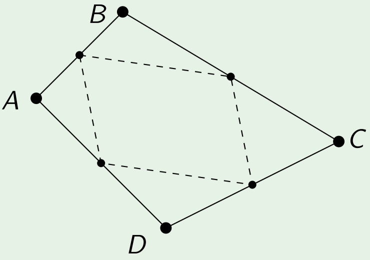

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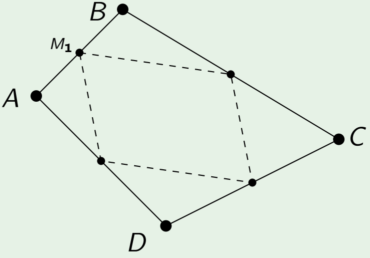

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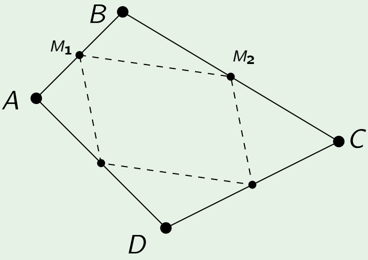

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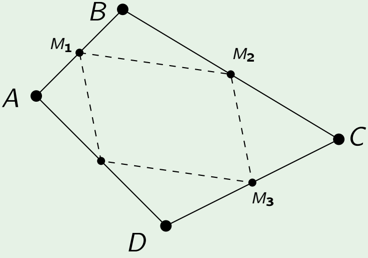

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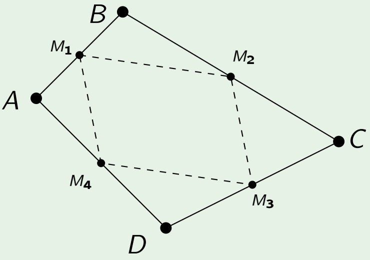

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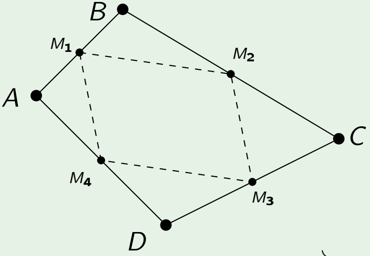

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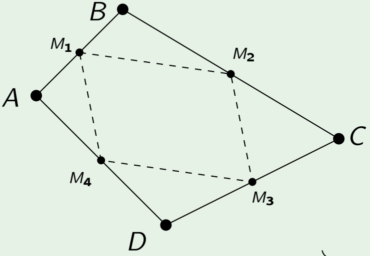

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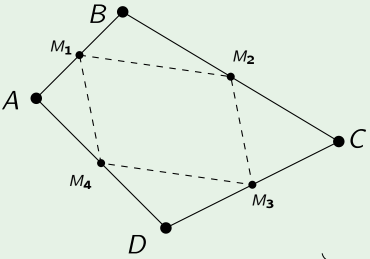

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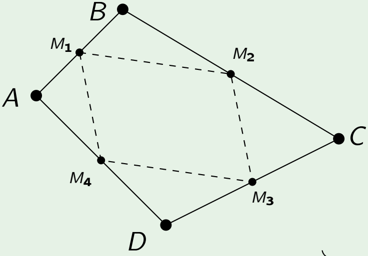

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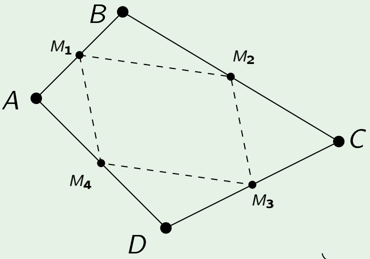

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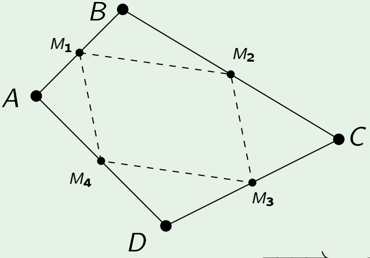

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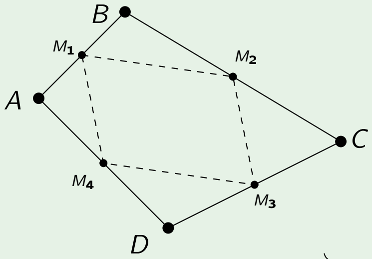

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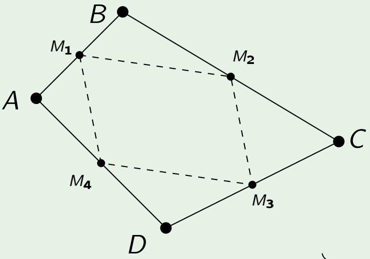

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M_4 the midpoint of $\overrightarrow{DA}$.

It suffices to prove that $\overrightarrow{M_1M_2} = \overrightarrow{M_4M_3}$.

$$\begin{aligned} \overrightarrow{M_1M_2} &= \overrightarrow{M_1B} + \overrightarrow{BM_2} \\ &= \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC} \\ &= \tfrac{1}{2}\overrightarrow{AC} \end{aligned} \qquad \begin{aligned} \overrightarrow{M_4M_3} &= \overrightarrow{M_4D} + \overrightarrow{DM_3} \\ &= \tfrac{1}{2}\overrightarrow{AD} + \tfrac{1}{2}\overrightarrow{DC} \\ &= \tfrac{1}{2}\overrightarrow{AC} \end{aligned}$$

Since $\overrightarrow{M_1M_2} = \overrightarrow{M_4M_3}$, the points M_1, M_2, M_3, M_4 are the vertices of a parallelogram.