A First Course in

# LINEAR ALGEBRA

# Lecture Notes

for Math 1503

# Matrices: Matrix Arithmetic





## A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

In addition we recognize the following contributors. All new content contributed is released under the same license as noted below.


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## Matrices - Basic Definitions and Notation

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Let $m$ and $n$ be positive integers.

- An $m \times n$ matrix is a rectangular array of numbers having $m$ rows and $n$ columns. Such a matrix is said to have size $m \times n$.



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General notation for an $m \times n$ matrix, $A$:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$



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5. **Negative of a Matrix:** for an $m \times n$ matrix $A$, its negative is denoted $-A$ and $-A = (-1)A$.
6. **Subtraction:** for $m \times n$ matrices $A$ and $B$, $A - B = A + (-1)B$.

## Matrix Addition

### Definition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A + B = C$ where $C$ is the $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$



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### Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 \\ 6 & 1 \end{bmatrix}$. Then,

$$\begin{aligned} A + B &= \begin{bmatrix} 1+0 & 3+-2 \\ 2+6 & 5+1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 8 & 6 \end{bmatrix} \end{aligned}$$

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4. There exists an $m \times n$ matrix $-A$ such that $A + (-A) = 0$. (existence of an additive inverse).



## Scalar Multiplication

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Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & -2 \\ 0 & 4 & 5 \end{bmatrix}$.



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**Example**

Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & -2 \\ 0 & 4 & 5 \end{bmatrix}$.

Then

$$\begin{aligned} 3A &= \begin{bmatrix} 3(2) & 3(0) & 3(-1) \\ 3(3) & 3(1) & 3(-2) \\ 3(0) & 3(4) & 3(5) \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 & -3 \\ 9 & 3 & -6 \\ 0 & 12 & 15 \end{bmatrix} \end{aligned}$$

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   (scalar multiplication distributes over matrix addition).

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4. $1A = A$. (existence of a multiplicative identity).



## Example

$$2\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} + 4\begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ 1 & -1 \end{bmatrix} =$$

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## Problem

Let $A$ and $B$ be $m \times n$ matrices. Simplify the expression

$$2[9(A-B)+7(2B-A)]-2[3(2B+A)-2(A+3B)-5(A+B)]$$

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### Solution

$$\begin{aligned} & 2[9(A-B)+7(2B-A)] - 2[3(2B+A)-2(A+3B)-5(A+B)] \\ = \; & 2(9A-9B+14B-7A) - 2(6B+3A-2A-6B-5A-5B) \\ = \; & 2(2A+5B) - 2(-4A-5B) \\ = \; & 12A+20B \end{aligned}$$



## Vectors

### Definitions

A row matrix or column matrix is often called a vector, and such matrices are referred to as row vectors and column vectors, respectively. If $X$ is a row vector of size $1 \times n$, and $Y$ is a column vector of size $m \times 1$, then we write

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

## Vector form of a system of linear equations

### Definition

Consider the system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$



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Such a system can be expressed in vector form or as a vector equation by using linear combinations of column vectors:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

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Express the following system of linear equations in vector form.

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 3x_3 & = & -6 \\ & - & x_2 & + & 5x_3 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & = & 1 \end{array}$$



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## Solution

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

## Matrix Vector Multiplication

### Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix with columns $A_1, A_2, \ldots, A_n$, written $A = \left[\begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array}\right]$, and let $X$ be an $n \times 1$ column vector,

$$X = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right]$$



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Then the product of matrix $A$ and (column) vector $X$ is the $m \times 1$ column vector given by

$$\left[\begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] = x_1A_1 + x_2A_2 + \cdots + x_nA_n = \sum_{j=1}^{n} x_jA_j$$

that is, $AX$ is a linear combination of the columns of $A$.

## Matrix Vector Multiplication

### Problem

Compute the product $AX$ for

$$A = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \text{ and } X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



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### Solution

$$AX = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$$

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Compute $AY$ for

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$$



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### Solution

$$AY = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}$$

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Such a system can be expressed in matrix form using matrix vector multiplication,

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Thus a system of linear equations can be expresses as a matrix equation $AX = B$, where $A$ is the coefficient matrix, $B$ is the constant matrix, and $X$ is the matrix of variables.

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### Solution

$$\begin{bmatrix} 2 & 4 & -3 \\ 0 & -1 & 5 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

## Matrix and Vector Equations

### Theorem

1. Every system of $m$ linear equations in $n$ variables can be written in the form $AX = B$ where $A$ is the coefficient matrix, $X$ is the matrix of variables, and $B$ is the constant matrix.



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2. The system $AX = B$ is consistent (i.e., has at least one solution) if and only if $B$ is a linear combination of the columns of $A$.
3. The vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a solution to the system $AX = B$ if and only if $x_1, x_2, \ldots, x_n$ are a solution to the vector equation
$$x_1A_1 + x_2A_2 + \cdots x_nA_n = B$$
where $A_1, A_2, \ldots, A_n$ are the columns of $A$.



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**Proof.**

(a) One first checks that $(x_1, \ldots, x_n)$ is a solution to the original system if and only if $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a solution to $AX = B$.

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This depends on the way that the matrix arithmetics (addition, multiplication by scalars, multiplication) was defined.



## Proof continued

**Proof.**

(b) Once (a) is taken care of, it gives a one-to-one correspondence between the set of solutions to the original system and the set of solutions to $AX = B$:

$$(x_1, \ldots, x_n) \mapsto \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

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(b) Once (a) is taken care of, it gives a one-to-one correspondence between the set of solutions to the original system and the set of solutions to $AX = B$:

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This *is* (3), and it implies that the two sets have the same cardinality, and (2) follows. □



### Problem

Let

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Express $B$ as a linear combination of the columns $A_1, A_2, A_3, A_4$ of $A$, or show that this is impossible.

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## Solution

Solve the system $AX = B$ where $X$ is a column vector with four entries.



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## Solution

Solve the system $AX = B$ where $X$ is a column vector with four entries. Do so by putting the **augmented matrix** $\left[\begin{array}{c|c} A & B \end{array}\right]$ in reduced row-echelon form.

## Problem

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Express $B$ as a linear combination of the columns $A_1, A_2, A_3, A_4$ of $A$, or show that this is impossible.

## Solution

Solve the system $AX = B$ where $X$ is a column vector with four entries. Do so by putting the **augmented matrix** $\left[\begin{array}{c|c} A & B \end{array}\right]$ in reduced row-echelon form.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 1 \\ 2 & -1 & 0 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & \frac{1}{7} \\ 0 & 1 & 0 & 1 & -\frac{5}{7} \\ 0 & 0 & 1 & -1 & \frac{3}{7} \end{array}\right]$$



## Problem

Let
$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Express $B$ as a linear combination of the columns $A_1, A_2, A_3, A_4$ of $A$, or show that this is impossible.

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Since there are infinitely many solutions ($x_4$ is assigned a parameter), choose any value for $x_4$.

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$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 1 \\ 2 & -1 & 0 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & \frac{1}{7} \\ 0 & 1 & 0 & 1 & -\frac{5}{7} \\ 0 & 0 & 1 & -1 & \frac{3}{7} \end{array}\right]$$

Since there are infinitely many solutions ($x_4$ is assigned a parameter), choose any value for $x_4$. Choosing $x_4 = 0$ (which is the simplest thing to do) gives us

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{5}{7}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{7}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{7}A_1 - \frac{5}{7}A_2 + \frac{3}{7}A_3 + 0A_4.$$



## Matrix Multiplication

### Definition (Product of two matrices)

Let $A$ be an $m \times n$ matrix and let $B = \begin{bmatrix} B_1 & B_2 & \cdots & B_p \end{bmatrix}$ be an $n \times p$ matrix, whose columns are $B_1, B_2, \ldots, B_p$. The product of $A$ and $B$ is the matrix

$$AB = A\begin{bmatrix} B_1 & B_2 & \cdots & B_p \end{bmatrix} = \begin{bmatrix} AB_1 & AB_2 & \cdots & AB_p \end{bmatrix}$$

i.e., the first column of $AB$ is $AB_1$, the second column of $AB$ is $AB_2$, etc. Note that $AB$ has size $m \times p$.

## Problem

Find the product $AB$ of matrices

$$A = \left[\begin{array}{rrr} -1 & 0 & 3 \\ 2 & -1 & 1 \end{array}\right] \text{ and } B = \left[\begin{array}{rrr} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{array}\right]$$



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## Solution

$AB$ has columns

$$AB_1 = \left[\begin{array}{rrr} -1 & 0 & 3 \\ 2 & -1 & 1 \end{array}\right] \left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right], AB_2 = \left[\begin{array}{rrr} -1 & 0 & 3 \\ 2 & -1 & 1 \end{array}\right] \left[\begin{array}{r} 1 \\ -2 \\ 0 \end{array}\right],$$

$$\text{and } AB_3 = \left[\begin{array}{rrr} -1 & 0 & 3 \\ 2 & -1 & 1 \end{array}\right] \left[\begin{array}{r} 2 \\ 4 \\ 0 \end{array}\right]$$

## Problem

Find the product $AB$ of matrices

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

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$$\text{and } AB_3 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

Thus, $AB = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}$.



# Compatibility for Matrix Multiplication

## Definition

Let $A$ and $B$ be matrices, and suppose that $A$ is $m \times n$.

- In order for the product $AB$ to exist, the number of rows in $B$ must be equal to the number of columns in $A$, implying that $B$ is an $n \times p$ matrix for some $p$.
- When defined, $AB$ is an $m \times p$ matrix.

If the product is defined, then $A$ and $B$ are said to be compatible for (matrix) multiplication.

## Example

As we saw in the previous problem

$$\overset{2\times 3}{\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}} \overset{3\times 3}{\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}} = \overset{2\times 3}{\begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}}$$



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Note that the product

$$\overset{3\times 3}{\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}} \overset{2\times 3}{\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}}$$

does not exist.

## Multiplication by the Zero Matrix

### Example

Compute the product $A0$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the $2 \times 2$ zero matrix given by $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$



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### Solution

*In this product, we compute*

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

*Hence,* $A0 = 0$.

## Definition (The $(i, j)$-entry of a product)

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the $(i, j)$-entry of $AB$ is given by

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$



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## Example

Using the above definition, the $(2, 3)$-entry of the product

$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

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Using the above definition, the $(2,3)$-entry of the product

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is computed using the second row of the first matrix, and the third column of the second matrix,



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## Example

Using the above definition, the $(2,3)$-entry of the product

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is computed using the second row of the first matrix, and the third column of the second matrix, resulting in

$$2 \times 2 + (-1) \times 4 + 1 \times 0 = 4 - 4 + 0 = 0.$$

## Questions on Matrix Multiplication

Given matrices $A$ and $B$, is $AB = BA$?



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The product $AB$ is defined if and only if $n = m'$.
The product $BA$ is defined if and only if $m = n'$.

Therefore the equation $AB = BA$ makes sense if and only if $A$ is an $m \times n$ matrix and $B$ is an $n \times m$ matrix for some—possibly different—$m$ and $n$.

## Questions on Matrix Multiplication

Given matrices $A$ and $B$, is $AB = BA$?

Suppose $A$ is an $m \times n$ matrix and $B$ is an $m' \times n'$ matrix.
The product $AB$ is defined if and only if $n = m'$.
The product $BA$ is defined if and only if $m = n'$.

Therefore the equation $AB = BA$ makes sense if and only if $A$ is an $m \times n$ matrix and $B$ is an $n \times m$ matrix for some—possibly different—$m$ and $n$.

So the right question is:

Given matrices $A$ and $B$ such that both $AB$ and $BA$ are defined, is $AB = BA$?



## Matrix Multiplication is Not Commutative

### Problem

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 1 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & -2 & 1 & -3 \end{bmatrix}$$

- Does $AB$ exist? If so, compute it.
- Does $BA$ exist? If so, compute it.

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- Does $AB$ exist? If so, compute it.
- Does $BA$ exist? If so, compute it.

### Solution

$$AB = \begin{bmatrix} 7 & -5 & 4 & -6 \\ -3 & 3 & -6 & 0 \\ -11 & 7 & -2 & 12 \end{bmatrix}$$

# Matrix Multiplication is Not Commutative

## Problem

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$$A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 1 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & -2 & 1 & -3 \end{bmatrix}$$

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## Solution

$$AB = \begin{bmatrix} 7 & -5 & 4 & -6 \\ -3 & 3 & -6 & 0 \\ -11 & 7 & -2 & 12 \end{bmatrix}$$

$BA$ does not exist



## Problem

Let

$$G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Does $GH$ exist? If so, compute it.
- Does $HG$ exist? If so, compute it.

## Problem

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## Solution

$$GH = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$HG = \begin{bmatrix} 1 \end{bmatrix}$$



## Problem

Let

$$G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Does $GH$ exist? If so, compute it.
- Does $HG$ exist? If so, compute it.

## Solution

$$GH = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$HG = \begin{bmatrix} 1 \end{bmatrix}$$

In this example, $GH$ and $HG$ both exist, but they are not equal. They aren't even the same size!

## Problem

Let

$$P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$$

- Does $PQ$ exist? If so, compute it.
- Does $QP$ exist? If so, compute it.



## Problem

Let

$$P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$$

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- Does $QP$ exist? If so, compute it.

## Solution

$$PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}$$



## Problem

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- Does $PQ$ exist? If so, compute it.
- Does $QP$ exist? If so, compute it.

## Solution

$$PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$QP = \begin{bmatrix} 1 & -1 \\ 6 & -3 \end{bmatrix}$$

In this example, $PQ$ and $QP$ both exist and are the same size, but $PQ \neq QP$.

### Fact

The three preceding problems illustrate an important property of matrix multiplication.

> *In general, matrix multiplication is not commutative, i.e., the order of the matrices in the product is important.*

In other words, in general $AB \neq BA$.



### Problem

Let

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Does $UV$ exist? If so, compute it.
- Does $VU$ exist? If so, compute it.

### Problem

Let

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Does $UV$ exist? If so, compute it.
- Does $VU$ exist? If so, compute it.

### Solution



### Problem

Let

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- Does $UV$ exist? If so, compute it.
- Does $VU$ exist? If so, compute it.

### Solution

$$UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

## Problem

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$$U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Does $UV$ exist? If so, compute it.
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## Solution

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- Does $UV$ exist? If so, compute it.
- Does $VU$ exist? If so, compute it.

## Solution

$$UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

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In this particular example, the matrices commute, i.e., $UV = VU$.

## Properties of Matrix Multiplication

### Theorem

Let $A$, $B$, and $C$ be matrices of the appropriate sizes, and let $r \in \mathbb{R}$ be a scalar. Then the following properties hold.



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1. $A(B+C) = AB + AC$.
   (matrix multiplication distributes over matrix addition).

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2. $(B+C)A = BA + CA$.
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3. $A(BC) = (AB)C$. (matrix multiplication is associative).

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4. $r(AB) = (rA)B = A(rB)$.



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3. $A(BC) = (AB)C$. (matrix multiplication is associative).
4. $r(AB) = (rA)B = A(rB)$.

This applies to matrix-vector multiplication as well, since a vector is a row matrix or a column matrix.

## Problem

*Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ be three $n \times n$ matrices. For $1 \leq i, j \leq n$ write down a formula for the $(i, j)$-entry of each of the following matrices.*

1. *AB*
2. *BA*
3. *A+C*
4. *A(BC)*
5. *(AB)C*
6. *(A+B)*
7. *C(A+B)*



# Elementary Proofs

## Problem

Let $A$ and $B$ be $m \times n$ matrices, and let $C$ be an $n \times p$ matrix. Prove that if $A$ and $B$ commute with $C$, then $A + B$ commutes with $C$.

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### Proof.

We are given that $AC = CA$ and $BC = CB$. Consider $(A + B)C$.

$$(A + B)C \quad =$$



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$$(A + B)C \quad = \quad AC + BC$$

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$$\begin{aligned}(A+B)C &= AC + BC \\ &= CA + CB\end{aligned}$$



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## Proof.

We are given that $AC = CA$ and $BC = CB$. Consider $(A+B)C$.

$$\begin{aligned} (A+B)C &= AC + BC \\ &= CA + CB \\ &= C(A+B) \end{aligned}$$

Since $(A+B)C = C(A+B)$, $A+B$ commutes with $C$. $\square$



## Problem

Let $A, B$ and $C$ be $n \times n$ matrices, and suppose that both $A$ and $B$ commute with $C$, i.e., $AC = CA$ and $BC = CB$. Show that $AB$ commutes with $C$.

## Problem

Let $A, B$ and $C$ be $n \times n$ matrices, and suppose that both $A$ and $B$ commute with $C$, i.e., $AC = CA$ and $BC = CB$. Show that $AB$ commutes with $C$.

## Proof.

We must show that $(AB)C = C(AB)$ given that $AC = CA$ and $BC = CB$.



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We must show that $(AB)C = C(AB)$ given that $AC = CA$ and $BC = CB$.

$$(AB)C = A(BC) \quad \text{(matrix multiplication is associative)}$$

## Problem

Let $A, B$ and $C$ be $n \times n$ matrices, and suppose that both $A$ and $B$ commute with $C$, i.e., $AC = CA$ and $BC = CB$. Show that $AB$ commutes with $C$.

## Proof.

We must show that $(AB)C = C(AB)$ given that $AC = CA$ and $BC = CB$.

$$\begin{aligned} (AB)C &= A(BC) \quad \text{(matrix multiplication is associative)} \\ &= A(CB) \quad (B \text{ commutes with } C) \end{aligned}$$



## Problem

Let $A, B$ and $C$ be $n \times n$ matrices, and suppose that both $A$ and $B$ commute with $C$, i.e., $AC = CA$ and $BC = CB$. Show that $AB$ commutes with $C$.

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We must show that $(AB)C = C(AB)$ given that $AC = CA$ and $BC = CB$.

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Let $A, B$ and $C$ be $n \times n$ matrices, and suppose that both $A$ and $B$ commute with $C$, i.e., $AC = CA$ and $BC = CB$. Show that $AB$ commutes with $C$.

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Therefore, $AB$ commutes with $C$. □



## Definition (Matrix Transpose)

If $A$ is an $m \times n$ matrix, then its transpose, denoted $A^T$, is the $n \times m$ whose $i^{th}$ row is the $i^{th}$ column of $A$, $1 \leq i \leq n$; i.e., if $A = [a_{ij}]$, then

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To prove each these properties, you only need to compute the $(i, j)$-entries of the matrices on the left-hand side and the right-hand side. And you can do it!



## Problem

Find the matrix $A$ if $\left(A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$.

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\Rightarrow A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix} &= \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 8 \end{bmatrix} \\
\Rightarrow A &= \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 8 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 3 & 3 \\ -2 & -1 & -4 \end{bmatrix}
\end{aligned}$$



# Symmetric Matrices

## Definition

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### Examples

$$\begin{bmatrix} 2 & -3 \\ -3 & 17 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 5 \\ 0 & 2 & 11 \\ 5 & 11 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 5 & -1 \\ 2 & 1 & -3 & 0 \\ 5 & -3 & 2 & -7 \\ -1 & 0 & -7 & 4 \end{bmatrix}$$

are symmetric matrices, and each is symmetric about its main diagonal.

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Since $(A^T + 2B)^T = A^T + 2B$, $A^T + 2B$ is symmetric. $\square$



# Skew Symmetric Matrices

## Definition

An $n \times n$ matrix $A$ is said to be skew symmetric if $A^T = -A$.

## Example (Skew Symmetric Matrices)

$$\left[\begin{array}{rr} 0 & 2 \\ -2 & 0 \end{array}\right], \left[\begin{array}{rrr} 0 & 9 & 4 \\ -9 & 0 & -3 \\ -4 & 3 & 0 \end{array}\right]$$

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We must show that $(A - A^T)^T = -(A - A^T)$. Using the properties of matrix addition, scalar multiplication, and transposition

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T).$$



## The $n \times n$ Identity Matrix

### Definition

For each $n \geq 2$, the $n \times n$ identity matrix, denoted $I_n$, is the matrix having ones on its main diagonal and zeros elsewhere, and is defined for all $n \geq 2$.

### Example

$$I_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], I_3 = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

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### Definition

Let $n \geq 2$. For each $j$, $1 \leq j \leq n$, we denote by $E_j$ the $j^{\text{th}}$ column of $I_n$.

### Example

When $n = 3$, $E_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.



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## Proof

The $(i,j)$-entry of $AI_n$ is the product of the $i^{th}$ row of $A = [a_{ij}]$, namely $\left[\begin{array}{cccccc} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \end{array}\right]$ with the $j^{th}$ column of $I_n$, namely $E_j$. Since $E_j$ has a one in row $j$ and zeros elsewhere,

$$\left[\begin{array}{cccccc} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \end{array}\right] E_j = a_{ij}$$

Since this is true for all $i \leq m$ and all $j \leq n$, $AI_n = A$.
The proof of $I_m A = A$ is analogous—work it out!



Instead of $AI_n$ and $I_m A$ we often write $AI$ and $IA$, respectively, since the size of the identity matrix is clear from the context: the sizes of $A$ and $I$ must be compatible for matrix multiplication.

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Thus

$$AI = A \text{ and } IA = A$$

which is why $I$ is called an identity matrix – it is an identity for matrix multiplication.



# Matrix Inverses

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Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

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so $B$ is an inverse of $A$.

## Does every square matrix have an inverse?



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No! Take e.g. the zero matrix $\mathbf{0_n}$ (all entries of $\mathbf{O_n}$ are equal to 0)

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for all $n \times n$ matrices $A$:

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is an inverse of $A$. Then

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix}$$

which is never equal to $I_2$. (Why?)



## Uniqueness of an Inverse

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If $A$ is a square matrix and $B$ and $C$ are inverses of $A$, then $B = C$.

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Since $B$ and $C$ are inverses of $A$, $AB = I = BA$ and $AC = I = CA$. Then

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so $B = C$. $\square$



## Example (revisited)

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$, we saw that

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The preceding theorem tells us that $B$ is the inverse of $A$, rather than just an inverse of $A$.

## Definitions

Let $A$ be a square matrix, i.e., an $n \times n$ matrix.

- The inverse of $A$, if it exists, is denoted $A^{-1}$, and

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- If $A$ has an inverse, then we say that $A$ is invertible.

## Finding the inverse of a $2 \times 2$ matrix

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Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.



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Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then there is a formula for $A^{-1}$:

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$$\begin{aligned} AA^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad - bc} \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \\ &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & -bc + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

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This can easily be verified by computing the products $AA^{-1}$ and $A^{-1}A$.

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad - bc} \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \\ &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & -bc + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Showing that $A^{-1}A = I_2$ is left as an exercise.



# Finding the inverse of an $n \times n$ matrix

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Suppose that $A$ is any $n \times n$ matrix.

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## Solution

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Although the formula for the inverse of a $2 \times 2$ matrix is quicker and easier to use than the matrix inversion algorithm, the general formula for the inverse an $n \times n$ matrix, $n \geq 3$ (which we will see later), is more complicated and difficult to use than the matrix inversion algorithm. To find inverses of square matrices that are not $2 \times 2$, the matrix inversion algorithm is the most efficient method to use.



## The Matrix Inversion Algorithm

Let $A$ be an $n \times n$ matrix. To find $A^{-1}$, if it exists,

- take the $n \times 2n$ matrix
$$\left[\begin{array}{c|c} A & I_n \end{array}\right]$$
obtained by augmenting $A$ with the $n \times n$ identity matrix, $I_n$.
- Perform elementary row operations to transform $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ into a reduced row-echelon matrix.

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## Theorem (Matrix Inverses)

*Let $A$ be an $n \times n$ matrix. Then the following conditions are equivalent.*

1. *$A$ is invertible.*
2. *the reduced row-echelon form on $A$ is $I$.*
3. *$\left[\begin{array}{c|c} A & I_n \end{array}\right]$ can be transformed into $\left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$ using the Matrix Inversion Algorithm.*



## Problem

Find, if possible, the inverse of $\left[\begin{array}{rrr} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{array}\right]$.

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$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$



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From this, we see that $A$ has no inverse.

## Problem

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find the inverse of $A$, if it exists.



## Solution (continued)

*Using the matrix inversion algorithm (fill in the operations)*

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$$\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{rrr|rrr} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$



## Solution (continued)

*Using the matrix inversion algorithm (fill in the operations)*

$$\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{rrr|rrr} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 4 & -7 & 1 & -3 & 0 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 25 & -3 & 5 & 4 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right] \rightarrow$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{10}{25} & 0 & -\frac{5}{25} \\ 0 & 1 & 0 & \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right] = \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

## Solution (continued)

Therefore, $A^{-1}$ exists, and

$$A^{-1} = \begin{bmatrix} \frac{10}{25} & 0 & -\frac{5}{25} \\ \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 10 & 0 & -5 \\ 1 & -10 & 7 \\ -3 & 5 & 4 \end{bmatrix}$$



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You can check your work by computing $AA^{-1}$ and $A^{-1}A$.

## Systems of Linear Equations and Inverses

Suppose that a system of $n$ linear equations in $n$ variables is written in matrix form as $AX = B$, and suppose that $A$ is invertible.



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The system of linear equations

$$\begin{aligned} 2x - 7y &= 3 \\ 5x - 18y &= 8 \end{aligned}$$

can be written in matrix form as $AX = B$:

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You can check that $A^{-1} = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}$.



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$$X = A^{-1}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$



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You should verify that $x = -2$, $y = -1$ is a solution to the system.

The last example illustrates another method for solving a system of linear equations when **the coefficient matrix is square and invertible**.



The last example illustrates another method for solving a system of linear equations when **the coefficient matrix is square and invertible**. Unless that coefficient matrix is $2 \times 2$, this is generally **NOT** an efficient method for solving a system of linear equations.

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Let $A, B$ and $C$ be matrices, and suppose that $A$ is invertible.

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### Problem

Find square matrices $A, B$ and $C$ for which $AB = AC$ but $B \neq C$.

## Inverses of Transposes and Products

### Example

Suppose $A$ is an invertible matrix. Then

$$A^T(A^{-1})^T =$$



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# Inverses of Transposes and Products

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This means that $(AB)^{-1} = B^{-1}A^{-1}$.

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The previous two examples prove the first two parts of the following theorem.



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2. If $A$ and $B$ are invertible matrices, then $AB$ is invertible and

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3. If $A_1, A_2, \ldots, A_k$ are invertible, then $A_1A_2\cdots A_k$ is invertible and

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$$

(the third part is proved by iterating the above, or, more formally, by using the mathematical induction)

## Properties of Inverses

### Theorem

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4. If $A$ is invertible and $p \in \mathbb{R}$ is nonzero, then $pA$ is invertible, and $(pA)^{-1} = \frac{1}{p}A^{-1}$.

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-A^T &= \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ -1 & -\frac{5}{2} \end{bmatrix} \\
A &= \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{2} & \frac{5}{2} \end{bmatrix}
\end{aligned}
$$

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True or false? Justify your answer.

$$\text{If } A^3 = 4I, \text{ then } A \text{ is invertible.}$$



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Therefore $A$ is invertible, and $A^{-1} = \frac{1}{4}A^2$.



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## Theorem

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## Proof of Theorem:

(1) $\Rightarrow$ (2) The rank of $A$ is the number of leading 1s in the RREF of $A$. Since the size of $A$ is $n \times n$, rank $(A) = n$ is equivalent to $A$ being row-equivalent to $I_n$.

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(4) $\Leftrightarrow$ (7): $CA = I$ if and only if $A^T C^T = I$; hence (4) for $A$ is equivalent to (7) for $A^T$.

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(4) $\Leftrightarrow$ (7): $CA = I$ if and only if $A^T C^T = I$; hence (4) for $A$ is equivalent to (7) for $A^T$.
We already know that $A^{-1}$ exists if and only if $(A^T)^{-1}$ exists.



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### Theorem

*If $A$ and $B$ are $n \times n$ matrices such that $AB = I$, then $BA = I$. Furthermore, $A$ and $B$ are invertible, with $B = A^{-1}$ and $A = B^{-1}$.*



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### Important Fact

In the second Theorem, it is essential that the matrices be square.

## Theorem

*If A and B are matrices such that $AB = I$ and $BA = I$, then A and B are square matrices (of the same size).*



## Example

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$.

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This example illustrates why "an inverse" of a non-square matrix doesn't make sense. If $A$ is $m \times n$ and $B$ is $n \times m$, where $m \neq n$, then even if $AB = I$, it will never be the case that $BA = I$.



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### Definition

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## Elementary Row Operations.

- **Type I:** Interchange two rows.
- **Type II:** Multiply a row by a nonzero number.
- **Type III:** Add a (nonzero) multiple of one row to a different row.



## Example

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

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$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix}$$



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We are interested in the effect that (left) multiplication of $A$ by $E$, $F$ and $G$ has on the matrix $A$.

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We are interested in the effect that (left) multiplication of $A$ by $E$, $F$ and $G$ has on the matrix $A$. Computing $EA$, $FA$, and $GA$ ...



## Example (continued)

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 3 & 3 \\ 2 & 2 \end{bmatrix}$$

## Example (continued)

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 3 & 3 \\ 2 & 2 \end{bmatrix}$$

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## Example (continued)

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$$GA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$$

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Notice that $EA$ is the matrix obtained from $A$ by interchanging row 2 and row 4,



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## Example (continued)

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Notice that $EA$ is the matrix obtained from $A$ by interchanging row 2 and row 4, which is the same row operation used to obtain $E$ from $I_4$. What about $FA$ and $GA$?



# Multiplication by an Elementary Matrix

## Theorem

Let $A$ be an $m \times n$ matrix, and suppose that $B$ is obtained from $A$ by performing a single elementary row operation.

## Multiplication by an Elementary Matrix

### Theorem

Let $A$ be an $m \times n$ matrix, and suppose that $B$ is obtained from $A$ by performing a single elementary row operation. Then $B = EA$ where $E$ is the elementary matrix obtained from $I_m$ by performing the same elementary operation on $I_m$ as was performed on $A$.



### Problem

Let

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$$

Find elementary matrices $E$ and $F$ so that $C = FEA$.

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**Note.** The statement of the problem implies that $C$ can be obtained from $A$ by a sequence of two elementary row operations, represented by elementary matrices $E$ and $F$.



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where $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



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$A \to EA \to F(EA) = C$, so $C = FEA$, i.e.,
$$\begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$$

You can check your work by doing the matrix multiplication.



# Inverses of Elementary Matrices

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Without using the matrix inversion algorithm, find the inverse of the elementary matrix

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$$G^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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$$G^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Check by computing $G^{-1}G$.

## Example (continued)

Similarly,

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



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and

$$F^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## The Form $B = UA$

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or, more concisely, $B = UA$ where $U = E_kE_{k-1}\cdots E_2E_1$.

To find $U$ so that $B = UA$, we could find $E_1, E_2, \ldots, E_k$ and multiply these together (in the correct order), but there is an easier method for finding $U$.



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Let $A$ be an $m \times n$ matrix. We write

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1. there exists an invertible $m \times m$ matrix $U$ such that $B = UA$;
2. $U$ can be computed by performing elementary row operations on $\left[\begin{array}{c|c} A & I_m \end{array}\right]$ to transform it into $\left[\begin{array}{c|c} B & U \end{array}\right]$;



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3. $U = E_k E_{k-1} \cdots E_2 E_1$, where $E_1, E_2, \ldots, E_k$ are elementary matrices corresponding, in order, to the elementary row operations used to obtain $B$ from $A$.

## Problem

Let $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$, and let $R$ be the reduced row-echelon form of $A$.
Find a matrix $U$ so that $R = UA$.



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## Solution

$$\left[\begin{array}{rrr|rr} 3 & 0 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 0 & -3 & -2 & -2 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & -1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rr} 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & -1 \end{array}\right]$$

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Starting with $\left[\begin{array}{c|c} A & I \end{array}\right]$, we've obtained $\left[\begin{array}{c|c} R & U \end{array}\right]$.



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Therefore $R = UA$, where

$$U = \left[\begin{array}{rr} \frac{1}{3} & 0 \\ \frac{2}{3} & -1 \end{array}\right]$$

## A Matrix as a Product of Elementary Matrices

### Example

Let

$$A = \begin{bmatrix} 1 & 2 & -4 \\ -3 & -6 & 13 \\ 0 & -1 & 2 \end{bmatrix}$$



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Notice that the reduced row-echelon form of $A$ equals $I_3$. Now find the matrices $E_1, E_2, E_3, E_4$ and $E_5$.



### Example (continued)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

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$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$



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$$E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that

$$\begin{aligned} (E_5(E_4(E_3(E_2(E_1A))))) &= I \\ (E_5E_4E_3E_2E_1)A &= I \end{aligned}$$

and therefore

$$A^{-1} = E_5E_4E_3E_2E_1$$



## Example (continued)

Since $A^{-1} = E_5E_4E_3E_2E_1$,

$$\begin{aligned} A^{-1} &= E_5E_4E_3E_2E_1 \\ (A^{-1})^{-1} &= (E_5E_4E_3E_2E_1)^{-1} \\ A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} \end{aligned}$$

This example illustrates the following result.

## Example (continued)

Since $A^{-1} = E_5E_4E_3E_2E_1$,

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This example illustrates the following result.

## Theorem

Let $A$ be an $n \times n$ matrix. Then, $A^{-1}$ exists if and only if $A$ can be written as the product of elementary matrices.



## Problem

Express $A = \left[\begin{array}{rr} 4 & 1 \\ -3 & 2 \end{array}\right]$ as a product of elementary matrices.

## Problem

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

## Solution

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$



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## Solution

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \underset{E_1}{\longrightarrow} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix}$$

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## Problem

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

## Solution

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \xrightarrow[E_1]{} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \xrightarrow[E_2]{} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix}$$



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and

$$E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix},$$

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and

$$E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix}, E_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

Since $E_4E_3E_2E_1A = I$, $A^{-1} = E_4E_3E_2E_1$, and hence

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$

## Solution (continued)

Therefore,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}^{-1}$$



## Solution (continued)

Therefore,

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Check your work by computing the product.



## Problem

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## Theorem

If $A$ is an $m \times n$ matrix and $R$ and $S$ are reduced row-echelon forms of $A$, then $R = S$.

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One result that we have assumed in all our work involving reduced row-echelon matrices is the following.

### Theorem

If $A$ is an $m \times n$ matrix and $R$ and $S$ are reduced row-echelon forms of $A$, then $R = S$.

This theorem ensures that the reduced row-echelon form of a matrix is unique,



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Is the $n \times n$ identity matrix an elementary matrix? Justify your answer.

One result that we have assumed in all our work involving reduced row-echelon matrices is the following.

### Theorem

If $A$ is an $m \times n$ matrix and $R$ and $S$ are reduced row-echelon forms of $A$, then $R = S$.

This theorem ensures that the reduced row-echelon form of a matrix is unique,
and its proof follows from the results about elementary matrices.

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