A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Linear Transformations: One to One and Onto

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Range of a Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation given by $T(\vec{x}) = A\vec{x}$. Consider all the vectors of the form $A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$. This set of vectors is called the range or image of T.
We denote this set as $T\mathbb{R}^n$, $T(\mathbb{R}^n)$ or $\mathrm{Im}(T)$. Notice that these vectors $A(\vec{x})$ are in $\mathbb{R}^m$.

The Form $A\vec{x}$

Theorem

Let A be an $m \times n$ matrix where $A_1, ..., A_n$ denote the columns of A. Then for a vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$,

$$A\vec{x} = \sum_{k=1}^{n} x_k A_k$$

Therefore $A(\mathbb{R}^n)$ is the collection of all linear combinations of these products.

Injections

Definition

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\vec{x}_1$ and $\vec{x}_2$ be in $\mathbb{R}^n$. We say that T is an injection or is one-to-one (sometimes written as 1-1) if $\vec{x}_1 \neq \vec{x}_2$ implies that

$$T(\vec{x}_1) \neq T(\vec{x}_2).$$

Equivalently, if $T(\vec{x}_1) = T(\vec{x}_2)$, then $\vec{x}_1 = \vec{x}_2$. Thus, T is one-to-one if two distinct vectors are never transformed into the same vector.

Theorem

Let A be an $m \times n$ matrix and let $\vec{x}$ be a vector of length n. Then the transformation induced by A, T_A, is one-to-one if and only if $A\vec{x} = 0$ implies $\vec{x} = 0$.

Since every linear transformation is induced by a matrix A, in order to show that T is one to one, it suffices to show that $A\vec{x} = 0$ has a unique solution.

Problem

Show that the transformation defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

is one-to-one.

Solution

Since T is a matrix transformation induced by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, it follows from the previous theorem that all we need to show is that $A\vec{x} = 0$ has the unique solution $\vec{x} = 0$. We do this in the standard way, by taking the augmented matrix of the system $A\vec{x} = 0$ and putting it in reduced row-echelon form.

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

From this we see that the system has unique solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and therefore T is a one-to-one.

Surjections

Definition

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. We say that T is a surjection or onto if, for every $\vec{b} \in \mathbb{R}^m$ there exists an $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ 0 \end{bmatrix} \text{ for all } \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2.$$

Then T is not onto. To see why, choose $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. Then there is no vector $\vec{x} \in \mathbb{R}^2$ so that $T(\vec{x}) = \vec{b}$; applying T to any vector results in a vector whose second entry is 0, and the second entry of $\vec{b}$ is 1.

Example (continued)

Consider the system $A\vec{x} = \vec{b}$, where A is the matrix induced by T,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then the augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

which is already in reduced row-echelon form. The fact that this system is inconsistent implies that T is not onto.

Theorem

Let A be an $m \times n$ matrix. Then the transformation T_A, induced by A, is onto if and only if $A\vec{x} = \vec{b}$ is consistent for every vector $\vec{b}$ in $\mathbb{R}^m$.

Problem

Show that the transformation defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

is onto.

Solution

Since $T : \mathbb{R}^2 \to \mathbb{R}^2$, we must show that for every $\vec{b} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, the system $A\vec{x} = \vec{b}$ is consistent.

Putting the augmented matrix of $A\vec{x} = \vec{b}$ into row-echelon form,

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 3 & 5 & b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & -1 & b-3a \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & 3a-b \end{array}\right].$$

We see that the system is consistent for all values of a and b, and therefore T is onto.

Onto but not one-to-one

Problem

Let T be the linear transformation induced by $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \end{bmatrix}$. Show that T_A is onto but not one-to-one.

Solution

Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \end{bmatrix} = R$$

For every $\vec{b}$ in $\mathbb{R}^2$, the rank of the augmented matrix $[A|\vec{b}]$ is equal to two, which is the rank of A. Therefore, the system $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$, so T_A is onto.

Since A has rank two, $A\vec{x} = 0$ has infinitely many solutions, so $\vec{x} = 0$ is not the only solution. Therefore, T_A is not one-to-one.

One-to-one but not onto

Problem

Let T be the linear transformation induced by $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ -1 & 2 \end{bmatrix}$. Show that T_A is one-to-one but not onto.

Solution

Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R$$

There exist vectors $\vec{b} \in \mathbb{R}^3$ for which the rank of $[A|\vec{b}]$ will be equal to three, while the rank of A is only two. Therefore, the system $A\vec{x} = \vec{b}$ is not consistent for every $\vec{b}$, so T_A is not onto.

Since A has rank two, every variable in $A\vec{x} = 0$ is a leading variable, so $\vec{x} = 0$ is the unique solution. Therefore, T_A is one-to-one.

One-to-one and onto

Problem

Let T be the linear transformation induced by $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. Show that T_A is one-to-one and onto.

Solution

Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = R$$

In this case, A is invertible, so $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for every $\vec{b}$ in $\mathbb{R}^2$. Therefore T_A is both one-to-one and onto.

Neither one-to-one nor onto

Problem

Let T be the linear transformation induced by $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Show that T_A is neither one-to-one nor onto.

Solution

Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

Since A has rank two, the augmented matrix $[A|\vec{b}]$ will have rank three for some choice of $\vec{b} \in \mathbb{R}^3$, resulting in $A\vec{x} = \vec{b}$ being inconsistent. Therefore, T_A is not onto.

The augmented matrix $[A|0]$ has rank two, so the system $A\vec{x} = 0$ has a non-leading variable, and hence does not have unique solution $\vec{x} = 0$. Therefore, T_A is not one-to-one.