A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Spectral Theory: 7.1 Eigenvalues and Eigenvectors

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Motivation: Calculating Powers of a Matrix

Example

Let $A = \left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array} \right]$. Find A^{100}.

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where D is a diagonal matrix.

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Theorem

If A is an $n \times n$ matrix and P is an invertible $n \times n$ matrix such that $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$ for each $k = 1, 2, 3, \ldots$

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Answer

Eigenvalues and eigenvectors.

Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix, λ a real number, and $X \neq 0$ an n-vector. If $AX = \lambda X$, then λ is an eigenvalue of A, and X is an eigenvector of A corresponding to λ, or a λ-eigenvector.

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Since $X \neq 0$, X is a nontrivial solution to the linear system with coefficient matrix $\lambda I - A$, and therefore the matrix $\lambda I - A$ is not invertible. Since a matrix is invertible if and only if its determinant is not equal to zero, it follows that

$$\det(\lambda I - A) = 0.$$

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Finding Eigenvalues and Eigenvectors

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Procedure:

Let A be an $n \times n$ matrix.

- **Eigenvalues:** Find λ by solving the equation

$$c_A(x) = \det(xI - A) = 0$$

- **Eigenvectors:** For each λ, find $X \neq 0$ by finding the basic solutions to

$$(A - \lambda I)X = 0$$

- **Check:** For each pair of λ, X check that $AX = \lambda X$.

Example (continued)

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so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$.

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so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$.
The 2-eigenvectors of A (meaning the eigenvectors of A corresponding to $\lambda_1 = 2$) are found by solving the homogeneous system $(2I - A)X = 0$.

Example (continued)

For $A = \left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array} \right]$, we've found

$$c_A(x) = x^2 - 7x + 10 = (x-2)(x-5),$$

so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$.
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Solve the system in the standard way, by putting the augmented matrix of the system in reduced row-echelon form.

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However, since eigenvectors are nonzero, the 2-eigenvectors of A are all vectors

$$X = t\left[\begin{array}{c} 1 \\ 1 \end{array}\right] \text{ where } t \in \mathbb{R} \text{ and } t \neq 0.$$

Example (continued)

To find the 5-eigenvectors of $A = \left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array} \right]$ solve the homogeneous system $(5I - A)X = 0$, with coefficient matrix

$$5I - A = 5 \left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \right] - \left[\begin{array}{rr} 4 & -2 \\ -1 & 3 \end{array} \right] = \left[\begin{array}{rr} 1 & 2 \\ 1 & 2 \end{array} \right].$$

$$\left[\begin{array}{rr|r} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{rr|r} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right].$$

Therefore the 5-eigenvectors of A are the vectors

$$X = \left[\begin{array}{r} -2s \\ s \end{array} \right] = s \left[\begin{array}{r} -2 \\ 1 \end{array} \right] \text{ where } s \in \mathbb{R} \text{ and } s \neq 0.$$

Basic Eigenvectors

Definition

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$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is is a basic eigenvector of A corresponding to the eigenvalue 2.

$X = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basic eigenvector of A corresponding to the eigenvalue 5.

Eigenvalues with multiplicity greater than one

Problem

Find the characteristic polynomial and eigenvalues of the matrix

$$A = \left[\begin{array}{rrr} 4 & 1 & 2 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{array} \right].$$

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Therefore, A has eigenvalues 1 and 4, with 4 being an eigenvalue of multiplicity two.

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The general solution is $X = \left[\begin{array}{r} -s \\ s \\ s \end{array}\right]$ where $s \in \mathbb{R}$. We get a basic eigenvector by choosing $s = 1$ (in fact, any nonzero value of s gives us a basic eigenvector).

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Therefore, $X = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is a (basic) eigenvector of A corresponding to $\lambda_1 = 1$.

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We can obtain other pairs of basic 4-eigenvectors for A by taking any nonzero scalar multiple of X_1, and any nonzero scalar multiple of X_2.

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Notice that every 4-eigenvector of A is a nonzero linear combination of basic 4-eigenvectors.

Problem

For

$$A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix},$$

find $c_A(x)$, the eigenvalues of A, and basic eigenvector(s) for each eigenvalue.

LECTURE 2

Eigenvalues and eigenvectors (review)

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(It always has a nontrivial solution.)
4. λ-eigenvectors are the (nontrivial) solutions to this system.

Similar Matrices

Definition

Let A, and B be $n \times n$ matrices. Suppose there exists an invertible matrix P such that

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Let A and B be similar matrices, so that $A = P^{-1}BP$ where A, B are $n \times n$ matrices and P is invertible. Then A and B have the same eigenvalues.

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Assume $BX = \lambda X$. Let $Y = P^{-1}X$. Then

$$AY = (P^{-1}BP)P^{-1}X = P^{-1}BX = P^{-1}\lambda X = \lambda Y.$$

Using Similar and Elementary Matrices

Problem

Find the eigenvalues for the matrix

$$A = \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix}$$

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Solution

We will use elementary matrices to simplify A before finding the eigenvalues. Left multiply A by $E(2 \times 2 + 3)$, and right multiply by the inverse of $E(2 \times 2 + 3)$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

Notice that the resulting matrix and A are similar matrices (with $E(2 \times 2 + 3)$ playing the role of P) so they have the same eigenvalues.

Solution (continued)

We do this step again, on the resulting matrix above.

$$\left[\begin{array}{rrr} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{rrr} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{array}\right] \left[\begin{array}{rrr} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{rrr} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{array}\right] = B$$

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Again by properties of similar matrices, the resulting matrix here (labeled B) has the same eigenvalues as our original matrix A. The advantage is that it is much simpler to find the eigenvalues of B than A.

Finding these eigenvalues follows the usual procedure and is left as an exercise.

Example (Triangular Matrices)

Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 5 & 1 & -2 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

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Therefore the eigenvalues of A are $2, 5, 0$ and -4, exactly the entries on the main diagonal of A.

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$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 0 & 5 & 1 & -2 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

The characteristic polynomial of A is

$$c_A(x) = \det(xI - A) = \det \begin{bmatrix} x-2 & 1 & 0 & -3 \\ 0 & x-5 & -1 & 2 \\ 0 & 0 & x & -7 \\ 0 & 0 & 0 & x+4 \end{bmatrix} = (x-2)(x-5)x(x+4).$$

Therefore the eigenvalues of A are $2, 5, 0$ and -4, exactly the entries on the main diagonal of A.

Eigenvalues of Triangular Matrices

If A is an $n \times n$ upper triangular (or lower triangular) matrix, then the eigenvalues of A are the entries on the main diagonal of A.

Geometric Interpretation of Eigenvalues and Eigenvectors

Example

Recall that in $\mathbb{R}^2$, reflection in the x-axis is a linear transformation that transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$.

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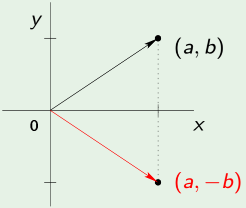

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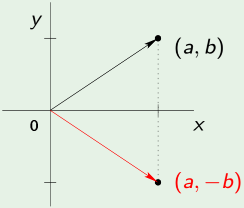

Let A be the matrix that induces reflection in the x-axis.

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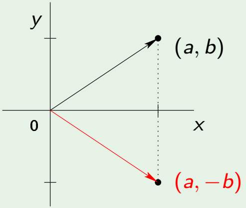

Let A be the matrix that induces reflection in the x-axis. If λ were an eigenvalue of A and X a corresponding eigenvector, then $AX = \lambda X$ implies that, geometrically, reflecting X in the x-axis is the same as changing X to a vector parallel to X.

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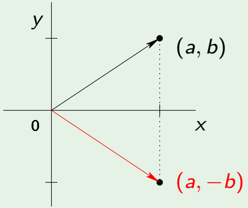

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How could this be possible?

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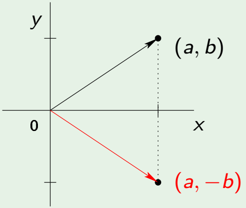

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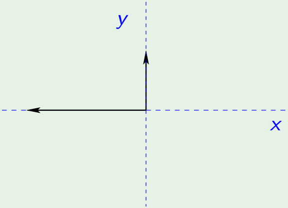
Can you picture what an eigenvector of A would look like?

Example (continued)



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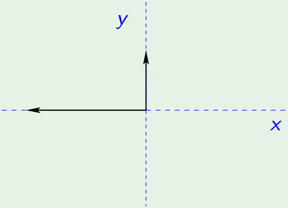

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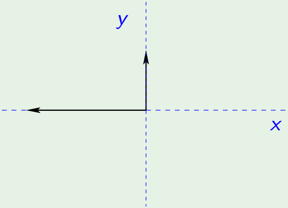

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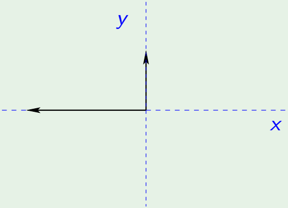

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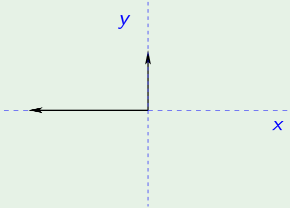

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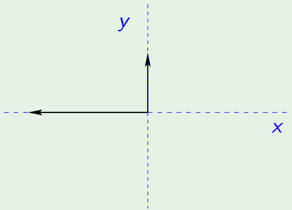

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This makes sense, since we know that reflection in the x-axis is induced by the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which has eigenvalues 1 and -1.

Example

In $\mathbb{R}^2$, reflection in the y-axis is a linear transformation that transforms $\left[\begin{array}{c} a \\ b \end{array}\right]$ to $\left[\begin{array}{r} -a \\ b \end{array}\right]$.

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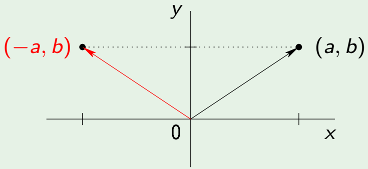

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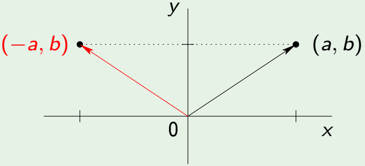

Let A be the matrix that induces reflection in the y-axis.

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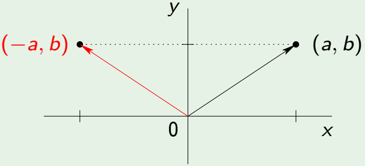

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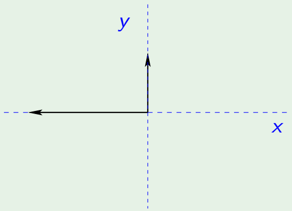

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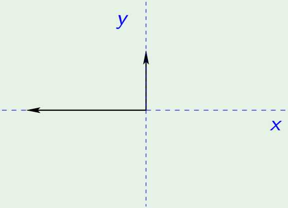

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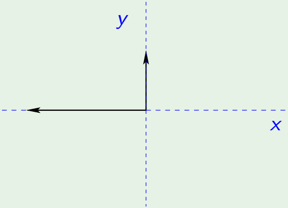

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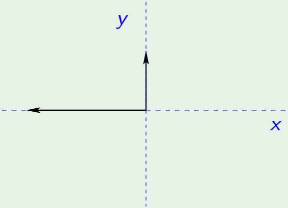

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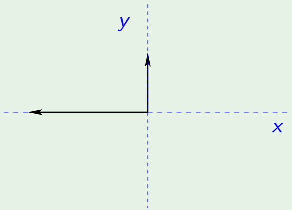

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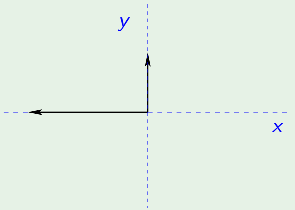

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This makes sense, since we know that reflection in the y-axis is induced by the matrix

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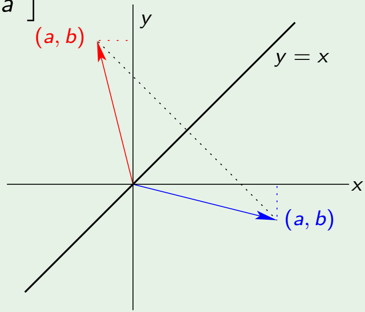
Reflection in the line $y = x$ is a linear transformation that transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.



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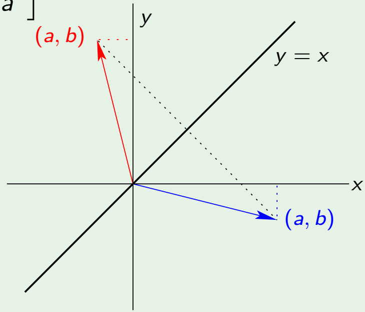

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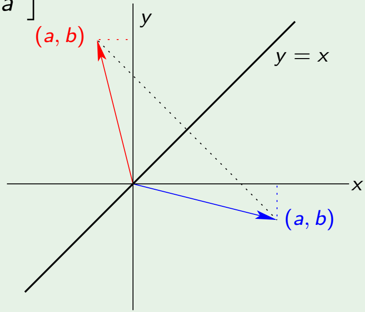

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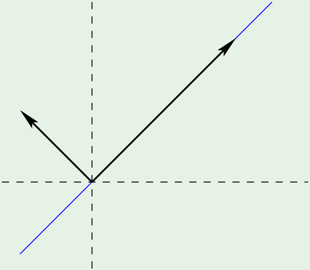

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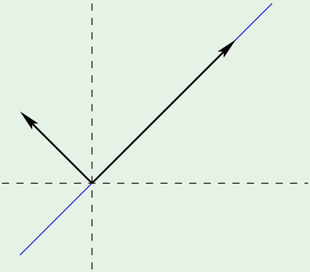

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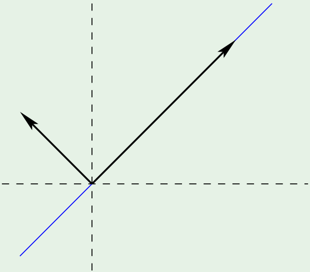

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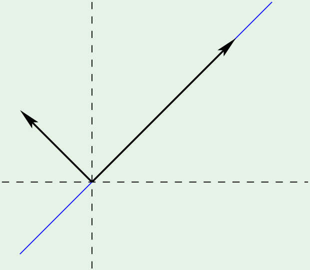

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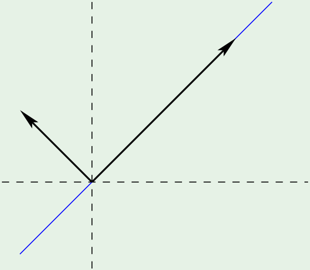

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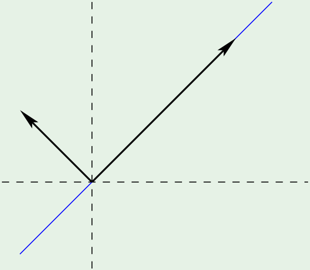

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Therefore, 1 and -1 are eigenvalues of A.

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Therefore, 1 and -1 are eigenvalues of A; in fact, these are the only two eigenvalues of A and each has multiplicity one. This follows from the fact that A is a 2×2 matrix, so it's characteristic polynomial has degree two.

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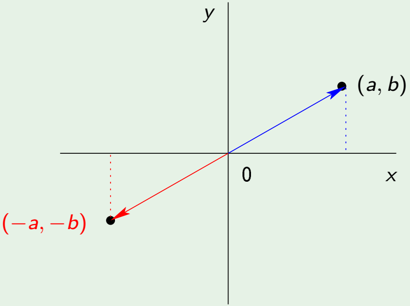

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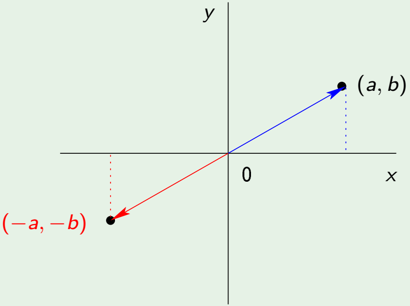

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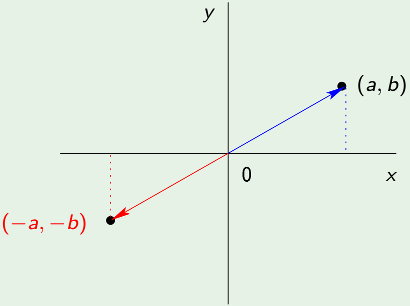

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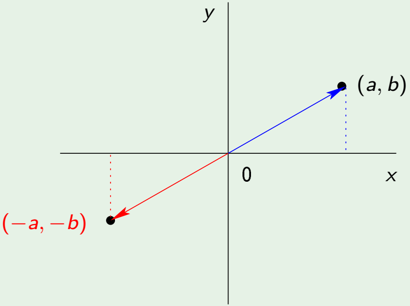

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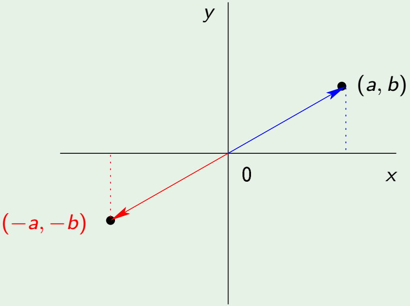

R_π is a linear transformation that transforms X to $-X$.

Let A denote the matrix that induces rotation through π. Then $AX = -X$ for every nonzero vector X, meaning that every nonzero vector of $\mathbb{R}^2$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -1$.

Example (Rotation through $\pi/2$)

We denote by $R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$ counterclockwise rotation about the origin through an angle of $\pi/2$.

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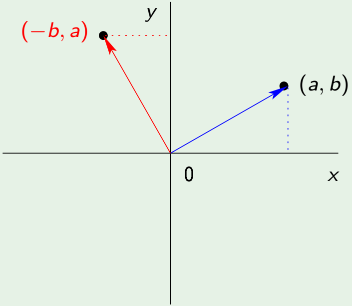

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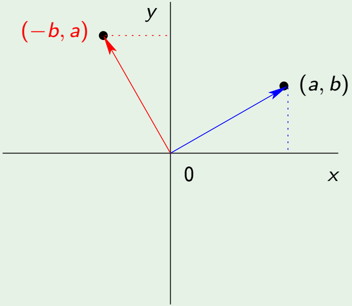

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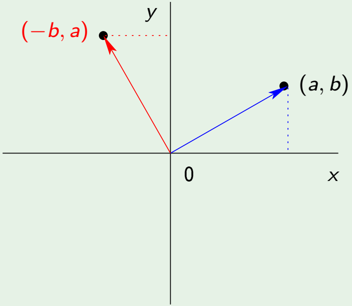

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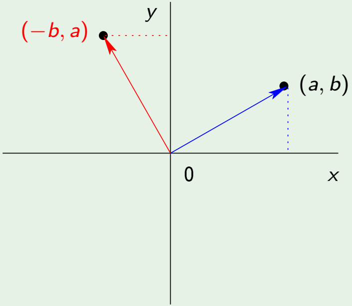

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