A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

5.5 (partial) and 5.7: Linear Transformations: One to One and Onto, Kernel, and Image

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Injections

Definition

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\vec{x}_1$ and $\vec{x}_2$ be in $\mathbb{R}^n$. We say that T is an injection or is one-to-one (sometimes written as 1-1) if $\vec{x}_1 \neq \vec{x}_2$ implies that

$$T(\vec{x}_1) \neq T(\vec{x}_2).$$

Equivalently, if $T(\vec{x}_1) = T(\vec{x}_2)$, then $\vec{x}_1 = \vec{x}_2$. Thus, T is one-to-one if two distinct vectors are never transformed into the same vector.

Theorem

Let A be an $m \times n$ matrix and let $\vec{x}$ be a vector of length n. Then the transformation induced by A, T_A, is one-to-one if and only if $A\vec{x} = 0$ implies $\vec{x} = 0$.

Since every linear transformation is induced by a matrix A, in order to show that T is one to one, it suffices to show that $A\vec{x} = 0$ has a unique solution.

Problem

Show that the transformation defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is one-to-one.

Solution

Since T is a matrix transformation induced by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, it follows from the previous theorem that all we need to show is that $A\vec{x} = 0$ has the unique solution $\vec{x} = 0$. We do this in the standard way, by taking the augmented matrix of the system $A\vec{x} = 0$ and putting it in reduced row-echelon form.

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

From this we see that the system has unique solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and therefore T is a one-to-one.

Not one-to-one

Problem

Let T be the linear transformation induced by $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \end{bmatrix}$. Show that T_A is not one-to-one.

Solution

Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \end{bmatrix} = R$$

Since A has rank two, $A\vec{x} = 0$ has infinitely many solutions, so $\vec{x} = 0$ is not the only solution. Therefore, T_A is not one-to-one.

One-to-one

Problem

Let T be the linear transformation induced by $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ -1 & 2 \end{bmatrix}$. Show that T_A is one-to-one.

Solution

Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R$$

Since A has rank two, every variable in $A\vec{x} = 0$ is a leading variable, so $\vec{x} = 0$ is the unique solution. Therefore, T_A is one-to-one.

One-to-one and onto

Problem

Let T be the linear transformation induced by $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. Show that T_A is one-to-one and onto.

Solution

Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = R$$

In this case, A is invertible, so $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for every $\vec{b}$ in $\mathbb{R}^2$. Therefore T_A is both one-to-one and onto.

Neither one-to-one nor onto

Problem

Let T be the linear transformation induced by $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Show that T_A is neither one-to-one nor onto.

Solution

Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

Since A has rank two, the augmented matrix $[A|\vec{b}]$ will have rank three for some choice of $\vec{b} \in \mathbb{R}^3$, resulting in $A\vec{x} = \vec{b}$ being inconsistent. Therefore, T_A is not onto.

The augmented matrix $[A|0]$ has rank two, so the system $A\vec{x} = 0$ has a non-leading variable, and hence does not have unique solution $\vec{x} = 0$. Therefore, T_A is not one-to-one.

Kernel and Image

Definition (Kernel)

Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$, and let $T : V \mapsto W$ be a linear transformation.
Then the kernel of T, $\ker(T)$, consists of all $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$.

$$\ker(T) = \left\{ \vec{v} \in V : T(\vec{v}) = \vec{0} \right\}$$

Definition (Image)

Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$, and let $T : V \mapsto W$ be a linear transformation.
Then the image of T, $\mathrm{im}(T)$, consists of all $\vec{w} \in W$ such that $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$.

$$\mathrm{im}(T) = \{ T(\vec{v}) : \vec{v} \in V \}$$

Problem to Try

Problem

Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$, and let $T : V \mapsto W$ be a linear transformation.

Show that $\ker(T)$ is a subspace of V and $\mathrm{im}(T)$ is a subspace of W.

Example

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b+c \\ c-a \end{bmatrix}$$

Then T is a linear transformation. Find a basis for $\ker(T)$ and $\mathrm{im}(T)$.

Solution

You can (and should!) verify that T is a linear transformation.

Solution (continued)

Kernel of T: We look for all vectors $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \vec{0}$.

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b+c \\ c-a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives a system of equations:

$$\begin{aligned} a+b+c &= 0 \\ c-a &= 0 \end{aligned}$$

The general solution is

$$\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \left\{\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} : t \in \mathbb{R}\right\} = \left\{t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} : t \in \mathbb{R}\right\}$$

And therefore a basis for the kernel is $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$.

Solution (continued)

Image of T: We can write the image as

$$\begin{aligned}
\operatorname{im}(T) &= \left\{ \left[\begin{array}{c} a+b+c \\ c-a \end{array} \right] : a,b,c \in \mathbb{R} \right\} \\
&= \left\{ \left[\begin{array}{c} a \\ -a \end{array} \right] + \left[\begin{array}{c} b \\ 0 \end{array} \right] + \left[\begin{array}{c} c \\ c \end{array} \right] : a,b,c \in \mathbb{R} \right\} \\
&= \left\{ a\left[\begin{array}{c} 1 \\ -1 \end{array} \right] + b\left[\begin{array}{c} 1 \\ 0 \end{array} \right] + c\left[\begin{array}{c} 1 \\ 1 \end{array} \right] : a,b,c \in \mathbb{R} \right\}
\end{aligned}$$

Thus $\operatorname{im}(T) = \operatorname{span}\left\{ \left[\begin{array}{c} 1 \\ -1 \end{array} \right], \left[\begin{array}{c} 1 \\ 0 \end{array} \right], \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \right\}$.

These vectors are not linearly independent, but the first two are so a basis for the image of T is

$$\left\{ \left[\begin{array}{c} 1 \\ -1 \end{array} \right], \left[\begin{array}{c} 1 \\ 0 \end{array} \right] \right\}.$$

Kernel and One to One

The kernel of a linear transformation gives important information about whether the transformation is one to one. Recall that a linear transformation T is one to one if and only if $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$.

Theorem (Dimension Theorem)

Let $T : V \mapsto W$ be a linear transformation where V is a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$.
Then T is one to one if and only if $\ker(T) = \left\{ \vec{0} \right\}$.

Dimension of the Kernel and Image

Theorem

Let $T : V \mapsto W$ be a linear transformation where V is a subspace of $\mathbb{R}^n$ and W is a subspace of $\mathbb{R}^m$. Suppose further that the dimension of V is k. Then

$$k = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$

Corollary

Let T, V, W be defined as above, with $\dim(V) = k$. Then

$$\begin{aligned} \dim(\ker(T)) &\leq k &\leq n \\ \dim(\operatorname{im}(T)) &\leq k &\leq n \end{aligned}$$

Example (Revisited)

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be defined by

$$T\left(\left[\begin{array}{c} a \\ b \\ c \end{array}\right]\right) = \left[\begin{array}{c} a+b+c \\ c-a \end{array}\right]$$

Find the dimension of $\ker(T)$ and $\operatorname{im}(T)$.

Solution

We already know that a basis for the kernel of T is given by

$$\left\{\left[\begin{array}{r} 1 \\ -2 \\ 1 \end{array}\right]\right\}$$

Therefore $\dim(\ker(T)) = 1$.

Solution (continued)

We also found a basis for the image of T as

$$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

and this of course shows $\dim(\mathrm{im}(T)) = 2$.

But we could have found the dimension of $\mathrm{im}(T)$ without finding a basis. That's because since the dimension of $\mathbb{R}^3$ is 3, and the dimension of $\ker(T)$ is 1, we get by the Dimension Theorem that:

$$\begin{aligned} \dim(\mathrm{im}(T)) &= \dim(\mathbb{R}^3) - \dim(\ker(T)) \\ &= 3 - 1 = 2 \end{aligned}$$