A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

5.9: The General Solution to a Linear System

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Example

Suppose that a system of linear equations can be written as

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 5 + 2x_2 - x_4 \\ x_2 \\ -4 - 3x_4 \\ x_4 \end{bmatrix}$$

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Setting $x_2 = 2$ and $x_4 = -1$ gives us the particular solution

$$\vec{x}_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ -1 \\ -1 \end{bmatrix}.$$

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Definition

Let $A\vec{x} = \vec{b}$ be a system of equations. Then, the corresponding system given by $A\vec{x} = \vec{0}$, found by replacing $\vec{b}$ with $\vec{0}$ is the associated homogeneous system.

Associated Homogeneous System

Example

Consider the system given by
$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 2 \\ -x_1 & + & 4x_2 & + & 5x_3 & - & 2x_4 & = & 3 \\ x_1 & + & 6x_2 & + & 3x_3 & + & 4x_4 & = & 4 \end{array}.$$

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We can write this as

$$\left[\begin{array}{rrrr} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] = \left[\begin{array}{c} 2 \\ 3 \\ 4 \end{array}\right]$$

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whose associated homogeneous system can be written as:

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Solutions to Systems as Linear Combinations of Vectors

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Basic Solutions to Homogeneous Systems

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$$\begin{aligned} x_1 &= \tfrac{9}{5}s - \tfrac{14}{5}t \\ x_2 &= -\tfrac{4}{5}s - \tfrac{1}{5}t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

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The General Solution to a Linear System

Theorem

Let T be a linear transformation given by $T(\vec{x}) = A\vec{x}$. Suppose that $\vec{x}_p$ is a particular solution of the system of linear equations given by $T(\vec{x}) = \vec{b}$, i.e., $T(\vec{x}_p) = \vec{b}$. Then if $\vec{y}$ is any other solution of $T(\vec{x}) = \vec{b}$, it can be written in the form $\vec{y} = \vec{x}_p + \vec{x}_0$ for some (particular) solution $\vec{x}_0$ to the associated homogeneous system $A\vec{x} = 0$.

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The theorem implies that if $\vec{x}_p$ is a particular solution to $A\vec{x} = \vec{b}$ and $\vec{x}_h$ is the general solution to the associated homogeneous system $A\vec{x} = 0$, then $\vec{x}_p + \vec{x}_h$ is the general solution to $A\vec{x} = \vec{b}$.

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The theorem implies that if $\vec{x}_p$ is a particular solution to $A\vec{x} = \vec{b}$ and $\vec{x}_h$ is the general solution to the associated homogeneous system $A\vec{x} = 0$, then $\vec{x}_p + \vec{x}_h$ is the general solution to $A\vec{x} = \vec{b}$. Furthermore, the general solution to $A\vec{x} = \vec{b}$ can always be written in the form $\vec{x}_p + \vec{x}_h$ where $\vec{x}_p$ is a particular solution to $A\vec{x} = \vec{b}$ and $\vec{x}_h$ is the general solution to $A\vec{x} = 0$.

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The system of linear equations $A\vec{x} = \vec{b}$, with

$$A = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 0 & 1 & -1 & 1 \\ -1 & 1 & -3 & 0 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

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Notice that $\vec{x}_p = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution to $A\vec{x} = \vec{b}$ (obtained by setting

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Notice that $\vec{x}_p = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution to $A\vec{x} = \vec{b}$ (obtained by setting $s = t = 0$), while $\vec{x}_h = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $s, t \in \mathbb{R}$ is the the general solution to the associated homogeneous system $A\vec{x} = 0$.

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$$\vec{y} = \left[\begin{array}{c} 1 - 2s - t \\ 2 + s - t \\ s \\ t \end{array}\right] = \left[\begin{array}{r} 1 \\ 2 \\ 0 \\ 0 \end{array}\right] + s\left[\begin{array}{r} -2 \\ 1 \\ 1 \\ 0 \end{array}\right] + t\left[\begin{array}{r} -1 \\ -1 \\ 0 \\ 1 \end{array}\right], \quad s, t \in \mathbb{R}.$$