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LINEAR ALGEBRA

Lecture Notes

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$\mathbb{R}^n$: Subspaces and Basis

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A First Course in Linear Algebra

Lecture Notes

Current Lecture Notes Revision: Version 2017 — Revision A

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Definition

The vector space $\mathbb{R}^n$ consists of the set $\mathbb{R}^n$ written as column matrices, along with the (matrix) operations of addition and scalar multiplication. Unless stated otherwise, $\mathbb{R}^n$ means the vector space $\mathbb{R}^n$.

A rigorous definition of an abstract vector space will be given after we have studied properties of the vector space $\mathbb{R}^n$.

We are not yet ready to formally define the term “subspace”. However, in the context of $\mathbb{R}^n$, we rely on the Subspace Test to determine whether or not a subset of $\mathbb{R}^n$ is a subspace.

Theorem (Subspace Test)

A subset V of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

1. the zero vector of $\mathbb{R}^n$, $\vec{0}_n$, is in V;
2. V is closed under addition, i.e., for all $\vec{u}, \vec{w} \in V$, $\vec{u} + \vec{w} \in V$;
3. V is closed under scalar multiplication, i.e., for all $\vec{u} \in V$ and $k \in \mathbb{R}$, $k\vec{u} \in V$.

The subset $V = \left\{\vec{0}_n\right\}$ is a subspace of $\mathbb{R}^n$ (verify this), as is the set $\mathbb{R}^n$ itself. Any other subspace of $\mathbb{R}^n$ is a **proper** subspace of $\mathbb{R}^n$.

Notation

If V is a subset of $\mathbb{R}^n$, we write $V \subseteq \mathbb{R}^n$.

Example

In $\mathbb{R}^3$, the line L through the origin that is parallel to the vector $\vec{d} = \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}$ has (vector) equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}, t \in \mathbb{R}$, so

$$L = \left\{ t\vec{d} \mid t \in \mathbb{R} \right\}.$$

Claim. L is a subspace of $\mathbb{R}^3$.

- First: $\vec{0}_3 \in L$ since $0\vec{d} = \vec{0}_3$.
- Suppose $\vec{u}, \vec{v} \in L$. Then by definition, $\vec{u} = s\vec{d}$ and $\vec{v} = t\vec{d}$, for some $s, t \in \mathbb{R}$. Thus
$$\vec{u} + \vec{v} = s\vec{d} + t\vec{d} = (s+t)\vec{d}.$$
Since $s + t \in \mathbb{R}$, $\vec{u} + \vec{v} \in L$; i.e., L is closed under addition.

Example 5 (continued)

- Suppose $\vec{u} \in L$ and $k \in \mathbb{R}$ (k is a scalar). Then $\vec{u} = t\vec{d}$, for some $t \in \mathbb{R}$, so

$$k\vec{u} = k(t\vec{d}) = (kt)\vec{d}.$$

  Since $kt \in \mathbb{R}$, $k\vec{u} \in L$; i.e., L is closed under scalar multiplication.

Therefore, L is a subspace of $\mathbb{R}^3$.

Note that there is nothing special about the vector $\vec{d}$ used in this example; the same proof works for any nonzero vector $\vec{d} \in \mathbb{R}^3$, so any line through the origin is a subspace of $\mathbb{R}^3$.

Example

In $\mathbb{R}^3$, let M denote the plane through the origin having equation $3x - 2y + z = 0$; then M has normal vector $\vec{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$. If $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then

$$M = \left\{ \vec{u} \in \mathbb{R}^3 \mid \vec{n} \bullet \vec{u} = 0 \right\},$$

where $\vec{n} \bullet \vec{u}$ is the dot product of vectors $\vec{n}$ and $\vec{u}$.

Claim. M is a subspace of $\mathbb{R}^3$.

- First: $\vec{0}_3 \in M$ since $\vec{n} \bullet \vec{0}_3 = 0$.
- Suppose $\vec{u}, \vec{v} \in M$. Then by definition, $\vec{n} \bullet \vec{u} = 0$ and $\vec{n} \bullet \vec{v} = 0$, so

  $$\vec{n} \bullet (\vec{u} + \vec{v}) = \vec{n} \bullet \vec{u} + \vec{n} \bullet \vec{v} = 0 + 0 = 0,$$

  and thus $(\vec{u} + \vec{v}) \in M$; i.e., M is closed under addition.

Example 6 (continued)

- Suppose $\vec{u} \in M$ and $k \in \mathbb{R}$. Then $\vec{n} \bullet \vec{u} = 0$, so

  $$\vec{n} \bullet (k\vec{u}) = k(\vec{n} \bullet \vec{u}) = k(0) = 0,$$

  and thus $k\vec{u} \in M$; i.e., M is closed under scalar multiplication.

Therefore, M is a subspace of $\mathbb{R}^3$.

As in the previous example, there is nothing special about the plane chosen for this example; any plane through the origin is a subspace of $\mathbb{R}^3$.

Problem

Is $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \text{ and } 2a - b = c + 2d \right\}$ a subspace of $\mathbb{R}^4$? Justify your answer.

Solution 1

The zero vector of $\mathbb{R}^4$ is the vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ with $a = b = c = d = 0$.

In this case, $2a - b = 2(0) + 0 = 0$ and $c + 2d = 0 + 2(0) = 0$, so $2a - b = c + 2d$. Therefore, $\vec{0}_4 \in V$.

Solution (continued)

Suppose

$$\vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \text{ are in } V.$$

Then $2a_1 - b_1 = c_1 + 2d_1$ and $2a_2 - b_2 = c_2 + 2d_2$. Now

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{bmatrix},$$

and

$$\begin{aligned} 2(a_1 + a_2) - (b_1 + b_2) &= (2a_1 - b_1) + (2a_2 - b_2) \\ &= (c_1 + 2d_1) + (c_2 + 2d_2) \\ &= (c_1 + c_2) + 2(d_1 + d_2). \end{aligned}$$

Therefore, $\vec{v}_1 + \vec{v}_2 \in V$.

Solution (continued)

Finally, suppose
$$\vec{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in V \text{ and } k \in \mathbb{R}.$$

Then $2a - b = c + 2d$. Now
$$k\vec{v} = k \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \\ kd \end{bmatrix},$$

and
$$2ka - kb = k(2a - b) = k(c + 2d) = kc + 2kd.$$

Therefore, $k\vec{v} \in V$.

It follows from the **Subspace Test** that V is a subspace of $\mathbb{R}^4$.

Definition

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$. The eigenspace of A corresponding to λ is the set

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}.$$

Note that

$$\begin{aligned} E_\lambda(A) &= \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}, \\ &= \left\{\vec{x} \in \mathbb{R}^n \mid \lambda\vec{x} - A\vec{x} = \vec{0}_n\right\} \\ &= \left\{\vec{x} \in \mathbb{R}^n \mid (\lambda I - A)\vec{x} = \vec{0}_n\right\} \end{aligned}$$

showing that

$$E_\lambda(A) = \mathrm{null}(\lambda I - A).$$

It follows that

- if λ is not an eigenvalue of A, then $E_\lambda(A) = \{\vec{0}_n\}$;
- the nonzero vectors of $E_\lambda(A)$ are the eigenvectors of A corresponding to λ;
- the eigenspace of A corresponding to λ is a subspace of $\mathbb{R}^n$.

Theorem

Let V be a nonempty collection of vectors in $\mathbb{R}^n$. Then V is a subspace of $\mathbb{R}^n$ if and only if there exist vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in V such that

$$V = \operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$$

Furthermore, let W be another subspace of $\mathbb{R}^n$ and suppose $\{\vec{u}_1, \cdots, \vec{u}_k\} \in W$. Then it follows that V is a subset of W.

Proof.

We first show that if V is a subspace, then it can be written as $V = \operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

Pick a vector $\vec{u}_1$ in V. If $V = \operatorname{span}\{\vec{u}_1\}$, then you have found your list of vectors and are done.

Proof (continued).

If $V \neq \text{span}\{\vec{u}_1\}$, then there exists $\vec{u}_2$ a vector of V which is not in $\text{span}\{\vec{u}_1\}$. Consider $\text{span}\{\vec{u}_1, \vec{u}_2\}$. If $V = \text{span}\{\vec{u}_1, \vec{u}_2\}$, we are done. Otherwise, pick $\vec{u}_3$ not in $\text{span}\{\vec{u}_1, \vec{u}_2\}$. Continue this way. Note that since V is a subspace, these spans are each contained in V. The process must stop with $\vec{u}_k$ for some $k \leq n$, and thus $V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

Now suppose $V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$, we must show this is a subspace. So let $\sum_{i=1}^k c_i \vec{u}_i$ and $\sum_{i=1}^k d_i \vec{u}_i$ be two vectors in V, and let a and b be two scalars. Then

$$a \sum_{i=1}^{k} c_i \vec{u}_i + b \sum_{i=1}^{k} d_i \vec{u}_i = \sum_{i=1}^{k} (ac_i + bd_i) \vec{u}_i$$

which is one of the vectors in $\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$ and is therefore contained in V. This shows that $\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$ has the properties of a subspace.

□

Proof (continued).

To prove that $V \subseteq W$, we prove that if $\vec{u}_i \in V$, then $\vec{u}_i \in W$.

Suppose $\vec{u} \in V$. Then $\vec{u} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_k\vec{u}_k$ for some $a_i \in \mathbb{R}$, $1 \leq i \leq k$.

Since W contain each $\vec{u}_i$ and W is a vector space, it follows that $a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_k\vec{u}_k \in W$. $\square$

Problem

Is $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \text{ and } 2a - b = c + 2d \right\}$ a subspace of $\mathbb{R}^4$?
Justify your answer.

Solution 2

Let $\vec{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in V$. Since $2a - b = c + 2d$, $c = 2a - b - 2d$, and thus

$$V = \left\{ \begin{bmatrix} a \\ b \\ 2a - b - 2d \\ d \end{bmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Therefore V is a subspace of $\mathbb{R}^4$.

Bases and Dimension

Theorem (Exchange Theorem)

Suppose $\{\vec{u}_1, \cdots, \vec{u}_r\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, and each $\vec{u}_k$ is contained in span $\{\vec{v}_1, \cdots, \vec{v}_s\}$ Then $s \geq r$.

In words, spanning sets have at least as many vectors as linearly independent sets.

Definition

Let V be a subspace of $\mathbb{R}^n$. Then $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is a **basis** for V if the following two conditions hold.

1. span $\{\vec{u}_1, \cdots, \vec{u}_k\} = V$
2. $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent

Example

The subset $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$, called the standard basis of $\mathbb{R}^n$. (We've already seen that $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is linearly independent and that $\mathbb{R}^n = \text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.)

Example

In a previous problem, we saw that $\mathbb{R}^4 = \text{span}(S)$ where

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

S is also linearly independent (prove this). Therefore, S is a basis of $\mathbb{R}^4$.

Theorem (Invariance Theorem)

Let V be a subspace of $\mathbb{R}^n$ with two bases B_1 and B_2. Suppose B_1 contains s vectors and B_2 contains r vectors. Then $s = r$.

Proof.

This follows right away from the Exchange Theorem. Indeed observe that $B_1 = \{\vec{u}_1, \cdots, \vec{u}_s\}$ is a spanning set for V while $B_2 = \{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent, so $s \geq r$. Similarly $B_2 = \{\vec{v}_1, \cdots, \vec{v}_r\}$ is a spanning set for V while $B_1 = \{\vec{u}_1, \cdots, \vec{u}_s\}$ is linearly independent, so $r \geq s$. □

Definition

Let V be a subspace of $\mathbb{R}^n$. Then the **dimension** of V, written $\dim(V)$ is defined to be the number of vectors in a basis.

Problem

In $\mathbb{R}^n$, what is the dimension of the subspace $\{\vec{0}_n\}$?

Solution

The only basis of the zero subspace is the empty set, $\emptyset$: (i) the empty set is (trivially) independent, and (ii) any linear combination of no vectors is the zero vector. Therfore, the zero subspace has dimension zero.

Example

Since $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$, $\mathbb{R}^n$ has dimension n.
This is why the Cartesian plane, $\mathbb{R}^2$, is called 2-dimensional, and $\mathbb{R}^3$ is called 3-dimensional.

Problem

Let

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = d - c \right\}.$$

Show that U is a subspace of $\mathbb{R}^4$, find a basis of U, and find $\dim(U)$.

Solution

The condition $a - b = d - c$ is equivalent to the condition $a = b - c + d$, so we may write

$$U = \left\{ \begin{bmatrix} b - c + d \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \right\} = \left\{ b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; b, c, d \in \mathbb{R} \right\}$$

This shows that U is a subspace of $\mathbb{R}^4$, since $U = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ where

$$\begin{aligned} \vec{x}_1 &= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T \\ \vec{x}_2 &= \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}^T \\ \vec{x}_3 &= \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T. \end{aligned}$$

Solution (continued)

Furthermore,

$$\left\{ \left[\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} -1 \\ 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array}\right] \right\}$$

is linearly independent, as can be seen by taking the reduced row-echelon form of the matrix whose columns are $\vec{x}_1, \vec{x}_2$ and $\vec{x}_3$.

$$\left[\begin{array}{rrr} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Since every column of the reduced row-echelon form matrix has a leading one, the columns are linearly independent.

Therefore $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is linearly independent and spans U, so is a basis of U, and hence U has dimension three.

Example

Suppose that $B_1 = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$ and that A is an $n \times n$ invertible matrix. Let $B_2 = \{A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_n\}$, and let

$$V = \left[\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{array} \right].$$

Since B is a basis of $\mathbb{R}^n$, B is independent (also a spanning set of $\mathbb{R}^n$); thus V is invertible. Now, because A and V are invertible, so is

$$AV = \left[\begin{array}{cccc} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \end{array} \right].$$

Therefore, the columns of AV are independent and span $\mathbb{R}^n$. Since the columns of AV are the vectors of B_2, B_2 is a basis of $\mathbb{R}^n$.

Properties of Bases

Theorem

Let V be a subspace of $\mathbb{R}^n$. Then there exists a basis of V with $\dim(V) \leq n$.

Example

Previously , we showed that

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = d - c \right\}$$

is a subspace of $\mathbb{R}^4$, and that $\dim(V) = 3$. Also, it is easy to verify that

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix} \right\},$$

is an independent subset of V.

S can be extended to a basis of V. To do so, find a vector in V that is **not** in $\mathrm{span}(S)$.

(continued)

$$\begin{bmatrix} 1 & 2 & ? \\ 1 & 3 & ? \\ 1 & 3 & ? \\ 1 & 2 & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 3 & -1 \\ 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, S can be extended to the basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\},$$

of V.

Problem

Let W be the subspace

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 19 \\ -8 \\ 8 \end{bmatrix}, \begin{bmatrix} -6 \\ -15 \\ 6 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Find a basis for W which consists of a subset of the given vectors.

Final Answer

A basis for W is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Theorem

The following properties hold in $\mathbb{R}^n$:

- Suppose $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent. Then $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is a basis for $\mathbb{R}^n$.
- Suppose $\{\vec{u}_1, \cdots, \vec{u}_m\}$ spans $\mathbb{R}^n$. Then $m \geq n$.
- If $\{\vec{u}_1, \cdots, \vec{u}_n\}$ spans $\mathbb{R}^n$, then $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent.

Question

What is the significance of this result?

Answer

Let V be a subspace of $\mathbb{R}^n$ and suppose $B \subseteq V$.

- If B spans V and $|B| = \dim(V)$, then B is also independent, and hence B is a basis of V.
- If B is independent and $|B| = \dim(V)$, then B also spans V, and hence B is a basis of V.

Therefore if $|B| = \dim(V)$, it is sufficient to prove that B is either independent or spans V in order to prove it is a basis.

Theorem

Let V and W be subspaces of $\mathbb{R}^n$, and suppose that $W \subseteq V$. Then $\dim(W) \leq \dim(V)$ with equality when $W = V$.

Theorem

Let W be any non-zero subspace $\mathbb{R}^n$ and let $W \subseteq V$ where V is also a subspace of $\mathbb{R}^n$. Then every basis of W can be extended to a basis for V.