A First Course in

LINEAR ALGEBRA

Lecture Notes

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$\mathbb{R}^n$: Linear Independence

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A First Course in Linear Algebra

Lecture Notes

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These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Definition

A set of non-zero vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if whenever

$$\sum_{i=1}^{k} a_i \vec{u}_i = \vec{0}$$

it follows that each $a_i = 0$. A set that is **not** linearly independent is called dependent.

Example

Is $S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$ linearly independent?

Suppose that a linear combination of these vectors vanishes, i.e., there exist $a, b, c \in \mathbb{R}$ so that

$$a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example 2 (continued)

Solve the homogeneous system of three equation in three variables:

$$\left[\begin{array}{rrr|r} -1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 5 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system has solutions $a = -2r, b = -3r, c = r$ for $r \in \mathbb{R}$, so it has nontrivial solutions. Therefore S is dependent. In particular, when $r = 1$ we see that

$$-2\left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right] - 3\left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right] + \left[\begin{array}{r} 1 \\ 3 \\ 5 \end{array}\right] = \left[\begin{array}{r} 0 \\ 0 \\ 0 \end{array}\right],$$

i.e., this is a nontrivial linear combination that vanishes.

Example

Consider the set $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} \subseteq \mathbb{R}^n$, and suppose $a_1, a_2, \ldots, a_n \in \mathbb{R}$ are such that

$$a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots a_n\vec{e}_n = \vec{0}_n.$$

Since

$$a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots a_n\vec{e}_n = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

the only linear combination that equals $\vec{0}_n$ is the one with $a_1 = a_2 = \cdots = a_n = 0$. Therefore, $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is linearly independent.

Problem

Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be an independent set of $\mathbb{R}^n$. Is $\{\vec{u}+\vec{v}, 2\vec{u}+\vec{w}, \vec{v}-5\vec{w}\}$ linearly independent?

Solution

Suppose $a(\vec{u}+\vec{v}) + b(2\vec{u}+\vec{w}) + c(\vec{v}-5\vec{w}) = \vec{0}_n$ for some $a, b, c \in \mathbb{R}$. Then

$$(a+2b)\vec{u} + (a+c)\vec{v} + (b-5c)\vec{w} = \vec{0}_n.$$

Since $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent,

$$\begin{aligned} a+2b &= 0 \\ a+c &= 0 \\ b-5c &= 0. \end{aligned}$$

This system of three equations in three variables has the unique solution $a = b = c = 0$. Therefore, $\{\vec{u}+\vec{v}, 2\vec{u}+\vec{w}, \vec{v}-5\vec{w}\}$ is independent.

Problem

Let U be a set of vectors in $\mathbb{R}^n$ suppose that $\vec{0}_n \in U$. Then U is linearly dependent.

Example

Let $\vec{u} \in \mathbb{R}^n$ and let $U = \{\vec{u}\}$.

1. If $\vec{u} = \vec{0}_n$, then S is dependent (see the previous Problem).
2. If $\vec{u} \neq \vec{0}_n$, then S is independent: if $a\vec{u} = \vec{0}_n$ for some $a \in \mathbb{R}$, then $a = 0$.

Example

$A = \begin{bmatrix} 0 & 1 & -1 & 2 & 5 & 1 \\ 0 & 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is an row-echelon form matrix. Treat the nonzero rows of A as transposes of vectors in $\mathbb{R}^6$:

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \\ 5 \\ 1 \end{bmatrix} \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix},$$

and suppose that $a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 = \vec{0}_6$ for some $a, b, c \in \mathbb{R}$.

Example 7 (continued)

This results in a system of six equations in three variables, whose augmented matrix is

$$\left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

The solution to the system is easily determined to be $a = b = c = 0$, so the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is independent. In *a slight abuse of terminology*, we say that the nonzero rows of A are independent.

In general, the nonzero rows of any row-echelon form matrix form an independent set of (row) vectors.

Theorem

Let $U \subseteq \mathbb{R}^n$ be an independent set. Then any vector $\vec{x} \in \mathrm{span}(U)$ can be written uniquely as a linear combination of vectors of U.

Proof.

To prove this theorem, we will show that two linear combinations of vectors in U that equal $\vec{x}$ must be the same. Let $U = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$. Suppose that there is a vector $\vec{x} \in \mathrm{span}(U)$ such that

$$\begin{aligned} \vec{x} &= s_1\vec{u}_1 + s_2\vec{u}_2 + \cdots + s_k\vec{u}_k, \text{ for some } s_1, s_2, \ldots, s_k \in \mathbb{R}, \text{ and} \\ \vec{x} &= t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots + t_k\vec{u}_k, \text{ for some } t_1, t_2, \ldots, t_k \in \mathbb{R}. \end{aligned}$$

Then $\vec{0}_n = \vec{x} - \vec{x} = (s_1 - t_1)\vec{u}_1 + (s_2 - t_2)\vec{u}_2 + \cdots + (s_k - t_k)\vec{u}_k$.
Since U is independent, $s_i - t_i = 0$ for all i, $1 \leq i \leq k$.

Therefore, $s_i = t_i$ for all i, $1 \leq i \leq k$, and the representation is unique. □

Problem

Suppose that $\vec{u}, \vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^3$, and that $\{\vec{v}, \vec{w}\}$ is independent. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent if and only if $\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}$.

Solution

If $\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}$, then there exist $a, b \in \mathbb{R}$ so that $\vec{u} = a\vec{v} + b\vec{w}$. Then $\vec{u} - a\vec{v} - b\vec{w} = \vec{0}_3$, so $\vec{u} - a\vec{v} - b\vec{w}$ is a nontrivial linear combination of $\{\vec{u}, \vec{v}, \vec{w}\}$ that vanishes, and thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is dependent.

Now suppose that $\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}$, and suppose that there exist $a, b, c \in \mathbb{R}$ such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}_3$. If $a \neq 0$, then $\vec{u} = -\frac{b}{a}\vec{v} - \frac{c}{a}\vec{w}$, and $\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}$, a contradiction. Therefore, $a = 0$, and $b\vec{v} + c\vec{w} = \vec{0}_3$. Since $\{\vec{v}, \vec{w}\}$ is independent, $b = c = 0$, and thus $a = b = c = 0$.

Therefore, $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent.

Theorem

Suppose A is an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \in \mathbb{R}^m$. Then

1. The columns of A form a linearly independent set if and only if $A\vec{x} = \vec{0}_m$ implies $\vec{x} = \vec{0}_n$.
2. The columns of A span $\mathbb{R}^m$ if and only if $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^m$.

How is this theorem useful?

Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^n$.

1. Are $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ linearly independent?
2. Do $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ span $\mathbb{R}^n$?

To answer both question, simply let A be a matrix whose columns are the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^n$. Find R, an row-echelon form of A.

- The answer to the first question is "yes" if and only if each column of R has a leading one. Why?
- The answer to the second question is "yes" if and only if each row of R has a leading one. Why?

Problem

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Solution

Let $A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix}$. Apply row operations to get R, an row-echelon form form of A:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last row of R consists only of zeros, $R\vec{x} = \vec{e}_4$ has no solution $\vec{x} \in \mathbb{R}^4$, implying that there is a $\vec{b} \in \mathbb{R}^4$ so that $A\vec{x} = \vec{b}$ has no solution $\vec{x} \in \mathbb{R}^4$. Therefore $\mathbb{R}^4 \neq \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$.

Theorem

Let A be an invertible $n \times n$ matrix. Then the columns of A are independent and span $\mathbb{R}^n$. Similarly, the rows of A are independent and span the set of all $1 \times n$ vectors.

This theorem also allows us to determine if a matrix is invertible. If an $n \times n$ matrix A has columns which are independent, or span $\mathbb{R}^n$, then it follows that A is invertible. If it has rows that are independent, or span the set of all $1 \times n$ vectors, then A is invertible.

Problem (Again!)

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $span\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Solution

Let $A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$.

The columns of A span $\mathbb{R}^4$ if and only if A is invertible. Since $\det A = 0$ (row 2 is (-1) times row 1), A is not invertible, and thus $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ does not span $\mathbb{R}^4$.