A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Determinants: Basic Techniques and Properties

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Determinant of a 2×2 Matrix

Definition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the determinant of A is defined as

$$\det A = ad - bc$$

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Notation. For $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we often write $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, i.e., use vertical bars instead of square brackets.

How do we find the determinant of an $n \times n$ matrix?

The determinant of an $n \times n$ matrix is defined recursively, using determinants of $(n-1) \times (n-1)$ submatrices, and requires some new definitions and notation.

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Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **sign** of the (i, j) position is $(-1)^{i+j}$. Thus the sign is 1 if $(i + j)$ is even, and -1 if $(i + j)$ is odd.

The Minor of a Matrix

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The ij^{th} minor of A, denoted as $minor\,(A)_{ij}$, is the determinant of the $n-1 \times n-1$ matrix which results from deleting the i^{th} row and the j^{th} column of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

For any matrix A, $minor\,(A)_{ij}$ is found by first removing the i^{th} row and j^{th} column, and taking the determinant of the remaining matrix.

The Minor of a Matrix

Example

Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

Find $minor(A)_{12}$.

The Minor of a Matrix

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$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

Find $minor(A)_{12}$.

Solution

First, remove the 1^{st} row and 2^{nd} column from A.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

The Minor of a Matrix

Example

Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

Find $minor(A)_{12}$.

Solution

First, remove the 1^{st} row and 2^{nd} column from A.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

The resulting matrix is $A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$

Solution (continued)

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Using our previous definition, we can calculate the determinant of this matrix to be

$$(2)(6) - (5)(1) = 12 - 5 = 7$$

Solution (continued)

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$$

Using our previous definition, we can calculate the determinant of this matrix to be

$$(2)(6) - (5)(1) = 12 - 5 = 7$$

Therefore, $minor(A)_{12} = 7$.

The Cofactors of a Matrix

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$$\mathrm{cof}(A)_{ij} = (-1)^{i+j} minor\,(A)_{ij}$$

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Example (continued)

Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

Find $\operatorname{cof}(A)_{12}$.

Solution

By the definition, we know that $\mathrm{cof}(A)_{12} = (-1)^{1+2} minor\,(A)_{12}$

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Therefore, $\text{cof}(A)_{12} = (-1)^{1+2} \mathit{minor}\,(A)_{12} = (-1)^3 7 = -7$

Cofactor Expansion

Using these definitions, we can now define the determinant of the $n \times n$ matrix A:

Definition

$\det A = a_{11}\mathrm{cof}(A)_{11} + a_{12}\mathrm{cof}(A)_{12} + a_{13}\mathrm{cof}(A)_{13} + \cdots + a_{1n}\mathrm{cof}(A)_{1n}$

This is called the cofactor expansion of $\det A$ along row 1.

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This is called the cofactor expansion of $\det A$ along row 1.

We may also write:

$$\det(A) = \sum_{j=1}^{n} a_{ij}\mathrm{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij}\mathrm{cof}(A)_{ij}$$

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The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

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The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

Cofactor expansion is also called Laplace Expansion.

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Example

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Solution

Using cofactor expansion along row 1,

$$\det A = 1\mathrm{cof}_{11}(A) + 1\mathrm{cof}_{12}(A) + 3\mathrm{cof}_{13}(A)$$

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Example

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Solution

Using cofactor expansion along row 1,

$$\begin{aligned} \det A &= 1\mathrm{cof}_{11}(A) + 1\mathrm{cof}_{12}(A) + 3\mathrm{cof}_{13}(A) \\ &= 1(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} \end{aligned}$$

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Using cofactor expansion along row 1,

$$\begin{aligned} \det A &= 1\mathrm{cof}_{11}(A) + 1\mathrm{cof}_{12}(A) + 3\mathrm{cof}_{13}(A) \\ &= 1(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} \\ &= 1(24-2) - 1(12-5) + 3(4-20) \end{aligned}$$

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Using cofactor expansion along row 1,

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Using cofactor expansion along row 1,

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Cofactor Expansion

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Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$. Find $\det A$.

Solution

Using cofactor expansion along row 1,

$$\begin{aligned}
\det A &= 1\mathrm{cof}_{11}(A) + 1\mathrm{cof}_{12}(A) + 3\mathrm{cof}_{13}(A) \\
&= 1(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} \\
&= 1(24-2) - 1(12-5) + 3(4-20) \\
&= 22 - 7 + 3(-16) \\
&= 22 - 7 - 48 \\
&= -33
\end{aligned}$$

Solution (continued)

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

Now try cofactor expansion along column 2.

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$$\begin{aligned}
\det A &= 1\text{cof}_{12}(A) + 4\text{cof}_{22}(A) + 2\text{cof}_{32}(A) \\
&= 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 4(-1)^4 \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} + 2(-1)^5 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \\
&= -1(12-5) + 4(6-15) - 2(1-6) \\
&= -(7) + 4(-9) - 2(-5) \\
&= -7 - 36 + 10 \\
&= -33
\end{aligned}$$

Solution (continued)

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

Now try cofactor expansion along column 2.

$$\begin{aligned}
\det A &= 1\mathrm{cof}_{12}(A) + 4\mathrm{cof}_{22}(A) + 2\mathrm{cof}_{32}(A) \\
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&= -1(12-5) + 4(6-15) - 2(1-6) \\
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\end{aligned}$$

We get the same answer!

The Determinant is Well Defined

Theorem

The determinant of an $n \times n$ matrix A can be computed using cofactor expansion along any row or column of A.

The Determinant is Well Defined

Theorem

The determinant of an $n \times n$ matrix A can be computed using cofactor expansion along any row or column of A.

What is the significance of this theorem?

This theorem allows us to choose any row or column for computing cofactor expansion, which gives us the opportunity to save ourselves some work!

Problem

Let $A = \left[\begin{array}{rrrr} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{array}\right]$. Find $\det A$.

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Solution

Cofactor expansion along row 1 yields

$$\det A \quad = \quad 0 \times \mathrm{cof}(A)_{11} + 1 \times \mathrm{cof}(A)_{12} + (-2) \times \mathrm{cof}(A)_{13} + 1 \times \mathrm{cof}(A)_{14}$$

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Let $A = \left[\begin{array}{rrrr} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{array}\right]$. Find $\det A$.

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$$\begin{aligned} \det A &= 0 \times \operatorname{cof}(A)_{11} + 1 \times \operatorname{cof}(A)_{12} + (-2) \times \operatorname{cof}(A)_{13} + 1 \times \operatorname{cof}(A)_{14} \\ &= \operatorname{cof}(A)_{12} - 2 \times \operatorname{cof}(A)_{13} + \operatorname{cof}(A)_{14}, \end{aligned}$$

Problem

Let $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Find $\det A$.

Solution

Cofactor expansion along row 1 yields

$$\begin{aligned} \det A &= 0 \times \mathrm{cof}(A)_{11} + 1 \times \mathrm{cof}(A)_{12} + (-2) \times \mathrm{cof}(A)_{13} + 1 \times \mathrm{cof}(A)_{14} \\ &= \mathrm{cof}(A)_{12} - 2 \times \mathrm{cof}(A)_{13} + \mathrm{cof}(A)_{14}, \end{aligned}$$

whereas cofactor expansion along, row 3 yields

$$\det A = 0 \times \mathrm{cof}(A)_{31} + 1 \times \mathrm{cof}(A)_{32} + (-1) \times \mathrm{cof}(A)_{33} + 0 \times \mathrm{cof}(A)_{34}$$

Problem

Let $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Find $\det A$.

Solution

Cofactor expansion along row 1 yields

$$\begin{aligned} \det A &= 0 \times \mathrm{cof}(A)_{11} + 1 \times \mathrm{cof}(A)_{12} + (-2) \times \mathrm{cof}(A)_{13} + 1 \times \mathrm{cof}(A)_{14} \\ &= \mathrm{cof}(A)_{12} - 2 \times \mathrm{cof}(A)_{13} + \mathrm{cof}(A)_{14}, \end{aligned}$$

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$$\begin{aligned} \det A &= 0 \times \mathrm{cof}(A)_{31} + 1 \times \mathrm{cof}(A)_{32} + (-1) \times \mathrm{cof}(A)_{33} + 0 \times \mathrm{cof}(A)_{34} \\ &= 1\mathrm{cof}(A)_{32} + (-1)\mathrm{cof}(A)_{33}, \end{aligned}$$

Problem

Let $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Find $\det A$.

Solution

Cofactor expansion along row 1 yields

$$\begin{aligned} \det A &= 0 \times \mathrm{cof}(A)_{11} + 1 \times \mathrm{cof}(A)_{12} + (-2) \times \mathrm{cof}(A)_{13} + 1 \times \mathrm{cof}(A)_{14} \\ &= \mathrm{cof}(A)_{12} - 2 \times \mathrm{cof}(A)_{13} + \mathrm{cof}(A)_{14}, \end{aligned}$$

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i.e., in the first case we must compute three cofactors, but in the second case we need only compute two cofactors.

Solution (continued)

Therefore, we can save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\det A = 1 \times \text{cof}(A)_{32} + (-1) \times \text{cof}(A)_{33}$$

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Therefore, we can save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1 \times \mathrm{cof}(A)_{32} + (-1) \times \mathrm{cof}(A)_{33} \\ &= 1 \times (-1)^5 \begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1) \times (-1)^6 \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \end{aligned}$$

Solution (continued)

Therefore, we can save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \left[\begin{array}{rrrr} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{array}\right]$$

$$\begin{aligned}
\det A &= 1 \times \operatorname{cof}(A)_{32} + (-1) \times \operatorname{cof}(A)_{33} \\
&= 1 \times (-1)^5 \left|\begin{array}{rrr} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{array}\right| + (-1) \times (-1)^6 \left|\begin{array}{rrr} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{array}\right| \\
&= -\left|\begin{array}{rrr} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{array}\right| - \left|\begin{array}{rrr} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{array}\right|
\end{aligned}$$

Solution (continued)

Therefore, we can save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1 \times \operatorname{cof}(A)_{32} + (-1) \times \operatorname{cof}(A)_{33} \\ &= 1 \times (-1)^5 \begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1) \times (-1)^6 \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\ &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \end{aligned}$$

Each of the two determinants above can easily be evaluated using cofactor expansion along column 2.

Solution (continued)

$$\det A = -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix}$$

Solution (continued)

$$\begin{aligned}\det A &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\ &= -(-2)(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix}\end{aligned}$$

Solution (continued)

$$\begin{aligned}
\det A &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\
&= -(-2)(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\
&= -2(10-21) + 1(10-21)
\end{aligned}$$

Solution (continued)

$$
\begin{aligned}
\det A &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\
&= -(-2)(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\
&= -2(10-21) + 1(10-21) \\
&= -2(-11) + (-11)
\end{aligned}
$$

Solution (continued)

$$\begin{aligned}
\det A &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\
&= -(-2)(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\
&= -2(10-21) + 1(10-21) \\
&= -2(-11) + (-11) \\
&= 22 - 11
\end{aligned}$$

Solution (continued)

$$\begin{aligned}
\det A &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\
&= -(-2)(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\
&= -2(10-21) + 1(10-21) \\
&= -2(-11) + (-11) \\
&= 22 - 11 \\
&= 11.
\end{aligned}$$

Therefore, $\det A = 11$.

A Row or Column of Zeros

Example

Let

$$A = \begin{bmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{bmatrix}.$$

A Row or Column of Zeros

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By choosing column 3 for cofactor expansion, we get $\det A = 0$,

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By choosing column 3 for cofactor expansion, we get $\det A = 0$, i.e.,

$$\det A = 0 \times \mathrm{cof}(A)_{13} + 0 \times \mathrm{cof}(A)_{23} + 0 \times \mathrm{cof}(A)_{33} + 0 \times \mathrm{cof}(A)_{43}$$

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Example

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By choosing column 3 for cofactor expansion, we get $\det A = 0$, i.e.,

$$\det A = 0 \times \mathrm{cof}(A)_{13} + 0 \times \mathrm{cof}(A)_{23} + 0 \times \mathrm{cof}(A)_{33} + 0 \times \mathrm{cof}(A)_{43} = 0.$$

Important Fact

If A is an $n \times n$ matrix with a row or column of zeros, then $\det A = 0$.

Determinants of a Triangular Matrices

Definitions

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Definitions

1. An $n \times n$ matrix A is called upper triangular if all entries **below** the main diagonal are zero.
2. An $n \times n$ matrix A is called lower triangular if all entries **above** the main diagonal are zero.
3. An $n \times n$ matrix A is called triangular if it is upper triangular or lower triangular.

Determinants of a Triangular Matrices

Definitions

1. An $n \times n$ matrix A is called upper triangular if all entries **below** the main diagonal are zero.
2. An $n \times n$ matrix A is called lower triangular if all entries **above** the main diagonal are zero.
3. An $n \times n$ matrix A is called triangular if it is upper triangular or lower triangular.

Theorem

If $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then

$$\det A = a_{11} \times a_{22} \times a_{33} \times \cdots \times a_{nn},$$

i.e., $\det A$ is the product of the entries of the main diagonal of A.

Example

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

Example

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} = 1 \times \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix}$$

Example

$$
\begin{aligned}
\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} &= 1 \times \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix} \\
&= 1 \times 5 \times \det \begin{bmatrix} 9 \end{bmatrix}
\end{aligned}
$$

Example

$$
\begin{aligned}
\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} &= 1 \times \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix} \\
&= 1 \times 5 \times \det \begin{bmatrix} 9 \end{bmatrix} \\
&= 1 \times 5 \times 9
\end{aligned}
$$

Example

$$\begin{aligned}
\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} &= 1 \times \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix} \\
&= 1 \times 5 \times \det \begin{bmatrix} 9 \end{bmatrix} \\
&= 1 \times 5 \times 9 \\
&= 45.
\end{aligned}$$

Example

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} = 1 \times \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix}$$
$$= 1 \times 5 \times \det \begin{bmatrix} 9 \end{bmatrix}$$
$$= 1 \times 5 \times 9$$
$$= 45.$$

Notice that 45 is the product of the entries on the main diagonal.

$$\begin{bmatrix} \color{blue}{1} & 2 & 3 \\ 0 & \color{blue}{5} & 6 \\ 0 & 0 & \color{blue}{9} \end{bmatrix}$$

Elementary Row Operations and Determinants

Theorem

Let A be an $n \times n$ matrix and B be an $n \times n$ matrix as defined below.

Elementary Row Operations and Determinants

Theorem

Let A be an $n \times n$ matrix and B be an $n \times n$ matrix as defined below.

1. Let B be a matrix which results from switching two rows of A. Then $\det(B) = -\det(A)$.
2. Let B be a matrix which results from multiplying some row of A by a scalar k. Then $\det(B) = k\det(A)$.
3. Let B be a matrix which results from adding a multiple of a row to another row. Then $\det(A) = \det(B)$.
4. If A contains a row which is a multiple of another row of A, then $\det(A) = 0$

Elementary Row Operations and Determinants

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4. If A contains a row which is a multiple of another row of A, then $\det(A) = 0$

An analogous theorem holds for elementary column operation. If A is a matrix, then an elementary column operation on A is simply the corresponding elementary row operation performed on the transpose of A, A^T.

Computing the Determinant

Example

$$\det \begin{bmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix}$$

Computing the Determinant

Example

$$\det \begin{bmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix} =$$

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Example

$$\det \begin{bmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix} =$$

Example

$$\det \begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

Example

$$\det \begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} =$$

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$$\det \begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} \quad =$$

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Example

$$\det \begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} =$$

Problem

If $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 4$, find $\det \begin{bmatrix} -b_1 & -b_2 & -b_3 \\ a_1 + 2b_1 & a_2 + 2b_2 & a_3 + 2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix}$.

Problem

If $\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 4$, find $\det\begin{bmatrix} -b_1 & -b_2 & -b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix}$.

Solution

$$\begin{vmatrix} -b_1 & -b_2 & -b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix}$$

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Solution

$$\begin{aligned}
\begin{vmatrix} -b_1 & -b_2 & -b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix} &= (-1)(3)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\end{aligned}$$

Problem

If $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 4$, find $\det \begin{bmatrix} -b_1 & -b_2 & -b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix}$.

Solution

$$\begin{aligned}
\begin{vmatrix} -b_1 & -b_2 & -b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix} &= (-1)(3)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)(-1)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)(-1) \times 4
\end{aligned}$$

Problem

If $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 4$, find $\det \begin{bmatrix} -b_1 & -b_2 & -b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix}$.

Solution

$$\begin{aligned} \begin{vmatrix} -b_1 & -b_2 & -b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix} &= (-1)(3) \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= (-3) \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= (-3)(-1) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= (-3)(-1) \times 4 \\ &= 12. \end{aligned}$$

Lecture II

Problem

Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{bmatrix}$. Find $\det A$.

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$$\det A = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix}$$

Lecture II

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Let $A = \left[\begin{array}{ccc} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{array} \right]$. Find $\det A$.

Solution

$$\det A = \left| \begin{array}{ccc} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{array} \right| = \left| \begin{array}{ccc} 3 & 5 & 9 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{array} \right|$$

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Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{bmatrix}$. Find $\det A$.

Solution

$$\det A = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 9 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = 0.$$

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Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{bmatrix}$. Find $\det A$.

Solution

$$\det A = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 9 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = 0.$$

Notice:

$$\text{row2} + \text{row3} - 2(\text{row1}) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Hence the determinant equals 0.

Determinants and Scalar Multiplication

Problem

Suppose A is a 3×3 matrix with $\det A = 7$. What is $\det(-2A)$?

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Write $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

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$$\det(-2A) = \left|\begin{array}{ccc} -2a_{11} & -2a_{12} & -2a_{13} \\ -2a_{21} & -2a_{22} & -2a_{23} \\ -2a_{31} & -2a_{32} & -2a_{33} \end{array}\right|$$

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$$= (-2)(-2)\left|\begin{array}{rrr} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -2a_{31} & -2a_{32} & -2a_{33} \end{array}\right|$$

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$$= (-2)^3 \det A$$

Determinants and Scalar Multiplication

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$$\det(-2A) = \begin{vmatrix} -2a_{11} & -2a_{12} & -2a_{13} \\ -2a_{21} & -2a_{22} & -2a_{23} \\ -2a_{31} & -2a_{32} & -2a_{33} \end{vmatrix} = (-2)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ -2a_{21} & -2a_{22} & -2a_{23} \\ -2a_{31} & -2a_{32} & -2a_{33} \end{vmatrix}$$

$$= (-2)(-2)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -2a_{31} & -2a_{32} & -2a_{33} \end{vmatrix} = (-2)(-2)(-2)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (-2)^3 \det A = (-8) \times 7$$

Determinants and Scalar Multiplication

Problem

Suppose A is a 3×3 matrix with $\det A = 7$. What is $\det(-2A)$?

Solution

Write $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then $-2A = \begin{bmatrix} -2a_{11} & -2a_{12} & -2a_{13} \\ -2a_{21} & -2a_{22} & -2a_{23} \\ -2a_{31} & -2a_{32} & -2a_{33} \end{bmatrix}$.

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$$= (-2)^3 \det A = (-8) \times 7 = -56.$$

Solution (continued)

Think about the matrix $-2A$ as the matrix obtained from A be multiplying each of its rows by -2.

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$\det A = (-2) \times (-2) \times (-2) \times 7$

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Think about the matrix $-2A$ as the matrix obtained from A be multiplying each of its rows by -2. This involves three elementary row operations, each of which contributes a factor of -2 to the determinant, and thus
$\det A = (-2) \times (-2) \times (-2) \times 7 = (-2)^3 \times 7.$

Solution (continued)

Think about the matrix $-2A$ as the matrix obtained from A be multiplying each of its rows by -2. This involves three elementary row operations, each of which contributes a factor of -2 to the determinant, and thus
$\det A = (-2) \times (-2) \times (-2) \times 7 = (-2)^3 \times 7$.

Theorem

If A is an $n \times n$ matrix and k is any scalar, then

$$\det(kA) = k^n \det A.$$

Determinants of Inverses, Products, and Transposes

Theorem

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. In this case,

$$\det(A^{-1}) = \frac{1}{\det A}.$$

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Theorem

Let A and B be $n \times n$ matrices. Then

$$\det(AB) = \det A \times \det B.$$

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An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. In this case,

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Theorem

Let A and B be $n \times n$ matrices. Then

$$\det(AB) = \det A \times \det B.$$

Theorem

If A is an $n \times n$ matrix, then the determinant of its transpose is given by

$$\det(A^T) = \det A.$$

Problem

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

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Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

Solution

$$\det A = \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix}$$

Problem

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

Solution

$$\det A = \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix} = c \begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 2 & c \end{vmatrix}$$

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Since A is invertible when $\det(A) \neq 0$, A is invertible for all $c \neq 0, 3, -3$.

Problem

Suppose A, B and C are 4×4 matrices with

$$\det A = -1, \det B = 2, \text{ and } \det C = 1.$$

Find $\det(2A^2(B^{-1})(C^T)^3 B(A^{-1}))$.

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$$\begin{aligned}\det(2A^2(B^{-1})(C^T)^3B(A^{-1})) &= 2^4(\det A)^2\frac{1}{\det B}(\det C)^3(\det B)\frac{1}{\det A} \\ &= 16(\det A)(\det C)^3\end{aligned}$$

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&= 16 \times (-1) \times 1^3 \\
&= -16.
\end{aligned}$$

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Let A be an $n \times n$ matrix. Find all conditions that ensure $\det(-A) = \det A$.

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When is this possible?

1. $(-1)^n \det A = \det A$ whenever $\det A = 0$.
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When is this possible?

1. $(-1)^n \det A = \det A$ whenever $\det A = 0$.
2. If $\det A \neq 0$, then $(-1)^n \det A = \det A$ only if $(-1)^n = 1$, i.e., only if n is even.

Therefore, $\det(-A) = \det A$ only if $\det A = 0$ or n is even.

The Cofactor Matrix

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The cofactor matrix of A, is the matrix

$$[\mathrm{cof}(A)_{ij}]\,,$$

i.e., the matrix whose (i,j)-entry is the (i,j)-cofactor of A.

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Reminder: the (i, j)-cofactor

$$\mathrm{cof}(A)_{ij} = (-1)^{i+j} \mathit{minor}\,(A)_{ij},$$

where $\mathit{minor}\,(A)_{ij}$ is the determinant of the matrix obtained from A by deleting row i and column j.

Problem

Find the cofactor matrix $[\mathrm{cof}(A)_{ij}]$ of the matrix

$$A = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}.$$

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For each i and j, $1 \leq i, j \leq 3$, we need to compute $\mathrm{cof}(A)_{ij}$, so there are 9 cofactors to compute.

$$\mathrm{cof}(A)_{11} \quad = \quad (-1)^{1+1} \det A_{11}$$

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Find the cofactor matrix $[\mathrm{cof}(A)_{ij}]$ of the matrix

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Find the cofactor matrix $[\mathrm{cof}(A)_{ij}]$ of the matrix

$$A = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}.$$

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For each i and j, $1 \leq i, j \leq 3$, we need to compute $\mathrm{cof}(A)_{ij}$, so there are 9 cofactors to compute.

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\end{aligned}$$

Solution (continued)

Computing the six remaining cofactors results in the cofactor matrix

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The Adjugate

Definition

If A is an $n \times n$ matrix, then the adjugate of A is defined by

$$\text{adj } A = \left[\text{ cof}(A)_{ij} \text{ } \right]^T,$$

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Example

$A = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}$, has cofactor matrix $\begin{bmatrix} -6 & -4 & 6 \\ 18 & 16 & -24 \\ -27 & -25 & 36 \end{bmatrix}$.

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Therefore, the adjugate of A is

$$\text{adj } A = \begin{bmatrix} -6 & -4 & 6 \\ 18 & 16 & -24 \\ -27 & -25 & 36 \end{bmatrix}^T = \begin{bmatrix} -6 & 18 & -27 \\ -4 & 16 & -25 \\ 6 & -24 & 36 \end{bmatrix}.$$

Problem

Find adj A when $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

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Solution

$$\text{adj } A = \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix}.$$

The Adjugate of a 2×2 Matrix

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Let A be a 2×2 matrix, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

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We've seen this matrix before: if $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A} \left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right]$$

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Example (continued)

Observe that, regardless of the value of $\det A$,

$$A(\text{adj } A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example (continued)

Observe that, regardless of the value of $\det A$,

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Therefore we have $A(\text{adj } A) = (\det A)I$.

The Adjugate Formula

Theorem

If A is an $n \times n$ matrix, then

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Inverting a matrix using the adjugate

Except in the case of a 2×2 matrix, the adjugate formula is a very inefficient method for computing the inverse of a matrix; the matrix inversion algorithm is much more practical. However, the adjugate formula is of theoretical significance.

Proof of the Adjugate Formula

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Recall that the (i, j)-entry of $\mathrm{adj}(A)$ is equal to $\mathrm{cof}(A)_{ji}$.

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for all i. If $i \neq j$ then this matrix has its ith column equal to its jth column, and therefore

$$a_{ij} = 0 \qquad \text{if } i \neq j.$$

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Even if A is not invertible, $\det(\text{adj } A) = (\det A)^{n-1}$, but the proof is more complicated.

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If A is an $n \times n$ invertible matrix, then the solution to $AX = B$ can be given in terms of determinants of matrices.

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$$x_i = \frac{\det A_i}{\det A}$$

Example (Cramer's Rule)

Solve the following system of linear equations using Cramer's Rule.

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and thus $x_1 = \frac{4}{-4} = -1$.

Example (continued)

Secondly, $x_2 = \frac{\det A_2}{\det A}$ where $\det A = -4$ and

$$\det A_2 = \begin{vmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -14,$$

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You can check this by substituting these values into the original system.

Polynomial Interpolation

Problem

Given data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of y corresponding to $x = \frac{3}{2}$.

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$$\begin{aligned}
p(0) &= r_0 = 1 \\
p(1) &= r_0 + r_1 + r_2 + r_3 = 2 \\
p(2) &= r_0 + 2r_1 + 4r_2 + 8r_3 = 5 \\
p(3) &= r_0 + 3r_1 + 9r_2 + 27r_3 = 10
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$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & 10 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

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\vdots & \vdots & \vdots \\
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The fact that the x_i are distinct guarantees that the coefficient matrix

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

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has determinant not equal to zero, and so the system has a unique solution, i.e., there is a unique interpolating polynomial for the data points.