A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

$\mathbb{R}^n$: The Dot Product

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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The Dot Product

Definition

Let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. The dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \bullet \vec{v} = x_1x_2 + y_1y_2 + z_1z_2,$$

i.e., $\vec{u} \bullet \vec{v}$ is a scalar.

Problem

Find $\vec{u} \bullet \vec{v}$ for $\vec{u} = \begin{bmatrix} 1 & 2 & 0 & -1 \end{bmatrix}^T, \vec{v} = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^T$.

Solution

$$\begin{aligned} \vec{u} \bullet \vec{v} &= (1)(0) + (2)(1) + (0)(2) + (-1)(3) \\ &= 0 + 2 + 0 + -3 = -1 \end{aligned}$$

Note

If

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

are in $\mathbb{R}^n$, then another way to think about the dot product $\vec{u} \bullet \vec{v}$ is as the 1×1 matrix

$$\vec{u}^T\vec{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1v_1 + u_2v_2 + \cdots + u_nv_n \end{bmatrix}$$

which is treated as a scalar given by $u_1v_1 + u_2v_2 + \cdots + u_nv_n$

Properties of the Dot Product

Theorem

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and let $k \in \mathbb{R}$.

1. $\vec{u} \bullet \vec{v}$ is a real number
2. $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$
3. $\vec{u} \bullet \vec{0} = 0$
4. $\vec{u} \bullet \vec{u} = \|\vec{u}\|^2$
5. $(k\vec{u}) \bullet \vec{v} = k(\vec{u} \bullet \vec{v}) = \vec{u} \bullet (k\vec{v})$
6. $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$
   $\vec{u} \bullet (\vec{v} - \vec{w}) = \vec{u} \bullet \vec{v} - \vec{u} \bullet \vec{w}$

Since, for $\vec{u} \in \mathbb{R}^n$, $\vec{u} \bullet \vec{u} = \|\vec{u}\|^2$, we have an alternate (but equivalent) expression for the length of $\vec{u}$:

$$\|\vec{u}\| = \sqrt{\vec{u} \bullet \vec{u}}.$$

Length of a Vector

We can use the properties of the dot product to find the length of a vector.

Problem

Find the length of the vector $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 2 \end{bmatrix}$.

Solution

By the properties of the dot product, $\|\vec{u}\|^2 = \vec{u} \bullet \vec{u}$.

$$\begin{aligned} \vec{u} \bullet \vec{u} &= (1)(1) + (3)(3) + (5)(5) + (2)(2) \\ &= 1 + 9 + 25 + 4 \\ &= 39 \end{aligned}$$

Therefore, $\|\vec{u}\| = \sqrt{\vec{u} \bullet \vec{u}} = \sqrt{39}$

Two Important Inequalities

Theorem

The **Cauchy-Schwarz Inequality** is given as follows. For $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u} \bullet \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

Equality is obtained if one vector is a scalar multiple of the other.

Theorem

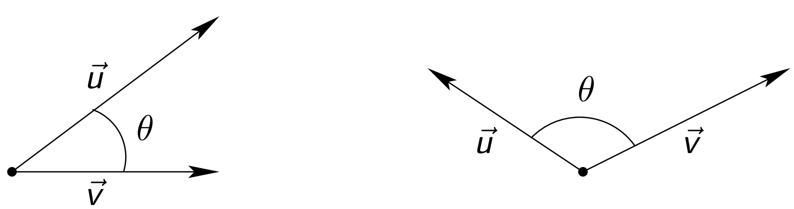
The **Triangle Inequality** is given as follows. For $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Equality is obtained if one vector is a non-negative scalar multiple of the other.

The Included Angle

Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$), positioned so they have the same tail. Then there is a unique angle θ between $\vec{u}$ and $\vec{v}$ with $0 \leq \theta \leq \pi$. This angle θ is called the included angle.



Theorem

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let θ denote the angle between $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \bullet \vec{v} = \|\vec{u}\| \ \|\vec{v}\| \cos\theta.$$

Finding the included angle for nonzero vectors

As a consequence of the Theorem, if $\vec{u}$ and $\vec{v}$ are nonzero vectors with included angle θ, then $||\vec{u}|| \neq 0$ and $||\vec{v}|| \neq 0$, and

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{||\vec{u}||\,||\vec{v}||}.$$

1. If $0 \leq \theta < \frac{\pi}{2}$, then $\cos\theta > 0$, implying that $\vec{u} \bullet \vec{v} > 0$. Conversely, if $\vec{u} \bullet \vec{v} > 0$, then $0 \leq \theta < \frac{\pi}{2}$.
2. If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$, implying that $\vec{u} \bullet \vec{v} = 0$. Conversely, if $\vec{u} \bullet \vec{v} = 0$, then $\theta = \frac{\pi}{2}$.
3. If $\frac{\pi}{2} < \theta \leq \pi$, then $\cos\theta < 0$, implying that $\vec{u} \bullet \vec{v} < 0$. Conversely, if $\vec{u} \bullet \vec{v} < 0$, then $\frac{\pi}{2} < \theta \leq \pi$.

Included Angle

Problem

Find the angle between $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Solution

$\vec{u} \bullet \vec{v} = 1$, $||\vec{u}|| = \sqrt{2}$ and $||\vec{v}|| = \sqrt{2}$.
Therefore,

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{||\vec{u}||\ ||\vec{v}||} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

Since $0 \leq \theta \leq \pi$, $\theta = \frac{\pi}{3}$.

Therefore, the angle between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{3}$.

Problem

Find the included angle for $\vec{u} = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$.

Solution

$$\vec{u} \bullet \vec{v} = -9, \ \ \|\vec{u}\| = \sqrt{54} = 3\sqrt{6}, \ \text{ and } \ \|\vec{v}\| = \sqrt{6}.$$

Let θ denote the included angle for $\vec{u}$ and $\vec{v}$. Then

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-9}{3\sqrt{6} \times \sqrt{6}} = \frac{-9}{18} = -\frac{1}{2}.$$

Since $0 \leq \theta \leq \pi$, the included angle is $\theta = \frac{2\pi}{3}$.

Problem

Find the included angle for $\vec{u} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$.

Solution

Let θ denote included angle.

$$\vec{u} \bullet \vec{v} = 0.$$

Regardless of $\|\vec{u}\|$ and $\|\vec{v}\|$, $\cos\theta = 0$, and therefore the included angle is $\theta = \frac{\pi}{2}$.

Orthogonal Vectors

Definition

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal, also called perpendicular, if and only if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\theta = \frac{\pi}{2}$.

Theorem

Nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \bullet \vec{v} = 0$.

Proof

We have $\vec{u} \perp \vec{v}$ if and only if $\|\vec{u} - \vec{v}\| = \|\vec{u} + \vec{v}\|$ (see the picture).
This is equivalent to

$$(\vec{u} - \vec{v}) \bullet (\vec{u} - \vec{v}) = (\vec{u} + \vec{v}) \bullet (\vec{u} + \vec{v})$$

which gives $-2\vec{u} \bullet \vec{v} = 2\vec{u} \bullet \vec{v}$ and therefore $\vec{u} \bullet \vec{v} = 0$.

Problem

Find all vectors $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both $\vec{u} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Solution

There are infinitely many such vectors.
Since $\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{w}$,

$$\begin{aligned} \vec{v} \bullet \vec{u} &= -x - 3y + 2z = 0 \\ \vec{v} \bullet \vec{w} &= y + z = 0 \end{aligned}$$

Solution (continued)

This is a homogeneous system of two linear equation in three variables.

$$\left[\begin{array}{rrr|r} -1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$\left[\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$ implies that $\vec{v} = \left[\begin{array}{r} 5t \\ -t \\ t \end{array}\right]$ for $t \in \mathbb{R}$.

Therefore, $\vec{v} = t\left[\begin{array}{r} 5 \\ -1 \\ 1 \end{array}\right]$ for all $t \in \mathbb{R}$.

Problem

Are $A(4,-7,9)$, $B(6,4,4)$ and $C(7,10,-6)$ the vertices of a right angle triangle?

Solution

$$\overrightarrow{AB} = \left[\begin{array}{r} 2 \\ 11 \\ -5 \end{array}\right], \overrightarrow{AC} = \left[\begin{array}{r} 3 \\ 17 \\ -15 \end{array}\right], \overrightarrow{BC} = \left[\begin{array}{r} 1 \\ 6 \\ -10 \end{array}\right]$$

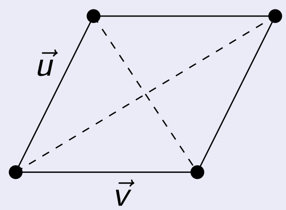
- $\overrightarrow{AB} \bullet \overrightarrow{AC} = 6 + 187 + 75 \neq 0$.
- $\overrightarrow{BA} \bullet \overrightarrow{BC} = (-\overrightarrow{AB}) \bullet \overrightarrow{BC} = -2 - 66 - 50 \neq 0$.
- $\overrightarrow{CA} \bullet \overrightarrow{CB} = (-\overrightarrow{AC}) \bullet (-\overrightarrow{BC}) = \overrightarrow{AC} \bullet \overrightarrow{BC} = 3 + 102 + 150 \neq 0$.

None of the angles is $\frac{\pi}{2}$, and therefore the triangle is not a right angle triangle.

Problem

A rhombus is a parallelogram with sides of equal length. Prove that the diagonals of a rhombus are perpendicular.

Solution



Define the parallelogram (rhombus) by vectors $\vec{u}$ and $\vec{v}$.

Then the diagonals are $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.

Show that $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular.

$$\begin{aligned}
(\vec{u}+\vec{v}) \bullet (\vec{u}-\vec{v}) &= \vec{u}\bullet\vec{u} - \vec{u}\bullet\vec{v} + \vec{v}\bullet\vec{u} - \vec{v}\bullet\vec{v} \\
&= \|\vec{u}\|^2 - \vec{u}\bullet\vec{v} + \vec{u}\bullet\vec{v} - \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 - \|\vec{v}\|^2 \\
&= 0, \text{ since } \|\vec{u}\| = \|\vec{v}\|.
\end{aligned}$$

Therefore, the diagonals are perpendicular.

Projections

Theorem

Given nonzero vectors $\vec{v}$ and $\vec{u}$ in $\mathbb{R}^n$(for $n = 2, 3 \ldots$), there exist unique vectors $\vec{v}_{||}, \vec{v}_{\perp}$ such that $\vec{v}$ can be written as a sum

$$\vec{v} = \vec{v}_{||} + \vec{v}_{\perp}$$

where $\vec{v}_{||}$ is parallel to $\vec{u}$ and $\vec{v}_{\perp}$ is orthogonal to $\vec{u}$.

$\vec{v}_{||}$ is the projection of $\vec{v}$ onto $\vec{u}$, written $\vec{v}_{||} = \text{proj}_{\vec{u}}\vec{v}$ and $\vec{v}_{\perp} = \vec{v} - \vec{v}_{||}$.

Projections

A formula for $\text{proj}_{\vec{u}}\vec{v}$

The defining properties of $\vec{v}_{||}$ and $\vec{v}_{\perp}$ are

1. $\vec{v}_{||}$ is parallel to $\vec{u}$;
2. $\vec{v}_{\perp}$ is orthogonal to $\vec{u}$;
3. $\vec{v}_{||} + \vec{v}_{\perp} = \vec{v}$.

Since $\vec{v}_{||}$ is parallel to $\vec{u}$, $\vec{v}_{||} = t\vec{u}$ for some $t \in \mathbb{R}$. Furthermore, $\vec{v}_{\perp} = \vec{v} - \vec{v}_{||}$ and $\vec{v}_{\perp}$ is orthogonal to $\vec{u}$, so

$$0 = \vec{v}_{\perp} \bullet \vec{u} = (\vec{v} - \vec{v}_{||}) \bullet \vec{u} = (\vec{v} - t\vec{u}) \bullet \vec{u} = \vec{v} \bullet \vec{u} - t(\vec{u} \bullet \vec{u}).$$

Since $\vec{u} \neq \vec{0}$, it follows that $t = \frac{\vec{v} \bullet \vec{u}}{||\vec{u}||^2}$. Therefore

$$\vec{v}_{||} = \left(\frac{\vec{v} \bullet \vec{u}}{||\vec{u}||^2}\right)\vec{u}, \quad \text{and} \quad \vec{v}_{\perp} = \vec{v} - \left(\frac{\vec{v} \bullet \vec{u}}{||\vec{u}||^2}\right)\vec{u}.$$

Projections

Theorem

Let $\vec{v}$ and $\vec{u}$ be vectors with $\vec{u} \neq \vec{0}$.

1. $\text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{v} \bullet \vec{u}}{||\vec{u}||^2}\right)\vec{u}$
2. $\vec{v} - \left(\frac{\vec{v} \bullet \vec{u}}{||\vec{u}||^2}\right)\vec{u}$ is orthogonal to $\vec{u}$.

Problem

Let $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$. Find vectors $\vec{v}_{||}$ and $\vec{v}_{\perp}$ so that $\vec{v} = \vec{v}_{||} + \vec{v}_{\perp}$, with $\vec{v}_{||}$ parallel to $\vec{u}$ and $\vec{v}_{\perp}$ orthogonal to $\vec{u}$.

Solution

$$\vec{v}_{||} = \text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2}\right)\vec{u} = \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 15/11 \\ 5/11 \\ -5/11 \end{bmatrix}.$$

$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{||} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 7 \\ -16 \\ 5 \end{bmatrix} = \begin{bmatrix} 7/11 \\ -16/11 \\ 5/11 \end{bmatrix}.$$

Distance from a Point to a Line

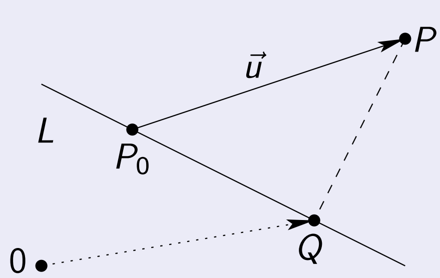
Problem

Let $P = (3, 2, -1)$ be a point in $\mathbb{R}^3$ and L a line with equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}.$$

Find the shortest distance from P to L, and find the point Q on L that is closest to P.

Solution



Let $P_0 = (2, 1, 3)$ be a point on L, and let $\vec{d} = \begin{bmatrix} 3 & -1 & -2 \end{bmatrix}^T$. Then $\overrightarrow{P_0Q} = \text{proj}_{\vec{d}}\overrightarrow{P_0P}$, $\overrightarrow{0Q} = \overrightarrow{0P_0} + \overrightarrow{P_0Q}$, and the shortest distance from P to L is the length of $\overrightarrow{QP}$, where $\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q}$.

Solution (continued)

$\overrightarrow{P_0P} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$, $\vec{d} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$.

$$\overrightarrow{P_0Q} = \text{proj}_{\vec{d}}\overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P} \bullet \vec{d}}{\|\vec{d}\|^2}\right)\vec{d} = \frac{10}{14}\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix}.$$

Therefore,

$$\overrightarrow{0Q} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 29 \\ 2 \\ 11 \end{bmatrix},$$

so $Q = Q\left(\frac{29}{7}, \frac{2}{7}, \frac{11}{7}\right)$.

Solution (continued)

Finally, the shortest distance from $P(3, 2, -1)$ to L is the length of $\overrightarrow{QP}$, where

$$\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} -4 \\ 6 \\ -9 \end{bmatrix}.$$

Therefore the shortest distance from P to L is

$$\|\overrightarrow{QP}\| = \frac{2}{7}\sqrt{(-4)^2 + 6^2 + (-9)^2} = \frac{2}{7}\sqrt{133}.$$