A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

5.1, 5.2, and 5.4 (partial)
Linear Transformations and Matrices

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Transformation by Matrix Multiplication

Example

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. By matrix multiplication, A transforms vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$.

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$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$$

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For example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Transformations

Definition

A transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$, sometimes written

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m,$$

and is called a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

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What do we mean by a function?

Informally, a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a rule that assigns exactly one vector of $\mathbb{R}^m$ to each vector of $\mathbb{R}^n$.
We use the notation $T(\vec{x})$ to mean the transformation T applied to the vector $\vec{x}$.

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Definition

If T acts by matrix multiplication of a matrix A (such as the previous example), we call T a matrix transformation, and write $T_A(\vec{x}) = A\vec{x}$.

Equality of Transformations

Definition

Suppose $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are transformations. Then $S = T$ if and only if $S(\vec{x}) = T(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$.

Specifying the Action of a Transformation

Example

$T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

is a transformation

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is a transformation that transforms the vector $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ in $\mathbb{R}^3$ into the vector

$$T \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 4+7 \\ 1-7 \\ 7-4 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ -6 \\ 3 \end{bmatrix}.$$

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A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

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1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (preservation of addition)
2. $T(a\vec{x}) = aT(\vec{x})$ (preservation of scalar multiplication)

Properties of Linear Transformations

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Suppose $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are vectors in $\mathbb{R}^n$ and

$$\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k$$

for some $a_1, a_2, \ldots, a_k \in \mathbb{R}$.

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3. $$\begin{aligned} T(\vec{y}) &= T(a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k) \\ &= a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + \cdots + a_kT(\vec{x}_k), \end{aligned}$$

   i.e., T preserves linear combinations.

Problem

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} \text{ and } T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}.$$

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The only way it is possible to solve this problem is if

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i.e., if there exist $a, b \in \mathbb{R}$ so that

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}.$$

Solution (continued)

To find a and b, solve the system of three equations in two variables:

$$\left[\begin{array}{cc|c} 1 & 4 & -7 \\ 3 & 0 & 3 \\ 1 & 5 & -9 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $a = 1$, $b = -2$, and

$$\left[\begin{array}{r} -7 \\ 3 \\ -9 \end{array}\right] = \left[\begin{array}{c} 1 \\ 3 \\ 1 \end{array}\right] - 2\left[\begin{array}{c} 4 \\ 0 \\ 5 \end{array}\right].$$

Solution (continued)

We now use that fact that linear transformations preserve linear combinations, implying that

$$T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}\right)$$

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Therefore, $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}$.

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Final Answer

$$T\begin{bmatrix} 2 \\ 5 \\ -3 \\ -7 \end{bmatrix} = \begin{bmatrix} -11 \\ 6 \\ -5 \end{bmatrix}.$$

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Proof.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation induced by the $m \times n$ matrix A, i.e., $T(\vec{x}) = A\vec{x}$ for each $\vec{x} \in \mathbb{R}^n$.

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$$T(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = T(\vec{x}) + T(\vec{y}),$$

proving that T preserves addition.

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proving that T preserves scalar multiplication.

Since T preserves addition and scalar multiplication T is a linear transformation. $\square$

Some Special Matrix Transformations

Example (The Zero Transformation)

If A is the $m \times n$ matrix of zeros, then the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ induced by A is called the zero transformation because for every vector $\vec{x}$ in $\mathbb{R}^n$

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Example (The Identity Transformation)

The transformation of $\mathbb{R}^n$ induced by I_n, the $n \times n$ identity matrix, is called the identity transformation because for every vector $\vec{x}$ in $\mathbb{R}^n$,

$$T(\vec{x}) = I_n\vec{x} = \vec{x}.$$

The identity transformation on $\mathbb{R}^n$ is usually written as $1_{\mathbb{R}^n}$.

Example (Revisited)

Recall $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

Not all transformations are matrix transformations!

Example

Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \left[\begin{array}{r} 1 \\ -1 \end{array} \right] \text{ for all } \vec{x} \in \mathbb{R}^2.$$

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Why is T not a matrix transformation?

Example (continued)

We have $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

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Since every matrix transformation is a linear transformation,

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Since every matrix transformation is a linear transformation, we consider $T(\vec{0}$, where $\vec{0}$ is the zero vector of $\mathbb{R}^2$.

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violating one of the properties of a linear transformation.

Example (continued)

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violating one of the properties of a linear transformation.

Therefore, T is not a linear transformation, and hence is not a matrix transformation.
Can you see any other reasons why T is not a matrix transformation?

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Matrix Transformations

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Corollary

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

Matrix and Linear Transformations

Good News!

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There is an easy way to find the matrix of T!

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Definition

The set of columns $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of I_n is called the standard basis of $\mathbb{R}^n$.

Matrix and Linear Transformations

Good News!

There is an easy way to find the matrix of T!

Definition

The set of columns $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of I_n is called the standard basis of $\mathbb{R}^n$.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
Then T is a matrix transformation.
Furthermore, T is induced by the unique matrix

$$A = \left[\begin{array}{cccc} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{array}\right],$$

where $\vec{e}_j$ is the j^{th} column of I_n, and $T(\vec{e}_j)$ is the j^{th} column of A.

Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ x-y \end{bmatrix}$$

for each $\vec{x} \in \mathbb{R}^2$. Find the matrix, A, of T.

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To find A, we must find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$.

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$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 2(0) \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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The columns $T(\vec{e}_1)$ and $T(\vec{e}_2)$ become the columns of A, so

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix},$$

and $T(\vec{x}) = A\vec{x}$ for every $\vec{x} \in \mathbb{R}^2$, so A is the matrix for T.

Find the Matrix of T

Problem

Sometimes T is not defined so nicely for us. Suppose T is given as

$$T\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; T\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find the matrix A of T.

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Solution

We need to write $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of the vectors provided. So we reduce the augmented matrix having $\vec{e}_1$ and $\vec{e}_2$ as the third and fourth columns:

$$\left[\begin{array}{cc|cc} 1 & 1 & \vec{e}_1 & \vec{e}_2 \\ 5 & 4 & & \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right]$$

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$$\left[\begin{array}{cc|cc}1&1&\vec{e}_1&\vec{e}_2\\5&4&&\end{array}\right]=\left[\begin{array}{cc|cc}1&1&1&0\\5&4&0&1\end{array}\right]\rightarrow\cdots\rightarrow\left[\begin{array}{cc|cc}1&0&-4&1\\0&1&5&-1\end{array}\right]$$

Solution (continued)

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|rr} 1 & 0 & -4 & 1 \\ 0 & 1 & 5 & -1 \end{array}\right]$$

From this we can see that

$$\vec{e_1} = \left[\begin{array}{c} 1 \\ 0 \end{array}\right] = -4\left[\begin{array}{c} 1 \\ 5 \end{array}\right] + 5\left[\begin{array}{c} 1 \\ 4 \end{array}\right]$$

and

$$\vec{e_2} = \left[\begin{array}{c} 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} 1 \\ 5 \end{array}\right] - \left[\begin{array}{c} 1 \\ 4 \end{array}\right]$$

Solution (continued)

So

$$T(\vec{e_1}) = T\left(-4\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$$

Solution (continued)

So

$$
\begin{aligned}
T(\vec{e_1}) &= T\left(-4\left[\begin{array}{c}1\\5\end{array}\right]+5\left[\begin{array}{c}1\\4\end{array}\right]\right)\\
&= -4T\left[\begin{array}{c}1\\5\end{array}\right]+5T\left[\begin{array}{c}1\\4\end{array}\right]
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Solution (continued)

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Solution (continued)

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Solution (continued)

The matrix of T is the matrix whose first column is $T(\vec{e_1})$, and second column is $T(\vec{e_2})$:

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The matrix of T is the matrix whose first column is $T(\vec{e_1})$, and second column is $T(\vec{e_2})$:

$$A = \left[\begin{array}{cc} T(\vec{e_1}) & T(\vec{e_2}) \end{array}\right] = \left[\begin{array}{rr} 11 & -2 \\ 2 & 0 \end{array}\right]$$

Determining if a Transformation is Linear

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a transformation defined by $T \left[\begin{array}{c} x \\ y \end{array} \right] = \left[\begin{array}{c} 2x \\ y \\ -x + 2y \end{array} \right]$.

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Since

$$A\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{rr} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 2x \\ y \\ -x+2y \end{array}\right] = T\left[\begin{array}{c} x \\ y \end{array}\right],$$

T is a matrix transformation, and is therefore a linear transformation.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by $T \left[\begin{array}{c} x \\ y \end{array} \right] = \left[\begin{array}{c} xy \\ x + y \end{array} \right]$.

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However,

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We see from this that if $x = 0$ or $y = 0$, then $xy = 0$, so $A\begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix}$.

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i.e., $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So T is not a linear transformation.

(Any more reasons?)

Rotations in $\mathbb{R}^2$

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The transformation

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denotes counterclockwise rotation about the origin through an angle of θ.

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Rotation through an angle of θ preserves scalar multiplication.

Rotation through an angle of θ preserves vector addition.

R_θ is a linear transformation

Since R_θ preserves addition and scalar multiplication, R_θ is a linear transformation, and hence a matrix transformation.

The matrix that induces R_θ can be found by computing $R_\theta(E_1)$ and $R_\theta(E_2)$, where

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The Matrix for R_θ

The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and is induced by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

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We denote by

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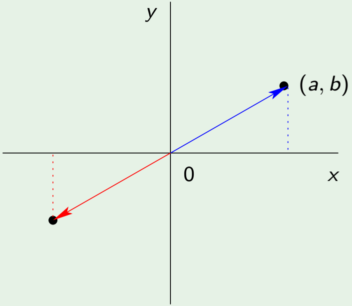

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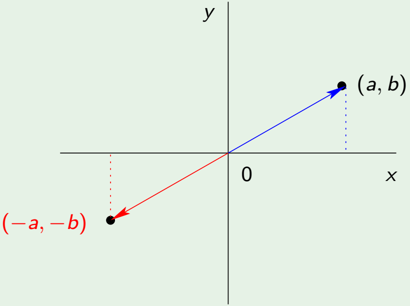

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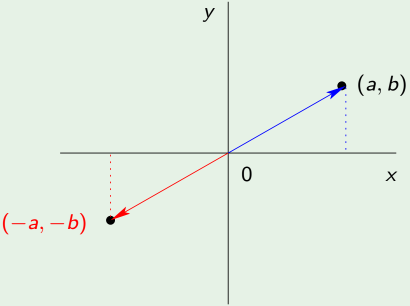

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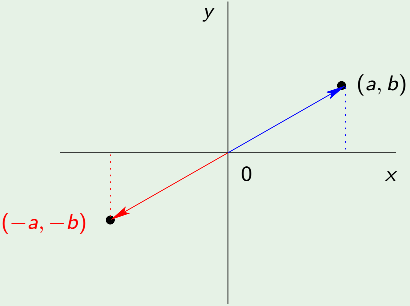

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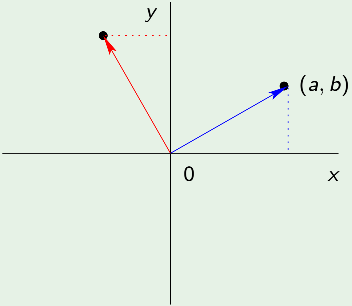

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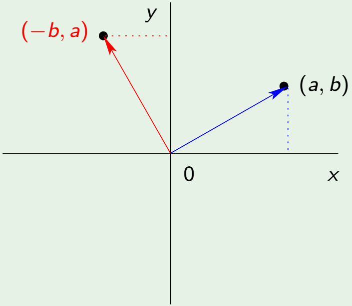

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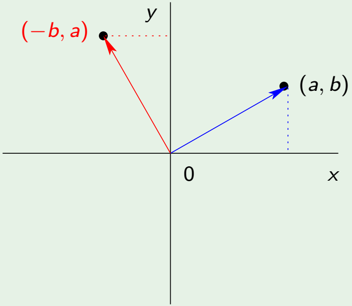

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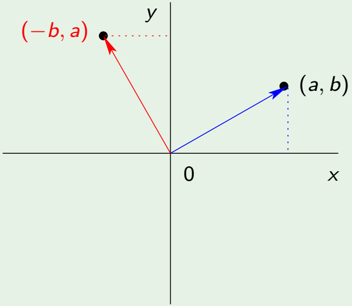

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Reflection in $\mathbb{R}^2$

Example

In $\mathbb{R}^2$, reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

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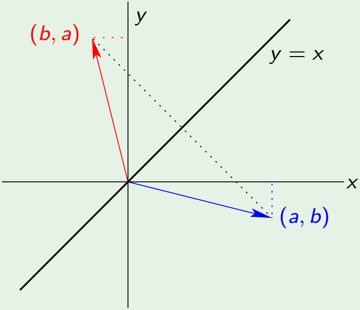
Reflection in the line $y = x$ transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.



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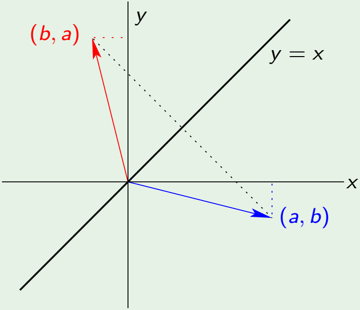

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