A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

$\mathbb{R}^n$: Planes

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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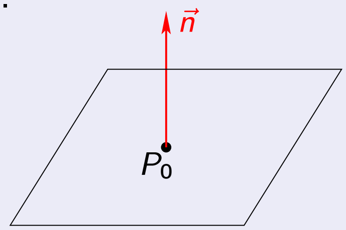
Equations of Planes

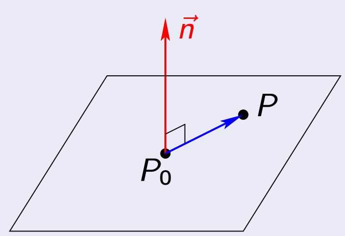
Given a point P_0 and a nonzero vector $\vec{n}$, there is a unique plane containing P_0 and orthogonal to $\vec{n}$.



Definition

A nonzero vector $\vec{n}$ is a normal vector to a plane if and only if $\vec{n} \bullet \vec{v} = 0$ for every vector $\vec{v}$ in the plane, i.e., $\vec{n}$ is orthogonal to every vector in the plane.

Consider a plane containing a point P_0 and orthogonal to vector $\vec{n}$, and let P be an arbitrary point on this plane. Then $\vec{n} \bullet \overrightarrow{P_0P} = 0$,



or, equivalently,

$$\vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0,$$

and is called a vector equation of the plane. The vector equation can also be written as

$$\vec{n} \bullet \overrightarrow{0P} = \vec{n} \bullet \overrightarrow{0P_0}.$$

Suppose a plane contains a fixed point $P_0 = (x_0, y_0, z_0)$ and has normal vector

$$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Let $P = (x, y, z)$ denote an arbitrary point on the plane. Since $\vec{n} \bullet \overrightarrow{0P} = \vec{n} \bullet \overrightarrow{0P_0}$,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}.$$

Thus

$$ax + by + cz = ax_0 + by_0 + cz_0,$$

where $d = ax_0 + by_0 + cz_0$ is simply a scalar.
A scalar equation of the plane has the form

$$ax + by + cz = d, \text{ where } a, b, c, d \in \mathbb{R}.$$

Problem

Find an equation of the plane containing $P_0(1, -1, 0)$ and orthogonal to $\vec{n} = \begin{bmatrix} -3 & 5 & 2 \end{bmatrix}^T$.

Solution

A vector equation of this plane is

$$\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} x-1 \\ y+1 \\ z \end{bmatrix} = 0.$$

Thus, a scalar equation of this plane is

$$-3x + 5y + 2z = -3(1) + 5(-1) + 2(0) = -8,$$

i.e., the plane has scalar equation

$$-3x + 5y + 2z = -8.$$

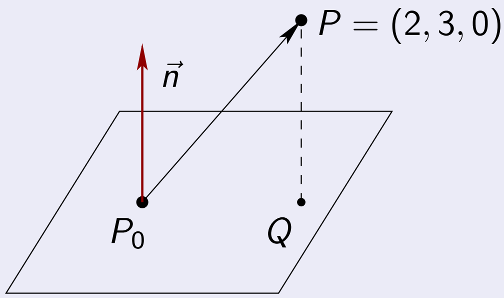
Shortest distance from a point to a plane

Problem

Find the shortest distance from the point $P = (2, 3, 0)$ to the plane with equation $5x + y + z = -1$, and find the point Q on the plane that is closest to P.

(wb example)

Solution

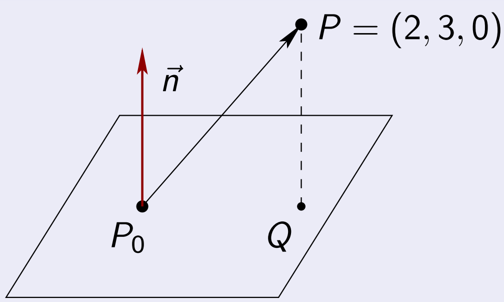


Pick an arbitrary point P_0 on the plane.

Then $\overrightarrow{QP} = \text{proj}_{\vec{n}} \overrightarrow{P_0P}$,
$\|\overrightarrow{QP}\|$ is the shortest distance,
and $\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP}$.

$\vec{n} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Choose $P_0 = (0, 0, -1)$. Then $\overrightarrow{P_0P} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Solution (continued)



$$\overrightarrow{P_0P} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \text{ and } \vec{n} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

$$\overrightarrow{QP} = \text{proj}_{\vec{n}} \overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P} \bullet \vec{n}}{\|\vec{n}\|^2} \right) \vec{n} = \frac{14}{27} \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

Since $\|\overrightarrow{QP}\| = \frac{14}{27}\sqrt{27} = \frac{14\sqrt{3}}{9}$, the shortest distance from P to the plane is $\frac{14\sqrt{3}}{9}$.

Solution (continued)

To find Q, we have

$$\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} - \frac{14}{27}\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} -16 \\ 67 \\ -14 \end{bmatrix}.$$

Therefore $Q = \left(-\frac{16}{27}, \frac{67}{27}, -\frac{14}{27}\right)$.

The Cross Product

Definition

Let $\vec{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$. Then

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}.$$

Note. $\vec{u} \times \vec{v}$ is a vector that is orthogonal to both $\vec{u}$ and $\vec{v}$.

A helpful way to remember (once we cover determinants):

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \text{ where } \vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Computing the Cross Product

Problem

Find $\vec{u} \times \vec{v}$ for $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

Solution

We will use the equation:

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Therefore,

$$\vec{u} \times \vec{v} = \begin{bmatrix} (-1)(1) - (2)(-2) \\ -((1)(1) - (2)(3)) \\ (1)(-2) - (-1)(3) \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

Properties of the Cross Product

Theorem

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be in $\mathbb{R}^3$.

1. $\vec{u} \times \vec{v}$ is a vector.
2. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.
3. $\vec{u} \times \vec{0} = \vec{0}$ and $\vec{0} \times \vec{u} = \vec{0}$.
4. $\vec{u} \times \vec{u} = \vec{0}$.
5. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$.
6. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$ for any scalar k.
7. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
8. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$.

Problem

Find all vectors orthogonal to both $\vec{u} = \begin{bmatrix} -1 & -3 & 2 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$.

Solution

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & -1 & 0 \\ \vec{j} & -3 & 1 \\ \vec{k} & 2 & 1 \end{vmatrix} = -5\vec{i} + \vec{j} - \vec{k} = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}.$$

Any scalar multiple of $\vec{u} \times \vec{v}$ is also orthogonal to both $\vec{u}$ and $\vec{v}$, so

$$t \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}, t \in \mathbb{R},$$

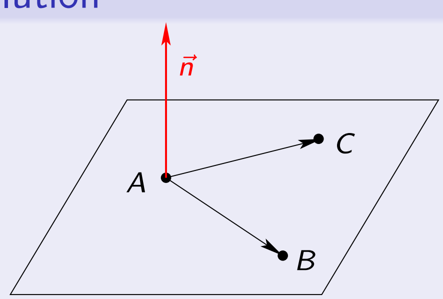
gives all vectors orthogonal to both $\vec{u}$ and $\vec{v}$.

Problem

Let $A = (1, -1, 2)$, $B = (2, 0, -1)$ and $C = (0, -2, 3)$ be points in $\mathbb{R}^3$. These points do no all lie on the same line (how can you tell?). Find an equation for the plane containing A, B, and C.

(wb example)

Solution



$\overrightarrow{AB}$ and $\overrightarrow{AC}$ lie in the plane, so $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ is a normal to the plane.

$$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \overrightarrow{AC} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \text{ and } \vec{n} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}.$$

$$\text{Therefore } -2x + 2y = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} = -4$$

i.e. $-2x + 2y = -4$ is an equation of the plane.

Lecture Part II

Distance between Skew Lines

Problem

Given two lines

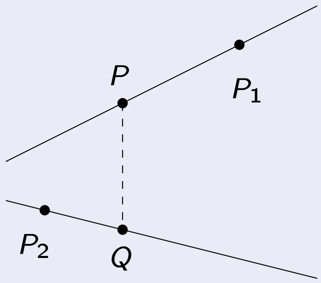
$$L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } L_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

A. Find the shortest distance between L_1 and L_2.

B. Find the shortest distance between L_1 and L_2, and find the points P on L_1 and Q on L_2 that are closest together.

Solution A

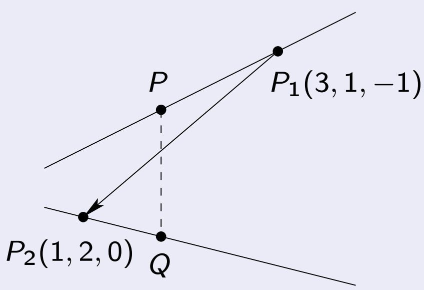
Solution



Choose $P_1(3,1,-1)$ on L_1 and $P_2(1,2,0)$ on L_2.

Let $\vec{d_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{d_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ denote direction vectors for L_1 and L_2, respectively.

Solution (continued)



$$\vec{d_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{d_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The shortest distance between L_1 and L_2 is the length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{n} = \vec{d_1} \times \vec{d_2}$.

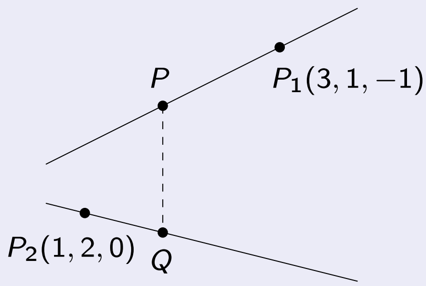
$$\overrightarrow{P_1P_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \text{ and } \vec{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

$$\text{proj}_{\vec{n}}\overrightarrow{P_1P_2} = \left(\frac{\overrightarrow{P_1P_2} \bullet \vec{n}}{\|\vec{n}\|^2} \right) \vec{n}, \text{ and } \|\text{proj}_{\vec{n}}\overrightarrow{P_1P_2}\| = \frac{|\overrightarrow{P_1P_2} \bullet \vec{n}|}{\|\vec{n}\|}.$$

Therefore, the shortest distance between L_1 and L_2 is $\frac{|-8|}{\sqrt{14}} = \frac{4}{7}\sqrt{14}$.

Solution B

Solution



$\vec{d_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{d_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix};$

$\overrightarrow{0P} = \begin{bmatrix} 3+s \\ 1+s \\ -1-s \end{bmatrix}$ for some $s \in \mathbb{R}$;

$\overrightarrow{0Q} = \begin{bmatrix} 1+t \\ 2 \\ 2t \end{bmatrix}$ for some $t \in \mathbb{R}$.

Now $\overrightarrow{PQ} = \begin{bmatrix} -2-s+t & 1-s & 1+s+2t \end{bmatrix}^T$ is orthogonal to both L_1 and L_2, so

$$\overrightarrow{PQ} \bullet \vec{d_1} = 0 \text{ and } \overrightarrow{PQ} \bullet \vec{d_2} = 0,$$

i.e.,

$$\begin{aligned} -2-3s-t &= 0 \\ s+5t &= 0. \end{aligned}$$

Solution B

Solution (continued)

This system has unique solution $s = -\frac{5}{7}$ and $t = \frac{1}{7}$. Therefore,

$$P = \left(\frac{16}{7}, \frac{2}{7}, -\frac{2}{7}\right) \text{ and } Q = \left(\frac{8}{7}, 2, \frac{2}{7}\right).$$

The shortest distance between L_1 and L_2 is $\|\overrightarrow{PQ}\|$. Since

$$P = \left(\frac{16}{7}, \frac{2}{7}, -\frac{2}{7}\right) \text{ and } Q = \left(\frac{8}{7}, 2, \frac{2}{7}\right),$$

$$\overrightarrow{PQ} = \frac{1}{7}\begin{bmatrix} 8 \\ 14 \\ 2 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 16 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -8 \\ 12 \\ 4 \end{bmatrix},$$

Solution B

Solution (continued)

$$\overrightarrow{PQ} = \frac{1}{7}\begin{bmatrix} -8 \\ 12 \\ 4 \end{bmatrix}$$

Therefore

$$\|\overrightarrow{PQ}\| = \frac{1}{7}\sqrt{224} = \frac{4}{7}\sqrt{14}.$$

The shortest distance between L_1 and L_2 is $\frac{4}{7}\sqrt{14}$.

Area and Volume

The Lagrange Identity

If $\vec{u}, \vec{v} \in \mathbb{R}^3$, then

$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \bullet \vec{v})^2.$$

Proof.

Write $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, and work out all the terms. □

The length of the cross product

As a consequence of the Lagrange Identity and the fact that

$$\vec{u} \bullet \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta,$$

we have

$$\begin{aligned} ||\vec{u} \times \vec{v}||^2 &= ||\vec{u}||^2||\vec{v}||^2 - (\vec{u} \bullet \vec{v})^2 \\ &= ||\vec{u}||^2||\vec{v}||^2 - ||\vec{u}||^2||\vec{v}||^2 \cos^2\theta \\ &= ||\vec{u}||^2||\vec{v}||^2(1 - \cos^2\theta) \\ &= ||\vec{u}||^2||\vec{v}||^2 \sin^2\theta. \end{aligned}$$

Taking square roots on both sides yields,

$$||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta.$$

Note that since $0 \leq \theta \leq \pi$, $\sin\theta \geq 0$.

If $\theta = 0$ or $\theta = \pi$, then $\sin\theta = 0$, and $||\vec{u} \times \vec{v}|| = 0$. This is consistent with our earlier observation that if $\vec{u}$ and $\vec{v}$ are parallel, then $\vec{u} \times \vec{v} = \vec{0}$.

Area of a Parallelogram

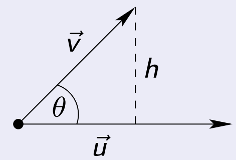
Theorem

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^3$ with included angle θ.

1. $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta$, and is the area of the parallelogram defined by $\vec{u}$ and $\vec{v}$.
2. $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$.

Proof of area of parallelogram.

The area of the parallelogram defined by $\vec{u}$ and $\vec{v}$ is $||\vec{u}||h$, where h is the height of the parallelogram.



$\sin\theta = \frac{h}{||\vec{v}||}$, implying that $h = ||\vec{v}|| \sin\theta$. Therefore, the area is

$$||\vec{u}||||\vec{v}|| \sin\theta.$$

□

Area of a Triangle

Problem

Find the area of the triangle having vertices $A(3,-1,2)$, $B(1,1,0)$ and $C(1,2,-1)$.

Solution

The area of the triangle is half the area of the parallelogram defined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\overrightarrow{AB} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$. Therefore

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix},$$

so the area of the triangle is $\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{2}$.

The Box Product

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. Then

$$\begin{aligned} \vec{u} \bullet (\vec{v} \times \vec{w}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \bullet \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ -(v_1 w_3 - v_3 w_1) \\ v_1 w_2 - v_2 w_1 \end{bmatrix} \\ &= u_1(v_2 w_3 - v_3 w_2) - u_2(v_1 w_3 - v_3 w_1) + u_3(v_1 w_2 - v_2 w_1) \\ &= u_1 \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}. \end{aligned}$$

The Box Product

Theorem

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. Then the box product is

$$\vec{u} \bullet (\vec{v} \times \vec{w}) = \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}.$$

Shorthand: $\vec{u} \bullet (\vec{v} \times \vec{w}) = \det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$.

Theorem

The order of the box product is defined as follows:

$$(\vec{u} \times \vec{v}) \bullet \vec{w} = \vec{u} \bullet (\vec{v} \times \vec{w}).$$

The Volume of a Parallelepiped

Theorem

The volume of the parallelepiped determined by the three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ is

$$|\vec{u} \bullet (\vec{v} \times \vec{w})|.$$

Problem

Find the volume of the parallelepiped determined by the vectors

$\vec{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Solution

The volume of the parallelepiped is $|\vec{u} \bullet (\vec{v} \times \vec{w})|$.
Since $\vec{u} \bullet (\vec{v} \times \vec{w}) = \det \left[\begin{array}{ccc} \vec{u} & \vec{v} & \vec{w} \end{array} \right]$, and

$$\det \left[\begin{array}{rrr} 2 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 2 & 1 \end{array} \right] = -2,$$

the volume of the parallelepiped is $|-2| = 2$ cubic units.