A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

$\mathbb{R}^n$: Parametric Lines

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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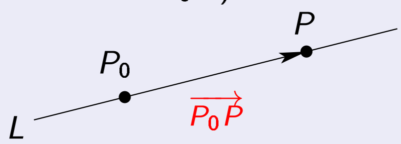
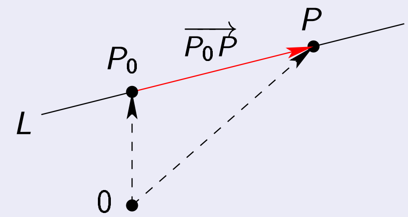
Lines in R^3

Let P_0 and P be two (different) points in $\mathbb{R}^3$. Then there is a unique line L in $\mathbb{R}^3$ containing these two points, and the vector $\overrightarrow{P_0P}$ is a direction vector for L (since L is parallel to $\overrightarrow{P_0P}$).





To describe the line L by an equation, we need to specify every point on L, and we do this by specifying the position vector of every point on L. For instance,

$$\overrightarrow{0P} = \overrightarrow{0P_0} + \overrightarrow{P_0P}.$$

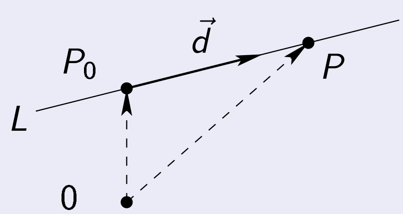
In fact, if Q is any point on L, then the vector $\overrightarrow{P_0Q}$ is parallel to $\overrightarrow{P_0P}$, i.e., $\overrightarrow{P_0Q} = t\overrightarrow{P_0P}$ for some $t \in \mathbb{R}$. Therefore,

$$\overrightarrow{0Q} = \overrightarrow{0P_0} + t\overrightarrow{P_0P}.$$

Notice that the point P is used only to get a vector parallel to the line L, i.e., a direction vector for L.

The Vector Equation of a Line

For a line L in $\mathbb{R}^3$ containing a point $P_0 = (x_0, y_0, z_0)$ and having direction vector $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,



the position vector of an arbitrary point $P = (x, y, z)$ on L is

$$\overrightarrow{0P} = \overrightarrow{0P_0} + t\vec{d}, t \in \mathbb{R}.$$

With the vectors written in component form, the equation is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}, t \in \mathbb{R}.$$

Definition

Let $\vec{p_0}$ be the position vector of a point (x_0, y_0, z_0) in $\mathbb{R}^3$, let $\vec{d}$ be a nonzero vector in $\mathbb{R}^3$, and let L denote the line containing (x_0, y_0, z_0) and having direction vector $\vec{d}$. If we denote by $\vec{p}$ the position vector of an arbitrary point on the line L, then

$$\vec{p} = \vec{p_0} + t\vec{d}, \ \ t \in \mathbb{R},$$

is a vector equation of the line L. Writing $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ t \in \mathbb{R}$$

is a vector equation of L written in component form.

Brief Aside: Lines in $\mathbb{R}^2$

Given a point (x_0, y_0) is $\mathbb{R}^2$ and a nonzero vector $\vec{d}$ in $\mathbb{R}^2$, the line L containing (x_0, y_0) and having direction vector $\vec{d}$ has vector equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \end{bmatrix}, \ t \in \mathbb{R},$$

where $\begin{bmatrix} x \\ y \end{bmatrix}$ is the position vector of an arbitrary point on L, and $\vec{d} = \begin{bmatrix} a \\ b \end{bmatrix}$.
For a fixed (but arbitrary) value of t, we get

$$x = x_0 + ta \ \text{ and } \ y = y_0 + tb.$$

Since $\vec{d}$ is nonzero, not both a and b are zero. Assume $a \neq 0$. Then $t = \frac{x-x_0}{a}$.
Substituting this into $y = y_0 + tb$ yields

$$y = y_0 + \frac{x - x_0}{a} b \qquad \text{or} \qquad y - y_0 = \frac{b}{a}(x - x_0).$$

You (hopefully) recognize this as the equation of a line given a point (x_0, y_0) and a slope $\frac{b}{a}$.

Example

The line in $\mathbb{R}^3$ containing the point $(-5, 1, 3)$ and having direction vector

$$\vec{d} = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} \text{ has equation } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix}, \ t \in \mathbb{R}.$$

An equation of a line is not unique

In the above example, $\vec{d}$ is a direction vector for L. However, any nonzero scalar multiple of $\vec{d}$ is also a direction vector for L. Furthermore, instead of using the point $(-5, 1, 3)$, we could use any other point on L. Since $(-3, 0, 10)$ is a point on L (setting $t = 1$ is the above equation), L is also described by the equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 10 \end{bmatrix} + t \begin{bmatrix} -4 \\ 2 \\ -14 \end{bmatrix}, \ t \in \mathbb{R}.$$

An equation of a line given two points

Example

Let L be a line in $\mathbb{R}^3$ containing the points $P(2, -1, 7)$ and $P_0(-3, 4, 5)$. Then a direction vector for L is

$$\overrightarrow{PP_0} = \begin{bmatrix} -3 - 2 \\ 4 - (-1) \\ 5 - 7 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ -2 \end{bmatrix}.$$

Therefore, a vector equation of L is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} + t \begin{bmatrix} -5 \\ 5 \\ -2 \end{bmatrix}.$$

An equation of a line given a point and a parallel line

Problem

Find an equation for the line L through $P(4,-7,1)$ and parallel to the line M with equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}.$$

Solution

Any direction vector for M is a direction vector for L, so $\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$ is a direction vector for L. Therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

is an equation for L.

Parametric equations of a line

Definition

Let L be a line with direction vector $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and containing the point $P_0 = (x_0, y_0, z_0)$. From a vector equation of L,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

we obtain parametric equations of L given by

$$\begin{array}{rcl} x & = & x_0 + ta \\ y & = & y_0 + tb \\ z & = & z_0 + tc \end{array}, \quad t \in \mathbb{R}.$$

The intersection of two lines

Problem

Given two lines L_1 and L_2, find the point of intersection, if it exists.

$$L_1: \begin{array}{rcl} x & = & 3+t \\ y & = & 1-2t \\ z & = & 3+3t \end{array} \qquad L_2: \begin{array}{rcl} x & = & 4+2s \\ y & = & 6+3s \\ z & = & 1+s \end{array}$$

Solution

This problem is impossible to solve if you use the same parameter name in both equations. Lines L_1 and L_2 intersect if and only if there are values $s, t \in \mathbb{R}$ such that

$$\begin{array}{rcl} 3+t & = & 4+2s \\ 1-2t & = & 6+3s \\ 3+3t & = & 1+s \end{array}$$

i.e., if and only if the system

$$\begin{array}{rcl} 2s-t & = & -1 \\ 3s+2t & = & -5 \\ s-3t & = & 2 \end{array}$$

is consistent.

Solution (continued)

$$\left[\begin{array}{rr|r} 2 & -1 & -1 \\ 3 & 2 & -5 \\ 1 & -3 & 2 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rr|r} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Since the system in consistent, L_1 and L_2 intersect when $s=-1$ and $t=-1$.

Using the equation for L_1

$$\begin{array}{rcl} x & = & 3+t \\ y & = & 1-2t \\ z & = & 3+3t \end{array}$$

and setting $t=-1$, the point of intersection is

$$P(3+(-1), 1-2(-1), 3+3(-1)) = P(2,3,0).$$

Note. You can check your work by setting $s=-1$ in the equation for L_2.

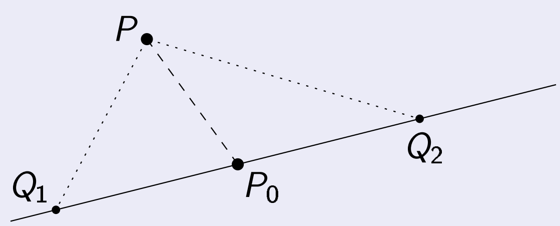
Problem

Find equations for the lines through $P(1,0,1)$ that meet the line

$$L: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

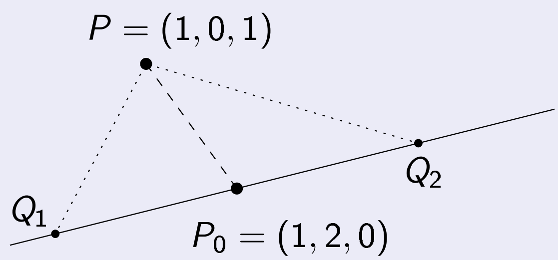
at points distance three from $P_0(1,2,0)$.

Solution



Find points Q_1 and Q_2 on L that are distance three from P_0, and then find equations for the lines through P and Q_1, and through P and Q_2.

Solution (continued)



$\vec{d} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ is a direction for L.

First, $||\vec{d}|| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$, so

$$\overrightarrow{0Q_1} = \overrightarrow{0P_0} + 1\vec{d}, \text{ and } \overrightarrow{0Q_2} = \overrightarrow{0P_0} - 1\vec{d}.$$

Thus

$$\overrightarrow{0Q_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \text{ and } \overrightarrow{0Q_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix},$$

and $Q_1 = (3,1,2)$ and $Q_2 = (-1,3,-2)$.

Solution (continued)

Equations for the lines:

- The line through $P(1,0,1)$ and $Q_1(3,1,2)$ has equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overrightarrow{0P} + t\overrightarrow{PQ_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

- The line through $P(1,0,1)$ and $Q_2(-1,3,-2)$ has equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overrightarrow{0P} + t\overrightarrow{PQ_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}.$$

Changing Between Forms of a Line

Example

Suppose the **symmetric form of a line** is

$$\frac{x-2}{3} = \frac{y-1}{2} = z+3$$

Find the parametric and vector forms of the line.

Solution

Let $t = \frac{x-2}{3}$, $t = \frac{y-1}{2}$, and $t = z+3$. Then, solving for x, y and z, we find the parametric equation of the line.

$$\begin{aligned} x &= 2+3t \\ y &= 1+2t \\ z &= -3+t \end{aligned}$$

Solution (continued)

We can write this in the vector form of the line.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$