A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Linear Transformations

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

In addition we recognize the following contributors. All new content contributed is released under the same license as noted below.

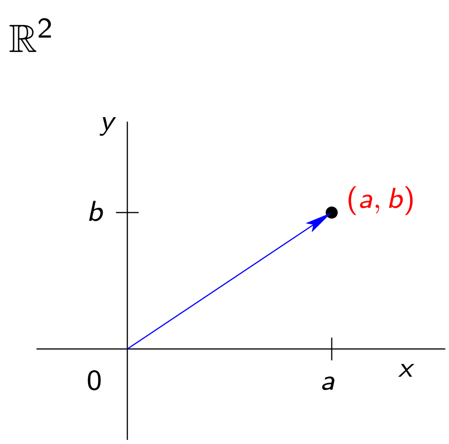
- **Tim Alderson, University of New Brunswick**
- **Ilijas Farah, York University**
- **Ken Kuttler, Brigham Young University**
- **Asia Weiss, York University**

Notation and Terminology

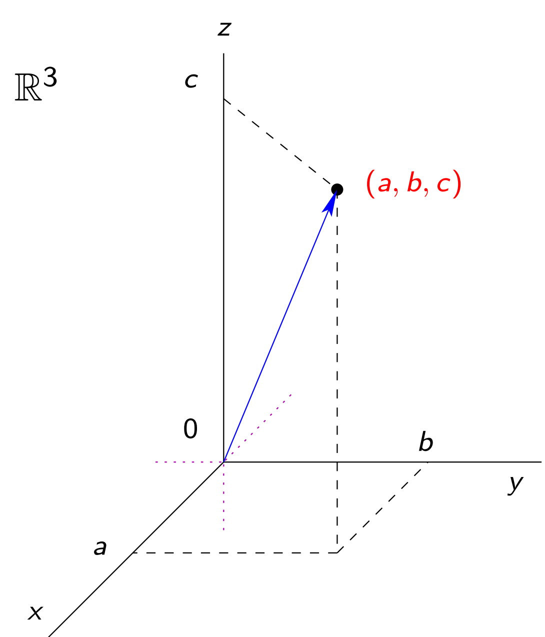
- We have already used $\mathbb{R}$ to denote the set of real numbers.
- We use $\mathbb{R}^2$ to the denote the set of all column vectors of length two, and we use $\mathbb{R}^3$ to the denote the set of all column vectors of length three (the length of a vector is the number of entries it contains).
- In general, we write $\mathbb{R}^n$ for the set of all column vectors of length n.

$\mathbb{R}^2$ and $\mathbb{R}^3$

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have convenient geometric interpretations as **position vectors** of points in the 2-dimensional (Cartesian) plane and in 3-dimensional space, respectively.



The vector $\begin{bmatrix} a \\ b \end{bmatrix}$.



The vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Transformation by Matrix Multiplication

Example

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. By matrix multiplication, A transforms vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$.

Consider the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Transforming this vector by A looks like:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$$

For example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Transformations

Definition

A transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$, sometimes written

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m,$$

and is called a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $m = n$, then we say T is a transformation of $\mathbb{R}^n$.

What do we mean by a function?

Informally, a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a rule that assigns exactly one vector of $\mathbb{R}^m$ to each vector of $\mathbb{R}^n$.

We use the notation $T(\vec{x})$ to mean the transformation T applied to the vector $\vec{x}$.

Definition

If T acts by matrix multiplication of a matrix A (such as the previous example), we call T a matrix transformation, and write $T_A(\vec{x}) = A\vec{x}$.

Equality of Transformations

Definition

Suppose $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are transformations. Then $S = T$ if and only if $S(\vec{x}) = T(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$.

Specifying the Action of a Transformation

Example

$T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

is a transformation that transforms the vector $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ in $\mathbb{R}^3$ into the vector

$$T \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 4+7 \\ 1-7 \\ 7-4 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ -6 \\ 3 \end{bmatrix}.$$

Linear Transformations

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (preservation of addition)
2. $T(a\vec{x}) = aT(\vec{x})$ (preservation of scalar multiplication)

Properties of Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\vec{x} \in \mathbb{R}^n$. Since T preserves scalar multiplication,

1. $T(0\vec{x}) = 0T(\vec{x})$ implying $T(0) = 0$, so T preserves the zero vector.
2. $T((-1)\vec{x}) = (-1)T(\vec{x})$, implying $T(-\vec{x}) = -T(\vec{x})$, so T preserves the negative of a vector.

Suppose $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are vectors in $\mathbb{R}^n$ and

$$\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k$$

for some $a_1, a_2, \ldots, a_k \in \mathbb{R}$. Then

3. $$\begin{aligned} T(\vec{y}) &= T(a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k) \\ &= a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + \cdots + a_kT(\vec{x}_k), \end{aligned}$$

   i.e., T preserves linear combinations.

Problem

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that

$$T\begin{bmatrix}1\\3\\1\end{bmatrix} = \begin{bmatrix}4\\4\\0\\-2\end{bmatrix} \text{ and } T\begin{bmatrix}4\\0\\5\end{bmatrix} = \begin{bmatrix}4\\5\\-1\\5\end{bmatrix}. \text{ Find } T\begin{bmatrix}-7\\3\\-9\end{bmatrix}.$$

Solution

The only way it is possible to solve this problem is if

$$\begin{bmatrix}-7\\3\\-9\end{bmatrix} \text{ is a linear combination of } \begin{bmatrix}1\\3\\1\end{bmatrix} \text{ and } \begin{bmatrix}4\\0\\5\end{bmatrix},$$

i.e., if there exist $a, b \in \mathbb{R}$ so that

$$\begin{bmatrix}-7\\3\\-9\end{bmatrix} = a\begin{bmatrix}1\\3\\1\end{bmatrix} + b\begin{bmatrix}4\\0\\5\end{bmatrix}.$$

Solution (continued)

To find a and b, solve the system of three equations in two variables:

$$\left[\begin{array}{cc|c}1 & 4 & -7\\3 & 0 & 3\\1 & 5 & -9\end{array}\right] \to \cdots \to \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 0\end{array}\right]$$

Thus $a = 1$, $b = -2$, and

$$\begin{bmatrix}-7\\3\\-9\end{bmatrix} = \begin{bmatrix}1\\3\\1\end{bmatrix} - 2\begin{bmatrix}4\\0\\5\end{bmatrix}.$$

Solution (continued)

We now use that fact that linear transformations preserve linear combinations, implying that

$$
\begin{aligned}
T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} &= T\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}\right) \\
&= T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}
\end{aligned}
$$

Therefore, $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}$.

Problem

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation such that

$$
T\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \text{ and } T\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}. \text{ Find } T\begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix}.
$$

Final Answer

$$
T\begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ -3 \end{bmatrix}.
$$

Matrix Transformations

Theorem

Every matrix transformation is a linear transformation.

Proof.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation induced by the $m \times n$ matrix A, i.e., $T(\vec{x}) = A\vec{x}$ for each $\vec{x} \in \mathbb{R}^n$. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and let $a \in \mathbb{R}$. Then

$$T(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = T(\vec{x}) + T(\vec{y}),$$

proving that T preserves addition. Also,

$$T(a\vec{x}) = A(a\vec{x}) = a(A\vec{x}) = aT(\vec{x}),$$

proving that T preserves scalar multiplication.

Since T preserves addition and scalar multiplication T is a linear transformation. □

Some Special Matrix Transformations

Example (The Zero Transformation)

If A is the $m \times n$ matrix of zeros, then the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ induced by A is called the zero transformation because for every vector $\vec{x}$ in $\mathbb{R}^n$

$$T(\vec{x}) = A\vec{x} = 0\vec{x} = \vec{0}.$$

Note that the first zero is the matrix A, while the second zero is the zero vector of $\mathbb{R}^m$. The zero transformation is usually written as $T = 0$.

Example (The Identity Transformation)

The transformation of $\mathbb{R}^n$ induced by I_n, the $n \times n$ identity matrix, is called the identity transformation because for every vector $\vec{x}$ in $\mathbb{R}^n$,

$$T(\vec{x}) = I_n\vec{x} = \vec{x}.$$

The identity transformation on $\mathbb{R}^n$ is usually written as $1_{\mathbb{R}^n}$.

Example (Revisited)

Recall $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

Is T a matrix transformation?

Consider $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, then

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix} = T \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

So in this case T is a matrix transformation!

Not all transformations are matrix transformations!

Example

Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for all } \vec{x} \in \mathbb{R}^2.$$

Why is T not a matrix transformation?

Example (continued)

We have $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \left[\begin{array}{r} 1 \\ -1 \end{array}\right] \text{ for all } \vec{x} \in \mathbb{R}^2.$$

Since every matrix transformation is a linear transformation, we consider $T(0)$, where 0 is the zero vector of $\mathbb{R}^2$.

$$T\left[\begin{array}{r} 0 \\ 0 \end{array}\right] = \left[\begin{array}{r} 0 \\ 0 \end{array}\right] + \left[\begin{array}{r} 1 \\ -1 \end{array}\right] = \left[\begin{array}{r} 1 \\ -1 \end{array}\right] \neq \left[\begin{array}{r} 0 \\ 0 \end{array}\right],$$

violating one of the properties of a linear transformation.

Therefore, T is not a linear transformation, and hence is not a matrix transformation.