A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Linear Transformations: Special Linear Transformations in $\mathbb{R}^2$

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Recall: Matrix Transformation

Definition

Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\vec{x}) = A\vec{x} \text{ for each } \vec{x} \in \mathbb{R}^n$$

is called the matrix transformation induced by A.

Rotations in $\mathbb{R}^2$

Definition

The transformation

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denotes counterclockwise rotation about the origin through an angle of θ.

Rotation through an angle of θ preserves scalar multiplication.

Rotation through an angle of θ preserves vector addition.

R_θ is a linear transformation

Since R_θ preserves addition and scalar multiplication, R_θ is a linear transformation, and hence a matrix transformation.

The matrix that induces R_θ can be found by computing $R_\theta(E_1)$ and $R_\theta(E_2)$, where

$$E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$R_\theta(E_1) = R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

and

$$R_\theta(E_2) = R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

The Matrix for R_θ

The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and is induced by the matrix

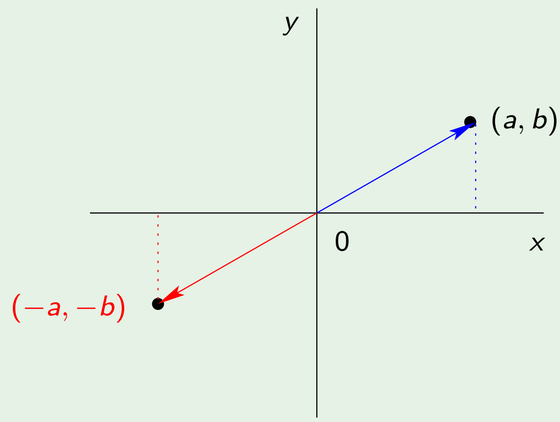
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Example (Rotation through π)

We denote by

$$R_\pi : \mathbb{R}^2 \to \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of π.



We see that $R_\pi \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so R_π is a matrix transformation.

Rotation

Problem

The transformation $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes a counterclockwise rotation about the origin through an angle of $\frac{\pi}{2}$ radians. Find the matrix of $R_{\frac{\pi}{2}}$.

Solution

First,

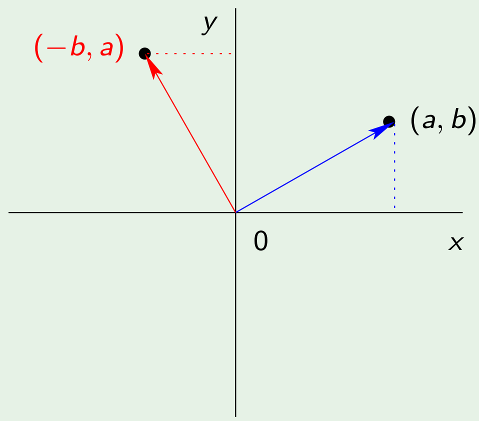
$$R_{\frac{\pi}{2}} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

Furthermore $R_{\frac{\pi}{2}}$ is a matrix transformation, and the matrix it is induced by is

$$\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example (Rotation through $\pi/2$)

We denote by
$$R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$$
counterclockwise rotation about the origin through an angle of $\pi/2$.



We see that $R_{\pi/2}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$, so $R_{\pi/2}$ is a matrix transformation.

Reflection in $\mathbb{R}^2$

Example

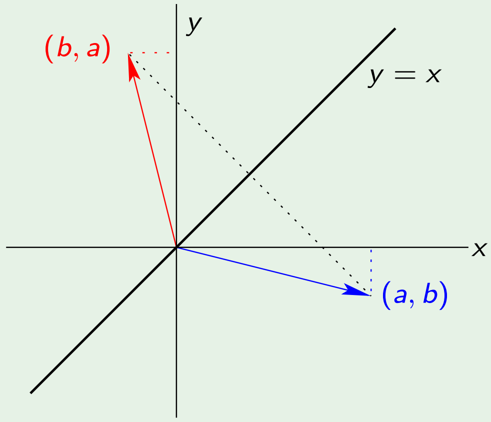
In $\mathbb{R}^2$, reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because
$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}.$$

Example

In $\mathbb{R}^2$, reflection in the y-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} -a \\ b \end{bmatrix}$. This is a matrix transformation because
$$\begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}.$$

Example

Reflection in the line $y = x$ transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.

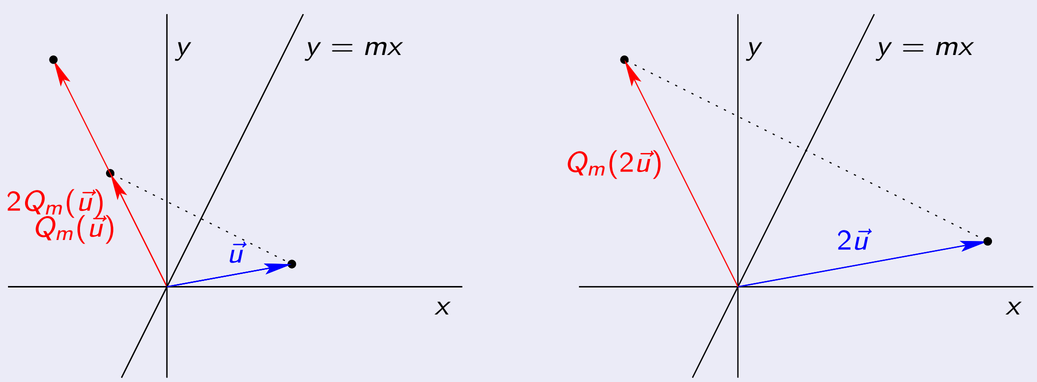


This is a matrix transformation because

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Reflection in $y = mx$ preserves scalar multiplication

Let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = mx$, and let $\vec{u} \in \mathbb{R}^2$.



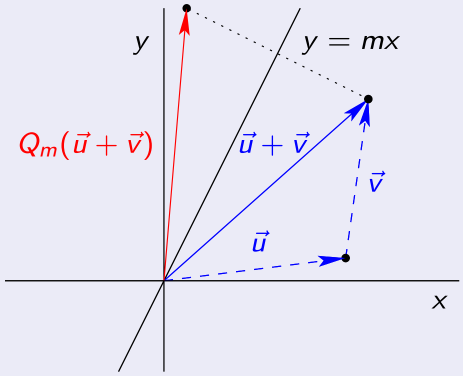
The figure indicates that $Q_m(2\vec{u}) = 2Q_m(\vec{u})$. In general, for any scalar k,

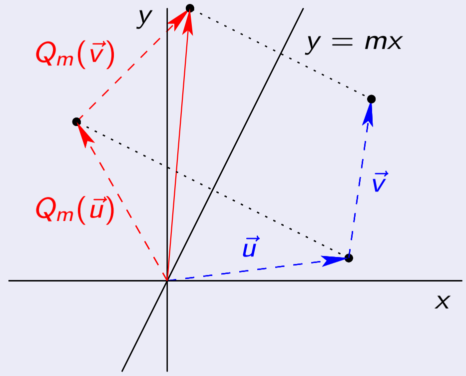
$$Q_m(kX) = kQ_m(X),$$

i.e., Q_m preserves scalar multiplication.

Reflection in $y = mx$ preserves vector addition

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$.





The figure indicates that

$$Q_m(\vec{u}) + Q_m(\vec{v}) = Q_m(\vec{u} + \vec{v}),$$

i.e., Q_m preserves vector addition.

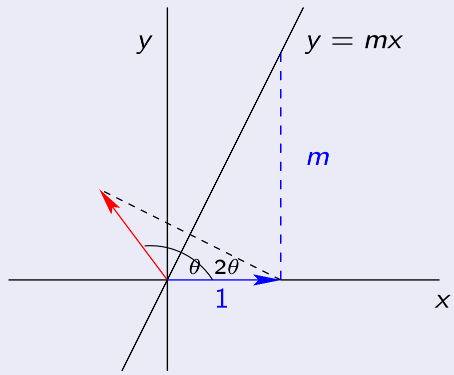
Q_m is a linear transformation

Since Q_m preserves addition and scalar multiplication, Q_m is a linear transformation, and hence a matrix transformation.

The matrix that induces Q_m can be found by computing $Q_m(E_1)$ and $Q_m(E_2)$, where

$$E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

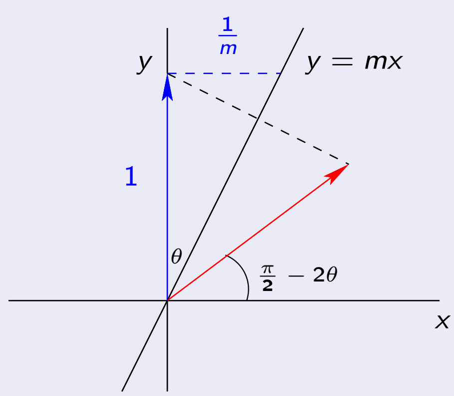
$Q_m(E_1)$



$$\cos\theta = \frac{1}{\sqrt{1+m^2}} \quad \text{and} \quad \sin\theta = \frac{m}{\sqrt{1+m^2}}$$

$$Q_m(E_1) = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta \\ 2\sin\theta\cos\theta \end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 \\ 2m \end{bmatrix}$$

$Q_m(E_2)$



$$\cos\theta = \frac{m}{\sqrt{1+m^2}} \quad \text{and} \quad \sin\theta = \frac{1}{\sqrt{1+m^2}}$$

$$\begin{aligned} Q_m(E_2) &= \begin{bmatrix} \cos(\frac{\pi}{2} - 2\theta) \\ \sin(\frac{\pi}{2} - 2\theta) \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2}\cos(2\theta) + \sin\frac{\pi}{2}\sin(2\theta) \\ \sin\frac{\pi}{2}\cos(2\theta) - \cos\frac{\pi}{2}\sin(2\theta) \end{bmatrix} \\ &= \begin{bmatrix} \sin(2\theta) \\ \cos(2\theta) \end{bmatrix} = \begin{bmatrix} 2\sin\theta\cos\theta \\ \cos^2\theta - \sin^2\theta \end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix} 2m \\ m^2-1 \end{bmatrix} \end{aligned}$$

The Matrix for Reflection in $y = mx$

The transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, reflection in the line $y = mx$, is a linear transformation and is induced by the matrix

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Multiple Actions

Problem

Find the rotation or reflection that equals reflection in the x-axis followed by rotation through an angle of $\frac{\pi}{2}$.

Solution

Let Q_0 denote the reflection in the x-axis, and $R_{\frac{\pi}{2}}$ denote the rotation through an angle of $\frac{\pi}{2}$. We want to find the matrix for the transformation $R_{\frac{\pi}{2}} \circ Q_0$.

Q_0 is induced by $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R_{\frac{\pi}{2}}$ is induced by

$$B = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution

Hence $R_{\frac{\pi}{2}} \circ Q_0$ is induced by

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Notice that $BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a reflection matrix.

How do we know this?

Solution (continued)

Compare BA to

$$Q_m = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Now, since $1 - m^2 = 0$, we know that $m = 1$ or $m = -1$. But $\frac{2m}{1+m^2} = 1 > 0$, so $m > 0$, implying $m = 1$.

Therefore,

$$R_{\frac{\pi}{2}} \circ Q_0 = Q_1,$$

reflection in the line $y = x$.

Reflection followed by Reflection

Problem

Find the rotation or reflection that equals reflection in the line $y = -x$ followed by reflection in the y-axis.

Solution

We must find the matrix for the transformation $Q_Y \circ Q_{-1}$.

Q_{-1} is induced by

$$A = \frac{1}{2}\left[\begin{array}{rr} 0 & -2 \\ -2 & 0 \end{array}\right] = \left[\begin{array}{rr} 0 & -1 \\ -1 & 0 \end{array}\right],$$

and Q_Y is induced by

$$B = \left[\begin{array}{rr} -1 & 0 \\ 0 & 1 \end{array}\right].$$

Therefore, $Q_Y \circ Q_{-1}$ is induced by BA.

Solution (continued)

$$BA = \left[\begin{array}{rr} -1 & 0 \\ 0 & 1 \end{array}\right]\left[\begin{array}{rr} 0 & -1 \\ -1 & 0 \end{array}\right] = \left[\begin{array}{rr} 0 & 1 \\ -1 & 0 \end{array}\right].$$

What transformation does BA induce?

Rotation through an angle θ such that

$$\cos\theta = 0 \text{ and } \sin\theta = -1.$$

Therefore, $Q_Y \circ Q_{-1} = R_{-\frac{\pi}{2}} = R_{\frac{3\pi}{2}}$.

Summary

In general,

- The composite of two rotations is a rotation

$$R_\theta \circ R_\eta = R_{\theta+\eta}$$

- The composite of two reflections is a rotation.

$$Q_m \circ Q_n = R_\theta$$

where θ is $2\times$ the angle between lines $y = mx$ and $y = nx$.

- The composite of a reflection and a rotation is a reflection.

$$R_\theta \circ Q_n = Q_m \circ Q_n \circ Q_n = Q_m$$