A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Matrices: Matrix Arithmetic

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

In addition we recognize the following contributors. All new content contributed is released under the same license as noted below.

- **Tim Alderson, University of New Brunswick**
- **Ilijas Farah, York University**
- **Ken Kuttler, Brigham Young University**
- **Asia Weiss, York University**

Matrices - Basic Definitions and Notation

Definitions

Let m and n be positive integers.

- An $m \times n$ matrix is a rectangular array of numbers having m rows and n columns. Such a matrix is said to have size $m \times n$.
- A row matrix (or row) is a $1 \times n$ matrix, and a column matrix (or column) is an $m \times 1$ matrix.
- A square matrix is an $n \times n$ matrix.
- The (i, j)-entry of a matrix is the entry in row i and column j. For a matrix A, the (i, j)-entry of A is often written as a_{ij}.

General notation for an $m \times n$ matrix, A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

Matrices – Properties and Operations

1. **Equality:** two matrices are equal if and only if they have the same size and the corresponding entries are equal.
2. **Zero Matrix:** an $m \times n$ matrix with all entries equal to zero.
3. **Addition:** matrices must have the same size; add corresponding entries.
4. **Scalar Multiplication:** multiply each entry of the matrix by the scalar.
5. **Negative of a Matrix:** for an $m \times n$ matrix A, its negative is denoted $-A$ and $-A = (-1)A$.
6. **Subtraction:** for $m \times n$ matrices A and B, $A - B = A + (-1)B$.

Matrix Addition

Definition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A + B = C$ where C is the $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$

Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & -2 \\ 6 & 1 \end{bmatrix}$. Then,

$$A + B \quad =$$

Theorem (Properties of Matrix Addition)

Let A, B and C be $m \times n$ matrices. Then the following properties hold.

1. $A + B = B + A$ (matrix addition is commutative).
2. $(A + B) + C = A + (B + C)$ (matrix addition is associative).
3. There exists an $m \times n$ zero matrix, 0, such that $A + 0 = A$. (existence of an additive identity).
4. There exists an $m \times n$ matrix $-A$ such that $A + (-A) = 0$. (existence of an additive inverse).

Scalar Multiplication

Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let k be a scalar. Then $kA = [ka_{ij}]$.

Example

Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & -2 \\ 0 & 4 & 5 \end{bmatrix}$.

Then

$$3A =$$

Theorem (Properties of Scalar Multiplication)

Let A, B be $m \times n$ matrices and let $k, p \in \mathbb{R}$ (scalars). Then the following properties hold.

1. $k(A + B) = kA + kB$.
   (scalar multiplication distributes over matrix addition).
2. $(k + p)A = kA + pA$.
   (addition distributes over scalar multiplication).
3. $k(pA) = (kp)A$. (scalar multiplication is associative).
4. $1A = A$. (existence of a multiplicative identity).

Example

$$2\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} + 4\begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ 1 & -1 \end{bmatrix} =$$

Problem

Let A and B be $m \times n$ matrices. Simplify the expression

$$2[9(A-B)+7(2B-A)]-2[3(2B+A)-2(A+3B)-5(A+B)]$$

Solution

$$\begin{aligned} & 2[9(A-B)+7(2B-A)]-2[3(2B+A)-2(A+3B)-5(A+B)] \\ = \; & 2(9A-9B+14B-7A)-2(6B+3A-2A-6B-5A-5B) \\ = \; & 2(2A+5B)-2(-4A-5B) \\ = \; & 12A+20B \end{aligned}$$

Vectors

Definitions

A row matrix or column matrix is often called a vector, and such matrices are referred to as row vectors and column vectors, respectively. If X is a row vector of size $1 \times n$, and Y is a column vector of size $m \times 1$, then we write

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Vector form of a system of linear equations

Definition

Consider the system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Such a system can be expressed in vector form or as a vector equation by using linear combinations of column vectors:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Vector form of a system of linear equations

Problem

Express the following system of linear equations in vector form.

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & - & 3x_3 & = & -6 \\ & - & x_2 & + & 5x_3 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & = & 1 \end{array}$$

Solution

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

Matrix Vector Multiplication

Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix with columns $A_1, A_2, \ldots, A_n$, written $A = \left[\begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \right]$, and let X be an $n \times 1$ column vector,

$$X = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right]$$

Then the product of matrix A and (column) vector X is the $m \times 1$ column vector given by

$$\left[\begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n = \sum_{j=1}^{n} x_j A_j$$

that is, AX is a linear combination of the columns of A. Notice how this is a generalization of the dot product between vectors.

Matrix Vector Multiplication

Problem

Compute the product AX for

$$A = \left[\begin{array}{cc} 1 & 4 \\ 5 & 0 \end{array} \right] \text{ and } X = \left[\begin{array}{c} 2 \\ 3 \end{array} \right]$$

Solution

$$AX = \left[\begin{array}{cc} 1 & 4 \\ 5 & 0 \end{array} \right] \left[\begin{array}{c} 2 \\ 3 \end{array} \right] = 2 \left[\begin{array}{c} 1 \\ 5 \end{array} \right] + 3 \left[\begin{array}{c} 4 \\ 0 \end{array} \right] = \left[\begin{array}{c} 2 \\ 10 \end{array} \right] + \left[\begin{array}{c} 12 \\ 0 \end{array} \right] = \left[\begin{array}{c} 14 \\ 10 \end{array} \right]$$

Matrix Vector Multiplication

Problem

Compute AY for

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$$

Solution

$AY =$

Matrix form of a system of linear equations

Definition

Consider the system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Such a system can be expressed in matrix form using matrix vector multiplication,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Thus a system of linear equations can be expresses as a matrix equation $AX = B$, where A is the coefficient matrix, B is the constant matrix, and X is the matrix of variables.

Matrix form of a system of linear equations

Problem

Express the following system of linear equations in matrix form.

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 3x_3 & = & -6 \\ & - & x_2 & + & 5x_3 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & = & 1 \end{array}$$

Solution

$$\begin{bmatrix} 2 & 4 & -3 \\ 0 & -1 & 5 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

Matrix and Vector Equations

Theorem

1. Every system of m linear equations in n variables can be written in the form $AX = B$ where A is the coefficient matrix, X is the matrix of variables, and B is the constant matrix.
2. The system $AX = B$ is consistent (i.e., has at least one solution) if and only if B is a linear combination of the columns of A.
3. The vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a solution to the system $AX = B$ if and only if $x_1, x_2, \ldots, x_n$ are a solution to the vector equation

$$x_1A_1 + x_2A_2 + \cdots x_nA_n = B$$

where $A_1, A_2, \ldots, A_n$ are the columns of A.

Proof of the Theorem (a sketch)

Every statement that deserves to be called a theorem deserves a proof, and the theorem from the previous slide is no exception. In this particular case the proof is straightforward (i.e. uneventful).

Proof.

(a) One first checks that $(x_1, \ldots, x_n)$ is a solution to the original system if and only if $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a solution to $AX = B$.

This depends on the way that the matrix arithmetics (addition, multiplication by scalars, multiplication) was defined.

Proof continued

Proof.

(b) Once (a) is taken care of, it gives a one-to-one correspondence between the set of solutions to the original system and the set of solutions to $AX = B$:

$$(x_1, \ldots, x_n) \mapsto \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

This *is* (3), and it implies that the two sets have the same cardinality, and (2) follows. □

Problem

Let

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Express B as a linear combination of the columns A_1, A_2, A_3, A_4 of A, or show that this is impossible.

Solution

Solve the system $AX = B$ where X is a column vector with four entries. Do so by putting the **augmented matrix** $\left[\begin{array}{c|c} A & B \end{array}\right]$ in reduced row-echelon form.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 1 \\ 2 & -1 & 0 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & \frac{1}{7} \\ 0 & 1 & 0 & 1 & -\frac{5}{7} \\ 0 & 0 & 1 & -1 & \frac{3}{7} \end{array}\right]$$

Since there are infinitely many solutions (x_4 is assigned a parameter), choose any value for x_4. Choosing $x_4 = 0$ (which is the simplest thing to do) gives us

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{5}{7}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{7}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{7}A_1 - \frac{5}{7}A_2 + \frac{3}{7}A_3 + 0A_4.$$

Matrix Multiplication

Definition (Product of two matrices)

Let A be an $m \times n$ matrix and let $B = \left[\begin{array}{cccc} B_1 & B_2 & \cdots & B_p \end{array}\right]$ be an $n \times p$ matrix, whose columns are $B_1, B_2, \ldots, B_p$. The product of A and B is the matrix

$$AB = A\left[\begin{array}{cccc} B_1 & B_2 & \cdots & B_p \end{array}\right] = \left[\begin{array}{cccc} AB_1 & AB_2 & \cdots & AB_p \end{array}\right]$$

i.e., the first column of AB is AB_1, the second column of AB is AB_2, etc. Note that AB has size $m \times p$.

Definition (The (i, j)-entry of a product)

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the (i, j)-entry of AB is given by

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

(Note: This can simply be viewed as the dot product of the i'th row of A with the j'th column of B.)

Example

Using the above definition, the $(2, 3)$-entry of the product

$$\left[\begin{array}{rrr} -1 & 0 & 3 \\ 2 & -1 & 1 \end{array}\right] \left[\begin{array}{rrr} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{array}\right]$$

is computed using the second row of the first matrix, and the third column of the second matrix, resulting in

$$2(2) + (-1)(4) + 1(0) = 4 - 4 + 0 = 0.$$

Problem

Find the product AB of matrices

$$A = \left[\begin{array}{rrr} -1 & 0 & 3 \\ 2 & -1 & 1 \end{array}\right] \text{ and } B = \left[\begin{array}{rrr} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{array}\right]$$

Solution

Compatibility for Matrix Multiplication

Definition

Let A and B be matrices, and suppose that A is $m \times n$.

- In order for the product AB to exist, the number of rows in B must be equal to the number of columns in A, implying that B is an $n \times p$ matrix for some p.
- When defined, AB is an $m \times p$ matrix.

If the product is defined, then A and B are said to be compatible for (matrix) multiplication.

Example

As we saw in the previous problem

$$\overset{2\times 3}{\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}} \overset{3\times 3}{\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}} = \overset{2\times 3}{\begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}}$$

Note that the product

$$\overset{3\times 3}{\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}} \overset{2\times 3}{\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}}$$

does not exist.

Multiplication by the Zero Matrix

Example

Compute the product $A0$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the 2×2 zero matrix given by $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Solution

In this product, we compute

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A0 = 0$.

Questions on Matrix Multiplication

Given matrices A and B, is $AB = BA$?

Suppose A is an $m \times n$ matrix and B is an $m' \times n'$ matrix.
The product AB is defined if and only if $n = m'$.
The product BA is defined if and only if $m = n'$.

Therefore the equation $AB = BA$ makes sense if and only if A is an $m \times n$ matrix and B is an $n \times m$ matrix for some—possibly different—m and n.

So the right question is:

Given matrices A and B such that both AB and BA are defined, is $AB = BA$?

Matrix Multiplication is Not Commutative

Problem

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 1 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & -2 & 1 & -3 \end{bmatrix}$$

- Does AB exist? If so, compute it.
- Does BA exist? If so, compute it.

Solution

$$AB = \begin{bmatrix} 7 & -5 & 4 & -6 \\ -3 & 3 & -6 & 0 \\ -11 & 7 & -2 & 12 \end{bmatrix}$$

BA does not exist

Problem

Let

$$G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Does GH exist? If so, compute it.
- Does HG exist? If so, compute it.

Solution

In this example, GH and HG both exist, but they are not equal. They aren't even the same size!

Problem

Let

$$P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$$

- Does PQ exist? If so, compute it.
- Does QP exist? If so, compute it.

Solution

$$PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$QP = \begin{bmatrix} 1 & -1 \\ 6 & -3 \end{bmatrix}$$

In this example, PQ and QP both exist and are the same size, but $PQ \neq QP$.

Fact

The three preceding problems illustrate an important property of matrix multiplication.

> *In general, matrix multiplication is not commutative, i.e., the order of the matrices in the product is important.*

In other words, in general $AB \neq BA$.

Problem

Let

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Does UV exist? If so, compute it.
- Does VU exist? If so, compute it.

Solution

$$UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$VU = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

In this particular example, the matrices commute, i.e., $UV = VU$.

Properties of Matrix Multiplication

Theorem

Let A, B, and C be matrices of the appropriate sizes, and let $r \in \mathbb{R}$ be a scalar. Then the following properties hold.

1. $A(B + C) = AB + AC$.
   (matrix multiplication distributes over matrix addition).
2. $(B + C)A = BA + CA$.
   (matrix multiplication distributes over matrix addition).
3. $A(BC) = (AB)C$. (matrix multiplication is associative).
4. $r(AB) = (rA)B = A(rB)$.

This applies to matrix-vector multiplication as well, since a vector is a row matrix or a column matrix.

Elementary Proofs

Problem

Let A and B be $m \times n$ matrices, and let C be an $n \times p$ matrix. Prove that if A and B commute with C, then $A + B$ commutes with C.

Proof.

$\square$

Problem

Let A, B and C be $n \times n$ matrices, and suppose that both A and B commute with C, i.e., $AC = CA$ and $BC = CB$. Show that AB commutes with C.

Proof.

We must show that $(AB)C = C(AB)$ given that $AC = CA$ and $BC = CB$.

$$\begin{aligned}
(AB)C &= A(BC) && \text{(matrix multiplication is associative)} \\
&= A(CB) && \text{(B commutes with C)} \\
&= (AC)B && \text{(matrix multiplication is associative)} \\
&= (CA)B && \text{(A commutes with C)} \\
&= C(AB) && \text{(matrix multiplication is associative)}
\end{aligned}$$

Therefore, AB commutes with C. $\square$

Definition (Matrix Transpose)

If A is an $m \times n$ matrix, then its transpose, denoted A^T, is the $n \times m$ whose i^{th} row is the i^{th} column of A, $1 \leq i \leq n$; i.e., if $A = [a_{ij}]$, then

$$A^T = [a_{ij}]^T = [a_{ji}]$$

i.e., the (i,j)-entry of A^T is the (j,i)-entry of A.

Theorem (Properties of the Transpose of a Matrix)

Let A and B be $m \times n$ matrices, C be a $n \times p$ matrix, and $r \in \mathbb{R}$ a scalar. Then

1. $(A^T)^T = A$
2. $(rA)^T = rA^T$
3. $(A+B)^T = A^T + B^T$
4. $(AC)^T = C^T A^T$

To prove each these properties, you only need to compute the (i,j)-entries of the matrices on the left-hand side and the right-hand side. And you can do it!

Problem

Find the matrix A if $\left(A + 3\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 8 \end{bmatrix}$.

Solution

Symmetric Matrices

Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The entries $a_{11}, a_{22}, a_{33}, \ldots$ are called the main diagonal of A.

Definition

The matrix A is called symmetric if and only if $A^T = A$. Note that this immediately implies that A is a **square** matrix.

Examples

$$\left[\begin{array}{rr} 2 & -3 \\ -3 & 17 \end{array}\right], \left[\begin{array}{rrr} -1 & 0 & 5 \\ 0 & 2 & 11 \\ 5 & 11 & -3 \end{array}\right], \left[\begin{array}{rrrr} 0 & 2 & 5 & -1 \\ 2 & 1 & -3 & 0 \\ 5 & -3 & 2 & -7 \\ -1 & 0 & -7 & 4 \end{array}\right]$$

are symmetric matrices, and each is symmetric about its main diagonal.

Problem

Show that if A and B are symmetric matrices, then $2A - B$ is symmetric.

Proof.

$\square$

Skew Symmetric Matrices

Definition

An $n \times n$ matrix A is said to be skew symmetric if $A^T = -A$.

Example (Skew Symmetric Matrices)

$$\left[\begin{array}{rr} 0 & 2 \\ -2 & 0 \end{array}\right], \left[\begin{array}{rrr} 0 & 9 & 4 \\ -9 & 0 & -3 \\ -4 & 3 & 0 \end{array}\right]$$

Problem

Show that if A is a square matrix, then $A - A^T$ is skew-symmetric.

Solution

We must show that $(A - A^T)^T = -(A - A^T)$. Using the properties of matrix addition, scalar multiplication, and transposition

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T).$$

The $n \times n$ Identity Matrix

Definition

For each $n \geq 2$, the $n \times n$ identity matrix, denoted I_n, is the matrix having ones on its main diagonal and zeros elsewhere, and is defined for all $n \geq 2$.

Example

$$I_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], I_3 = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Definition

Let $n \geq 2$. For each j, $1 \leq j \leq n$, we denote by E_j the j^{th} column of I_n.

Example

When $n = 3$, $E_1 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right], E_2 = \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right], E_3 = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$.

Theorem

Let A be an $m \times n$ matrix Then $AI_n = A$ and $I_m A = A$.

Proof

The (i,j)-entry of AI_n is the product of the i^{th} row of $A = [a_{ij}]$, namely $\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \end{bmatrix}$ with the j^{th} column of I_n, namely E_j. Since E_j has a one in row j and zeros elsewhere,

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \end{bmatrix} E_j = a_{ij}$$

Since this is true for all $i \leq m$ and all $j \leq n$, $AI_n = A$.
The proof of $I_m A = A$ is analogous—work it out!

Instead of AI_n and $I_m A$ we often write AI and IA, respectively, since the size of the identity matrix is clear from the context: the sizes of A and I must be compatible for matrix multiplication.
Thus

$$AI = A \text{ and } IA = A$$

which is why I is called an identity matrix – it is an identity for matrix multiplication.

Matrix Inverses

Definition

Let A be an $n \times n$ matrix. Then B is an inverse of A if and only if $AB = I_n$ and $BA = I_n$. Note that since A and I_n are both $n \times n$, B **must also be** an $n \times n$ matrix.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so B is an inverse of A.

Does every square matrix have an inverse?

No! Take e.g. the zero matrix $\mathbf{0_n}$ (all entries of $\mathbf{O_n}$ are equal to 0)

$$A\mathbf{0_n} = \mathbf{0_n}A = \mathbf{O_n}$$

for all $n \times n$ matrices A: The (i,j)-entry of $\mathbf{O_n}A$ is equal to $\sum_{k=1}^{n} 0a_{kj} = 0$.

Does every **nonzero** square matrix have an inverse?

Example

Does the matrix
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$
have an inverse?

No! To see this, suppose
$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
is an inverse of A. Then
$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix}$$
which is never equal to I_2. (Why?)

Uniqueness of an Inverse

Theorem

If A is a square matrix and B and C are inverses of A, then $B = C$.

Proof.

Since B and C are inverses of A, $AB = I = BA$ and $AC = I = CA$. Then
$$B = BI = B(AC) = (BA)C = IC = C$$
so $B = C$. □

Example (revisited)

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$, we saw that

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The preceding theorem tells us that B is the inverse of A, rather than just an inverse of A.

Definitions

Let A be a square matrix, i.e., an $n \times n$ matrix.

- The inverse of A, if it exists, is denoted A^{-1}, and

$$AA^{-1} = I = A^{-1}A$$

- If A has an inverse, then we say that A is invertible (or nonsingular).

Finding the inverse of a 2×2 matrix

Example

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then there is a formula for A^{-1}:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This can easily be verified by computing the products AA^{-1} and $A^{-1}A$.

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad - bc} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \\ &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & -bc + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Showing that $A^{-1}A = I_2$ is left as an exercise.

Finding the inverse of an $n \times n$ matrix

Problem

Suppose that A is any $n \times n$ matrix.

- How do we know whether or not A^{-1} exists?
- If A^{-1} exists, how do we find it?

Solution

The matrix inversion algorithm.

Although the formula for the inverse of a 2×2 matrix is quicker and easier to use than the matrix inversion algorithm, the general formula for the inverse an $n \times n$ matrix, $n \geq 3$ (which we will see later), is more complicated and difficult to use than the matrix inversion algorithm. To find inverses of square matrices that are not 2×2, the matrix inversion algorithm is the most efficient method to use.

The Matrix Inversion Algorithm

Let A be an $n \times n$ matrix. To find A^{-1}, if it exists,

- take the $n \times 2n$ matrix
$$\left[\begin{array}{c|c} A & I_n \end{array}\right]$$
obtained by augmenting A with the $n \times n$ identity matrix, I_n.
- Perform elementary row operations to transform $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ into a reduced row-echelon matrix.

Theorem (Matrix Inverses)

Let A be an $n \times n$ matrix. Then the following conditions are equivalent.

1. *A is invertible.*
2. *the reduced row-echelon form on A is I.*
3. *$\left[\begin{array}{c|c} A & I_n \end{array}\right]$ can be transformed into $\left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$ using the Matrix Inversion Algorithm.*

Problem

Find, if possible, the inverse of $\left[\begin{array}{rrr} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{array}\right]$.

Solution

Using the matrix inversion algorithm (fill in the operations)

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

From this, we see that A has no inverse.

Problem

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find the inverse of A, if it exists.

Solution (continued)

Using the matrix inversion algorithm (fill in the operations)

$$\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{rrr|rrr} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 4 & -7 & 1 & -3 & 0 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 25 & -3 & 5 & 4 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right] \rightarrow$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{10}{25} & 0 & -\frac{5}{25} \\ 0 & 1 & 0 & \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right] = \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

Solution (continued)

Therefore, A^{-1} exists, and

$$A^{-1} = \begin{bmatrix} \frac{10}{25} & 0 & -\frac{5}{25} \\ \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 10 & 0 & -5 \\ 1 & -10 & 7 \\ -3 & 5 & 4 \end{bmatrix}$$

You can check your work by computing AA^{-1} and $A^{-1}A$.

Systems of Linear Equations and Inverses

Suppose that a system of n linear equations in n variables is written in matrix form as $AX = B$, and suppose that A is invertible.

Example

The system of linear equations

$$\begin{aligned} 2x - 7y &= 3 \\ 5x - 18y &= 8 \end{aligned}$$

can be written in matrix form as $AX = B$:

$$\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

You can check that $A^{-1} = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}$.

Example (continued)

Since A^{-1} exists and has the property that $A^{-1}A = I$, we obtain the following.

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B \\ (A^{-1}A)X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

i.e., $AX = B$ has the unique solution given by $X = A^{-1}B$. Therefore,

$$X = A^{-1}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

You should verify that $x = -2$, $y = -1$ is a solution to the system.

The last example illustrates another method for solving a system of linear equations when **the coefficient matrix is square and invertible**. Unless that coefficient matrix is 2×2, this is generally **NOT** an efficient method for solving a system of linear equations.

Example

Let A, B and C be matrices, and suppose that A is invertible.

1. If $AB = AC$, then
$$\begin{aligned} A^{-1}(AB) &= A^{-1}(AC) \\ (A^{-1}A)B &= (A^{-1}A)C \\ IB &= IC \\ B &= C \end{aligned}$$

2. If $BA = CA$, then
$$\begin{aligned} (BA)A^{-1} &= (CA)A^{-1} \\ B(AA^{-1}) &= C(AA^{-1}) \\ BI &= CI \\ B &= C \end{aligned}$$

Problem

Find square matrices A, B and C for which $AB = AC$ but $B \neq C$.

Inverses of Transposes and Products

Example

Suppose A is an invertible matrix. Then
$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
and
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$
This means that $(A^T)^{-1} = (A^{-1})^T$.

Example

Suppose A and B are invertible $n \times n$ matrices. Then
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$
and
$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$
This means that $(AB)^{-1} = B^{-1}A^{-1}$.

Inverses of Transposes and Products

The previous two examples prove the first two parts of the following theorem.

Theorem

1. If A is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$.
2. If A and B are invertible matrices, then AB is invertible and
$$(AB)^{-1} = B^{-1}A^{-1}$$
3. If $A_1, A_2, \ldots, A_k$ are invertible, then $A_1A_2\cdots A_k$ is invertible and
$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$$
(the third part is proved by iterating the above, or, more formally, by using mathematical induction)

Properties of Inverses

Theorem

1. I is invertible, and $I^{-1} = I$.
2. If A is invertible, so is A^{-1}, and $(A^{-1})^{-1} = A$.
3. If A is invertible, so is A^k, and $(A^k)^{-1} = (A^{-1})^k$.
(A^k means A multiplied by itself k times)
4. If A is invertible and $p \in \mathbb{R}$ is nonzero, then pA is invertible, and $(pA)^{-1} = \frac{1}{p}A^{-1}$.

Example

Given $(3I - A^T)^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, we wish to find the matrix A. Taking inverses of both sides of the equation:

Example (continued)

Problem

True or false? Justify your answer.

$$\text{If } A^3 = 4I, \text{ then } A \text{ is invertible.}$$

Solution

A Fundamental Result

Theorem

Let A be an $n \times n$ matrix, and let X, B be $n \times 1$ vectors. The following conditions are equivalent.

1. *The rank of A is n.*
2. *A can be transformed to I_n by elementary row operations.*
3. *A is invertible.*
4. *There exists an $n \times n$ matrix C with the property that $CA = I_n$.*
5. *The system $AX = B$ has a unique solution X for any choice of B.*
6. *$AX = 0$ has only the trivial solution, $X = 0$.*
7. *There exists an $n \times n$ matrix C with the property that $AC = I_n$.*

Proof of Theorem:

(1) $\Rightarrow$ (2) The rank of A is the number of leading 1s in the RREF of A. Since the size of A is $n \times n$, rank $(A) = n$ is equivalent to A being row-equivalent to I_n.
(2) $\Rightarrow$ (3): Matrix inversion algorithm.
(3) $\Rightarrow$ (4): $C = A^{-1}$.
(4) $\Rightarrow$ (5): $X = CB$.
(5) $\Rightarrow$ (6): Take $B = 0$.
(6) $\Rightarrow$ (1): If rank of A is $< n$, then there are non-leading variables in the RREF of $[A|0]$. Hence $AX = 0$ has infinitely many solutions.
(4) $\Leftrightarrow$ (7): $CA = I$ if and only if $A^T C^T = I$; hence (4) for A is equivalent to (7) for A^T.
We already know that A^{-1} exists if and only if $(A^T)^{-1}$ exists.

The following is an important and useful consequence of the theorem.

Theorem

If A and B are $n \times n$ matrices such that $AB = I$, then $BA = I$. Furthermore, A and B are invertible, with $B = A^{-1}$ and $A = B^{-1}$.

Important Fact

In the second Theorem, it is essential that the matrices be square.

Theorem

If A and B are matrices such that $AB = I$ and $BA = I$, then A and B are square matrices (of the same size).

Example

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \neq I_3$$

This example illustrates why "an inverse" of a non-square matrix doesn't make sense. If A is $m \times n$ and B is $n \times m$, where $m \neq n$, then even if $AB = I$, it will never be the case that $BA = I$.