A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

5.5 (partial) and 5.7: Linear Transformations: One to One and Onto, Kernel, and Image

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Injections

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Equivalently, if $T\left(\vec{x}_1\right) = T\left(\vec{x}_2\right)$, then $\vec{x}_1 = \vec{x}_2$. Thus, T is one-to-one if two distinct vectors are never transformed into the same vector.

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Theorem

Let A be an $m \times n$ matrix and let $\vec{x}$ be a vector of length n. Then the transformation induced by A, T_A, is one-to-one if and only if $A\vec{x} = 0$ implies $\vec{x} = 0$ (equivalently, iff A has full column rank).

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Since every linear transformation is induced by a matrix A, in order to show that T is one to one, it suffices to show that $A\vec{x} = 0$ has a unique solution.

Problem

Show that the transformation defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is one-to-one.

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Solution

Since T is a matrix transformation induced by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and A has full column rank (=2), it follows that $A\vec{x} = 0$ has the unique solution $\vec{x} = 0$.

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Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

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Then T is not onto. To see why, choose $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. Then there is no vector $\vec{x} \in \mathbb{R}^2$ so that $T(\vec{x}) = \vec{b}$; applying T to any vector results in a vector whose second entry is 0, and the second entry of $\vec{b}$ is 1.

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$$A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right].$$

Then the augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

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which is already in reduced row-echelon form.

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Let A be an $m \times n$ matrix.

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Theorem

Let A be an $m \times n$ matrix. Then the transformation T_A, induced by A, is onto if and only if $A\vec{x} = \vec{b}$ is consistent for every vector $\vec{b}$ in $\mathbb{R}^m$.

Problem

Show that the transformation defined by

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T is a matrix transformation, $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

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Since $\det(A) = -1 \neq 0$, A is invertible.

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Since $\det(A) = -1 \neq 0$, A is invertible.

Therefore, for every choice of $\vec{b}$, the system $A\vec{x} = \vec{b}$ has an unique solution (namely $\vec{x} = A^{-1}\vec{b}$) . So T is onto.

Not one-to-one

Problem

Let T be the linear transformation induced by $A = \left[\begin{array}{rrr} 1 & 2 & -1 \\ 3 & 5 & 0 \end{array}\right]$. Show that T_A is not one-to-one.

Not one-to-one

Problem

Let T be the linear transformation induced by $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \end{bmatrix}$. Show that T_A is not one-to-one.

Solution

Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \end{bmatrix} = R$$

Since A has rank two, $A\vec{x} = 0$ has infinitely many solutions, so $\vec{x} = 0$ is not the only solution. Therefore, T_A is not one-to-one.

One-to-one

Problem

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Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = R$$

Since A has rank two, every variable in $A\vec{x} = 0$ is a leading variable, so $\vec{x} = 0$ is the unique solution. Therefore, T_A is one-to-one.

One-to-one and onto

Problem

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Solution

Let R be a row-echelon form of A.

$$A = \left[\begin{array}{rr} 1 & -1 \\ 2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{rr} 1 & -1 \\ 0 & 1 \end{array}\right] = R$$

In this case, A is invertible, so $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for every $\vec{b}$ in $\mathbb{R}^2$. Therefore T_A is both one-to-one and onto.

Neither one-to-one nor onto

Problem

Let T be the linear transformation induced by $A = \left[\begin{array}{rrr} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{array}\right]$. Show that T_A is neither one-to-one nor onto.

Neither one-to-one nor onto

Problem

Let T be the linear transformation induced by $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Show that T_A is neither one-to-one nor onto.

Solution

Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

Since A has rank two, the augmented matrix $[A|\vec{b}]$ will have rank three for some choice of $\vec{b} \in \mathbb{R}^3$, resulting in $A\vec{x} = \vec{b}$ being inconsistent. Therefore, T_A is not onto.

Neither one-to-one nor onto

Problem

Let T be the linear transformation induced by $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Show that T_A is neither one-to-one nor onto.

Solution

Let R be a row-echelon form of A.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

Since A has rank two, the augmented matrix $[A|\vec{b}]$ will have rank three for some choice of $\vec{b} \in \mathbb{R}^3$, resulting in $A\vec{x} = \vec{b}$ being inconsistent. Therefore, T_A is not onto.

The augmented matrix $[A|0]$ has rank two, so the system $A\vec{x} = 0$ has a non-leading variable, and hence does not have unique solution $\vec{x} = 0$. Therefore, T_A is not one-to-one.

Matrix of a One to One or Onto Transformation

Theorem (5.34, Matrix of a One to One or Onto Transformation)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation induced by the $m \times n$ matrix A. Then T is one to one if and only if the rank of A is n. T is onto if and only if the rank of A is m.

Kernel and Image

Definition (Kernel)

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$$\ker(T) = \left\{ \vec{v} \in V : T(\vec{v}) = \vec{0} \right\}$$

If A is the matrix corresponding to T, then $\ker(T)$ is the null-space of A.

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$$\ker(T) = \left\{ \vec{v} \in V : T(\vec{v}) = \vec{0} \right\}$$

If A is the matrix corresponding to T, then $\ker(T)$ is the null-space of A.

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Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$, and let $T : V \mapsto W$ be a linear transformation.
Then the kernel of T, $\ker(T)$, consists of all $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$.

$$\ker(T) = \left\{ \vec{v} \in V : T(\vec{v}) = \vec{0} \right\}$$

If A is the matrix corresponding to T, then $\ker(T)$ is the null-space of A.

Definition (Image)

Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$, and let $T : V \mapsto W$ be a linear transformation.
Then the image of T, $\mathrm{im}(T)$, consists of all $\vec{w} \in W$ such that $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$.

Kernel and Image

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Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$, and let $T : V \mapsto W$ be a linear transformation.
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If A is the matrix corresponding to T, then $\ker(T)$ is the null-space of A.

Definition (Image)

Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$, and let $T : V \mapsto W$ be a linear transformation.
Then the image of T, $\operatorname{im}(T)$, consists of all $\vec{w} \in W$ such that $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$.

$$\operatorname{im}(T) = \{T(\vec{v}) : \vec{v} \in V\}$$

If A is the matrix corresponding to T, then $\operatorname{im}(T)$ is the column space of A.

Example

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be defined by

$$T\left(\left[\begin{array}{c} a \\ b \\ c \end{array}\right]\right) = \left[\begin{array}{c} a+b+c \\ c-a \end{array}\right]$$

Then T is a linear transformation. Find a basis for $\ker(T)$ and $\operatorname{im}(T)$.

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Then T is a linear transformation. Find a basis for $\ker(T)$ and $\mathrm{im}(T)$.

Solution

You can (and should!) verify that T is a linear transformation.

Solution (continued)

Kernel of T: We look for all vectors $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \vec{0}$.

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b+c \\ c-a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution (continued)

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This gives a system of equations:

$$\begin{aligned} a+b+c &= 0 \\ c-a &= 0 \end{aligned}$$

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This gives a system of equations:

$$\begin{aligned} a+b+c &= 0 \\ c-a &= 0 \end{aligned}$$

The general solution is

$$\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

And therefore a basis for the kernel is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Solution (continued)

Image of T: We can write the image as

$$\begin{aligned}
\operatorname{im}(T) &= \left\{ \begin{bmatrix} a+b+c \\ c-a \end{bmatrix} : a,b,c \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \end{bmatrix} : a,b,c \in \mathbb{R} \right\} \\
&= \left\{ a\begin{bmatrix} 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}
\end{aligned}$$

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&= \left\{ \left[\begin{array}{c} a \\ -a \end{array} \right] + \left[\begin{array}{c} b \\ 0 \end{array} \right] + \left[\begin{array}{c} c \\ c \end{array} \right] : a, b, c \in \mathbb{R} \right\} \\
&= \left\{ a \left[\begin{array}{c} 1 \\ -1 \end{array} \right] + b \left[\begin{array}{c} 1 \\ 0 \end{array} \right] + c \left[\begin{array}{c} 1 \\ 1 \end{array} \right] : a, b, c \in \mathbb{R} \right\}
\end{aligned}$$

Thus $\mathrm{im}(T) = \mathrm{span} \left\{ \left[\begin{array}{c} 1 \\ -1 \end{array} \right], \left[\begin{array}{c} 1 \\ 0 \end{array} \right], \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \right\}$.

Solution (continued)

Image of T: We can write the image as

$$
\begin{aligned}
\mathrm{im}(T) &= \left\{ \begin{bmatrix} a+b+c \\ c-a \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \\
&= \left\{ a \begin{bmatrix} 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}
\end{aligned}
$$

Thus $\mathrm{im}(T) = \mathrm{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

These vectors are not linearly independent, but the first two are so a basis for the image of T is

$$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Kernel and One to One

The kernel of a linear transformation gives important information about whether the transformation is one to one. Recall that a linear transformation T is one to one if and only if $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$.

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Theorem

Let $T : V \mapsto W$ be a linear transformation where V is a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$.
Then T is one to one if and only if $\ker(T) = \left\{ \vec{0} \right\}$.

Dimension of the Kernel and Image

Theorem (Dimension Theorem)

Let $T : V \mapsto W$ be a linear transformation where V is a subspace of $\mathbb{R}^n$ and W is a subspace of $\mathbb{R}^m$. Suppose further that the dimension of V is k. Then

$$k = \dim(\ker(T)) + \dim(\mathrm{im}(T))$$

Example (Revisited)

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be defined by

$$T\left(\left[\begin{array}{c} a \\ b \\ c \end{array}\right]\right) = \left[\begin{array}{c} a+b+c \\ c-a \end{array}\right]$$

Find the dimension of $\ker(T)$ and $\operatorname{im}(T)$.

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We already know that a basis for the kernel of T is given by

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Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be defined by

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Find the dimension of $\ker(T)$ and $\mathrm{im}(T)$.

Solution

We already know that a basis for the kernel of T is given by

$$\left\{\left[\begin{array}{r} 1 \\ -2 \\ 1 \end{array}\right]\right\}$$

Therefore $\dim(\ker(T)) = 1$.

Solution (continued)

We also found a basis for the image of T as

$$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

and this of course shows $\dim(\mathrm{im}(T)) = 2$.

Solution (continued)

We also found a basis for the image of T as

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and this of course shows $\dim(\operatorname{im}(T)) = 2$.

But we could have found the dimension of $\operatorname{im}(T)$ without finding a basis. That's because since the dimension of $\mathbb{R}^3$ is 3, and the dimension of $\ker(T)$ is 1, we get by the Dimension Theorem that:

$$\begin{aligned} \dim(\operatorname{im}(T)) &= \dim(\mathbb{R}^3) - \dim(\ker(T)) \\ &= 3 - 1 = 2 \end{aligned}$$