A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

6.3: Complex Numbers; Roots of Complex Numbers

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Roots of Complex Numbers

Definition

Let z and q be complex numbers, and let n be a positive integer. Then z is called an n^{th} root of q if $z^n = q$.

De Moivre's Theorem and its implication

If θ is any angle and n is a positive integer, $\left(e^{i\theta}\right)^n = e^{in\theta}$. This implies that for any real number $r > 0$ and any positive integer n,

$$(re^{i\theta})^n = r^n e^{in\theta}.$$

This leads to the following result.

Corollary

Let q be a nonzero complex number and n a positive integer. Then $z^n = q$ has exactly n complex solutions, i.e., q has exactly n complex n^{th} roots.

Example

For any positive real number a, $z^2 = a$ has two complex (in this case, real) solution, $z = \sqrt{a}$ and $z = -\sqrt{a}$. This is equivalent to the statement that a has two complex (in this case, real) square roots.

- One particular example: 25 has two square roots, 5 and -5, and these are the two solutions to $z^2 = 25$.
- But we all knew that. A more interesting example is that -1 has no real square roots, but suddenly it has two (complex) square roots, i and $-i$. These are the two (complex) solutions to $z^2 = 1$.

Cube Roots

Example

To find the (three) cube roots of i, we solve the equation $z^3 = i$. To do so, we express both z and i in polar form: convert i to polar form, and write $z = re^{i\theta}$, giving us

$$(re^{i\theta})^3 = e^{\pi i/2}.$$

Thus $r^3 e^{3i\theta} = 1e^{\pi i/2}$, implying that $r^3 = 1$ and $3\theta = \frac{\pi}{2}$.

- Since r is a non-negative real number, $r^3 = 1$ implies that $r = 1$.
- The statement $3\theta = \frac{\pi}{2}$ is not completely correct. The problem that arises is that the argument, $\frac{\pi}{2}$ is not unique. Instead, we could have written

  $$i = e^{5\pi i/2} \text{ or } i = e^{9\pi i/2} \text{ or } i = e^{-3\pi i/2}.$$

  In fact, there are infinitely many choices for the argument. The important thing to notice is that any two different arguments differ by a multiple of $2\pi i$, and thus we may write

  $$3\theta = \frac{\pi}{2} + 2\pi k, \ \ k \in \mathbb{Z}.$$

  ($\mathbb{Z}$ denotes the set of integers: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$).

Example (continued)

Dividing both sides of $3\theta = \frac{\pi}{2} + 2\pi k$ by 3:

$$\theta = \frac{\pi}{6} + \frac{2}{3}\pi k = \frac{(1+4k)\pi}{6},$$

where k is any integer. The Corollary to De Moivre's Theorem tells us that there are only three different cube roots. These are obtained by using $k = 0$, $k = 1$, and $k = 2$, resulting in three values of θ:

$$\frac{\pi}{6}, \frac{5\pi}{6}, \text{ and } \frac{9\pi}{6} = \frac{3\pi}{2}.$$

Thus the cube roots of i are

$$e^{\pi i/6}, e^{5\pi i/6}, \text{ and } e^{3\pi i/2}.$$

We now convert these to Cartesian form.

Example (continued)

$$
\begin{aligned}
e^{\pi i/6} &= \frac{\sqrt{3}}{2} + \frac{1}{2}i, \\
e^{5\pi i/6} &= -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \\
e^{3\pi i/2} &= -i.
\end{aligned}
$$

You can check your work by computing the cube of each of these.

This process is summarized in the following procedure.

Finding Roots of a Complex Number

Let z be a complex number. We wish to find the n^{th} roots of z, that is all w such that $w^n = z$.
There are n distinct n^{th} roots, $w_1, w_2, \ldots, w_{n-1}$, and they can be found as follows:.

1. Express both z and w in polar form $z = re^{i\theta}, w = se^{i\phi}$. Then $w^n = z$ becomes:

   $$(se^{i\theta})^n = s^n e^{in\phi} = re^{i\theta}$$

   We need to solve for s and ϕ.
2. Solve the following two equations:

$$
\begin{aligned}
s^n &= r \\
e^{in\phi} &= e^{i\theta}
\end{aligned}
\tag{1}
$$

Continued

3. The solution to $s^n = r$ is $s = \sqrt[n]{r}$ (since s must be positive).
4. The solutions to $e^{in\phi} = e^{i\theta}$ are given by:

$$n\phi = \theta + 2\pi k, \text{ for } k = 0, 1, 2, \cdots, n-1$$

or

$$\phi = \frac{\theta + 2k\pi}{n}, \text{ for } k = 0, 1, 2, \cdots, n-1$$

5. Using the solutions r, θ to the equations given in (1) construct the n^{th} roots of the form $z = re^{i\theta}$.
6. So the solutions to $w^n = z$ are

$$w_k = \sqrt[n]{r} \cdot e^{\frac{\theta + 2k\pi}{n} i}, \quad k = 0, 1, 2, \ldots, n-1$$

Problem

Find all 4'th roots of $z = 2(\sqrt{3}i - 1)$, and express each in the form $a + bi$.

Solution

1. Convert $z = 2(\sqrt{3}i - 1) = -2 + 2\sqrt{3}i$ to polar form:

$$|z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{16} = 4.$$

If θ is an argument for $-2 + 2\sqrt{3}i$, then determine θ by sketching z, or as follows

$$\cos\theta = \frac{-2}{4} = -\frac{1}{2} \text{ and } \sin\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}, \text{ so } \theta = \frac{2\pi}{3}.$$

Thus $z^4 = 4e^{2\pi i/3}$. Let $z = re^{i\theta}$.

Solution (continued)

2. The solutions are

$$\begin{aligned} w_k &= \sqrt[n]{r} \cdot e^{\frac{\theta+2k\pi}{n} i}, \quad k = 0, 1, 2, \ldots, n-1 \\ &= \sqrt[4]{4} \cdot e^{\frac{\theta+2k\pi}{4} i}, \quad k = 0, 1, 2, 3 \\ &= \sqrt{2} e^{\frac{\frac{2\pi}{3}+2k\pi}{4} i}, \quad k = 0, 1, 2, 3 \\ &= \sqrt{2} e^{\frac{\pi+3k\pi}{6} i}, \quad k = 0, 1, 2, 3 \\ &= \sqrt{2} e^{\frac{\pi}{6} i}, e^{\frac{2\pi}{3} i}, e^{\frac{7\pi}{6} i}, e^{\frac{5\pi}{3} i} \end{aligned}$$

Solution (continued)

5. Converting to Cartesian form:

$$\begin{array}{llll} k = 0: & w_0 = \sqrt{2} e^{\pi i/6} & = \sqrt{2}(\frac{(\sqrt{3}}{2} + \frac{1}{2} i) & = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2} i \\ k = 1: & w_1 = \sqrt{2} e^{2\pi i/3} & = \sqrt{2}(-\frac{1}{2} + \frac{\sqrt{3}}{2} i) & = -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2} i \\ k = 2: & w_2 = \sqrt{2} e^{7\pi i/6} & = \sqrt{2}(-\frac{\sqrt{3}}{2} - \frac{1}{2} i) & = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2} i \\ k = 3: & w_3 = \sqrt{2} e^{5\pi i/3} & = \sqrt{2}(\frac{1}{2} - \frac{\sqrt{3}}{2} i) & = \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2} i \end{array}$$

Therefore, the fourth roots of $2(\sqrt{3}i - 1)$ are:

$$\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2} i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2} i, -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2} i, \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2} i.$$

Roots of Unity

Definition

A complex number z is a root of unity if there exists a positive integer n so that $z^n = 1$.

Problem

Since $z = 1 = 1e^{0i}$, the 6'th roots are

$$w_k = \sqrt[6]{1} \cdot e^{\frac{0+2k\pi}{6} i}, \quad k = 0, 1, 2, \ldots, 5$$

$$= e^{\frac{k\pi}{3} i}, \quad k = 0, 1, 2, \ldots, 5$$

Find the sixth roots of unity, i.e., all solutions to $z^6 = 1$.

Solution (continued)

$w_k = e^{\frac{k\pi}{3} i}, \quad k = 0, 1, 2, \ldots, 5$. Converting these to Cartesian form:

k	w_k
0	$e^{0i} = 1$
1	$e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$
2	$e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$
3	$e^{\pi i} = -1$
4	$e^{4\pi i/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$
5	$e^{5\pi i/3} = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

If you graph these six point in the complex plane, you'll see that they result in six equally spaced points on the unit circle, one of them being $(1, 0)$.

Roots of Unity

For any integer $n \geq 1$, the (complex) solutions to $w^n = 1$ are

$$w_k = e^{2\pi ki/n} \text{ for } k = 0, 1, 2, \ldots, n-1.$$

Furthermore, the n^{th} roots of unity correspond to n equally spaced points on the unit circle, one of them being $(1, 0)$.