A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Linear Transformations and Matrices

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

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Transformation by Matrix Multiplication

Example

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. By matrix multiplication, A transforms vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$.

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Consider the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Transforming this vector by A looks like:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$$

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For example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Transformations

Definition

A transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$, sometimes written

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m,$$

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What do we mean by a function?

Informally, a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a rule that assigns exactly one vector of $\mathbb{R}^m$ to each vector of $\mathbb{R}^n$.

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Definition

If T acts by matrix multiplication of a matrix A (such as the previous example), we call T a matrix transformation, and write $T_A(\vec{x}) = A\vec{x}$.

Equality of Transformations

Definition

Suppose $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are transformations. Then $S = T$ if and only if $S(\vec{x}) = T(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$.

Specifying the Action of a Transformation

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$T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

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is a transformation that transforms the vector $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ in $\mathbb{R}^3$ into the vector

$$T \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 4+7 \\ 1-7 \\ 7-4 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ -6 \\ 3 \end{bmatrix}.$$

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A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

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1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (preservation of addition)
2. $T(a\vec{x}) = aT(\vec{x})$ (preservation of scalar multiplication)

Properties of Linear Transformations

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Suppose $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ are vectors in $\mathbb{R}^n$ and

$$\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k$$

for some $a_1, a_2, \ldots, a_k \in \mathbb{R}$.

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3. $$\begin{aligned} T(\vec{y}) &= T(a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k) \\ &= a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + \cdots + a_kT(\vec{x}_k), \end{aligned}$$

   i.e., T preserves linear combinations.

Problem

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that

$$T\left[\begin{array}{r} 1 \\ 3 \\ 1 \end{array}\right] = \left[\begin{array}{r} 4 \\ 4 \\ 0 \\ -2 \end{array}\right] \text{ and } T\left[\begin{array}{r} 4 \\ 0 \\ 5 \end{array}\right] = \left[\begin{array}{r} 4 \\ 5 \\ -1 \\ 5 \end{array}\right].$$

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The only way it is possible to solve this problem is if

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i.e., if there exist $a, b \in \mathbb{R}$ so that

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}.$$

Solution (continued)

To find a and b, solve the system of three equations in two variables:

$$\left[\begin{array}{rr|r} 1 & 4 & -7 \\ 3 & 0 & 3 \\ 1 & 5 & -9 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $a = 1$, $b = -2$, and

$$\left[\begin{array}{r} -7 \\ 3 \\ -9 \end{array}\right] = \left[\begin{array}{r} 1 \\ 3 \\ 1 \end{array}\right] - 2\left[\begin{array}{r} 4 \\ 0 \\ 5 \end{array}\right].$$

Solution (continued)

We now use that fact that linear transformations preserve linear combinations, implying that

$$T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}\right)$$

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Therefore, $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}$.

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Final Answer

$$T\begin{bmatrix} 2 \\ 5 \\ -3 \\ -7 \end{bmatrix} = \begin{bmatrix} -11 \\ 6 \\ -5 \end{bmatrix}.$$

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proving that T preserves addition.

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Since T preserves addition and scalar multiplication T is a linear transformation. $\square$

Some Special Matrix Transformations

Example (The Zero Transformation)

If A is the $m \times n$ matrix of zeros, then the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ induced by A is called the zero transformation because for every vector $\vec{x}$ in $\mathbb{R}^n$

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The transformation of $\mathbb{R}^n$ induced by I_n, the $n \times n$ identity matrix, is called the identity transformation because for every vector $\vec{x}$ in $\mathbb{R}^n$,

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The identity transformation on $\mathbb{R}^n$ is usually written as $1_{\mathbb{R}^n}$.

Example (Revisited)

Recall $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

Not all transformations are matrix transformations!

Example

Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \left[\begin{array}{r} 1 \\ -1 \end{array}\right] \text{ for all } \vec{x} \in \mathbb{R}^2.$$

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Why is T not a matrix transformation?

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$$T \left[\begin{array}{c} 0 \\ 0 \end{array} \right]$$

Example (continued)

We have $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

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$$T\left[\begin{array}{c} 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] + \left[\begin{array}{r} 1 \\ -1 \end{array}\right]$$

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We have $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

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$$T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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$$T \left[\begin{array}{c} 0 \\ 0 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \end{array} \right] + \left[\begin{array}{r} 1 \\ -1 \end{array} \right] = \left[\begin{array}{r} 1 \\ -1 \end{array} \right] \neq \left[\begin{array}{c} 0 \\ 0 \end{array} \right],$$

violating one of the properties of a linear transformation.

Example (continued)

We have $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for all } \vec{x} \in \mathbb{R}^2.$$

Since every matrix transformation is a linear transformation, we consider $T(0)$, where 0 is the zero vector of $\mathbb{R}^2$.

$$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

violating one of the properties of a linear transformation.

Therefore, T is not a linear transformation, and hence is not a matrix transformation.
Can you see any other reasons why T is not a matrix transformation?

Matrix Transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then we can find an $n \times m$ matrix A such that

$$T(\vec{x}) = A\vec{x}$$

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Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then we can find an $n \times m$ matrix A such that

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In this case, we say that T is induced, or determined, by A and we write

$$T_A(\vec{x}) = A\vec{x}$$

Corollary

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

Matrix and Linear Transformations

Good News!

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There is an easy way to find the matrix of T!

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There is an easy way to find the matrix of T!

Definition

The set of columns $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of I_n is called the standard basis of $\mathbb{R}^n$.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Matrix and Linear Transformations

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Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation.

Matrix and Linear Transformations

Good News!

There is an easy way to find the matrix of T!

Definition

The set of columns $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of I_n is called the standard basis of $\mathbb{R}^n$.

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation. Furthermore, T is induced by the unique matrix

$$A = \left[\begin{array}{cccc} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{array} \right],$$

where $\vec{e}_j$ is the j^{th} column of I_n, and $T(\vec{e}_j)$ is the j^{th} column of A.

Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T \left[\begin{array}{c} x \\ y \end{array} \right] = \left[\begin{array}{c} x + 2y \\ x - y \end{array} \right]$$

for each $\vec{x} \in \mathbb{R}^2$. Find the matrix, A, of T.

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To find A, we must find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$.

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$$T \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$$

Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ x-y \end{bmatrix}$$

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To find A, we must find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$.

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2(0) \\ 1-0 \end{bmatrix}$$

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To find A, we must find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$.

$$T\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} 1+2(0) \\ 1-0 \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \text{ and } T\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$$

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$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ x-y \end{bmatrix}$$

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To find A, we must find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$.

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$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}$$

for each $\vec{x} \in \mathbb{R}^2$. Find the matrix, A, of T.

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To find A, we must find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$.

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for each $\vec{x} \in \mathbb{R}^2$. Find the matrix, A, of T.

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To find A, we must find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$.

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The columns $T(\vec{e}_1)$ and $T(\vec{e}_2)$ become the columns of A, so

$$A = \left[\begin{array}{rr} 1 & 2 \\ 1 & -1 \end{array} \right],$$

and $T(\vec{x}) = A\vec{x}$ for every $\vec{x} \in \mathbb{R}^2$.

Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T \left[\begin{array}{c} x \\ y \end{array} \right] = \left[\begin{array}{c} x + 2y \\ x - y \end{array} \right]$$

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To find A, we must find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$.

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and $T(\vec{x}) = A\vec{x}$ for every $\vec{x} \in \mathbb{R}^2$. Therefore A is the matrix for T.

Find the Matrix of T

Problem

Sometimes T is not defined so nicely for us. Suppose T is given as

$$T\left[\begin{array}{c} 1 \\ 5 \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \end{array}\right],\; T\left[\begin{array}{c} 1 \\ 4 \end{array}\right] = \left[\begin{array}{c} 3 \\ 2 \end{array}\right]$$

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Solution

We need to write $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of the vectors provided. So we reduce the augmented matrix having $\vec{e}_1$ and $\vec{e}_2$ as the third and fourth columns:

$$\left[\begin{array}{cc|cc} 1 & 1 & \vec{e}_1 & \vec{e}_2 \\ 5 & 4 & & \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right]$$

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$$\left[\begin{array}{cc|cc} 1 & 1 & \vec{e}_1 & \vec{e}_2 \\ 5 & 4 & & \end{array} \right] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|rr} 1 & 0 & -4 & 1 \\ 0 & 1 & 5 & -1 \end{array} \right]$$

Solution (continued)

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|rr} 1 & 0 & -4 & 1 \\ 0 & 1 & 5 & -1 \end{array}\right]$$

From this we can see that

$$\vec{e_1} = \left[\begin{array}{c} 1 \\ 0 \end{array}\right] = -4\left[\begin{array}{c} 1 \\ 5 \end{array}\right] + 5\left[\begin{array}{c} 1 \\ 4 \end{array}\right]$$

and

$$\vec{e_2} = \left[\begin{array}{c} 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} 1 \\ 5 \end{array}\right] - \left[\begin{array}{c} 1 \\ 4 \end{array}\right]$$

Solution (continued)

So

$$T(\vec{e_1}) = T\left(-4\left[\begin{array}{c} 1 \\ 5 \end{array}\right] + 5\left[\begin{array}{c} 1 \\ 4 \end{array}\right]\right)$$

Solution (continued)

So

$$\begin{aligned} T(\vec{e_1}) &= T\left(-4\left[\begin{array}{c} 1 \\ 5 \end{array}\right] + 5\left[\begin{array}{c} 1 \\ 4 \end{array}\right]\right) \\ &= -4T\left[\begin{array}{c} 1 \\ 5 \end{array}\right] + 5T\left[\begin{array}{c} 1 \\ 4 \end{array}\right] \end{aligned}$$

Solution (continued)

So

$$
\begin{aligned}
T(\vec{e_1}) &= T\left(-4\left[\begin{array}{c}1\\5\end{array}\right]+5\left[\begin{array}{c}1\\4\end{array}\right]\right)\\
&= -4T\left[\begin{array}{c}1\\5\end{array}\right]+5T\left[\begin{array}{c}1\\4\end{array}\right]\\
&= -4\left[\begin{array}{c}1\\2\end{array}\right]+5\left[\begin{array}{c}3\\2\end{array}\right]
\end{aligned}
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Solution (continued)

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$$T(\vec{e_1}) = T\left(-4\begin{bmatrix}1\\5\end{bmatrix} + 5\begin{bmatrix}1\\4\end{bmatrix}\right)$$

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and

$$T(\vec{e_2}) = T\begin{bmatrix}0\\1\end{bmatrix}$$

$$= T\left(\begin{bmatrix}1\\5\end{bmatrix} - \begin{bmatrix}1\\4\end{bmatrix}\right)$$

Solution (continued)

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$$T(\vec{e_1}) = T\left(-4\begin{bmatrix}1\\5\end{bmatrix}+5\begin{bmatrix}1\\4\end{bmatrix}\right)$$

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$$= T\begin{bmatrix}1\\5\end{bmatrix}-T\begin{bmatrix}1\\4\end{bmatrix}$$

Solution (continued)

So

$$T(\vec{e_1}) = T\left(-4\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$$

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Solution (continued)

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$$
\begin{aligned}
T(\vec{e_1}) &= T\left(-4\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) \\
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&= \begin{bmatrix} 11 \\ 2 \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
T(\vec{e_2}) &= T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= T\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) \\
&= T\begin{bmatrix} 1 \\ 5 \end{bmatrix} - T\begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ 0 \end{bmatrix}
\end{aligned}
$$

Solution (continued)

The matrix of T is the matrix whose first column is $T(\vec{e_1})$, and second column is $T(\vec{e_2})$:

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The matrix of T is the matrix whose first column is $T(\vec{e_1})$, and second column is $T(\vec{e_2})$:

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Determining if a Transformation is Linear

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a transformation defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x + 2y \end{bmatrix}$.

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If T were a linear transformation, then T would be induced by the matrix

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

Determining if a Transformation is Linear

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a transformation defined by $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x + 2y \end{bmatrix}$.

One way to show that T is a linear transformation is to show that it preserves addition and scalar multiplication. However, now that we know that linear transformations are matrix transformations, we can use this to our advantage.

If T were a linear transformation, then T would be induced by the matrix

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} T \begin{bmatrix} 1 \\ 0 \end{bmatrix} & T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}.$$

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Since

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x+2y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix},$$

T is a matrix transformation, and is therefore a linear transformation.

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Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by $T \left[\begin{array}{c} x \\ y \end{array} \right] = \left[\begin{array}{c} xy \\ x + y \end{array} \right]$.

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i.e., $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq T\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore, T is not a linear transformation.

Rotations in $\mathbb{R}^2$

Definition

The transformation

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denotes counterclockwise rotation about the origin through an angle of θ.

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Rotation through an angle of θ preserves scalar multiplication.

Rotation through an angle of θ preserves vector addition.

R_θ is a linear transformation

Since R_θ preserves addition and scalar multiplication, R_θ is a linear transformation, and hence a matrix transformation.

The matrix that induces R_θ can be found by computing $R_\theta(E_1)$ and $R_\theta(E_2)$, where

$$E_1 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right] \text{ and } E_2 = \left[\begin{array}{c} 0 \\ 1 \end{array} \right].$$

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The Matrix for R_θ

The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and is induced by the matrix

$$\left[\begin{array}{cr} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right].$$

Example (Rotation through π)

We denote by

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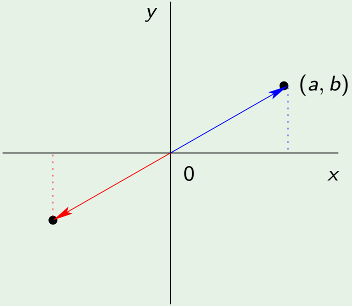

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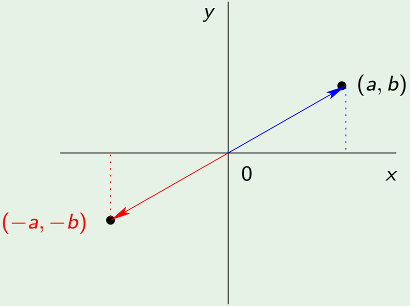

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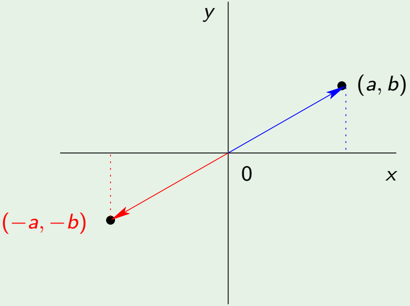

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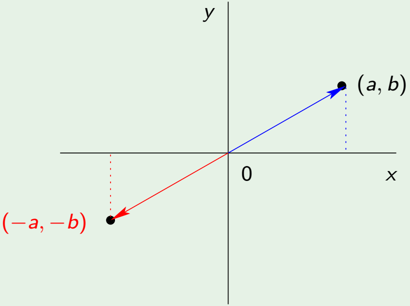

We see that $R_\pi \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so R_π is a matrix transformation.

Rotation

Problem

The transformation $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes a counterclockwise rotation about the origin through an angle of $\frac{\pi}{2}$ radians. Find the matrix of $R_{\frac{\pi}{2}}$.

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Solution

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Furthermore $R_{\frac{\pi}{2}}$ is a matrix transformation, and the matrix it is induced by is

$$\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

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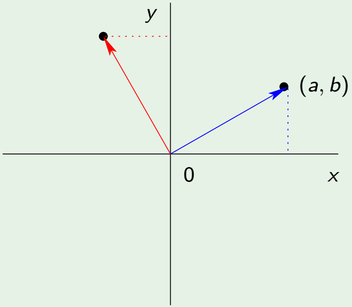

Example (Rotation through $\pi/2$)

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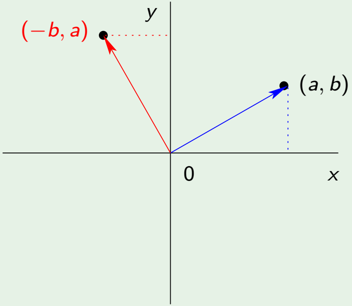

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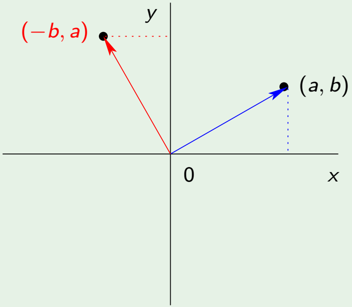

We see that $R_{\pi/2} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} =$

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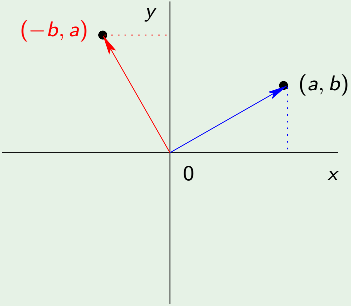

We see that $R_{\pi/2} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so $R_{\pi/2}$ is a matrix transformation.

Reflection in $\mathbb{R}^2$

Example

In $\mathbb{R}^2$, reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

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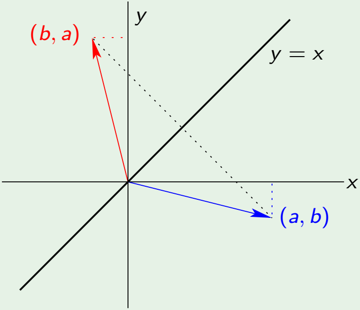
In $\mathbb{R}^2$, reflection in the y-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} -a \\ b \end{bmatrix}$. This is a matrix transformation because

$$\begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example

Reflection in the line $y = x$ transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.



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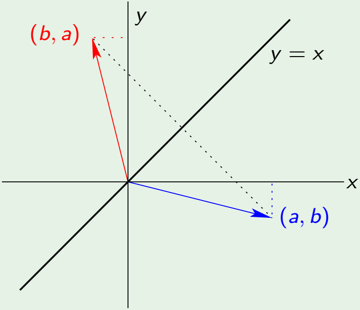

This is a matrix transformation because

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$