A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Linear Transformations: Kernel and Image

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

In addition we recognize the following contributors. All new content contributed is released under the same license as noted below.

- **Tim Alderson, University of New Brunswick**
- **Ilijas Farah, York University**
- **Ken Kuttler, Brigham Young University**
- **Asia Weiss, York University**

Kernel and Image

Definition (Kernel)

Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$, and let $T : V \mapsto W$ be a linear transformation.
Then the kernel of T, $\ker(T)$, consists of all $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$.

$$\ker(T) = \left\{ \vec{v} \in V : T(\vec{v}) = \vec{0} \right\}$$

Definition (Image)

Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$, and let $T : V \mapsto W$ be a linear transformation.
Then the image of T, $\mathrm{im}(T)$, consists of all $\vec{w} \in W$ such that $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$.

$$\mathrm{im}(T) = \{ T(\vec{v}) : \vec{v} \in V \}$$

Problem to Try

Problem

Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$, and let $T : V \mapsto W$ be a linear transformation.

Show that $\ker(T)$ is a subspace of V and $\mathrm{im}(T)$ is a subspace of W.

Example

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be defined by

$$T\left(\left[\begin{array}{c} a \\ b \\ c \end{array}\right]\right) = \left[\begin{array}{c} a+b+c \\ c-a \end{array}\right]$$

Then T is a linear transformation. Find a basis for $\ker(T)$ and $\mathrm{im}(T)$.

Solution

You can (and should!) verify that T is a linear transformation.

Solution (continued)

Kernel of T: We look for all vectors $\vec{x} \in \mathbb{R}^3$ such that $T(\vec{x}) = \vec{0}$.

$$T\left(\left[\begin{array}{c} a \\ b \\ c \end{array}\right]\right) = \left[\begin{array}{c} a+b+c \\ c-a \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

This gives a system of equations:

$$\begin{array}{rcl} a+b+c & = & 0 \\ c-a & = & 0 \end{array}$$

The general solution is

$$\left(\left[\begin{array}{c} a \\ b \\ c \end{array}\right]\right) = \left\{\left[\begin{array}{c} t \\ -2t \\ t \end{array}\right] : t \in \mathbb{R}\right\} = \left\{t\left[\begin{array}{c} 1 \\ -2 \\ 1 \end{array}\right] : t \in \mathbb{R}\right\}$$

And therefore a basis for the kernel is $\left\{\left[\begin{array}{c} 1 \\ -2 \\ 1 \end{array}\right]\right\}$.

Solution (continued)

Image of T: We can write the image as

$$\begin{aligned}\text{im}(T) &= \left\{ \begin{bmatrix} a+b+c \\ c-a \end{bmatrix} : a,b,c \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \end{bmatrix} : a,b,c \in \mathbb{R} \right\} \\ &= \left\{ a\begin{bmatrix} 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}\end{aligned}$$

Thus $\text{im}(T) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

These vectors are not linearly independent, but the first two are so a basis for the image of T is

$$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Kernel and One to One

The kernel of a linear transformation gives important information about whether the transformation is one to one. Recall that a linear transformation T is one to one if and only if $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$.

Theorem (Dimension Theorem)

Let $T : V \mapsto W$ be a linear transformation where V is a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$.
Then T is one to one if and only if $\ker(T) = \left\{\vec{0}\right\}$.

Dimension of the Kernel and Image

Theorem

Let $T : V \mapsto W$ be a linear transformation where V is a subspace of $\mathbb{R}^n$ and W is a subspace of $\mathbb{R}^m$. Suppose further that the dimension of V is k. Then

$$k = \dim(\ker(T)) + \dim(\mathrm{im}(T))$$

Corollary

Let T, V, W be defined as above, with $\dim(V) = k$. Then

$$\begin{aligned} \dim(\ker(T)) &\leq k &\leq n \\ \dim(\mathrm{im}(T)) &\leq k &\leq n \end{aligned}$$

Example (Revisited)

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ be defined by

$$T\left(\left[\begin{array}{c} a \\ b \\ c \end{array}\right]\right) = \left[\begin{array}{c} a+b+c \\ c-a \end{array}\right]$$

Find the dimension of $\ker(T)$ and $\mathrm{im}(T)$.

Solution

We already know that a basis for the kernel of T is given by

$$\left\{\left[\begin{array}{r} 1 \\ -2 \\ 1 \end{array}\right]\right\}$$

Therefore $\dim(\ker(T)) = 1$.

Solution (continued)

We also found a basis for the image of T as

$$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

and this of course shows $\dim(\mathrm{im}(T)) = 2$.

But we could have found the dimension of $\mathrm{im}(T)$ without finding a basis. That's because since the dimension of $\mathbb{R}^3$ is 3, and the dimension of $\ker(T)$ is 1, we get by the Dimension Theorem that:

$$\begin{aligned} \dim(\mathrm{im}(T)) &= \dim(\mathbb{R}^3) - \dim(\ker(T)) \\ &= 3 - 1 = 2 \end{aligned}$$