A First Course in

LINEAR ALGEBRA

Lecture Notes

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$\mathbb{R}^n$: Row, Column, and Null Space

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A First Course in Linear Algebra

Lecture Notes

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Definitions

Let A be an $m \times n$ matrix. The **column space** of A, written $\text{col}(A)$, is the span of the columns. The **row space** of A, written $\text{row}(A)$, is the span of the rows.

Notation

Let A and B be $m \times n$ matrices. We write $A \to B$ if B can be obtained from A by a sequence of elementary row operations. Note that $A \to B$ if and only if $B \to A$.

Lemma

Let A and B be $m \times n$ matrices such that A can be carried to B by elementary row [column] operations. Then $\text{row}(A) = \text{row}(B)$ [$\text{col}(A) = \text{col}(B)$].

Proof.

We will prove that the above is true for row operations, which can be easily applied to column operations. Let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$ denote the rows of A.

- If B is obtained from A by a interchanging two rows of A, then A and B have exactly the same rows, so $\text{row}(B) = \text{row}(A)$.
- Suppose $p \neq 0$, and for some j, $1 \leq j \leq m$, B is obtained from A by multiplying row j by p. Then

$$\text{row}(B) = \text{span}\{\vec{r}_1, \ldots, p\vec{r}_j, \ldots, \vec{r}_m\}.$$

  Since $\{\vec{r}_1, \ldots, p\vec{r}_j, \ldots, \vec{r}_m\} \subseteq \text{row}(A)$, it follows that $\text{row}(B) \subseteq \text{row}(A)$. Conversely, since $\{\vec{r}_1, \ldots, \vec{r}_m\} \subseteq \text{row}(B)$, it follows that $\text{row}(A) \subseteq \text{row}(B)$. Therefore, $\text{row}(B) = \text{row}(A)$.

Proof (continued).

- Suppose $p \neq 0$, and suppose that for some i and j, $1 \leq i, j \leq m$, B is obtained from A by adding p time row j to row i. Without loss of generality, we may assume $i < j$.

  Then
  $$\mathrm{row}(B) = \mathrm{span}\{\vec{r}_1, \ldots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \ldots, \vec{r}_j, \ldots, \vec{r}_m\}.$$
  Since
  $$\{\vec{r}_1, \ldots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \ldots, \vec{r}_m\} \subseteq \mathrm{row}(A),$$
  it follows that $\mathrm{row}(B) \subseteq \mathrm{row}(A)$.

  Conversely, since
  $$\{\vec{r}_1, \ldots, \vec{r}_m\} \subseteq \mathrm{row}(B),$$
  it follows that $\mathrm{row}(A) \subseteq \mathrm{row}(B)$. Therefore, $\mathrm{row}(B) = \mathrm{row}(A)$.

Corollary

Let A be an $m \times n$ matrix, U an invertible $m \times m$ matrix, and V an invertible $n \times n$ matrix. Then $\text{row}(UA) = \text{row}(A)$ and $\text{col}(AV) = \text{col}(A)$.

Proof.

Since U is invertible, U is a product of elementary matrices, implying that $A \to UA$ by a sequence of elementary row operations. Then $\text{row}(UA) = \text{row}(A)$.

Now consider AV: $\text{col}(AV) = \text{row}((AV)^T) = \text{row}(V^T A^T)$ and V^T is invertible (a matrix is invertible if and only if its transpose is invertible). It follows from the first part of this Corollary that

$$\text{row}(V^T A^T) = \text{row}(A^T).$$

But $\text{row}(A^T) = \text{col}(A)$, and therefore $\text{col}(AV) = \text{col}(A)$. □

Lemma

Let A be an $m \times n$ matrix and let R be its row-echelon form . Then the nonzero rows of R form a basis of row(R), and consequently of row(A).

A variation of this lemma provides a basis of col(A). Suppose A is row reduced to its row-echelon form R. Identify the pivot columns of R (columns which have leading ones), and take the corresponding columns of A. It turns out that this forms a basis of col(A).

Example

Let

$$R = \begin{bmatrix} 1 & 2 & 2 & -2 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Since the nonzero rows of R are linearly independent, they form a basis of $\text{row}(R)$.
2. Let $B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\} \subseteq \mathbb{R}^5$. Then B is linearly independent and spans $\text{col}(R)$, and thus is a basis of $\text{col}(R)$. Now let X denote the set of columns of R that contain the leading ones. Then X is a linearly independent subset of $\text{col}(R)$ with $4 = \dim(\text{col}(R))$ vectors. It follows that X spans $\text{col}(R)$, and therefore is a basis of $\text{col}(R)$.

Definition

Previously, we defined rank(A) to be the number of leading entries in the row-echelon form of A. Using an understanding of dimension and row space, we can now define rank as follows:

$$\text{rank}(A) = \dim(\text{row}(A))$$

Theorem (Rank Theorem)

Let A be an $m \times n$ matrix. Then $\dim(\text{col}(A))$, the dimension of the column space, is equal to the dimension of the row space, $\dim(\text{row}(A))$.

Problem

Find the rank of the following matrix and describe the column and row spaces.

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{bmatrix}$$

Solution

The reduced row-echelon form of A is

$$\begin{bmatrix} 1 & 0 & -9 & 9 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the rank is 2.

Solution continued

Notice that the first two columns of R are pivot columns. We find the corresponding columns of A to create a basis for $\operatorname{col}(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}$$

We know that the nonzero rows of R create a basis of $\operatorname{row}(A)$. For the above matrix, the row space equals

$$\operatorname{row}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 & -9 & 9 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 5 & -3 & 0 \end{bmatrix} \right\}$$

Corollary

Let A be a matrix. Then the following are true:

1. $rank(A) = rank(A^T)$.
2. For A of size $m \times n$, $rank(A) \leq m$ and $rank(A) \leq n$.
3. For A of size $n \times n$, A is invertible if and only if $rank(A) = n$.
4. For invertible matrices B and C of appropriate size, $rank(A) = rank(BA) = rank(AC)$.

Theorem

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}(A) = n$.
2. $\text{row}(A) = \mathbb{R}^n$, i.e., the rows of A span $\mathbb{R}^n$.
3. The columns of A are independent in $\mathbb{R}^m$.
4. The $n \times n$ matrix $A^T A$ is invertible.
5. There exists an $n \times m$ matrix C so that $CA = I_n$.
6. If $A\vec{x} = \vec{0}_m$ for some $\vec{x} \in \mathbb{R}^n$, then $\vec{x} = \vec{0}_n$.

Theorem

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}(A) = m$.
2. $\text{col}(A) = \mathbb{R}^m$, i.e., the columns of A span $\mathbb{R}^m$.
3. The rows of A are independent in $\mathbb{R}^n$.
4. The $m \times m$ matrix AA^T is invertible.
5. There exists an $n \times m$ matrix C so that $AC = I_m$.
6. The system $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

Definitions

Let A be an $m \times n$ matrix. The null space of A is defined as

$$\text{null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}_m\},$$

and the image space of A is defined as

$$\text{im}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$

Note. Since A is $m \times n$ and $\vec{x} \in \mathbb{R}^n$, $A\vec{x} \in \mathbb{R}^m$, so $\text{im}(A) \subseteq \mathbb{R}^m$ while $\text{null}(A) \subseteq \mathbb{R}^n$.

Problem

Prove that if A is an $m \times n$ matrix, then $\mathrm{null}(A)$ is a subspace of $\mathbb{R}^n$.

Proof.

- Since $A\vec{0}_n = \vec{0}_m$, $\vec{0}_n \in \mathrm{null}(A)$.
- Let $\vec{x}, \vec{y} \in \mathrm{null}(A)$. Then $A\vec{x} = \vec{0}_m$ and $A\vec{y} = \vec{0}_m$, so
$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}_m + \vec{0}_m = \vec{0}_m,$$
and thus $\vec{x} + \vec{y} \in \mathrm{null}(A)$.
- Let $\vec{x} \in \mathrm{null}(A)$ and $k \in \mathbb{R}$. Then $A\vec{x} = \vec{0}_m$, so
$$A(k\vec{x}) = k(A\vec{x}) = k\vec{0}_m = \vec{0}_m,$$
and thus $k\vec{x} \in \mathrm{null}(A)$.

Therefore, $\mathrm{null}(A)$ is a subspace of $\mathbb{R}^n$. $\square$

Problem

Prove that if A is an $m \times n$ matrix, then $\text{im}(A)$ is a subspace of $\mathbb{R}^m$.

Proof.

- Since $\vec{0}_n \in \mathbb{R}^n$ and $A\vec{0}_n = \vec{0}_m$, $\vec{0}_m \in \text{im}(A)$.
- Let $\vec{x}, \vec{y} \in \text{im}(A)$. Then $\vec{x} = A\vec{u}$ and $\vec{y} = A\vec{v}$ for some $\vec{u}, \vec{v} \in \mathbb{R}^n$, so
$$\vec{x} + \vec{y} = A\vec{u} + A\vec{v} = A(\vec{u} + \vec{v}).$$
Since $\vec{u} + \vec{v} \in \mathbb{R}^n$, it follows that $\vec{x} + \vec{y} \in \text{im}(A)$.
- Let $\vec{x} \in \text{im}(A)$ and $k \in \mathbb{R}$. Then $\vec{x} = A\vec{u}$ for some $\vec{u} \in \mathbb{R}^n$, and thus
$$k\vec{x} = k(A\vec{u}) = A(k\vec{u}).$$
Since $k\vec{u} \in \mathbb{R}^n$, it follows that $k\vec{x} \in \text{im}(A)$.

Therefore, $\text{im}(A)$ is a subspace of $\mathbb{R}^m$. $\square$

Theorem

Let A be an $m \times n$ matrix such that $\text{rank}(A) = r$. Then the system $A\vec{x} = \vec{0}_m$ has $n - r$ basic solutions, providing a basis of $\text{null}(A)$ with $\dim(\text{null}(A)) = n - r$.

Outline of Proof.

- We have already seen that $\text{null}(A)$ is spanned by any set of basic solutions to $A\vec{x} = \vec{0}_m$, so it is enough to prove that $\dim(\text{null}(A)) = n - r$.
- Suppose $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is a basis of $\text{null}(A)$ (show $k = n - r$).
- Extend $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ to a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k, \ldots \vec{x}_n\}$ of $\mathbb{R}^n$.
- Consider the set $\{A\vec{x}_1, A\vec{x}_2, \ldots, A\vec{x}_k, \ldots A\vec{x}_n\}$ (a subset of $\mathbb{R}^m$).
- Then $A\vec{x}_j = \vec{0}_m$ for $1 \leq j \leq k$ since $\vec{x}_1, \ldots, \vec{x}_k \in \text{null}(A)$.
- To complete the proof, show $S = \{A\vec{x}_{k+1}, \ldots A\vec{x}_n\}$ is a basis of $\text{im}(A)$, by showing S is independent, and that S spans $\text{im}(A)$.
- Since $\text{im}(A) = \text{col}(A)$, $\dim(\text{im}(A)) = r$, implying $n - k = r$. Therefore, $k = n - r$.

Example

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$

Find $null(A)$ *and* $im(A)$.

Solution

T find $\text{null}(A)$, we need solutions to:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 3 & 3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution is given by

$$t \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R}$$

Therefore,

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution continued

Finally $\mathrm{im}\,(A)$ is just $\{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$, that is $\mathrm{im}\,(A) = \mathrm{col}(A)$.

Notice from the above calculation that that the first two columns of the reduced row-echelon form are pivot columns. Therefore

$$\mathrm{im}\,(A) = \mathrm{col}(A) = \mathrm{span}\left\{ \left[\begin{array}{r} 1 \\ 0 \\ 2 \end{array}\right], \left[\begin{array}{r} 2 \\ -1 \\ 3 \end{array}\right] \right\}$$

Problem

Let A be a 5×6 matrix. Can the columns of A be independent? Can the rows of A be independent? Justify your answer.

Solution

The rank of the matrix is at most five; since there are six columns, the columns can not be independent. However, the rows could be independent: take a 5×6 matrix whose first five columns are the columns of the 5×5 identity matrix.

Problem

Let A be an 5×9 matrix. Is it possible that $\dim(\text{null}(A)) = 3$? Justify your answer.

Solution

As a consequence of the Rank Theorem, we have rank $(A) \leq 5$, so $\dim(\text{im}(A)) \leq 5$. Since $\dim(\text{null}(A)) = 9 - \dim(\text{im}(A))$, it follows that

$$\dim(\text{null}(A)) \geq 9 - 5 = 4.$$

Therefore, it is not possible that $\dim(\text{null}(A)) = 3$.