A First Course in

LINEAR ALGEBRA

Lecture Notes

for Math 1503

Determinants: Basic Techniques and Properties

A First Course in Linear Algebra

Lecture Slides

These lecture slides were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text A First Course in Linear Algebra based on K. Kuttler's original text.

In addition we recognize the following contributors. All new content contributed is released under the same license as noted below.

- **Tim Alderson, University of New Brunswick**
- **Ilijas Farah, York University**
- **Ken Kuttler, Brigham Young University**
- **Asia Weiss, York University**

Determinant of a 2×2 Matrix

Definition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the determinant of A is defined as

$$\det A = ad - bc$$

Notation. For $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we often write $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, i.e., use vertical bars instead of square brackets.

What about the determinant of an $n \times n$ matrix for other values of n?

How do we find the determinant of an $n \times n$ matrix?

The determinant of an $n \times n$ matrix is defined recursively, using determinants of $(n-1) \times (n-1)$ submatrices, and requires some new definitions and notation.

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The sign of the (i, j) position is $(-1)^{i+j}$. Thus the sign is 1 if $(i + j)$ is even, and -1 if $(i + j)$ is odd.

The Minor of a Matrix

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The ij^{th} minor of A, denoted as $minor\,(A)_{ij}$, is the determinant of the $n-1 \times n-1$ matrix which results from deleting the i^{th} row and the j^{th} column of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

For any matrix A, $minor\,(A)_{ij}$ is found by first removing the i^{th} row and j^{th} column, and taking the determinant of the remaining matrix.

The Minor of a Matrix

Example

Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

Find $minor(A)_{12}$.

Solution

First, remove the 1^{st} row and 2^{nd} column from A.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

The resulting matrix is $A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$

Solution (continued)

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$$

Using our previous definition, we can calculate the determinant of this matrix to be

$$(2)(6) - (5)(1) = 12 - 5 = 7$$

Therefore, $minor(A)_{12} = 7$.

The Cofactors of a Matrix

Definition

The ij^{th} cofactor of A is

$$\text{cof}(A)_{ij} = (-1)^{i+j} minor\,(A)_{ij}$$

Example (continued)

Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$$

Find $\text{cof}(A)_{12}$.

Solution

By the definition, we know that $\text{cof}(A)_{12} = (-1)^{1+2} minor\,(A)_{12}$

From earlier, we know that $minor\,(A)_{12} = 7$.

Therefore, $\text{cof}(A)_{12} = (-1)^{1+2} minor\,(A)_{12} = (-1)^3 7 = -7$

Cofactor Expansion

Using these definitions, we can now define the determinant of the $n \times n$ matrix A:

Definition

$\det A = a_{11}\text{cof}(A)_{11} + a_{12}\text{cof}(A)_{12} + a_{13}\text{cof}(A)_{13} + \cdots + a_{1n}\text{cof}(A)_{1n}$
This is called the cofactor expansion of $\det A$ along row 1.

In other words,

$$\det(A) = \sum_{j=1}^{n} a_{ij}\text{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij}\text{cof}(A)_{ij}$$

The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

Cofactor expansion is also called Laplace Expansion.

Cofactor Expansion

Example

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$. Find $\det A$.

Solution

Using cofactor expansion along row 1,

$$\begin{aligned}
\det A &= 1\mathrm{cof}_{11}(A) + 1\mathrm{cof}_{12}(A) + 3\mathrm{cof}_{13}(A) \\
&= 1(-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} \\
&= 1(24-2) - 1(12-5) + 3(4-20) \\
&= 22 - 7 + 3(-16) \\
&= 22 - 7 - 48 \\
&= -33
\end{aligned}$$

Solution (continued)

$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 2 & 6 \end{bmatrix}$

Now try cofactor expansion along column 2.

$$\begin{aligned}
\det A &= 1\mathrm{cof}_{12}(A) + 4\mathrm{cof}_{22}(A) + 2\mathrm{cof}_{32}(A) \\
&= 1(-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} + 4(-1)^4 \begin{vmatrix} 1 & 3 \\ 5 & 6 \end{vmatrix} + 2(-1)^5 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \\
&= -1(12-5) + 4(6-15) - 2(1-6) \\
&= -(7) + 4(-9) - 2(-5) \\
&= -7 - 36 + 10 \\
&= -33
\end{aligned}$$

We get the same answer!

The Determinant is Well Defined

Theorem

The determinant of an $n \times n$ matrix A can be computed using cofactor expansion along any row or column of A.

What is the significance of this theorem?

This theorem allows us to choose any row or column for computing cofactor expansion, which gives us the opportunity to save ourselves some work!

Problem

Let $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Find $\det A$.

Solution

Cofactor expansion along row 1 yields

$$\begin{aligned} \det A &= 0 \times \text{cof}(A)_{11} + 1 \times \text{cof}(A)_{12} + (-2) \times \text{cof}(A)_{13} + 1 \times \text{cof}(A)_{14} \\ &= \text{cof}(A)_{12} - 2 \times \text{cof}(A)_{13} + \text{cof}(A)_{14}, \end{aligned}$$

whereas cofactor expansion along, row 3 yields

$$\begin{aligned} \det A &= 0 \times \text{cof}(A)_{31} + 1 \times \text{cof}(A)_{32} + (-1) \times \text{cof}(A)_{33} + 0 \times \text{cof}(A)_{34} \\ &= 1\text{cof}(A)_{32} + (-1)\text{cof}(A)_{33}, \end{aligned}$$

i.e., in the first case we must compute three cofactors, but in the second case we need only compute two cofactors.

Solution (continued)

Therefore, we can save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1 \times \text{cof}(A)_{32} + (-1) \times \text{cof}(A)_{33} \\ &= 1 \times (-1)^5 \begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1) \times (-1)^6 \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\ &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \end{aligned}$$

Each of the two determinants above can easily be evaluated using cofactor expansion along column 2.

Solution (continued)

$$\begin{aligned} \det A &= -\begin{vmatrix} 0 & -2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\ &= -(-2)(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} - 1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\ &= -2(10-21) + 1(10-21) \\ &= -2(-11) + (-11) \\ &= 22 - 11 \\ &= 11. \end{aligned}$$

Therefore, $\det A = 11$.

A Row or Column of Zeros

Example

Let

$$A = \begin{bmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{bmatrix}.$$

By choosing column 3 for cofactor expansion, we get $\det A = 0$, i.e.,

$$\det A = 0 \times \mathrm{cof}(A)_{13} + 0 \times \mathrm{cof}(A)_{23} + 0 \times \mathrm{cof}(A)_{33} + 0 \times \mathrm{cof}(A)_{43} = 0.$$

Important Fact

If A is an $n \times n$ matrix with a row or column of zeros, then $\det A = 0$.

Determinants of a Triangular Matrices

Definitions

1. An $n \times n$ matrix A is called upper triangular if all entries **below** the main diagonal are zero.
2. An $n \times n$ matrix A is called lower triangular if all entries **above** the main diagonal are zero.
3. An $n \times n$ matrix A is called triangular if it is upper triangular or lower triangular.

Theorem

If $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then

$$\det A = a_{11} \times a_{22} \times a_{33} \times \cdots \times a_{nn},$$

i.e., $\det A$ is the product of the entries of the main diagonal of A.

Example

$$\det\begin{bmatrix}1 & 2 & 3\\ 0 & 5 & 6\\ 0 & 0 & 9\end{bmatrix} = 1\times\det\begin{bmatrix}5 & 6\\ 0 & 9\end{bmatrix}$$
$$= 1\times 5\times\det\begin{bmatrix}9\end{bmatrix}$$
$$= 1\times 5\times 9$$
$$= 45.$$

Notice that 45 is the product of the entries on the main diagonal.

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 5 & 6\\ 0 & 0 & 9\end{bmatrix}$$

Elementary Row Operations and Determinants

Example

Let

$$A = \begin{bmatrix}2 & 0 & -3\\ 0 & 4 & 0\\ 1 & 0 & -2\end{bmatrix}.$$

Computing $\det A$ by cofactor expansion along row (or column) 2 yields

$$\det A = 4(-1)^4\begin{vmatrix}2 & -3\\ 1 & -2\end{vmatrix} = 4(-1) = -4.$$

Let B_1, B_2, and B_3 be obtained from A by interchanging rows 1 and 2, multiplying row 3 by -3, and adding twice row 1 to row 3, respectively, i.e.,

$$B_1 = \begin{bmatrix}2 & 0 & -3\\ 1 & 0 & -2\\ 0 & 4 & 0\end{bmatrix}, B_2 = \begin{bmatrix}2 & 0 & -3\\ 0 & 4 & 0\\ -3 & 0 & 6\end{bmatrix}, B_3 = \begin{bmatrix}2 & 0 & -3\\ 0 & 4 & 0\\ 5 & 0 & -8\end{bmatrix}.$$

We are interested in how elementary operations affect the determinant. Computing $\det B_1$, $\det B_2$, and $\det B_3$:

Example (continued)

$$\det B_1 = 4(-1)^5 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = (-4)(-1) = 4 = (-1)\det A.$$

$$\det B_2 = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ -3 & 6 \end{vmatrix} = 4(12-9) = 4 \times 3 = 12 = -3\det A.$$

$$\det B_3 = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ 5 & -8 \end{vmatrix} = 4(-16+15) = 4(-1) = -4 = \det A.$$

The general effects of elementary row operations on the determinant are summarized in the next theorem.

Theorem

Let A be an $n \times n$ matrix and B be an $n \times n$ matrix as defined below.

1. Let B be a matrix which results from switching two rows of A. Then $\det(B) = -\det(A)$.
2. Let B be a matrix which results from multiplying some row of A by a scalar k. Then $\det(B) = k\det(A)$.
3. Let B be a matrix which results from adding a multiple of a row to another row. Then $\det(A) = \det(B)$.
4. If A contains a row which is a multiple of another row of A, then $\det(A) = 0$

An analogous theorem holds for elementary column operation. If A is a matrix, then an elementary column operation on A is simply the corresponding elementary row operation performed on the transpose of A, A^T.

Computing the Determinant

Example

$$
\begin{aligned}
\det\begin{bmatrix} -3 & 5 & -6 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix} &= \begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix} && \text{row } 1 + 3\times(\text{row } 2) \\
&= \begin{vmatrix} 0 & 2 & 3 \\ 1 & -1 & 3 \\ 0 & -2 & -5 \end{vmatrix} && \text{row } 3 - 2\times(\text{row } 2) \\
&= (1)(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ -2 & -5 \end{vmatrix} && \text{cofactor expansion: column } 1 \\
&= -(-10+6) \\
&= 4.
\end{aligned}
$$

Example

$$
\begin{aligned}
\det\begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} &= \begin{vmatrix} 7 & 5 & 10 & 0 \\ -1 & -3 & 8 & 0 \\ 6 & 4 & 15 & 0 \\ 1 & 1 & 2 & -1 \end{vmatrix} \\
&= (-1)(-1)^{8}\begin{vmatrix} 7 & 5 & 10 \\ -1 & -3 & 8 \\ 6 & 4 & 15 \end{vmatrix} \\
&= -\begin{vmatrix} 0 & -16 & 66 \\ -1 & -3 & 8 \\ 0 & -14 & 63 \end{vmatrix} \\
&= -(-1)(-1)^{3}\begin{vmatrix} -16 & 66 \\ -14 & 63 \end{vmatrix} \\
&= -\begin{vmatrix} -2 & 3 \\ -14 & 63 \end{vmatrix} \\
&= -(-126+42) \\
&= 84.
\end{aligned}
$$

Problem

If $\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 4$, find $\det \begin{bmatrix} -b_1 & -b_2 & -b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix}$.

Solution

$$\begin{aligned}
\begin{vmatrix} -b_1 & -b_2 & -b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{vmatrix} &= (-1)(3)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)(-1)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (-3)(-1) \times 4 \\
&= 12.
\end{aligned}$$

Problem

Let $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{bmatrix}$. Find $\det A$.

Solution

$$\det A = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 9 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 3 & 5 & 9 \\ 1 & 2 & 4 \end{vmatrix} = 0.$$

Notice:

$$\text{row2} + \text{row3} - 2(\text{row1}) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Hence the determinant equals 0.

Problem

Let

$$A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{bmatrix}.$$

Show that $\det B = 9 \det A$.

Solution

$$\det B = \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} \xrightarrow{R_1+(-2)R_3} \begin{vmatrix} p-4x & q-4y & r-4z \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$\xrightarrow{R_2+(-2)R_1} \begin{vmatrix} p-4x & q-4y & r-4z \\ 9x & 9y & 9z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} \xrightarrow{\frac{1}{9}R_2} 9 \begin{vmatrix} p-4x & q-4y & r-4z \\ x & y & z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$\xrightarrow{R_1+(4)R_2} 9 \begin{vmatrix} p & q & r \\ x & y & z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} \xrightarrow{R_3+(-2)R_2} 9 \begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix}$$

$$\xrightarrow{R_1<->R_3} -9 \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} \xrightarrow{R_2<->R_3} 9 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 9 \det A.$$

Determinants and Scalar Multiplication

Problem

Suppose A is a 3×3 matrix with $\det A = 7$. What is $\det(-2A)$?

Solution

Write $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then $-2A = \begin{bmatrix} -2a_{11} & -2a_{12} & -2a_{13} \\ -2a_{21} & -2a_{22} & -2a_{23} \\ -2a_{31} & -2a_{32} & -2a_{33} \end{bmatrix}$.

$$\det(-2A) = \begin{vmatrix} -2a_{11} & -2a_{12} & -2a_{13} \\ -2a_{21} & -2a_{22} & -2a_{23} \\ -2a_{31} & -2a_{32} & -2a_{33} \end{vmatrix} = (-2)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ -2a_{21} & -2a_{22} & -2a_{23} \\ -2a_{31} & -2a_{32} & -2a_{33} \end{vmatrix}$$

$$= (-2)(-2)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -2a_{31} & -2a_{32} & -2a_{33} \end{vmatrix} = (-2)(-2)(-2)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (-2)^3 \det A = (-8) \times 7 = -56.$$

Solution (continued)

Think about the matrix $-2A$ as the matrix obtained from A be multiplying each of its rows by -2. This involves three elementary row operations, each of which contributes a factor of -2 to the determinant, and thus $\det A = (-2) \times (-2) \times (-2) \times 7 = (-2)^3 \times 7.$

Theorem

If A is an $n \times n$ matrix and k is any scalar, then

$$\det(kA) = k^n \det A.$$

Determinants of Inverses

Theorem

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. In this case,

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Example (Illustration of the above Theorem.)

Let $A = \begin{bmatrix} 2 & -3 \\ -6 & 4 \end{bmatrix}$. Then $\det A = 2(4) - (-3)(-6) = -10$. Since $\det A \neq 0$, the above Theorem tell us that A is invertible, and that $\det(A^{-1})$ should be equal to $-\frac{1}{10}$.

We can verify this directly. Using the formula for the inverse of a 2×2 matrix

$$A^{-1} = \frac{1}{-10} \begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}.$$

Therefore,

$$\det A^{-1} = \left(-\frac{1}{10}\right)^2 \begin{vmatrix} 4 & 3 \\ 6 & 2 \end{vmatrix} = \left(-\frac{1}{10}\right)^2 (8 - 18) = \frac{1}{100} \times (-10) = -\frac{1}{10}.$$

Problem

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

Solution

$$\begin{aligned} \det A = \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix} &= c \begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 2 & c \end{vmatrix} \\ &= c(10 - c^2) - c \\ &= c(9 - c^2) \\ &= c(3 - c)(3 + c). \end{aligned}$$

Since A is invertible when $\det(A) \neq 0$, A is invertible for all $c \neq 0, 3, -3$.

Determinants of Products and Transposes

Theorem

Let A and B be $n \times n$ matrices. Then

$$\det(AB) = \det A \times \det B.$$

Theorem

If A is an $n \times n$ matrix, then the determinant of its transpose is given by

$$\det(A^T) = \det A.$$

Problem

Suppose A, B and C are 4×4 matrices with

$$\det A = -1, \det B = 2, \text{ and } \det C = 1.$$

Find $\det(2A^2(B^{-1})(C^T)^3B(A^{-1}))$.

Solution

$$\begin{aligned}
\det(2A^2(B^{-1})(C^T)^3B(A^{-1})) &= 2^4(\det A)^2\frac{1}{\det B}(\det C)^3(\det B)\frac{1}{\det A} \\
&= 16(\det A)(\det C)^3 \\
&= 16 \times (-1) \times 1^3 \\
&= -16.
\end{aligned}$$

Problem

Suppose A is a 3×3 matrix. Find $\det A$ and $\det B$ if
$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$

Solution

First,
$$\begin{aligned}
\det(2A^{-1}) &= -4 \\
2^3 \det(A^{-1}) &= -4 \\
\frac{1}{\det A} &= \frac{-4}{8} = -\frac{1}{2}.
\end{aligned}$$

Therefore, $\det A = -2$. Using this fact,
$$\begin{aligned}
\det(A^3(B^{-1})^T) &= -4 \\
(\det A)^3 \det(B^{-1}) &= -4 \\
(-2)^3 \det(B^{-1}) &= -4 \\
(-8) \det(B^{-1}) &= -4 \\
\frac{1}{\det B} &= \frac{-4}{-8} = \frac{1}{2}
\end{aligned}$$

Therefore, $\det B = 2$.

Problem

Let A be an $n \times n$ matrix. Find all conditions that ensure $\det(-A) = \det A$.

Solution

Since $\det(-A) = (-1)^n \det A$, $\det(-A) = \det A$ if and only if
$$(-1)^n \det A = \det A.$$

When is this possible?

1. $(-1)^n \det A = \det A$ whenever $\det A = 0$.
2. If $\det A \neq 0$, then $(-1)^n \det A = \det A$ only if $(-1)^n = 1$, i.e., only if n is even.

Therefore, $\det(-A) = \det A$ only if $\det A = 0$ or n is even.

Using Row Operations

Problem

Let

$$A = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 6 & 0 \end{bmatrix}$$

Find $\det(A)$.

Solution

We could solve this using cofactor expansion. However, we can also use row operations to simplify A first.
First, switch rows 1 and 2 to obtain matrix B.

$$B = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 1 & 2 \\ 2 & 6 & 0 \end{bmatrix}$$

Then, $\det(B) = -\det(A)$, which we can write as $\det(A) = -\det(B)$.

Solution (continued)

Now, multiply row 3 by $\frac{1}{2}$ to obtain matrix C.

$$C = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

Then, $\det(C) = \frac{1}{2}\det(B) = -\frac{1}{2}\det(A)$.
Now, subtract 5 times row 1 from row 2, and 1 times row 1 from row 3 to obtain matrix D.

$$D = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -8 \\ 0 & 0 & -2 \end{bmatrix}$$

Then, $\det(D) = \det(C) = -\frac{1}{2}\det(A)$. Hence, $\det(A) = -2\det(D)$.

Solution (continued)

Now we can use cofactor expansion to find $\det(D)$.

$$\det(D) = (1)(-1)^{1+1} \begin{vmatrix} -14 & -8 \\ 0 & -2 \end{vmatrix} = 28$$

Similarly, since D is triangular, we can find the determinant by multiplying the entries on the main diagonal.
Then

$$\det(A) = -2\det(D) = -2(28) = -56$$

The Cofactor Matrix

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The cofactor matrix of A, is the matrix

$$[\mathrm{cof}(A)_{ij}],$$

i.e., the matrix whose (i,j)-entry is the (i,j)-cofactor of A.

Reminder: the (i,j)-cofactor

$$\mathrm{cof}(A)_{ij} = (-1)^{i+j} \mathit{minor}\,(A)_{ij},$$

where $\mathit{minor}\,(A)_{ij}$ is the determinant of the matrix obtained from A by deleting row i and column j.

Problem

Find the cofactor matrix $[\text{cof}(A)_{ij}]$ of the matrix

$$A = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}.$$

Solution

For each i and j, $1 \leq i, j \leq 3$, we need to compute $\text{cof}(A)_{ij}$, so there are 9 cofactors to compute.

$$\begin{aligned}
\text{cof}(A)_{11} &= (-1)^{1+1} \det A_{11} = \begin{vmatrix} 9 & 7 \\ 6 & 4 \end{vmatrix} = 9 \times 4 - 6 \times 7 = 36 - 42 = -6. \\
\text{cof}(A)_{12} &= (-1)^{1+2} \det A_{12} = \begin{vmatrix} 1 & 7 \\ 0 & 4 \end{vmatrix} = -(4 - 0) = -4. \\
\text{cof}(A)_{13} &= (-1)^{1+3} \det A_{12} = \begin{vmatrix} 1 & 9 \\ 0 & 6 \end{vmatrix} = (6 - 0) = 6.
\end{aligned}$$

Solution (continued)

Computing the six remaining cofactors results in the cofactor matrix

$$\begin{bmatrix} -6 & -4 & 6 \\ 18 & 16 & -24 \\ -27 & -25 & 36 \end{bmatrix}.$$

The Adjugate

Definition

If A is an $n \times n$ matrix, then the adjugate of A is defined by

$$\text{adj } A = \left[\text{ cof}(A)_{ij} \text{ } \right]^T,$$

where $\text{cof}(A)_{ij}$ is the (i,j)-cofactor of A, i.e., *adj* A is the transpose of the cofactor matrix.

Example

$A = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}$, has cofactor matrix $\begin{bmatrix} -6 & -4 & 6 \\ 18 & 16 & -24 \\ -27 & -25 & 36 \end{bmatrix}$.

Therefore, the adjugate of A is

$$\text{adj } A = \begin{bmatrix} -6 & -4 & 6 \\ 18 & 16 & -24 \\ -27 & -25 & 36 \end{bmatrix}^T = \begin{bmatrix} -6 & 18 & -27 \\ -4 & 16 & -25 \\ 6 & -24 & 36 \end{bmatrix}.$$

Problem

Find adj A when $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

Solution

$$\text{adj } A = \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix}.$$

The Adjugate of a 2×2 Matrix

Example

Let A be a 2×2 matrix, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\begin{aligned} \text{adj } A = \begin{bmatrix} \text{cof}(A)_{11} & \text{cof}(A)_{12} \\ \text{cof}(A)_{21} & \text{cof}(A)_{22} \end{bmatrix}^T &= \begin{bmatrix} (-1)^2 d & (-1)^3 c \\ (-1)^3 b & (-1)^4 a \end{bmatrix}^T \\ &= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \end{aligned}$$

We've seen this matrix before: if $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A} \text{adj } A.$$

Example (continued)

Observe that, regardless of the value of $\det A$,

$$\begin{aligned} A(\text{adj } A) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= (\det A) I_2. \end{aligned}$$

Example

In an earlier example, we saw that $A = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix}$, has adjugate $\text{adj } A = \begin{bmatrix} -6 & 18 & -27 \\ -4 & 16 & -25 \\ 6 & -24 & 36 \end{bmatrix}$. Computing $A(\text{adj } A)$ we see that

$$A(\text{adj } A) = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{bmatrix} \begin{bmatrix} -6 & 18 & -27 \\ -4 & 16 & -25 \\ 6 & -24 & 36 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}.$$

Note that

$$\det A = \begin{vmatrix} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -36 & -25 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{vmatrix} = -\begin{vmatrix} -36 & -25 \\ 6 & 4 \end{vmatrix} = -6.$$

Therefore we have $A(\text{adj } A) = (\det A)I$.

The Adjugate Formula

Theorem

If A is an $n \times n$ matrix, then

$$A(\text{adj } A) = (\det A)I = (\text{adj}A)A.$$

Furthermore, if $\det A \neq 0$, then we get a formula for A^{-1}, i.e.,

$$A^{-1} = \frac{1}{\det A}\text{adj } A.$$

Inverting a matrix using the adjugate

Except in the case of a 2×2 matrix, the adjugate formula is a very inefficient method for computing the inverse of a matrix; the matrix inversion algorithm is much more practical. However, the adjugate formula is of theoretical significance.

Proof of the Adjugate Formula

Example

Recall that the (i,j)-entry of $\mathrm{adj}(A)$ is equal to $\mathrm{cof}(A)_{ji}$. Let us compute the (i,j)-entry of $B = A \cdot \mathrm{adj}(A)$:

$$b_{ij} = \sum_{k=1}^{n} a_{ik}\mathrm{cof}(A)_{ki}$$

By the cofactor expansion theorem, b_{ij} is equal to the determinant of matrix C obtained from A by replacing its jth column by $a_{i1}, a_{i2}, \ldots a_{in}$ — i.e., its ith column.

If $i = j$ then this matrix is A and therefore

$$a_{ii} = \det A$$

for all i. If $i \neq j$ then this matrix has its ith column equal to its jth column, and therefore

$$a_{ij} = 0 \qquad \text{if } i \neq j.$$

Using the Adjugate to Find the Inverse of a Matrix

Example

Let $A = \left[\begin{array}{rrr} 4 & 0 & 3 \\ 1 & 9 & 7 \\ 0 & 6 & 4 \end{array}\right]$. As we saw earlier, $\det A = -6 \neq 0$, so A is invertible, and

$$\mathrm{adj}\ A = \left[\begin{array}{rrr} -6 & 18 & -27 \\ -4 & 16 & -25 \\ 6 & -24 & 36 \end{array}\right].$$

$$A^{-1} = \frac{1}{\det A}\mathrm{adj}\ A = \frac{1}{-6}\left[\begin{array}{rrr} -6 & 18 & -27 \\ -4 & 16 & -25 \\ 6 & -24 & 36 \end{array}\right] = \left[\begin{array}{rrr} 1 & -3 & \frac{9}{2} \\ \frac{2}{3} & -\frac{8}{3} & \frac{25}{6} \\ -1 & 4 & -6 \end{array}\right].$$

You can check this by computing AA^{-1}. You could also check by using the Matrix Inversion Algorithm to find A^{-1} (though this is more work).

Problem

Let A be an $n \times n$ invertible matrix. Show that $\det(\text{adj } A) = (\det A)^{n-1}$.

Solution

Using the adjugate formula,

$$\begin{aligned} A(\text{adj } A) &= (\det A)I \\ \det(A(\text{adj } A)) &= \det((\det A)I) \\ (\det A) \times \det(\text{adj } A) &= (\det A)^n(\det I) \\ (\det A) \times \det(\text{adj } A) &= (\det A)^n \end{aligned}$$

Since A is invertible, $\det A \neq 0$, so we divide both sides of the last equation by $\det A$ to obtain

$$\det(\text{adj } A) = (\det A)^{n-1}.$$

Even if A is not invertible, $\det(\text{adj } A) = (\det A)^{n-1}$, but the proof is more complicated.

Cramer's Rule

If A is an $n \times n$ invertible matrix, then the solution to $AX = B$ can be given in terms of determinants of matrices.

Theorem (Cramer's Rule)

Let A be an $n \times n$ invertible matrix, and consider the system $AX = B$, where $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$. We define A_i to be the matrix obtained from A by replacing column i with B. Then for each value of i, $1 \leq i \leq n$,

$$x_i = \frac{\det A_i}{\det A}$$

Example (Cramer's Rule)

Solve the following system of linear equations using Cramer's Rule.

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & x_3 & = & -1 \\ 5x_1 & + & 2x_2 & & & = & 2 \\ x_1 & + & x_2 & - & x_3 & = & 1 \end{array}$$

First, $x_1 = \frac{\det A_1}{\det A}$, where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

Computing the determinants of these two matrices,

$$\det A = -4 \text{ and } \det A_1 = 4,$$

and thus $x_1 = \frac{4}{-4} = -1$.

Example (continued)

Secondly, $x_2 = \frac{\det A_2}{\det A}$ where $\det A = -4$ and

$$\det A_2 = \begin{vmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -14,$$

and thus $x_2 = \frac{-14}{4} = \frac{7}{2}$. Finally, $x_3 = \frac{\det A_3}{\det A}$, where $\det A = -4$ and

$$\det A_3 = \begin{vmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -6,$$

and thus $x_3 = \frac{-6}{-4} = \frac{3}{2}$. Therefore, the solution to the system is given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{7}{2} \\ \frac{3}{2} \end{bmatrix}.$$

You can check this by substituting these values into the original system.

Polynomial Interpolation

Problem

Given data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of y corresponding to $x = \frac{3}{2}$.

Solution

We want to find the coefficients r_0, r_1, r_2 and r_3 of

$$p(x) = r_0 + r_1x + r_2x^2 + r_3x^3$$

so that $p(0) = 1$, $p(1) = 2$, $p(2) = 5$, and $p(3) = 10$.

$$\begin{aligned} p(0) &= r_0 = 1 \\ p(1) &= r_0 + r_1 + r_2 + r_3 = 2 \\ p(2) &= r_0 + 2r_1 + 4r_2 + 8r_3 = 5 \\ p(3) &= r_0 + 3r_1 + 9r_2 + 27r_3 = 10 \end{aligned}$$

Solution (continued)

Solve this system of four equations in the four variables r_0, r_1, r_2 and r_3.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & 10 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore $r_0 = 1$, $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, and so

$$p(x) = 1 + x^2.$$

The estimate for the value of y corresponding to $x = \frac{3}{2}$ is

$$y = p\left(\frac{3}{2}\right) = 1 + \left(\frac{3}{2}\right)^2 = \frac{13}{4},$$

resulting in the point $(\frac{3}{2}, \frac{13}{4})$.

Theorem

Given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial $p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$ such that $p(x_i) = y_i$ for $i = 1, 2, \ldots, n$. The polynomial $p(x)$ is called the interpolating polynomial for the data points.

To find $p(x)$, set up a system of n linear equations in the n variables $r_0, r_1, r_2, \ldots, r_{n-1}$.

$$\begin{array}{ccc} r_0 + r_1 x_1 + r_2 x_1^2 + \cdots + r_{n-1} x_1^{n-1} & = & y_1 \\ r_0 + r_1 x_2 + r_2 x_2^2 + \cdots + r_{n-1} x_2^{n-1} & = & y_2 \\ \vdots & \vdots & \vdots \\ r_0 + r_1 x_n + r_2 x_n^2 + \cdots + r_{n-1} x_n^{n-1} & = & y_n \end{array}$$

The fact that the x_i are distinct guarantees that the coefficient matrix

$$\left[\begin{array}{ccccc} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{array}\right]$$

has determinant not equal to zero, and so the system has a unique solution, i.e., there is a unique interpolating polynomial for the data points.