CALCULUS
Early Transcendentals

An Open Text

by David Guichard, Gregory Hartman, Et al.

Adapted for

Math1003, Math 1013 & Math 2513

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at

University of New Brunswick Saint John

Calculus – Early Transcendentals

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Version 2016 — Revision A

Original text: Originally written by David Guichard of Whitman College, the single variable material is a modification and expansion of notes written and generously released by Neal Koblitz of the University of Washington. That version also contains some content, exercises, and examples from *Elementary Calculus: An Approach Using Infinitesimals*, written by H. Jerome Keisler of the University of Wisconsin under a Creative Commons license. Albert Schueller, Barry Balof, and Mike Wills all of Whitman College, have also contributed content.

Edits 2012-2014: The majority of the text was modified and expanded by Michael Cavers of the University of Calgary.

Edits 2014: Mark Blenkinsop of Carleton University further augmented and edited the content; in particular, the addition of a section on Linear and Higher Order Approximations.

Edits 2015: The text has been additionally augmented and edited by Joseph Ling of the University of Calgary; specifically, new exercises, proofs for theorems, revised arrangement of topics in the chapter on Applications of Derivatives, as well as additional explanation of content and examples in the section on Continuity.

Edits 2016: The majority of the text was modified and expanded by Tim Alderson of the University of New Brunswick Saint John. Augmented most sections of the text, adding new problems and answers. A significant amount of material was included from Apex Calculus, by Gregory Hartman, (http://www.apexcalculus.com/).
Other material was adapted from Active Calculus, by Matt Boelkins, and from the FreeCalc Project.

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Introduction and Review

The emphasis in this course is on problems—doing calculations and story problems. To master problem solving one needs a tremendous amount of practice doing problems. The more problems you do the better you will be at doing them, as patterns will start to emerge in both the problems and in successful approaches to them. You will learn quickly and effectively if you devote some time to doing problems every day.

Typically the most difficult problems are story problems, since they require some effort before you can begin calculating. Here are some pointers for doing story problems:

1. Carefully read each problem twice before writing anything.
2. Assign letters to quantities that are described only in words; draw a diagram if appropriate.
3. Decide which letters are constants and which are variables. A letter stands for a constant if its value remains the same throughout the problem.
4. Using mathematical notation, write down what you know and then write down what you want to find.
5. Decide what category of problem it is (this might be obvious if the problem comes at the end of a particular chapter, but will not necessarily be so obvious if it comes on an exam covering several chapters).
6. Double check each step as you go along; don't wait until the end to check your work.
7. Use common sense; if an answer is out of the range of practical possibilities, then check your work to see where you went wrong.

1. Review

Success in calculus depends on your background in algebra, trigonometry, analytic geometry and functions. In this chapter, we review many of the concepts you will need to know to succeed in this course.

1.1 Algebra

1.1.1. Sets and Number Systems

A **set** can be thought of as any collection of *distinct* objects considered as a whole. Typically, sets are represented using **set-builder notation** and are surrounded by braces. Recall that $(,)$ are called **parentheses** or **round brackets**; $[,]$ are called **square brackets**; and $\{,\}$ are called **braces** or **curly brackets**.

Example 1.1: Sets

The collection $\{a, b, 1, 2\}$ *is a set. It consists of the collection of four distinct objects, namely,* a*,* b*,* 1 *and* 2*.*

Let S be any set. We use the notation $\mathbf{x} \in \mathbf{S}$ to mean that x is an element *inside* of the set S, and the notation $\mathbf{x} \notin \mathbf{S}$ to mean that x is *not* an element of the set S.

Example 1.2: Set Membership

If $S = \{a, b, c\}$*, then* $a \in S$ *but* $d \notin S$*.*

The **intersection** between two sets S and T is denoted by $S \cap T$ and is the collection of all elements that belong to *both* S and T. The **union** between two sets S and T is denoted by $S \cup T$ and is the collection of all elements that belong to *either* S or T (or both).

Example 1.3: Union and Intersection

Let $S = \{a, b, c\}$ and $T = \{b, d\}$. Then $S \cap T = \{b\}$ and $S \cup T = \{a, b, c, d\}$. Note that we do not write the element b twice in $S \cup T$ even though b is in both S and T.

Numbers can be classified into sets called **number systems**.

$\mathbb{N}$	the **natural** numbers	$\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	the **integers**	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$\mathbb{Q}$	the **rational** numbers	Ratios of integers: $\left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$
$\mathbb{R}$	the **real** numbers	Can be written using a finite or infinite *decimal expansion*
$\mathbb{C}$	the **complex** numbers	These allow us to solve equations such as $x^2 + 1 = 0$

In the table, the set of rational numbers is written using set-builder notation. The colon, :, used in this manner means *such that*. Often times, a vertical bar | may also be used to mean *such that*. The expression $\left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$ can be read out loud as *the set of all fractions p over q such that p and q are both integers and q is not equal to zero.*

Example 1.4: Rational Numbers

The numbers $-\frac{3}{4}$, 2.647, 17, $0.\bar{7}$ *are all rational numbers. You can think of rational numbers as fractions of one integer over another. Note that 2.647 can be written as a fraction:*

$$2.647 = 2.647 \times \frac{1000}{1000} = \frac{2647}{1000}.$$

Also note that in the expression $0.\bar{7}$*, the bar over the* 7 *indicates that the* 7 *is repeated forever:*

$$0.77777777\ldots = \frac{7}{9}.$$

All rational numbers are real numbers with the property that their decimal expansion either *terminates* after a finite number of digits or begins to *repeat* the same finite sequence of digits over and over. Real numbers that are not rational are called **irrational**.

Example 1.5: Irrational Numbers

Some of the most common irrational numbers include:

- $\sqrt{2}$. *Can you prove this is irrational? (The proof uses a technique called contradiction.)*
- π. *Recall that π (**pi**) is defined as the ratio of the circumference of a circle to its*

diameter and can be approximated by 3.14159265.

- *e. Sometimes called Euler's number, e can be approximated by* 2.718281828459. *We will review the definition of e in a later chapter.*

Let S and T be two sets. If every element of S is also an element of T, then we say S is a **subset** of T and write $S \subseteq T$. Furthermore, if S is a subset of T but not equal to T, we often write $S \subset T$. The five sets of numbers in the table give an increasing sequence of sets:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

That is, all natural numbers are also integers, all integers are also rational numbers, all rational numbers are also real numbers, and all real numbers are also complex numbers.

1.1.2. Law of Exponents

The Law of Exponents is a set of rules for simplifying expressions that governs the combination of exponents (powers). Recall that $\sqrt[n]{\ }$ denotes the nth root. For example $\sqrt[3]{8} = 2$ represents that the cube root of 8 is equal to 2.

Definition 1.1: Law of Exponents

Definitions

If m, n are positive integers, then:

1. $x^n = x \cdot x \cdot \ldots \cdot x$ *(n times).*
2. $x^0 = 1$, *for* $x \neq 0$.
3. $x^{-n} = \dfrac{1}{x^n}$, *for* $x \neq 0$.
4. $x^{m/n} = \sqrt[n]{x^m}$ or $\left(\sqrt[n]{x}\right)^m$, *for* $x \geq 0$.

Combining

1. $x^a x^b = x^{a+b}$.
2. $\dfrac{x^a}{x^b} = x^{a-b}$, *for* $x \neq 0$.
3. $\left(x^a\right)^b = x^{ab} = x^{ba} = \left(x^b\right)^a$.

Distributing

1. $(xy)^a = x^a y^a$, *for* $x \geq 0$, $y \geq 0$.
2. $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$, *for* $x \geq 0$, $y > 0$.

In the next example, the word *simplify* means *to make simpler* or to write the expression more compactly.

Example 1.6: Laws of Exponents

Simplify the following expression as much as possible assuming $x, y > 0$:

$$\frac{3x^{-2}y^3x}{y^2\sqrt{x}}.$$

Solution. Using the Law of Exponents, we have:

$$\begin{aligned}
\frac{3x^{-2}y^3x}{y^2\sqrt{x}} &= \frac{3x^{-2}y^3x}{y^2x^{\frac{1}{2}}}, \quad \text{since } \sqrt{x} = x^{\frac{1}{2}}, \\
&= \frac{3x^{-2}yx}{x^{\frac{1}{2}}}, \quad \text{since } \frac{y^3}{y^2} = y, \\
&= \frac{3y}{x^{\frac{3}{2}}}, \quad \text{since } \frac{x^{-2}x}{x^{\frac{1}{2}}} = \frac{x^{-1}}{x^{\frac{1}{2}}} = x^{-\frac{3}{2}} = \frac{1}{x^{\frac{3}{2}}}, \\
&= \frac{3y}{\sqrt{x^3}}, \quad \text{since } x^{\frac{3}{2}} = \sqrt{x^3}.
\end{aligned}$$

An answer of $3yx^{-3/2}$ is equally acceptable, and such an expression may prove to be computationally simpler, although a positive exponent may be preferred. ♣

1.1.3. The Quadratic Formula and Completing the Square

The technique of **completing the square** allows us to solve quadratic equations and also to determine the center of a circle/ellipse or the vertex of a parabola.

The main idea behind completing the square is to turn:

$$ax^2 + bx + c$$

into

$$a(x-h)^2 + k.$$

One way to complete the square is to use the following formula:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + c.$$

But this formula is a bit complicated, so some students prefer following the steps outlined in the next example.

Example 1.7: Completing the Square

Solve $2x^2 + 12x - 32 = 0$ by completing the square.

Solution. In this instance, we will *not* divide by 2 first (usually you would) in order to demonstrate what you should do when the 'a' value is not 1.

$2x^2 + 12x - 32 = 0$	Start with original equation.
$2x^2 + 12x = 32$	Move the number over to the other side.
$2(x^2 + 6x) = 32$	Factor out the a from the $ax^2 + bx$ expression.
$6 \quad \rightarrow \quad \frac{6}{2} = 3 \quad \rightarrow \quad 3^2 = 9$	Take the number in front of x, **divide by** 2, then **square** it.
$2(x^2 + 6x + 9) = 32 + 2 \cdot 9$	Add the result to both sides, taking $a = 2$ into account.
$2(x+3)^2 = 50$	Factor the resulting perfect square trinomial.
	You have now completed the square!
$(x+3)^2 = 25 \quad \rightarrow \quad x = 2 \text{ or } x = -8$	To solve for x, simply divide by $a = 2$ and take square roots.

♣

Suppose we want to solve for x in the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$. The solution(s) to this equation are given by the **quadratic formula**.

Key Idea 1.1.0: The Quadratic Formula

The solutions to $ax^2 + bx + c = 0$ (with $a \neq 0$) are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Proof. To prove the quadratic formula we use the technique of *completing the square*. The general technique involves taking an expression of the form $x^2 + rx$ and trying to find a number we can add so that we end up with a perfect square (that is, $(x+n)^2$). It turns out if you add $(r/2)^2$ then you can factor it as a perfect square.

For example, suppose we want to solve for x in the equation $ax^2 + bx + c = 0$, where $a \neq 0$. Then we can move c to the other side and divide by a (remember, $a \neq 0$ so we can divide by it) to get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

To write the left side as a perfect square we use what was mentioned previously. We have $r = (b/a)$ in this case, so we must add $(r/2)^2 = (b/2a)^2$ to both sides

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2.$$

We know that the left side can be factored as a perfect square

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2.$$

The right side simplifies by using the exponent rules and finding a common denominator

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}.$$

Taking the square root we get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{-4ac + b^2}{4a^2}},$$

which can be rearranged as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In essence, the quadratic formula is just completing the square. ♣

1.1.4. Inequalities, Intervals and Solving Basic Inequalities

Inequality Notation

Recall that we use the symbols $<, >, \leq, \geq$ when writing an inequality. In particular,

- $a < b$ means a is to the *left* of b (that is, a is *strictly less* than b),
- $a \leq b$ means a is to the *left of or the same* as b (that is, a is *less than or equal* to b),
- $a > b$ means a is to the *right* of b (that is, a is *strictly greater* than b),
- $a \geq b$ means a is to the *right of or the same* as b (that is, a is *greater than or equal* to b).

To keep track of the difference between the symbols, some students use the following mnemonic.

Key Idea 1.1.0: Mnemonic

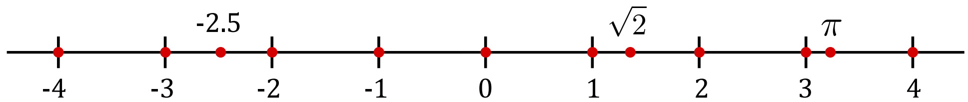
The $<$ symbol looks like a slanted L which stands for "*L*ess than".

Example 1.8: Inequalities

The following expressions are true:

$$1 < 2, \quad -5 < -2, \quad 1 \leq 2, \quad 1 \leq 1, \quad 4 \geq \pi > 3, \quad 7.23 \geq -7.23.$$

The real numbers are ordered and are often illustrated using the **real number line**:



Intervals

Assume a, b are real numbers with $a < b$ (i.e., a is strictly less than b). An **interval** is a set of every real number between two indicated numbers and may or may not contain the two numbers themselves. When describing intervals we use both round brackets and square brackets.

(1) Use of round brackets in intervals: (,). The notation **(a,b)** is what we call the **open interval from a to b** and consists of all the numbers between a and b, but does *not* include a or b. Using set-builder notation we write this as:

$$(a, b) = \{x \in \mathbb{R} \, : \, a < x < b\}.$$

We read $\{x \in \mathbb{R} \, : \, a < x < b\}$ as "the set of real numbers x such that x is greater than a and less than b" On the real number line we represent this with the following diagram:

Note that the circles on a and b are not shaded in, we call these **open circles** and use them to denote that a,b are *omitted* from the set.

(2) Use of square brackets in intervals: [,]. The notation **[a,b]** is what we call the **closed interval from a to b** and consists of all the numbers between a and b and *including* a and b. Using set-builder notation we write this as

$$[a, b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}.$$

On the real number line we represent this with the following diagram:

Note that the circles on a and b are shaded in, we call these **closed circles** and use them to denote that a and b are *included* in the set.

To keep track of when to shade a circle in, you may find the following mnemonic useful:

Key Idea 1.1.0: Mnemonic

The round brackets (,) and non-shaded circle both form an "O" shape which stands for "Open and Omit".

Taking combinations of round and square brackets, we can write different possible types of intervals (we assume $a < b$):

$(a, b) = \{x \in \mathbb{R} : a < x < b\}$ 	$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ 	$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ 
$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ 	$(a, \infty) = \{x \in \mathbb{R} : x > a\}$ 	$[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$ 
$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$ 	$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$ 	$(-\infty, \infty) = \mathbb{R} =$ all real numbers

Note: Any set which is bound at positive and/or negative infinity is an open interval.

Inequality Rules

Before solving inequalities, we start with the properties and rules of inequalities.

Key Idea 1.1.0: Inequality Rules

Add/subtract a number to both sides:

- If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

Adding two inequalities of the same type:

- If $a < b$ and $c < d$, then $a + c < b + d$.
  Add the left sides together, add the right sides together.

Multiplying by a positive number:

- Let $c > 0$. If $a < b$, then $c \cdot a < c \cdot b$.

Multiplying by a negative number:

- Let $c < 0$. If $a < b$, then $c \cdot a > c \cdot b$.
  Note that we reversed the inequality symbol!

Similar rules hold for each of $\leq$, $>$ and $\geq$.

Solving Basic Inequalities

We can use the inequality rules to solve some simple inequalities.

Example 1.9: Basic Inequality

Find all values of x satisfying

$$3x + 1 > 2x - 3.$$

Write your answer in both interval and set-builder notation. Finally, draw a number line indicating your solution set.

Solution. Subtracting $2x$ from both sides gives $x + 1 > -3$. Subtracting 1 from both sides gives $x > -4$. Therefore, the solution is the interval $(-4, \infty)$. In set-builder notation the solution may be written as $\{x \in \mathbb{R} : x > -4\}$. We illustrate the solution on the number line as follows:

♣

Sometimes we need to split our inequality into two cases as the next example demonstrates.

Example 1.10: Double Inequalities

Solve the inequality

$$4 > 3x - 2 \geq 2x - 1.$$

Solution. We need both $4 > 3x - 2$ and $3x - 2 \geq 2x - 1$ to be true:

$$\begin{array}{ccc} 4 > 3x - 2 & \textit{and} & 3x - 2 \geq 2x - 1, \\ 6 > 3x & \textit{and} & x - 2 \geq -1, \\ 2 > x & \textit{and} & x \geq 1, \\ x < 2 & \textit{and} & x \geq 1. \end{array}$$

Thus, we require $x \geq 1$ but also $x < 2$ to be true. This gives all the numbers between 1 and 2, including 1 but not including 2. That is, the solution to the inequality $4 > 3x - 2 \geq 2x - 1$ is the interval $[1, 2)$. In set-builder notation this is the set $\{x \in \mathbb{R} : 1 \leq x < 2\}$. ♣

Example 1.11: Positive Inequality

Solve $4x - x^2 > 0$.

Solution. We provide two methods to solve this inequality.

First method. Factor $4x - x^2$ as $x(4-x)$. The product of two numbers is positive when either both are positive or both are negative, i.e., if either $x > 0$ and $4-x > 0$, or else $x < 0$ and $4-x < 0$. The latter alternative is impossible, since if x is negative, then $4-x$ is greater than 4, and so cannot be negative. As for the first alternative, the condition $4-x > 0$ can be rewritten (adding x to both sides) as $4 > x$, so we need: $x > 0$ and $4 > x$ (this is sometimes combined in the form $4 > x > 0$, or, equivalently, $0 < x < 4$). In interval notation, this says that the solution is the interval $(0, 4)$.

Second method. Write $4x - x^2$ as $-(x^2 - 4x)$, and then complete the square, obtaining

$$-\Big((x-2)^2 - 4\Big) = 4 - (x-2)^2.$$

For this to be positive we need $(x-2)^2 < 4$, which means that $x-2$ must be less than 2 and greater than -2: $-2 < x - 2 < 2$. Adding 2 to everything gives $0 < x < 4$.

Both of these methods are equally correct; you may use either in a problem of this type. ♣

We next present another method to solve more complicated looking inequalities. In the next example we will solve a rational inequality by using a number line and test points. We follow the guidelines below.

Key Idea 1.1.0: Guidelines for Solving Rational Inequalities

1. Move everything to *one side* to get a 0 on the other side.
2. If needed, combine terms using a *common denominator*.
3. *Factor* the numerator and denominator.
4. Identify points where either the numerator or denominator is 0. Such points are called **split points**.
5. Draw a *number line* and indicate your split points on the number line. Draw *closed/open circles* for each split point depending on if that split point satisfies the inequality (division by zero is not allowed).
6. The split points will split the number line into subintervals. For each subinterval pick a *test point* and see if the expression in Step 3 is positive or negative. Indicate this with a $+$ or $-$ symbol on the number line for that subinterval.
7. Now write your answer in set-builder notation. Use the union symbol $\cup$ if you have multiple intervals in your solution.

Example 1.12: Rational Inequality

Write the solution to the following inequality using interval notation:

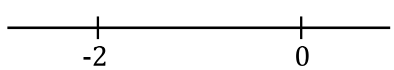
$$\frac{2-x}{2+x} \geq 1.$$

Solution. One method to solve this inequality is to multiply both sides by $2+x$, but because we do not know if $2+x$ is positive or negative we must split it into two cases (*Case 1:* $2+x>0$ and *Case 2:* $2+x<0$).

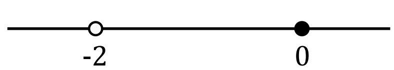
Instead we follow the guidelines for solving rational inequalities:

$$\begin{aligned}
\text{Start with original problem:} \quad & \frac{2-x}{2+x} \geq 1 \\
\text{Move everything to one side:} \quad & \frac{2-x}{2+x} - 1 \geq 0 \\
\text{Find a common denominator:} \quad & \frac{2-x}{2+x} - \frac{2+x}{2+x} \geq 0 \\
\text{Combine fractions:} \quad & \frac{(2-x)-(2+x)}{2+x} \geq 0 \\
\text{Expand numerator:} \quad & \frac{2-x-2-x}{2+x} \geq 0 \\
\text{Simplify numerator:} \quad & \frac{-2x}{2+x} \geq 0 \quad (*)
\end{aligned}$$

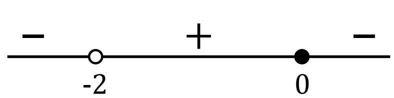
Now we have the numerator and denominator in fully factored form. The split points are $x=0$ (makes the numerator 0) and $x=-2$ (makes the denominator 0). Let us draw a number line with the split points indicated on it:



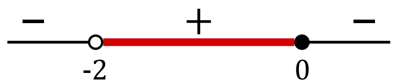
The point $x=0$ is included since if we sub $x=0$ into (*) we get $0 \geq 0$ which is true. The point $x=-2$ is not included since we cannot divide by zero. We indicate this with open/closed circles on the number line (remember that open means omit):



Now choosing a test point from each of the three subintervals we can determine if the expression $\frac{-2x}{2+x}$ is positive or negative. When $x=-3$, it is negative. When $x=-1$, it is positive. When $x=1$, it is negative. Indicating this on the number line gives:



Since we wish to solve $\frac{-2x}{2+x} \geq 0$, we look at where the $+$ signs are and shade that area on the number line:



Since there is a closed circle at 0, we include it. Therefore, the solution is $(-2, 0]$. ♣

Example 1.13: Rational Inequality

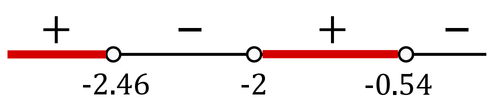
Write the solution to the following inequality using interval notation:

$$\frac{2}{x+2} > 3x+3.$$

Solution. We provide a brief outline of the solution. By subtracting $(3x+3)$ from both sides and using a common denominator of $x+2$, we can collect like terms and simplify to get:

$$\frac{-(3x^2+9x+4)}{x+2} > 0.$$

The denominator is zero when $x = -2$. Using the quadratic formula, the numerator is zero when $x = \frac{-9\pm\sqrt{33}}{6}$ (these two numbers are approximately -2.46 and -0.54). Since the inequality uses "$>$" and $0 > 0$ is false, we do not include any of the split points in our solution. After choosing suitable test points and determining the sign of $\frac{-(3x^2+9x+4)}{x+2}$ we have



Looking where the $+$ symbols are located gives the solution:

$$\left(-\infty, \frac{-9-\sqrt{33}}{6}\right) \bigcup \left(-2, \frac{-9+\sqrt{33}}{6}\right).$$

When writing the final answer we use *exact* expressions for numbers in mathematics, not approximations (unless stated otherwise). ♣

1.1.5. The Absolute Value

The **absolute value** of a number x is written as $|x|$ and represents the *distance* x is from zero. Mathematically, we define it as follows:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Thus, if x is a negative real number, then $-x$ is a positive real number. The absolute value does *not* just turn minuses into pluses. That is, $|2x-1| \neq 2x+1$. You should be familiar with the following properties.

Key Idea 1.1.0: Absolute Value Properties

1. $|x| \geq 0$.
2. $|xy| = |x||y|$.
3. $|1/x| = 1/|x|$ when $x \neq 0$.
4. $|-x| = |x|$.
5. $|x + y| \leq |x| + |y|$. This is called the **triangle inequality**.
6. $\sqrt{x^2} = |x|$.

Example 1.14: $\sqrt{x^2} = |x|$

Observe that $\sqrt{(-3)^2}$ gives an answer of 3, not -3.

When solving inequalities with absolute values, the following are helpful.

Case 1: $a > 0$.

- $|x| = a$ has solutions $x = \pm a$.
- $|x| \leq a$ means $x \geq -a$ **and** $x \leq a$ (that is, $-a \leq x \leq a$).
- $|x| < a$ means $x < -a$ **and** $x < a$ (that is, $-a < x < a$).
- $|x| \geq a$ means $x \leq -a$ **or** $x \geq a$.
- $|x| > a$ means $x < -a$ **or** $x > a$.

Case 2: $a < 0$.

- $|x| = a$ has no solutions.
- Both $|x| \leq a$ and $|x| < a$ have no solutions.
- Both $|x| \geq a$ and $|x| > a$ have solution set $\{x | x \in \mathbb{R}\}$.

Case 3: $a = 0$.

- $|x| = 0$ has solution $x = 0$.
- $|x| < 0$ has no solutions.
- $|x| \leq 0$ has solution $x = 0$.
- $|x| > 0$ has solution set $\{x \in \mathbb{R} | x \neq 0\}$.
- $|x| \geq 0$ has solution set $\{x | x \in \mathbb{R}\}$.

1.1.6. Solving Inequalities that Contain Absolute Values

We start by solving an equality that contains an absolute value. To do so, we recall that if $a \geq 0$ then the solution to $|x| = a$ is $x = \pm a$. In cases where we are not sure if the right side is positive or negative, we must perform a check at the end.

Example 1.15: Absolute Value Equality

Solve for x in $|2x+3| = 2-x$.

Solution. This means that either:

$$\begin{array}{ccc} 2x+3=+(2-x) & or & 2x+3=-(2-x) \\ 2x+3=2-x & or & 2x+3=-2+x \\ 3x=-1 & or & x=-5 \\ x=-1/3 & or & x=-5 \end{array}$$

Since we do not know if the right side "$a = 2-x$" is positive or negative, we must perform a check of our answers omit any that are incorrect.

If $x = -1/3$, then we have $LS = |2(-1/3)+3| = |-2/3+3| = |7/3| = 7/3$ and $RS = 2-(-1/3) = 7/3$. In this case $LS = RS$, so $x = -1/3$ is a solution.

If $x = -5$, then we have $LS = |2(-5)+3| = |-10+3| = |-7| = 7$ and $RS = 2-(-5) = 2+5 = 7$. In this case $LS = RS$, so $x = -5$ is a solution. ♣

We next look at absolute values and inequalities.

Example 1.16: Absolute Value Inequality

Solve $|x-5| < 7$.

Solution. This simply means $-7 < x-5 < 7$. Adding 5 to each gives $-2 < x < 12$. Therefore the solution is the interval $(-2, 12)$. ♣

In some questions you must be careful when multiplying by a negative number as in the next problem.

Example 1.17: Absolute Value Inequality

Solve $|2-z| < 7$.

Solution. This simply means $-7 < 2-z < 7$. Subtracting 2 gives: $-9 < -z < 5$. Now multiplying by -1 gives: $9 > z > -5$. *Remember to reverse the inequality signs!* We can rearrange this as $-5 < z < 9$. Therefore the solution is the interval $(-5, 9)$. ♣

Example 1.18: Absolute Value Inequality

Solve $|2 - z| \geq 7$.

Solution. Recall that for $a > 0$, $|x| \geq a$ means $x \leq -a$ or $x \geq a$. Thus, either $2 - z \leq -7$ *or* $2 - z \geq 7$. Either $9 \leq z$ *or* $-5 \geq z$. Either $z \geq 9$ *or* $z \leq -5$. In interval notation, either z is in $[9, \infty)$ *or* z is in $(-\infty, -5]$. All together, we get our solution to be: $(-\infty, -5] \cup [9, \infty)$. ♣

In the previous two examples the *only* difference is that one had $<$ in the question and the other had $\geq$. Combining the two solutions gives the *entire* real number line!

Example 1.19: Absolute Value Inequality

Solve $0 < |x - 5| \leq 7$.

Solution. We split this into two cases.

(1) For $0 < |x-5|$ note that we always have that an absolute value is positive or zero (i.e., $0 \leq |x-5|$ is always true). So, for this part, we need to avoid $0 = |x - 5|$ from occurring. Thus, x *cannot* be 5, that is, $x \neq 5$.

(2) For $|x - 5| \leq 7$, we have $-7 \leq x - 5 \leq 7$. Adding 5 to each gives $-2 \leq x \leq 12$. Therefore the solution to $|x - 5| \leq 7$ is the interval $[-2, 12]$.

To combine (1) and (2) we need combine $x \neq 5$ with $x \in [-2, 12]$. Omitting 5 from the interval $[-2, 12]$ gives our solution to be: $[-2, 5) \cup (5, 12]$. ♣

Exercises for 1.1

1.1.1 *Simplify the following expressions as much as possible assuming $x, y > 0$:*

(a) $\dfrac{x^3 y^{-1/3}}{\sqrt[3]{y^2} x^2}$

(b) $\dfrac{3x^{-1/3} y^{-2} \sqrt[3]{x^4}}{\sqrt{9x} y^{-3}}$

(c) $\left(\dfrac{16x^2 y}{x^4}\right)^{1/2} \dfrac{\sqrt[3]{x^2}}{2\sqrt{y}}$

1.1.2 *Find the constants a, b, c if the expression*

$$\frac{4x^{-1} y^2 \sqrt[3]{x}}{2x\sqrt{y}}$$

is written in the form ax^by^c.

1.1.3 *Find the roots of the quadratic equation*

$$x^2 - 2x - 24 = 0.$$

1.1.4 *Solve the equation*

$$\frac{x}{4x-16} - 2 = \frac{1}{x-3}.$$

1.1.5 *Solve the following inequalities. Write your answer as a union of intervals.*

(a) $3x + 1 > 6$

(b) $0 \leq 7x - 1 < 1$

(c) $\dfrac{x^2(x-1)}{(x+2)(x+3)^3} \leq 0$

(d) $x^2 + 1 > 0$

(e) $x^2 + 1 < 0$

(f) $x^2 + 1 > 2x$

(g) $x^3 > 4x$

(h) $x^3 \geq 4x^2$

(i) $\dfrac{1}{x} > 2$

(j) $\dfrac{x}{x+2} \leq \dfrac{2}{x-1}$

1.1.6 *Solve the equation* $|6x + 2| = 1$.

1.1.7 *Find solutions to the following absolute value inequalities. Write your answer as a union of intervals.*

(a) $|x| \geq 2$

(b) $|x - 3| \leq 1$

(c) $|2x + 5| \geq 4$

(d) $|x + 2| < 3x - 6$

(e) $|2x + 5| + 4 \geq 1$

(f) $5 < |x + 1| < 8$

1.1.8 *Solve the equation* $\sqrt{1-x} + x = 1$.

1.2 Analytic Geometry

In what follows, we use the notation (x_1, y_1) to represent a point in the (x, y) coordinate system, also called the (x, y)-plane. Previously, we used (a, b) to represent an open interval. Notation often gets reused and abused in mathematics, but thankfully, it is usually clear from the context what we mean.

In the (x, y) coordinate system we normally write the x-axis horizontally, with positive numbers to the right of the origin, and the y-axis vertically, with positive numbers above the origin. That is, unless stated otherwise, we take "rightward" to be the positive x-direction and "upward" to be the positive y-direction. In a purely mathematical situation, we normally choose the same scale for the x- and y-axes. For example, the line joining the origin to the point (a, a) makes an angle of 45° with the x-axis (and also with the y-axis).

In applications, often letters other than x and y are used, and often different scales are chosen in the horizontal and vertical directions.

Example 1.20: Data Plot

Suppose you drop a coin from a window, and you want to study how its height above the ground changes from second to second. It is natural to let the letter t denote the time (the number of seconds since the object was released) and to let the letter h denote the height. For each t (say, at one-second intervals) you have a corresponding height h. This information can be tabulated, and then plotted on the (t, h) coordinate plane, as shown in figure 1.2.

seconds	0	1	2	3	4
meters	80	75.1	60.4	35.9	1.6

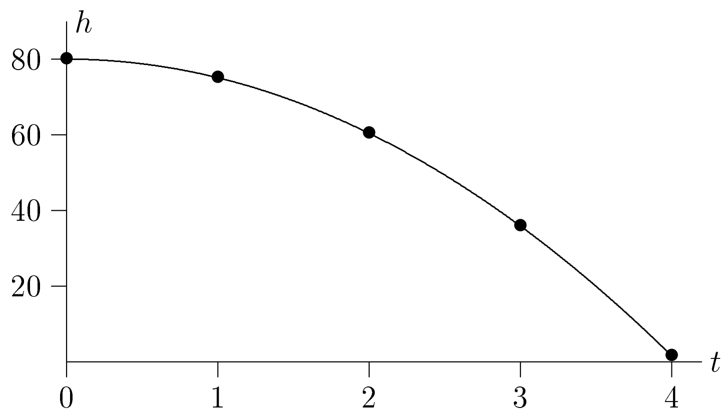


Figure 1.1: A data plot, height versus time.

We use the word "quadrant" for each of the four regions into which the plane is divided by the axes: the first quadrant is where points have both coordinates positive, or the "northeast" portion of the plot, and the second, third, and fourth quadrants are counted off counterclockwise, so the second quadrant is the northwest, the third is the southwest, and the fourth is the southeast.

Suppose we have two points A and B in the (x, y)-plane. We often want to know the change in x-coordinate (also called the "horizontal distance") in going from A to B. This is often written Δx, where the meaning of Δ (a capital delta in the Greek alphabet) is "change in". Thus, Δx can be read as "change in x" although it usually is read as "delta x". The point is that Δx denotes a single number, and should not be interpreted as "delta times x". Similarly, the "change in y" is written Δy and represents the difference between the y-coordinates of the two points. It is the vertical distance you have to move in going from A to B.

Example 1.21: Change in x and y

If $A = (2, 1)$ and $B = (3, 3)$ the change in x is

$$\Delta x = 3 - 2 = 1$$

while the change in y is

$$\Delta y = 3 - 1 = 2.$$

The general formulas for the change in x and the change in y between a point (x_1, y_1) and a point (x_2, y_2) are:

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1.$$

Note that either or both of these might be negative.

In what follows, we use the notation (x_1, y_1) to represent a point in the (x, y) coordinate system, also called the (x, y)-plane. Previously, we used (a, b) to represent an open interval. Notation often gets reused and abused in mathematics, but thankfully, it is usually clear from the context what we mean.

In the (x, y) coordinate system we normally write the x-axis horizontally, with positive numbers to the right of the origin, and the y-axis vertically, with positive numbers above the origin. That is, unless stated otherwise, we take "rightward" to be the positive x-direction and "upward" to be the positive y-direction. In a purely mathematical situation, we normally choose the same scale for the x- and y-axes. For example, the line joining the origin to the point (a, a) makes an angle of 45° with the x-axis (and also with the y-axis).

In applications, often letters other than x and y are used, and often different scales are chosen in the horizontal and vertical directions.

Example 1.22: Data Plot

Suppose you drop a coin from a window, and you want to study how its height above the ground changes from second to second. It is natural to let the letter t denote the time (the number of seconds since the object was released) and to let the letter h denote the height. For each t (say, at one-second intervals) you have a corresponding height h. This information can be tabulated, and then plotted on the (t, h) coordinate plane, as shown in figure 1.2.

We use the word "quadrant" for each of the four regions into which the plane is divided by the axes: the first quadrant is where points have both coordinates positive, or the "northeast" portion of the plot, and the second, third, and fourth quadrants are counted off counterclockwise, so the second quadrant is the northwest, the third is the southwest, and the fourth is the southeast.

Suppose we have two points A and B in the (x, y)-plane. We often want to know the change in x-coordinate (also called the "horizontal distance") in going from A to B. This is often written Δx, where the meaning of Δ (a capital delta in the Greek alphabet) is "change in". Thus, Δx can be read as "change in x" although it usually is read as "delta x". The point is that Δx denotes a single number, and should not be interpreted as "delta times x". Similarly, the "change in y" is written Δy and represents the difference between the y-coordinates of the two points. It is the vertical distance

seconds	0	1	2	3	4
meters	80	75.1	60.4	35.9	1.6

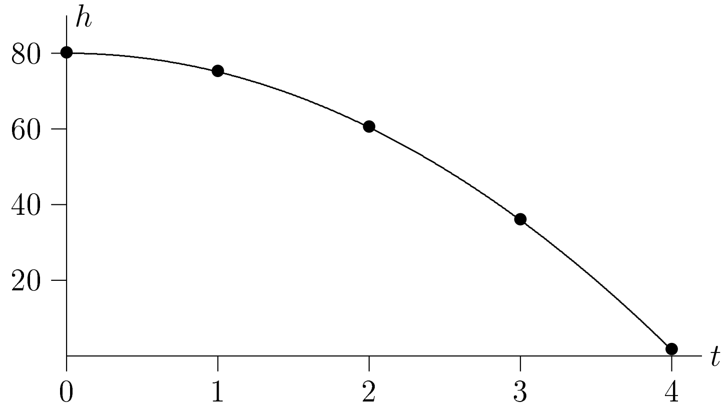


Figure 1.2: A data plot, height versus time.

you have to move in going from A to B.

Example 1.23: Change in x and y

If $A = (2, 1)$ *and* $B = (3, 3)$ *the change in* x *is*

$$\Delta x = 3 - 2 = 1$$

while the change in y *is*

$$\Delta y = 3 - 1 = 2.$$

The general formulas for the change in x and the change in y between a point (x_1, y_1) and a point (x_2, y_2) are:

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1.$$

Note that either or both of these might be negative.

1.2.1. Lines

If we have two *distinct* points $A(x_1, y_1)$ and $B(x_2, y_2)$, then we can draw one and only one straight line through both points. By the **slope** of this line we mean the ratio of Δy to Δx. The slope is often denoted by the letter m.

Key Idea 1.2.0: Slope Formula

The slope of the line joining the points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{\Delta y}{\Delta x} = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{\text{rise}}{\text{run}}.$$

Example 1.24: Slope of a Line Joining Two Points

The line joining the two points $(1, -2)$ *and* $(3, 5)$ *has slope* $m = \dfrac{5 - (-2)}{3 - 1} = \dfrac{7}{2}$.

The most familiar form of the equation of a straight line is:

$$y = mx + b.$$

Here m is the slope of the line: if you increase x by 1, the equation tells you that you have to increase y by m; and if you increase x by Δx, then y increases by $\Delta y = m\Delta x$. The number b is called the **y-intercept**, because it is where the line crosses the y-axis (when $x = 0$). If you know two points on a line, the formula $m = (y_2 - y_1)/(x_2 - x_1)$ gives you the slope. Once you know a point and the slope, then the y-intercept can be found by substituting the coordinates of either point in the equation: $y_1 = mx_1 + b$, i.e., $b = y_1 - mx_1$. Alternatively, one can use the **"point-slope" form** of the equation of a straight line: start with $(y - y_1)/(x - x_1) = m$ and then multiply to get

$$(y - y_1) = m(x - x_1),$$

the point-slope form. Of course, this may be further manipulated to get $y = mx - mx_1 + y_1$, which is essentially the **"$y = mx + b$" form**.

It is possible to find the equation of a line between two points directly from the relation $m = (y - y_1)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)$, which says "the slope measured between the point (x_1, y_1) and the point (x_2, y_2) is the same as the slope measured between the point (x_1, y_1) and any other point (x, y) on the line." For example, if we want to find the equation of the line joining our earlier points $A(2, 1)$ and $B(3, 3)$, we can use this formula:

$$m = \frac{y - 1}{x - 2} = \frac{3 - 1}{3 - 2} = 2, \qquad \text{so that} \qquad y - 1 = 2(x - 2), \qquad \text{i.e.,} \qquad y = 2x - 3.$$

Of course, this is really just the point-slope formula, except that we are not computing m in a separate step. We summarize the three common forms of writing a straight line below:

Key Idea 1.2.0: Slope-Intercept Form of a Straight Line

An equation of a line with slope m and y-intercept b is:

$$y = mx + b.$$

Key Idea 1.2.0: Point-Slope Form of a Straight Line

An equation of a line passing through (x_1, y_1) and having slope m is:

$$y - y_1 = m(x - x_1).$$

Key Idea 1.2.0: General Form of a Straight Line

Any line can be written in the form

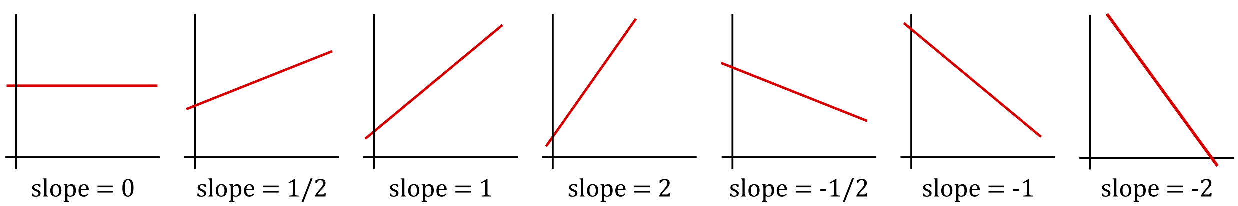
$$Ax + By + C = 0,$$

where A, B, C are constants and A, B are not both 0.

The slope m of a line in the form $y = mx + b$ tells us the direction in which the line is pointing. If m is positive, the line goes into the 1st quadrant as you go from left to right. If m is large and positive, it has a steep incline, while if m is small and positive, then the line has a small angle of inclination. If m is negative, the line goes into the 4th quadrant as you go from left to right. If m is a large negative number (large in absolute value), then the line points steeply downward. If m is negative but small in absolute value, then it points only a little downward.

If $m = 0$, then the line is horizontal and its equation is simply $y = b$.

All of these possibilities are illustrated below.



There is one type of line that cannot be written in the form $y = mx + b$, namely, vertical lines. A vertical line has an equation of the form $x = a$. Sometimes one says that a vertical line has an "infinite" slope.

It is often useful to find the x-intercept of a line $y = mx + b$. This is the x-value when $y = 0$. Setting $mx + b$ equal to 0 and solving for x gives: $x = -b/m$.

Example 1.25: Finding x-intercepts

To find x-intercept(s) of the line $y = 2x - 3$ we set $y = 0$ and solve for x:

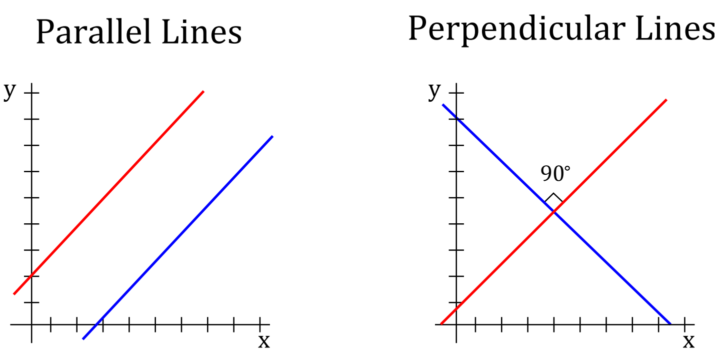
$$0 = 2x - 3 \qquad \rightarrow \qquad x = \frac{3}{2}.$$

Thus, the line has an x-intercept of $3/2$.

It is often necessary to know if two lines are parallel or perpendicular. Let m_1 and m_2 be the slopes of the nonvertical lines L_1 and L_2. Then:

- L_1 and L_2 are **parallel** if and only if $m_1 = m_2$.
- L_1 and L_2 are **perpendicular** if and only if $m_2 = \dfrac{-1}{m_1}$.

In the case of perpendicular lines, we say their slopes are *negative reciprocals*. Below is a visual representation of a pair of parallel lines and a pair of perpendicular lines.



Example 1.26: Equation of a Line

For each part below, find an equation of a line satisfying the requirements:

(a) *Through the two points $(0, 3)$ and $(-2, 4)$.*

(b) *With slope 7 and through point $(1, -2)$.*

(c) *With slope 2 and y-intercept 4.*

(d) *With x-intercept 8 and y-intercept -3.*

(e) *Through point $(5, 3)$ and parallel to the line $2x + 4y + 2 = 0$.*

(f) *With y-intercept 4 and perpendicular to the line $y = -\frac{2}{3}x + 3$.*

Solution. (a) We use the *slope formula* on $(x_1, y_1) = (0, 3)$ and $(x_2, y_2) = (-2, 4)$ to find m:

$$m = \frac{(4) - (3)}{(-2) - (0)} = \frac{1}{-2} = -\frac{1}{2}.$$

Now using the *point-slope formula* we get an equation to be:

$$y - 3 = -\frac{1}{2}(x - 0) \quad \rightarrow \quad y = -\frac{1}{2}x + 3.$$

(b) Using the *point-slope formula* with $m = 7$ and $(x_1, y_1) = (1, -2)$ gives:

$$y - (-2) = 7(x - 1) \quad \rightarrow \quad y = 7x - 9.$$

(c) Using the *slope-intercept formula* with $m = 2$ and $b = 4$ we get $y = 2x + 4$.

(d) Note that the intercepts give us two points: $(x_1, y_1) = (8, 0)$ and $(x_2, y_2) = (0, -3)$. Now follow the steps in part (a):

$$m = \frac{-3 - 0}{0 - 8} = \frac{3}{8}$$

. Using the *point-slope formula* we get an equation to be:

$$y - (-3) = \frac{3}{8}(x - 0) \quad \rightarrow \quad y = \frac{3}{8}x - 3$$

.

(e) The line $2x + 4y + 2 = 0$ can be written as:

$$4y = -2x - 2 \quad \rightarrow \quad y = -\frac{1}{2}x - \frac{1}{2}.$$

This line has slope $-1/2$. Since our line is *parallel* to it, we have $m = -1/2$. Now we have a point $(x_1, y_1) = (5, 3)$ and slope $m = -1/2$, thus, the *point-slope formula* gives:

$$y - 3 = -\frac{1}{2}(x - 5).$$

(f) The line $y = -\frac{2}{3}x + 3$ has slope $m = -2/3$. Since our line is perpendicular to it, the slope of our line is the *negative reciprocal*, hence, $m = 3/2$. Now we have $b = 4$ and $m = 3/2$, thus by the *slope-intercept formula*, an equation of the line is

$$y = \frac{3}{2}x + 4.$$

♣

Example 1.27: Parallel and Perpendicular Lines

Are the two lines $7x + 2y + 3 = 0$ and $6x - 4y + 2 = 0$ perpendicular? Are they parallel? If they are not parallel, what is their point of intersection?

Solution. The first line is:

$$7x + 2y + 3 = 0 \quad \to \quad 2y = -7x - 3 \quad \to \quad y = -\frac{7}{2}x - \frac{3}{2}.$$

It has slope $m_1 = -7/2$. The second line is:

$$6x - 4y + 2 = 0 \quad \to \quad -4y = -6x - 2 \quad \to \quad y = \frac{3}{2}x + \frac{1}{2}.$$

It has slope $m_2 = 3/2$. Since $m_1 \cdot m_2 \neq -1$ (they are not negative reciprocals), the lines are not perpendicular. Since $m_1 \neq m_2$ the lines are not parallel.

We find points of intersection by setting y-values to be the same and solving. In particular, we have

$$-\frac{7}{2}x - \frac{3}{2} = \frac{3}{2}x + \frac{1}{2}.$$

Solving for x gives $x = -2/5$. Then substituting this into either equation gives $y = -1/10$. Therefore, the lines intersect at the point $(-2/5, -1/10)$. ♣

1.2.2. Distance between Two Points and Midpoints

Given two points (x_1, y_1) and (x_2, y_2), recall that their horizontal distance from one another is $\Delta x = x_2 - x_1$ and their vertical distance from one another is $\Delta y = y_2 - y_1$. Actually, the word "distance" normally denotes "positive distance". Δx and Δy are *signed* distances, but this is clear from context. The (positive) distance from one point to the other is the length of the hypotenuse of a right triangle with legs $|\Delta x|$ and $|\Delta y|$, as shown in figure 1.3. The Pythagorean Theorem states that the distance between the two points is the square root of the sum of the squares of the horizontal and vertical sides:

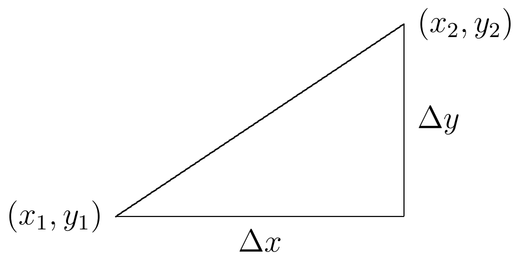


Figure 1.3: Distance between two points (here, Δx and Δy are positive).

Key Idea 1.2.0: Distance Formula

The distance between points (x_1, y_1) and (x_2, y_2) is

$$\text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 1.28: Distance Between Two Points

The distance, d, between points $A(2,1)$ and $B(3,3)$ is

$$d = \sqrt{(3-2)^2 + (3-1)^2} = \sqrt{5}.$$

As a special case of the distance formula, suppose we want to know the distance of a point (x, y) to the origin. According to the distance formula, this is

$$\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}.$$

A point (x, y) is at a distance r from the origin if and only if $\sqrt{x^2 + y^2} = r$, or, if we square both sides: $x^2 + y^2 = r^2$. As illustrated in the next section, this is the equation of the circle of radius, r, centered at the origin.

Furthermore, given two points we can determine the **midpoint** of the line segment joining the two points.

Key Idea 1.2.0: Midpoint Formula

The midpoint of the line segment joining two points (x_1, y_1) and (x_2, y_2) is the point with coordinates:

$$\text{midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 1.29: Midpoint of a Line Segment

Find the midpoint of the line segment joining the given points: $(1,0)$ *and* $(5,-2)$.

Solution. Using the *midpoint formula* on $(x_1, y_1) = (1,0)$ and $(x_2, y_2) = (5,-2)$ we get:

$$\left(\frac{(1) + (5)}{2}, \frac{(0) + (-2)}{2}\right) = (3,-1).$$

Thus, the midpoint of the line segment occurs at $(3,-1)$. ♣

1.2.3. Conics

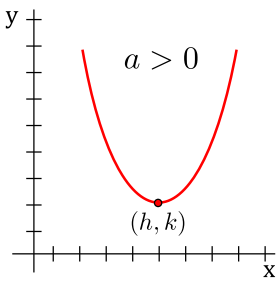
In this section we review equations of parabolas, circles, ellipses and hyperbolas. We will give the equations of various conics in **standard form** along with a sketch. A useful mnemonic is the following.

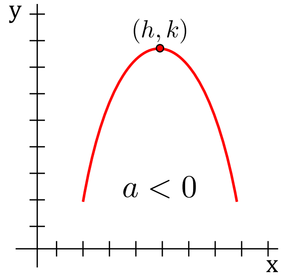
Key Idea 1.2.0: Mnemonic

In each conic formula presented, the terms '$x-h$' and '$y-k$' will always appear. The point (h,k) will alway represent either the centre or vertex of the particular conic.

Vertical Parabola: The equation of a vertical parabola is:

$$y-k=a(x-h)^2$$

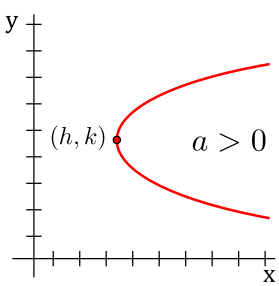


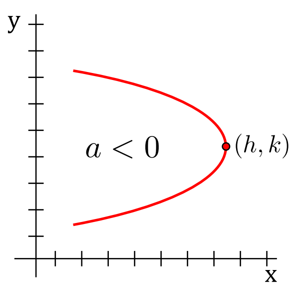


- (h,k) is the *vertex* of the parabola.
- a is the vertical *stretch factor.*
- If $a>0$, the parabola opens *upward.*
- If $a<0$, the parabola opens *downward.*

Horizontal Parabola: The equation of a horizontal parabola is:

$$x-h=a(y-k)^2$$

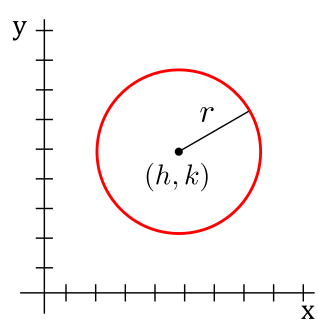




- (h,k) is the *vertex* of the parabola.
- a is the horizontal *stretch factor.*
- If $a>0$, the parabola opens *right.*
- If $a<0$, the parabola opens *left.*

Circle: The equation of a circle is:

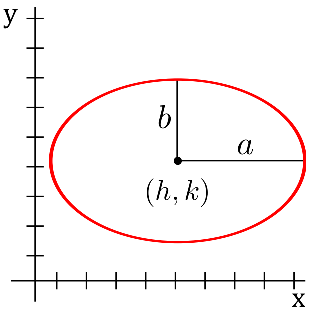
$$(x-h)^2+(y-k)^2=r^2$$



- (h,k) is the *centre* of the circle.
- r is the *radius* of the circle.

Ellipse: The equation of an ellipse is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$



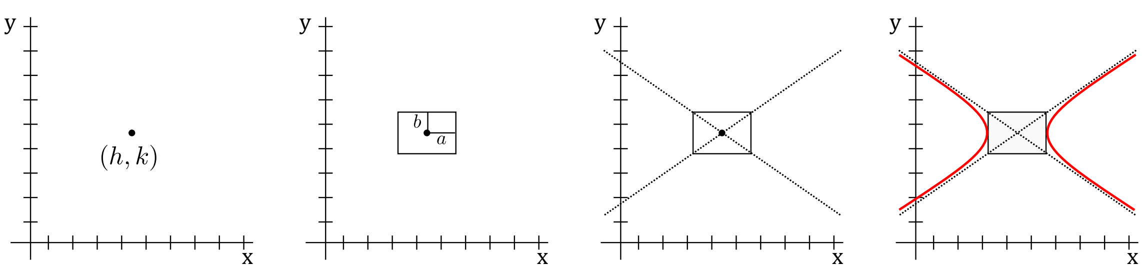
- (h,k) is the *centre* of the ellipse.
- a is the *horizontal distance* from the centre to the edge of the ellipse.
- b is the *vertical distance* from the centre to the edge of the ellipse.

Horizontal Hyperbola: The equation of a horizontal hyperbola is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

- (h,k) is the *centre* of the hyperbola.
- a is the *horizontal distance* from the centre to the edge of the box.
- a,b are the *reference box* values. The box has a centre of (h,k).
- b is the *vertical distance* from the centre to the edge of the box.

Given the equation of a horizontal hyperbola, one may sketch it by first placing a dot at the point (h,k). Then draw a box around (h,k) with horizontal distance a and vertical distance b to the edge of the box. Then draw dotted lines (called the **asymptotes** of the hyperbola) through the corners of the box. Finally, sketch the hyperbola in a horizontal direction as illustrated below.

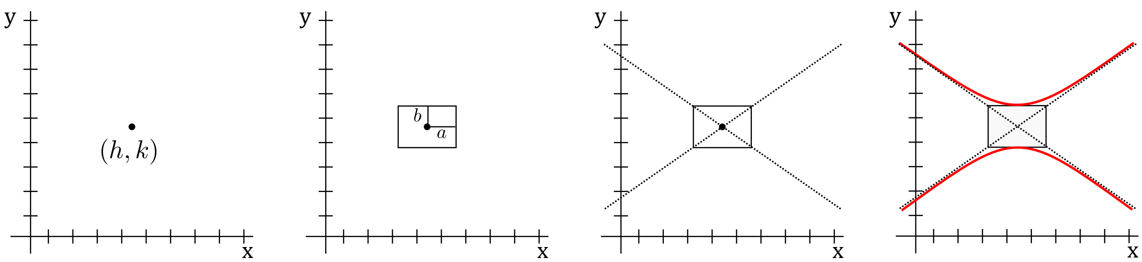


Vertical Hyperbola: The equation of a vertical hyperbola is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=-1$$

- (h, k) is the *centre* of the hyperbola.
- a, b are the *reference box* values. The box has a centre of (h, k).
- a is the *horizontal distance* from the centre to the edge of the box.
- b is the *vertical distance* from the centre to the edge of the box.

Given the equation of a vertical hyperbola, one may sketch it by following the same steps as with a horizontal hyperbola, but sketching the hyperbola going in a vertical direction.



Key Idea 1.2.0: Determining the Type of Conic

An equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

gives rise to a graph that can be generated by performing a conic section (parabolas, circles, ellipses, hyperbolas). Note that the Bxy term involves conic rotation. The Dx, Ex, and F terms affect the vertex and centre. For simplicity, we will omit the Bxy term. To determine the type of graph we focus our analysis on the values of A and C.

- If $A = C$, the graph is a *circle*.
- If $AC > 0$ (and $A \neq C$), the graph is an *ellipse*.
- If $AC = 0$, the graph is a *parabola*.
- If $AC < 0$, the graph is a *hyperbola*.

Example 1.30: Center and Radius of a Circle

Find the centre and radius of the circle $y^2 + x^2 - 12x + 8y + 43 = 0$.

Solution. We need to complete the square twice, once for the x terms and once for the y terms. We'll do both at the same time. First let's collect the terms with x together, the terms with y together, and move the number to the other side.

$$(x^2 - 12x) + (y^2 + 8y) = -43$$

We add 36 to both sides for the x term ($-12 \to \frac{-12}{2} = -6 \to (-6)^2 = 36$), and 16 to both sides for the y term ($8 \to \frac{8}{2} = 4 \to (4)^2 = 16$):

$$(x^2 - 12x + 36) + (y^2 + 8y + 16) = -43 + 36 + 16$$

Factoring gives:

$$(x-6)^2+(y+4)^2=3^2.$$

Therefore, the centre of the circle is $(6,-4)$ and the radius is 3. ♣

Example 1.31: Type of Conic

What type of conic is $4x^2-y^2-8x+8=0$? Put it in standard form.

Solution. Here we have $A=4$ and $C=-1$. Since $AC<0$, the conic is a hyperbola. Let us complete the square for the x and y terms. First let's collect the terms with x together, the terms with y together, and move the number to the other side.

$$(4x^2-8x)-y^2=-8$$

Now we factor out 4 from the x terms.

$$4(x^2-2x)-y^2=-8$$

Notice that we don't need to complete the square for the y terms (it is already completed!). To complete the square for the x terms we add **1** ($-2 \rightarrow \frac{-2}{2}=-1 \rightarrow (-1)^2=1$), taking into consideration that the a value is 4:

$$4(x^2-2x+1)-y^2=-8+4\cdot 1$$

Factoring gives:

$$4(x-1)^2-y^2=-4$$

A hyperbola in standard form has ± 1 on the right side and a positive x^2 on the left side, thus, we must divide by 4:

$$(x-1)^2-\frac{y^2}{4}=-1$$

Now we can see that the equation represents a vertical hyperbola with centre $(1,0)$ (and with a value $\sqrt{1}=1$, and b value $\sqrt{4}=2$). ♣

Example 1.32: Equation of Parabola

Find an equation of the parabola with vertex $(1,-1)$ that passes through the points $(-4,24)$ and $(7,35)$.

Solution. We first need to determine if it is a vertical parabola or horizontal parabola. See figure 1.4 for a sketch of the three points $(1,-1)$, $(-4,24)$ and $(7,35)$ in the xy-plane. Note that the vertex is $(1,-1)$. Given the location of the vertex, the parabola cannot open downwards. It also cannot open left or right (because the vertex is between the other two points - if it were to open to the right, every other point would need to be to the right of the vertex; if it were to open to the left, every other point would need to be to the left of the vertex). Therefore, the parabola must open upwards and it is a vertical parabola. It has an equation of

$$y-k=a(x-h)^2.$$

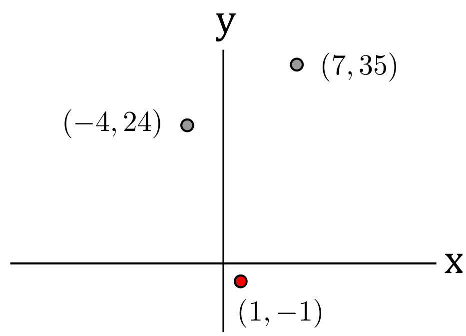


Figure 1.4: Figure for Example 1.32

As the vertex is $(h, k) = (1, -1)$ we have:

$$y - (-1) = a(x - 1)^2$$

To determine a, we substitute one of the points into the equation and solve. Let us substitute the point $(x, y) = (-4, 24)$ into the equation:

$$24 - (-1) = a(-4 - 1)^2 \quad \rightarrow \quad 25 = 25a \quad \rightarrow \quad a = 1.$$

Therefore, the equation of the parabola is:

$$y + 1 = (x - 1)^2.$$

Note that if we substituted $(7, 35)$ into the equation instead, we would also get $a = 1$. ♣

Exercises for 1.2

1.2.1 *Find the equation of the line in the form $y = mx + b$:*

(a) through $(1, 1)$ *and* $(-5, -3)$

(b) through $(-1, 2)$ *with slope* -2

(c) through $(-1, 1)$ *and* $(5, -3)$

(d) through $(2, 5)$ *and parallel to the line* $3x + 9y + 6 = 0$

(e) with x-intercept 5 and perpendicular to the line $y = 2x + 4$

1.2.2 *Change the following equations to the form $y = mx+b$, graph the line, and find the y-intercept and x-intercept.*

(a) $y - 2x = 2$

(b) $x + y = 6$

(c) $x = 2y - 1$

(d) $3 = 2y$

(e) $2x + 3y + 6 = 0$

1.2.3 *Determine whether the lines* $3x + 6y = 7$ *and* $2x + 4y = 5$ *are parallel.*

1.2.4 *Suppose a triangle in the* (x, y)*-plane has vertices* $(-1, 0)$, $(1, 0)$ *and* $(0, 2)$*. Find the equations of the three lines that lie along the sides of the triangle in* $y = mx + b$ *form.*

1.2.5 *Let* x *stand for temperature in degrees Celsius (centigrade), and let* y *stand for temperature in degrees Fahrenheit. A temperature of* $0°C$ *corresponds to* $32°F$*, and a temperature of* $100°C$ *corresponds to* $212°F$*. Find the equation of the line that relates temperature Fahrenheit* y *to temperature Celsius* x *in the form* $y = mx + b$*. Graph the line, and find the point at which this line intersects* $y = x$*. What is the practical meaning of this point?*

1.2.6 *A car rental firm has the following charges for a certain type of car: \$25 per day with 100 free miles included, \$0.15 per mile for more than 100 miles. Suppose you want to rent a car for one day, and you know you'll use it for more than 100 miles. What is the equation relating the cost* y *to the number of miles* x *that you drive the car?*

1.2.7 *A photocopy store advertises the following prices: 5c per copy for the first 20 copies, 4c per copy for the 21st through 100th copy, and 3c per copy after the 100th copy. Let* x *be the number of copies, and let* y *be the total cost of photocopying. (a) Graph the cost as* x *goes from 0 to 200 copies. (b) Find the equation in the form* $y = mx + b$ *that tells you the cost of making* x *copies when* x *is more than 100.*

1.2.8 *Market research tells you that if you set the price of an item at \$1.50, you will be able to sell 5000 items; and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Let* x *be the number of items you can sell, and let* P *be the price of an item. (a) Express* P *linearly in terms of* x*, in other words, express* P *in the form* $P = mx + b$*. (b) Express* x *linearly in terms of* P*.*

1.2.9 *An instructor gives a 100-point final exam, and decides that a score 90 or above will be a grade of 4.0, a score of 40 or below will be a grade of 0.0, and between 40 and 90 the grading will be linear. Let* x *be the exam score, and let* y *be the corresponding grade. Find a formula of the form* $y = mx + b$ *which applies to scores* x *between 40 and 90.*

1.2.10 *Find the distance between the pairs of points:*

(a) $(-1, 1)$ *and* $(1, 1)$.

(b) $(5, 3)$ *and* $(-7, -2)$.

(c) $(1, 1)$ *and the origin.*

1.2.11 *Find the midpoint of the line segment joining the point* $(20, -10)$ *to the origin.*

1.2.12 *Find the equation of the circle of radius 3 centered at:*

(a) $(0, 0)$

(b) $(5, 6)$

(c) $(-5, -6)$

(d) $(0, 3)$

(e) $(0, -3)$

(f) $(3, 0)$

1.2.13 *For each pair of points* $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *find an equation of the circle with center at* A *that goes through* B*.*

(a) $A(2, 0)$, $B(4, 3)$

(b) $A(-2, 3)$, $B(4, 3)$

1.2.14 *Determine the type of conic and sketch it.*

(a) $x^2 + y^2 + 10y = 0$

(b) $9x^2 - 90x + y^2 + 81 = 0$

(c) $6x + y^2 - 8y = 0$

1.2.15 *Find the standard equation of the circle passing through* $(-2, 1)$ *and tangent to the line* $3x - 2y = 6$ *at the point* $(4, 3)$*. Sketch. (Hint: The line through the center of the circle and the point of tangency is perpendicular to the tangent line.)*

1.3 Trigonometry

In this section we review the definitions of trigonometric functions.

1.3.1. Angles and Sectors of Circles

Mathematicians tend to deal mostly with **radians** and we will see later that some formulas are more elegant when using radians (rather than degrees). The relationship between degrees and radians is:

$$\pi \text{ rad} = 180^\circ.$$

Using this formula, some common angles can be derived:

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$	$270°$	$360°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Example 1.33: Degrees to Radians

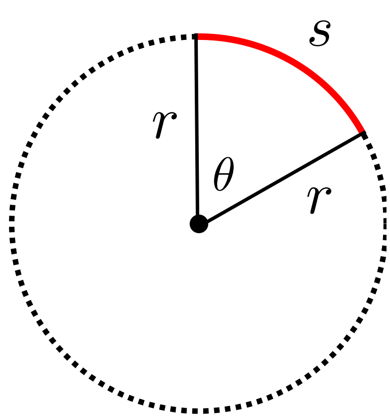
To convert $45°$ *to radians, multiply by* $\frac{\pi}{180°}$ *to get* $\frac{\pi}{4}$.

Example 1.34: Radians to Degrees

To convert $\frac{5\pi}{6}$ *radians to degrees, multiply by* $\frac{180°}{\pi}$ *to get* $150°$.

From now on, unless otherwise indicated, we will *always* use radian measure.

In the diagram below is a sector of a circle with **central angle** θ and radius r **subtending** an arc with length s.



When θ is measure in radians, we have the following formula relating θ, s and r:

$$\theta = \frac{s}{r} \quad \text{or} \quad s = r\theta.$$

Key Idea 1.3.0: Sector Area

The area of the sector is equal to:

$$\text{Sector Area} = \frac{1}{2}r^2\theta.$$

Example 1.35: Angle Subtended by Arc

If a circle has radius 3 cm, then an angle of 2 rad is subtended by an arc of 6 cm ($s = r\theta = 3 \cdot 2 = 6$).

Example 1.36: Area of Circle

If we substitute $\theta = 2\pi$ *(a complete revolution) into the sector area formula we get the area of a circle:*

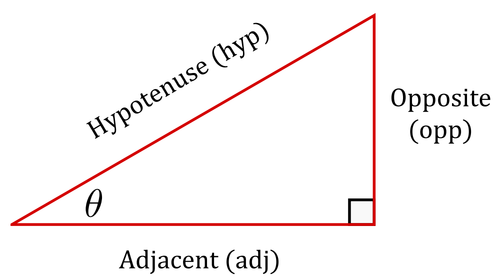
$$A = \frac{1}{2}r^2(2\pi) = \pi r^2.$$

1.3.2. Trigonometric Functions

There are six basic trigonometric functions:

- Sine (abbreviated by sin)
- Cosine (abbreviated by cos)
- Tangent (abbreviated by tan)
- Cosecant (abbreviated by csc)
- Secant (abbreviated by sec)
- Cotangent (abbreviated by cot)

We first describe trigonometric functions in terms of ratios of two sides of a *right angle triangle* containing the angle θ.



With reference to the above triangle, for an acute angle θ (that is, $0 \leq \theta < \pi/2$), the six trigonometric functions can be described as follows:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

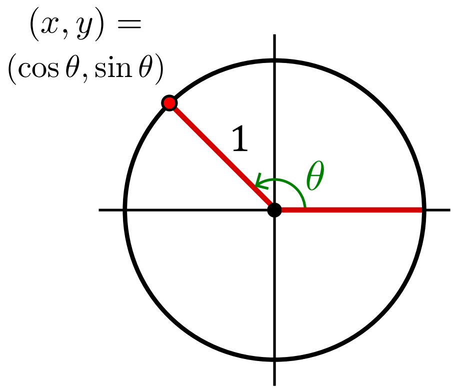
$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Key Idea 1.3.0: Mnemonic

The mnemonic *SOH CAH TOA* is useful in remembering how trigonometric functions of acute angles relate to the sides of a right triangle.

This description does not apply to *obtuse* or *negative angles*. To define the six basic trigonometric functions we first define sine and cosine as the lengths of various line segments from a unit circle, and then we define the remaining four basic trigonometric functions in terms of sine and cosine.

Take a line originating at the origin (making an angle of θ with the positive half of the x-axis) and suppose this line intersects the unit circle at the point (x, y). The x- and y-coordinates of this point of intersection are equal to $\cos\theta$ and $\sin\theta$, respectively.



For angles greater than 2π or less than -2π, simply continue to rotate around the circle. In this way, sine and cosine become periodic functions with period 2π:

$$\sin\theta = \sin(\theta + 2\pi k) \qquad \cos\theta = \cos(\theta + 2\pi k)$$

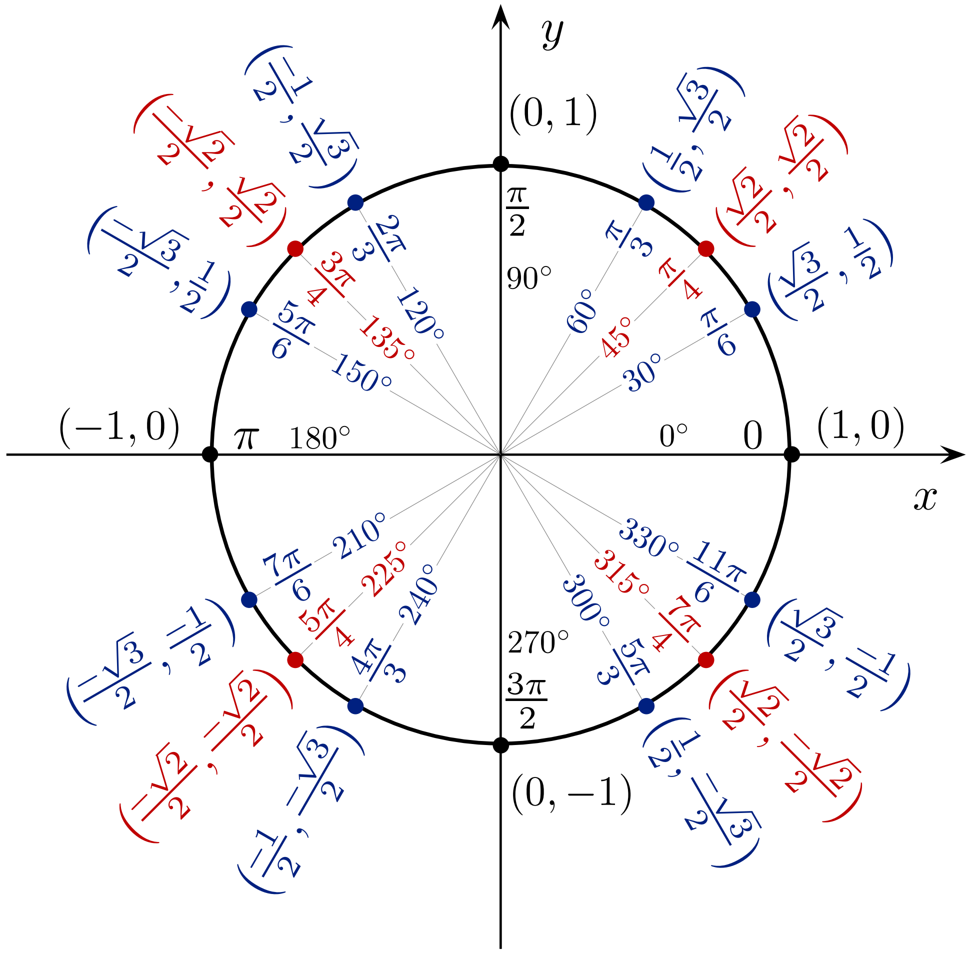
for any angle θ and any integer k.

Above, only sine and cosine were defined directly by the circle. We now define the remaining four basic trigonometric functions in terms of the functions $\sin\theta$ and $\cos\theta$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

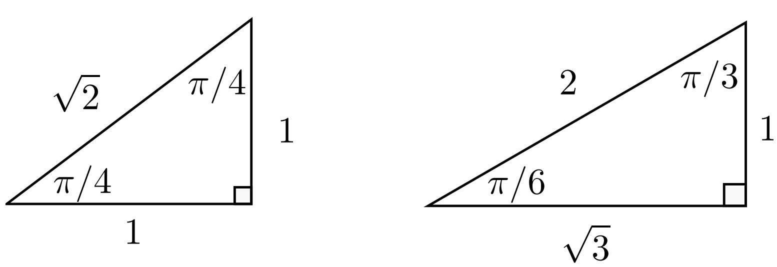
1.3.3. Computing Exact Trigonometric Ratios

The **unit circle** is often used to determine the *exact* value of a particular trigonometric function.



Reading from the unit circle one can see that $\cos 5\pi/6 = -\sqrt{3}/2$ and $\sin 5\pi/6 = 1/2$ (remember the that the x-coordinate is $\cos\theta$ and the y-coordinate is $\sin\theta$). However, we don't always have access to the unit circle. In this case, we can compute the exact trigonometric ratios for $\theta = 5\pi/6$ by using **special triangles** and the **CAST rule** described below.

The first special triangle has angles of $45°, 45°, 90°$ (i.e., $\pi/4, \pi/4, \pi/2$) with side lengths $1, 1, \sqrt{2}$, while the second special triangle has angles of $30°, 60°, 90°$ (i.e., $\pi/6, \pi/3, \pi/2$) with side lengths $1, 2, \sqrt{3}$. They are classically referred to as the $1-1-\sqrt{2}$ triangle, and the $1-2-\sqrt{3}$ triangle, respectively, shown below.



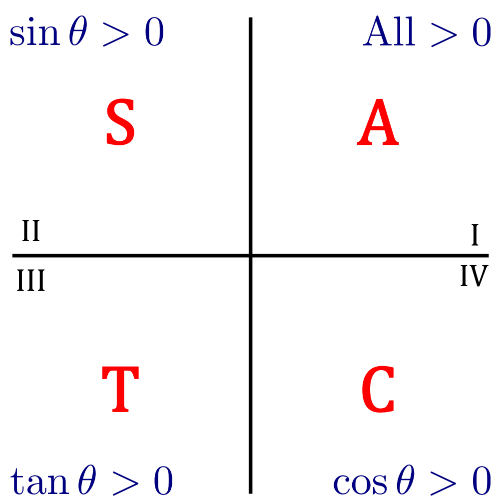
Key Idea 1.3.0: Mnemonic

The first triangle should be easy to remember. To remember the second triangle, place the largest number (2) across from the largest angle ($90° = \pi/2$). Place the smallest number (1) across from the smallest angle ($30° = \pi/6$). Place the middle number ($\sqrt{3} \approx 1.73$) across from the middle angle ($60° = \pi/3$). Double check using the Pythagorean Theorem that the sides satisfy $a^2 + b^2 = c^2$.

The special triangles allow us to compute the exact value (excluding the sign) of trigonometric ratios, but to determine the sign, we can use the *CAST rule*.

Key Idea 1.3.0: The CAST Rule

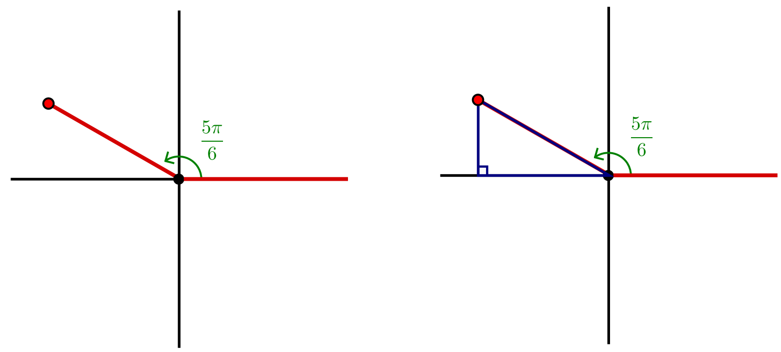
The CAST rule says that in quadrant I all three of $\sin\theta$, $\cos\theta$, $\tan\theta$ are positive. In quadrant II, only $\sin\theta$ is positive, while $\cos\theta$, $\tan\theta$ are negative. In quadrant III, only $\tan\theta$ is positive, while $\sin\theta$, $\cos\theta$ are negative. In quadrant IV, only $\cos\theta$ is positive, while $\sin\theta$, $\tan\theta$ are negative. To remember this, simply label the quadrants by the letters C-A-S-T starting in the bottom right and labelling counter-clockwise.



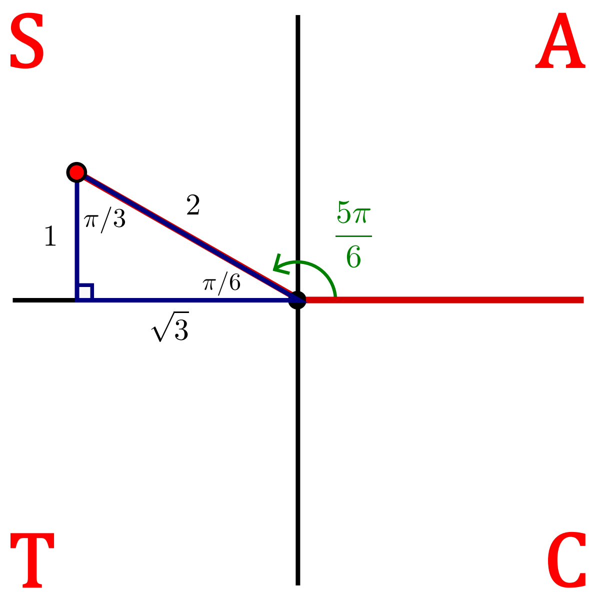
Example 1.37: Determining Trigonometric Ratios Without Unit Circle

Determine $\sin 5\pi/6$, $\cos 5\pi/6$, $\tan 5\pi/6$, $\sec 5\pi/6$, $\csc 5\pi/6$ *and* $\cot 5\pi/6$ *exactly by using the special triangles and CAST rule.*

Solution. We start by drawing the xy-plane and indicating our angle of $5\pi/6$ in standard position (positive angles rotate *counterclockwise* while negative angles rotate *clockwise*). Next, we drop a perpendicular to the x-axis (never drop it to the y-axis!).



Notice that we can now figure out the angles in the triangle. Since $180° = \pi$, we have an interior angle of $\pi - 5\pi/6 = \pi/6$ inside the triangle. As the *angles of a triangle add up to* $180° = \pi$, the other angle must be $\pi/3$. This gives one of our special triangles. We label it accordingly and add the CAST rule to our diagram.



From the above figure we see that $5\pi/6$ lies in quadrant II where $\sin\theta$ is positive and $\cos\theta$ and $\tan\theta$ are negative. This gives us the *sign* of $\sin\theta$, $\cos\theta$ and $\tan\theta$. To determine the *value* we use the special triangle and SOH CAH TOA.

Using $\sin\theta = opp/hyp$ we find a value of $1/2$. But $\sin\theta$ is positive in quadrant II, therefore,

$$\sin\frac{5\pi}{6} = +\frac{1}{2}.$$

Using $\cos\theta = adj/hyp$ we find a value of $\sqrt{3}/2$. But $\cos\theta$ is negative in quadrant II, therefore,

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}.$$

Using $\tan\theta = opp/adj$ we find a value of $1/\sqrt{3}$. But $\tan\theta$ is negative in quadrant II, therefore,

$$\tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}.$$

To determine $\sec\theta$, $\csc\theta$ and $\cot\theta$ we use the definitions:

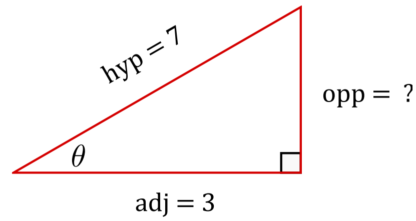
$$\csc\frac{5\pi}{6} = \frac{1}{\sin\frac{5\pi}{6}} = +2, \qquad \sec\frac{5\pi}{6} = \frac{1}{\cos\frac{5\pi}{6}} = -\frac{2}{\sqrt{3}}, \qquad \cot\frac{5\pi}{6} = \frac{1}{\tan\frac{5\pi}{6}} = -\sqrt{3}.$$

♣

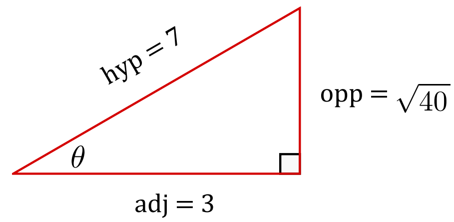
Example 1.38: CAST Rule

If $\cos\theta = 3/7$ and $3\pi/2 < \theta < 2\pi$, then find $\cot\theta$.

Solution. We first draw a right angle triangle. Since $\cos\theta = adj/hyp = 3/7$, we let the adjacent side have length 3 and the hypotenuse have length 7.



Using the Pythagorean Theorem, we have $3^2 + (\text{opp})^2 = 7^2$. Thus, the opposite side has length $\sqrt{40}$.



To find $\cot\theta$ we use the definition:

$$\cot\theta = \frac{1}{\tan\theta}.$$

Since we are given $3\pi/2 < \theta < 2\pi$, we are in the fourth quadrant. By the CAST rule, $\tan\theta$ is negative in this quadrant. As $\tan\theta = opp/adj$, it has a value of $\sqrt{40}/3$, but by the CAST rule it is negative, that is,

$$\tan\theta = -\frac{\sqrt{40}}{3}.$$

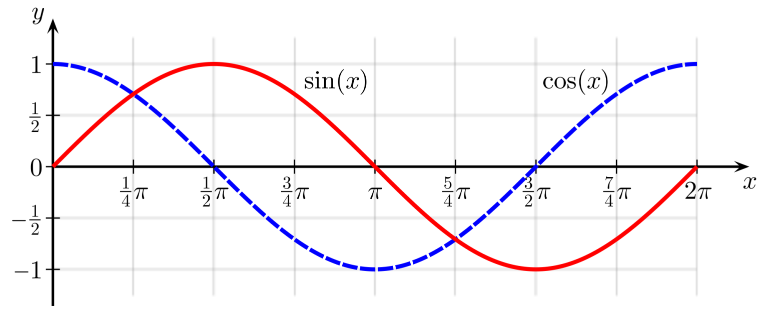
Therefore,

$$\cot\theta = -\frac{3}{\sqrt{40}}.$$

♣

1.3.4. Graphs of Trigonometric Functions

The graph of the functions $\sin x$ and $\cos x$ can be visually represented as:

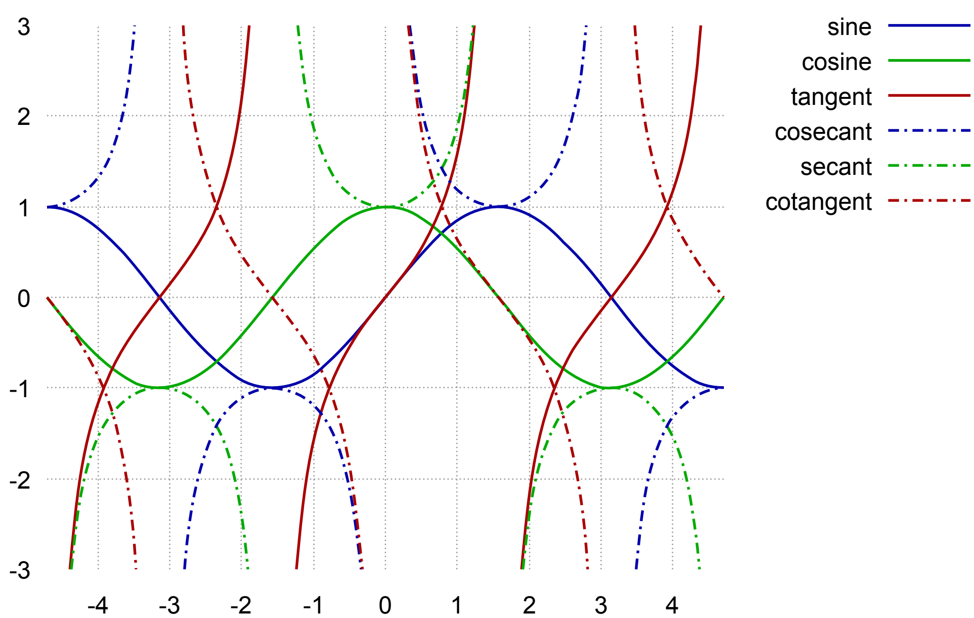


Both $\sin x$ and $\cos x$ have domain $(-\infty, \infty)$ and range $[-1, 1]$. That is,

$$-1 \leq \sin x \leq 1 \qquad -1 \leq \cos x \leq 1.$$

The zeros of $\sin x$ occur at the integer multiples of π, that is, $\sin x = 0$ whenever $x = n\pi$, where n is an integer. Similarly, $\cos x = 0$ whenever $x = \pi/2 + n\pi$, where n is an integer.

The six basic trigonometric functions can be visually represented as:



Both tangent and cotangent have range $(-\infty, \infty)$, whereas cosecant and secant have range $(-\infty, -1] \cup [1, \infty)$. Each of these functions is periodic. Tangent and cotangent have period π, whereas sine, cosine, cosecant and secant have period 2π.

1.3.5. Trigonometric Identities

There are numerous trigonometric identities, including those relating to shift/periodicity, Pythagoras type identities, double-angle formulas, half-angle formulas and addition formulas. We list these below.

1. **Shifts and periodicity**

$\sin(\theta+2\pi) = \sin\theta$	$\cos(\theta+2\pi) = \cos\theta$	$\tan(\theta+2\pi) = \tan\theta$
$\sin(\theta+\pi) = -\sin\theta$	$\cos(\theta+\pi) = -\cos\theta$	$\tan(\theta+\pi) = \tan\theta$
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$	$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$	$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

2. **Pythagoras type formulas**
   - $\sin^2\theta + \cos^2\theta = 1$
   - $\tan^2\theta + 1 = \sec^2\theta$
   - $1 + \cot^2\theta = \csc^2\theta$
3. **Double-angle formulas**
   - $\sin(2\theta) = 2\sin\theta\cos\theta$
   - $$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta. \end{aligned}$$
4. **Half-angle formulas**
   - $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$
   - $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$
5. **Addition formulas**
   - $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$
   - $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$
   - $\tan(\theta+\phi) = \dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$
   - $\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$
   - $\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

Example 1.39: Double Angle

Find all values of x with $0 \le x \le \pi$ such that $\sin 2x = \sin x$.

Solution. Using the double-angle formula $\sin 2x = 2\sin x\cos x$ we have:

$$2\sin x\cos x = \sin x$$

$$2\sin x\cos x - \sin x = 0$$

$$\sin x(2\cos x - 1) = 0$$

Thus, either $\sin x = 0$ or $\cos x = 1/2$. For the first case when $\sin x = 0$, we get $x = 0$ or $x = \pi$. For the second case when $\cos x = 1/2$, we get $x = \pi/3$ (use the special triangles and CAST rule to get this). Thus, we have three solutions: $x = 0$, $x = \pi/3$, $x = \pi$. ♣

Exercises for 1.3

1.3.1 *Find all values of* θ *such that* $\sin(\theta) = -1$*; give your answer in radians.*

1.3.2 *Find all values of* θ *such that* $\cos(2\theta) = 1/2$*; give your answer in radians.*

1.3.3 *Compute the following:*

(a) $\sin(3\pi)$

(b) $\sec(5\pi/6)$

(c) $\cos(-\pi/3)$

(d) $\csc(4\pi/3)$

(e) $\tan(7\pi/4)$

(f) $\cot(13\pi/4)$

1.3.4 *If* $\sin\theta = \frac{3}{5}$ *and* $\frac{\pi}{2} < \theta < \pi$*, then find* $\sec\theta$*.*

1.3.5 *Suppose that* $\tan\theta = x$ *and* $\pi < \theta < \frac{3\pi}{2}$*, find* $\sin\theta$ *and* $\cos\theta$ *in terms of* x*.*

1.3.6 *Find an angle* θ *such that* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ *and* $\sin\theta = \sin\frac{23\pi}{7}$*.*

1.3.7 *Use an angle sum identity to compute* $\cos(\pi/12)$*.*

1.3.8 *Use an angle sum identity to compute* $\tan(5\pi/12)$*.*

1.3.9 *Verify the following identities*

(a) $\cos^2(t)/(1-\sin(t)) = 1+\sin(t)$

(b) $2\csc(2\theta) = \sec(\theta)\csc(\theta)$

(c) $\sin(3\theta) - \sin(\theta) = 2\cos(2\theta)\sin(\theta)$

1.3.10 *Sketch the following functions:*

(a) $y = 2\sin(x)$

(b) $y = \sin(3x)$

(c) $y = \sin(-x)$

1.3.11 *Find all of the solutions of* $2\sin(t) - 1 - \sin^2(t) = 0$ *in the interval* $[0, 2\pi]$*.*

1.4 Additional Exercises

These problems require a comprehensive knowledge of the skills reviewed in this chapter. They are not in any particular order. A proficiency in these skills will help you a long way as your learn the calculus material in the following chapters.

1.4.1 *Rationalize the denominator for each of the following expressions. That is, re-write the expression in such a way that no square roots appear in the denominator. Also, simplify your answers if possible.*

(a) $\dfrac{1}{\sqrt{2}}$

(b) $\dfrac{3h}{\sqrt{x+h+1}-\sqrt{x+1}}$

1.4.2 *Solve the following equations.*

(a) $2-5(x-3)=4-10x$

(b) $2x^2-5x=3$

(c) $x^2-x-3=0$

(d) $x^2+x+3=0$

(e) $\sqrt{x^2+9}=2x$

1.4.3 *By means of counter-examples, show why it is wrong to say that the following equations hold for all real numbers for which the expressions are defined.*

(a) $(x-2)^2=x^2-2^2$

(b) $\dfrac{1}{x+h}=\dfrac{1}{x}+\dfrac{1}{h}$

(c) $\sqrt{x^2+y^2}=x+y$

1.4.4 *Find an equation of the line passing through the point* $(-2,5)$ *and parallel to the line* $x+3y-2=0$.

1.4.5 *Solve* $\dfrac{x^2-1}{3x-1}\leq 1$.

1.4.6 *Explain why the following expression never represents a real number (for any real number* x*):* $\sqrt{x-2}+\sqrt{1-x}$.

1.4.7 *Simplify the expression* $\dfrac{[3(x+h)^2+4]-[3x^2+4]}{h}$ *as much as possible.*

1.4.8 *Simplify the expression* $\dfrac{\frac{x+h}{2(x+h)-1}-\frac{x}{2x-1}}{h}$ *as much as possible.*

1.4.9 *Simplify the expression* $-\sin x(\cos x+3\sin x)-\cos x(-\sin x+3\cos x)$.

1.4.10 *Solve the equation* $\cos x=\frac{\sqrt{3}}{2}$ *on the interval* $0\le x\le 2\pi$.

1.4.11 *Find an angle* θ *such that* $0\le\theta\le\pi$ *and* $\cos\theta=\cos\frac{38\pi}{5}$.

1.4.12 *What can you say about* $\dfrac{|x|+|4-x|}{x-2}$ *when* x *is a large (positive) number?*

1.4.13 *Find an equation of the circle with centre in* $(-2,3)$ *and passing through the point* $(1,-1)$.

1.4.14 *Find the centre and radius of the circle described by* $x^2+y^2+6x-4y+12=3$.

1.4.15 *If* $y=9x^2+6x+7$, *find all possible values of* y.

1.4.16 *Simplify* $\left(\dfrac{3x^2y^3z^{-1}}{18x^{-1}yz^3}\right)^2$.

1.4.17 *If* $y=\dfrac{3x+2}{1-4x}$, *then what is* x *in terms of* y*?*

1.4.18 *Divide* x^2+3x-5 *by* $x+2$ *to obtain the quotient and the remainder. Equivalently, find polynomial* $Q(x)$ *and constant* R *such that*

$$\frac{x^2+3x-5}{x+2}=Q(x)+\frac{R}{x+2}.$$

2. Functions

2.1 What is a Function?

A **function** $y = f(x)$ is a rule for determining y when we're given a value of x. For example, the rule $y = f(x) = 2x + 1$ is a function. Any line $y = mx + b$ is called a **linear** function. The graph of a function looks like a curve above (or below) the x-axis, where for any value of x the rule $y = f(x)$ tells us how far to go above (or below) the x-axis to reach the curve.

Functions can be defined in various ways: by an algebraic formula or several algebraic formulas, by a graph, or by an experimentally determined table of values. In the latter case, the table gives a bunch of points in the plane, which we might then interpolate with a smooth curve, if that makes sense.

Given a value of x, a function must give at most one value of y. Thus, vertical lines are not functions. For example, the line $x = 1$ has infinitely many values of y if $x = 1$. It is also true that if x is any number (not 1) there is no y which corresponds to x, but that is not a problem—only multiple y values is a problem.

One test to identify whether or not a curve in the (x, y) coordinate system is a function is the following.

Theorem 2.1: The Vertical Line Test

A curve in the (x, y) coordinate system represents a function if and only if no vertical line intersects the curve more than once.

In addition to lines, another familiar example of a function is the parabola $y = f(x) = x^2$. We can draw the graph of this function by taking various values of x (say, at regular intervals) and plotting the points $(x, f(x)) = (x, x^2)$. Then connect the points with a smooth curve. (See figure 2.1.)

The two examples $y = f(x) = 2x + 1$ and $y = f(x) = x^2$ are both functions which can be evaluated at *any* value of x from negative infinity to positive infinity. For many functions, however, it only makes sense to take x in some interval or outside of some "forbidden" region. The interval of x-values at which we're allowed to evaluate the function is called the **domain** of the function.

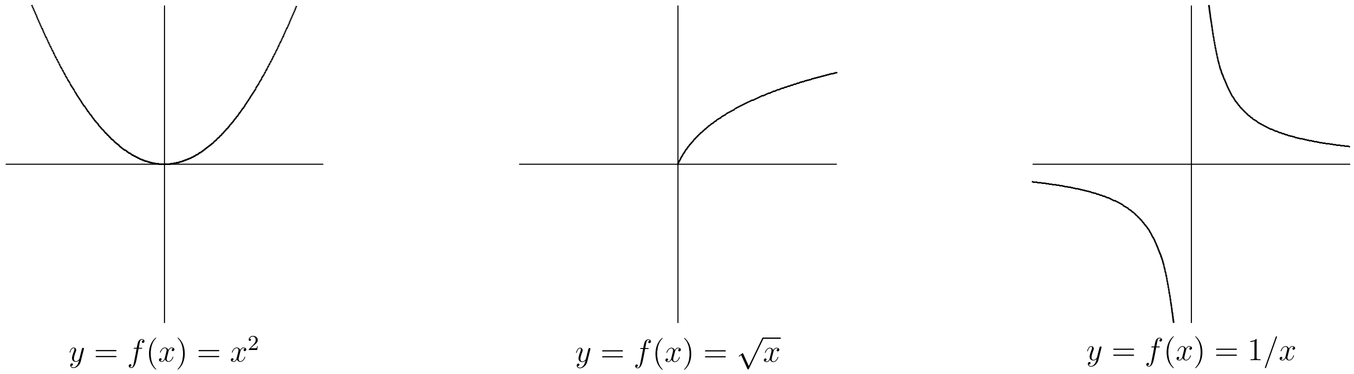


Figure 2.1: Some graphs.

Example 2.1: Domain of the Square-Root Function

The square-root function $y = f(x) = \sqrt{x}$ is the rule which says, given an x-value, take the nonnegative number whose square is x. This rule only makes sense if $x \geq 0$. We say that the domain of this function is $x \geq 0$, or more formally $\{x \in \mathbb{R} : x \geq 0\}$. Alternately, we can use interval notation, and write that the domain is $[0, \infty)$. The fact that the domain of $y = \sqrt{x}$ is $[0, \infty)$ means that in the graph of this function (see figure 2.1) we have points (x, y) only above x-values on the right side of the x-axis.

Another example of a function whose domain is not the entire x-axis is: $y = f(x) = 1/x$, the reciprocal function. We cannot substitute $x = 0$ in this formula. The function makes sense, however, for any nonzero x, so we take the domain to be: $\{x \in \mathbb{R} : x \neq 0\}$. The graph of this function does not have any point (x, y) with $x = 0$. As x gets close to 0 from either side, the graph goes off toward infinity. We call the vertical line $x = 0$ an **asymptote**.

To summarize, two reasons why certain x-values are excluded from the domain of a function are the following.

Key Idea 2.1.0: Restrictions for the Domain of a Function

1. We cannot divide by zero, and
2. We cannot take the square root of a negative number.

We will encounter some other ways in which functions might be undefined later.

Another reason why the domain of a function might be restricted is that in a given situation the x-values outside of some range might have no practical meaning. For example, if y is the area of a square of side x, then we can write $y = f(x) = x^2$. In a purely mathematical context the domain of the function $y = x^2$ is all of $\mathbb{R}$. However, in the story-problem context of finding areas of squares, we restrict the domain to positive values of x, because a square with negative or zero side makes no sense.

In a problem in pure mathematics, we usually take the domain to be all values of x at which the formulas can be evaluated. However, in a story problem there might be further restrictions on the domain because only certain values of x are of interest or make practical sense.

In a story problem, we often use letters other than x and y. For example, the volume V of a sphere is a function of the radius r, given by the formula $V = f(r) = \frac{4}{3}\pi r^3$. Also, letters different from f may be used. For example, if y is the velocity of something at time t, we may write $y = v(t)$ with the letter v (instead of f) standing for the velocity function (and t playing the role of x).

The letter playing the role of x is called the **independent variable**, and the letter playing the role of y is called the **dependent variable** (because its value "depends on" the value of the independent variable). In story problems, when one has to translate from English into mathematics, a crucial step is to determine what letters stand for variables. If only words and no letters are given, then we have to decide which letters to use. Some letters are traditional. For example, almost always, t stands for time.

Example 2.2: Open Box

An open-top box is made from an $a \times b$ rectangular piece of cardboard by cutting out a square of side x from each of the four corners, and then folding the sides up and sealing them with duct tape. Find a formula for the volume V of the box as a function of x, and find the domain of this function.

Solution. The box we get will have height x and rectangular base of dimensions $a - 2x$ by $b - 2x$. Thus,

$$V = f(x) = x(a - 2x)(b - 2x).$$

Here a and b are constants, and V is the variable that depends on x, i.e., V is playing the role of y.

This formula makes mathematical sense for any x, but in the story problem the domain is much less. In the first place, x must be positive. In the second place, it must be less than half the length of either of the sides of the cardboard. Thus, the domain is

$$\left\{ x \in \mathbb{R} \,:\, 0 < x < \frac{1}{2}(\text{minimum of } a \text{ and } b) \right\}.$$

In interval notation we write: the domain is the interval $(0, \min(a, b)/2)$. You might think about whether we could allow 0 or (the minimum of a and b) to be in the domain. They make a certain physical sense, though we normally would not call the result a box. If we were to allow these values, what would the corresponding volumes be? Does that volume make sense? ♣

Example 2.3: Circle of Radius r Centered at the Origin

Is the circle of radius r centered at the origin the graph of a a function?

Solution. The equation for this circle is usually given in the form $x^2 + y^2 = r^2$. To write the equation in the form $y = f(x)$ we solve for y, obtaining $y = \pm\sqrt{r^2 - x^2}$. But *this is not a function*, because when we substitute a value in $(-r, r)$ for x there are two corresponding values of y. To get a function, we must choose one of the two signs in front of the square root. If we choose the positive sign, for example, we get the upper semicircle $y = f(x) = \sqrt{r^2 - x^2}$ (see figure 2.2). The domain of this function is the interval $[-r, r]$, i.e., x must be between $-r$ and r (including the endpoints). If x is outside of that interval, then $r^2 - x^2$ is negative, and we cannot take the square root. In terms of the graph, this just means that there are no points on the curve whose x-coordinate is greater than r or less than $-r$. ♣

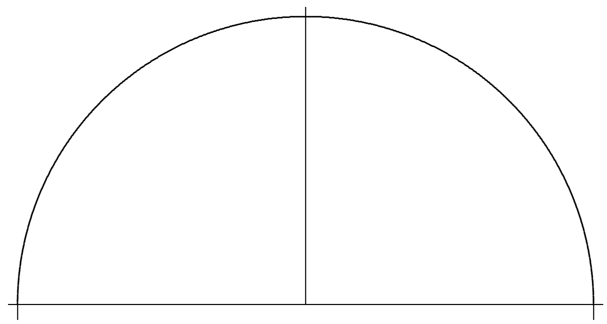


Figure 2.2: Upper semicircle $y = \sqrt{r^2 - x^2}$.

Example 2.4: Domain

Find the domain of

$$y = f(x) = \frac{1}{\sqrt{4x - x^2}}.$$

Solution. To answer this question, we must rule out the x-values that make $4x - x^2$ negative (because we cannot take the square root of a negative number) and also the x-values that make $4x - x^2$ zero (because if $4x - x^2 = 0$, then when we take the square root we get 0, and we cannot divide by 0). In other words, the domain consists of all x for which $4x - x^2$ is strictly positive. The inequality $4x - x^2 > 0$ was solved in Example 1.11. In interval notation, the domain is the interval $(0, 4)$. ♣

A function does not always have to be given by a single formula as the next example demonstrates.

Example 2.5: Piecewise Velocity

Suppose that $y = v(t)$ is the velocity function for a car which starts out from rest (zero velocity) at time $t = 0$; then increases its speed steadily to 20 m/sec, taking 10 seconds to do this; then travels at constant speed 20 m/sec for 15 seconds; and finally applies the brakes to decrease speed steadily to 0, taking 5 seconds to do this. The formula for $y = v(t)$ is different in each of the three time intervals: first $y = 2x$, then $y = 20$, then $y = -4x + 120$. The graph of this function is shown in figure 2.3.

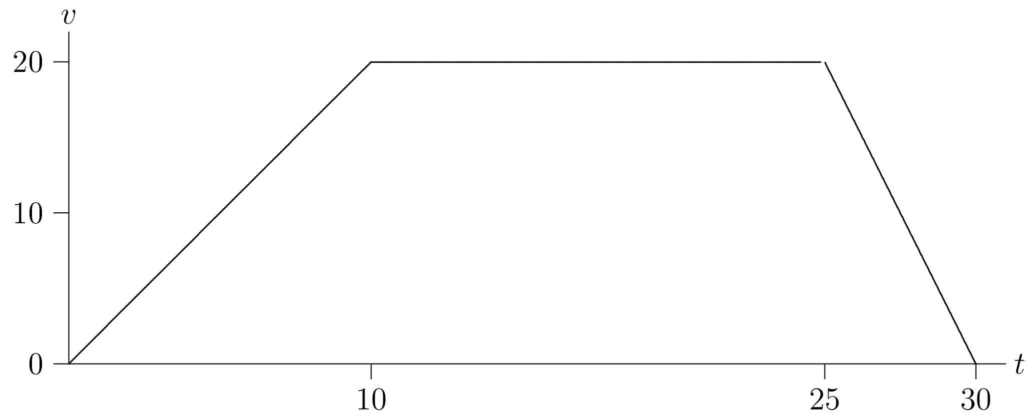


Figure 2.3: A velocity function.

Exercises for 2.1

2.1.1 *Find the domain of each of the following functions:*

(a) $y = x^2 + 1$

(b) $y = f(x) = \sqrt{2x - 3}$

(c) $y = f(x) = 1/(x + 1)$

(d) $y = f(x) = 1/(x^2 - 1)$

(e) $y = f(x) = \sqrt{-1/x}$

(f) $y = f(x) = \sqrt[3]{x}$

(g) $y = f(x) = \sqrt{r^2 - (x - h)^2}$, *where r and h are positive constants.*

(h) $y = f(x) = \sqrt[4]{x}$

(i) $y = \sqrt{1 - x^2}$

(j) $y = f(x) = \sqrt{1 - (1/x)}$

(k) $y = f(x) = 1/\sqrt{1 - (3x)^2}$

(l) $y = f(x) = \sqrt{x} + 1/(x - 1)$

(m) $y = f(x) = 1/(\sqrt{x} - 1)$

2.1.2 *A farmer wants to build a fence along a river. He has 500 feet of fencing and wants to enclose a rectangular pen on three sides (with the river providing the fourth side). If x is the length of the side perpendicular to the river, determine the area of the pen as a function of x. What is the domain of this function?*

2.1.3 *A can in the shape of a cylinder is to be made with a total of 100 square centimeters of material in the side, top, and bottom; the manufacturer wants the can to hold the maximum possible volume. Write the volume as a function of the radius r of the can; find the domain of the function.*

2.1.4 *A can in the shape of a cylinder is to be made to hold a volume of one liter (1000 cubic centimeters). The manufacturer wants to use the least possible material for the can. Write the surface area of the can (total of the top, bottom, and side) as a function of the radius r of the can; find the domain of the function.*

2.2 Transformations and Compositions

2.2.1. Tranformations

Transformations are operations we can apply to a function in order to obtain a *new* function. The most common transformations include translations, stretches and reflections. We summarize these below.

Function	Conditions	How to graph $F(x)$ given the graph of $f(x)$
$F(x) = f(x) + c$	$c > 0$	Shift $f(x)$ upwards by c units
$F(x) = f(x) - c$	$c > 0$	Shift $f(x)$ downwards by c units
$F(x) = f(x + c)$	$c > 0$	Shift $f(x)$ to the left by c units
$F(x) = f(x - c)$	$c > 0$	Shift $f(x)$ to the right by c units
$F(x) = -f(x)$		Reflect $f(x)$ about the x-axis
$F(x) = f(-x)$		Reflect $f(x)$ about the y-axis
$F(x) = \|f(x)\|$		Take the part of the graph of $f(x)$ that lies below the x-axis and reflect it about the x-axis

For horizontal and vertical stretches, different resources use different terminology and notation. Use the one you are most comfortable with! Below, both a, b are positive numbers. Note that we only use the term *stretch* in this case:

Function	Conditions	How to graph $F(x)$ given the graph of $f(x)$
$F(x) = af(x)$	$a > 0$	Stretch $f(x)$ vertically by a factor of a
$F(x) = f(bx)$	$b > 0$	Stretch $f(x)$ horizontally by a factor of $1/b$

In the next case, we use both the terms *stretch* and *shrink*. We also split up vertical stretches into two cases ($0 < a < 1$ and $a > 1$), and split up horizontal stretches into two cases ($0 < b < 1$ and $b > 1$). Note that having $0 < a < 1$ is the same as having $1/c$ with $c > 1$. Also note that *stretching by a factor of* $1/c$ is the same as *shrinking by a factor* c.

Function	Conditions	How to graph $F(x)$ given the graph of $f(x)$
$F(x) = cf(x)$	$c > 1$	Stretch $f(x)$ vertically by a factor of c
$F(x) = (1/c)f(x)$	$c > 1$	Shrink $f(x)$ vertically by a factor of c
$F(x) = f(cx)$	$c > 1$	Shrink $f(x)$ horizontally by a factor of c
$F(x) = f(x/c)$	$c > 1$	Stretch $f(x)$ horizontally by a factor of c

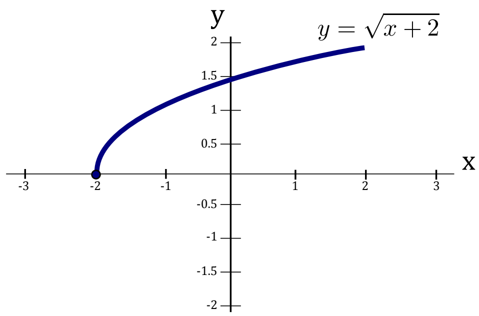
Some resources keep the condition $0 < c < 1$ rather than using $1/c$. This is illustrated in the next table.

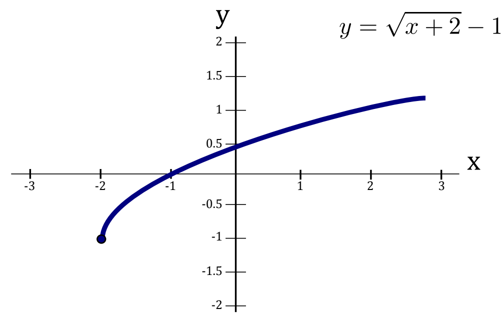
Function	Conditions	How to graph $F(x)$ given the graph of $f(x)$
$F(x) = df(x)$	$d > 1$	Stretch $f(x)$ vertically by a factor of d
$F(x) = df(x)$	$0 < d < 1$	Shrink $f(x)$ vertically by a factor of $1/d$
$F(x) = f(dx)$	$d > 1$	Shrink $f(x)$ horizontally by a factor of d
$F(x) = f(dx)$	$0 < d < 1$	Stretch $f(x)$ horizontally by a factor of $1/d$

Example 2.6: Transformations and Graph Sketching

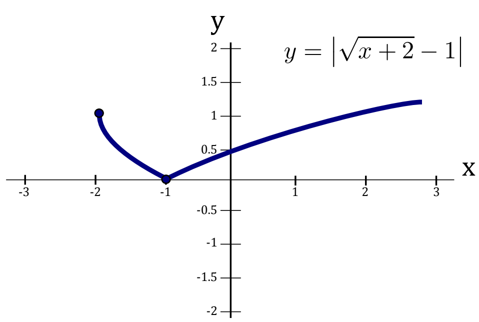
In this example we will use appropriate transformations to sketch the graph of the function $y = |\sqrt{x+2} - 1| - 1$.

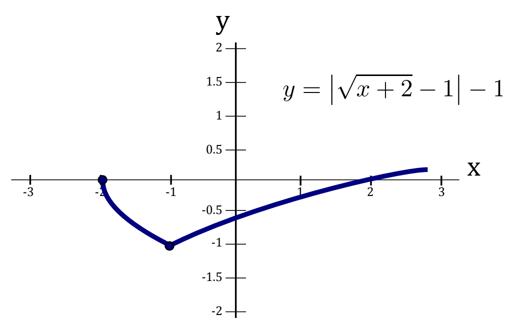
Solution. We start with the graph of a function we know how to sketch, in particular, $y = \sqrt{x}$: To obtain the graph of the function $y = \sqrt{x+2}$ from the graph $y = \sqrt{x}$, we must shift $y = \sqrt{x}$ to the left by 2 units. To obtain the graph of the function $y = \sqrt{x+2} - 1$ from the graph $y = \sqrt{x+2}$, we must shift $y = \sqrt{x+2}$ downwards by 1 unit.





To obtain the graph of the function $y = |\sqrt{x+2} - 1|$ from the graph $y = \sqrt{x+2} - 1$, we must take the part of the graph of $y = \sqrt{x+2} - 1$ that lies below the x-axis and reflect it (upwards) about the x-axis. Finally, to obtain the graph of the function $y = |\sqrt{x+2} - 1| - 1$ from the graph $y = |\sqrt{x+2} - 1|$, we must shift $y = |\sqrt{x+2} - 1|$ downwards by 1 unit:





♣

2.2.2. Combining Two Functions

Let f and g be two functions. Then we can form new functions by adding, subtracting, multiplying, or dividing. These new functions, $f + g$, $f - g$, fg and f/g, are defined in the usual way.

Key Idea 2.2.0: Operations on Functions

$$(f+g)(x) = f(x) + g(x) \qquad (f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Suppose D_f is the domain of f and D_g is the domain of g. Then the domains of $f+g$, $f-g$ and fg are the same and are equal to the intersection $D_f \cap D_g$ (that is, everything that is in *common* to both the domain of f and the domain of g). Since division by zero is *not allowed*, the domain of f/g is $\{x \in D_f \cap D_g : g(x) \neq 0\}$.

Another way to combine two functions f and g together is a procedure called composition.

Key Idea 2.2.0: Function Composition

Given two functions f and g, the **composition** of f and g, denoted by $f \circ g$, is defined as:

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is $\{x \in D_g : g(x) \in D_f\}$, that is, it contains all values x in the domain of g such that $g(x)$ is in the domain of f.

Example 2.7: Domain of a Composition

Let $f(x) = x^2$ and $g(x) = \sqrt{x}$. Find the domain of $f \circ g$.

Solution. The domain of f is $D_f = \{x \in \mathbb{R}\}$. The domain of g is $D_g = \{x \in \mathbb{R} : x \geq 0\}$. The function $(f \circ g)(x) = f(g(x))$ is:

$$f(g(x)) = \left(\sqrt{x}\right)^2 = x.$$

Typically, $h(x) = x$ would have a domain of $\{x \in \mathbb{R}\}$, but since it came from a **composed function**, we must consider $g(x)$ when looking at the domain of $f(g(x))$. Thus, the domain of $f \circ g$ is $\{x \in \mathbb{R} : x \geq 0\}$. ♣

Example 2.8: Combining Two Functions

Let $f(x) = x^2 + 3$ and $g(x) = x - 2$. Find $f+g$, $f-g$, fg, f/g, $f \circ g$ and $g \circ f$. Also, determine the domains of these new functions.

Solution. For $f+g$ we have:

$$(f+g)(x) = f(x) + g(x) = (x^2+3) + (x-2) = x^2 + x + 1.$$

For $f-g$ we have:

$$(f-g)(x) = f(x) - g(x) = (x^2+3) - (x-2) = x^2 + 3 - x + 2 = x^2 - x + 5.$$

For fg we have:

$$(fg)(x) = f(x) \cdot g(x) = (x^2+3)(x-2) = x^3 - 2x^2 + 3x - 6.$$

For f/g we have:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2+3}{x-2}.$$

For $f \circ g$ we have:

$$(f \circ g)(x) = f(g(x)) = f(x-2) = (x-2)^2 + 3 = x^2 - 4x + 7.$$

For $g \circ f$ we have:

$$(g \circ f)(x) = g(f(x)) = g(x^2+3) = (x^2+3) - 2 = x^2 + 1.$$

The domains of $f+g$, $f-g$, fg, $f \circ g$ and $g \circ f$ is $\{x \in \mathbb{R}\}$, while the domain of f/g is $\{x \in \mathbb{R} \,:\, x \neq 2\}$. ♣

As in the above problem, $f \circ g$ and $g \circ f$ are generally different functions.

Exercises for 2.2

2.2.1 *Starting with the graph of $y = \sqrt{x}$, the graph of $y = 1/x$, and the graph of $y = \sqrt{1-x^2}$ (the upper unit semicircle), sketch the graph of each of the following functions:*

(a) $f(x) = \sqrt{x-2}$

(b) $f(x) = -1 - 1/(x+2)$

(c) $f(x) = 4 + \sqrt{x+2}$

(d) $y = f(x) = x/(1-x)$

(e) $y = f(x) = -\sqrt{-x}$

(f) $f(x) = 2 + \sqrt{1-(x-1)^2}$

(g) $f(x) = -4 + \sqrt{-(x-2)}$

(h) $f(x) = 2\sqrt{1-(x/3)^2}$

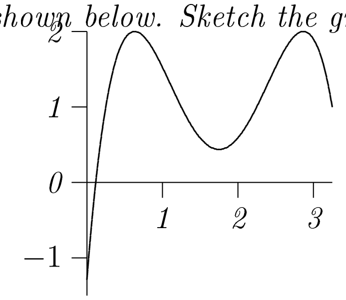
(i) $f(x) = 1/(x+1)$

(j) $f(x) = 4 + 2\sqrt{1-(x-5)^2/9}$

(k) $f(x) = 1 + 1/(x-1)$

(l) $f(x) = \sqrt{100 - 25(x-1)^2} + 2$

2.2.2 *The graph of $f(x)$ is shown below. Sketch the graphs of the following functions.*



(a) $y = f(x-1)$

(b) $y = 1 + f(x+2)$

(c) $y = 1 + 2f(x)$

(d) $y = 2f(3x)$

(e) $y = 2f(3(x-2)) + 1$

(f) $y = (1/2)f(3x-3)$

(g) $y = f(1 + x/3) + 2$

(h) $y = |f(x) - 2|$

2.2.3 *Suppose $f(x) = 3x - 9$ and $g(x) = \sqrt{x}$. What is the domain of the composition $(g \circ f)(x)$?*

2.3 Exponential Functions

An **exponential function** is a function of the form $f(x) = a^x$, where a is a constant. Examples are 2^x, 10^x and $(1/2)^x$. To more formally define the exponential function we look at various kinds of input values.

It is obvious that $a^5 = a \cdot a \cdot a \cdot a \cdot a$ and $a^3 = a \cdot a \cdot a$, but when we consider an exponential function a^x we can't be limited to substituting integers for x. What does $a^{2.5}$ or $a^{-1.3}$ or a^{π} mean? And is it really true that $a^{2.5}a^{-1.3} = a^{2.5-1.3}$? The answer to the first question is actually quite difficult, so we will evade it; the answer to the second question is "yes."

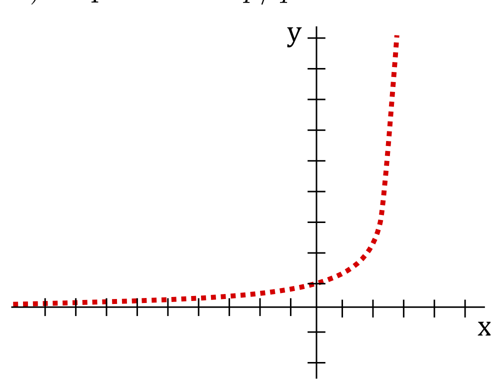
We'll evade the full answer to the hard question, but we have to know something about exponential functions. You need first to understand that since it's not "obvious" what 2^x should mean, we are really free to make it mean whatever we want, so long as we keep the behavior that *is* obvious, namely, when x is a positive integer. What else do we want to be true about 2^x? We want the properties of the previous two paragraphs to be true for all exponents: $2^x 2^y = 2^{x+y}$ and $(2^x)^y = 2^{xy}$.

After the positive integers, the next easiest number to understand is 0: $2^0 = 1$. You have presumably learned this fact in the past; why is it true? It is true precisely because we want $2^a 2^b = 2^{a+b}$ to be true about the function 2^x. We need it to be true that $2^0 2^x = 2^{0+x} = 2^x$, and this only works if $2^0 = 1$. The same argument implies that $a^0 = 1$ for any a.

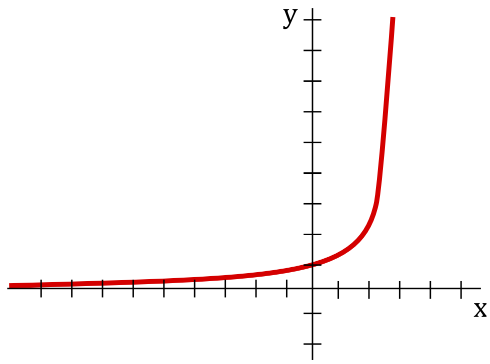
The next easiest set of numbers to understand is the negative integers: for example, $2^{-3} = 1/2^3$. We know that whatever 2^{-3} means it must be that $2^{-3}2^3 = 2^{-3+3} = 2^0 = 1$, which means that 2^{-3} must be $1/2^3$. In fact, by the same argument, once we know what 2^x means for some value of x, 2^{-x} must be $1/2^x$ and more generally $a^{-x} = 1/a^x$.

Next, consider an exponent $1/q$, where q is a positive integer. We want it to be true that $(2^x)^y = 2^{xy}$, so $(2^{1/q})^q = 2$. This means that $2^{1/q}$ is a q-th root of 2, $2^{1/q} = \sqrt[q]{2}$. This is all we need to understand that $2^{p/q} = (2^{1/q})^p = (\sqrt[q]{2})^p$ and $a^{p/q} = (a^{1/q})^p = (\sqrt[q]{a})^p$.

What's left is the hard part: what does 2^x mean when x cannot be written as a fraction, like $x = \sqrt{2}$ or $x = \pi$? What we know so far is how to assign meaning to 2^x whenever $x = p/q$. If we were to graph a^x (for some $a > 1$) at points $x = p/q$ then we'd see something like this:

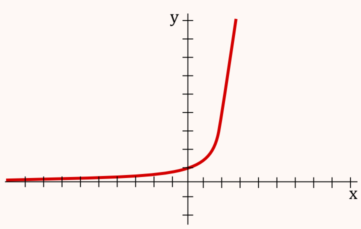


This is a poor picture, but it illustrates a series of individual points above the rational numbers on the x-axis. There are really a lot of "holes" in the curve, above $x = \pi$, for example. But (this is the hard part) it is possible to prove that the holes can be "filled in", and that the resulting function, called a^x, really does have the properties we want, namely that $a^x a^y = a^{x+y}$ and $(a^x)^y = a^{xy}$. Such a graph would then look like this:



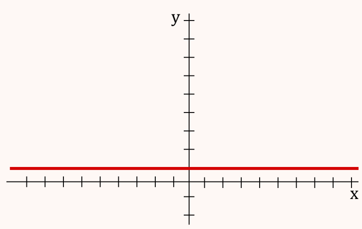
Key Idea 2.3.0: Three Types of Exponential Functions

There are *three kinds* of exponential functions $f(x) = a^x$ depending on whether $a > 1$, $a = 1$ or $0 < a < 1$:

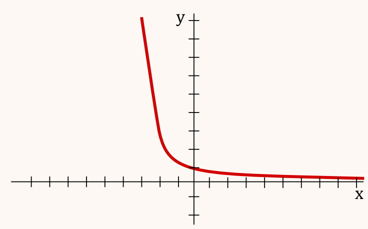


$f(x) = a^x$

$a > 1$



$f(x) = 1^x$



$f(x) = a^x$

$0 < a < 1$

Properties of Exponential Functions

The first thing to note is that if $a < 0$ then problems can occur. Observe that if $a = -1$ then $(-1)^x$ is not defined for every x. For example, $x = 1/2$ is a square root and gives $(-1)^{1/2} = \sqrt{-1}$ which is not a real number.

Key Idea 2.3.0: Exponential Function Properties

- *Only defined for positive a:* a^x is only defined for all real x if $a > 0$
- *Always positive:* $a^x > 0$, for all x
- *Exponent rules:*
  1. $a^x a^y = a^{x+y}$
  2. $\dfrac{a^x}{a^y} = a^{x-y}$
  3. $(a^x)^y = a^{xy} = a^{yx} = (a^y)^x$
  4. $a^x b^x = (ab)^x$
- *Long term behaviour:* If $a > 1$, then $a^x \to \infty$ as $x \to \infty$ and $a^x \to 0$ as $x \to -\infty$.

The last property can be observed from the graph. If $a > 1$, then as x gets larger and larger, so does a^x. On the other hand, as x gets large and negative, the function approaches the x-axis, that is, a^x approaches 0.

Example 2.9: Reflection of Exponential

Determine an equation of the function after reflecting $y = 2^x$ about the line $x = -2$.

Solution. First reflect about the y-axis to get $y = 2^{-x}$. Now shift by $2 \times 2 = 4$ units to the *left* to get $y = 2^{-(x+4)}$. Side note: Can you see why this sequence of transformations is the same as reflection in the line $x = -2$? Can you come up with a general rule for these types of reflections?

♣

Example 2.10: Determine the Exponential Function

Determine the exponential function $f(x) = ka^x$ that passes through the points $(1, 6)$ and $(2, 18)$.

Solution. We substitute our two points into the equation to get:

$$x = 1, y = 6 \to 6 = ka^1$$

$$x = 2, y = 18 \to 18 = ka^2$$

This gives us $6 = ka$ and $18 = ka^2$. The first equation is $k = 6/a$ and subbing this into the second gives: $18 = (6/a)a^2$. Thus, $18 = 6a$ and $a = 3$. Now we can see from $6 = ka$ that $k = 2$. Therefore, the exponential function is

$$f(x) = 2 \cdot 3^x.$$

♣

There is one base that is so important and convenient that we give it a special symbol. This number is denoted by $e = 2.71828\ldots$ (and is an irrational number). Its *importance* stems from the fact that it simplifies many formulas of Calculus and also shows up in other fields of mathematics.

Example 2.11: Domain of Function with Exponential

Find the domain of $f(x) = \dfrac{1}{\sqrt{e^x+1}}$.

Solution. For domain, we cannot divide by zero or take the square root of negative numbers. Note that one of the properties of exponentials is that they are always positive! Thus, $e^x + 1 > 0$ (in fact, as $e^x > 0$ we actually have that $e^x + 1$ is at least one). Therefore, $e^x + 1$ is never zero nor negative, and gives no restrictions on x. Thus, the domain is $\mathbb{R}$. ♣

Exercises for 2.3

2.3.1 *Determine an equation of the function* $y = a^x$ *passing through the point* $(3, 8)$.

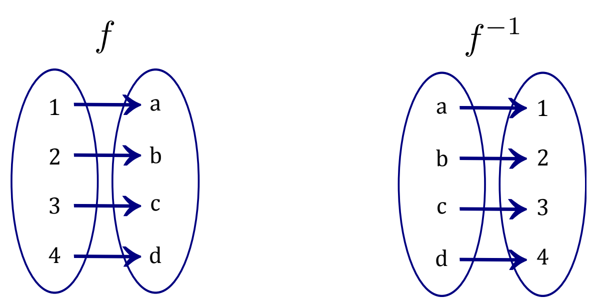
2.3.2 *Find the* y*-intercept of* $f(x) = 4^x + 6$.

2.3.3 *Find the* y*-intercept of* $f(x) = 2\left(\frac{1}{2}\right)^x$.

2.3.4 *Find the domain of* $y = e^{-x} + e^{\frac{1}{x}}$.

2.4 Inverse Functions

In mathematics, an *inverse* is a function that serves to "undo" another function. That is, if $f(x)$ produces y, then putting y into the inverse of f produces the output x. A function f that has an inverse is called invertible and the inverse is denoted by f^{-1}. It is best to illustrate inverses using an arrow diagram:



Notice how f maps 1 to a, and f^{-1} undoes this, that is, f^{-1} maps a back to 1. Don't confuse $f^{-1}(x)$ with exponentiation: the inverse f^{-1} is *different* from $\frac{1}{f(x)}$.

Not every function has an inverse. It is easy to see that if a function $f(x)$ is going to have an inverse, then $f(x)$ *never* takes on the same value twice. We give this property a special name.

A function $f(x)$ is called **one-to-one** if every element of the range corresponds to *exactly* one element of the domain. Similar to the Vertical Line Test (VLT) for functions, we have the Horizontal Line Test (HLT) for the one-to-one property.

Theorem 2.2: The Horizontal Line Test

A function is one-to-one if and only if there is no horizontal line that intersects its graph more than once.

Example 2.12: Parabola is Not One-to-one

The parabola $f(x) = x^2$ it not one-to-one because it does not satisfy the horizontal line test. For example, the horizontal line $y = 1$ intersects the parabola at two points, when $x = -1$ and $x = 1$.

We now formally define the inverse of a function.

Definition 2.1: Inverse of a Function

Let $f(x)$ and $g(x)$ be two one-to-one functions. If $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ then we say that $f(x)$ and $g(x)$ are **inverses** *of each other. We denote $g(x)$ (the inverse of $f(x)$) by $g(x) = f^{-1}(x)$.*

Thus, if f maps x to y, then f^{-1} maps y back to x. This gives rise to the *cancellation formulas*:

$$f^{-1}(f(x)) = x, \quad \text{for every } x \text{ in the domain of } f(x),$$
$$f(f^{-1}(x)) = x, \quad \text{for every } x \text{ in the domain of } f^{-1}(x).$$

Example 2.13: Finding the Inverse at Specific Values

If $f(x) = x^9 + 2x^7 + x + 1$, find $f^{-1}(5)$ and $f^{-1}(1)$.

Solution. Rather than trying to compute a formula for f^{-1} and then computing $f^{-1}(5)$, we can simply find a number c such that f evaluated at c gives 5. Note that subbing in some simple values ($x = -3, -2, 1, 0, 1, 2, 3$) and evaluating $f(x)$ we eventually find that $f(1) = 1^9 + 2(1^7) + 1 + 1 = 5$ and $f(0) = 1$. Therefore, $f^{-1}(5) = 1$ and $f^{-1}(1) = 0$. ♣

To compute the equation of the inverse of a function we use the following *guidelines*.

Key Idea 2.4.0: Guidelines for Computing Inverses

1. Write down $y = f(x)$.
2. Solve for x in terms of y.
3. Switch the x's and y's.
4. The result is $y = f^{-1}(x)$.

Example 2.14: Finding the Inverse Function

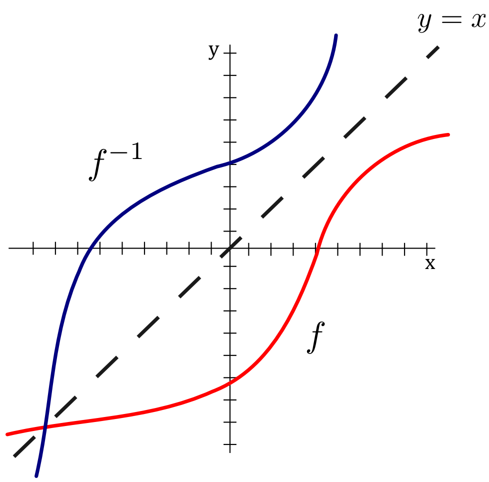
We find the inverse of the function $f(x) = 2x^3 + 1$.

Solution. Starting with $y = 2x^3 + 1$ we solve for x as follows:

$$y - 1 = 2x^3 \quad \to \quad \frac{y-1}{2} = x^3 \quad \to \quad x = \sqrt[3]{\frac{y-1}{2}}.$$

Therefore, $f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$. ♣

This example shows how to find the inverse of a function *algebraically*. But what about finding the inverse of a function *graphically*? Step 3 (switching x and y) gives us a good graphical technique to find the inverse, namely, for each point (a, b) where $f(a) = b$, sketch the point (b, a) for the inverse. More formally, to obtain $f^{-1}(x)$ *reflect* the graph $f(x)$ about the line $y = x$.



Exercises for 2.4

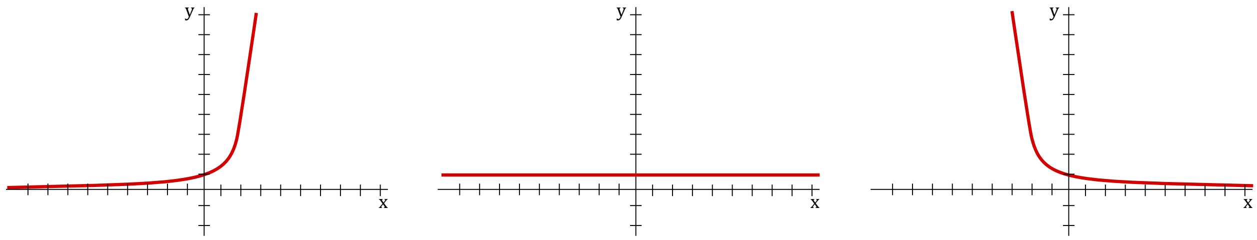
2.4.1 *Is the function* $f(x) = |x|$ *one-to-one?*

2.4.2 *If* $h(x) = e^x + x + 1$, *find* $h^{-1}(2)$.

2.4.3 *Find a formula for the inverse of the function* $f(x) = \frac{x+2}{x-2}$.

2.5 Logarithms

Recall the *three kinds* of exponential functions $f(x) = a^x$ depending on whether $0 < a < 1$, $a = 1$ or $a > 1$:



$f(x) = a^x$ $a > 1$ $\qquad$ $f(x) = 1^x$ $\qquad$ $f(x) = a^x$ $0 < a < 1$

So long as $a \neq 1$, the function $f(x) = a^x$ satisfies the horizontal line test and therefore has an inverse. We call the *inverse of* a^x the **logarithmic function with base a** and denote it by $\log_a$. In particular,

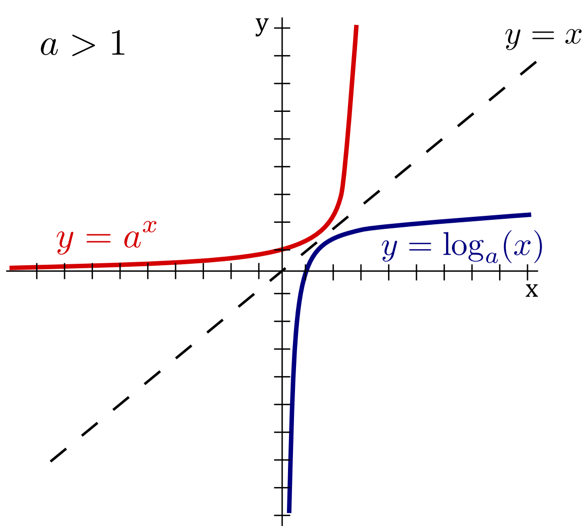
$$\log_a x = y \iff a^y = x.$$

The *cancellation formulas* for logs are:

$$\log_a(a^x) = x, \quad \text{for every } x \in \mathbb{R},$$

$$a^{\log_a(x)} = x, \quad \text{for every } x > 0.$$

Since the function $f(x) = a^x$ for $a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$, the logarithmic function has domain $(0, \infty)$ and range $\mathbb{R}$. For the most part, we only focus on logarithms with a base larger than 1 (i.e., $a > 1$) as these are the most important.



Notice that every logarithm passes through the point $(1, 0)$ in the same way that every exponential function passes through the point $(0, 1)$.

Some properties of logarithms are as follows.

Key Idea 2.5.0: Logarithm Properties

Let A, B be positive numbers and $b > 0$ ($b \neq 1$) be a base.

- $\log_b(AB) = \log_b A + \log_b B$,
- $\log_b\left(\frac{A}{B}\right) = \log_b A - \log_b B$,
- $\log_b(A^n) = n \log_b A$, where n is any real number.

Example 2.15: Compute Lorarithms

To compute $\log_2(24) - \log_2(3)$ *we can do the following:*

$$\log_2(24) - \log_2(3) = \log_2\left(\frac{24}{3}\right) = \log_2(8) = 3,$$

since $2^3 = 8$.

The Natural Logarithm

As mentioned earlier for exponential functions, the number $e \approx 2.71828\ldots$ is the most convenient base to use in Calculus. For this reason we give the logarithm with base e a special name: **the natural logarithm**. We also give it special notation:

$$\log_e x = \ln x.$$

You may pronounce ln as either: "el - en", "lawn", or refer to it as "natural log". The above properties of logarithms also apply to the natural logarithm.

Often we need to turn a logarithm (in a different base) into a natural logarithm. This gives rise to the *change of base formula*.

Key Idea 2.5.0: Change of Base Formula

$$\log_a x = \frac{\ln x}{\ln a}.$$

Example 2.16: Combine Logarithms

Write $\ln A + 2\ln B - \ln C$ as a single logarithm.

Solution. Using properties of logarithms, we have,

$$\begin{aligned} \ln A + 2\ln B - \ln C &= \ln A + \ln B^2 - \ln C \\ &= \ln(AB^2) - \ln C \\ &= \ln \frac{AB^2}{C} \end{aligned}$$

♣

Example 2.17: Solve Exponential Equations using Logarithms

If $e^{x+2} = 6e^{2x}$, then solve for x.

Solution. Taking the natural logarithm of both sides and noting the cancellation formulas (along with $\ln e = 1$), we have:

$$\begin{aligned} e^{x+2} &= 6e^{2x} \\ \ln e^{x+2} &= \ln(6e^{2x}) \\ x+2 &= \ln 6 + \ln e^{2x} \\ x+2 &= \ln 6 + 2x \\ x &= 2 - \ln 6 \end{aligned}$$

♣

Example 2.18: Solve Logarithm Equations using Exponentials

If $\ln(2x-1) = 2\ln(x)$, then solve for x.

Solution. "Taking e" of both sides and noting the cancellation formulas, we have:

$$\begin{aligned} e^{\ln(2x-1)} &= e^{2\ln(x)} \\ (2x-1) &= e^{\ln(x^2)} \\ 2x-1 &= x^2 \\ x^2 - 2x + 1 &= 0 \\ (x-1)^2 &= 0 \end{aligned}$$

Therefore, the solution is $x = 1$. ♣

Exercises for 2.5

2.5.1 *Expand* $\log_{10}((x+45)^7(x-2))$.

2.5.2 *Expand* $\log_2 \dfrac{x^3}{3x-5+(7/x)}$.

2.5.3 *Write* $\log_2 3x + 17\log_2(x-2) - 2\log_2(x^2+4x+1)$ *as a single logarithm.*

2.5.4 *Solve* $\log_2(1+\sqrt{x}) = 6$ *for* x.

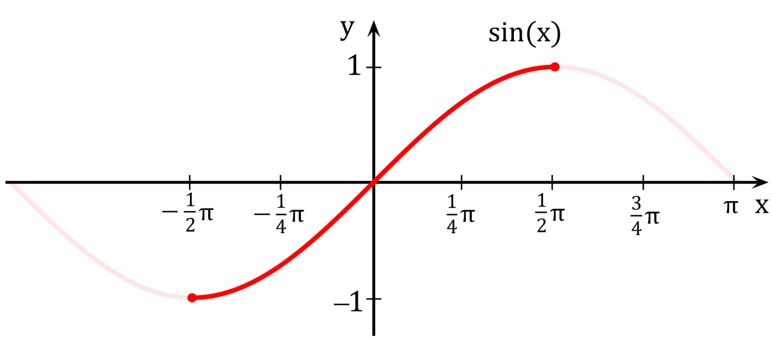
2.5.5 *Solve* $2^{x^2} = 8$ *for* x.

2.5.6 *Solve* $\log_2(\log_3(x)) = 1$ *for* x.

2.6 Inverse Trigonometric Functions

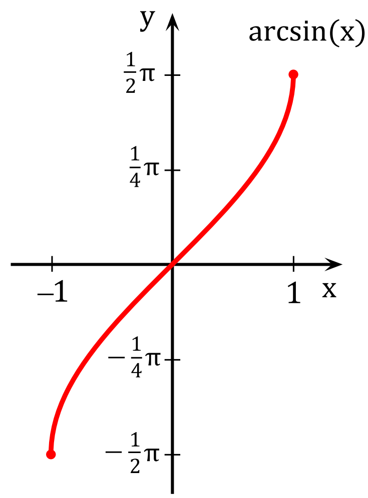
The trigonometric functions frequently arise in problems, and often it is necessary to invert the functions, for example, to find an angle with a specified sine. Of course, there are many angles with the same sine, so the sine function doesn't actually have an inverse that reliably "undoes" the sine function. If you know that $\sin x = 0.5$, you can't reverse this to discover x, that is, you can't solve for x, as there are infinitely many angles with sine 0.5. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values of the sine exactly once. If we "discard" all other angles, the resulting function does have a proper inverse.

The sine takes on all values between -1 and 1 exactly once on the interval $[-\pi/2, \pi/2]$.



If we truncate the sine, keeping only the interval $[-\pi/2, \pi/2]$, then this truncated sine has an inverse function. We call this the inverse sine or the arcsine, and write it in one of two common notation:

$y = \arcsin(x)$, or $y = \sin^{-1}(x)$.



Recall that a function and its inverse undo each other in either order, for example, $(\sqrt[3]{x})^3 = x$ and $\sqrt[3]{x^3} = x$. This does not work with the sine and the "inverse sine" because the inverse sine is the inverse of the truncated sine function, not the real sine function. It is true that $\sin(\arcsin(x)) = x$, that is, the sine undoes the arcsine. It is not true that the arcsine undoes the sine, for example, $\sin(5\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$, so doing first the sine then the arcsine does not get us back where we started. This is because $5\pi/6$ is not in the domain of the truncated sine. If we start with an angle between $-\pi/2$ and $\pi/2$ then the arcsine does reverse the sine: $\sin(\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$.

Example 2.19: Arcsine of Common Values

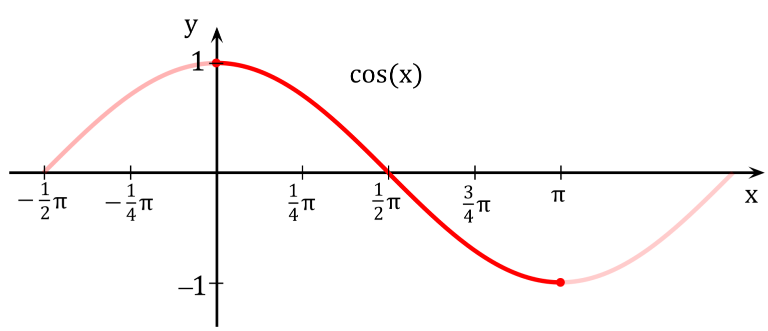
Compute $\sin^{-1}(0), \quad \sin^{-1}(1) \quad$ *and* $\quad \sin^{-1}(-1)$.

Solution. These come directly from the graph of $y = \arcsin x$:

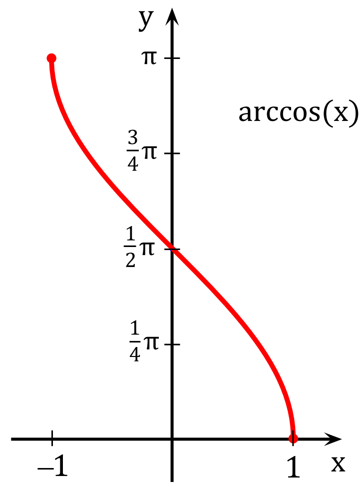
$$\sin^{-1}(0) = 0 \qquad \sin^{-1}(1) = \frac{\pi}{2} \qquad \sin^{-1}(-1) = -\frac{\pi}{2}$$

♣

We can do something similar for the cosine function. As with the sine, we must first truncate the cosine so that it can be inverted, in particular, we use the interval $[0, \pi]$.



Note that the truncated cosine uses a different interval than the truncated sine, so that if $y = \arccos(x)$ we know that $0 \le y \le \pi$.



Example 2.20: Arccosine of Common Values

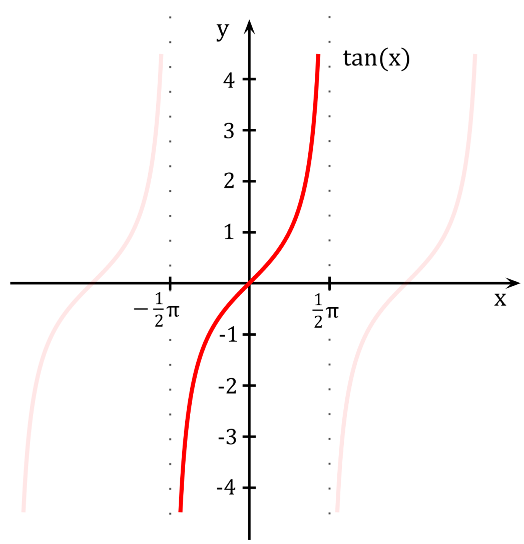
Compute $\cos^{-1}(0)$, $\cos^{-1}(1)$ *and* $\cos^{-1}(-1)$.

Solution. These come directly from the graph of $y = \arccos x$:

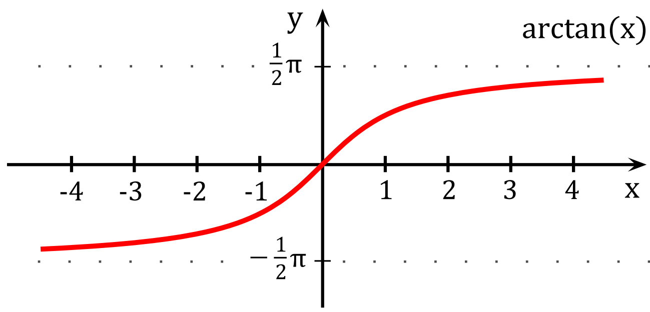
$$\cos^{-1}(0) = \frac{\pi}{2} \qquad \cos^{-1}(1) = 0 \qquad \cos^{-1}(-1) = \pi$$

♣

Finally we look at the tangent; the other trigonometric functions also have "partial inverses" but the sine, cosine and tangent are enough for most purposes. The truncated tangent uses an interval of $(-\pi/2, \pi/2)$.



Reflecting the truncated tangent in the line $y = x$ gives the arctangent function.



Example 2.21: Arctangent of Common Values

Compute $\tan^{-1}(0)$*. What value does* $\tan^{-1} x$ *approach as* x *gets larger and larger? What value does* $\tan^{-1} x$ *approach as* x *gets large (and negative)?*

Solution. These come directly from the graph of $y = \arctan x$. In particular, $\tan^{-1}(0) = 0$. As x gets larger and larger, $\tan^{-1} x$ approaches a value of $\frac{\pi}{2}$, whereas, as x gets large but negative, $\tan^{-1} x$ approaches a value of $-\frac{\pi}{2}$. ♣

The cancellation rules are tricky since we restricted the domains of the trigonometric functions in order to obtain inverse trig functions:

Key Idea 2.6.0: Cancellation Rules

$$\sin(\sin^{-1} x) = x, \quad x \in [-1, 1] \qquad \sin^{-1}(\sin x) = x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos(\cos^{-1} x) = x, \quad x \in [-1, 1] \qquad \cos^{-1}(\cos x) = x, \quad x \in [0, \pi]$$

$$\tan(\tan^{-1} x) = x, \quad x \in (-\infty, \infty) \qquad \tan^{-1}(\tan x) = x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Example 2.22: Arcsine of $1/2$

Find $\sin^{-1}(1/2)$.

Solution. Since $\sin^{-1}(x)$ outputs values in $[-\pi/2,\ \pi/2]$, the answer must be in this interval. Let $\theta = \sin^{-1}(1/2)$. We need to compute θ. Take the sine of both sides to get $\sin\theta = \sin(\sin^{-1}(1/2)) = 1/2$ by the cancellation rule. There are many angles θ that work, but we want the one in the interval $[-\pi/2,\ \pi/2]$. Thus, $\theta = \pi/6$ and hence, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. ♣

Example 2.23: Arccosine and the Cancellation Rule

Compute $\cos^{-1}(\cos(0)), \quad \cos^{-1}(\cos(\pi)), \quad \cos^{-1}(\cos(2\pi)), \quad \cos^{-1}(\cos(3\pi))$.

Solution. Since $\cos^{-1}(x)$ outputs values in $[0,\ \pi]$, the answers must be in this interval. The first two we can cancel using the cancellation rules:

$$\cos^{-1}(\cos(0)) = 0 \qquad \text{and} \qquad \cos^{-1}(\cos(\pi)) = \pi.$$

The third one we cannot cancel since $2\pi \notin [0, \pi]$:

$$\cos^{-1}(\cos(2\pi)) \quad \text{is NOT equal to} \quad 2\pi.$$

But we know that cosine is a 2π-periodic function, so $\cos(2\pi) = \cos(0)$:

$$\cos^{-1}(\cos(2\pi)) = \cos^{-1}(\cos(0)) = 0$$

Similarly with the fourth one, we can NOT cancel yet since $3\pi \notin [0, \pi]$. Using $\cos(3\pi) = \cos(3\pi - 2\pi) = \cos(\pi)$:

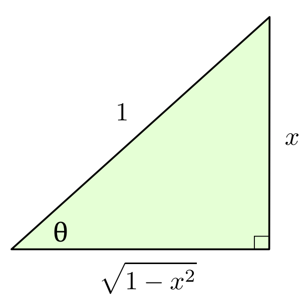
$$\cos^{-1}(\cos(3\pi)) = \cos^{-1}(\cos(\pi)) = \pi.$$

♣

Example 2.24: The Triangle Technique

Rewrite the expression $\cos(\sin^{-1} x)$ *without trig functions. Note that the domain of this function is all* $x \in [-1, 1]$.

Solution. Let $\theta = \sin^{-1} x$. We need to compute $\cos\theta$. Taking the sine of both sides gives $\sin\theta = \sin(\sin^{-1}(x)) = x$ by the cancellation rule. We then draw a right triangle using $\sin\theta = x/1$:



If z is the remaining side, then by the Pythagorean Theorem:

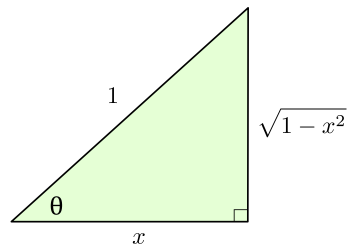
$$z^2 + x^2 = 1 \qquad \to \qquad z^2 = 1 - x^2 \qquad \to \qquad z = \pm\sqrt{1-x^2}$$

and hence $z = +\sqrt{1-x^2}$ since $\theta \in [-\pi/2,\ \pi/2]$. Thus, $\cos\theta = \sqrt{1-x^2}$ by SOH CAH TOA, so, $\cos(\sin^{-1} x) = \sqrt{1-x^2}$. ♣

Example 2.25: The Triangle Technique 2

For $x \in (0,1)$*, rewrite the expression* $\sin(2\cos^{-1} x)$*. Compute* $\sin(2\cos^{-1}(1/2))$.

Solution. Let $\theta = \cos^{-1} x$ so that $\cos\theta = x$. The question now asks for us to compute $\sin(2\theta)$. We then draw a right triangle using $\cos\theta = x/1$:



To find $\sin(2\theta)$ we use the double angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$. But $\sin\theta = \sqrt{1-x^2}$, for $\theta \in [0,\pi]$, and $\cos\theta = x$. Therefore, $\sin(2\cos^{-1} x) = 2x\sqrt{1-x^2}$. When $x = 1/2$ we have $\sin(2\cos^{-1}(1/2)) = \dfrac{\sqrt{3}}{2}$. ♣

In Figure 2.4 we show the restrictions of the domains of the standard trigonometric functions that allow them to be invertible.

Function	Domain	Range	Inverse Function	Domain	Range
$\sin x$	$[-\pi/2, \pi/2]$	$[-1,1]$	$\sin^{-1} x$	$[-1,1]$	$[-\pi/2,\pi/2]$
$\cos x$	$[0,\pi]$	$[-1,1]$	$\cos^{-1}(x)$	$[-1,1]$	$[0,\pi]$
$\tan x$	$(-\pi/2,\pi/2)$	$(-\infty,\infty)$	$\tan^{-1}(x)$	$(-\infty,\infty)$	$(-\pi/2,\pi/2)$
$\csc x$	$[-\pi/2,0)\cup(0,\pi/2]$	$(-\infty,-1]\cup[1,\infty)$	$\csc^{-1} x$	$(-\infty,-1]\cup[1,\infty)$	$(0,\pi/2]\cup(\pi,3\pi/2]$
$\sec x$	$[0,\pi/2)\cup(\pi/2,\pi]$	$(-\infty,-1]\cup[1,\infty)$	$\sec^{-1}(x)$	$(-\infty,-1]\cup[1,\infty)$	$[0,\pi/2)\cup[\pi,3\pi/2]$
$\cot x$	$(0,\pi)$	$(-\infty,\infty)$	$\cot^{-1}(x)$	$(-\infty,\infty)$	$(0,\pi)$

Figure 2.4: Domains and ranges of the trigonometric and inverse trigonometric functions.

Exercises for 2.6

2.6.1 *Compute the following:*

(a) $\sin^{-1}(\sqrt{3}/2)$

(b) $\cos^{-1}(-\sqrt{2}/2)$

2.6.2 *Compute the following:*

(a) $\sin^{-1}(\sin(\pi/4))$

(b) $\sin^{-1}(\sin(17\pi/3))$

(c) $\cos\left(\cos^{-1}(1/3)\right)$

(d) $\tan\left(\cos^{-1}(-4/5)\right)$

2.6.3 *Rewrite the expression* $\tan\left(\cos^{-1} x\right)$ *without trigonometric functions. What is the domain of this function?*

2.7 Hyperbolic Functions

The **hyperbolic functions** are a set of functions that have many applications to mathematics, physics, and engineering. Among many other applications, they are used to describe the formation of satellite rings around planets, to describe the shape of a rope hanging from two points, and have application to the theory of special relativity. This section defines the hyperbolic functions and describes many of their properties, especially their usefulness to calculus.

These functions are sometimes referred to as the "hyperbolic trigonometric functions" as there are many, many connections between them and the standard trigonometric functions. Figure 2.6 demonstrates one such connection. Just as cosine and sine are used to define points on the circle defined by $x^2 + y^2 = 1$, the functions **hyperbolic cosine** and **hyperbolic sine** are used to define points on the hyperbola $x^2 - y^2 = 1$. This is a bit surprising given our initial definitions.

Definition 2.2: Hyperbolic Functions

1. $\cosh x = \dfrac{e^x + e^{-x}}{2}$
2. $\sinh x = \dfrac{e^x - e^{-x}}{2}$
3. $\tanh x = \dfrac{\sinh x}{\cosh x}$
4. $\text{sech} x = \dfrac{1}{\cosh x}$
5. $\text{csch} x = \dfrac{1}{\sinh x}$
6. $\coth x = \dfrac{\cosh x}{\sinh x}$

These hyperbolic functions are graphed in Figure 2.5. In the graphs of $\cosh x$ and $\sinh x$, graphs of $e^x/2$ and $e^{-x}/2$ are included with dashed lines. As x gets "large," $\cosh x$ and $\sinh x$ each act like $e^x/2$; when x is a large negative number, $\cosh x$ acts like $e^{-x}/2$ whereas $\sinh x$ acts like $-e^{-x}/2$.

Notice the domains of $\tanh x$ and $\text{sech} x$ are $(-\infty, \infty)$, whereas both $\coth x$ and $\text{csch} x$ have vertical asymptotes at $x = 0$. Also note the ranges of these functions, especially $\tanh x$: as $x \to \infty$, both $\sinh x$ and $\cosh x$ approach $e^{-x}/2$, hence $\tanh x$ approaches 1.

The following example explores some of the properties of these functions that bear remarkable resemblance to the properties of their trigonometric counterparts.

Pronunciation Note:

"cosh" rhymes with "gosh,"

"sinh" rhymes with "pinch," or alternatively is pronounced"shine" and

"tanh" rhymes with "ranch.", or alternatively is pronounced"tank"

Notice that cosh is even (that is, $\cosh(-x) = \cosh(x)$) while sinh is odd ($\sinh(-x) = -\sinh(x)$), and $\cosh x + \sinh x = e^x$. Also, for all x, $\cosh x > 0$, while $\sinh x = 0$ if and only if $e^x - e^{-x} = 0$, which is true precisely when $x = 0$.

Theorem 2.3: Range of Hyperbolic Cosine

The range of $\cosh x$ *is* $[1, \infty)$.

Proof. Let $y = \cosh x$. We solve for x:

$$\begin{array}{rcl} y &=& \dfrac{e^x + e^{-x}}{2} \\ 2y &=& e^x + e^{-x} \\ 2ye^x &=& e^{2x} + 1 \\ 0 &=& e^{2x} - 2ye^x + 1 \\ e^x &=& \dfrac{2y \pm \sqrt{4y^2 - 4}}{2} \\ e^x &=& y \pm \sqrt{y^2 - 1} \end{array}$$

From the last equation, we see $y^2 \geq 1$, and since $y \geq 0$, it follows that $y \geq 1$.
Now suppose $y \geq 1$, so $y \pm \sqrt{y^2 - 1} > 0$. Then $x = \ln(y \pm \sqrt{y^2 - 1})$ is a real number, and $y = \cosh x$, so y is in the range of $\cosh(x)$. ♣

The graphs can be generated based on our knowledge of the exponential function.

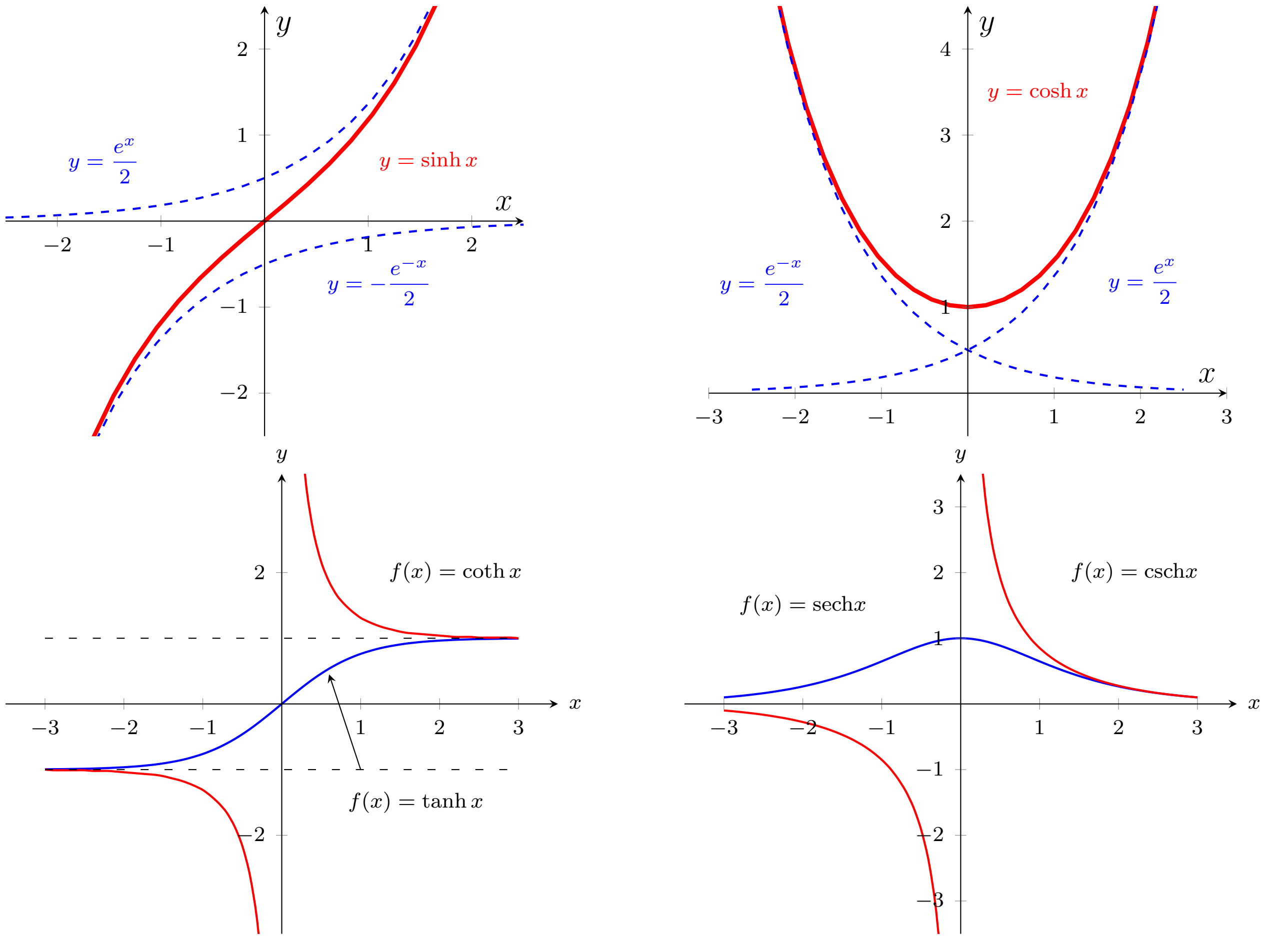


Figure 2.5: Graphs of the Hyperbolic Functions

Certainly the hyperbolic functions do not closely resemble the trigonometric functions graphically. But they do have analogous properties, beginning with the following identity.

Theorem 2.4: Hyperbolic Identity

For all x in $\mathbb{R}$, $\cosh^2 x - \sinh^2 x = 1$.

Proof. The proof is a straightforward computation:

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1.$$

♣

This immediately gives two additional identities:

$$1 - \tanh^2 x = \mathrm{sech}^2 x \qquad \text{and} \qquad \coth^2 x - 1 = \mathrm{csch}^2 x.$$

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of $x^2 - y^2 = 1$ is a hyperbola with asymptotes $x = \pm y$ whose x-intercepts are ± 1. If (x, y) is a point on the right half of the hyperbola, and if we let $x = \cosh\theta$, then $y = \pm\sqrt{x^2 - 1} = \pm\sqrt{\cosh^2 x - 1} = \pm\sinh\theta$. So for some suitable θ, $\cosh\theta$ and $\sinh\theta$ are the coordinates of a typical point on the hyperbola. In fact, it turns out that θ is twice the area shown in the first graph of figure 2.6. Even this is analogous to trigonometry; $\cos\theta$ and $\sin\theta$ are the coordinates of a typical point on the unit circle, and θ is twice the area shown in the second graph of Figure 2.6.

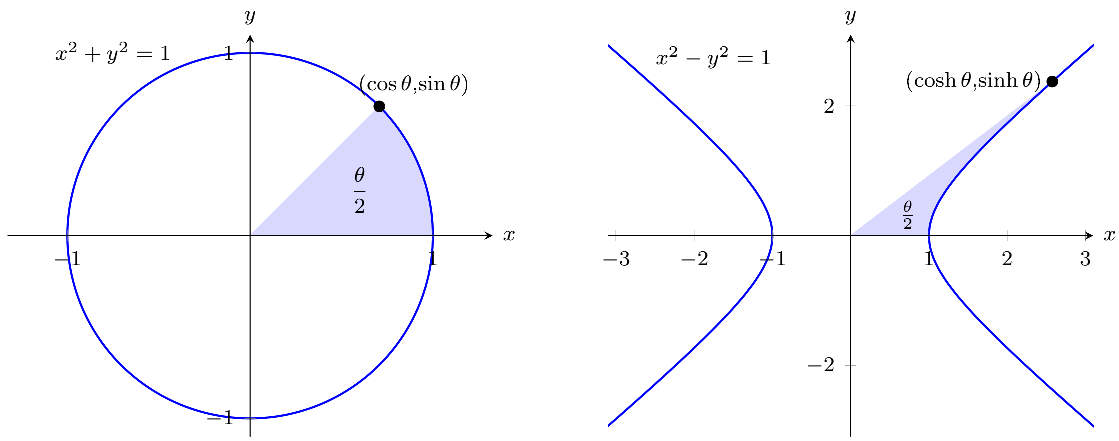


Figure 2.6: Using trigonometric functions to define points on a circle and hyperbolic functions to define points on a hyperbola. The area of the shaded regions are included in them.

Example 2.26: Computing Hyperbolic Tangent

Use the definition of the hyperbolic functions to rewrite the following expressions.

1. $\tanh^2 x + \text{sech}^2 x$
2. $2\cosh x \sinh x$
3. $\tanh(\ln 2)$.

Solution.

1. $$\tanh^2 x + \text{sech}^2 x = \frac{\sinh^2 x}{\cosh^2 x} + \frac{1}{\cosh^2 x} = \frac{\sinh^2 x + 1}{\cosh^2 x} \qquad \text{Now use Theorem 2.4.}$$
$$= \frac{\cosh^2 x}{\cosh^2 x} = 1.$$
So $\tanh^2 x + \text{sech}^2 x = 1$.

2. $$\begin{aligned} 2\cosh x \sinh x &= 2\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right) \\ &= 2\cdot\frac{e^{2x}-e^{-2x}}{4} \\ &= \frac{e^{2x}-e^{-2x}}{2} = \sinh(2x). \end{aligned}$$
Thus $2\cosh x \sinh x = \sinh(2x)$.

3. $$\tanh(\ln 2) = \frac{\sinh(\ln 2)}{\cosh(\ln 2)} = \frac{\frac{e^{\ln 2} - e^{-\ln 2}}{2}}{\frac{e^{\ln 2} + e^{-\ln 2}}{2}} = \frac{\frac{2-(1/2)}{2}}{\frac{2+(1/2)}{2}} = \frac{2-(1/2)}{2+(1/2)} = \frac{3}{5}$$

   Thus $\tanh(\ln 2) = \frac{3}{5}$.

♣

The following summarizes some of the important identities relating to hyperbolic functions. Each can be verified by referring back to Definition 2.2.

Basic Identities

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\tanh^2 x + \text{sech}^2 x = 1$
3. $\coth^2 x - \text{csch}^2 x = 1$
4. $\cosh 2x = \cosh^2 x + \sinh^2 x$
5. $\sinh 2x = 2 \sinh x \cosh x$
6. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
7. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Since $\cosh x > 0$, $\sinh x$ is increasing and hence one-to-one, so $\sinh x$ has an inverse, $\text{arcsinh} x$. Also, $\sinh x > 0$ when $x > 0$, so $\cosh x$ is injective on $[0, \infty)$ and has a (partial) inverse, $\text{arccosh} x$. The other hyperbolic functions have inverses as well, though $\text{arcsech} x$ is only a partial inverse.

Exercises for 2.7

2.7.1 *Verify the given identity using Definition 2.2, as done in Example 2.26.*

(a) $\coth^2 x - \text{csch}^2 x = 1.$

(b) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(c) $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

(d) $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

2.7.2 *Show that the range of* $\sinh x$ *is all real numbers. (Hint: show that if* $y = \sinh x$ *then* $x = \ln(y + \sqrt{y^2 + 1})$*.)*

2.7.3 *Show that the range of* $\tanh x$ *is* $(-1, 1)$*. What are the ranges of* coth, sech, *and* csch*? (Use the fact that they are reciprocal functions.)*

2.7.4 *Prove that for every* $x, y \in \mathbb{R}$, $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$*. Obtain a similar identity for* $\sinh(x - y)$*.*

2.7.5 *Prove that for every* $x, y \in \mathbb{R}$, $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$*. Obtain a similar identity for* $\cosh(x - y)$*.*

2.7.6 *Show that* $\sinh(2x) = 2 \sinh x \cosh x$ *and* $\cosh(2x) = \cosh^2 x + \sinh^2 x$ *for every* x*. Conclude also that* $(\cosh(2x) - 1)/2 = \sinh^2 x$*.*

2.8 Additional Exercises

2.8.1 *If* $f(x) = \dfrac{1}{x - 1}$*, then which of the following is equal to* $f\left(\dfrac{1}{x}\right)$?

(a) $f(x)$

(b) $-f(x)$

(c) $xf(x)$

(d) $-xf(x)$

(e) $\dfrac{f(x)}{x}$

(f) $-\dfrac{f(x)}{x}$

2.8.2 *If* $f(x) = \dfrac{x}{x + 3}$*, then find and simplify* $\dfrac{f(x) - f(2)}{x - 2}$.

2.8.3 *If* $f(x) = x^2$*, then find and simplify* $\dfrac{f(3 + h) - f(3)}{h}$.

2.8.4 *What is the domain of*

(a) $f(x) = \dfrac{\sqrt{x - 2}}{x^2 - 9}$?

(b) $g(x) = \dfrac{\sqrt[3]{x - 2}}{x^2 - 9}$?

2.8.5 *Suppose that* $f(x) = x^3$ *and* $g(x) = x$. *What is the domain of* $\dfrac{f}{g}$*?*

2.8.6 *Suppose that* $f(x) = 3x - 4$. *Find a function* g *such that* $(g \circ f)(x) = 5x + 2$.

2.8.7 *Which of the following functions is one-to-one?*

(a) $f(x) = x^2 + 4x + 3$

(b) $g(x) = |x| + 2$

(c) $h(x) = \sqrt[3]{x+1}$

(d) $F(x) = \cos x, \; -\pi \le x \le \pi$

(e) $G(x) = e^x + e^{-x}$

2.8.8 *What is the inverse of* $f(x) = \ln\left(\dfrac{e^x}{e^x - 1}\right)$*? What is the domain of* f^{-1}*?*

2.8.9 *Solve the following equations.*

(a) $e^{2-x} = 3$

(b) $e^{x^2} = e^{4x-3}$

(c) $\ln\left(1 + \sqrt{x}\right) = 2$

(d) $\ln(x^2 - 3) = \ln 2 + \ln x$

2.8.10 *Find the exact value of* $\sin^{-1}\left(-\sqrt{2}/2\right) - \cos^{-1}\left(-\sqrt{2}/2\right)$.

2.8.11 *Find* $\sin^{-1}\left(\sin(23\pi/5)\right)$.

2.8.12 *It can be proved that* $f(x) = x^3 + x + e^{x-1}$ *is one-to-one. What is the value of* $f^{-1}(3)$*?*

2.8.13 *Sketch the graph of* $f(x) = \begin{cases} -x & \text{if } x \le 0 \\ \tan^{-1} x & \text{if } x > 0 \end{cases}$

3. Limits

Calculus means “a method of calculation or reasoning.” When one computes the sales tax on a purchase, one employs a simple calculus. When one finds the area of a polygonal shape by breaking it up into a set of triangles, one is using another calculus. Proving a theorem in geometry employs yet another calculus.

Despite the wonderful advances in mathematics that had taken place into the first half of the 17^{th} century, mathematicians and scientists were keenly aware of what they *could not do.* (This is true even today.) In particular, two important concepts eluded mastery by the great thinkers of that time: area and rates of change.

Area seems innocuous enough; areas of circles, rectangles, parallelograms, etc., are standard topics of study for students today just as they were then. However, the areas of *arbitrary* shapes could not be computed, even if the boundary of the shape could be described exactly.

Rates of change were also important. When an object moves at a constant rate of change, then “distance = rate $\times$ time.” But what if the rate is not constant – can distance still be computed? Or, if distance is known, can we discover the rate of change?

It turns out that these two concepts were related. Two mathematicians, Sir Isaac Newton and Gottfried Leibniz, are credited with independently formulating a system of computing that solved the above problems and showed how they were connected. Their system of reasoning was “a” calculus. However, as the power and importance of their discovery took hold, it became known to many as “the” calculus. Today, we generally shorten this to discuss “calculus.”

The foundation of “the calculus” is the *limit.* It is a tool to describe a particular behavior of a function. This chapter begins our study of the limit by approximating its value graphically and numerically. After a formal definition of the limit, properties are established that make “finding limits” tractable. Once the limit is understood, then the problems of area and rates of change can be approached.

3.1 The tangent problem

Consider the computer- generated plot of the function $f(x) = \frac{x^2}{2}$ below. The line drawn in blue in Figure 3.1 appears to just touch the graph of the function $y = \frac{x^2}{2}$ at the point $P = (2, f(2)) = (2, \frac{2^2}{2}) = (2, 2)$. In mathematical language we say that the blue line is tangent to the graph of $y = \frac{x^2}{2}$ (“tangent” comes from Latin, “to touch”). We shall give a formal definition of a tangent line later in Section **??**; until then, we shall refer to a “tangent line” informally, relying on the reader’s intuition.

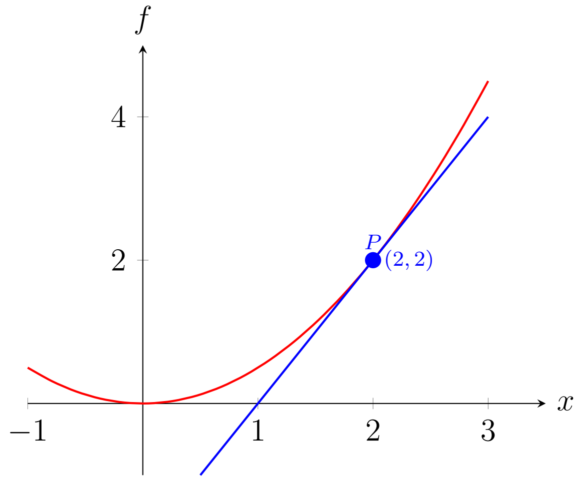


Figure 3.1

We are seeking an equation for the tangent through $P = (2, 2) = (2, f(2))$ drawn in 3.1. We know one point on the tangent line - namely the point $P = (2, 2)$. Recall that every non-vertical line has equation

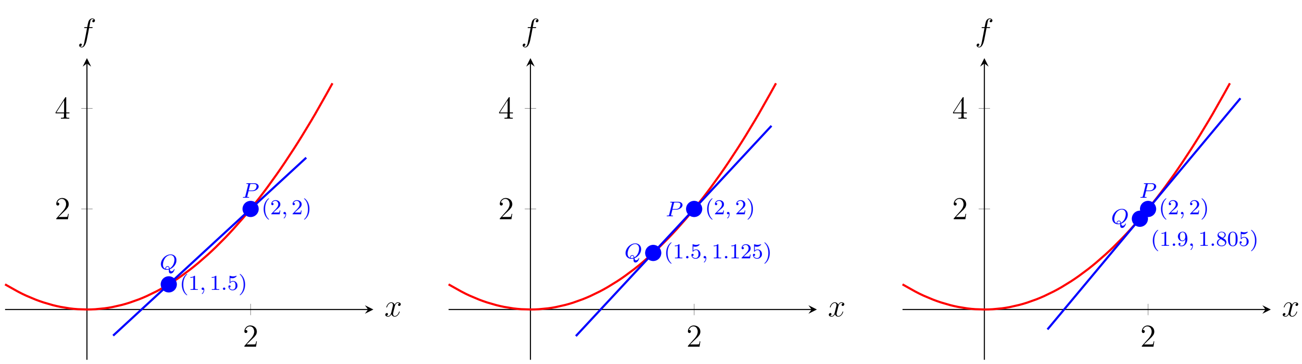
$$y = mx + c$$

for some numbers m and c, where m is called the slope of the line and c is called the y-intercept of the line. As the tangent line passes through the point $P = (2, 2)$, it has equation $y - 2 = m(x - 2)$ for some slope m that we yet need to define.

It is natural to approximate the tangent line using secant lines passing through the point $P = (2, 2)$ and nearby points $Q = (t, f(t)) = (t, \frac{t^2}{2})$ lying on the graph of $f(x)$. The line passing through $P = (2, 2)$ and $Q = (t, f(t))$ has slope $m_{PQ} := \frac{f(t)-2}{t-2}$ and therefore has equation

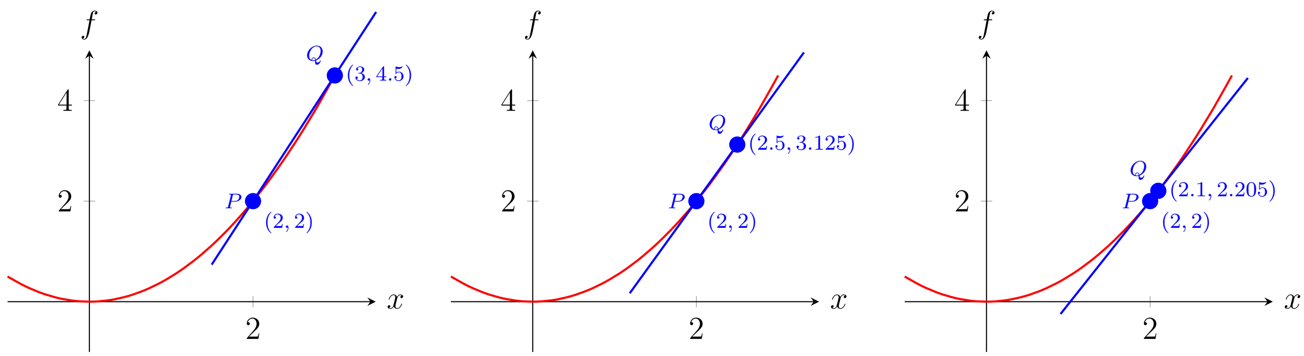
$$y - 2 = m_{PQ}(x - 2), \qquad \text{where } m_{PQ} = \left(\frac{f(t) - 2}{t - 2}\right).$$

As the equation of the tangent line is $y - 2 = m(x - 2)$, we choose to approximate m by the numbers m_{PQ} as Q gets close to the point P. On the other hand, the point $Q = (t, f(t))$ gets closer to $P = (x, f(x))$ as t gets closer to x.



(a) $t = 1$, $m_{PQ} = 1.5$ **(b)** $t = 1.5$, $m_{PQ} = 1.75$ **(c)** $t = 1.9$, $m_{PQ} = 1.95$

Figure 3.2: Slopes of secant lines as Q approaches P from the left.



(a) $t = 3$, $m_{PQ} = 2.5$ **(b)** $t = 2.5$, $m_{PQ} = 2.25$ **(c)** $t = 2.1$, $m_{PQ} = 2.05$

Figure 3.3: Slopes of secant lines as Q approaches P from the right.

In Table 3.1, we have computed the values of $f(t)$ and $m_{PQ} = \frac{f(t)-2}{t-2}$ for various values of t close to $x = 2$. In Figures 3.2-3.3, and in Table 3.1 we see that as t gets closer to 2, m_{PQ} appear to get closer to the number 2, and indeed, we can say that m_{PQ} "gets infinitely close to 2 as t approaches 2" and so we can define $m = 2$. Then the equation of the equation of the tangent line at P becomes

$$y - 2 = 2(x - 2) \quad ,$$

which is exactly the equation of the line plotted in blue in Figure 3.1.

x	$f(x) = \frac{t^2}{2}$	$t - 2$	$f(t) - f(2)$	$m_{PQ} = \frac{f(t)-f(2)}{t-2}$
0	0	-2	-2	1
1	0.5	-1	-1.5	1.5
1.5	1.125	-0.5	-0.875	1.75
1.9	1.805	-0.1	-0.195	1.95
1.99	1.98005	-0.01	-0.01995	1.995
1.999	1.998	-0.001	-0.0019995	1.9995
2.001	2.002	0.001	0.0020005	2.0005
2.01	2.02005	0.01	0.02005	2.005
2.1	2.205	0.1	0.205	2.05
2.5	3.125	0.5	1.125	2.25
3	4.5	1	2.5	2.5
4	8	2	6	3

Table 3.1: Values of $f(x)$ and slope of line through P, Q.

In order to give a strict definition of tangent, we need to define formally what it means for m_{PQ} to "get infinitely close" to the number 2. The colloquial phrase "to get infinitely close to," corresponds to the mathematical notion of taking limits.

Velocity:

A similar idea is behind our notion of *velocity*. We can see immediately how fast we are going, that is we can find the *instantaneous velocity* of a car we are driving just by looking at the speedometer. But what is that actually measuring? The *average velocity* makes sense, it's the distance travelled over a specific interval of time. But the speedometer doesn't measure that, it has a reading at each instant, and does not calculate an average velocity.

But the idea of instantaneous velocity of a moving car, at a specific time, has to be interpreted as another limiting process, taking average velocities over shorter and shorter time intervals:

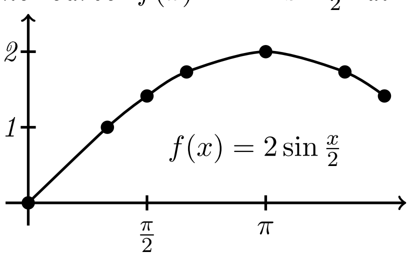
$$velocity\ v := \lim_{\Delta t \to 0} \frac{\Delta distance}{\Delta t}$$

where Δt means "the change in time", $\Delta = change\,in$.

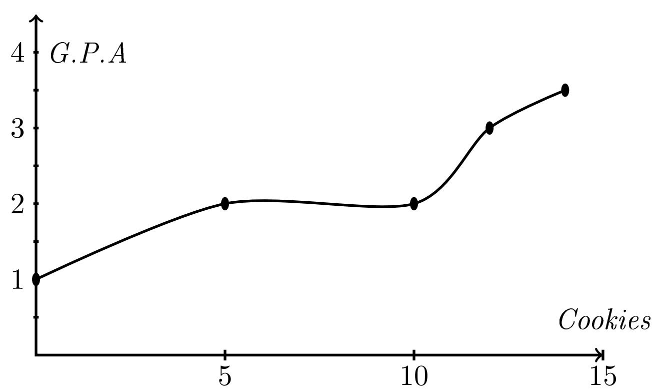
If you plot the distance on the vertical axis, and time on the horizontal, then you have the same picture as before, and the velocity is the slope of that graph.

Exercises for Section 3.1

3.1.1 *Estimate the slope of the tangent line to the curve* $f(x) = 2\sin\frac{x}{2}$ *at the point* $(\frac{\pi}{2}, \sqrt{2})$.



3.1.2 *The following graphic shows 5 students and their grades posted compared to the number of cookies that they made for me. Estimate the G.P.A. of a student who makes me 17 cookies.*



3.1.3 *Gravity on Earth dictates that a projectile with initial velocity of* v_0 *and starting height of* h_0 *has height (in feet) at time* t *given by*

$$h(t) = -16t^2 + v_0 t + h_0.$$

A rock dropped off the Twin Falls bridge takes 6 seconds to reach the river below. Find the average velocity of the rock.

3.1.4 *If a rock is thrown upward on the planet Oz with a velocity of 20 m/s, its height in meters* t *seconds later is given by* $y = 20t - 10t^2$. *Find the average velocity over the given time intervals:*

- $[1, 2]$
- $[1, 1.5]$
- $[1, 1.1]$
- $[1, 1.01]$

Estimate the instantaneous velocity when $t = 1$.

3.2 The Limit

The value a function f approaches as its input x approaches some value is said to be the limit of f. Limits are essential to the study of calculus and, as we will see, are used in defining continuity, derivatives, and integrals.

Consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

Notice that $x = 1$ does not belong to the domain of $f(x)$. Regardless, we would like to know how $f(x)$ behaves close to the point $x = 1$. We start with a table of values:

x	$f(x)$
0.5	1.5
0.9	1.9
0.99	1.99
1.01	2.01
1.1	2.1
1.5	2.5

It appears that for values of x close to 1 we have that $f(x)$ is close to 2. In fact, we can make the values of $f(x)$ as close to 2 as we like by taking x sufficiently close to 1. We express this by saying *the limit of the function $f(x)$ as x approaches 1 is equal to 2* and use the notation:

$$\lim_{x \to 1} f(x) = 2.$$

Definition 3.1: Limit (Useable Definition)

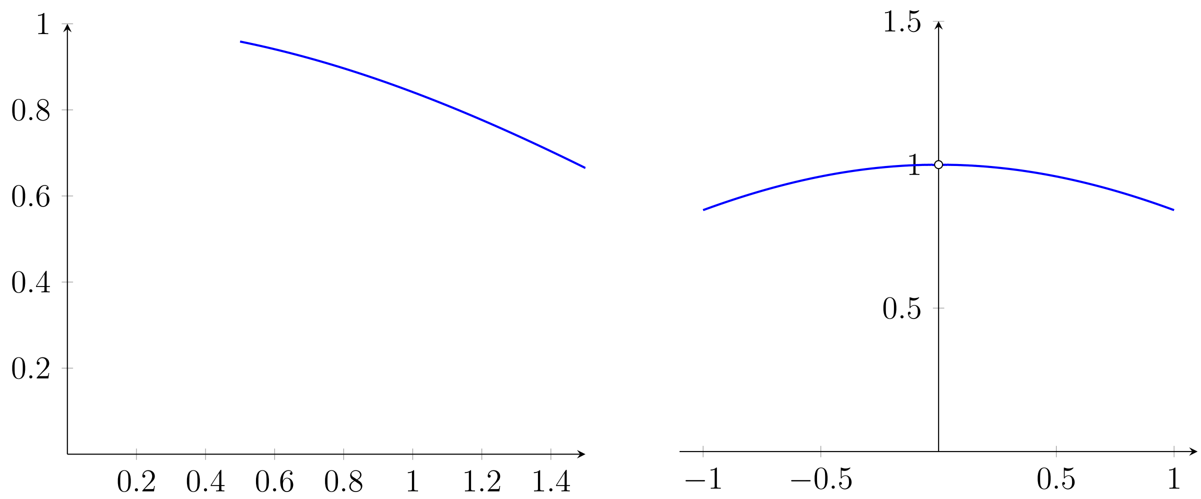
In general, we will write

$$\lim_{x \to a} f(x) = L,$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side of a) but not equal to a.

We read the expression $\lim_{x \to a} f(x) = L$ as "*the limit of $f(x)$ as x approaches a is equal to L*". When evaluating a limit, you are essentially answering the following question: What number does the function *approach* while x gets closer and closer to a (but *not equal* to a)? The phrase *but not equal to a* in the definition of a limit means that when finding the limit of $f(x)$ as x approaches a we never actually consider $x = a$. In fact, as we just saw in the example above, a may not even belong to the domain of f. All that matters for limits is what happens to f close to a, not necessarily what happens to f at a.

Consider the function $y = \frac{\sin x}{x}$. When x is near the value 1, what value (if any) is y near?One might think first to look at a graph of this function to approximate the appropriate y values. Consider Figure **??**, where $y = \frac{\sin x}{x}$ is graphed. For values of x near 1, it seems that y takes on values near 0.85. In fact, when $x = 1$, then $y = \frac{\sin 1}{1} \approx 0.84$, so it makes sense that when x is "near" 1, y will be "near" 0.84.



(a) $\sin(x)/x$ **near** $x = 1$**.**

(b) $\sin(x)/x$ **near** $x = 0$**.**

Figure 3.4

Consider this again at a different value for x. When x is near 0, what value (if any) is y near? By considering Figure 3.4 (a), one can see that it seems that y takes on values near 1. But what happens when $x = 0$? We have

$$y \to \frac{\sin 0}{0} \to \text{“}\ \frac{0}{0}\ \text{”}.$$

The expression "0/0" has no value; it is *indeterminate.* Such an expression gives no information about what is going on with the function nearby. We cannot find out how y behaves near $x = 0$ for this function simply by letting $x = 0$.

Finding a limit entails understanding how a function behaves near a particular value of x. We approximated

$$\lim_{x\to 1} \frac{\sin x}{x} \approx 0.84 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin x}{x} \approx 1.$$

(We *approximated* these limits, hence used the "≈" symbol, since we are working with the pseudo-definition of a limit, not the actual definition.)

Once we have the true definition of a limit, we will find limits *analytically*; that is, exactly using a variety of mathematical tools. For now, we will *approximate* limits both graphically and numerically. Graphing a function can provide a good approximation, though often not very precise. Numerical methods can provide a more accurate approximation. We have already approximated limits graphically, so we now turn our attention to numerical approximations.

Consider again $\lim_{x\to 1} \sin(x)/x$. To approximate this limit numerically, we can create a table of x and $f(x)$ values where x is "near" 1. This is done in Table 3.2.

Notice that for values of x near 1, we have $\sin(x)/x$ near 0.841. The $x = 1$ row is in bold to highlight the fact that when considering limits, we are *not* concerned with the value of the function at that particular x value; we are only concerned with the values of the function when x is *near* 1.

x	$\sin(x)/x$
0.9	0.870363
0.99	0.844471
0.999	0.841772
1	**0.841471**
1.001	0.84117
1.01	0.838447
1.1	0.810189

Table 3.2: Values of $\sin(x)/x$ **with** x **near 1.**

Now approximate $\lim_{x\to 0} \sin(x)/x$ numerically. We already approximated the value of this limit as 1 graphically in Figure 3.4 (a). Table 3.3 shows the value of $\sin(x)/x$ for values of x near 0. Ten places after the decimal point are shown to highlight how close to 1 the value of $\sin(x)/x$ gets as x takes on values very near 0. We include the $x = 0$ row in bold again to stress that we are not concerned with the value of our function at $x = 0$, only on the behaviour of the function *near* 0.

x	$\sin(x)/x$
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
0	**not defined**
0.001	0.9999998333
0.01	0.9999833334
0.1	0.9983341665

Table 3.3: Values of $\sin(x)/x$ **with** x **near 1.**

This numerical method gives confidence to say that 1 is a good approximation of $\lim_{x\to 0} \sin(x)/x$; that is,

$$\lim_{x\to 0} \sin(x)/x \approx 1.$$

Later we will be able to prove that the limit is *exactly* 1.

We now consider several examples that allow us explore different aspects of the limit concept.

Example 3.1: Approximating the value of a limit

Use graphical and numerical methods to approximate

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3}.$$

Solution. To graphically approximate the limit, graph

$$y = (x^2 - x - 6)/(6x^2 - 19x + 3)$$

on a small interval that contains 3. To numerically approximate the limit, create a table of values where the x values are near 3. This is done in Figure 3.5 and Table 3.4, respectively.

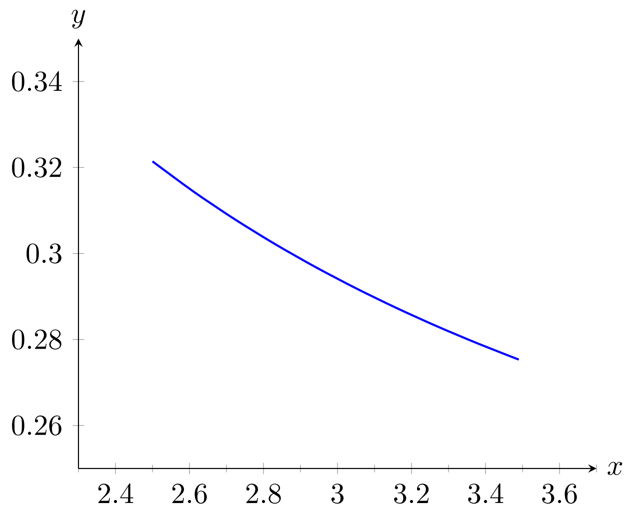


Figure 3.5

x	$\frac{x^2-x-6}{6x^2-19x+3}$
2.9	0.29878
2.99	0.294569
2.999	0.294163
3	**not defined**
3.001	0.294073
3.01	0.293669
3.1	0.289773

Table 3.4

The graph shows that when x is near 3, the value of y is very near 0.3. By considering values of x near 3, we see that $y = 0.294$ is a better approximation. The graph and the table imply that

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3} \approx 0.294.$$

♣

This example may bring up a few questions about approximating limits (and the nature of limits themselves).

1. If a graph does not produce as good an approximation as a table, why bother with it?
2. How many values of x in a table are "enough?" In the previous example, could we have just used $x = 3.001$ and found a fine approximation?

Graphs are useful since they give a visual understanding concerning the behavior of a function. Sometimes a function may act "erratically" near certain x values which is hard to discern numerically but very plain graphically. Since graphing utilities are very accessible, it makes sense to make proper use of them.

Since tables and graphs are used only to *approximate* the value of a limit, there is not a firm answer to how many data points are "enough." Include enough so that a trend is clear, and use values (when possible) both less than and greater than the value in question. In Example **??**, we used both values less than and greater than 3. Had we used just $x = 3.001$, we might have been tempted to conclude that the limit had a value of 0.3. While this is not far off, we could do better. Using values "on both sides of 3" helps us identify trends.

Example 3.2: Approximating the value of a limit

Graphically and numerically approximate the limit of $f(x)$ as x approaches 0, where

$$f(x) = \begin{cases} x+1 & x < 0 \\ -x^2+1 & x > 0 \end{cases}.$$

Solution. Again we graph $f(x)$ and create a table of its values near $x = 0$ to approximate the limit. Note that this is a piecewise defined function, so it behaves differently on either side of 0. Figure 3.6 shows a graph of $f(x)$, and on either side of 0 it seems the y values approach 1. Note that $f(0)$ is not actually defined, as indicated in the graph with the open circle.

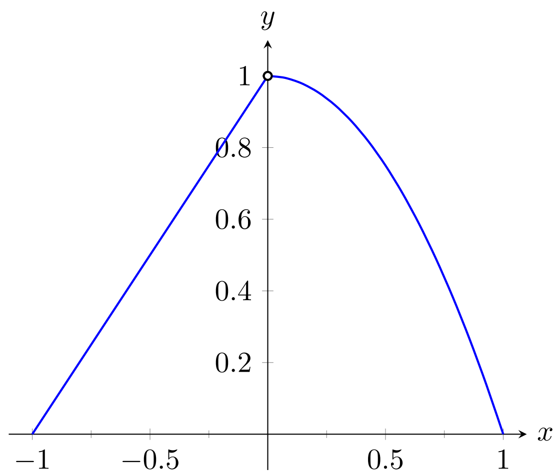


Figure 3.6

x	$f(x)$
-0.1	0.9
-0.01	0.99
-0.001	0.999
0.001	0.999999
0.01	0.9999
0.1	0.99

Table 3.5

Table **??** shows values of $f(x)$ for values of x near 0. It is clear that as x takes on values very near 0, $f(x)$ takes on values very near 1. It turns out that if we let $x = 0$ for either "piece" of $f(x)$, 1 is returned; this is significant and we'll return to this idea later.

The graph and table allow us to say that $\lim_{x\to 0} f(x) \approx 1$; in fact, we are probably very sure it *equals* 1. ♣

Identifying When Limits Do Not Exist

A function may not have a limit for all values of x. That is, we cannot say $\lim_{x\to c} f(x) = L$ for some numbers L for all values of c, for there may not be a number that $f(x)$ is approaching. There are three ways in which a limit may fail to exist.

1. The function $f(x)$ may approach different values on either side of c.
2. The function may grow without upper or lower bound as x approaches c.
3. The function may oscillate as x approaches c.

We'll explore each of these in turn.

Example 3.3: Different Values Approached From Left and Right

Explore why $\lim_{x\to 1} f(x)$ *does not exist, where*

$$f(x) = \begin{cases} x^2 - 2x + 3 & x \leq 1 \\ x & x > 1 \end{cases}.$$

Solution. A graph of $f(x)$ around $x = 1$ and a table are given Figure 3.7 and Table 3.6, respectively. It is clear that as x approaches 1, $f(x)$ does not seem to approach a single number. Instead, it seems as though $f(x)$ approaches two different numbers. When considering values of x less than 1 (approaching 1 from the left), it seems that $f(x)$ is approaching 2; when considering values of x greater than 1 (approaching 1 from the right), it seems that $f(x)$ is approaching 1. Recognizing this behavior is important; we'll study this in greater depth later. Right now, it suffices to say that the limit does not exist since $f(x)$ is not approaching one particular value as x approaches 1.

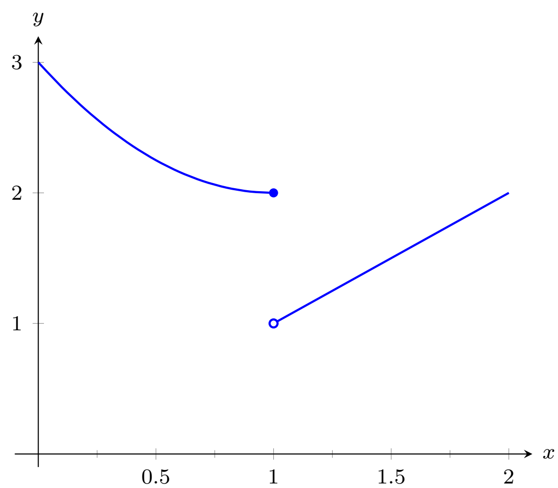


Figure 3.7: Observing no limit as $x \to 1$ in Example 3.3.

x	$f(x)$
0.9	2.01
0.99	2.0001
0.999	2.000001
1.001	1.001
1.01	1.01
1.1	1.1

Table 3.6: Values of $f(x)$ near $x = 1$ in Example 3.3.

♣

Example 3.4: The Function Grows Without Bound

Explore why $\lim_{x\to 1} \frac{1}{(x-1)^2}$ *does not exist.*

Solution. A graph and table of $f(x) = 1/(x-1)^2$ are given in Figure 3.8 and Table 3.7, respectively. Both show that as $x \to 1$, $f(x)$ grows larger and larger.

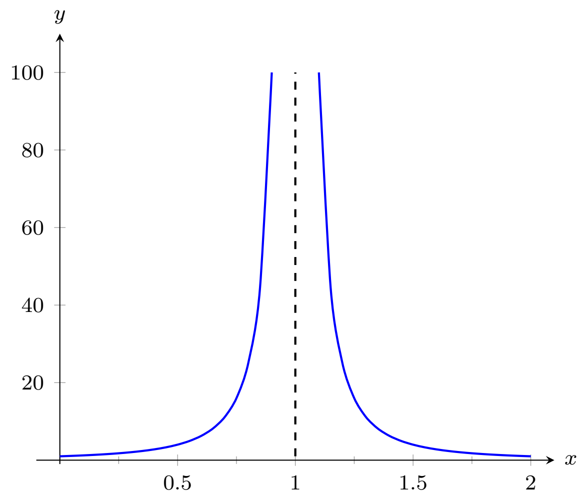


Figure 3.8: Observing no limit as $x \to 1$ in Example 3.4.

x	$f(x)$
0.9	100.
0.99	10000.
0.999	$1. \times 10^6$
1.001	$1. \times 10^6$
1.01	10000.
1.1	100.

Table 3.7: Values of $f(x)$ near $x = 1$ in Example 3.4.

We can deduce this on our own, without the aid of the graph and table. If x is near 1, then $(x-1)^2$ is very small, and:

$$\frac{1}{\text{very small number}} = \text{very large number.}$$

Since $f(x)$ is not approaching a single number, we conclude that

$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

does not exist. ♣

Example 3.5: The Function Oscillates

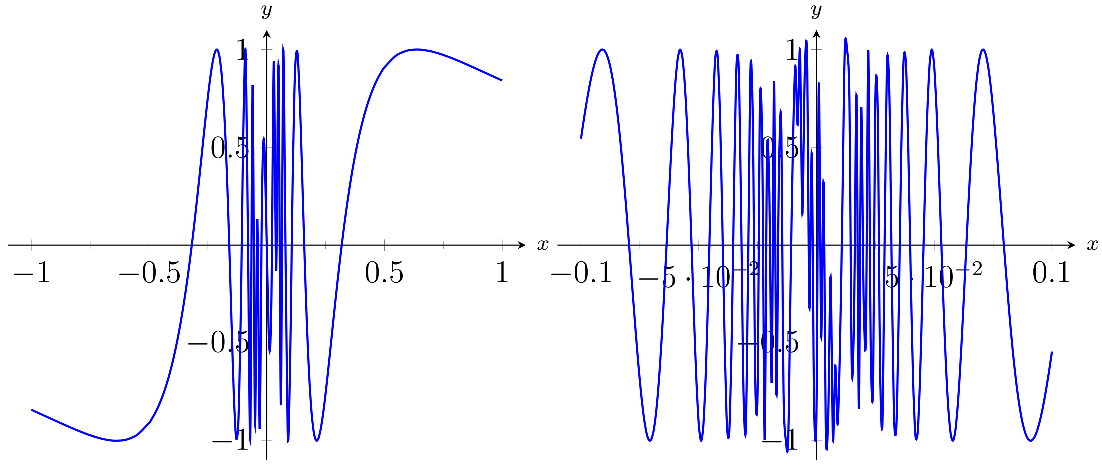
Explore why $\lim_{x\to 0} \sin(1/x)$ *does not exist.*

Solution. Two graphs of $f(x) = \sin(1/x)$ are given in Figures 3.9. Figure 3.9(a) shows $f(x)$ on the interval $[-1, 1]$; notice how $f(x)$ seems to oscillate near $x = 0$. One might think that despite the oscillation, as x approaches 0, $f(x)$ approaches 0. However, Figure 3.9(b) zooms in on $\sin(1/x)$, on the interval $[-0.1, 0.1]$. Here the oscillation is even more pronounced. Finally, in the table in Figure 3.9(c), we see $\sin(x)/x$ evaluated for values of x near 0. As x approaches 0, $f(x)$ does not appear to approach any value.

It can be shown that in reality, as x approaches 0, $\sin(1/x)$ takes on all values between -1 and 1 infinite times! Because of this oscillation,

$\lim_{x\to 0} \sin(1/x)$ does not exist.

♣



(a) **(b)**

0.1	-0.544021
0.01	-0.506366
0.001	0.82688
0.0001	-0.305614
1×10^{-5}	0.0357488
1×10^{-6}	-0.349994
1×10^{-7}	0.420548

(c)

Figure 3.9: Observing that $f(x) = \sin(1/x)$ **has no limit as** $x \to 0$

3.2.1. Limits of Difference Quotients

We have approximated limits of functions as x approached a particular number. We will consider another important kind of limit after explaining a few key ideas.

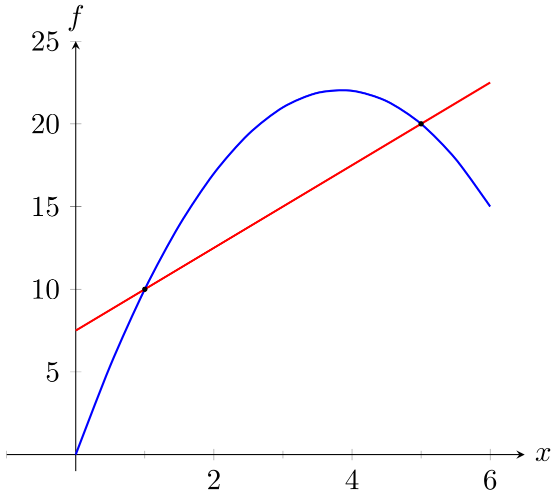


Figure 3.10: Interpreting a difference quotient as the slope of a secant line.

Let $f(x)$ represent the position function, in feet, of some particle that is moving in a straight line, where x is measured in seconds. Let's say that when $x = 1$, the particle is at position 10 ft., and when $x = 5$, the particle is at 20 ft. Another way of expressing this is to say

$$f(1) = 10 \quad \text{and} \quad f(5) = 20.$$

Since the particle traveled 10 feet in 4 seconds, we can say the particle's *average velocity* was 2.5 ft/s. We write this calculation using a "quotient of differences," or, a *difference quotient*:

$$\frac{f(5) - f(1)}{5 - 1} = \frac{10}{4} = 2.5\text{ft/s}.$$

This difference quotient can be thought of as the familiar "rise over run" used to compute the slopes of lines. In fact, that is essentially what we are doing: given two points on the graph of f, we are finding the slope of the *secant line* through those two points. See Figure 3.10.

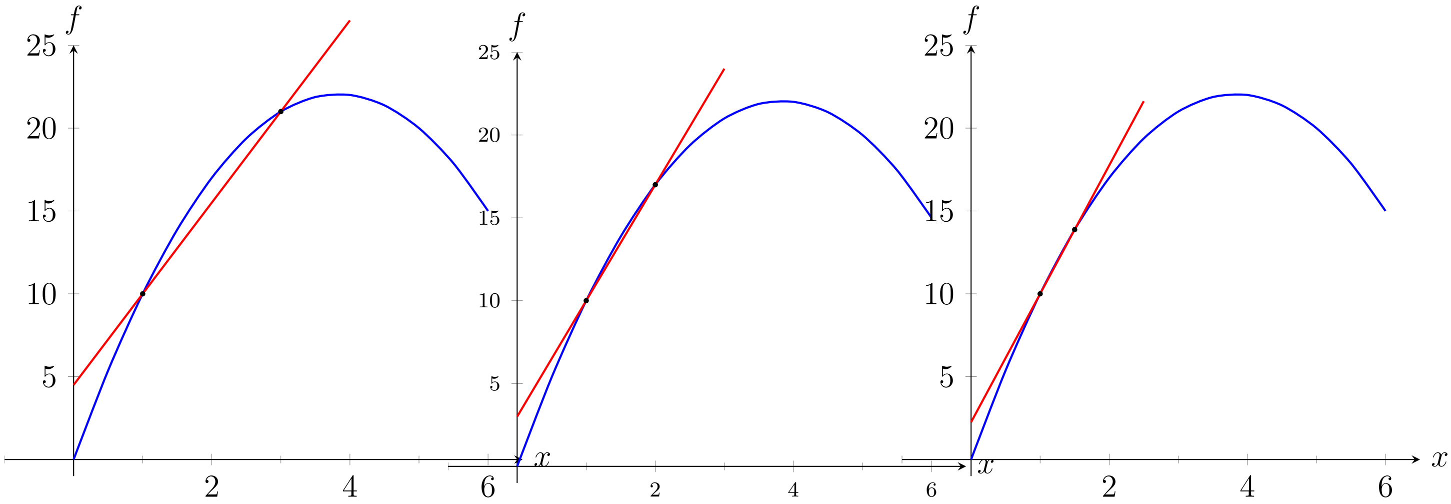
Now consider finding the average speed on another time interval. We again start at $x = 1$, but consider the position of the particle h seconds later. That is, consider the positions of the particle when $x = 1$ and when $x = 1 + h$. The difference quotient is now

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

Let $f(x) = -1.5x^2 + 11.5x$; note that $f(1) = 10$ and $f(5) = 20$, as in our discussion. We can compute this difference quotient for all values of h (even negative values!) except $h = 0$, for then we get "0/0," the indeterminate form introduced earlier. For all values $h \neq 0$, the difference quotient computes the average velocity of the particle over an interval of time of length h starting at $x = 1$.

For small values of h, i.e., values of h close to 0, we get average velocities over very short time periods and compute secant lines over small intervals. See Figure 3.11. This leads us to wonder what the limit of the difference quotient is as h approaches 0. That is,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \ ?$$



(a) $h = 2$ (b) $h = 1$ (c) $h = .5$

Figure 3.11: Secant lines of $f(x)$ at $x = 1$ and $x = 1 + h$, for shrinking values of h (i.e., $h \to 0$).

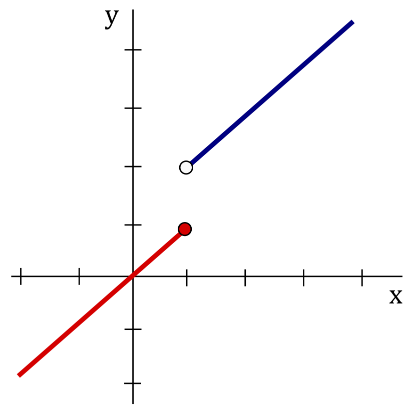
As we do not yet have a true definition of a limit nor an exact method for computing it, we settle for approximating the value. While we could graph the difference quotient (where the x-axis would represent h values and the y-axis would represent values of the difference quotient) we settle for making a table. See Figure 3.8. The table gives us reason to assume the value of the limit is about 8.5.

h	$\frac{f(1+h)-f(1)}{h}$
-0.5	9.25
-0.1	8.65
-0.01	8.515
0.01	8.485
0.1	8.35
0.5	7.75

Table 3.8: The difference quotient evaluated at values of h near 0.

One-sided limits

Consider the following piecewise defined function:



$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1 \end{cases}$$

Observe from the graph that as x gets closer and closer to 1 from the *left*, then $f(x)$ approaches $+1$. Similarly, as x gets closer and closer 1 from the *right*, then $f(x)$ approaches $+2$. We use the following notation to indicate this:

$$\lim_{x \to 1^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 2.$$

Definition 3.2: Left and Right-Hand Limit (Useable Definition)

In general, we will write

$$\lim_{x \to a^-} f(x) = L,$$

*if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a. This is called the **left-hand limit** of $f(x)$ as x approaches a. Similarly, we write*

$$\lim_{x \to a^+} f(x) = L,$$

*if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x greater than a. This is called the **right-hand limit** of $f(x)$ as x approaches a.*

Practically speaking, when evaluating a left-hand limit at a, we consider only values of x "to the left of a," on the real number line i.e., where $x < a$. The admittedly imperfect notation $x \to a^-$ is used to imply that we look at values of x to the left of a. The notation has nothing to do with positive or negative values of either x or a. A similar statement holds for evaluating right-hand limits; there we consider only values of x to the right of a on the real number line, i.e., $x > a$.

We practice evaluating left and right-hand limits through a series of examples.

Example 3.6: Evaluating one sided limits

Let $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 3 - x & 1 < x < 2 \end{cases}$, *as shown in Figure 3.12. Find each of the following:*

1. $\lim\limits_{x \to 1^-} f(x)$
2. $\lim\limits_{x \to 1^+} f(x)$
3. $\lim\limits_{x \to 1} f(x)$
4. $f(1)$
5. $\lim\limits_{x \to 0^+} f(x)$
6. $f(0)$
7. $\lim\limits_{x \to 2^-} f(x)$
8. $f(2)$

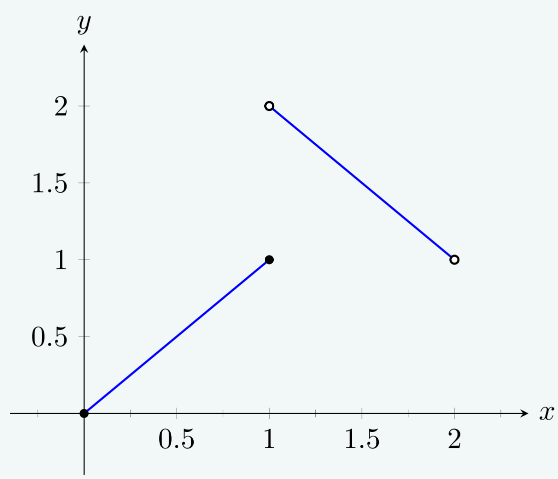


Figure 3.12: A graph of f in Example 3.6.

Solution. For these problems, the visual aid of the graph effective in evaluating the limits.

1. As x goes to 1 *from the left*, we see that $f(x)$ is approaching the value of 1. Therefore $\lim\limits_{x \to 1^-} f(x) = 1$.
2. As x goes to 1 *from the right*, we see that $f(x)$ is approaching the value of 2. Recall that it does not matter that there is an "open circle" there; we are evaluating a limit, not the value of the function. Therefore $\lim\limits_{x \to 1^+} f(x) = 2$.
3. *The* limit of f as x approaches 1 does not exist since the function does not approach one particular value, but two different values from the left and the right.
4. Using the definition and by looking at the graph we see that $f(1) = 1$.
5. As x goes to 0 from the right, we see that $f(x)$ is also approaching 0. Therefore $\lim\limits_{x \to 0^+} f(x) = 0$. Note we cannot consider a left-hand limit at 0 as f is not defined for values of $x < 0$.
6. Using the definition and the graph, $f(0) = 0$.
7. As x goes to 2 from the left, we see that $f(x)$ is approaching the value of 1. Therefore $\lim\limits_{x \to 2^-} f(x) = 1$.
8. The graph and the definition of the function show that $f(2)$ is not defined.

♣

Note how the left and right-hand limits were different at $x = 1$. This, of course, causes *the* limit to not exist. The following theorem states what is fairly intuitive: *the* limit exists precisely when the left and right-hand limits are equal.

Theorem 3.1: Limits and One Sided Limits

Let f be a function defined on an open interval I containing c. Then

$$\lim_{x\to c} f(x) = L$$

if, and only if,

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

The phrase "if, and only if" means the two statements are *equivalent*: they are either both true or both false. If the limit equals L, then the left and right hand limits both equal L. If the limit is not equal to L, then at least one of the left and right-hand limits is not equal to L (it may not even exist).

One thing to consider in Examples 3.6 – 3.9 is that the value of the function may/may not be equal to the value(s) of its left/right-hand limits, even when these limits agree.

Example 3.7: Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} 2-x & 0 < x < 1 \\ (x-2)^2 & 1 < x < 2 \end{cases}$, *as shown in Figure 3.13. Evaluate the following.*

1. $\lim_{x\to 1^-} f(x)$
2. $\lim_{x\to 1^+} f(x)$
3. $\lim_{x\to 1} f(x)$
4. $f(1)$
5. $\lim_{x\to 0^+} f(x)$
6. $f(0)$
7. $\lim_{x\to 2^-} f(x)$
8. $f(2)$

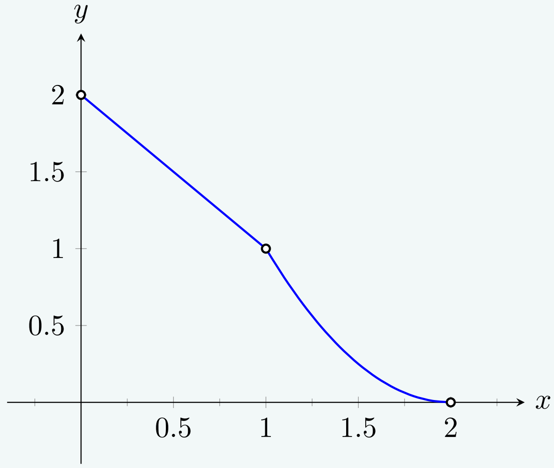


Figure 3.13: A graph of f from Example 3.7

Solution. Again we will evaluate each using both the definition of f and its graph.

1. As x approaches 1 from the left, we see that $f(x)$ approaches 1. Therefore $\lim_{x\to 1^-} f(x) = 1$.
2. As x approaches 1 from the right, we see that again $f(x)$ approaches 1. Therefore $\lim_{x\to 1+} f(x) = 1$.
3. *The* limit of f as x approaches 1 exists and is 1, as f approaches 1 from both the right and left. Therefore $\lim_{x\to 1} f(x) = 1$.
4. $f(1)$ is not defined. Note that 1 is not in the domain of f as defined by the problem, which is indicated on the graph by an open circle when $x = 1$.
5. As x goes to 0 from the right, $f(x)$ approaches 2. So $\lim_{x\to 0^+} f(x) = 2$.
6. $f(0)$ is not defined as 0 is not in the domain of f.
7. As x goes to 2 from the left, $f(x)$ approaches 0. So $\lim_{x\to 2^-} f(x) = 0$.
8. $f(2)$ is not defined as 2 is not in the domain of f.

♣

Example 3.8: Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} (x-1)^2 & 0 \le x \le 2, x \ne 1 \\ 1 & x = 1 \end{cases}$, *as shown in Figure 3.14. Evaluate the following.*

1. $\lim_{x\to 1^-} f(x)$

2. $\lim_{x\to 1^+} f(x)$

3. $\lim_{x\to 1} f(x)$

4. $f(1)$

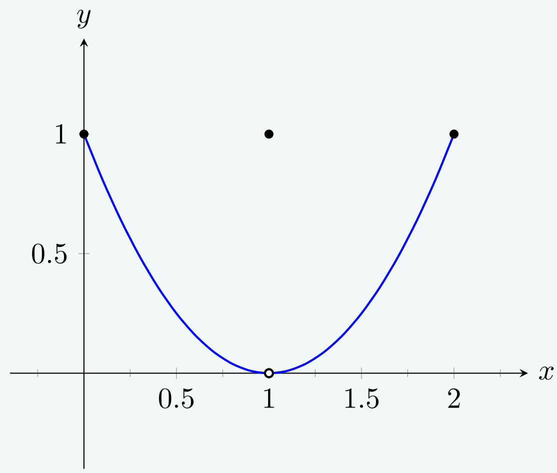


Figure 3.14: Graphing f in Example 3.8

Solution. It is clear by looking at the graph that both the left and right-hand limits of f, as x approaches 1, is 0. Thus it is also clear that *the* limit is 0; i.e., $\lim_{x\to 1} f(x) = 0$. It is also clearly stated that $f(1) = 1$.

♣

Example 3.9: Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \end{cases}$, *as shown in Figure 3.15. Evaluate the following.*

1. $\lim_{x\to 1^-} f(x)$
2. $\lim_{x\to 1^+} f(x)$
3. $\lim_{x\to 1} f(x)$
4. $f(1)$

Solution. It is clear from the definition of the function and its graph that all of the following are equal:

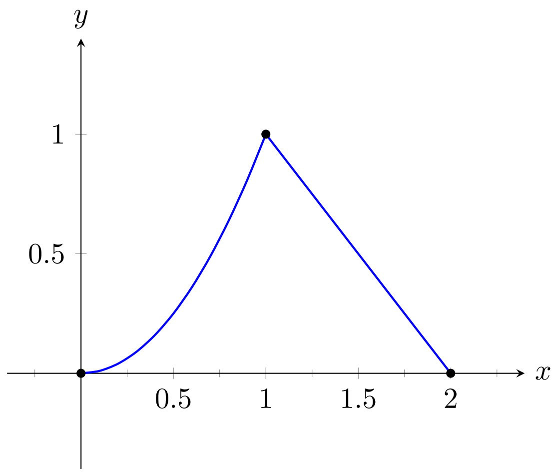


Figure 3.15: Graphing f in Example 3.9

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1) = 1.$$

♣

In Examples **??** – **??** we were asked to find both $\lim_{x\to 1} f(x)$ and $f(1)$. Consider the following table:

	$\lim_{x\to 1} f(x)$	$f(1)$
Example 3.6	does not exist	1
Example 3.7	1	not defined
Example 3.8	0	1
Example 3.9	1	1

Only in Example 3.9 do both the function and the limit exist and agree. This seems "nice;" in fact, it seems "normal." This is in fact an important situation which we explore in the next section, entitled "Continuity." In short, a *continuous function* is one in which when a function approaches a value as $x \to c$ (i.e., when $\lim_{x\to c} f(x) = L$), it actually *attains* that value at c. Such functions behave nicely as they are very predictable.

Proper understanding of limits is key to understanding calculus. With limits, we can accomplish seemingly impossible mathematical things, like adding up an infinite number of numbers (and not get infinity) and finding the slope of a line between two points, where the "two points" are actually the same point. These are not just mathematical curiosities; they allow us to link position, velocity and acceleration together, connect cross-sectional areas to volume, find the work done by a variable force, and much more.

In the next section we give the formal definition of the limit and begin our study of finding limits analytically.

Exercises for Section 3.2

3.2.1

(a) *In your own words, what does it mean to "find the limit of $f(x)$ as x approaches 3"?*

(b) *An expression of the form $\frac{0}{0}$ is called ___.*

(c) *T/F: The limit of $f(x)$ as x approaches 5 is $f(5)$.*

(d) *Describe three situations where* $\lim\limits_{x\to c} f(x)$ *does not exist.*

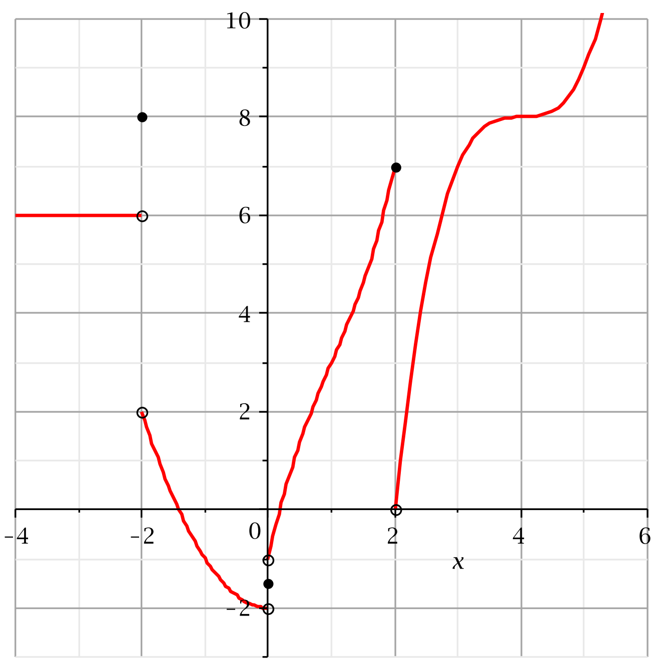
(e) *In your own words, what is a difference quotient?*

(f) *T/F: If* $\lim\limits_{x\to 1^-} f(x) = 5$, *then* $\lim\limits_{x\to 1} f(x) = 5$

(g) *T/F: If* $\lim\limits_{x\to 1^-} f(x) = 5$, *then* $\lim\limits_{x\to 1^+} f(x) = 5$

(h) *T/F: If* $\lim\limits_{x\to 1} f(x) = 5$, *then* $\lim\limits_{x\to 1^-} f(x) = 5$

3.2.2 *Evaluate the expressions by reference to this graph:*



(a) $\lim\limits_{x\to 4} f(x)$

(b) $\lim\limits_{x\to -3} f(x)$

(c) $\lim\limits_{x\to 0} f(x)$

(d) $\lim\limits_{x\to 0^-} f(x)$

(e) $\lim\limits_{x\to 0^+} f(x)$

(f) $f(-2)$

(g) $\lim\limits_{x\to 2^-} f(x)$

(h) $\lim\limits_{x\to -2^-} f(x)$

(i) $\lim\limits_{x\to 0} f(x+1)$

(j) $f(0)$

(k) $\lim\limits_{x\to 1^-} f(x-4)$

(l) $\lim\limits_{x\to 0^+} f(x-2)$

3.2.3 *Approximate the given limits both numerically and graphically.*

(a) -1

(b) $\lim\limits_{x\to 0} x^3 - 3x^2 + x - 5$

(c) $\lim\limits_{x\to 0} \dfrac{x+1}{x^2+3x}$

(d) $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-4x+3}$

(e) $\lim\limits_{x\to -1} \dfrac{x^2+8x+7}{x^2+6x+5}$

(f) $\lim\limits_{x\to 2} \dfrac{x^2+7x+10}{x^2-4x+4}$

(g) $\lim\limits_{x\to 2} f(x)$, *where*

$$f(x) = \begin{cases} x+2 & x \le 2 \\ 3x-5 & x > 2 \end{cases}.$$

(h) $\lim\limits_{x\to 3} f(x)$, *where*

$$f(x) = \begin{cases} x^2 - x + 1 & x \le 3 \\ 2x+1 & x > 3 \end{cases}.$$

(i) $\lim\limits_{x\to 0} f(x)$, *where*

$$f(x) = \begin{cases} \cos x & x \le 0 \\ x^2 + 3x + 1 & x > 0 \end{cases}.$$

(j) $\lim\limits_{x\to \pi/2} f(x)$, *where*

$$f(x) = \begin{cases} \sin x & x \le \pi/2 \\ \cos x & x > \pi/2 \end{cases}.$$

3.2.4 *A function f and a value a are given. Approximate the limit of the difference quotient,* $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, *using* $h = \pm 0.1, \pm 0.01$.

(a) $f(x) = -7x + 2, \quad a = 3$

(b) $f(x) = 9x + 0.06, \quad a = -1$

(c) $f(x) = x^2 + 3x - 7, \quad a = 1$

(d) $f(x) = \dfrac{1}{x+1}, \quad a = 2$

(e) $f(x) = -4x^2 + 5x - 1, \quad a = -3$

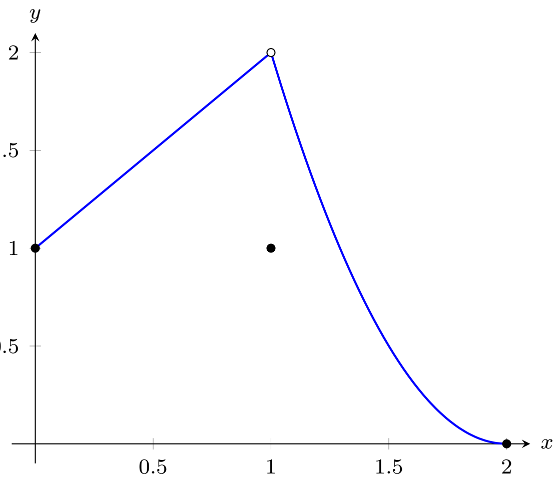
(f) $f(x) = \ln x, \quad a = 5$

(g) $f(x) = \sin x, \quad a = \pi$

(h) $f(x) = \cos x, \quad a = \pi$

3.2.5

(a)



(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

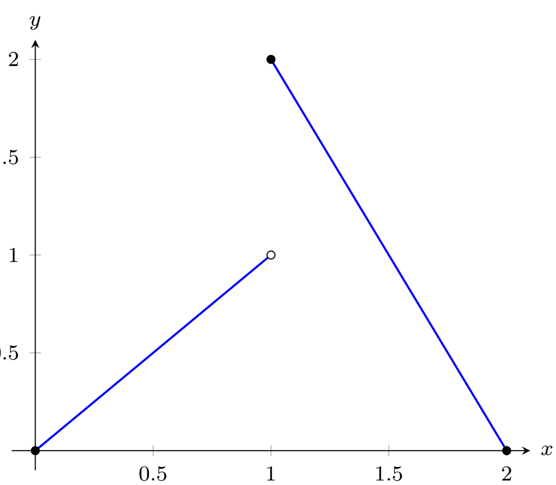
(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 0^-} f(x)$

(f) $\lim\limits_{x\to 0^+} f(x)$

(b)



(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

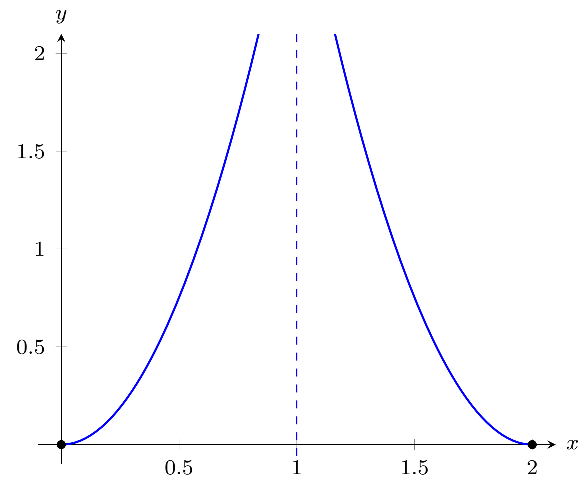
(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 2^-} f(x)$

(f) $\lim\limits_{x\to 2^+} f(x)$

(c)



(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

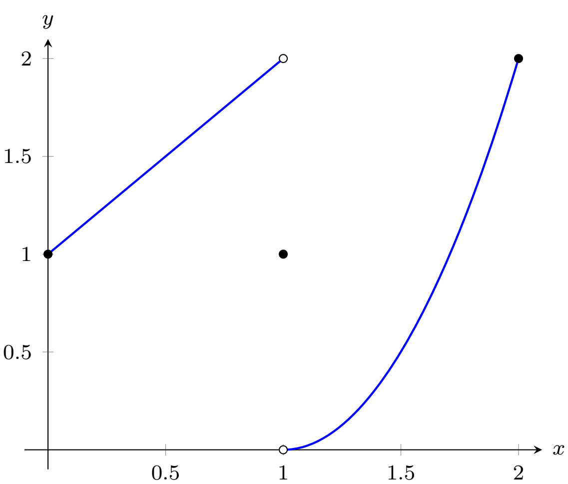
(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 2^-} f(x)$

(f) $\lim\limits_{x\to 0^+} f(x)$

(d)



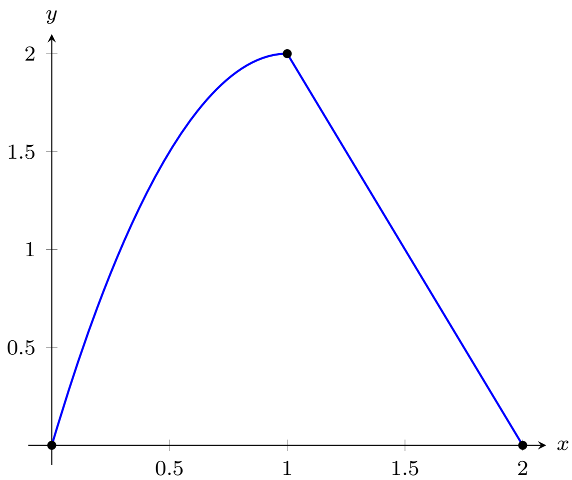
(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e)



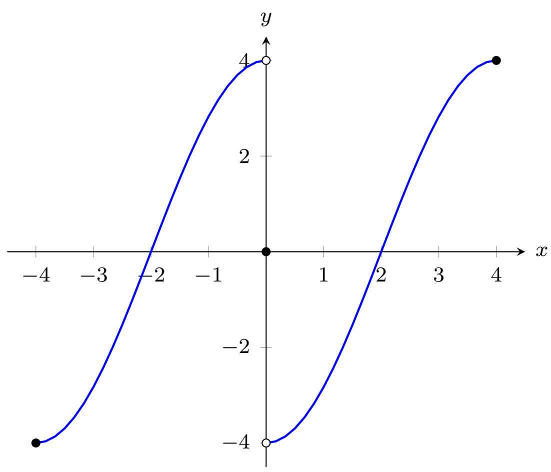
(a) $\lim_{x\to 1^-} f(x)$

(b) $\lim_{x\to 1^+} f(x)$

(c) $\lim_{x\to 1} f(x)$

(d) $f(1)$

(f)



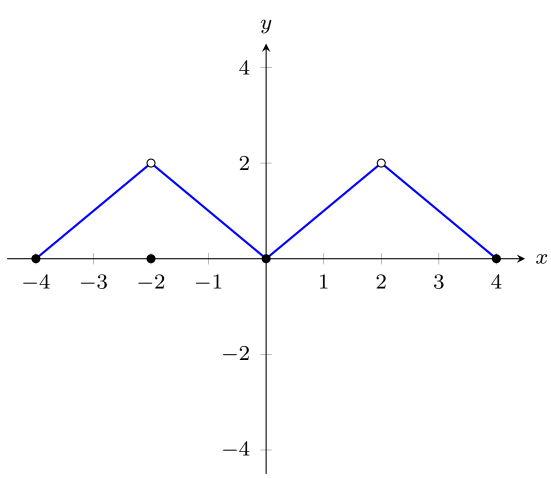
(a) $\lim_{x\to 0^-} f(x)$

(b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$

(d) $f(0)$

(g)



(a) $\lim_{x\to -2^-} f(x)$

(b) $\lim_{x\to -2^+} f(x)$

(c) $\lim_{x\to -2} f(x)$

(d) $f(-2)$

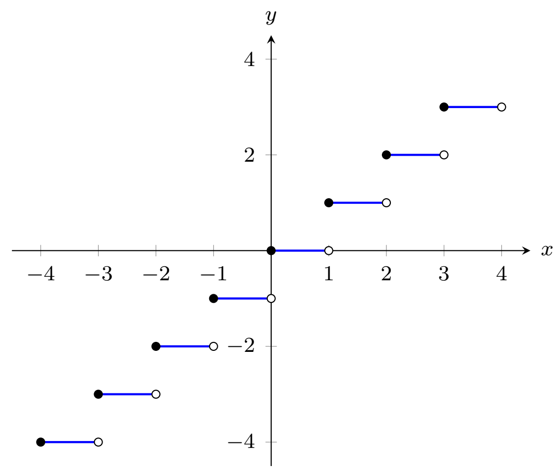
(e) $\lim_{x\to 2^-} f(x)$

(f) $\lim_{x\to 2^+} f(x)$

(g) $\lim_{x\to 2} f(x)$

(h) $f(2)$

(h)



Let $-3 \le a \le 3$ be an integer.

(a) $\lim_{x\to a^-} f(x)$

(b) $\lim_{x\to a^+} f(x)$

(c) $\lim_{x\to a} f(x)$

(d) $f(a)$

3.2.6

(a)

(b)

(c)

(d)

3.2.7 *Use a calculator to estimate* $\lim_{x\to 0} \frac{\sin x}{x}$, *where x is in radians.*

3.2.8 *Use a calculator to estimate* $\lim_{x\to 0} \frac{\tan(3x)}{\tan(5x)}$, *where x is in radians.*

3.2.9 *Use a calculator to estimate* $\lim_{x\to 1^+} \frac{|x-1|}{1-x^2}$ *and* $\lim_{x\to 1^-} \frac{|x-1|}{1-x^2}$.

3.3 Precise Definition of a Limit

This section introduces the formal definition of a limit. Many refer to this as "the epsilon–delta," definition, referring to the letters ϵ and δ of the Greek alphabet.

Before we give the actual definition, let's consider a few informal ways of describing a limit. Given a function $y = f(x)$ and an x-value, c, we say that "the limit of the function f, as x approaches c, is a value L":

1. if "y tends to L" as "x tends to c."

2. if "y approaches L" as "x approaches c."

3. if "y is near L" whenever "x is near c."

The problem with these definitions is that the words "tends," "approach," and especially "near" are not exact. In what way does the variable x tend to, or approach, c? How near do x and y have to be to c and L, respectively?

The definition we describe in this section comes from formalizing **3**. A quick restatement gets us closer to what we want:

3′. If x is within a certain *tolerance level* of c, then the corresponding value $y = f(x)$ is within a certain *tolerance level* of L.

The traditional notation for the x-tolerance is the lowercase Greek letter delta, or δ, and the y-tolerance is denoted by lowercase epsilon, or ϵ. One more rephrasing of **3′** nearly gets us to the actual definition:

3″. If x is within δ units of c, then the corresponding value of y is within ϵ units of L.

We can write "x is within δ units of c" mathematically as

$$|x - c| < \delta, \qquad \text{which is equivalent to} \qquad c - \delta < x < c + \delta.$$

Letting the symbol "$\longrightarrow$" represent the word "implies," we can rewrite **3″** as

$$|x - c| < \delta \longrightarrow |y - L| < \epsilon \qquad \text{or} \qquad c - \delta < x < c + \delta \longrightarrow L - \epsilon < y < L + \epsilon.$$

The point is that δ and ϵ, being tolerances, can be any positive (but typically small) values. Finally, we have the formal definition of the limit with the notation seen in the previous section.

Definition 3.3: The Limit of a Function f

Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. The limit of $f(x)$, as x approaches c, is L, denoted by

$$\lim_{x \to c} f(x) = L,$$

means that given any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \neq c$, if $|x-c| < \delta$, then $|f(x)-L| < \epsilon$.

(Mathematicians often enjoy writing ideas without using any words. Here is the wordless definition of the limit:

$$\lim_{x \to c} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \; s.t. \; 0 < |x - c| < \delta \longrightarrow |f(x) - L| < \epsilon.)$$

Note the order in which ϵ and δ are given. In the definition, the y-tolerance ϵ is given *first* and then the limit will exist **if** we can find an x-tolerance δ that works.

An example will help us understand this definition. Note that the explanation is long, but it will take one through all steps necessary to understand the ideas.

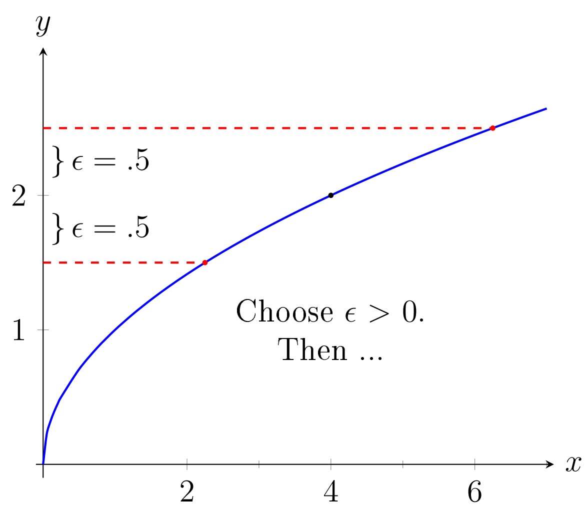
Example 3.10: Evaluating a limit using the definition

Show that $\lim_{x \to 4} \sqrt{x} = 2$.

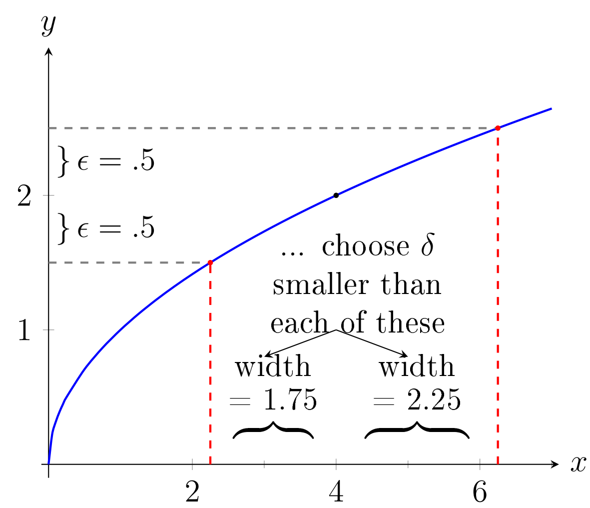
Solution. Before we use the formal definition, let's try some numerical tolerances. What if the y tolerance is 0.5, or $\epsilon = 0.5$? How close to 4 does x have to be so that y is within 0.5 units of 2, i.e., $1.5 < y < 2.5$? In this case, we can proceed as follows:

$$\begin{aligned} 1.5 < \; & y < 2.5 \\ 1.5 < \; & \sqrt{x} < 2.5 \\ 1.5^2 < \; & x < 2.5^2 \\ 2.25 < \; & x < 6.25. \end{aligned}$$

So, what is the desired x tolerance? Remember, we want to find a symmetric interval of x values, namely $4 - \delta < x < 4 + \delta$. The lower bound of 2.25 is 1.75 units from 4; the upper bound of 6.25 is 2.25 units from 4. We need the smaller of these two distances; we must have $\delta \leq 1.75$. See Figure 3.16.



(a)



(b) With $\epsilon = 0.5$, we pick any $\delta < 1.75$.

Figure 3.16: Illustrating the $\epsilon - \delta$ process.

NOTE: The concept of a **one-sided limit** can also be made precise.

Definition 3.4: One-sided Limit

Suppose that $f(x)$ is a function. We say that $\lim_{x\to a^-} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < a - x < \delta$, $|f(x) - L| < \epsilon$. We say that $\lim_{x\to a^+} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < x - a < \delta$, $|f(x) - L| < \epsilon$.

Given the y tolerance $\epsilon = 0.5$, we have found an x tolerance, $\delta \le 1.75$, such that whenever x is within δ units of 4, then y is within ϵ units of 2. That's what we were trying to find.

Let's try another value of ϵ.

What if the y tolerance is 0.01, i.e., $\epsilon = 0.01$? How close to 4 does x have to be in order for y to be within 0.01 units of 2 (or $1.99 < y < 2.01$)? Again, we just square these values to get $1.99^2 < x < 2.01^2$, or

$$3.9601 < x < 4.0401.$$

What is the desired x tolerance? In this case we must have $\delta \le 0.0399$, which is the minimum distance from 4 of the two bounds given above.

What we have so far: if $\epsilon = 0.5$, then $\delta \le 1.75$ and if $\epsilon = 0.01$, then $\delta \le 0.0399$. A pattern is not easy to see, so we switch to general ϵ try to determine δ symbolically. We start by assuming $y = \sqrt{x}$ is within ϵ units of 2:

$$\begin{aligned}
|y-2| &< \epsilon & & \\
-\epsilon < y-2 &< \epsilon & &\text{(Definition of absolute value)}\\
-\epsilon < \sqrt{x}-2 &< \epsilon & &(y=\sqrt{x})\\
2-\epsilon < \sqrt{x} &< 2+\epsilon & &\text{(Add 2)}\\
(2-\epsilon)^2 < x &< (2+\epsilon)^2 & &\text{(Square all)}\\
4-4\epsilon+\epsilon^2 < x &< 4+4\epsilon+\epsilon^2 & &\text{(Expand)}\\
4-(4\epsilon-\epsilon^2) < x &< 4+(4\epsilon+\epsilon^2). & &\text{(Rewrite in the desired form)}
\end{aligned}$$

The "desired form" in the last step is "$4-$ *something* $< x < 4+$ *something*." Since we want this last interval to describe an x tolerance around 4, we have that either $\delta \le 4\epsilon - \epsilon^2$ or $\delta \le 4\epsilon + \epsilon^2$, whichever is smaller:

$$\delta \le \min\{4\epsilon - \epsilon^2, 4\epsilon + \epsilon^2\}.$$

Since $\epsilon > 0$, the minimum is $\delta \le 4\epsilon - \epsilon^2$. That's the formula: given an ϵ, set $\delta \le 4\epsilon - \epsilon^2$.

We can check this for our previous values. If $\epsilon = 0.5$, the formula gives $\delta \le 4(0.5) - (0.5)^2 = 1.75$ and when $\epsilon = 0.01$, the formula gives $\delta \le 4(0.01) - (0.01)^2 = 0.399$.

So given any $\epsilon > 0$, set $\delta \le 4\epsilon - \epsilon^2$. Then if $|x-4| < \delta$ (and $x \ne 4$), then $|f(x) - 2| < \epsilon$, satisfying the definition of the limit. We have shown formally (and finally!) that $\lim_{x\to 4} \sqrt{x} = 2$. ♣

The previous example was a little long in that we sampled a few specific cases of ϵ before handling the general case. Normally this is not done. The previous example is also a bit unsatisfying in that $\sqrt{4} = 2$; why work so hard to prove something so obvious? Many ϵ-δ proofs are long and difficult to do. In this section, we will focus on examples where the answer is, frankly, obvious, because the non–obvious examples are even harder. In the next section we will learn some theorems that allow us to evaluate limits *analytically*, that is, without using the ϵ-δ definition.

Example 3.11: Evaluating a limit using the definition

Show that $\lim_{x \to 2} x^2 = 4$.

Solution. Let's do this example symbolically from the start. Let $\epsilon > 0$ be given; we want $|y - 4| < \epsilon$, i.e., $|x^2 - 4| < \epsilon$. How do we find δ such that when $|x - 2| < \delta$, we are guaranteed that $|x^2 - 4| < \epsilon$?

This is a bit trickier than the previous example, but let's start by noticing that $|x^2 - 4| = |x - 2| \cdot |x + 2|$. Consider:

$$|x^2 - 4| < \epsilon \longrightarrow |x - 2| \cdot |x + 2| < \epsilon \longrightarrow |x - 2| < \frac{\epsilon}{|x + 2|}. \tag{3.1}$$

Could we not set $\delta = \dfrac{\epsilon}{|x + 2|}$?

We are close to an answer, but the catch is that δ must be a *constant* value (so it can't contain x). There is a way to work around this, but we do have to make an assumption. Remember that ϵ is supposed to be a small number, which implies that δ will also be a small value. In particular, we can (probably) assume that $\delta < 1$. If this is true, then $|x - 2| < \delta$ would imply that $|x - 2| < 1$, giving $1 < x < 3$.

Now, back to the fraction $\dfrac{\epsilon}{|x + 2|}$. If $1 < x < 3$, then $3 < x + 2 < 5$ (add 2 to all terms in the inequality). Taking reciprocals, we have

$$\begin{aligned}
\frac{1}{5} &< \frac{1}{|x + 2|} < \frac{1}{3} && \text{which implies} \\
\frac{1}{5} &< \frac{1}{|x + 2|} && \text{which implies} \\
\frac{\epsilon}{5} &< \frac{\epsilon}{|x + 2|}. && \qquad (3.2)
\end{aligned}$$

This suggests that we set $\delta \leq \dfrac{\epsilon}{5}$. To see why, let consider what follows when we assume $|x - 2| < \delta$:

$$\begin{aligned}
|x - 2| &< \delta \\
|x - 2| &< \frac{\epsilon}{5} && \text{(Our choice of } \delta\text{)} \\
|x - 2| \cdot |x + 2| &< |x + 2| \cdot \frac{\epsilon}{5} && \text{(Multiply by } |x + 2|\text{)} \\
|x^2 - 4| &< |x + 2| \cdot \frac{\epsilon}{5} && \text{(Combine left side)} \\
|x^2 - 4| &< |x + 2| \cdot \frac{\epsilon}{5} < |x + 2| \cdot \frac{\epsilon}{|x + 2|} = \epsilon && \text{(Using (3.2) as long as } \delta < 1\text{)}
\end{aligned}$$

We have arrived at $|x^2 - 4| < \epsilon$ as desired. Note again, in order to make this happen we needed δ to first be less than 1. That is a safe assumption; we want ϵ to be arbitrarily small, forcing δ to also be small.

We have also picked δ to be smaller than "necessary." We could get by with a slightly larger δ, as shown in Figure 3.17. The dashed outer lines show the boundaries defined by our choice of ϵ. The dotted inner lines show the boundaries defined by setting $\delta = \epsilon/5$. Note how these dotted lines are within the dashed lines. That is perfectly fine; by choosing x within the dotted lines we are guaranteed that $f(x)$ will be within ϵ of 4.

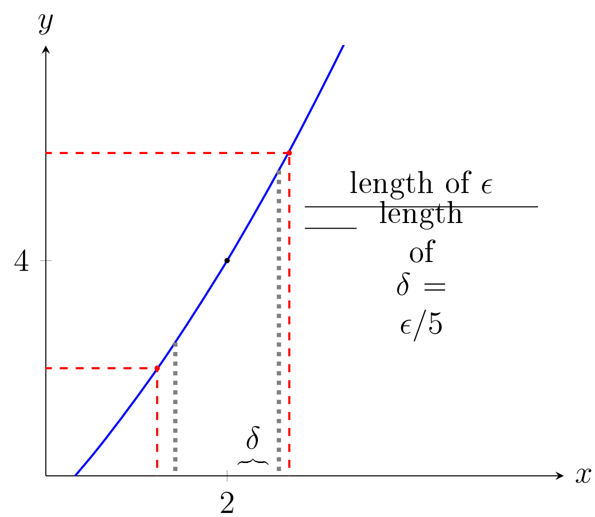


Figure 3.17: Choosing $\delta = \epsilon/5$ in Example ??.

In summary, given $\epsilon > 0$, set $\delta =\leq \epsilon/5$. Then $|x-2| < \delta$ implies $|x^2-4| < \epsilon$ (i.e. $|y-4| < \epsilon$) as desired. This shows that $\lim_{x\to 2} x^2 = 4$. Figure 3.17 gives a visualization of this; by restricting x to values within $\delta = \epsilon/5$ of 2, we see that $f(x)$ is within ϵ of 4. ♣

It probably seems obvious that $\lim_{x\to 2} x^2 = 4$, and it is worth examining more closely why it seems obvious. If we write $x^2 = x \cdot x$, and ask what happens when x approaches 2, we might say something like, "Well, the first x approaches 2, and the second x approaches 2, so the product must approach $2 \cdot 2$." In fact this is pretty much right on the money, except for that word "must." Is it really true that if x approaches a and y approaches b then xy approaches ab? It is, but it is not really obvious, since x and y might be quite complicated. The good news is that we can see that this is true once and for all, and then we don't have to worry about it ever again. When we say that x might be "complicated" we really mean that in practice it might be a function. Here is then what we want to know:

Theorem 3.2: Limit Product

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then $\lim_{x\to a} f(x)g(x) = LM$.

Proof. We must use the Precise Definition of a Limit to prove the Produce Law for Limits. So given any ϵ we need to find a δ so that $0 < |x-a| < \delta$ implies $|f(x)g(x) - LM| < \epsilon$. What do we have to work with? We know that we can make $f(x)$ close to L and $g(x)$ close to M, and we have to somehow connect these facts to make $f(x)g(x)$ close to LM.

We use, as is often the case, a little algebraic trick:

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \end{aligned}$$

$$\begin{aligned}&\le |f(x)(g(x)-M)| + |(f(x)-L)M| \\ &= |f(x)||g(x)-M| + |f(x)-L||M|.\end{aligned}$$

This is all straightforward except perhaps for the "$\le$". That is an example of the *triangle inequality*, which says that if a and b are any real numbers then $|a+b| \le |a| + |b|$. If you look at a few examples, using positive and negative numbers in various combinations for a and b, you should quickly understand why this is true. We will not prove it formally.

Suppose $\epsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there is a value δ_1 such that $0 < |x-a| < \delta_1$ implies $|f(x)-L| < \dfrac{\epsilon}{2(1+|M|)}$. This means that $0 < |x-a| < \delta_1$ implies $|f(x)-L||M| < |f(x)-L|(1+|M|) < \epsilon/2$.

Now we focus our attention on the other term in the inequality, $|f(x)||g(x)-M|$. We can make $|g(x)-M|$ smaller than any fixed number by making x close enough to a; unfortunately, $\epsilon/(2f(x))$ is not a fixed number, since x is a variable. Here we need another little trick, just like the one we used in analyzing x^2. We can find a δ_2 so that $|x-a| < \delta_2$ implies that $|f(x)-L| < 1$, meaning that $L-1 < f(x) < L+1$. This means that $|f(x)| < N$, where N is either $|L-1|$ or $|L+1|$, depending on whether L is negative or positive. The important point is that N doesn't depend on x. Finally, we know that there is a δ_3 so that $0 < |x-a| < \delta_3$ implies $|g(x)-M| < \epsilon/(2N)$. Let δ be the smallest of δ_1, δ_2, and δ_3. Then $|x-a| < \delta$ implies that $|f(x)-L| < \epsilon/(2(1+|M|))$, $|f(x)| < N$, and $|g(x)-M| < \epsilon/(2N)$. Then

$$\begin{aligned}|f(x)g(x)-LM| &\le |f(x)||g(x)-M| + |f(x)-L||M| \\ &< N\frac{\epsilon}{2N} + \frac{\epsilon}{2(1+|M|)}(1+|M|) \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This is just what we needed, so by the official definition, $\lim_{x\to a} f(x)g(x) = LM$. ♣

Examples 3.10 and 3.11 determine δ by using logic that is difficult to recreate as one learns this topic. For instance, Equation (3.2) is used based on the following facts:

1. We want $\delta \le \frac{\epsilon}{|x+2|}$. Since we cannot let δ vary according to x,
2. we notice that $|x+2| < 5$ for the values we are interested in, so
3. $\frac{\epsilon}{5} < \frac{\epsilon}{|x+2|}$ and setting $\delta < \frac{\epsilon}{5}$ ensures that $\delta < \frac{\epsilon}{|x+2|}$.

The following theorem offers some inequalities that are useful when creating δ–ϵ proofs.

Theorem 3.3: Power Function Inequalities

Let $x > y > 0$ and $n > 1$. The following inequalities hold:

- $x^n + y^n < (x+y)^n$
- $(x-y)^n < x^n - y^n$
- $\sqrt[n]{x+y} < \sqrt[n]{x} + \sqrt[n]{y}$
- $\sqrt[n]{x} - \sqrt[n]{y} < \sqrt[n]{x-y}$

Make note of the general pattern exhibited in these last two examples. In some sense, each starts out "backwards." That is, while we want to

1. start with $|x-c|<\delta$ and conclude that
2. $|f(x)-L|<\epsilon$,

we actually start by assuming

1. $|f(x)-L|<\epsilon$, then perform some algebraic manipulations to give an inequality of the form
2. $|x-c|<$ *something*.

When we have properly done this, the *something* on the "greater than" side of the inequality becomes our δ. We can refer to this as the "scratch–work" phase of our proof. Once we have δ, we can formally start with $|x-c|<\delta$ and use algebraic manipulations to conclude that $|f(x)-L|<\epsilon$, usually by using the same steps of our "scratch–work" in reverse order.

We highlight this process in the following example.

Example 3.12: Evaluating a limit using the definition

Prove that $\lim\limits_{x\to 1} x^3-2x=-1$.

Solution. We start our scratch–work by considering $|f(x)-(-1)|<\epsilon$:

$$\begin{aligned}
|f(x)-(-1)| &< \epsilon \\
|x^3-2x+1| &< \epsilon && \text{(Now factor)} \\
|(x-1)(x^2+x-1)| &< \epsilon \\
|x-1| &< \frac{\epsilon}{|x^2+x-1|}. && (3.3)
\end{aligned}$$

We are at the phase of saying that $|x-1|<$ *something*, where *something*$=\epsilon/|x^2+x-1|$. We want to turn that *something* into δ.

Since x is approaching 1, we are safe to assume that x is between 0 and 2. So

$$\begin{aligned}
0 &< x < 2 \\
0 &< x^2 < 4. && \text{(squared each term)}
\end{aligned}$$

Since $0<x<2$, we can add 0, x and 2, respectively, to each part of the inequality and maintain the inequality.

$$\begin{aligned}
0 &< x^2+x < 6 \\
-1 &< x^2+x-1 < 5. && \text{(subtracted 1 from each part)}
\end{aligned}$$

In Equation (3.3), we wanted $|x-1|<\epsilon/|x^2+x-1|$. The above shows that given any x in $[0,2]$, we know that

$$\begin{aligned}
x^2+x-1 &< 5 && \text{which implies that} \\
\frac{1}{5} &< \frac{1}{x^2+x-1} && \text{which implies that} \\
\frac{\epsilon}{5} &< \frac{\epsilon}{x^2+x-1}. && (3.4)
\end{aligned}$$

So we set $\delta \leq \epsilon/5$. This ends our scratch–work, and we begin the formal proof (which also helps us understand why this was a good choice of δ).

Given ϵ, let $\delta \leq \epsilon/5$. We want to show that when $|x-1| < \delta$, then $|(x^3-2x)-(-1)| < \epsilon$. We start with $|x-1| < \delta$:

$$\begin{aligned}
|x-1| &< \delta \\
|x-1| &< \frac{\epsilon}{5} \\
|x-1| &< \frac{\epsilon}{5} < \frac{\epsilon}{|x^2+x-1|} && \text{(for } x \text{ near 1, from Equation (3.4))} \\
|x-1| \cdot |x^2+x-1| &< \epsilon \\
|x^3-2x+1| &< \epsilon \\
|(x^3-2x)-(-1)| &< \epsilon,
\end{aligned}$$

which is what we wanted to show. Thus $\lim\limits_{x\to 1} x^3 - 2x = -1$. ♣

We illustrate evaluating limits once more.

Example 3.13: Evaluating a limit using the definition

Prove that $\lim\limits_{x\to 0} e^x = 1$.

Solution. Symbolically, we want to take the equation $|e^x - 1| < \epsilon$ and unravel it to the form $|x-0| < \delta$. Here is our scratch–work:

$$\begin{aligned}
|e^x - 1| &< \epsilon \\
-\epsilon < e^x - 1 &< \epsilon && \text{(Definition of absolute value)} \\
1-\epsilon < e^x &< 1+\epsilon && \text{(Add 1)} \\
\ln(1-\epsilon) < x &< \ln(1+\epsilon) && \text{(Take natural logs)}
\end{aligned}$$

Making the safe assumption that $\epsilon < 1$ ensures the last inequality is valid (i.e., so that $\ln(1-\epsilon)$ is defined). We can then set δ to be the minimum of $|\ln(1-\epsilon)|$ and $\ln(1+\epsilon)$; i.e.,

$$\delta = \min\{|\ln(1-\epsilon)|, \ln(1+\epsilon)\} = \ln(1+\epsilon).$$

Note: Recall $\ln 1 = 0$ and $\ln x < 0$ when $0 < x < 1$. So $\ln(1-\epsilon) < 0$, hence we consider its absolute value.

Now, we work through the actual the proof:

$$\begin{aligned}
|x-0| &< \delta \\
-\delta < x &< \delta && \text{(Definition of absolute value)} \\
-\ln(1+\epsilon) < x &< \ln(1+\epsilon). \\
\ln(1-\epsilon) < x &< \ln(1+\epsilon). && \text{(since } \ln(1-\epsilon) < -\ln(1+\epsilon))
\end{aligned}$$

The above line is true by our choice of δ and by the fact that since $|\ln(1-\epsilon)| > \ln(1+\epsilon)$ and $\ln(1-\epsilon) < 0$, we know $\ln(1-\epsilon) < -\ln(1+\epsilon)$.

$$1-\epsilon < e^x < 1+\epsilon \qquad \text{(Exponentiate)}$$

$$-\epsilon < e^x - 1 < \epsilon \qquad \text{(Subtract 1)}$$

In summary, given $\epsilon > 0$, let $\delta = \ln(1+\epsilon)$. Then $|x - 0| < \delta$ implies $|e^x - 1| < \epsilon$ as desired. We have shown that $\lim\limits_{x\to 0} e^x = 1$. ♣

We note that we could actually show that $\lim_{x\to c} e^x = e^c$ for any constant c. We do this by factoring out e^c from both sides, leaving us to show $\lim_{x\to c} e^{x-c} = 1$ instead. By using the substitution $u = x - c$, this reduces to showing $\lim_{u\to 0} e^u = 1$ which we just did in the last example. As an added benefit, this shows that in fact the function $f(x) = e^x$ is *continuous* at all values of x, an important concept we will define in Section 3.6.

This formal definition of the limit is not an easy concept grasp. Our examples are actually "easy" examples, using "simple" functions like polynomials, square–roots and exponentials. It is very difficult to prove, using the techniques given above, that $\lim\limits_{x\to 0}(\sin x)/x = 1$, as we approximated in the previous section.

There is hope. The next section shows how one can evaluate complicated limits using certain basic limits as building blocks. While limits are an incredibly important part of calculus (and hence much of higher mathematics), rarely are limits evaluated using the definition. Rather, the techniques of the following section are employed.

Exercises for Section 3.3

3.3.1

(a) *What is wrong with the following "definition" of a limit?*

> *"The limit of $f(x)$, as x approaches a, is K" means that given any $\delta > 0$ there exists $\epsilon > 0$ such that whenever $|f(x) - K| < \epsilon$, we have $|x - a| < \delta$.*

(b) *Which is given first in establishing a limit, the x–tolerance or the y–tolerance?*

(c) *T/F: ϵ must always be positive.*

(d) *T/F: δ must always be positive.*

3.3.2 *Prove the given limit using an $\epsilon - \delta$ proof.*

(a) $\lim\limits_{x\to 2} 5 = 5$

(b) $\lim\limits_{x\to 5} 3 - x = -2$

(c) $\lim\limits_{x\to 3} x^2 - 3 = 6$

(d) $\lim\limits_{x\to 2} x^3 - 1 = 7$

(e) $\lim\limits_{x\to 0} e^{2x} - 1 = 0$

(f) $\lim\limits_{x\to 0} \sin x = 0$ *(Hint: use the fact that $|\sin x| \leq |x|$, with equality only when $x = 0$.)*

(g) $\lim\limits_{x\to 4} x^2 + x - 5 = 15$

3.3.3 *Let ϵ be a small positive real number. How close to 2 must we hold x in order to be sure that $3x + 1$ lies within ϵ units of 7?*

3.4 Computing Limits: Limit Laws

Properties of limits

In Section 3.2 we explored the concept of the limit without a strict definition, meaning we could only make approximations. In the previous section we gave the definition of the limit and demonstrated how to use it to verify our approximations were correct. Thus far, our method of finding a limit is 1) make a really good approximation either graphically or numerically, and 2) verify our approximation is correct using a ϵ-δ proof.

Recognizing that ϵ-δ proofs are cumbersome, this section gives a series of theorems which allow us to find limits much more quickly and intuitively.

Suppose that $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = 3$. What is $\lim_{x\to 2}(f(x)+g(x))$? Intuition tells us that the limit should be 5, as we expect limits to behave in a nice way. The following theorem states that already established limits do behave nicely.

Theorem 3.4: Basic Limit Properties

Let a, c, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = K.$$

The following limits hold.

1. *Constants:* $\lim_{x\to a} c = c$
2. *Identity* $\lim_{x\to a} x = a$
3. *Sum/Difference Rules:* $\lim_{x\to a}(f(x) \pm g(x)) = L \pm K$
4. *Scalar Multiple Rule:* $\lim_{x\to a} c \cdot f(x) = cL$
5. *Limit Product Rule:* $\lim_{x\to a} f(x) \cdot g(x) = LK$
6. *Limit Quotient Rule:* $\lim_{x\to a} f(x)/g(x) = L/K,\ (K \neq 0)$
7. *Limit Power Rule:* $\lim_{x\to a} f(x)^n = L^n$
8. *Continuity of Roots:* $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$
9. *Compositions: Adjust our previously given limit assumptions to:*

$$\lim_{x\to a} g(x) = L \text{ and } \lim_{x\to L} f(x) = f(L).$$

Then $\lim_{x\to c} f(g(x)) = f(L)$.

We make a note about Property #8: when n is even, L must be greater than 0. If n is odd, then the statement is true for all L.

Regarding Property #9, note the special form of the condition on f: it is not enough to know that $\lim_{x\to L} f(x) = M$, though it is a bit tricky to see why. We have included an example in the exercise section to illustrate this tricky point for those who are interested. As we shall eventually see, many of the most familiar functions do have this property, so this result can therefore be applied.

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the "innermost bits" and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim_{x\to a} x = a$.

Example 3.14: Limit Properties

Compute $\lim_{x\to 1} \dfrac{x^2-3x+5}{x-2}$.

Solution. If we apply the theorem in all its gory detail, we get

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2-3x+5}{x-2} &= \frac{\lim_{x\to 1}(x^2-3x+5)}{\lim_{x\to 1}(x-2)} \\
&= \frac{(\lim_{x\to 1} x^2) - (\lim_{x\to 1} 3x) + (\lim_{x\to 1} 5)}{(\lim_{x\to 1} x) - (\lim_{x\to 1} 2)} \\
&= \frac{(\lim_{x\to 1} x)^2 - 3(\lim_{x\to 1} x) + 5}{(\lim_{x\to 1} x) - 2} \\
&= \frac{1^2 - 3\cdot 1 + 5}{1-2} \\
&= \frac{1-3+5}{-1} = -3
\end{aligned}$$

♣

Example 3.15: Using basic limit properties

Let

$$\lim_{x\to 2} f(x) = 2, \quad \lim_{x\to 2} g(x) = 3 \quad \text{and} \quad p(x) = 3x^2 - 5x + 7.$$

Find the following limits:

1. $\lim_{x\to 2} \big(f(x) + g(x)\big)$
2. $\lim_{x\to 2} \big(5f(x) + g(x)^2\big)$
3. $\lim_{x\to 2} p(x)$

Solution.

1. Using the Sum/Difference rule, we know that $\lim_{x\to 2} \big(f(x) + g(x)\big) = 2 + 3 = 5$.

2. Using the Scalar Multiple and Sum/Difference rules, we find that $\lim\limits_{x\to 2}\left(5f(x)+g(x)^2\right) = 5\cdot 2+3^2 = 19$.

3. Here we combine the Power, Scalar Multiple, Sum/Difference and Constant Rules. We show quite a few steps, but in general these can be omitted:

$$\begin{aligned}\lim_{x\to 2} p(x) &= \lim_{x\to 2}(3x^2-5x+7)\\ &= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 5x + \lim_{x\to 2} 7\\ &= 3\cdot 2^2 - 5\cdot 2 + 7\\ &= 9\end{aligned}$$

♣

Part 3 of the previous example demonstrates how the limit of a quadratic polynomial can be determined using the properties of Theorem 3.4. Not only that, recognize that

$$\lim_{x\to 2} p(x) = 9 = p(2);$$

i.e., the limit at 2 was found just by plugging 2 into the function. This holds true for all polynomials, and also for rational functions (which are quotients of polynomials), as stated in the following theorem.

Theorem 3.5: Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim\limits_{x\to c} p(x) = p(c)$

2. $\lim\limits_{x\to c} \dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$, *where* $q(c)\neq 0$.

Example 3.16: Finding a limit of a rational function

Using Theorem 3.5, find

$$\lim_{x\to -1}\frac{3x^2-5x+1}{x^4-x^2+3}.$$

Solution. Using Theorem 3.5, we can quickly state that

$$\begin{aligned}\lim_{x\to -1}\frac{3x^2-5x+1}{x^4-x^2+3} &= \frac{3(-1)^2-5(-1)+1}{(-1)^4-(-1)^2+3}\\ &= \frac{9}{3} = 3.\end{aligned}$$

♣

It was likely frustrating in Example 3.11 to do a lot of work to prove that

$$\lim_{x\to 2} x^2 = 4$$

as it seemed fairly obvious. The previous theorems state that many functions behave in such an "obvious" fashion, as demonstrated by the rational function in Example 3.16.

Polynomial and rational functions are not the only functions to behave in such a predictable way. The following theorem gives a list of functions whose behavior is particularly "nice" in terms of limits. In the next section, we will give a formal name to these functions that behave "nicely."

Theorem 3.6: Special Limits

Let c be a real number in the domain of the given function and let n be a positive integer. The following limits hold:

1. $\lim\limits_{x\to c} \sin x = \sin c$
2. $\lim\limits_{x\to c} \cos x = \cos c$
3. $\lim\limits_{x\to c} \tan x = \tan c$
4. $\lim\limits_{x\to c} \csc x = \csc c$
5. $\lim\limits_{x\to c} \sec x = \sec c$
6. $\lim\limits_{x\to c} \cot x = \cot c$
7. $\lim\limits_{x\to c} a^x = a^c \; (a > 0)$
8. $\lim\limits_{x\to c} \ln x = \ln c$
9. $\lim\limits_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

Example 3.17: Evaluating limits analytically

Evaluate the following limits.

1. $\lim\limits_{x\to \pi} \cos x$
2. $\lim\limits_{x\to 3} (\sec^2 x - \tan^2 x)$
3. $\lim\limits_{x\to \pi/2} \cos x \sin x$
4. $\lim\limits_{x\to 1} e^{\ln x}$
5. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

Solution.

1. This is a straightforward application of Theorem 3.6. $\lim\limits_{x\to \pi} \cos x = \cos \pi = -1$.

2. We can approach this in at least two ways. First, by directly applying Theorem 3.6, we have:

$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = \sec^2 3 - \tan^2 3.$$

Using the Pythagorean Theorem, this last expression is 1; therefore

$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = 1.$$

We can also use the Pythagorean Theorem from the start.

$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = \lim_{x\to 3} 1 = 1,$$

using the Constant limit rule. Either way, we find the limit is 1.

3. Applying the Product limit rule of Theorem 3.4 and Theorem 3.6 gives

$$\lim_{x\to \pi/2} \cos x \sin x = \cos(\pi/2) \sin(\pi/2) = 0 \cdot 1 = 0.$$

4. Again, we can approach this in two ways. First, we can use the exponential/logarithmic identity that $e^{\ln x} = x$ and evaluate $\lim_{x\to 1} e^{\ln x} = \lim_{x\to 1} x = 1$.

   We can also use the Composition limit rule of Theorem 3.4. Using Theorem 3.6, we have $\lim_{x\to 1} \ln x = \ln 1 = 0$. Applying the Composition rule,
$$\lim_{x\to 1} e^{\ln x} = \lim_{x\to 0} e^x = e^0 = 1.$$
   Both approaches are valid, giving the same result.

5. We encountered this limit in Section 3.2. Applying our theorems, we attempt to find the limit as
$$\lim_{x\to 0} \frac{\sin x}{x} \to \frac{\sin 0}{0} \to \text{“}\ \frac{0}{0}\ \text{”}.$$
   This, of course, violates a condition of Theorem 3.4, as the limit of the denominator is not allowed to be 0. Therefore, we are still unable to evaluate this limit with tools we currently have at hand.

♣

Our final theorem for this section will be motivated by the following example.

Example 3.18: Using algebra to evaluate a limit

Evaluate the following limit:
$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

Solution. We begin by attempting to apply Theorem 3.6 and substituting 1 for x in the quotient. This gives:
$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \text{“}\ \frac{0}{0}\ \text{”},$$
and indeterminate form. We cannot apply the theorem.

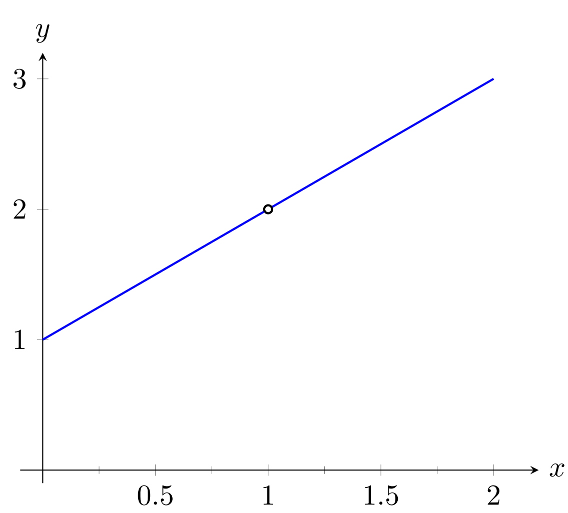


Figure 3.18: Graphing f in Example 3.18 to understand a limit.

By graphing the function, as in Figure 3.18, we see that the function seems to be linear, implying that the limit should be easy to evaluate. Recognize that the numerator of our quotient can be factored:

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}.$$

The function is not defined when $x = 1$, but for all other x,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1.$$

Clearly $\lim\limits_{x\to 1} x+1 = 2$. Recall that when considering limits, we are not concerned with the value of the function at 1, only the value the function approaches as x approaches 1. Since $(x^2-1)/(x-1)$ and $x+1$ are the same at all points except $x = 1$, they both approach the same value as x approaches 1. Therefore we can conclude that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

♣

The key to the above example is that the functions $y = (x^2-1)/(x-1)$ and $y = x+1$ are identical except at $x = 1$. Since limits describe a value the function is approaching, not the value the function actually attains, the limits of the two functions are always equal.

Theorem 3.7: Limits of Functions Equal At All But One Point

Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim\limits_{x\to c} g(x) = L$ for some real number L. Then

$$\lim_{x\to c} f(x) = L.$$

The Fundamental Theorem of Algebra tells us that when dealing with a rational function of the form $g(x)/f(x)$ and directly evaluating the limit $\lim\limits_{x\to c} \frac{g(x)}{f(x)}$ returns "0/0", then $(x-c)$ is a factor of both $g(x)$ and $f(x)$. One can then use algebra to factor this term out, cancel, then apply Theorem 3.7. We demonstrate this once more.

Example 3.19: Evaluating a limit using Theorem 3.7

Evaluate $\lim\limits_{x\to 3} \dfrac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15}$.

Solution. We begin by applying Theorem 3.6 and substituting 3 for x. This returns the familiar indeterminate form of "0/0". Since the numerator and denominator are each polynomials, we know that $(x-3)$ is factor of each. Using whatever method is most comfortable to you, factor out $(x-3)$ from each (using polynomial division, synthetic division, a computer algebra system, etc.). We find that

$$\frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} = \frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)}.$$

We can cancel the $(x-3)$ terms as long as $x \neq 3$. Using Theorem 3.7 we conclude:

$$\begin{aligned}
\lim_{x\to 3} \frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} &= \lim_{x\to 3} \frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)} \\
&= \lim_{x\to 3} \frac{(x^2+x-2)}{(2x^2+9x-5)} \\
&= \frac{10}{40} = \frac{1}{4}.
\end{aligned}$$

♣

Example 3.20: Left and Right Limit

Evaluate $\lim_{x\to 0} \dfrac{x}{|x|}$.

Solution. The function $f(x) = x/|x|$ is undefined at 0; when $x > 0$, $|x| = x$ and so $f(x) = 1$; when $x < 0$, $|x| = -x$ and $f(x) = -1$. Thus

$$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} -1 = -1$$

while

$$\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} 1 = 1.$$

The limit of $f(x)$ must be equal to both the left and right limits; since they are different, the limit $\lim_{x\to 0} \dfrac{x}{|x|}$ does not exist. ♣

Another of the most common algebraic tricks is called *rationalization*. Rationalizing makes use of the difference of squares formula $(a-b)(a+b) = a^2-b^2$. Here is an example.

Example 3.21: Rationalizing

Compute $\lim_{x\to -1} \dfrac{\sqrt{x+5}-2}{x+1}$.

Solution.

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} &= \lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} \cdot \frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} \\
&= \lim_{x\to -1} \frac{x+5-4}{(x+1)(\sqrt{x+5}+2)} \\
&= \lim_{x\to -1} \frac{x+1}{(x+1)(\sqrt{x+5}+2)} \\
&= \lim_{x\to -1} \frac{1}{\sqrt{x+5}+2} = \frac{1}{4}
\end{aligned}$$

At the very last step we have used the last two parts of Theorem **??**. ♣

We end this section by revisiting a limit first seen in Section 3.2, a limit of a difference quotient. Let $f(x) = -1.5x^2 + 11.5x$; we approximated the limit $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \approx 8.5$. We formally evaluate this limit in the following example.

Example 3.22: Evaluating the limit of a difference quotient

Let $f(x) = -1.5x^2 + 11.5x$; *find* $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$.

Solution. Since f is a polynomial, our first attempt should be to employ Theorem 3.6 and substitute 0 for h. However, we see that this gives us "0/0." Knowing that we have a rational function hints that some algebra will help. Consider the following steps:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0} \frac{-1.5(1+h)^2 + 11.5(1+h) - \big(-1.5(1)^2 + 11.5(1)\big)}{h} \\
&= \lim_{h \to 0} \frac{-1.5(1 + 2h + h^2) + 11.5 + 11.5h - 10}{h} \\
&= \lim_{h \to 0} \frac{-1.5h^2 + 8.5h}{h} \\
&= \lim_{h \to 0} \frac{h(-1.5h + 8.5)}{h} \\
&= \lim_{h \to 0} (-1.5h + 8.5) \quad \text{(using Theorem 3.7, as } h \neq 0\text{)} \\
&= 8.5 \quad \text{(using Theorem 3.6)}
\end{aligned}$$

This matches our previous approximation. ♣

This section contains several valuable tools for evaluating limits. One of the main results of this section is Theorem 3.6; it states that many functions that we use regularly behave in a very nice, predictable way.

Exercises for 3.4

3.4.1 *Explain in your own words, without using* ϵ-δ *formality, why* $\lim_{x \to c} b = b$.

3.4.2 *Explain in your own words, without using* ϵ-δ *formality, why* $\lim_{x \to c} x = c$.

3.4.3 *What does the text mean when it says that certain functions' "behavior is 'nice' in terms of limits"? What, in particular, is "nice"?*

3.4.4 *Using:*

$\lim_{x \to 9} f(x) = 6$ $\qquad$ $\lim_{x \to 6} f(x) = 9$

$\lim_{x \to 9} g(x) = 3$ $\qquad$ $\lim_{x \to 6} g(x) = 3$

evaluate the following limits, where possible. If it is not possible to know, state so.

(a) $\lim_{x\to 9}(f(x)+g(x))$

(b) $\lim_{x\to 9}(3f(x)/g(x))$

(c) $\lim_{x\to 9}\left(\frac{f(x)-2g(x)}{g(x)}\right)$

(d) $\lim_{x\to 6}\left(\frac{f(x)}{3-g(x)}\right)$

(e) $\lim_{x\to 9} g\big(f(x)\big)$

(f) $\lim_{x\to 6} f\big(g(x)\big)$

(g) $\lim_{x\to 6} g\big(f(f(x))\big)$

(h) $\lim_{x\to 6} f(x)g(x)-f^{\,2}(x)+g^2(x)$

3.4.5 *Using:*

$\lim_{x\to 1} f(x)=2 \qquad \lim_{x\to 10} f(x)=1$

$\lim_{x\to 1} g(x)=0 \qquad \lim_{x\to 10} g(x)=\pi$

evaluate the limits given, where possible. If it is not possible to know, state so.

(a) $\lim_{x\to 1} f(x)^{g(x)}$

(b) $\lim_{x\to 10} \cos\big(g(x)\big)$

(c) $\lim_{x\to 1} f(x)g(x)$

(d) $\lim_{x\to 1} g\big(5f(x)\big)$

3.4.6 *Compute the limits. If a limit does not exist, explain why.*

(a) $\lim_{x\to 3}\frac{x^2+x-12}{x-3}$

(b) $\lim_{x\to 1}\frac{x^2+x-12}{x-3}$

(c) $\lim_{x\to -4}\frac{x^2+x-12}{x-3}$

(d) $\lim_{x\to \pi}\frac{3x+1}{1-x}$

(e) $\lim_{x\to \pi}\frac{x^2+3x+5}{5x^2-2x-3}$

(f) $\lim_{x\to \pi}\left(\frac{x-3}{x-5}\right)^7$

(g) $\lim_{x\to \pi/4}\cos x\sin x$

(h) $\lim_{x\to 0}\ln x$

(i) $\lim_{x\to 3} 4^{x^3-8x}$

(j) $\lim_{x\to 2}\frac{x^2+x-12}{x-2}$

(k) $\lim_{x\to 1}\frac{\sqrt{x+8}-3}{x-1}$

(l) $\lim_{x\to 0^+}\sqrt{\frac{1}{x}+2}-\sqrt{\frac{1}{x}}$

(m) $\lim_{x\to 2} 3$

(n) $\lim_{x\to 4} 3x^3 - 5x$

(o) $\lim_{x\to 0} \dfrac{4x-5x^2}{x-1}$

(p) $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$

(q) $\lim_{x\to 0^+} \dfrac{\sqrt{2-x^2}}{x}$

(r) $\lim_{x\to 0^+} \dfrac{\sqrt{2-x^2}}{x+1}$

(s) $\lim_{x\to a} \dfrac{x^3-a^3}{x-a}$

(t) $\lim_{x\to 2} (x^2+4)^3$

3.4.7 *Let* $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ *and* $g(x) = 0$. *What are the values of* $L = \lim_{x\to 0} g(x)$ *and* $M = \lim_{x\to L} f(x)$*? Is it true that* $\lim_{x\to 0} f(g(x)) = M$*? What are some noteworthy differences between this example and part # 9 of Theorem 3.4?*

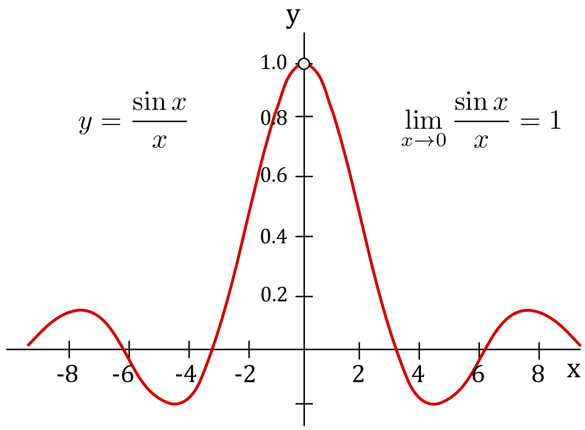
3.5 The Squeeze Theorem And Some Special Limits

The previous section could have been titled "Using Known Limits to Find Unknown Limits." By knowing certain limits of functions, we can find limits involving sums, products, powers, etc., of these functions. We further the development of such comparative tools with the Squeeze Theorem, a clever and intuitive way to find the value of some limits.

In this section we aim to compute the limit:

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

We start by analyzing the graph of $y=\dfrac{\sin x}{x}$:



Notice that $x=0$ is not in the domain of this function. Nevertheless, we can look at the limit as x approaches 0. From the graph we find that the limit is 1 (there is an open circle at $x=0$ indicating 0 is not in the domain). We just convinced you this limit formula holds true based on the graph, but how does one attempt to prove this limit more formally? To do this we employ some indirect reasoning embodied in the **Squeeze Theorem**.

Before stating this theorem formally, suppose we have functions f, g and h where g always takes on values between f and h; that is, for all x in an interval,

$$f(x)\le g(x)\le h(x).$$

If f and h have the same limit at c, and g is always "squeezed" between them, then g must have the same limit as well. That is what the Squeeze Theorem states.

Theorem 3.8: Squeeze Theorem

Let f, g and h be functions on an open interval I containing a such that for all x in I,

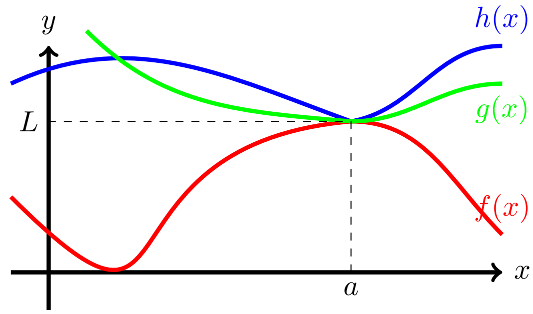
$$f(x)\le g(x)\le h(x).$$

If

$$\lim_{x\to a}f(x)=L=\lim_{x\to a}h(x),$$

then

$$\lim_{x\to a}g(x)=L.$$



It can take some work to figure out appropriate functions by which to "squeeze" the given function of which you are trying to evaluate a limit. However, that is generally the only place work is necessary; the theorem makes the "evaluating the limit part" very simple.

We use the Squeeze Theorem in the following example to finally prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Example 3.23: Using the Squeeze Theorem

Use the Squeeze Theorem to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Solution. We begin by considering the unit circle. Each point on the unit circle has coordinates $(\cos\theta, \sin\theta)$ for some angle θ as shown in Figure 3.19. Using similar triangles, we can extend the line from the origin through the point to the point $(1, \tan\theta)$, as shown. (Here we are assuming that $0 \leq \theta \leq \pi/2$. Later we will show that we can also consider $\theta \leq 0$.)

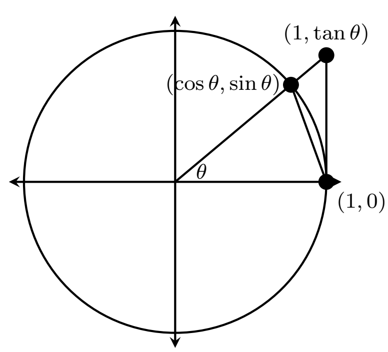
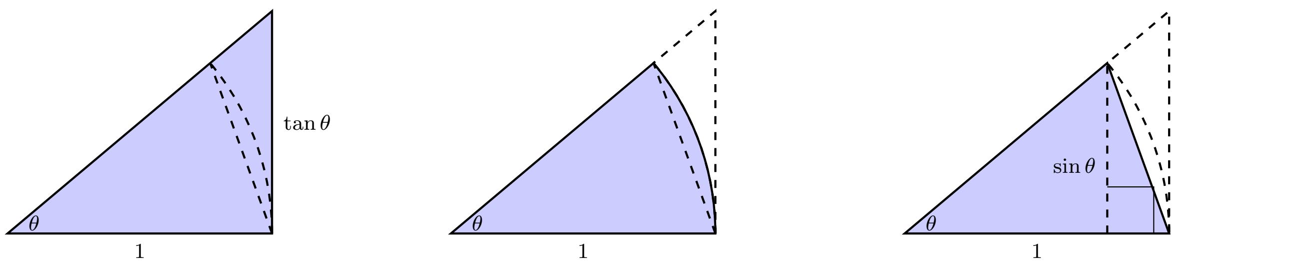


Figure 3.19: The unit circle and related triangles.

Figure 3.19 shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is $\frac{1}{2}\tan\theta$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is $\frac{1}{2}\sin\theta$. It is then clear from the diagram that we get the inequality in Figure 3.20.



(a) $\dfrac{\tan\theta}{2}$ **(b)** $\dfrac{\theta}{2}$ **(c)** $\dfrac{\sin\theta}{2}$

Figure 3.20: Demonstrating that $\dfrac{\tan\theta}{2} \geq \dfrac{\theta}{2} \geq \dfrac{\sin\theta}{2}$

Multiply all terms by $\dfrac{2}{\sin\theta}$, giving

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

(These inequalities hold for all values of θ near 0, even negative values, since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.)

Now take limits.

$$\lim_{\theta\to 0}\cos\theta \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq \lim_{\theta\to 0} 1$$

$$\cos 0 \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq 1$$

$$1 \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq 1$$

Clearly this means that $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

♣

Two notes about the previous example are worth mentioning. First, one might be discouraged by this application, thinking "I would *never* have come up with that on my own. This is too hard!" Don't be discouraged; within this text we will guide you in your use of the Squeeze Theorem. As one gains mathematical maturity, clever proofs like this are easier and easier to create.

Second, this limit tells us more than just that as x approaches 0, $\sin(x)/x$ approaches 1. Both x and $\sin x$ are approaching 0, but the *ratio* of x and $\sin x$ approaches 1, meaning that they are approaching 0 in essentially the same way. Another way of viewing this is: for small x, the functions $y = x$ and $y = \sin x$ are essentially indistinguishable.

We include this special limit, along with three others, in the following theorem.

Theorem 3.9: Special Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$
3. $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$
4. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

A short word on how to interpret the latter three limits. We know that as x goes to 0, $\cos x$ goes to 1. So, in the second limit, both the numerator and denominator are approaching 0. However, since the limit is 0, we can interpret this as saying that "$\cos x$ is approaching 1 faster than x is approaching 0."

In the third limit, inside the parentheses we have an expression that is approaching 1 (though never equaling 1), and we know that 1 raised to any power is still 1. At the same time, the power is growing toward infinity. What happens to a number near 1 raised to a very large power? In this particular case, the result approaches Euler's number, e, approximately 2.718.

In the fourth limit, we see that as $x \to 0$, e^x approaches 1 "just as fast" as $x \to 0$, resulting in a limit of 1.

Example 3.24: Limit of Other Trig Functions

Compute the following limit $\lim_{x \to 0} \frac{\sin 5x \cos x}{x}$.

Solution. We have

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin 5x \cos x}{x} &= \lim_{x \to 0} \frac{5 \sin 5x \cos x}{5x} \\
&= \lim_{x \to 0} 5 \cos x \left(\frac{\sin 5x}{5x} \right) \\
&= 5 \cdot (1) \cdot (1) \quad = \quad 5
\end{aligned}$$

since $\cos(0) = 1$ and $\lim_{x \to 0} \frac{\sin 5x}{5x} = 1$. ♣

Let's do a harder one now.

Example 3.25: Limit of Other Trig Functions

Compute the following limit: $\lim_{x \to 0} \frac{\tan^3 2x}{x^2 \sin 7x}$.

Solution. Recall that the $\tan^3(2x)$ means that $\tan(2x)$ is being raised to the third power.

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan^3(2x)}{x^2\sin(7x)} &= \lim_{x\to 0}\frac{(\sin(2x))^3}{x^2\sin(7x)\cos^3(2x)} && \text{Rewrite in terms of sin and cos}\\
&= \lim_{x\to 0}\frac{(2x)^3\left(\frac{\sin(2x)}{2x}\right)^3}{x^2(7x)\left(\frac{\sin(7x)}{7x}\right)\cos^3(2x)} && \text{Make sine terms look like: } \frac{\sin\theta}{\theta}\\
&= \lim_{x\to 0}\frac{8x^3(1)^3}{7x^3(1)(1^3)} && \text{Replace } \lim_{x\to 0}\frac{\sin nx}{nx} \text{ with 1. Also, } \cos(0)=1.\\
&= \lim_{x\to 0}\frac{8}{7} && \text{Cancel } x^3\text{'s.}\\
&= \frac{8}{7}.
\end{aligned}$$

♣

Example 3.26: Applying the Squeeze Theorem

Compute the following limit: $\lim_{x\to 0^+} x^3\cos\left(\frac{1}{\sqrt{x}}\right)$.

Solution. We use the *Squeeze Theorem* to evaluate this limit. We know that $\cos\alpha$ satisfies $-1\le\cos\alpha\le 1$ for any choice of α. Therefore we can write:

$$-1\le\cos\left(\frac{1}{\sqrt{x}}\right)\le 1$$

Since $x\to 0^+$ implies $x>0$, multiplying by x^3 gives:

$$-x^3\le x^3\cos\left(\frac{1}{\sqrt{x}}\right)\le x^3.$$

$$\lim_{x\to 0^+}(-x^3)\le\lim_{x\to 0^+}\left(x^3\cos\left(\frac{1}{\sqrt{x}}\right)\right)\le\lim_{x\to 0^+}x^3.$$

But using our rules we know that

$$\lim_{x\to 0^+}(-x^3)=0,\qquad \lim_{x\to 0^+}x^3=0$$

and the Squeeze Theorem says that the only way this can happen is if

$$\lim_{x\to 0^+}x^3\cos\left(\frac{1}{\sqrt{x}}\right)=0.$$

♣

When solving problems using the Squeeze Theorem it is also helpful to have the following theorem.

Theorem 3.10: Monotone Limits

If $f(x)\le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then $\lim_{x\to a}f(x)\le\lim_{x\to a}g(x)$.

Exercises for 3.5

3.5.1 *Compute the following limits.*

(a) $\lim\limits_{x\to 0} \dfrac{\sin(5x)}{x}$

(b) $\lim\limits_{x\to 0} \dfrac{\sin(7x)}{\sin(2x)}$

(c) $\lim\limits_{x\to 0} \dfrac{\cot(4x)}{\csc(3x)}$

(d) $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$

(e) $\lim\limits_{x\to \pi/4} \dfrac{\sin x - \cos x}{\cos(2x)}$

3.5.2 *For all* $x \geq 0$, $4x - 9 \leq f(x) \leq x^2 - 4x + 7$. *Find* $\lim\limits_{x\to 4} f(x)$.

3.5.3 *For all* x, $2x \leq g(x) \leq x^4 - x^2 + 2$. *Find* $\lim\limits_{x\to 1} g(x)$.

3.5.4 *Use the Squeeze Theorem where appropriate, to evaluate the given limit.*

(a) $\lim\limits_{x\to 0} x \sin\left(\dfrac{1}{x}\right)$

(b) $\lim\limits_{x\to 0} \sin x \cos\left(\dfrac{1}{x^2}\right)$

(c) $\lim\limits_{x\to 0} x \sin\left(\dfrac{1}{x}\right)$

(d) $\lim\limits_{x\to 3^+} f(x)$, *where* $6x - 9 \leq f(x) \leq x^2$ *on* $[0, 3]$.

3.5.5 *Use the Squeeze Theorem to show that* $\lim\limits_{x\to 0} x^4 \cos(2/x) = 0$.

3.5.6 *Find the value of* $\lim_{x\to\infty} \dfrac{3x + \sin x}{x + \cos x}$. *Justify your steps carefully.*

3.5.7 *The following exercises challenge your understanding of limits but can be evaluated using the knowledge gained in this section.*

(a) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$

(b) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{8x}$

(c) $\lim_{x\to 0} \frac{\ln(1+x)}{x}$

(d) $\lim_{x\to 0} \frac{\sin x}{x}$, *where x is measured in degrees, not radians.*

3.6 Continuity

The graph shown in Figure 3.21(a) represents a **continuous** function. Geometrically, this is because there are no jumps in the graphs. That is, if you pick a point on the graph and approach it from the left and right, the values of the function approach the value of the function at that point. For example, we can see that this is not true for function values near $x = 1$ on the graph in Figure 3.21(b) which is not continuous at that location.

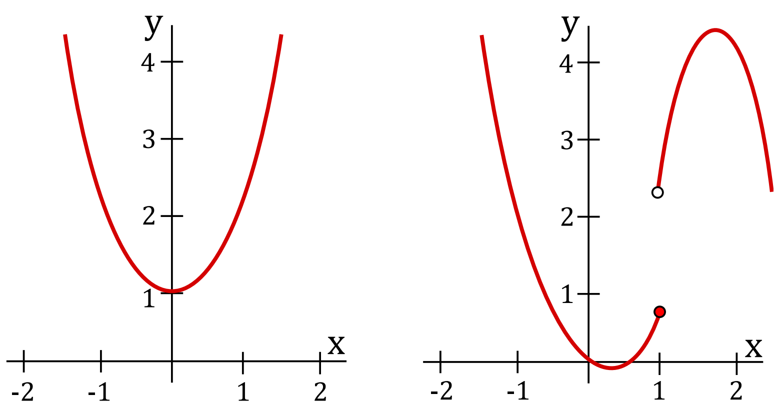


Figure 3.21: (a) A continuous function. (b) A function with a discontinuity at $x = 1$.

Definition 3.5: Continuous at a Point

A function f is **continuous at a point** *a if*

$$\lim_{x\to a} f(x) = f(a).$$

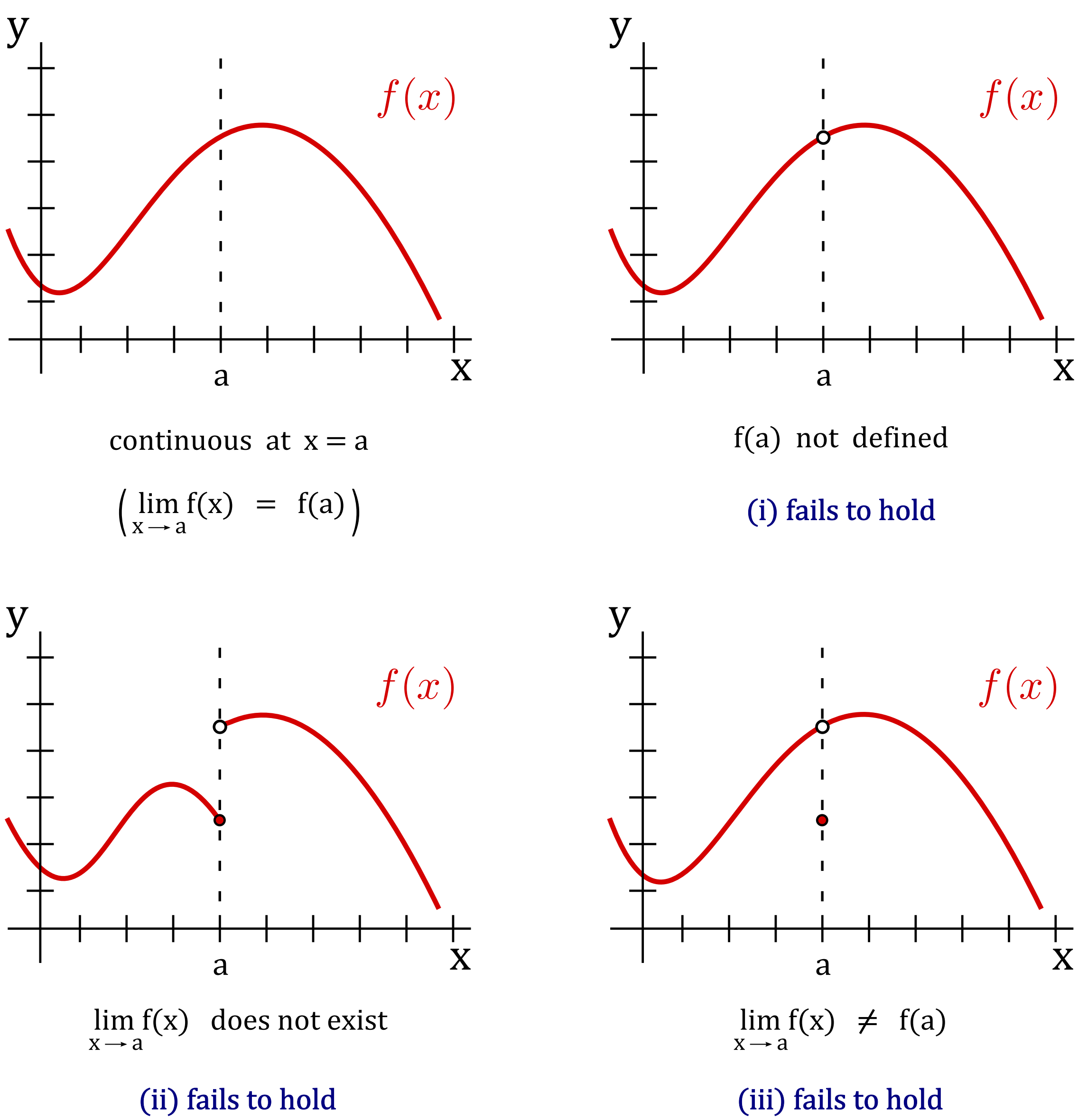
Some readers may prefer to think of continuity at a point as a three part definition. That is, a function $f(x)$ is continuous at $x = a$ if the following three conditions hold:

(i) $f(a)$ is defined (that is, a belongs to the domain of f),

(ii) $\lim_{x\to a} f(x)$ exists (that is, left-hand limit = right-hand limit),

(iii) $\lim_{x\to a} f(x) = f(a)$ (that is, the numbers from (i) and (ii) are equal).

The figures below show graphical examples of functions where either (i), (ii) or (iii) can fail to hold.



On the other hand, if f is defined on an open interval containing a, except perhaps at a, we can say that f is **discontinuous** at a if f is not continuous at a.

Graphically, you can think of continuity as being able to draw your function without having to lift your pencil off the paper. If your pencil has to jump off the page to continue drawing the function, then the function is not continuous at that point. This is illustrated in Figure 3.21(b) where if we tried to draw the function (from left to right) we need to lift our pencil off the page once we reach the point $x = 1$ in order to be able to continue drawing the function.

Definition 3.6: Continuity on an Open Interval

A function f is **continuous on an open interval** (a, b) *if it is continuous at every point in the interval.*

Furthermore, a function is **everywhere continuous** *if it is continuous on the entire real number line* $(-\infty, \infty)$..

Recall the function graphed in a previous section as shown in Figure 3.22.

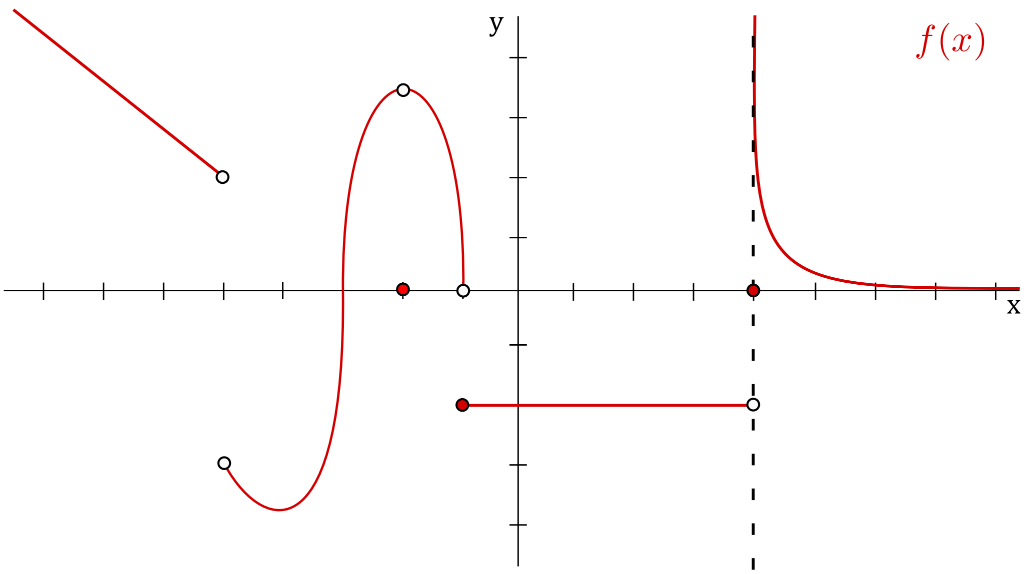
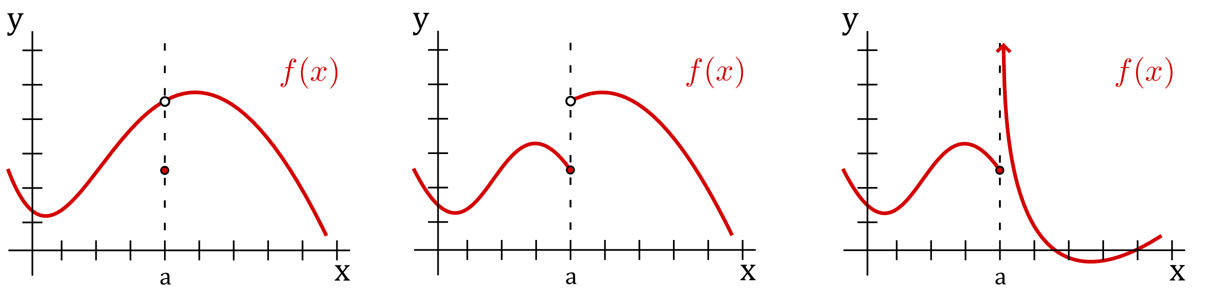


Figure 3.22: A function with discontinuities at $x = -5$, $x = -2$, $x = -1$ and $x = 4$.

We can draw this function without lifting our pencil *except* at the points $x = -5$, $x = -2$, $x = -1$, and $x = 4$. Thus, $f(x)$ is *continuous* at every real number *except* at these four numbers. At $x = -5$, $x = -2$, $x = -1$, and $x = 4$, the function $f(x)$ is *discontinuous*.

At $x = -2$ we have a **removable discontinuity** because we could remove this discontinuity simply by redefining $f(-2)$ to be 3.5. At $x = -5$ and $x = -1$ we have **jump discontinuities** because the function jumps from one value to another. From the right of $x = 4$, we have an **infinite discontinuity** because the function goes off to infinity.

Formally, we say $f(x)$ has a **removable discontinuity** at $x = a$ if $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$. Note that we do not require $f(a)$ to be defined in this case, that is, a need not belong to the domain of $f(x)$.



$\lim_{x \to a} f(x) \neq f(a)$ | $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$ | Either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$

REMOVABLE DISCONTINUITY | **JUMP DISCONTINUITY** | **INFINITE DISCONTINUITY**

Example 3.27: Continuous at a Point

What value of c will make the following function $f(x)$ continuous at 2?

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ c & \text{if } x = 2 \end{cases}$$

Solution. In order to be continuous at 2 we require

$$\lim_{x \to 2} f(x) = f(2)$$

to hold. We use the three part definition listed previously to check this.

1. First, $f(2) = c$, and c is some real number. Thus, $f(2)$ is defined.

2. Now, we must evaluate the limit. Rather than computing both one-sided limits, we just compute the limit directly. For x close to 2 (but not equal to 2) we can replace $f(x)$ with $\frac{x^2-x-2}{x-2}$ to get:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3.$$

Therefore the limit exists and equals 3.

3. Finally, for f to be continuous at 2, we need that the numbers in the first two items to be equal. Therefore, we require $c = 3$. Thus, when $c = 3$, $f(x)$ is continuous at 2, for any other value of c, $f(x)$ is discontinuous at 2. ♣

Example 3.28: Finding intervals of continuity

The floor function, $f(x) = \lfloor x \rfloor$, returns the largest integer smaller than the input x. (For example, $f(\pi) = \lfloor \pi \rfloor = 3$.) The graph of f in Figure 3.23 demonstrates why this is often called a "step function."

Give the intervals on which f is continuous.

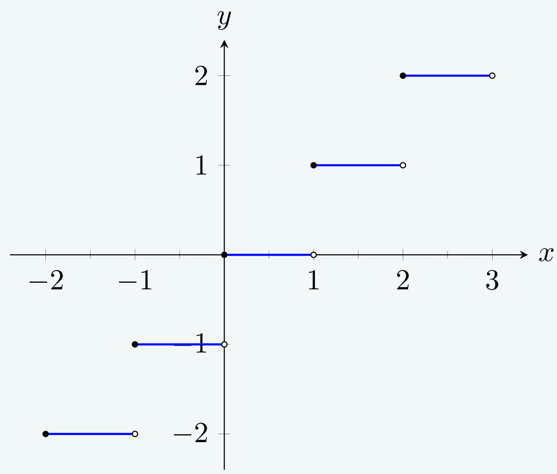


Figure 3.23: A graph of the step function in Example 3.28.

Solution. We examine the three criteria for continuity.

1. The limits $\lim_{x\to c} f(x)$ do not exist at the jumps from one "step" to the next, which occur at all integer values of c. Therefore the limits exist for all c except when c is an integer.
2. The function is defined for all values of c.
3. The limit $\lim\limits_{x\to c} f(x) = f(c)$ for all values of c where the limit exist, since each step consists of just a line.

We conclude that f is continuous everywhere except at integer values of c. So the intervals on which f is continuous are

$$\ldots, (-2,-1), (-1,0), (0,1), (1,2), \ldots.$$

♣

Our definition of continuity on an interval specifies the interval is an open interval. We can extend the definition of continuity to closed intervals by considering the appropriate one-sided limits at the endpoints.

Definition 3.7: Continuous from the Right and from the Left

A function f is **left continuous at a point** a *if*

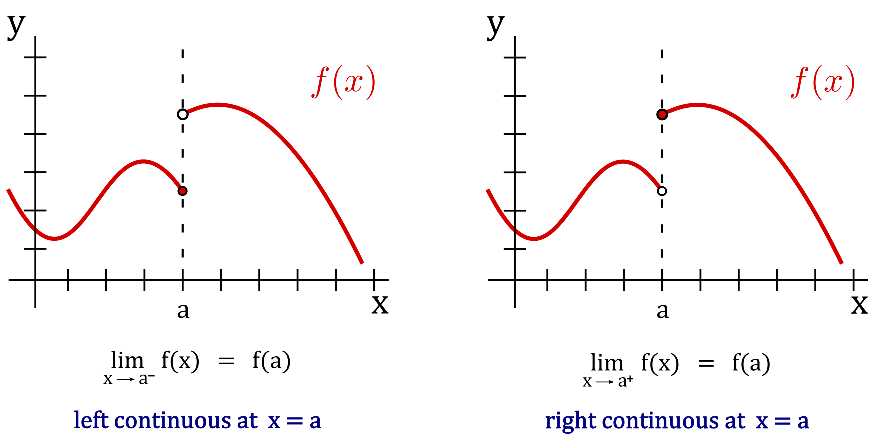
$$\lim_{x\to a^-} f(x) = f(a)$$

and **right continuous at a point** a *if*

$$\lim_{x\to a^+} f(x) = f(a).$$

If a function f is continuous at a, then it is both left and right continuous at a.

The above definition regarding left (or right) continuous functions is illustrated with the following figure:



One-sided limits allows us to extend the definition of continuity to closed intervals. The following definition means a function is continuous on a closed interval if it is continuous in the interior of the interval and possesses the appropriate one-sided continuity at the endpoints of the interval.

Definition 3.8: Continuity on a Closed Interval

A function f is **continuous on the closed interval** $[a, b]$ *if:*

(i) it is continuous on the open interval (a, b);

(ii) it is left continuous at point a:

$$\lim_{x \to a^-} f(x) = f(a);$$

and

(iii) it is right continuous at point b:

$$\lim_{x \to b^+} f(x) = f(b).$$

This definition can be extended to continuity on half-open intervals such as $(a, b]$ and $[a, b)$, and unbounded intervals.

Example 3.29: Continuity on Other Intervals

The function $f(x) = \sqrt{x}$ is continuous on the (closed) interval $[0, \infty)$.

The function $f(x) = \sqrt{4 - x}$ is continuous on the (closed) interval $(-\infty, 4]$.

The continuity of functions is preserved under the operations of addition, subtraction, multiplication and division (in the case that the function in the denominator is nonzero).

Theorem 3.11: Operations of Continuous Functions

If f and g are continous at a, and c is a constant, then the following functions are also continuous at a:

(i) $f \pm g$;

(ii) cf;

(iii) fg;

(iv) f/g (provided $g(a) \neq 0$).

Example 3.30: Determining intervals on which a function is continuous

State the interval(s) on which each of the following functions is continuous.

1. $f(x) = \sqrt{x-1} + \sqrt{5-x}$
2. $f(x) = x \sin x$
3. $f(x) = \tan x$
4. $f(x) = \sqrt{\ln x}$

Solution. We examine each in turn.

1. The square–root terms are continuous on the intervals $[1, \infty)$ and $(-\infty, 5]$, respectively. As f is continuous only where each term is continuous, f is continuous on $[1, 5]$, the intersection of these two intervals. A graph of f is given in Figure 3.24.

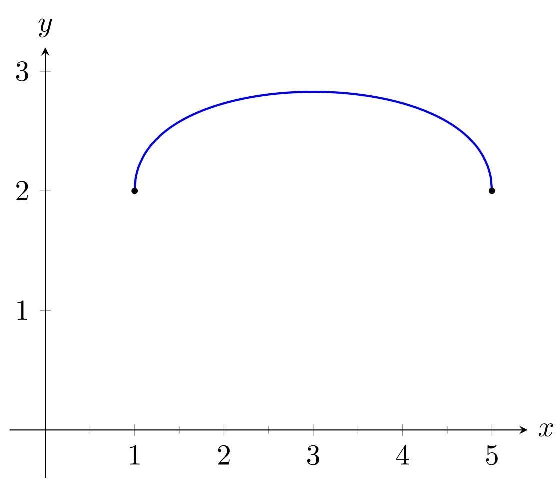


Figure 3.24: A graph of f in Example 3.30(a).

2. The functions $y = x$ and $y = \sin x$ are each continuous everywhere, hence their product is, too.

3. $f(x) = \tan x$ is continuous "on its domain." Its domain includes all real numbers except odd multiples of $\pi/2$. Thus $f(x) = \tan x$ is continuous on

$$\ldots \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots,$$

or, equivalently, on $D = \{x \in \mathbb{R} \mid x \neq n \cdot \frac{\pi}{2}$, n is an odd integer$\}$.

4. The domain of $y = \sqrt{x}$ is $[0, \infty)$. The range of $y = \ln x$ is $(-\infty, \infty)$, but if we restrict its domain to $[1, \infty)$ its range is $[0, \infty)$. So restricting $y = \ln x$ to the domain of $[1, \infty)$ restricts its output is $[0, \infty)$, on which $y = \sqrt{x}$ is defined. Thus the domain of $f(x) = \sqrt{\ln x}$ is $[1, \infty)$.

♣

The next theorem states that the inverse f^{-1} of a continuous function $f(x)$ is also continuous. This is not so surprising because the graph of $f^{-1}(x)$ is a reflection of the graph of $f(x)$ through the line $y = x$. If the graph of f has no "breaks" in it, then neither will the graph of f^{-1}.

Theorem 3.12: Continuity of Inverse Functions

If $f(x)$ is continuous on an interval I, with range R, then f^{-1} is continuous on every interval of the domain R.

One consequence of this is that the inverse trigonometric functions are all continuous on their respective domains. Below we list some common functions that are known to be continuous on every interval inside their domains.

Example 3.31: Common Types of Continuous Functions

- *Polynomials (for all x), e.g., $y = mx + b$, $y = ax^2 + bx + c$.*
- *Rational functions (except at points x which gives division by zero).*
- *Root functions $\sqrt[n]{x}$ (for all x if n is odd, and for $x \geq 0$ if n is even).*
- *Trigonometric functions*
- *Inverse trigonometric functions*
- *Exponential funtions*
- *Logarithmic functions*

For rational functions with removable discontinuities as a result of a zero, we can define a new function filling in these gaps to create a piecewise function that is continuous everywhere.

Continuous functions are where the *direct substitution property* hold. This fact can often be used to compute the limit of a continuous function.

Example 3.32: Evaluate a Limit

Evaluate the following limit: $\lim_{x\to\pi} \dfrac{\sqrt{x} + \sin x}{1 + x + \cos x}$.

Solution. We will use a continuity argument to justify that direct substitution can be applied. By the list above, $\sqrt{x}$, $\sin x$, 1, x and $\cos x$ are all continuous functions at π. Then $\sqrt{x} + \sin x$ and $1 + x + \cos x$ are both continuous at π. Finally,

$$\frac{\sqrt{x} + \sin x}{1 + x + \cos x}$$

is a continuous function at π since $1 + \pi + \cos\pi \neq 0$. Hence, we can directly substitute to get the limit:

$$\lim_{x\to\pi} \frac{\sqrt{x} + \sin x}{1 + x + \cos x} = \frac{\sqrt{\pi} + \sin\pi}{1 + \pi + \cos\pi} = \frac{\sqrt{\pi}}{\pi} = \frac{1}{\sqrt{\pi}}.$$

♣

Continuity is also preserved under the composition of functions.

Theorem 3.13: Continuity of Function Composition

If g is continuous at a and f is continuous at $g(a)$, then the composition function $f \circ g$ is continuous at a.

Example 3.33: Continuity with Composition of Functions

Determine where the following functions is continuous:

(a) $h(x) = \cos(x^2)$

(b) $H(x) = \ln(1 + \sin(x))$

Solution.

(a) The functions that make up the composition $h(x) = f(g(x))$ are $g(x) = x^2$ and $f(x) = \cos(x)$. The function g is continuous on $\mathbb{R}$ since it is a polynomial, and f is also continuous everywhere. Therefore, $h(x) = (f \circ g)(x)$ is continuous on $\mathbb{R}$ by Theorem 3.13.

(b) We know from Example 3.31 that $f(x) = \ln x$ is continuous and $g(x) = 1 + \sin x$ are continuous. Thus by Theorem 3.13, $H(x) = f(g(x))$ is continuous wherever it is defined. Now $\ln(1 + \sin x)$ is defined when $1+\sin x > 0$. Recall that $-1 \leq \sin x \leq 1$, so $1+\sin x > 0$ except when $\sin x = -1$, which happens when $x = \pm 3\pi/2, \pm 7\pi/2, \ldots$. Therefore, H has discontinuities when $x = 3\pi n/2$, $n = 1, 2, 3, \ldots$ and is continuous on the intervals between these values.

♣

Intermediate Value Theorem

Whether or not an equation *has* a solution is an important question in mathematics. Consider the following two questions:

Example 3.34: Motivation for the Intermediate Value Theorem

1. *Does $e^x + x^2 = 0$ have a solution?*
2. *Does $e^x + x = 0$ have a solution?*

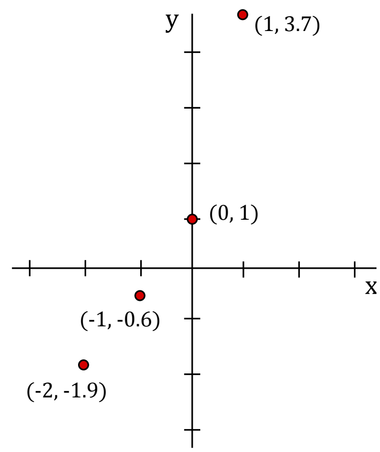
Solution.

1. The first question is easy to answer since for any exponential function we know that $a^x > 0$, and we also know that whenever you square a number you get a nonnegative answer: $x^2 \geq 0$. Hence, $e^x + x^2 > 0$, and thus, is never equal to zero. Therefore, the first equation has no solution.

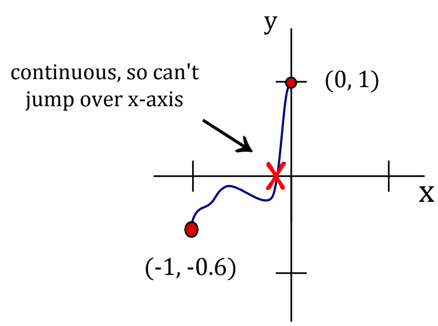
2. For the second question, it is difficult to see if $e^x + x = 0$ has a solution. If we tried to solve for x, we would run into problems. Let's make a table of values to see what kind of values we get (recall that $e \approx 2.7183$):

x	$e^x + x$
-2	$e^{-2} - 2 \approx -1.9$
-1	$e^{-1} - 1 \approx -0.6$
0	$e^0 + 0 = 1$
1	$e + 1 \approx 3.7$

Sketching this gives:



Let $f(x) = e^x + x$. Notice that if we choose $a = -1$ and $b = 0$ then we have $f(a) < 0$ and $f(b) > 0$. A point where the function $f(x)$ crosses the x-axis gives a solution to $e^x + x = 0$. Since $f(x) = e^x + x$ is continuous (both e^x and x are continuous), then the function *must* cross the x-axis somewhere between -1 and 0:



Therefore, our equation has a solution.

Note that by looking at smaller and smaller intervals (a, b) with $f(a) < 0$ and $f(b) > 0$, we can get a better and better approximation for a solution to $e^x + x = 0$. For example, taking the interval $(-0.4, -0.6)$ gives $f(-0.4) < 0$ and $f(-0.6) > 0$, thus, there is a solution to $f(x) = 0$ between -0.4 and -0.6. It turns out that the solution to $e^x + x = 0$ is $x \approx -0.56714$.

♣

We now generalize the argument used in the previous example. In that example we had a continuous function that went from negative to positive and hence, had to cross the x-axis at some point. In fact, we don't need to use the x-axis, any line $y = N$ will work so long as the function is continuous and below the line $y = N$ at some point and above the line $y = N$ at another point. This is known as the Intermediate Values Theorem and it is formally stated as follows:

Theorem 3.14: Intermediate Value Theorem

If f is continuous on the interval $[a, b]$ and N is between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there is a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem guarantees that if $f(x)$ is continuous and $f(a) < N < f(b)$, the line $y = N$ intersects the function at some point $x = c$. Such a number c is between a and b and has the property that $f(c) = N$ (see Figure 3.25(a)). We can also think of the theorem as saying if we draw the line $y = N$ between the lines $y = f(a)$ and $y = f(b)$, then the function cannot jump over the line $y = N$. On the other hand, if $f(x)$ is *not* continuous, then the theorem may *not* hold. See Figure 3.25(b) where there is no number c in (a, b) such that $f(c) = N$. Finally, we remark that there may be multiple choices for c (i.e., lots of numbers between a and b with y-coordinate N). See Figure 3.25(c) for such an example.

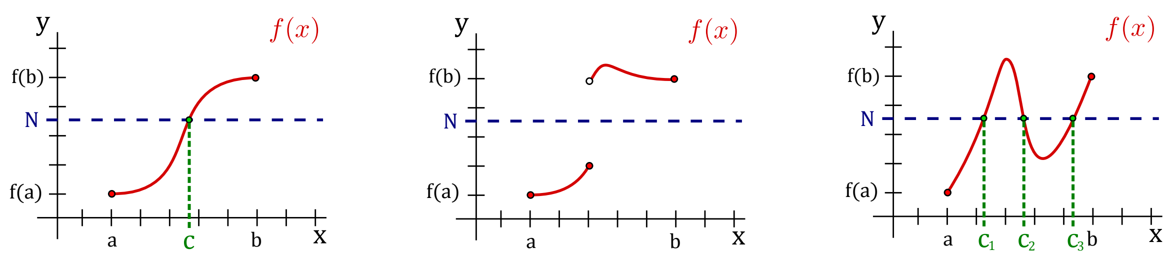


Figure 3.25: (a) A continuous function where IVT holds for a single value c. (b) A discontinuous function where IVT fails to hold. (c) A continuous function where IVT holds for multiple values in (a, b).

The Intermediate Value Theorem is most frequently used for $N = 0$.

Example 3.35: Intermediate Value Theorem

Show that there is a solution of $\sqrt[3]{x} + x = 1$ in the interval $(0, 8)$.

Solution. Let $f(x) = \sqrt[3]{x} + x - 1$, $N = 0$, $a = 0$, and $b = 8$. Since $\sqrt[3]{x}$, x and -1 are continuous on $\mathbb{R}$, and the sum of continuous functions is again continuous, we have that $f(x)$ is continuous on $\mathbb{R}$, thus in particular, $f(x)$ is continuous on $[0, 8]$. We have $f(a) = f(0) = \sqrt[3]{0} + 0 - 1 = -1$ and $f(b) = f(8) = \sqrt[3]{8} + 8 - 1 = 9$. Thus $N = 0$ lies between $f(a) = -1$ and $f(b) = 9$, so the conditions of the Intermediate Value Theorem are satisfied. So, there exists a number c in $(0, 8)$ such that $f(c) = 0$. This means that c satisfies $\sqrt[3]{c} + c - 1 = 0$, in otherwords, is a solution for the equation given.

Alternatively we can let $f(x) = \sqrt[3]{x} + x$, $N = 1$, $a = 0$ and $b = 8$. Then as before $f(x)$ is the sum of two continuous functions, so is also continuous everywhere, in particular, continuous on the interval $[0, 8]$. We have $f(a) = f(0) = \sqrt[3]{0} + 0 = 0$ and $f(b) = f(8) = \sqrt[3]{8} + 8 = 10$. Thus $N = 1$ lies between $f(a) = 0$ and $f(b) = 10$, so the conditions of the Intermediate Value Theorem are satisfied. So, there exists a number c in $(0, 8)$ such that $f(c) = 1$. This means that c satisfies $\sqrt[3]{c} + c = 1$, in otherwords, is a solution for the equation given. ♣

Example 3.36: Roots of Function

Explain why the function $f = x^3 + 3x^2 + x - 2$ has a root between 0 and 1.

Solution. By theorem **??**, f is continuous. Since $f(0) = -2$ and $f(1) = 3$, and 0 is between -2 and 3, there is a $c \in (0, 1)$ such that $f(c) = 0$. ♣

One important application of the Intermediate Value Theorem is root finding. Given a function f, we are often interested in finding values of x where $f(x) = 0$. These roots may be very difficult to find exactly. Good approximations can be found through successive applications of this theorem. Suppose through direct computation we find that $f(a) < 0$ and $f(b) > 0$, where $a < b$. The Intermediate Value Theorem states that there is a c in $[a, b]$ such that $f(c) = 0$. The theorem does not give us any clue as to where that value is in the interval $[a, b]$, just that it exists.

There is a technique that produces a good approximation of c. Let d be the midpoint of the interval $[a, b]$ and consider $f(d)$. There are three possibilities:

1. $f(d) = 0$ – we got lucky and stumbled on the actual value. We stop as we found a root.
2. $f(d) < 0$ Then we know there is a root of f on the interval $[d, b]$ – we have halved the size of our interval, hence are closer to a good approximation of the root.
3. $f(d) > 0$ Then we know there is a root of f on the interval $[a, d]$ – again,we have halved the size of our interval, hence are closer to a good approximation of the root.

Successively applying this technique is called the *Bisection Method* of root finding. We continue until the interval is sufficiently small. We demonstrate this in the following example.

Example 3.37: Using the Bisection Method

Approximate the root of $f(x) = x - \cos x$, accurate to three places after the decimal.

Solution. Consider the graph of $f(x) = x - \cos x$, shown in Figure 3.26. It is clear that the graph crosses the x-axis somewhere near $x = 0.8$. To start the Bisection Method, pick an interval that contains 0.8. We choose $[0.7, 0.9]$. Note that all we care about are signs of $f(x)$, not their actual value, so this is all we display.

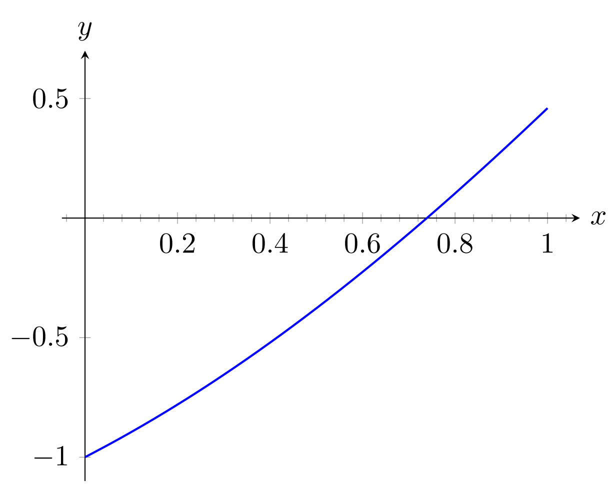


Figure 3.26: Graphing a root of $f(x) = x - \cos x$.

Iteration 1: $f(0.7) < 0$, $f(0.9) > 0$, and $f(0.8) > 0$. So replace 0.9 with 0.8 and repeat.

Iteration 2: $f(0.7) < 0$, $f(0.8) > 0$, and at the midpoint, 0.75, we have $f(0.75) > 0$. So replace 0.8 with 0.75 and repeat. Note that we don't need to continue to check the endpoints, just the midpoint. Thus we put the rest of the iterations in Table 3.27.

Iteration #	Interval	Midpoint Sign
1	$[0.7, 0.9]$	$f(0.8) > 0$
2	$[0.7, 0.8]$	$f(0.75) > 0$
3	$[0.7, 0.75]$	$f(0.725) < 0$
4	$[0.725, 0.75]$	$f(0.7375) < 0$
5	$[0.7375, 0.75]$	$f(0.7438) > 0$
6	$[0.7375, 0.7438]$	$f(0.7407) > 0$
7	$[0.7375, 0.7407]$	$f(0.7391) > 0$
8	$[0.7375, 0.7391]$	$f(0.7383) < 0$
9	$[0.7383, 0.7391]$	$f(0.7387) < 0$
10	$[0.7387, 0.7391]$	$f(0.7389) < 0$
11	$[0.7389, 0.7391]$	$f(0.7390) < 0$
12	$[0.7390, 0.7391]$	

Figure 3.27: Iterations of the Bisection Method of Root Finding

Notice that in the 12^{th} iteration we have the endpoints of the interval each starting with 0.739. Thus we have narrowed the zero down to an accuracy of the first three places after the decimal. Using a computer, we have

$$f(0.7390) = -0.00014, \quad f(0.7391) = 0.000024.$$

Either endpoint of the interval gives a good approximation of where f is 0. The Intermediate Value Theorem states that the actual zero is still within this interval. While we do not know its exact value, we know it starts with 0.739.

This type of exercise is rarely done by hand. Rather, it is simple to program a computer to run such an algorithm and stop when the endpoints differ by a preset small amount. One of the authors did write such a program and found the zero of f, accurate to 10 places after the decimal, to be 0.7390851332. While it took a few minutes to write the program, it took less than a thousandth of a second for the program to run the necessary 35 iterations. In less than 8 hundredths of a second, the zero was calculated to 100 decimal places (with less than 200 iterations).

♣

It is a simple matter to extend the Bisection Method to solve problems similar to "Find x, where $f(x) = 0$." For instance, we can find x, where $f(x) = 1$. It actually works very well to define a new function g where $g(x) = f(x) - 1$. Then use the Bisection Method to solve $g(x) = 0$.

Similarly, given two functions f and g, we can use the Bisection Method to solve $f(x) = g(x)$. Once again, create a new function h where $h(x) = f(x) - g(x)$ and solve $h(x) = 0$.

In Section 5.4.4 another equation solving method will be introduced, called Newton's Method. In many cases, Newton's Method is much faster. It relies on more advanced mathematics, though, so we will wait before introducing it.

This section formally defined what it means to be a continuous function. "Most" functions that we deal with are continuous, so often it feels odd to have to formally define this concept. Regardless, it is important, and forms the basis of the next chapter.

In the next section we examine one more aspect of limits: limits that involve infinity.

Exercises for 3.6

3.6.1 *Concepts.*

(a) *Given functions f and g on an interval I, how can the Bisection Method be used to find a value c where $f(c) = g(c)$?*

(b) *What is a "root" of a function?*

(c) *In your own words, describe what the Intermediate Value Theorem states.*

(d) *In your own words, describe what it means for a function to be continuous.*

(e) *T/F: The sum of continuous functions is also continuous.*

(f) *T/F: If f is continuous on $[0, 1)$ and $[1, 2)$, then f is continuous on $[0, 2)$.*

(g) *T/F: If f is continuous on $[a, b]$, then $\lim\limits_{x \to a^-} f(x) = f(a)$.*

(h) *T/F: If f is continuous at c, then $\lim\limits_{x \to c^+} f(x) = f(c)$.*

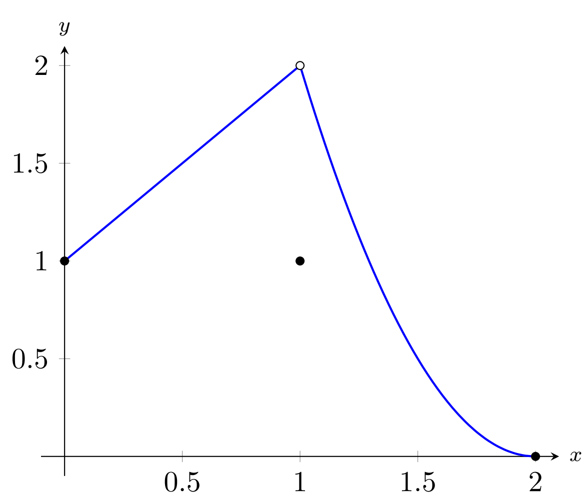
(i) *T/F: If f is continuous at c, then $\lim\limits_{x \to c} f(x)$ exists.*

(j) *T/F: If f is defined on an open interval containing c, and $\lim\limits_{x \to c} f(x)$ exists, then f is continuous at c.*

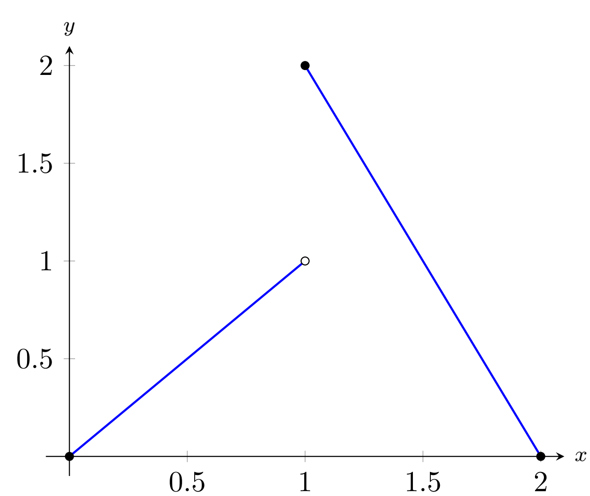
3.6.2

In the following exercises a graph of a function f is given along with a value a. Determine if f is continuous at a; if it is not, state why it is not.

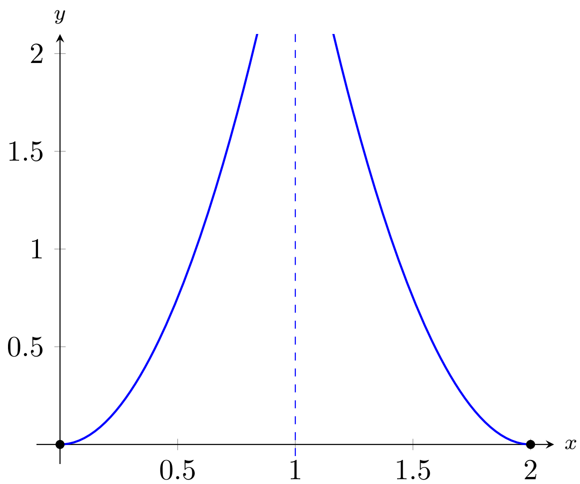
(a) $a = 1$



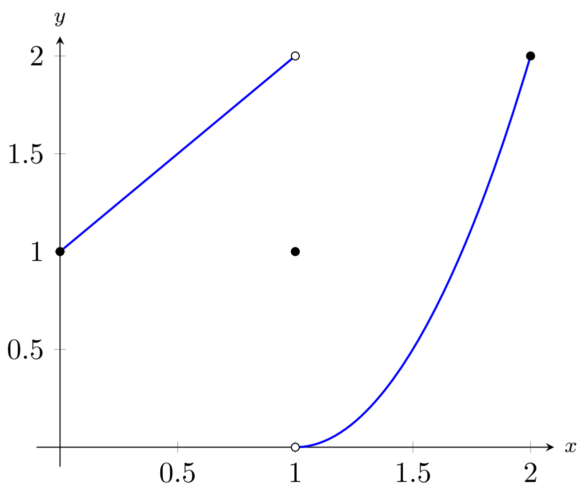
(b) $a = 1$



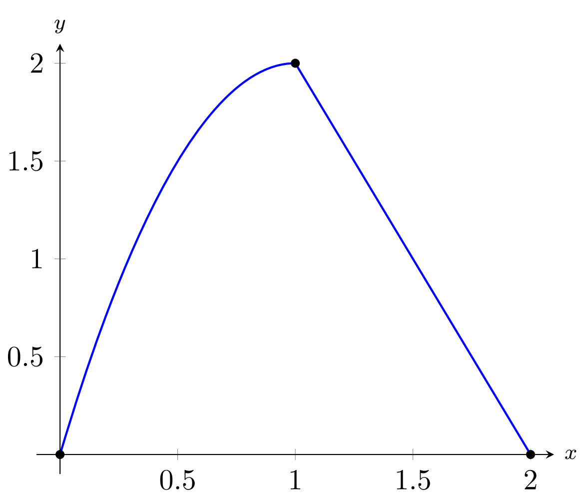
(c) $a = 1$



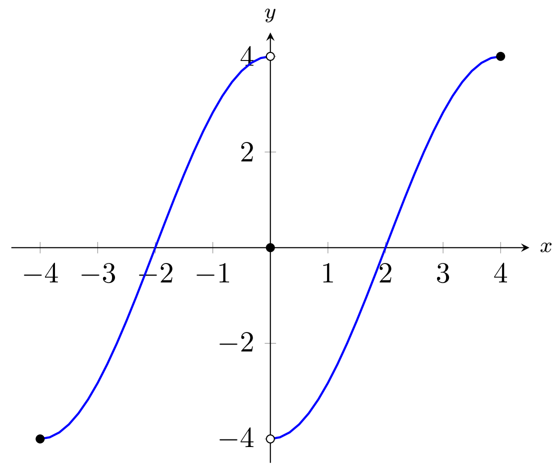
(d) $a = 0$



(e) $a = 1$



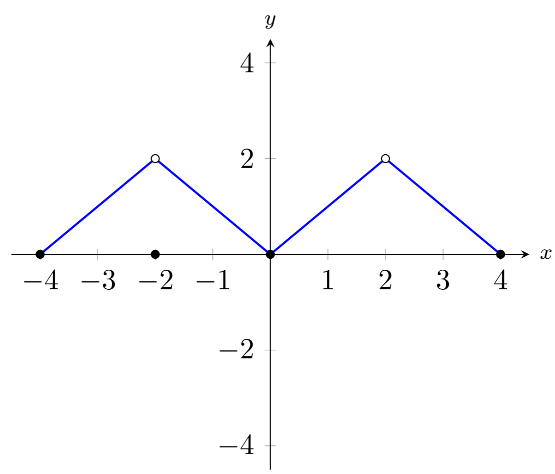
(f) $a = 4$



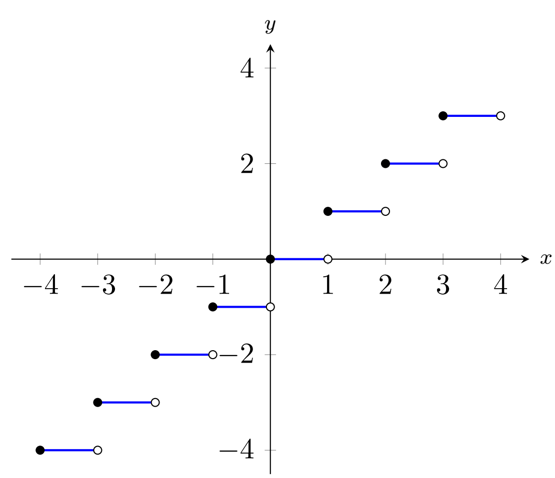
(g) *(a)* $a = -2$

(b) $a = 0$

(c) $a = 2$



(h) $a = 3$



3.6.3 *Determine if f is continuous at the indicated values. If not, explain why.*

(a) $f(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin x}{x} & x > 0 \end{cases}$

(a) $x = 0$

(b) $x = \pi$

(b) $f(x) = \begin{cases} x^3 - x & x < 1 \\ x - 2 & x \geq 1 \end{cases}$

(a) $x = 0$

(b) $x = 1$

(c) $f(x) = \begin{cases} \frac{x^2+5x+4}{x^2+3x+2} & x \neq -1 \\ 3 & x = -1 \end{cases}$

(a) $x = -1$

(b) $x = 10$

(d) $f(x) = \begin{cases} \frac{x^2-64}{x^2-11x+24} & x \neq 8 \\ 5 & x = 8 \end{cases}$

(a) $x = 0$

(b) $x = 8$

3.6.4 *Give the intervals on which the given function is continuous.*

(a) $f(x) = x^2 - 3x + 9$

(b) $g(x) = \sqrt{x^2 - 4}$

(c) $h(k) = \sqrt{1-k} + \sqrt{k+1}$

(d) $f(t) = \sqrt{5t^2 - 30}$

(e) $g(t) = \dfrac{1}{\sqrt{1-t^2}}$

(f) $g(x) = \dfrac{1}{1+x^2}$

(g) $f(x) = e^x$

(h) $g(s) = \ln s$

(i) $h(t) = \cos t$

(j) $f(k) = \sqrt{1 - e^k}$

(k) $f(x) = \sin(e^x + x^2)$

3.6.5 *Consider the function*

$$h(x) = \begin{cases} 2x-3, & \text{if } x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}$$

Show that it is continuous at the point $x = 0$. Is h a continuous function?

3.6.6 *Find the values of a that make the function $f(x)$ continuous for all real numbers.*

$$f(x) = \begin{cases} 4x+5, & \text{if } x \geq -2, \\ x^2+a, & \text{if } x < -2. \end{cases}$$

3.6.7 *Find the values of the constant c so that the function $g(x)$ is continuous on $(-\infty, \infty)$, where*

$$g(x) = \begin{cases} 2-2c^2x, & \text{if } x < -1, \\ 6-7cx^2, & \text{if } x \geq -1. \end{cases}$$

3.6.8

Let f be continuous on $[1,5]$ where $f(1) = -2$ and $f(5) = -10$. Does a value $1 < c < 5$ exist such that $f(c) = -9$? Why/why not?

3.6.9 *Let g be continuous on $[-3,7]$ where $g(0) = 0$ and $g(2) = 25$. Does a value $-3 < c < 7$ exist such that $g(c) = 15$? Why/why not?*

3.6.10 *Let f be continuous on $[-1,1]$ where $f(-1) = -10$ and $f(1) = 10$. Does a value $-1 < c < 1$ exist such that $f(c) = 11$? Why/why not?*

3.6.11 *Let h be a function on $[-1,1]$ where $h(-1) = -10$ and $h(1) = 10$. Does a value $-1 < c < 1$ exist such that $h(c) = 0$? Why/why not?*

3.6.12 *Use the Bisection Method to approximate, accurate to two decimal places, the value of the root of the given function in the given interval.*

(a) $f(x) = x^2 + 2x - 4$ *on* $[1, 1.5]$.

(b) $f(x) = \sin x - 1/2$ *on* $[0.5, 0.55]$

(c) $f(x) = e^x - 2$ *on* $[0.65, 0.7]$.

(d) $f(x) = \cos x - \sin x$ *on* $[0.7, 0.8]$.

3.6.13 *Let* $f(x) = \begin{cases} x^2-5 & x < 5 \\ 5x & x \geq 5 \end{cases}$.

(a) $\lim_{x \to 5^-} f(x)$

(b) $\lim_{x \to 5^+} f(x)$

(a) $\lim_{x \to 5} f(x)$

(b) $f(5)$

3.6.14

Numerically approximate the following limits:

(a) $\displaystyle\lim_{x\to -4/5^+} \frac{x^2-8.2x-7.2}{x^2+5.8x+4}$

(b) $\displaystyle\lim_{x\to -4/5^-} \frac{x^2-8.2x-7.2}{x^2+5.8x+4}$

3.7 Infinite Limits and Limits at Infinity

We occasionally want to know what happens to some quantity when a variable gets very large or "goes to infinity".

Example 3.38: Limit at Infinity

What happens to the function $\cos(1/x)$ *as* x *goes to infinity? It seems clear that as* x *gets larger and larger,* $1/x$ *gets closer and closer to zero, so* $\cos(1/x)$ *should be getting closer and closer to* $\cos(0) = 1$.

As with ordinary limits, this concept of "limit at infinity" can be made precise. Roughly, we want $\lim_{x\to\infty} f(x) = L$ to mean that we can make $f(x)$ as close as we want to L by making x large enough.

Definition 3.9: Limit at Infinity (Formal Definition)

If f *is a function, we say that* $\lim_{x\to\infty} f(x) = L$ *if for every* $\epsilon > 0$ *there is an* $N > 0$ *so that whenever* $x > N$, $|f(x) - L| < \epsilon$. *We may similarly define* $\lim_{x\to-\infty} f(x) = L$.

We include this definition for completeness, but we will not explore it in detail. Suffice it to say that such limits behave in much the same way that ordinary limits do; in particular there is a direct analog of Theorem **??**.

Example 3.39: Limit at Infinity

Compute $\displaystyle\lim_{x\to\infty} \frac{2x^2-3x+7}{x^2+47x+1}$.

Solution. As x goes to infinity both the numerator and denominator go to infinity. We divide the numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{2x^2-3x+7}{x^2+47x+1} = \lim_{x\to\infty} \frac{2-\dfrac{3}{x}+\dfrac{7}{x^2}}{1+\dfrac{47}{x}+\dfrac{1}{x^2}}.$$

Now as x approaches infinity, all the quotients with some power of x in the denominator approach zero, leaving 2 in the numerator and 1 in the denominator, so the limit again is 2. ♣

In the previous example, we *divided by the highest power of x that occurs in the denominator* in order to evaluate the limit. We illustrate another technique similar to this.

Example 3.40: Limit at Infinity

Compute the following limit:

$$\lim_{x\to\infty} \frac{2x^2+3}{5x^2+x}.$$

Solution. As x becomes large, both the numerator and denominator become large, so it isn't clear what happens to their ratio. The highest power of x in the denominator is x^2, therefore we will divide every term in both the numerator and denominator by x^2 as follows:

$$\lim_{x\to\infty} \frac{2x^2+3}{5x^2+x} = \lim_{x\to\infty} \frac{2+3/x^2}{5+1/x}.$$

Most of the limit rules from last lecture also apply to infinite limits, so we can write this as:

$$= \frac{\lim\limits_{x\to\infty} 2 + 3 \lim\limits_{x\to\infty} \dfrac{1}{x^2}}{\lim\limits_{x\to\infty} 5 + \lim\limits_{x\to\infty} \dfrac{1}{x}} = \frac{2+3(0)}{5+0} = \frac{2}{5}.$$

Note that we used the theorem above to get that $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$ and $\lim\limits_{x\to\infty} \dfrac{1}{x^2} = 0$.

A shortcut technique is to analyze only the *leading terms* of the numerator and denominator. A leading term is a term that has the highest power of x. If there are multiple terms with the same exponent, you must include all of them.

Top: The leading term is $2x^2$.

Bottom: The leading term is $5x^2$.

Now only looking at leading terms and ignoring the other terms we get:

$$\lim_{x\to\infty} \frac{2x^2+3}{5x^2+x} = \lim_{x\to\infty} \frac{2x^2}{5x^2} = \frac{2}{5}.$$

♣

We next look at limits whose value is infinity (or minus infinity).

Definition 3.10: Infinite Limit (Useable Definition)

In general, we will write

$$\lim_{x\to a} f(x) = \infty$$

if we can make the value of $f(x)$ arbitrarily large by taking x to be sufficiently close to a (on either side of a) but not equal to a. Similarly, we will write

$$\lim_{x\to a} f(x) = -\infty$$

if we can make the value of $f(x)$ arbitrarily large and negative by taking x to be sufficiently close to a (on either side of a) but not equal to a.

This definition can be modified for one-sided limits as well as limits with $x \to a$ replaced by $x \to \infty$ or $x \to -\infty$.

Example 3.41: Limit at Infinity

Compute the following limit: $\lim_{x\to\infty}(x^3 - x)$.

Solution. One might be tempted to write:

$$\lim_{x\to\infty} x^3 - \lim_{x\to\infty} x = \infty - \infty,$$

however, we do not know what $\infty - \infty$ is, as ∞ is not a real number and so cannot be treated like one. We instead write:

$$\lim_{x\to\infty}(x^3 - x) = \lim_{x\to\infty} x(x^2 - 1).$$

As x becomes arbitrarily large, then both x and $x^2 - 1$ become arbitrarily large, and hence their product $x(x^2 - 1)$ will also become arbitrarily large. Thus we see that

$$\lim_{x\to\infty}(x^3 - x) = \infty.$$

♣

Example 3.42: Limit at Infinity and Basic Functions

We can easily evaluate the following limits by observation:

1. $\lim_{x\to\infty} \frac{6}{\sqrt{x^3}} = 0$
2. $\lim_{x\to-\infty} x - x^2 = -\infty$
3. $\lim_{x\to\infty} x^3 + x = \infty$
4. $\lim_{x\to\infty} \cos(x) = DNE$
5. $\lim_{x\to\infty} e^x = \infty$
6. $\lim_{x\to-\infty} e^x = 0$
7. $\lim_{x\to 0^+} \ln x = -\infty$
8. $\lim_{x\to 0} \cos(1/x) = DNE$

Often, the shorthand notation $\frac{1}{0^+} = +\infty$ and $\frac{1}{0^-} = -\infty$ is used to represent the following two limits respectively:

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Using the above convention we can compute the following limits.

Example 3.43: Limit at Infinity and Basic Functions

Compute $\lim_{x\to 0^+} e^{1/x}$, $\lim_{x\to 0^-} e^{1/x}$ *and* $\lim_{x\to 0} e^{1/x}$.

Solution. We have:

$$\lim_{x\to 0^+} e^{\frac{1}{x}} = e^{\frac{1}{0^+}} = e^{+\infty} = \infty.$$

$$\lim_{x\to 0^-} e^{\frac{1}{x}} = e^{\frac{1}{0^-}} = e^{-\infty} = 0.$$

Thus, as left-hand limit $\neq$ right-hand limit,

$$\lim_{x\to 0} e^{\frac{1}{x}} = \text{DNE}.$$

♣

3.7.1. Vertical Asymptotes

The line $x = a$ is called a **vertical asymptote** of $f(x)$ if *at least one* of the following is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Example 3.44: Vertical Asymptotes

Find the vertical asymptotes of $f(x) = \dfrac{2x}{x-4}$.

Solution. In the definition of vertical asymptotes we need a certain limit to be $\pm\infty$. Candidates would be to consider values not in the domain of $f(x)$, such as $a = 4$. As x approaches 4 but is larger than 4 then $x-4$ is a small positive number and $2x$ is close to 8, so the quotient $2x/(x-4)$ is a large positive number. Thus we see that

$$\lim_{x\to 4^+} \frac{2x}{x-4} = \infty.$$

Thus, at least one of the conditions in the definition above is satisfied. Therefore $x = 4$ is a vertical asymptote. ♣

3.7.2. Horizontal Asymptotes

The line $y = L$ is a **horizontal asymptote** of $f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to -\infty} f(x) = L.$$

Example 3.45: Horizontal Asymptotes

Find the horizontal asymptotes of $f(x) = \dfrac{|x|}{x}$.

Solution. We must compute two infinite limits. First,

$$\lim_{x\to\infty} \frac{|x|}{x}.$$

Notice that for x arbitrarily large that $x > 0$, so that $|x| = x$. In particular, for x in the interval $(0,\infty)$ we have

$$\lim_{x\to\infty} \frac{|x|}{x} = \lim_{x\to\infty} \frac{x}{x} = 1.$$

Second, we must compute

$$\lim_{x\to-\infty} \frac{|x|}{x}.$$

Notice that for x arbitrarily large negative that $x < 0$, so that $|x| = -x$. In particular, for x in the interval $(-\infty, 0)$ we have

$$\lim_{x\to-\infty} \frac{|x|}{x} = \lim_{x\to-\infty} \frac{-x}{x} = -1.$$

Therefore there are two horizontal asymptotes, namely, $y = 1$ and $y = -1$. ♣

3.7.3. Slant Asymptotes

Some functions may have slant (or *oblique*) asymptotes, which are neither vertical nor horizontal. If

$$\lim_{x\to\infty} [f(x) - (mx+b)] = 0$$

then the straight line $y = mx + b$ is a **slant asymptote** to $f(x)$. Visually, the vertical distance between $f(x)$ and $y = mx + b$ is decreasing towards 0 and the curves do not intersect or cross at any point as x approaches infinity. Similarly when $x \to -\infty$.

Example 3.46: Slant Asymptote in a Rational Function

Find the slant asymptotes of $f(x) = \dfrac{-3x^2+4}{x-1}$.

Solution. Note that this function has no horizontal asymptotes since $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$.

In rational functions, slant asymptotes occur when the degree in the numerator is one greater than in the denominator. We use long division to rearrange the function:

$$\frac{-3x^2+4}{x-1} = -3x - 3 + \frac{1}{x-1}.$$

The part we're interested in is the resulting polynomial $-3x - 3$. This is the line $y = mx + b$ we were seeking, where $m = -3$ and $b = -3$. Notice that

$$\lim_{x\to\infty} \frac{-3x^2+4}{x-1} - (-3x-3) = \lim_{x\to\infty} \frac{1}{x-1} = 0$$

and

$$\lim_{x\to-\infty} \frac{-3x^2+4}{x-1} - (-3x-3) = \lim_{x\to-\infty} \frac{1}{x-1} = 0.$$

Thus, $y = -3x - 3$ is a slant asymptote of $f(x)$. ♣

Although rational functions are the most common type of function we encounter with slant asymptotes, there are other types of functions we can consider that present an interesting challenge.

Example 3.47: Slant Asymptote

Show that $y = 2x + 4$ is a slant asymptote of $f(x) = 2x - 3^x + 4$.

Solution. This is because

$$\lim_{x\to-\infty}[f(x) - (2x+4)] = \lim_{x\to-\infty}(-3^x) = 0.$$

♣

We note that $\lim_{x\to\infty}[f(x) - (2x+4)] = \lim_{x\to\infty}(-3^x) = -\infty$. So, the vertical distance between $y = f(x)$ and the line $y = 2x + 4$ decreases toward 0 only when $x \to -\infty$ and not when $x \to \infty$. The graph of f approaches the slant asymptote $y = 2x + 4$ only at the far left and not at the far right. One might ask if $y = f(x)$ approaches a slant asymptote when $x \to \infty$. The answer turns out to be no, but we will need to know something about the relative growth rates of the exponential functions and linear functions in order to prove this. Specifically, one can prove that when the base is greater than 1 the exponential functions grows faster than any power function as $x \to \infty$. This can be phrased like this: For any $a > 1$ and any $n > 0$,

$$\lim_{x\to\infty} \frac{a^x}{x^n} = \infty \text{ and } \lim_{x\to\infty} \frac{x^n}{a^x} = 0.$$

These facts are most easily proved with the aim of something called the L'Hôpital's Rule.

3.7.4. End Behaviour and Comparative Growth Rates

Let us now look at the last two subsections and go deeper. In the last two subsections we looked at horizontal and slant asymptotes. Both are special cases of the end behaviour of functions, and both concern situations where the graph of a function approaches a straight line as $x \to \infty$ or $-\infty$. But not all functions have this kind of end behviour. For example, $f(x) = x^2$ and $f(x) = x^3$ do not approach a straight line as $x \to \infty$ or $-\infty$. The best we can say with the notion of limit developed at this stage are that

$$\begin{aligned} \lim_{x\to\infty} x^2 &= \infty, \ \lim_{x\to-\infty} x^2 = \infty, \\ \lim_{x\to\infty} x^3 &= \infty, \ \lim_{x\to-\infty} x^3 = -\infty. \end{aligned}$$

Similarly, we can describe the end behaviour of transcendental functions such as $f(x) = e^x$ using limits, and in this case, the graph approaches a line as $x \to -\infty$ but not as $x \to \infty$.

$$\lim_{x\to-\infty} e^x = 0, \ \lim_{x\to\infty} e^x = \infty.$$

People have found it useful to make a finer distinction between these end behaviours all thus far captured by the symbols ∞ and $-\infty$. Specifically, we will see that the above functions have different growth rates at infinity. Some increases to infinty faster than others. Specifically,

Definition 3.11: Comparative Growth Rates

Suppose that f and g are two functions such that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. We say that $f(x)$ grows faster than $g(x)$ as $x \to \infty$ if the following holds:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty,$$

or equivalently,

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

Here are a few obvious examples:

Example 3.48:

Show that if $m > n$ are two positive integers, then $f(x) = x^m$ grows faster than $g(x) = x^n$ as $x \to \infty$.

Solution. Since $m > n$, $m - n$ is a positive integer. Therefore,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^m}{x^n} = \lim_{x\to\infty} x^{m-n} = \infty.$$

♣

Example 3.49:

Show that if $m > n$ are two positive integers, then any monic polynomial $P_m(x)$ of degree m grows faster than any monic polynomial $P_n(x)$ of degree n as $x \to \infty$. [Recall that a polynomial is monic if its leading coefficient is 1.]

Solution. By assumption, $P_m(x) = x^m +$ terms of degrees less than $m = x^m + a_{m-1}x^{m-1} + \ldots$, and $P_n(x) = x^n +$ terms of degrees less than $n = x^n + b_{n-1}x^{n-1} + \ldots$. Dividing the numerator and denominator by x^n, we get

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{m-n} + a_{m-1}x^{m-n-1} + \ldots}{1 + \frac{b_{n-1}}{x} + \ldots} = \lim_{x\to\infty} x^{m-n}\left(\frac{1 + \frac{a_{m-1}}{x} + \ldots}{1 + \frac{b_{n-1}}{x} + \ldots}\right) = \infty,$$

since the limit of the bracketed fraction is 1 and the limit of x^{m-n} is ∞, as we showed in Example 3.48.
♣

Example 3.50:

Show that a polynomial grows exactly as fast as its highest degree term as $x \to \infty$ or $-\infty$. That is, if $P(x)$ is any polynomial and $Q(x)$ is its highest degree term, then both limits

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} \text{ and } \lim_{x\to-\infty} \frac{P(x)}{Q(x)}$$

are finite and nonzero.

Solution. Suppose that $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $a_n \neq 0$. Then the highest degree term is $Q(x) = a_n x^n$. So,

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) = a_n \neq 0.$$

♣

Let's state a theorem we mentioned when we discussed the last example in the last subsection:

Theorem 3.15:

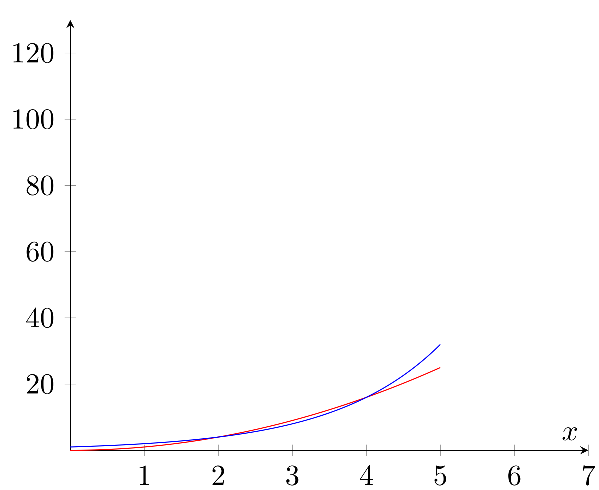
Let n be any positive integer and let $a > 1$. Then $f(x) = a^x$ grows faster than $g(x) = x^n$ as $x \to \infty$:

$$\lim_{x\to\infty} \frac{a^x}{x^n} = \infty, \quad \lim_{x\to\infty} \frac{x^n}{a^x} = 0.$$

In particular,

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty, \quad \lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

The easiest way to prove this is to use the L'Hôpital's Rule, which we will introduce in a later chapter. For now, one can plot and compare the graphs of an exponential function and a power function. Here is a comparison between $f(x) = x^2$ and $g(x) = 2^x$:



Notice also that as $x \to -\infty$, x^n grows in size but e^x does not. More specifically, $x^n \to \infty$ or $-\infty$ according as n is even or odd, while $e^x \to 0$. So, it is meaningless to compare their "growth" rates, although we can still calculate the limit

$$\lim_{x\to-\infty}\frac{e^x}{x^n}=0.$$

Let's see an application of our theorem.

Example 3.51:

Find the horizontal asymptote(s) of $f(x)=\dfrac{x^3+2e^x}{e^x-4x^2}$.

Solution. To find horizontal asymptotes, we calculate the limits of $f(x)$ as $x\to\infty$ and $x\to-\infty$. For $x\to\infty$, we divide the numerator and the denominator by e^x, and then we take limit to get

$$\lim_{x\to\infty}\frac{x^3+2e^x}{e^x-4x^2}=\lim_{x\to\infty}\frac{\frac{x^3}{e^x}+2}{1-4\frac{x^2}{e^x}}=\frac{0+2}{1-4(0)}=2.$$

For $x\to-\infty$, we divide the numerator and the denominator by x^2 to get

$$\lim_{x\to-\infty}\frac{x^3+2e^x}{e^x-4x^2}=\lim_{x\to-\infty}\frac{x+2\frac{e^x}{x^2}}{\frac{e^x}{x^2}-4}.$$

The denominator now approaches $0-4=-4$. The numerator has limit $-\infty$. So, the quotient has limit ∞:

$$\lim_{x\to-\infty}\frac{x+2\frac{e^x}{x^2}}{\frac{e^x}{x^2}-4}=\infty.$$

So, $y=2$ is a horizontal asymptote. The function $y=f(x)$ approaches the line $y=2$ as $x\to\infty$. And this is the only horizontal asymptote, since the function $y=f(x)$ does not approach any horizontal line as $x\to-\infty$. ♣

Since the growth rate of a polynomial is the same as that of its leading term, the following is obvious:

Example 3.52:

If $P(x)$ is any polynomial, then

$$\lim_{x\to\infty}\frac{P(x)}{e^x}=0.$$

Also, if r is any real number, then we can place it between two consecutive integers n and $n+1$. For example, $\sqrt{3}$ is between 1 and 2, e is between 2 and 3, and π is between 3 and 4. Then the following is totally within our expectation:

Example 3.53:

Prove that if $a > 1$ is any basis and $r > 0$ is any exponent, then $f(x) = a^x$ grows faster than $g(x) = x^r$ as $x \to \infty$.

Solution. Let r be between consecutive integers n and $n+1$. Then for all $x > 1$, $x^n \leq x^r \leq x^{n+1}$. Dividing by a^x, we get

$$\frac{x^n}{a^x} \leq \frac{x^r}{a^x} \leq \frac{x^{n+1}}{a^x}.$$

Since

$$\lim_{x\to\infty} \frac{x^n}{a^x} = 0.$$

♣

What about exponential functions with different bases? We recall from the graphs of the exponential functions that for any base $a > 1$,

$$\lim_{x\to\infty} a^x = \infty.$$

So, the exponential functions with bases greater than 1 all grow to infinity as $x \to \infty$. How do their growth rates compare?

Theorem 3.16:

If $1 < a < b$, then $f(x) = b^x$ grows faster than $g(x) = a^x$ as $x \to \infty$.

Proof. Proof. Since $a < b$, we have $\frac{b}{a} > 1$. So,

$$\lim_{x\to\infty} \frac{b^x}{a^x} = \lim_{x\to\infty} \left(\frac{b}{a}\right)^x = \infty.$$

♣

Another function that grows to infinity as $x \to \infty$ is $g(x) = \ln x$. Recall that the natural logarithmic function is the inverse of the exponential function $y = e^x$. Since e^x grows very fast as x increases, we should expect $\ln x$ to grow very slowly as x increases. The same applies to logarithmic functions with any basis $a > 1$. This is the content of the next theorem.

Theorem 3.17:

Let r be any positive real number and $a > 1$. Then

(a) $f(x) = x^r$ grows faster than $g(x) = \ln x$ as $x \to \infty$.

(b) $f(x) = x^r$ grows faster than $g(x) = \log_a x$ as $x \to \infty$.

Proof.

1. We use a change of variable. Letting $t = \ln x$, then $x = e^t$. So, $x \to \infty$ if and only if $t \to \infty$, and

$$\lim_{x\to\infty} \frac{\ln x}{x^r} = \lim_{t\to\infty} \frac{t}{(e^t)^r} = \lim_{t\to\infty} \frac{t}{(e^r)^t}.$$

Now, since $r > 0$, $a = e^r > 1$. So, a^t grows as t increases, and it grows faster than t as $t \to \infty$. Therefore,

$$\lim_{x\to\infty} \frac{\ln x}{x^r} = \lim_{t\to\infty} \frac{t}{(e^r)^t} = \lim_{t\to\infty} \frac{t}{a^t} = 0.$$

2. The change of base identity $\log_a x = \dfrac{\ln x}{\ln a}$ implies that $\log_a x$ is simply a constant multple of $\ln x$. The result now follows from (a).

♣

Exercises for 3.7

3.7.1 *Compute the following limits.*

(a) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+x} - \sqrt{x^2-x}$

(b) $\displaystyle\lim_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}}$

(c) $\displaystyle\lim_{t\to1^+} \frac{(1/t)-1}{t^2-2t+1}$

(d) $\displaystyle\lim_{t\to\infty} \frac{t+5-2/t-1/t^3}{3t+12-1/t^2}$

(e) $\displaystyle\lim_{y\to\infty} \frac{\sqrt{y+1}+\sqrt{y-1}}{y}$

(f) $\displaystyle\lim_{x\to0^+} \frac{3+x^{-1/2}+x^{-1}}{2+4x^{-1/2}}$

(g) $\displaystyle\lim_{x\to\infty} \frac{x+x^{1/2}+x^{1/3}}{x^{2/3}+x^{1/4}}$

(h) $\displaystyle\lim_{t\to\infty} \frac{1-\sqrt{\frac{t}{t+1}}}{2-\sqrt{\frac{4t+1}{t+2}}}$

(i) $\displaystyle\lim_{t\to\infty} \frac{1-\frac{t}{t-1}}{1-\sqrt{\frac{t}{t-1}}}$

(j) $\displaystyle\lim_{x\to-\infty} \frac{x+x^{-1}}{1+\sqrt{1-x}}$

(k) $\displaystyle\lim_{x\to1^+} \frac{\sqrt{x}}{x-1}$

(l) $\displaystyle\lim_{x\to\infty} \frac{x^{-1}+x^{-1/2}}{x+x^{-1/2}}$

(m) $\displaystyle\lim_{x\to\infty} \frac{x+x^{-2}}{2x+x^{-2}}$

(n) $\displaystyle\lim_{x\to\infty} \frac{5+x^{-1}}{1+2x^{-1}}$

(o) $\displaystyle\lim_{x\to\infty} \frac{4x}{\sqrt{2x^2+1}}$

(p) $\displaystyle\lim_{x\to\infty} (x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)$

(q) $\displaystyle\lim_{x\to0^+} (x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)$

(r) $\displaystyle\lim_{x\to2} \frac{x^3-6x-2}{x^3-4x}$

3.7.2 *The function* $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ *has two horizontal asymptotes. Find them and give a rough sketch of* f *with its horizontal asymptotes.*

3.7.3 *Find the vertical asymptotes of* $f(x) = \dfrac{\ln x}{x-2}$.

3.7.4 *Suppose that a falling object reaches velocity* $v(t) = 50(1-e^{-t/5})$ *at time* t*, where distance is measured in m and time s. What is the object's terminal velocity, i.e. the value of* $v(t)$ *as* t *goes to infinity?*

3.7.5 *Find the slant asymptote of* $f(x) = \dfrac{x^2+x+6}{x-3}$.

3.7.6 *Compute the following limits.*

(a) $\lim_{x \to -\infty} (2x^3 - x)$

(b) $\lim_{x \to \infty} \tan^{-1}(e^x)$

(c) $\lim_{x \to -\infty} \tan^{-1}(e^x)$

(d) $\lim_{x \to \infty} \dfrac{e^x + x^4}{x^3 + 5 \ln x}$

(e) $\lim_{x \to \infty} \dfrac{2^x + 5(3^x)}{3(2^x) - 3^x}$

(f) $\lim_{x \to -\infty} \dfrac{2^x + 5(3^x)}{3(2^x) - 3^x}$

(g) $\lim_{x \to 0^+} \sqrt{x} \ln x$ *[Hint: Let* $t = 1/x$*]*

4. Derivatives

4.1 The Rate of Change of a Function

Suppose that y is a function of x, say $y = f(x)$. It is often useful to know how sensitive the value of y is to small changes in x.

Example 4.1: Small Changes in x

Consider $y = f(x) = \sqrt{625 - x^2}$ (the upper semicircle of radius 25 centered at the origin), and let's compute the changes of y resulting from small changes of x around $x = 7$.

Solution. When $x = 7$, we find that $y = \sqrt{625 - 49} = 24$. Suppose we want to know how much y changes when x increases a little, say to 7.1 or 7.01.

In the case of a straight line $y = mx + b$, the slope $m = \Delta y / \Delta x$ measures the change in y per unit change in x. This can be interpreted as a measure of "sensitivity"; for example, if $y = 100x + 5$, a small change in x corresponds to a change one hundred times as large in y, so y is quite sensitive to changes in x.

Let us look at the same ratio $\Delta y / \Delta x$ for our function $y = f(x) = \sqrt{625 - x^2}$ when x changes from 7 to 7.1. Here $\Delta x = 7.1 - 7 = 0.1$ is the change in x, and

$$\begin{aligned}\Delta y = f(x + \Delta x) - f(x) &= f(7.1) - f(7) \\ &= \sqrt{625 - 7.1^2} - \sqrt{625 - 7^2} \\ &\approx 23.9706 - 24 = -0.0294.\end{aligned}$$

Thus, $\Delta y / \Delta x \approx -0.0294/0.1 = -0.294$. This means that y changes by less than one third the change in x, so apparently y is not very sensitive to changes in x at $x = 7$. We say "apparently" here because we don't really know what happens between 7 and 7.1. Perhaps y changes dramatically as x runs through the values from 7 to 7.1, but at 7.1 y just happens to be close to its value at 7. This is not in fact the case for this particular function, but we don't yet know why. ♣

The quantity $\Delta y / \Delta x \approx -0.294$ may be interpreted as the slope of the line through $(7, 24)$ and $(7.1, 23.9706)$, called a **chord** of the circle. In general, if we draw the chord from the point $(7, 24)$ to a nearby point on the semicircle $(7 + \Delta x,\ f(7 + \Delta x))$, the slope of this chord is the so-called **difference quotient**

$$\frac{f(7 + \Delta x) - f(7)}{\Delta x} = \frac{\sqrt{625 - (7 + \Delta x)^2} - 24}{\Delta x}.$$

For example, if x changes only from 7 to 7.01, then the difference quotient (slope of the chord) is approximately equal to $(23.997081 - 24)/0.01 = -0.2919$. This is slightly different than for the chord from $(7, 24)$ to $(7.1, 23.9706)$.

As Δx is made smaller (closer to 0), $7+\Delta x$ gets closer to 7 and the chord joining $(7, f(7))$ to $(7+\Delta x, f(7+\Delta x))$ shifts slightly, as shown in Figure 4.1. The chord gets closer and closer to the **tangent line** to the circle at the point $(7, 24)$. (The tangent line is the line that just grazes the circle at that point, i.e., it doesn't meet the circle at any second point.) Thus, as Δx gets smaller and smaller, the slope $\Delta y/\Delta x$ of the chord gets closer and closer to the slope of the tangent line. This is actually quite difficult to see when Δx is small, because of the scale of the graph. The values of Δx used for the figure are 1, 5, 10 and 15, not really very small values. The tangent line is the one that is uppermost at the right hand endpoint.

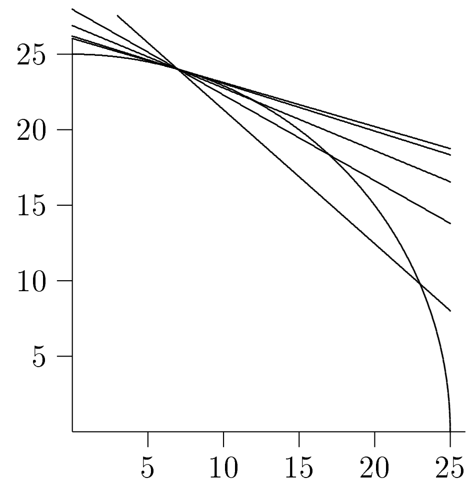


Figure 4.1: Chords approximating the tangent line.

So far we have found the slopes of two chords that should be close to the slope of the tangent line, but what is the slope of the tangent line exactly? Since the tangent line touches the circle at just one point, we will never be able to calculate its slope directly, using two "known" points on the line. What we need is a way to capture what happens to the slopes of the chords as they get "closer and closer" to the tangent line. Instead of looking at more particular values of Δx, let's see what happens if we do some algebra with the difference quotient using just Δx. The slope of a chord from $(7, 24)$ to a nearby point $(7+\Delta x, f(7+\Delta x))$ is given by

$$\begin{aligned}
\frac{f(7+\Delta x) - f(7)}{\Delta x} &= \frac{\sqrt{625-(7+\Delta x)^2} - 24}{\Delta x} \\
&= \left(\frac{\sqrt{625-(7+\Delta x)^2} - 24}{\Delta x}\right)\left(\frac{\sqrt{625-(7+\Delta x)^2} + 24}{\sqrt{625-(7+\Delta x)^2} + 24}\right) \\
&= \frac{625 - (7+\Delta x)^2 - 24^2}{\Delta x(\sqrt{625-(7+\Delta x)^2} + 24)} \\
&= \frac{49 - 49 - 14\Delta x - \Delta x^2}{\Delta x(\sqrt{625-(7+\Delta x)^2} + 24)} \\
&= \frac{\Delta x(-14 - \Delta x)}{\Delta x(\sqrt{625-(7+\Delta x)^2} + 24)} \\
&= \frac{-14 - \Delta x}{\sqrt{625-(7+\Delta x)^2} + 24}
\end{aligned}$$

Now, can we tell by looking at this last formula what happens when Δx gets very close to zero? The numerator clearly gets very close to -14 while the denominator gets very close to $\sqrt{625 - 7^2} + 24 = 48$.

The fraction is therefore very close to $-14/48 = -7/24 \cong -0.29167$. In fact, the slope of the tangent line is exactly $-7/24$.

What about the slope of the tangent line at $x = 12$? Well, 12 can't be all that different from 7; we just have to redo the calculation with 12 instead of 7. This won't be hard, but it will be a bit tedious. What if we try to do all the algebra without using a specific value for x? Let's copy from above, replacing 7 by x.

$$\begin{aligned}
\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x} \\
&= \frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x}\frac{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}} \\
&= \frac{625-(x+\Delta x)^2-625+x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{625-x^2-2x\Delta x-\Delta x^2-625+x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{\Delta x(-2x-\Delta x)}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{-2x-\Delta x}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}
\end{aligned}$$

Now what happens when Δx is very close to zero? Again it seems apparent that the quotient will be very close to

$$\frac{-2x}{\sqrt{625-x^2}+\sqrt{625-x^2}} = \frac{-2x}{2\sqrt{625-x^2}} = \frac{-x}{\sqrt{625-x^2}}.$$

Replacing x by 7 gives $-7/24$, as before, and now we can easily do the computation for 12 or any other value of x between -25 and 25.

So now we have a single expression, $-x/\sqrt{625-x^2}$, that tells us the slope of the tangent line for any value of x. This slope, in turn, tells us how sensitive the value of y is to small changes in the value of x.

The expression $-x/\sqrt{625-x^2}$ defines a new function called the **derivative** of the original function (since it is derived from the original function). If the original is referred to as f or y then the derivative is often written f' or y' (pronounced "f prime" or "y prime"). So in this case we might write $f'(x) = -x/\sqrt{625-x^2}$ or $y' = -x/\sqrt{625-x^2}$. At a particular point, say $x = 7$, we write $f'(7) = -7/24$ and we say that "f prime of 7 is $-7/24$" or "the derivative of f at 7 is $-7/24$."

To summarize, we compute the derivative of $f(x)$ by forming the difference quotient

$$\frac{f(x+\Delta x)-f(x)}{\Delta x},$$

which is the slope of a line, then we figure out what happens when Δx gets very close to 0.

At this point, we should note that the idea of letting Δx get closer and closer to 0 is precisely the idea of a limit that we discussed in the last chapter. The limit here is a limit as Δx approaches 0. Using limit notation, we can write $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

In the particular case of a circle, there's a simple way to find the derivative. Since the tangent to a circle at a point is perpendicular to the radius drawn to the point of contact, its slope is the negative reciprocal of the slope of the radius. The radius joining $(0, 0)$ to $(7, 24)$ has slope $24/7$. Hence, the tangent line has slope $-7/24$. In general, a radius to the point $(x, \sqrt{625-x^2})$ has slope $\sqrt{625-x^2}/x$, so the slope of the tangent line is $-x/\sqrt{625-x^2}$, as before. It is **NOT** always true that a tangent line is perpendicular to a line from the origin—don't use this shortcut in any other circumstance.

As above, and as you might expect, for different values of x we generally get different values of the derivative $f'(x)$. Could it be that the derivative always has the same value? This would mean that the slope of f, or the slope of its tangent line, is the same everywhere. One curve that always has the same slope is a line; it seems odd to talk about the tangent line to a line, but if it makes sense at all the tangent line must be the line itself. It is not hard to see that the derivative of $f(x) = mx + b$ is $f'(x) = m$.

Velocity

We started this section by saying, "It is often useful to know how sensitive the value of y is to small changes in x." We have seen one purely mathematical example of this, involving the function $f(x) = \sqrt{625 - x^2}$. Here is a more applied example.

With careful measurement it might be possible to discover that the height of a dropped ball t seconds after it is released is $h(t) = h_0 - kt^2$. (Here h_0 is the initial height of the ball, when $t = 0$, and k is some number determined by the experiment.) A natural question is then, "How fast is the ball going at time t?" We can certainly get a pretty good idea with a little simple arithmetic. To make the calculation more concrete, let's use units of meters and seconds and say that $h_0 = 100$ meters and $k = 4.9$. Suppose we're interested in the speed at $t = 2$. We know that when $t = 2$ the height is $100 - 4 \cdot 4.9 = 80.4$ meters. A second later, at $t = 3$, the height is $100 - 9 \cdot 4.9 = 55.9$ meters. The change in height during that second is $55.9 - 80.4 = -24.5$ meters. The negative sign means the height has decreased, as we expect for a falling ball, and the number 24.5 is the average speed of the ball during the time interval, in meters per second.

We might guess that 24.5 meters per second is not a terrible estimate of the speed at $t = 2$, but certainly we can do better. At $t = 2.5$ the height is $100 - 4.9(2.5)^2 = 69.375$ meters. During the half second from $t = 2$ to $t = 2.5$, the change in height is $69.375 - 80.4 = -11.025$ meters giving an average speed of $11.025/(1/2) = 22.05$ meters per second. This should be a better estimate of the speed at $t = 2$. So it's clear now how to get better and better approximations: compute average speeds over shorter and shorter time intervals. Between $t = 2$ and $t = 2.01$, for example, the ball drops 0.19649 meters in one hundredth of a second, at an average speed of 19.649 meters per second.

We still might reasonably ask for the precise speed at $t = 2$ (the *instantaneous* speed) rather than just an approximation to it. For this, once again, we need a limit. Let's calculate the average speed during the time interval from $t = 2$ to $t = 2 + \Delta t$ without specifying a particular value for Δt. The change in height during the time interval from $t = 2$ to $t = 2 + \Delta t$ is

$$\begin{aligned}
h(2 + \Delta t) - h(2) &= (100 - 4.9(2 + \Delta t)^2) - 80.4 \\
&= 100 - 4.9(4 + 4\Delta t + \Delta t^2) - 80.4 \\
&= 100 - 19.6 - 19.6\Delta t - 4.9\Delta t^2 - 80.4 \\
&= -19.6\Delta t - 4.9\Delta t^2 \\
&= -\Delta t(19.6 + 4.9\Delta t)
\end{aligned}$$

The average speed during this time interval is then

$$\frac{\Delta t(19.6 + 4.9\Delta t)}{\Delta t} = 19.6 + 4.9\Delta t.$$

When Δt is very small, this is very close to 19.6. Indeed, $\lim_{\Delta x \to 0}(19.6 + 4.9\Delta t) = 19.6$. So the exact speed at $t = 2$ is 19.6 meters per second.

At this stage we need to make a distinction between *speed* and *velocity*. Velocity is signed speed, that is, speed with a direction indicated by a sign (positive or negative). Our algebra above actually told us that the instantaneous velocity of the ball at $t = 2$ is -19.6 meters per second. The number 19.6 is the speed and the negative sign indicates that the motion is directed downwards (the direction of decreasing height).

In the language of the previous section, we might have started with $f(x) = 100 - 4.9x^2$ and asked for the slope of the tangent line at $x = 2$. We would have answered that question by computing

$$\lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x} = -19.6 - 4.9\Delta x = 19.6$$

The algebra is the same. Thus, the velocity of the ball is the value of the derivative of a certain function, namely, of the function that gives the position of the ball.

The upshot is that this problem, finding the velocity of the ball, is *exactly* the same problem mathematically as finding the slope of a curve. This may already be enough evidence to convince you that whenever some quantity is changing (the height of a curve or the height of a ball or the size of the economy or the distance of a space probe from earth or the population of the world) the *rate* at which the quantity is changing can, in principle, be computed in exactly the same way, by finding a derivative.

Exercises for Section 4.1

4.1.1 *Draw the graph of the function* $y = f(x) = \sqrt{169 - x^2}$ *between* $x = 0$ *and* $x = 13$. *Find the slope* $\Delta y/\Delta x$ *of the chord between the points of the circle lying over (a)* $x = 12$ *and* $x = 13$, *(b)* $x = 12$ *and* $x = 12.1$, *(c)* $x = 12$ *and* $x = 12.01$, *(d)* $x = 12$ *and* $x = 12.001$. *Now use the geometry of tangent lines on a circle to find (e) the exact value of the derivative* $f'(12)$. *Your answers to (a)–(d) should be getting closer and closer to your answer to (e).*

4.1.2 *Use geometry to find the derivative* $f'(x)$ *of the function* $f(x) = \sqrt{625 - x^2}$ *in the text for each of the following x: (a) 20, (b) 24, (c)* -7, *(d)* -15. *Draw a graph of the upper semicircle, and draw the tangent line at each of these four points.*

4.1.3 *Draw the graph of the function* $y = f(x) = 1/x$ *between* $x = 1/2$ *and* $x = 4$. *Find the slope of the chord between (a)* $x = 3$ *and* $x = 3.1$, *(b)* $x = 3$ *and* $x = 3.01$, *(c)* $x = 3$ *and* $x = 3.001$. *Now use algebra to find a simple formula for the slope of the chord between* $(3, f(3))$ *and* $(3 + \Delta x, f(3 + \Delta x))$. *Determine what happens when* Δx *approaches 0. In your graph of* $y = 1/x$, *draw the straight line through the point* $(3, 1/3)$ *whose slope is this limiting value of the difference quotient as* Δx *approaches 0.*

4.1.4 *Find an algebraic expression for the difference quotient* $\big(f(1 + \Delta x) - f(1)\big)/\Delta x$ *when* $f(x) = x^2 - (1/x)$. *Simplify the expression as much as possible. Then determine what happens as* Δx *approaches 0. That value is* $f'(1)$.

4.1.5 *Draw the graph of* $y = f(x) = x^3$ *between* $x = 0$ *and* $x = 1.5$. *Find the slope of the chord between (a)* $x = 1$ *and* $x = 1.1$, *(b)* $x = 1$ *and* $x = 1.001$, *(c)* $x = 1$ *and* $x = 1.00001$. *Then use algebra to find a simple formula for the slope of the chord between 1 and* $1 + \Delta x$. *(Use the expansion* $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.*) Determine what happens as* Δx *approaches 0, and in your graph of* $y = x^3$ *draw the straight line through the point* $(1, 1)$ *whose slope is equal to the value you just found.*

4.1.6 *Find an algebraic expression for the difference quotient* $(f(x + \Delta x) - f(x))/\Delta x$ *when* $f(x) = mx + b$. *Simplify the expression as much as possible. Then determine what happens as* Δx *approaches 0. That value is* $f'(x)$.

4.1.7 *Sketch the unit circle. Discuss the behavior of the slope of the tangent line at various angles around the circle. Which trigonometric function gives the slope of the tangent line at an angle θ? Why? Hint: think in terms of ratios of sides of triangles.*

4.1.8 *Sketch the parabola $y = x^2$. For what values of x on the parabola is the slope of the tangent line positive? Negative? What do you notice about the graph at the point(s) where the sign of the slope changes from positive to negative and vice versa?*

4.1.9 *An object is traveling in a straight line so that its position (that is, distance from some fixed point) is given by this table:*

time (seconds)	0	1	2	3
distance (meters)	0	10	25	60

Find the average speed of the object during the following time intervals: $[0,1]$, $[0,2]$, $[0,3]$, $[1,2]$, $[1,3]$, $[2,3]$. *If you had to guess the speed at $t = 2$ just on the basis of these, what would you guess?*

4.1.10 *Let $y = f(t) = t^2$, where t is the time in seconds and y is the distance in meters that an object falls on a certain airless planet. Draw a graph of this function between $t = 0$ and $t = 3$. Make a table of the average speed of the falling object between (a) 2 sec and 3 sec, (b) 2 sec and 2.1 sec, (c) 2 sec and 2.01 sec, (d) 2 sec and 2.001 sec. Then use algebra to find a simple formula for the average speed between time* 2 *and time* $2 + \Delta t$. *(If you substitute* $\Delta t = 1$, 0.1, 0.01, 0.001 *in this formula you should again get the answers to parts (a)–(d).) Next, in your formula for average speed (which should be in simplified form) determine what happens as Δt approaches zero. This is the instantaneous speed. Finally, in your graph of $y = t^2$ draw the straight line through the point* $(2, 4)$ *whose slope is the instantaneous velocity you just computed; it should of course be the tangent line.*

4.1.11 *If an object is dropped from an 80-meter high window, its height y above the ground at time t seconds is given by the formula $y = f(t) = 80 - 4.9t^2$. (Here we are neglecting air resistance; the graph of this function was shown in figure 1.2.) Find the average velocity of the falling object between (a) 1 sec and 1.1 sec, (b) 1 sec and 1.01 sec, (c) 1 sec and 1.001 sec. Now use algebra to find a simple formula for the average velocity of the falling object between 1 sec and $1 + \Delta t$ sec. Determine what happens to this average velocity as Δt approaches 0. That is the instantaneous velocity at time $t = 1$ second (it will be negative, because the object is falling).*

4.2 The Derivative Function

In Section 4.1, we have seen how to create, or derive, a new function $f'(x)$ from a function $f(x)$, and that this new function carries important information. In one example we saw that $f'(x)$ tells us how steep the graph of $f(x)$ is; in another we saw that $f'(x)$ tells us the velocity of an object if $f(x)$ tells us the position of the object at time x. As we said earlier, this same mathematical idea is useful whenever $f(x)$ represents some changing quantity and we want to know something about how it changes, or roughly, the "rate" at which it changes. Most functions encountered in practice are built up from a small collection of "primitive" functions in a few simple ways, for example, by adding or multiplying functions together to get new, more complicated functions. To make good use of the information provided by $f'(x)$ we need to be able to compute it for a variety of such functions.

We will begin to use different notations for the derivative of a function. While initially confusing, each is often useful so it is worth maintaining multiple versions of the same thing.

Consider again the function $f(x) = \sqrt{625 - x^2}$. We have computed the derivative $f'(x) = -x/\sqrt{625 - x^2}$, and have already noted that if we use the alternate notation $y = \sqrt{625 - x^2}$ then we might write $y' = -x/\sqrt{625 - x^2}$. Another notation is quite different, and in time it will become clear why it is often a useful one. Recall that to compute the the derivative of f we computed

$$\lim_{\Delta x \to 0} \frac{\sqrt{625 - (7 + \Delta x)^2} - 24}{\Delta x}.$$

The denominator here measures a distance in the x direction, sometimes called the "run", and the numerator measures a distance in the y direction, sometimes called the "rise," and "rise over run" is the slope of a line. Recall that sometimes such a numerator is abbreviated Δy, exchanging brevity for a more detailed expression. So in general, we define a derivative by the following equation.

Definition 4.1: Defnition of Derivative

The derivative of $y = f(x)$ with respect to x is

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Some textbooks use h in place of Δx in the definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

To recall the form of the limit, we sometimes say instead that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

In other words, dy/dx is another notation for the derivative, and it reminds us that it is related to an actual slope between two points. This notation is called **Leibniz notation**, after Gottfried Leibniz, who developed the fundamentals of calculus independently, at about the same time that Isaac Newton did. Again, since we often use f and $f(x)$ to mean the original function, we sometimes use df/dx and $df(x)/dx$ to refer to the derivative. If the function $f(x)$ is written out in full we often write the last of these something like this

$$f'(x) = \frac{d}{dx}\sqrt{625 - x^2}$$

with the function written to the side, instead of trying to fit it into the numerator.

Example 4.2: Derivative of $y = t^2$

Find the derivative of $y = f(t) = t^2$.

Solution. We compute

$$\begin{aligned} y' &= \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{(t+\Delta t)^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{t^2 + 2t\Delta t + \Delta t^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{2t\Delta t + \Delta t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} 2t + \Delta t = 2t. \end{aligned}$$

♣

Remember that Δt is a single quantity, not a "Δ" times a "t", and so Δt^2 is $(\Delta t)^2$ not $\Delta(t^2)$. Doing the same example using the second formula for the derivative with h in place of Δt gives the following. Note that we compute $f(t+h)$ by substituting $t+h$ in place of t everywhere we see t in the expression $f(t)$, *while making no other changes* (at least initially). For example, if $f(t) = t + \sqrt{(t+3)^2 - t}$ then $f(t+h) = (t+h) + \sqrt{((t+h)+3)^2 - (t+h)} = t + h + \sqrt{(t+h+3)^2 - t - h}$.

Example 4.3: Derivative of $y = t^2$

Find the derivative of $y = f(t) = t^2$.

Solution. We compute

$$\begin{aligned} f'(t) &= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{h \to 0} \frac{(t+h)^2 - t^2}{h} \\ &= \lim_{h \to 0} \frac{t^2 + 2th + h^2 - t^2}{h} \\ &= \lim_{h \to 0} \frac{2th + h^2}{h} \\ &= \lim_{h \to 0} 2t + h = 2t. \end{aligned}$$

♣

Example 4.4: Derivative

Find the derivative of $y = f(x) = 1/x$.

Solution. The computation:

$$\begin{aligned}
y' &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x}{x(x+\Delta x)} - \frac{x+\Delta x}{x(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x - x - \Delta x}{x(x+\Delta x)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta x}{x(x+\Delta x)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-1}{x(x+\Delta x)} = \frac{-1}{x^2}
\end{aligned}$$

♣

Note: If you happen to know some "derivative formulas" from an earlier course, for the time being you should pretend that you do not know them. In examples like the ones above and the exercises below, you are required to know how to find the derivative formula starting from basic principles. We will later develop some formulas so that we do not always need to do such computations, but we will continue to need to know how to do the more involved computations.

To recap, given any function f and any number x in the domain of f, we define $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ wherever this limit exists, and we call the number $f'(x)$ the derivative of f at x. Geometrically, $f'(x)$ is the slope of the tangent line to the graph of f at the point $(x, f(x))$. The following symbols also represent the derivative:

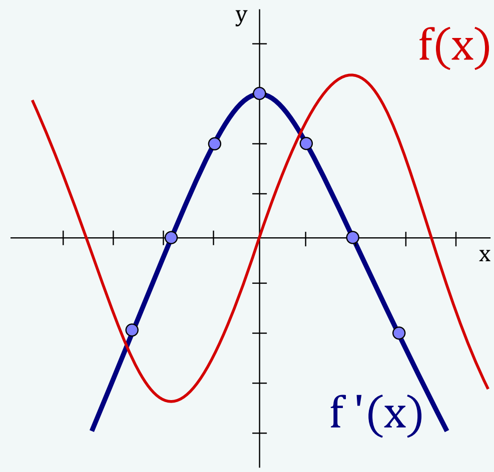
$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x).$$

The symbol d/dx is called a differential operator which means to take the derivative of the function $f(x)$ with respect to the variable x.

In the next example we emphasize the geometrical interpretation of derivative.

Example 4.5: Geometrical Interpretation of Derivative

Consider the function $f(x)$ given by the graph below. Verify that the graph of $f'(x)$ is indeed the *derivative of $f(x)$ by analyzing slopes of tangent lines to the graph at different points.*



Solution. We must think about the tangent lines to the graph of f, because the slopes of these lines are the values of $f'(x)$.

We start by checking the graph of f for horizontal tangent lines, since horizontal lines have a slope of 0. We find that the tangent line is horizontal at the points where x has the values -1.9 and 1.8 (approximately). At each of these values of x, we must have $f'(x) = 0$, which means that the graph of f' has an x-intercept (a point where the graph intersects the x-axis).

Note that horizontal tangent lines have a slope of zero and these occur approximately at the points $(-1.9, -3.2)$ and $(1.8, 3.2)$ of the graph. Therefore $f'(x)$ will cross the x-axis when $x = -1.9$ and $x = 1.8$.

Analyzing the slope of the tangent line of $f(x)$ at $x = 0$ gives approximately 3.0, thus, $f'(0) = 3.0$. Similarly, analyzing the slope of the tangent lines of $f(x)$ at $x = 1$ and $x = -1$ give approximately 2.0 for both, thus, $f'(1) = f'(-1) = 2.0$. ♣

In the next example we verify that the slope of a straight line is m.

Example 4.6: Derivative of a Linear Function

Let m, b be any two real numbers. Determine $f'(x)$ if $f(x) = mx + b$.

Solution. By the definition of derivative (using h in place of Δx) we have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(m(x+h)+b) - (mx+b)}{h}$$

$$= \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m.$$

This is not surprising. We know that $f'(x)$ always represents the slope of a tangent line to the graph of f. In this example, since the graph of f is a straight line $y = mx + b$ already, every tangent line is the same line $y = mx + b$. Since this line has a slope of m, we must have $f'(x) = m$. ♣

4.2.1. Differentiable

Now that we have introduced the derivative of a function at a point, we can begin to use the adjective **differentiable**.

Definition 4.2: Differentiable at a Point

A function f is differentiable at point a if $f'(a)$ exists.

Definition 4.3: Differentiable on an Interval

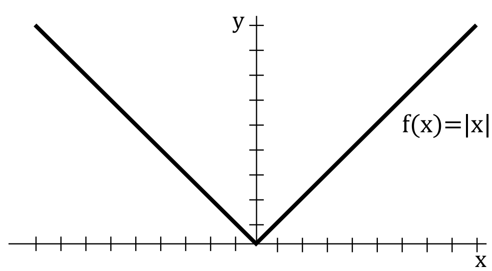
A function f is differentiable on an open interval if it is differentiable at every point in the interval.

Sometimes one encounters a point in the domain of a function $y = f(x)$ where there is **no derivative**, because there is no tangent line. In order for the notion of the tangent line at a point to make sense, the curve must be "smooth" at that point. This means that if you imagine a particle traveling at some steady speed along the curve, then the particle does not experience an abrupt change of direction. There are two types of situations you should be aware of—corners and cusps—where there's a sudden change of direction and hence no derivative.

Example 4.7: Derivative of the Absolute Value

Discuss the derivative of the absolute value function $y = f(x) = |x|$.

Solution. If x is positive, then this is the function $y = x$, whose derivative is the constant 1. (Recall that when $y = f(x) = mx + b$, the derivative is the slope m.) If x is negative, then we're dealing with the function $y = -x$, whose derivative is the constant -1. If $x = 0$, then the function has a corner, i.e., there is no tangent line. A tangent line would have to point in the direction of the curve—but there are *two* directions of the curve that come together at the origin.



We can summarize this as

$$y' = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \\ \text{undefined}, & \text{if } x = 0. \end{cases}$$

In particular, the absolute value function $f(x) = |x|$ is *not* differentiable at $x = 0$. ♣

We note that the following theorem can be proved using limits.

Theorem 4.1: Differentiable implies Continuity

If f is differentiable at a, then f is continuous at a.

Proof. Suppose that f is differentiable at a. That is,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. At this stage, we find it convenient to write this limit in an alternative form so that its connection with continuity can become more easily seen. If we let $x = a + h$, then $h = x - a$. Furthermore, $h \to 0$ is equivalent to $x \to a$. So,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

(This alternative formulation of the derivative is also standard. We will use it whenever we find it convenient to do so. You should get familiar with it.) Continuity at a can now be proved as follows:

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) \right) \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a) + \lim_{x \to a} f(a) \\ &= f'(a) \cdot (a - a) + f(a) \\ &= f(a). \end{aligned}$$

♣

However, if f is continuous at a it is *not* necessarily true that f is differentiable at a. For example, it was shown that $f(x) = |x|$ is not differentiable at $x = 0$ in the previous example, however, one can observe that $f(x) = |x|$ is continuous everywhere.

Example 4.8: Derivative of $y = x^{2/3}$

Discuss the derivative of the function $y = x^{2/3}$, shown in Figure 4.2.

Solution. We will later see how to compute this derivative; for now we use the fact that $y' = (2/3)x^{-1/3}$. Visually this looks much like the absolute value function, but it technically has a cusp, not a corner. The absolute value function has no tangent line at 0 because there are (at least) two obvious contenders—the tangent line of the left side of the curve and the tangent line of the right side. The function $y = x^{2/3}$ does not have a tangent line at 0, but unlike the absolute value function it can be said to have a single direction: as we approach 0 from either side the tangent line becomes closer and closer to a vertical line; the curve is vertical at 0. But as before, if you imagine traveling along the curve, an abrupt change in direction is required at 0: a full 180 degree turn. ♣

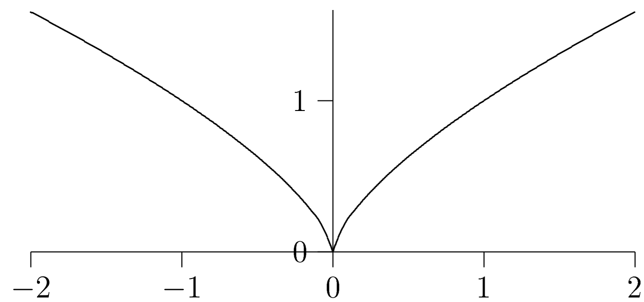


Figure 4.2: A cusp on $x^{2/3}$.

In practice we won't worry much about the distinction between these examples; in both cases the function has a "sharp point" where there is no tangent line and no derivative.

4.2.2. Second and Other Derivatives

If f is a differentiable function then its derivative f' is also a function and so we can take the derivative of f'. The new function, denoted by f'', is called the **second derivative** of f, since it is the derivative of the derivative of f.

The following symbols represent the second derivative:

$$f''(x) = y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

We can continue this process to get the third derivative of f.

In general, the n**th derivative** of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, then we write:

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

4.2.3. Velocities

Suppose $f(t)$ is a position function of an object, representing the displacement of the object from the origin at time t. In terms of derivatives, the **velocity of an object is:**

$$v(a) = f'(a)$$

The change of velocity with respect to time is called the **acceleration** and can be found as follows:

$$a(t) = v'(t) = f''(t).$$

Acceleration is the derivative of the velocity function and the second derivative of the position function.

Example 4.9: Position, Velocity and Acceleration

Suppose the position function of an object is $f(t) = t^2$ metres at t seconds. Find the velocity and acceleration of the object at time $t = 1s$.

Solution. By the definition of velocity and acceleration we need to compute $f'(t)$ and $f''(t)$. Using the definition of derivative, we have,

$$f'(t) = \lim_{h\to 0} \frac{(t+h)^2 - t^2}{h} = \lim_{h\to 0} \frac{2th + h^2}{h} = \lim_{h\to 0}(2t + h) = 2t.$$

Therefore, $v(t) = f'(t) = 2t$. Thus, the velocity at time $t = 1$ is $v(1) = 2\ m/s$. We now have that the acceleration at time t is:

$$a(t) = f''(t) = \lim_{h\to 0} \frac{2(t+h) - 2t}{h} = \lim_{h\to 0} \frac{2h}{h} = 2.$$

Therefore, $a(t) = 2$. Substituting $t = 1$ into the function $a(t)$ gives $a(1) = 2\ m/s^2$. ♣

Exercises for Section 4.2

4.2.1 *Find the derivatives of the following functions.*

(a) $y = f(x) = \sqrt{169 - x^2}$

(b) $y = f(t) = 80 - 4.9t^2$

(c) $y = f(x) = x^2 - (1/x)$

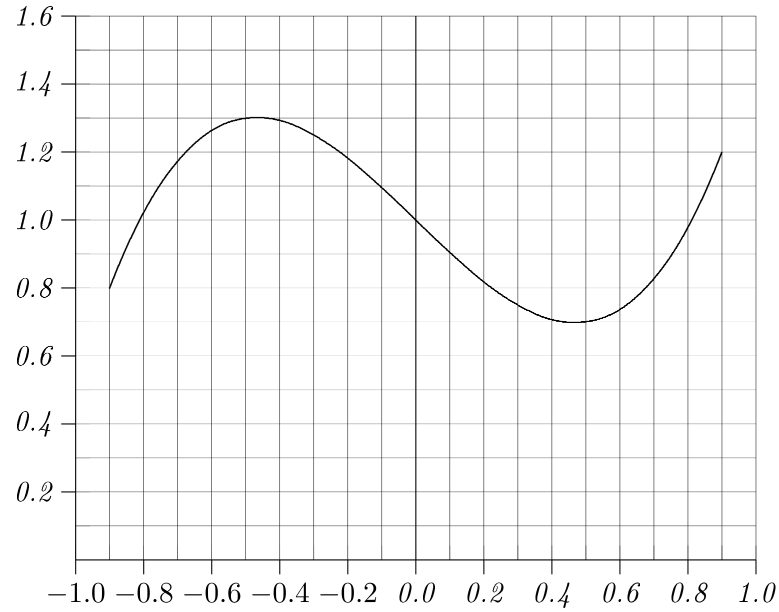
(d) $y = f(x) = ax^2 + bx + c$, *where* a, b, *and* c *are constants.*

(e) $y = f(x) = x^3$

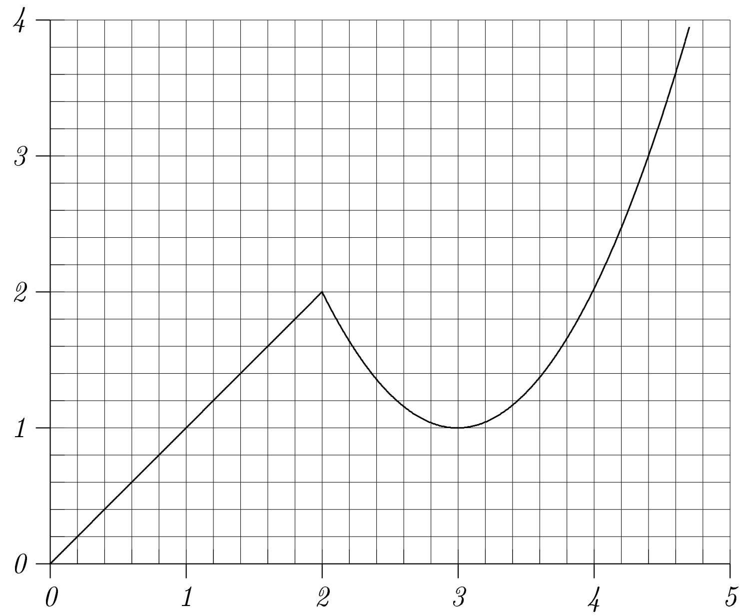
(f) $y = f(x) = 2/\sqrt{2x+1}$

(g) $y = g(t) = (2t - 1)/(t + 2)$

4.2.2 *Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero. Make sure you indicate any places* *where the derivative does not exist.*



4.2.3 *Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero. Make sure you indicate any places where the derivative does not exist.*



4.2.4 *Find an equation for the tangent line to the graph of $f(x) = 5 - x - 3x^2$ at the point $x = 2$*

4.2.5 *Find a value for a so that the graph of $f(x) = x^2 + ax - 3$ has a horizontal tangent line at $x = 4$.*

4.3 Derivative Rules

Using the definition of the derivative of a function is quite tedious. In this section we introduce a number of different shortcuts that can be used to compute the derivative. Recall that the *definition of derivative* is: Given any number x for which the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, we assign to x the number $f'(x)$.

Next, we give some basic *derivative rules* for finding derivatives without having to use the limit definition directly.

Theorem 4.2: Derivative of a Constant Function

Let c be a constant, then $\frac{d}{dx}(c) = 0$.

Proof. Let $f(x) = c$ be a constant function. By the definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

♣

Example 4.10: Derivative of a Constant Function

The derivative of $f(x) = 17$ *is* $f'(x) = 0$ *since the derivative of a constant is* 0.

Theorem 4.3: The Power Rule

If n is a positive integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

Proof. We use the formula:

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$

which can be verified by mulitplying out the right side. Let $f(x) = x^n$ be a power function for some positive integer n. Then at any number a we have:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) = na^{n-1}.$$

♣

It turns out that the Power Rule holds for any real number n (though it is a bit more difficult to prove).

Theorem 4.4: The Power Rule (General)

If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

Example 4.11: Derivative of a Power Function

By the power rule, the derivative of $g(x) = x^4$ *is* $g'(x) = 4x^3$.

Theorem 4.5: The Constant Multiple Rule

If c *is a constant and* f *is a differentiable function, then*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).$$

Proof. For convenience let $g(x) = cf(x)$. Then:

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$
$$= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x),$$

where c can be moved in front of the limit by the Limit Rules. ♣

Example 4.12: Derivative of a Multiple of a Function

By the constant multiple rule and the previous example, the derivative of $F(x) = 2\cdot(17+x^4)$ *is*

$$F'(x) = 2(4x^3) = 8x^3.$$

Theorem 4.6: The Sum/Difference Rule

If f *and* g *are both differentiable functions, then*

$$\frac{d}{dx}[f(x)\pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

Proof. For convenience let $r(x) = f(x)\pm g(x)$. Then:

$$\begin{aligned}
r'(x) &= \lim_{h\to 0}\frac{r(x+h)-r(x)}{h} \\
&= \lim_{h\to 0}\frac{[f(x+h)\pm g(x+h)]-[f(x)\pm g(x)]}{h} \\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\pm\frac{g(x+h)-g(x)}{h}\right] \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\pm\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x)
\end{aligned}$$

♣

Example 4.13: Derivative of a Sum/Difference of Functions

By the sum/difference rule, the derivative of $h(x) = 17 + x^4$ is

$$h'(x) = f'(x) + g'(x) = 0 + 4x^3 = 4x^3.$$

Theorem 4.7: The Product Rule

If f and g are both differentiable functions, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Proof. For convenience let $r(x) = f(x) \cdot g(x)$. As in the previous proof, we want to separate the functions f and g. The trick is to add and subtract $f(x+h)g(x)$ in the numerator. Then:

$$\begin{aligned}
r'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right] \\
&= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

♣

Example 4.14: Derivative of a Product of Functions

Find the derivative of $h(x) = (3x-1)(2x+3)$.

Solution. One way to do this question is to expand the expression. Alternatively, we use the product rule with $f(x) = 3x - 1$ and $g(x) = 2x + 3$. Note that $f'(x) = 3$ and $g'(x) = 2$, so,

$$h'(x) = (3) \cdot (2x+3) + (3x-1) \cdot (2) = 6x + 9 + 6x - 2 = 12x + 7.$$

♣

Theorem 4.8: The Quotient Rule

If f and g are both differentiable functions, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Proof. The proof is similar to the previous proof but the trick is to add and subtract the term $f(x)g(x)$ in the numerator. We omit the details. ♣

Example 4.15: Derivative of a Quotient of Functions

Find the derivative of $h(x) = \dfrac{3x-1}{2x+3}$.

Solution. By the quotient rule (using $f(x) = 3x - 1$ and $g(x) = 2x + 3$) we have:

$$\begin{aligned} h'(x) &= \frac{\frac{d}{dx}(3x-1)\cdot(2x+3) - (3x-1)\cdot\frac{d}{dx}(2x+3)}{(2x+3)^2} \\ &= \frac{3(2x+3)-(3x-1)(2)}{(2x+3)^2} = \frac{11}{(2x+3)^2}. \end{aligned}$$

♣

Example 4.16: Second Derivative

Find the second derivative of $f(x) = 5x^3 + 3x^2$.

Solution. We must differentiate $f(x)$ twice:

$$f'(x) = 15x^2 + 6x,$$
$$f''(x) = 30x + 6.$$

♣

Exercises for Section 4.3

4.3.1 *Find the derivatives of the following functions.*

(a) x^{100}

(b) x^{-100}

(c) $\dfrac{1}{x^5}$

(d) x^π

(e) $x^{3/4}$

(f) $x^{-9/7}$

(g) $5x^3 + 12x^2 - 15$

(h) $-4x^5 + 3x^2 - 5/x^2$

(i) $5(-3x^2 + 5x + 1)$

(j) $(x+1)(x^2+2x-3)$

(k) $(x+1)(x^2+2x-3)^{-1}$

(l) $x^3(x^3 - 5x + 10)$

(m) $(x^2+5x-3)(x^5)$

(n) $(x^2+5x-3)(x^{-5})$

(o) $(5x^3+12x^2-15)^{-1}$

4.3.2 *Find an equation for the tangent line to* $f(x) = x^3/4 - 1/x$ *at* $x = -2$.

4.3.3 *Find an equation for the tangent line to* $f(x) = 3x^2 - \pi^3$ *at* $x = 4$.

4.3.4 *Suppose the position of an object at time* t *is given by* $f(t) = -49t^2/10 + 5t + 10$. *Find a function giving the speed of the object at time* t. *The acceleration of an object is the rate at which its speed is* *changing, which means it is given by the derivative of the speed function. Find the acceleration of the object at time t.*

4.3.5 *Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of f, cf, f', and $(cf)'$ on the same diagram.*

4.3.6 *The general polynomial P of degree n in the variable x has the form $P(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \ldots + a_n x^n$. What is the derivative (with respect to x) of P?*

4.3.7 *Find a cubic polynomial whose graph has horizontal tangents at $(-2, 5)$ and $(2, 3)$.*

4.3.8 *Prove that $\frac{d}{dx}(cf(x)) = cf'(x)$ using the definition of the derivative.*

4.3.9 *Suppose that f and g are differentiable at x. Show that $f - g$ is differentiable at x using the two linearity properties from this section.*

4.3.10 *Use the product rule to compute the derivative of $f(x) = (2x - 3)^2$. Sketch the function. Find an equation of the tangent line to the curve at $x = 2$. Sketch the tangent line at $x = 2$.*

4.3.11 *Suppose that f, g, and h are differentiable functions. Show that $(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.*

4.3.12 *Compute the derivative of $\dfrac{x^3}{x^3 - 5x + 10}$.*

4.3.13 *Compute the derivative of $\dfrac{x^2 + 5x - 3}{x^5 - 6x^3 + 3x^2 - 7x + 1}$.*

4.3.14 *Compute the derivative of $\dfrac{x}{\sqrt{x - 625}}$.*

4.3.15 *Compute the derivative of $\dfrac{\sqrt{x - 5}}{x^{20}}$.*

4.3.16 *Find an equation for the tangent line to $f(x) = (x^2 - 4)/(5 - x)$ at $x = 3$.*

4.3.17 *Find an equation for the tangent line to $f(x) = (x - 2)/(x^3 + 4x - 1)$ at $x = 1$.*

4.3.18 *If $f'(4) = 5$, $g'(4) = 12$, $(fg)(4) = f(4)g(4) = 2$, and $g(4) = 6$, compute $f(4)$ and $\frac{d}{dx}\frac{f}{g}$ at 4.*

4.4 Derivative Rules for Trigonometric Functions

We next look at the derivative of the sine function. In order to prove the derivative formula for sine, we recall two limit computations from earlier:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0,$$

and the double angle formula

$$\sin(A+B) = \sin A \cos B + \sin B \cos A.$$

Theorem 4.9: Derivative of Sine Function

$(\sin x)' = \cos x$

Proof. Let $f(x) = \sin x$. Using the definition of derivative we have:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x
\end{aligned}$$

since

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

♣

A formula for the derivative of the *cosine function* can be found in a similar fashion:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Using the quotient rule we get formulas for the remaining trigonometric ratios. To summarize, here are the derivatives of the six trigonometric functions:

$\frac{d}{dx}(\sin(x)) = \cos(x)$	$\frac{d}{dx}(\tan(x)) = \sec^2(x)$	$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$
$\frac{d}{dx}(\cos(x)) = -\sin(x)$	$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$	$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$

Example 4.17: Derivative of Product of Trigonometric Functions

Find the derivative of $f(x) = \sin x \tan x$.

Solution. Using the Product Rule we obtain

$$f'(x) = \cos x \tan x + \sin x \sec^2 x.$$

♣

Exercises for Section 4.4

4.4.1 *Find the derivatives of the following functions.*

(a) $\sin x \cos x$ (b) $\cot x$ (c) $\csc x - x \tan x$

4.4.2 *Find the points on the curve $y = x + 2\cos x$ that have a horizontal tangent line.*

4.5 The Chain Rule

Let $h(x) = \sqrt{625 - x^2}$. The rules stated previously do not allow us to find $h'(x)$. However, $h(x)$ is a composition of two functions. Let $f(x) = \sqrt{x}$ and $g(x) = 625 - x^2$. Then we see that

$$h(x) = (f \circ g)(x).$$

From our rules we know that $f'(x) = \frac{1}{2}x^{-1/2}$ and $g'(x) = -2x$, thus it would be convenient to have a rule which allows us to differentiate $f \circ g$ in terms of f' and g'. This gives rise to the chain rule.

Key Idea 4.5.0: The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $h = f \circ g$ [recall $f \circ g$ is defined as $f(g(x))$] is differentiable at x and $h'(x)$ is given by:

$$h'(x) = f'(g(x)) \cdot g'(x).$$

The chain rule has a particularly simple expression if we use the Leibniz notation for the derivative. The quantity $f'(g(x))$ is the derivative of f with x replaced by g; this can be written df/dg. As usual, $g'(x) = dg/dx$. Then the chain rule becomes

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}.$$

This looks like trivial arithmetic, but it is not: dg/dx is not a fraction, that is, not literal division, but a single symbol that means $g'(x)$. Nevertheless, it turns out that what looks like trivial arithmetic, and is therefore easy to remember, is really true.

It will take a bit of practice to make the use of the chain rule come naturally—it is more complicated than the earlier differentiation rules we have seen.

Example 4.18: Chain Rule

Compute the derivative of $\sqrt{625-x^2}$.

Solution. We already know that the answer is $-x/\sqrt{625-x^2}$, computed directly from the limit. In the context of the chain rule, we have $f(x)=\sqrt{x}$, $g(x)=625-x^2$. We know that $f'(x)=(1/2)x^{-1/2}$, so $f'(g(x))=(1/2)(625-x^2)^{-1/2}$. Note that this is a two step computation: first compute $f'(x)$, then replace x by $g(x)$. Since $g'(x)=-2x$ we have

$$f'(g(x))g'(x)=\frac{1}{2\sqrt{625-x^2}}(-2x)=\frac{-x}{\sqrt{625-x^2}}.$$

♣

Example 4.19: Chain Rule

Compute the derivative of $1/\sqrt{625-x^2}$.

Solution. This is a quotient with a constant numerator, so we could use the quotient rule, but it is simpler to use the chain rule. The function is $(625-x^2)^{-1/2}$, the composition of $f(x)=x^{-1/2}$ and $g(x)=625-x^2$. We compute $f'(x)=(-1/2)x^{-3/2}$ using the power rule, and then

$$f'(g(x))g'(x)=\frac{-1}{2(625-x^2)^{3/2}}(-2x)=\frac{x}{(625-x^2)^{3/2}}.$$

♣

In practice, of course, you will need to use more than one of the rules we have developed to compute the derivative of a complicated function.

Example 4.20: Derivative of Quotient

Compute the derivative of

$$f(x)=\frac{x^2-1}{x\sqrt{x^2+1}}.$$

Solution. The "last" operation here is division, so to get started we need to use the quotient rule first. This gives

$$\begin{aligned} f'(x) &= \frac{(x^2-1)'x\sqrt{x^2+1}-(x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)} \\ &= \frac{2x^2\sqrt{x^2+1}-(x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}. \end{aligned}$$

Now we need to compute the derivative of $x\sqrt{x^2+1}$. This is a product, so we use the product rule:

$$\frac{d}{dx}x\sqrt{x^2+1}=x\frac{d}{dx}\sqrt{x^2+1}+\sqrt{x^2+1}.$$

Finally, we use the chain rule:

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{d}{dx}(x^2+1)^{1/2} = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

And putting it all together:

$$\begin{aligned} f'(x) &= \frac{2x^2\sqrt{x^2+1} - (x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}. \\ &= \frac{2x^2\sqrt{x^2+1} - (x^2-1)\left(x\frac{x}{\sqrt{x^2+1}} + \sqrt{x^2+1}\right)}{x^2(x^2+1)}. \end{aligned}$$

This can be simplified of course, but we have done all the calculus, so that only algebra is left. ♣

Example 4.21: Chain of Composition

Compute the derivative of $\sqrt{1+\sqrt{1+\sqrt{x}}}$.

Solution. Here we have a more complicated chain of compositions, so we use the chain rule twice. At the outermost "layer" we have the function $g(x) = 1+\sqrt{1+\sqrt{x}}$ plugged into $f(x) = \sqrt{x}$, so applying the chain rule once gives

$$\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} = \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\frac{d}{dx}\left(1+\sqrt{1+\sqrt{x}}\right).$$

Now we need the derivative of $\sqrt{1+\sqrt{x}}$. Using the chain rule again:

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}.$$

So the original derivative is

$$\begin{aligned} \frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} &= \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}. \\ &= \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}} \end{aligned}$$

♣

Using the chain rule, the power rule, and the product rule, it is possible to avoid using the quotient rule entirely.

Example 4.22: Derivative of Quotient without Quotient Rule

Compute the derivative of $f(x) = \dfrac{x^3}{x^2+1}$.

Solution. Write $f(x) = x^3(x^2+1)^{-1}$, then

$$\begin{aligned} f'(x) &= x^3 \frac{d}{dx}(x^2+1)^{-1} + 3x^2(x^2+1)^{-1} \\ &= x^3(-1)(x^2+1)^{-2}(2x) + 3x^2(x^2+1)^{-1} \\ &= -2x^4(x^2+1)^{-2} + 3x^2(x^2+1)^{-1} \\ &= \frac{-2x^4}{(x^2+1)^2} + \frac{3x^2}{x^2+1} \\ &= \frac{-2x^4}{(x^2+1)^2} + \frac{3x^2(x^2+1)}{(x^2+1)^2} \\ &= \frac{-2x^4+3x^4+3x^2}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2} \end{aligned}$$

Note that we already had the derivative on the second line; all the rest is simplification. It is easier to get to this answer by using the quotient rule, so there's a trade off: more work for fewer memorized formulas.

♣

Exercises for Section 4.5

Find the derivatives of the functions. For extra practice, and to check your answers, do some of these in more than one way if possible.

4.5.1 $x^4 - 3x^3 + (1/2)x^2 + 7x - \pi$

4.5.2 $x^3 - 2x^2 + 4\sqrt{x}$

4.5.3 $(x^2+1)^3$

4.5.4 $x\sqrt{169 - x^2}$

4.5.5 $(x^2 - 4x + 5)\sqrt{25 - x^2}$

4.5.6 $\sqrt{r^2 - x^2}$, *r is a constant*

4.5.7 $\sqrt{1 + x^4}$

4.5.8 $\dfrac{1}{\sqrt{5 - \sqrt{x}}}$.

4.5.9 $(1+3x)^2$

4.5.10 $\dfrac{(x^2+x+1)}{(1-x)}$

4.5.11 $\dfrac{\sqrt{25-x^2}}{x}$

4.5.12 $\sqrt{\dfrac{169}{x}-x}$

4.5.13 $\sqrt{x^3-x^2-(1/x)}$

4.5.14 $100/(100-x^2)^{3/2}$

4.5.15 $\sqrt[3]{x+x^3}$

4.5.16 $\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$

4.5.17 $(x+8)^5$

4.5.18 $(4-x)^3$

4.5.19 $(x^2+5)^3$

4.5.20 $(6-2x^2)^3$

4.5.21 $(1-4x^3)^{-2}$

4.5.22 $5(x+1-1/x)$

4.5.23 $4(2x^2-x+3)^{-2}$

4.5.24 $\dfrac{1}{1+1/x}$

4.5.25 $\dfrac{-3}{4x^2-2x+1}$

4.5.26 $(x^2+1)(5-2x)/2$

4.5.27 $(3x^2+1)(2x-4)^3$

4.5.28 $\dfrac{x+1}{x-1}$

4.5.29 $\dfrac{x^2-1}{x^2+1}$

4.5.30 $\dfrac{(x-1)(x-2)}{x-3}$

4.5.31 $\dfrac{2x^{-1}-x^{-2}}{3x^{-1}-4x^{-2}}$

4.5.32 $3(x^2+1)(2x^2-1)(2x+3)$

4.5.33 $\dfrac{1}{(2x+1)(x-3)}$

4.5.34 $((2x+1)^{-1}+3)^{-1}$

4.5.35 $(2x+1)^3(x^2+1)^2$

4.5.36 *Find an equation for the tangent line to* $f(x)=(x-2)^{1/3}/(x^3+4x-1)^2$ *at* $x=1$.

4.5.37 *Find an equation for the tangent line to* $y=9x^{-2}$ *at* $(3,1)$.

4.5.38 *Find an equation for the tangent line to* $(x^2-4x+5)\sqrt{25-x^2}$ *at* $(3,8)$.

4.5.39 *Find an equation for the tangent line to* $\dfrac{(x^2+x+1)}{(1-x)}$ *at* $(2,-7)$.

4.5.40 *Find an equation for the tangent line to* $\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$ *at* $(1,\sqrt{4+\sqrt{5}})$.

4.5.41 *Let* $y=f(x)$ *and* $x=g(t)$. *If* $g(1)=2$, $f(2)=3$, $g'(1)=4$ *and* $f'(2)=5$, *find the derivative of* $f\circ g$ *at 1.*

4.5.42 *Express the derivative of* $g(x)=x^2f(x^2)$ *in terms of* f *and the derivative of* f.

4.6 Derivatives of Exponential & Logarithmic Functions

As with the sine function, we don't know anything about derivatives that allows us to compute the derivatives of the exponential and logarithmic functions without going back to basics. Let's do a little work with the definition again:

$$\begin{aligned}
\frac{d}{dx}a^x &= \lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{a^x a^{\Delta x}-a^x}{\Delta x}\\
&= \lim_{\Delta x\to 0}a^x\frac{a^{\Delta x}-1}{\Delta x}
\end{aligned}$$

$$= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

There are two interesting things to note here: As in the case of the sine function we are left with a limit that involves Δx but not x, which means that if $\lim_{\Delta x \to 0}(a^{\Delta x} - 1)/\Delta x$ exists, then it is a constant number. This means that a^x has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is $\lim_{x \to 0} \sin x/x = 1$; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that $\lim_{\Delta x \to 0}(a^{\Delta x} - 1)/\Delta x$ even exists—does this fraction really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider $(2^x - 1)/x$ for some small values of x: 1, 0.828427124, 0.756828460, 0.724061864, 0.70838051, 0.70070877 when x is 1, 1/2, 1/4, 1/8, 1/16, 1/32, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider next $(3^x - 1)/x$: 2, 1.464101616, 1.264296052, 1.177621520, 1.13720773, 1.11768854, at the same values of x. It turns out to be true that in the limit this is about 1.1. Two examples don't establish a pattern, but if you do more examples you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a = 2$ and $a = 3$ the limit will be exactly 1; the value at which this happens is called e, so that

$$\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1.$$

As you might guess from our two examples, e is closer to 3 than to 2, and in fact $e \approx 2.718$.

Now we see that the function e^x has a truly remarkable property:

$$\begin{aligned}
\frac{d}{dx}e^x &= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{e^x e^{\Delta x} - e^x}{\Delta x} \\
&= \lim_{\Delta x \to 0} e^x \frac{e^{\Delta x} - 1}{\Delta x} \\
&= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} \\
&= e^x
\end{aligned}$$

That is, e^x is its own derivative, or in other words the slope of e^x is the same as its height, or the same as its second coordinate: The function $f(x) = e^x$ goes through the point (z, e^z) and has slope e^z there, no matter what z is. It is sometimes convenient to express the function e^x without an exponent, since complicated exponents can be hard to read. In such cases we use $\exp(x)$, e.g., $\exp(1 + x^2)$ instead of e^{1+x^2}.

What about the logarithm function? This too is hard, but as the cosine function was easier to do once the sine was done, so is the logarithm easier to do now that we know the derivative of the exponential function. Let's start with $\log_e x$, which as you probably know is often abbreviated $\ln x$ and called the "natural logarithm" function.

Consider the relationship between the two functions, namely, that they are inverses, that one "undoes" the other. Graphically this means that they have the same graph except that one is "flipped" or "reflected" through the line $y = x$:

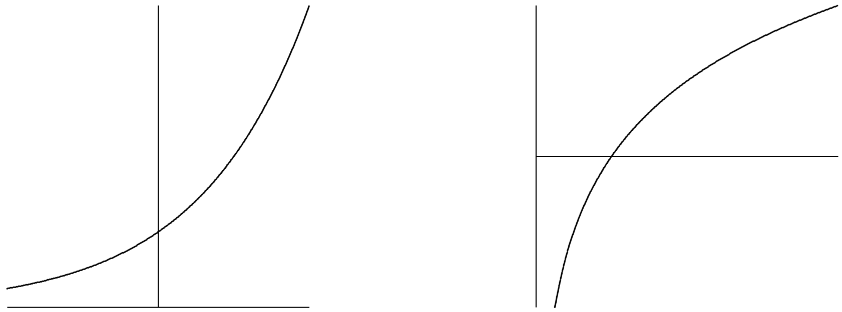


Figure 4.3: The exponential and logarithmic functions.

This means that the slopes of these two functions are closely related as well: For example, the slope of e^x is e at $x = 1$; at the corresponding point on the $\ln(x)$ curve, the slope must be $1/e$, because the "rise" and the "run" have been interchanged. Since the slope of e^x is e at the point $(1, e)$, the slope of $\ln(x)$ is $1/e$ at the point $(e, 1)$.

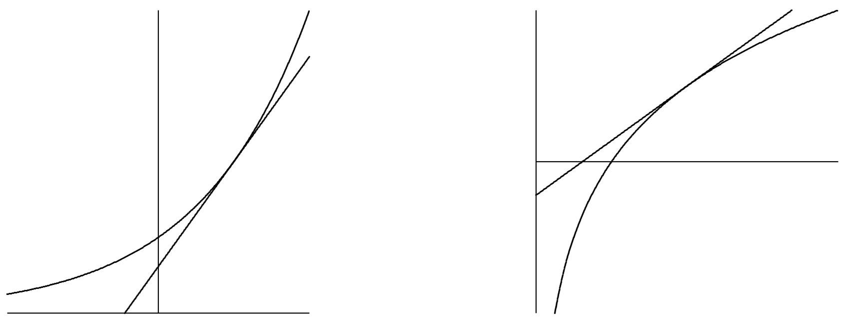


Figure 4.4: The exponential and logarithmic functions.

More generally, we know that the slope of e^x is e^z at the point (z, e^z), so the slope of $\ln(x)$ is $1/e^z$ at (e^z, z). In other words, the slope of $\ln x$ is the reciprocal of the first coordinate at any point; this means that the slope of $\ln x$ at $(x, \ln x)$ is $1/x$. The upshot is:

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof—it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon.

Note that $\ln x$ is defined only for $x > 0$. It is sometimes useful to consider the function $\ln |x|$, a function defined for $x \neq 0$. When $x < 0$, $\ln |x| = \ln(-x)$ and

$$\frac{d}{dx} \ln |x| = \frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Thus whether x is positive or negative, the derivative is the same.

What about the functions a^x and $\log_a x$? We know that the derivative of a^x is some constant times a^x itself, but what constant? Remember that "the logarithm is the exponent" and you will see that $a = e^{\ln a}$. Then

$$a^x = (e^{\ln a})^x = e^{x \ln a},$$

and we can compute the derivative using the chain rule:

$$\frac{d}{dx}a^x = \frac{d}{dx}(e^{\ln a})^x = \frac{d}{dx}e^{x\ln a} = (\ln a)e^{x\ln a} = (\ln a)a^x.$$

The constant is simply $\ln a$. Likewise we can compute the derivative of the logarithm function $\log_a x$. Since

$$x = e^{\ln x}$$

we can take the logarithm base a of both sides to get

$$\log_a(x) = \log_a(e^{\ln x}) = \ln x \log_a e.$$

Then

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e.$$

This is a perfectly good answer, but we can improve it slightly. Since

$$\begin{aligned} a &= e^{\ln a} \\ \log_a(a) &= \log_a(e^{\ln a}) = \ln a \log_a e \\ 1 &= \ln a \log_a e \\ \frac{1}{\ln a} &= \log_a e, \end{aligned}$$

we can replace $\log_a e$ to get

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

You may if you wish memorize the formulas.

Key Idea 4.6.0: Derivative Formulas for a^x and $\log_a x$

$$\frac{d}{dx}a^x = (\ln a)a^x \quad \text{and} \quad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

Because the "trick" $a = e^{\ln a}$ is often useful, and sometimes essential, it may be better to remember the trick, not the formula.

Example 4.23: Derivative of Exponential Function

Compute the derivative of $f(x) = 2^x$.

Solution.

$$\begin{aligned} \frac{d}{dx}2^x &= \frac{d}{dx}(e^{\ln 2})^x \\ &= \frac{d}{dx}e^{x\ln 2} \\ &= \left(\frac{d}{dx}x\ln 2\right)e^{x\ln 2} \\ &= (\ln 2)e^{x\ln 2} = 2^x \ln 2 \end{aligned}$$

♣

Example 4.24: Derivative of Exponential Function

Compute the derivative of $f(x) = 2^{x^2} = 2^{(x^2)}$.

Solution.

$$\begin{aligned}
\frac{d}{dx}2^{x^2} &= \frac{d}{dx}e^{x^2 \ln 2} \\
&= \left(\frac{d}{dx}x^2 \ln 2\right) e^{x^2 \ln 2} \\
&= (2 \ln 2) x e^{x^2 \ln 2} \\
&= (2 \ln 2) x 2^{x^2}
\end{aligned}$$

♣

Example 4.25: Power Rule

Recall that we have not justified the power rule except when the exponent is a positive or negative integer.

Solution. We can use the exponential function to take care of other exponents.

$$\begin{aligned}
\frac{d}{dx}x^r &= \frac{d}{dx}e^{r \ln x} \\
&= \left(\frac{d}{dx}r \ln x\right) e^{r \ln x} \\
&= (r\frac{1}{x})x^r \\
&= rx^{r-1}
\end{aligned}$$

♣

Exercises for Section 4.6

Find the derivatives of the functions.

4.6.1 3^{x^2}

4.6.2 $\frac{\sin x}{e^x}$

4.6.3 $(e^x)^2$

4.6.4 $\sin(e^x)$

4.6.5 $e^{\sin x}$

4.6.6 $x^{\sin x}$

4.6.7 $x^3 e^x$

4.6.8 $x + 2^x$

4.6.9 $(1/3)^{x^2}$

4.6.10 e^{4x}/x

4.6.11 $\ln(x^3 + 3x)$

4.6.12 $\ln(\cos(x))$

4.6.13 $\sqrt{\ln(x^2)}/x$

4.6.14 $\ln(\sec(x) + \tan(x))$

4.6.15 $x^{\cos(x)}$

4.6.16 $x \ln x$

4.6.17 $\ln(\ln(3x))$

4.6.18 $\dfrac{1 + \ln(3x^2)}{1 + \ln(4x)}$

4.6.19 *Find the value of a so that the tangent line to $y = \ln(x)$ at $x = a$ is a line through the origin. Sketch the resulting situation.*

4.6.20 *If $f(x) = \ln(x^3 + 2)$ compute $f'(e^{1/3})$.*

4.7 Implicit Differentiation

As we have seen, there is a close relationship between the derivatives of e^x and $\ln x$ because these functions are inverses. Rather than relying on pictures for our understanding, we would like to be able to exploit this relationship computationally. In fact this technique can help us find derivatives in many situations, not just when we seek the derivative of an inverse function.

We will begin by illustrating the technique to find what we already know, the derivative of $\ln x$. Let's write $y = \ln x$ and then $x = e^{\ln x} = e^y$, that is, $x = e^y$. We say that this equation defines the function $y = \ln x$ implicitly because while it is not an explicit expression $y = \ldots$, it is true that if $x = e^y$ then y is in fact the natural logarithm function. Now, for the time being, pretend that all we know of y is that $x = e^y$; what can we say about derivatives? We can take the derivative of both sides of the equation:

$$\frac{d}{dx}x = \frac{d}{dx}e^y.$$

Then using the chain rule on the right hand side:

$$1 = \left(\frac{d}{dx}y\right)e^y = y'e^y.$$

Then we can solve for y':

$$y' = \frac{1}{e^y} = \frac{1}{x}.$$

There is one little difficulty here. To use the chain rule to compute $d/dx(e^y) = y'e^y$ we need to know that the function y *has* a derivative. All we have shown is that *if* it has a derivative then that derivative must be $1/x$. When using this method we will always have to assume that the desired derivative exists, but fortunately this is a safe assumption for most such problems.

The example $y = \ln x$ involved an inverse function defined implicitly, but other functions can be defined implicitly, and sometimes a single equation can be used to implicitly define more than one function.

Here's a familiar example.

Example 4.26: Derivative of Circle Equation

The equation $r^2 = x^2 + y^2$ describes a circle of radius r. The circle is not a function $y = f(x)$ because for some values of x there are two corresponding values of y. If we want to work with a function, we can break the circle into two pieces, the upper and lower semicircles, each of which is a function. Let's call these $y = U(x)$ and $y = L(x)$; in fact this is a fairly simple example, and it's possible to give explicit expressions for these: $U(x) = \sqrt{r^2 - x^2}$ and $L(x) = -\sqrt{r^2 - x^2}$. But it's somewhat easier, and quite useful, to view both functions as given implicitly by $r^2 = x^2 + y^2$: both $r^2 = x^2 + U(x)^2$ and $r^2 = x^2 + L(x)^2$ are true, and we can think of $r^2 = x^2 + y^2$ as defining both $U(x)$ and $L(x)$.

Now we can take the derivative of both sides as before, remembering that y is not simply a variable but a function—in this case, y is either $U(x)$ or $L(x)$ but we're not yet specifying which one. When we take the derivative we just have to remember to apply the chain rule where y appears.

$$\begin{aligned} \frac{d}{dx}r^2 &= \frac{d}{dx}(x^2 + y^2) \\ 0 &= 2x + 2yy' \\ y' &= \frac{-2x}{2y} = -\frac{x}{y} \end{aligned}$$

Now we have an expression for y', but it contains y as well as x. This means that if we want to compute y' for some particular value of x we'll have to know or compute y at that value of x as well. It is at this point that we will need to know whether y is $U(x)$ or $L(x)$. Occasionally it will turn out that we can avoid explicit use of $U(x)$ or $L(x)$ by the nature of the problem.

Example 4.27: Slope of the Circle

Find the slope of the circle $4 = x^2 + y^2$ at the point $(1, -\sqrt{3})$.

Solution. Since we know both the x and y coordinates of the point of interest, we do not need to explicitly recognize that this point is on $L(x)$, and we do not need to use $L(x)$ to compute y – but we could. Using the calculation of y' from above,

$$y' = -\frac{x}{y} = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

It is instructive to compare this approach to others.

We might have recognized at the start that $(1, -\sqrt{3})$ is on the function $y = L(x) = -\sqrt{4 - x^2}$. We could then take the derivative of $L(x)$, using the power rule and the chain rule, to get

$$L'(x) = -\frac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{4 - x^2}}.$$

Then we could compute $L'(1) = 1/\sqrt{3}$ by substituting $x = 1$.

Alternately, we could realize that the point is on $L(x)$, but use the fact that $y' = -x/y$. Since the point is on $L(x)$ we can replace y by $L(x)$ to get

$$y' = -\frac{x}{L(x)} = -\frac{x}{\sqrt{4 - x^2}},$$

without computing the derivative of $L(x)$ explicitly. Then we substitute $x = 1$ and get the same answer as before. ♣

In the case of the circle it is possible to find the functions $U(x)$ and $L(x)$ explicitly, but there are potential advantages to using implicit differentiation anyway. In some cases it is more difficult or impossible to find an explicit formula for y and implicit differentiation is the only way to find the derivative.

Example 4.28: Derivative of Function defined Implicitly

Find the derivative of any function defined implicitly by $yx^2 + y^2 = x$.

Solution. We treat y as an unspecified function and use the chain rule:

$$\begin{aligned}
\frac{d}{dx}(yx^2 + y^2) &= \frac{d}{dx}x \\
(y \cdot 2x + y' \cdot x^2) + 2yy' &= 1 \\
y' \cdot x^2 + 2yy' &= -y \cdot 2x \\
y' &= \frac{-2xy}{x^2 + 2y}
\end{aligned}$$

♣

Example 4.29: Derivative of Function defined Implicitly

Find the derivative of any function defined implicitly by $yx^2 + e^y = x$.

Solution. We treat y as an unspecified function and use the chain rule:

$$\begin{aligned}
\frac{d}{dx}(yx^2 + e^y) &= \frac{d}{dx}x \\
(y \cdot 2x + y' \cdot x^2) + y'e^y &= 1 \\
y'x^2 + y'e^y &= 1 - 2xy \\
y'(x^2 + e^y) &= 1 - 2xy \\
y' &= \frac{1 - 2xy}{x^2 + e^y}
\end{aligned}$$

♣

You might think that the step in which we solve for y' could sometimes be difficult—after all, we're using implicit differentiation here because we can't solve the equation $yx^2 + e^y = x$ for y, so maybe after taking the derivative we get something that is hard to solve for y'. In fact, *this never happens.* All occurrences y' come from applying the chain rule, and whenever the chain rule is used it deposits a single y' multiplied by some other expression. So it will always be possible to group the terms containing y' together and factor out the y', just as in the previous example. If you ever get anything more difficult you have made a mistake and should fix it before trying to continue.

It is sometimes the case that a situation leads naturally to an equation that defines a function implicitly.

Example 4.30: Equation and Derivative of Ellipse

Discuss the equation and derivative of the ellispe.

Solution. Consider all the points (x, y) that have the property that the distance from (x, y) to (x_1, y_1) plus the distance from (x, y) to (x_2, y_2) is $2a$ (a is some constant). These points form an ellipse, which like a circle is not a function but can be viewed as two functions pasted together. Since we know how to write down the distance between two points, we can write down an implicit equation for the ellipse:

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2} = 2a.$$

Then we can use implicit differentiation to find the slope of the ellipse at any point, though the computation is rather messy. ♣

Example 4.31: Derivative of Function defined Implicitly

Find $\frac{dy}{dx}$ by implicit differentiation if

$$2x^3 + x^2y - y^9 = 3x + 4.$$

Solution. Differentiating both sides with respect to x gives:

$$6x^2 + \left(2xy + x^2\frac{dy}{dx}\right) - 9y^8\frac{dy}{dx} = 3,$$

$$x^2\frac{dy}{dx} - 9y^8\frac{dy}{dx} = 3 - 6x^2 - 2xy$$

$$\left(x^2 - 9y^8\right)\frac{dy}{dx} = 3 - 6x^2 - 2xy$$

$$\frac{dy}{dx} = \frac{3 - 6x^2 - 2xy}{x^2 - 9y^8}.$$

♣

In the previous examples we had functions involving x and y, and we thought of y as a function of x. In these problems we differentiated with respect to x. So when faced with x's in the function we differentiated as usual, but when faced with y's we differentiated as usual except we multiplied by a $\frac{dy}{dx}$ for that term because we were using Chain Rule.

In the following example we will assume that both x and y are functions of t and want to differentiate the equation with respect to t. This means that every time we differentiate an x we will be using the Chain Rule, so we must multiply by $\frac{dx}{dt}$, and whenever we differentiate a y we multiply by $\frac{dy}{dt}$.

Example 4.32: Derivative of Function of an Additional Variable

Thinking of x and y as functions of t, differentiate the following equation with respect to t:

$$x^2 + y^2 = 100.$$

Solution. Using the Chain Rule we have:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

♣

Example 4.33: Derivative of Function of an Additional Variable

If $y = x^3 + 5x$ and $\frac{dx}{dt} = 7$, find $\frac{dy}{dt}$ when $x = 1$.

Solution. Differentiating each side of the equation $y = x^3 + 5x$ with respect to t gives:

$$\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 5\frac{dx}{dt}.$$

When $x = 1$ and $\frac{dx}{dt} = 7$ we have:

$$\frac{dy}{dt} = 3(1^2)(7) + 5(7) = 21 + 35 = 56.$$

♣

Logarithmic Differentiation

Previously we've seen how to do the derivative of a number to a function $(a^{f(x)})'$, and also a function to a number $[(f(x))^n]'$. But what about the derivative of a function to a function $[(f(x))^{g(x)}]'$?

In this case, we use a procedure known as **logarithmic differentiation**.

Key Idea 4.7.0: Steps for Logarithmic Differentiation

- Take ln of both sides of $y = f(x)$ to get $\ln y = \ln f(x)$ and simplify using logarithm properties,
- Differentiate implicitly with respect to x and solve for $\frac{dy}{dx}$,
- Replace y with its function of x (i.e., $f(x)$).

Example 4.34: Logarithmic Differentiation

Differentiate $y = x^x$.

Solution. We take ln of both sides:

$$\ln y = \ln x^x.$$

Using log properties we have:

$$\ln y = x \ln x.$$

Differentiating implicitly gives:

$$\frac{y'}{y} = (1)\ln x + x\frac{1}{x}.$$

$$\frac{y'}{y} = \ln x + 1.$$

Solving for y' gives:

$$y' = y(1 + \ln x).$$

Replace $y = x^x$ gives:

$$y' = x^x(1 + \ln x).$$

Another method to find this derivative is as follows:

$$\begin{aligned} \frac{d}{dx}x^x &= \frac{d}{dx}e^{x\ln x} \\ &= \left(\frac{d}{dx}x\ln x\right)e^{x\ln x} \\ &= (x\frac{1}{x} + \ln x)x^x \\ &= (1 + \ln x)x^x \end{aligned}$$

♣

In fact, logarithmic differentiation can be used on more complicated products and quotients (not just when dealing with functions to the power of functions).

Example 4.35: Logarithmic Differentiation

Differentiate (assuming $x > 0$):

$$y = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}.$$

Solution. Using product & quotient rules for this problem is a complete nightmare! Let's apply logarithmic differentiation instead. Take ln of both sides:

$$\ln y = \ln\left(\frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}\right).$$

Applying log properties:

$$\ln y = \ln\left((x+2)^3(2x+1)^9\right) - \ln\left(x^8(3x+1)^4\right).$$

$$\ln y = \ln\left((x+2)^3\right) + \ln\left((2x+1)^9\right) - \left[\ln\left(x^8\right) + \ln\left((3x+1)^4\right)\right].$$

$$\ln y = 3\ln(x+2) + 9\ln(2x+1) - 8\ln x - 4\ln(3x+1).$$

Now, differentiating implicitly with respect to x gives:

$$\frac{y'}{y} = \frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}.$$

Solving for y' gives:

$$y' = y\left(\frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}\right).$$

Replace $y = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}$ gives:

$$y' = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}\left(\frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}\right).$$

♣

Exercises for Section 4.7

4.7.1 *Find a formula for the derivative y' at the point (x, y):*

(a) $y^2 = 1 + x^2$

(b) $x^2 + xy + y^2 = 7$

(c) $x^3 + xy^2 = y^3 + yx^2$

(d) $4\cos x \sin y = 1$

(e) $\sqrt{x} + \sqrt{y} = 9$

(f) $\tan(x/y) = x + y$

(g) $\sin(x + y) = xy$

(h) $\frac{1}{x} + \frac{1}{y} = 7$

4.7.2 *A hyperbola passing through $(8, 6)$ consists of all points whose distance from the origin is a constant more than its distance from the point (5,2). Find the slope of the tangent line to the hyperbola at $(8, 6)$.*

4.7.3 *The graph of the equation $x^2 - xy + y^2 = 9$ is an ellipse. Find the lines tangent to this curve at the two points where it intersects the x-axis. Show that these lines are parallel.*

4.7.4 *Repeat the previous problem for the points at which the ellipse intersects the y-axis.*

4.7.5 *If $y = \log_a x$ then $a^y = x$. Use implicit differentiation to find y'.*

4.7.6 *Find the points on the ellipse from the previous two problems where the slope is horizontal and where it is vertical.*

4.7.7 *Find an equation for the tangent line to $x^4 = y^2 + x^2$ at $(2, \sqrt{12})$. (This curve is the* **kampyle of Eudoxus**.*)*

4.7.8 *Find an equation for the tangent line to $x^{2/3} + y^{2/3} = a^{2/3}$ at a point (x_1, y_1) on the curve, with $x_1 \neq 0$ and $y_1 \neq 0$. (This curve is an* **astroid**.*)*

4.7.9 *Find an equation for the tangent line to $(x^2 + y^2)^2 = x^2 - y^2$ at a point (x_1, y_1) on the curve, with $x_1 \neq 0, -1, 1$. (This curve is a* **lemniscate**.*)*

4.7.10 *Two curves are* **orthogonal** *if at each point of intersection, the angle between their tangent lines is $\pi/2$. Two families of curves, $\mathcal{A}$ and $\mathcal{B}$, are* **orthogonal trajectories** *of each other if given any curve C in $\mathcal{A}$ and any curve D in $\mathcal{B}$ the curves C and D are orthogonal. For example, the family of horizontal lines in the plane is orthogonal to the family of vertical lines in the plane.*

(a) *Show that $x^2 - y^2 = 5$ is orthogonal to $4x^2 + 9y^2 = 72$. (Hint: You need to find the intersection points of the two curves and then show that the product of the derivatives at each intersection point is -1.)*

(b) *Show that $x^2 + y^2 = r^2$ is orthogonal to $y = mx$. Conclude that the family of circles centered at the origin is an orthogonal trajectory of the family of lines that pass through the origin.*

Note that there is a technical issue when $m = 0$. The circles fail to be differentiable when they cross the x-axis. However, the circles are orthogonal to the x-axis. Explain why. Likewise, the vertical line through the origin requires a separate argument.

(c) *For $k \neq 0$ and $c \neq 0$ show that $y^2 - x^2 = k$ is orthogonal to $yx = c$. In the case where k and c are both zero, the curves intersect at the origin. Are the curves $y^2 - x^2 = 0$ and $yx = 0$ orthogonal to each other?*

(d) *Suppose that $m \neq 0$. Show that the family of curves $\{y = mx + b \mid b \in \mathbb{R}\}$ is orthogonal to the family of curves $\{y = -(x/m) + c \mid c \in \mathbb{R}\}$.*

4.7.11 *Differentiate the function* $y = \dfrac{(x-1)^8(x-23)^{1/2}}{27x^6(4x-6)^8}$

4.7.12 *Differentiate the function $f(x) = (x+1)^{\sin x}$.*

4.7.13 *Differentiate the function* $g(x) = \dfrac{e^x(\cos x + 2)^3}{\sqrt{x^2+4}}$.

4.8 Derivatives of Inverse Functions

Suppose we wanted to find the *derivative of the inverse*, but do not have an actual formula for the inverse function? Then we can use the following derivative formula for the inverse evaluated at a.

Theorem 4.10: Derivatives of Inverse Functions

Let f be differentiable and one to one on an open interval I, where $f'(x) \neq 0$ for all x in I, let J be the range of f on I, let g be the inverse function of f, and let $f(a) = b$ for some a in I. Then g is a differentiable function on J, and in particular,

1. $\left(f^{-1}\right)'(b) = g'(b) = \dfrac{1}{f'(a)}$ *and* *2.* $\left(f^{-1}\right)'(x) = g'(x) = \dfrac{1}{f'(g(x))}$

To see why this is true, start with the function $y = f^{-1}(x)$. Write this as $x = f(y)$ and differentiate both sides implicitly with respect to x using the chain rule:

$$1 = f'(y) \cdot \frac{dy}{dx}.$$

Thus,

$$\frac{dy}{dx} = \frac{1}{f'(y)},$$

but $y = f^{-1}(x)$, thus,

$$\left[f^{-1}\right]'(x) = \frac{1}{f'\left[f^{-1}(x)\right]}.$$

At the point $x = a$ this becomes:

$$\left[f^{-1}\right]'(a) = \frac{1}{f'\left[f^{-1}(a)\right]}$$

In Section 4.6, we saw that $\dfrac{d}{dx}\big(\ln x\big) = \dfrac{1}{x}$. We can justify that now using Theorem 4.10, as shown in the example.

Example 4.36: Finding the derivative of $y = \ln x$

Use Theorem 4.10 to compute $\dfrac{d}{dx}\big(\ln x\big)$. View $y = \ln x$ as the inverse of $y = e^x$. Therefore, using our standard notation, let $f(x) = e^x$ and $g(x) = \ln x$. We wish to find $g'(x)$. Theorem 4.10 gives:

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{e^{\ln x}} \\ &= \frac{1}{x}. \end{aligned}$$

Example 4.37: Derivatives of Inverse Functions

Suppose $f(x) = x^5 + 2x^3 + 7x + 1$. *Find* $\left[f^{-1}\right]'(1)$.

Solution. First we should show that f^{-1} exists (i.e. that f is one-to-one). In this case the derivative $f'(x) = 5x^4 + 6x^2 + 7$ is strictly greater than 0 for all x, so f is strictly increasing and thus one-to-one.

It's difficult to find the inverse of $f(x)$ (and then take the derivative). Thus, we use the above formula evaluated at 1:

$$\left[f^{-1}\right]'(1) = \frac{1}{f'\left[f^{-1}(1)\right]}.$$

Note that to use this formula we need to know what $f^{-1}(1)$ is, and the derivative $f'(x)$. To find $f^{-1}(1)$ we make a table of values (plugging in $x = -3, -2, -1, 0, 1, 2, 3$ into $f(x)$) and see what value of x gives 1. We omit the table and simply observe that $f(0) = 1$. Thus,

$$f^{-1}(1) = 0.$$

Now we have:

$$\left[f^{-1}\right]'(1) = \frac{1}{f'(0)}.$$

And so, $f'(0) = 7$. Therefore,

$$\left[f^{-1}\right]'(1) = \frac{1}{7}.$$

♣

4.8.1. Derivatives of Inverse Trigonometric Functions

We can apply the technique used to find the derivative of f^{-1} above to find the derivatives of the inverse trigonometric functions.

In the following examples we will derive the formulae for the derivative of the inverse sine, inverse cosine and inverse tangent. The other three inverse trigonometric functions have been left as exercises at the end of this section.

Example 4.38: Derivative of Inverse Sine

Find the derivative of $\sin^{-1}(x)$.

Solution. Adopting the notation in Theorem 4.10, let $g(x) = \arcsin x$ and $f(x) = \sin x$. Thus $f'(x) = \cos x$. Applying the theorem, we have

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{\cos(\arcsin x)}. \end{aligned}$$

Alternatively, we could use the technique in the justification of Theorem 4.10. Write $y = \sin^{-1}(x)$, so $x = \sin(y)$ and $-\pi/2 \leq y \leq \pi/2$, and differentiate both sides with respect to x using the chain rule.

$$\frac{d}{dx}x = \frac{d}{dx}\sin(y)$$

$$\begin{aligned}
1 &= \cos(y)\frac{dy}{dx}\\
1 &= \cos\left(\sin^{-1}(x)\right)\frac{dy}{dx}\\
\frac{dy}{dx} &= \frac{1}{\cos\left(\sin^{-1}(x)\right)}
\end{aligned}$$

Although correct, this formula is cumbersome to use, and hardly enlightening. It can be simplified significantly with a bit of trigonometry. We have $y = \sin^{-1}(x)$, so $\sin(\theta) = x$, and we wish to find $\cos(y)$ in order to simplify our expression above. Construct a right angle triangle with angle y, opposite side length x and hypotenuse 1. The Pythagorean Theorem gives an adjacent side length of $\sqrt{1-x^2}$.

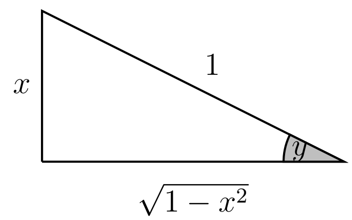


Figure 4.5: A right triangle defined by $y = \sin^{-1}(\frac{x}{1})$ with the length of the third leg found using the Pythagorean Theorem.

So, reading from the triangle we have $\cos\left(\sin^{-1}(x)\right) = \cos(y) = \sqrt{1-x^2}$. Note that we choose the non-negative square root $\sqrt{1-x^2}$ since $\cos(\theta) \geq 0$ when $-\pi/2 \leq \theta \leq \pi/2$ (the range of $\arcsin(x)$).

Finally, the derivative of inverse sine is

$$\left(\sin^{-1}(x)\right)' = \frac{1}{\sqrt{1-x^2}}$$

♣

Example 4.39: Derivative of Inverse Cosine

Find the derivative of $\cos^{-1}(x)$.

Solution. Let $y = \cos^{-1}(x)$, so $\cos(y) = x$ and $0 \leq y \leq \pi$. Next we differentiate implicitly:

$$\begin{aligned}
\frac{d}{dx}(\cos y) &= \frac{d}{dx}(x)\\
-\sin y \cdot \frac{dy}{dx} &= 1\\
\frac{dy}{dx} &= -\frac{1}{\sin y}
\end{aligned}$$

Since $\cos y = x$, we construct a triangle with angle y, adjacent side length x and hypotenuse 1.

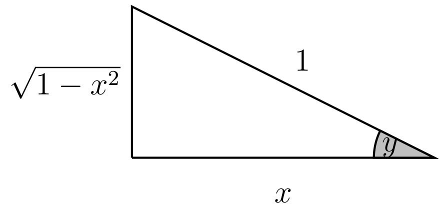


Figure 4.6: A right triangle defined by $y = \cos^{-1}(\frac{x}{1})$ with the length of the third leg found using the Pythagorean Theorem.

Solving for the opposite side length using Pythagorean Theorem we obtain $\sqrt{1-x^2}$. Using this triangle we can see that $\sin y = \sqrt{1-x^2}$ $(0 \le y \le \pi)$. Substituting this into the equation for dy/dx, we find that

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$$

. ♣

In the following example we explore an alternate method of finding the derivative.

Example 4.40: Derivative of Inverse Tangent

Find the derivative of $\tan^{-1}(x)$.

Solution. We begin with $\tan\left(\tan^{-1}(x)\right) = x$. Taking the derivative using the Chain Rule we obtain

$$\sec^2\left(\tan^{-1}(x)\right) \cdot \frac{d}{dx}\left(\tan^{-1}(x)\right) = 1,$$

which we rearrange to obtain

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{\sec^2\left(\tan^{-1}(x)\right)}.$$

Let $\tan^{-1}(x) = \theta$, then $\tan(\theta) = x$. We construct a triangle with angle θ, adjacent side 1 and opposite side x. The hypotenuse is $\sqrt{1+x^2}$ using Pythagorean theorem.

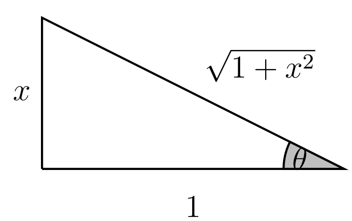


Figure 4.7: A right triangle defined by $\theta = \tan^{-1}(\frac{x}{1})$ with the length of the third leg found using the Pythagorean Theorem.

Then $\sec^2\left(\tan^{-1}(x)\right) = \sec^2(\theta) = (\sec(\theta))^2 = \left(\sqrt{1+x^2}\right)^2 = 1 + x^2$. Recall that $\sec(x) = 1/\cos(x)$. Finally, the derivative is

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}.$$

♣

Using similar techniques, we can find the derivatives of the remaining inverse trigonometric functions.

Theorem 4.11: Derivatives of Inverse Trigonometric Functions

The inverse trigonometric functions are differentiable on all open sets contained in their domains (as listed in Figure 2.4) and their derivatives are as follows:

1. $\dfrac{d}{dx}\big(\sin^{-1}(x)\big) = \dfrac{1}{\sqrt{1-x^2}}$
2. $\dfrac{d}{dx}\big(\sec^{-1}(x)\big) = \dfrac{1}{x\sqrt{x^2-1}}$
3. $\dfrac{d}{dx}\big(\tan^{-1}(x)\big) = \dfrac{1}{1+x^2}$
4. $\dfrac{d}{dx}\big(\cos^{-1}(x)\big) = -\dfrac{1}{\sqrt{1-x^2}}$
5. $\dfrac{d}{dx}\big(\csc^{-1}(x)\big) = -\dfrac{1}{x\sqrt{x^2-1}}$
6. $\dfrac{d}{dx}\big(\cot^{-1}(x)\big) = -\dfrac{1}{1+x^2}$

Note how the last three derivatives are merely the opposites of the first three, respectively. Because of this, the first three are used almost exclusively throughout this text.

4.8.2. Glossary of Derivatives of Elementary Functions

In this chapter we have defined the derivative, given rules to facilitate its computation, and given the derivatives of a number of standard functions. We restate the most important of these in the following theorem, intended to be a reference for further work.

Theorem 4.12: Glossary of Derivatives of Elementary Functions

Let u and v be differentiable functions, and let a, c and n be real numbers, $a > 0$, $n \neq 0$.

1. $\frac{d}{dx}\big(cu\big) = cu'$
2. $\frac{d}{dx}\big(u \pm v\big) = u' \pm v'$
3. $\frac{d}{dx}\big(u \cdot v\big) = uv' + u'v$
4. $\frac{d}{dx}\big(\frac{u}{v}\big) = \frac{u'v - uv'}{v^2}$
5. $\frac{d}{dx}\big(u(v)\big) = u'(v)v'$
6. $\frac{d}{dx}\big(c\big) = 0$
7. $\frac{d}{dx}\big(x\big) = 1$
8. $\frac{d}{dx}\big(x^n\big) = nx^{n-1}$
9. $\frac{d}{dx}\big(e^x\big) = e^x$
10. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$
11. $\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$
12. $\frac{d}{dx}\big(\log_a x\big) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}\big(\sin x\big) = \cos x$
14. $\frac{d}{dx}\big(\cos x\big) = -\sin x$
15. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$
16. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$
17. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$
18. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$
19. $\frac{d}{dx}\big(\sin^{-1} x\big) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\big(\cos^{-1} x\big) = -\frac{1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\big(\csc^{-1} x\big) = -\frac{1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\big(\sec^{-1} x\big) = \frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\big(\tan^{-1} x\big) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\big(\cot^{-1} x\big) = -\frac{1}{1+x^2}$

Exercises for 4.8

4.8.1 *Given $f(x) = 1 + \ln(x-2)$, first show that f^{-1} exists, then compute $\left[f^{-1}\right]'(1)$.*

4.8.2 *The* **inverse cotangent function**, *denoted by $\cot^{-1}(x)$, is defined to be the inverse of the restricted cotangent function: $\cot(x)$, $0 < x < \pi$. Find the derivative of $\cot^{-1}(x)$.*

4.8.3 *The* **inverse secant function**, *denoted by $\sec^{-1}(x)$, is defined to be the inverse of the restricted secant function: $\sec(x)$, $x \in [0, \pi/2) \cup [\pi, 3\pi/2)$. Find the derivative of $\sec^{-1}(x)$.*

4.8.4 *The* **inverse cosecant function**, *denoted by $\csc^{-1}(x)$, is defined to be the inverse of the restricted cosecant function: $\csc(x)$, $x \in (0, \pi/2] \cup (\pi, 3\pi/2]$. Find the derivative of $\csc^{-1}(x)$.*

4.8.5 *Suppose $f(x) = x^3 + 4x + 2$. Find the slope of the tangent line to the graph of $g(x) = xf^{-1}(x)$ at the point where $x = 7$.*

4.8.6 *Find the derivatives of $\sin^{-1}(x) + \cos^{-1}(x)$ and $(x^2+1)\tan^{-1}(x)$.*

4.8.7 *Differentiate $y = \sin^{-1}(x^2)$ and $y = \tan^{-1}(3x)$.*

4.9 Additional Exercises

4.9.1 *Find the derivatives of the following functions from definition.*

(a) $f(x) = (2x+3)^2$

(b) $g(x) = x^{3/2}$

4.9.2 *Let* $f(x) = \begin{cases} x^3 & \text{if } x \leq 1 \\ 5x - x^2 & \text{if } x > 1 \end{cases}$. *Use the definition of the derivative to find* $f'(1)$.

4.9.3 *Differentiate the following functions.*

(a) $y = 7x^4 - 7\pi^4 + \dfrac{1}{\pi \sqrt[3]{x}}$

(b) $f(x) = \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$

(c) $f(x) = |x-1| + |x+2|$

(d) $f(x) = x^2 \sin x \cos x$

(e) $y = \dfrac{x \sin x}{1 + \sin x}$

(f) $g(x) = \sqrt{2 + \dfrac{3}{\sqrt{x}}}$

(g) $y = \sqrt[3]{x^4 + x^2 + 1} + \dfrac{1}{(x^3 - x + 4)^5}$

(h) $y = \sin^3 x - \sin(x^3)$

(i) $F(x) = \sec^4 x + \tan^4 x$

(j) $y = \cos^2\left(\dfrac{1-x}{1+x}\right)$

(k) $y = \tan(\sin(x^2 + \sec^2 x))$

(l) $y = \dfrac{1}{2 + \sin \frac{\pi}{x}}$

4.9.4 *Differentiate the following functions.*

(a) $y = e^{3x} + e^{-x} + e^2$

(b) $y = e^{2x} \cos 3x$

(c) $f(x) = \tan(x + e^x)$

(d) $g(x) = \dfrac{e^x}{e^x + 2}$

(e) $y = \ln(2 + \sin x) - \sin(2 + \ln x)$

(f) $f(x) = e^{x^\pi} + x^{\pi^e} + \pi^{e^x}$

(g) $y = \log_a(b^x) + b^{\log_a x}$, *where a and b are positive real numbers and $a \neq 1$*

(h) $y = (x^2 + 1)^{x^3+1}$

(i) $y = (x^2 + e^x)^{1/\ln x}$

(j) $y = \dfrac{x\sqrt{x^2 + x + 1}}{(2 + \sin x)^4(3x + 5)^7}$

4.9.5 *Find* $\dfrac{dy}{dx}$ *if y is a differentiable function that satisfy the given equation.*

(a) $x^2 + xy + y^2 = 7$

(b) $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$

(c) $x^2 \sin y + y^3 = \cos x$

(d) $x^2 + xe^y = 2y + e^x$

4.9.6 *Differentiate the following functions.*

(a) $y = x \sin^{-1} x$

(b) $f(x) = \dfrac{\sin^{-1} x}{\cos^{-1} x}$

(c) $g(x) = \tan^{-1}\left(\dfrac{x}{a}\right)$, *where $a > 0$*

(d) $y = x \tan^{-1} x - \dfrac{1}{2}\ln(x^2 + 1)$

5. Applications of Derivatives

In this chapter we explore how to use derivative and differentiation to solve a variety of problems, some mathematical and some practical. We explore some applications which motivated and were formalized in the definition of the derivative, and look at a few clever uses of the tangent line (which has immediate geometric ties to the definition of the derivative).

5.1 Related Rates

When defining the derivative $f'(x)$, we define it to be exactly the rate of change of $f(x)$ with respect to x. Consequently, any question about rates of change can be rephrased as a question about derivatives. **When we calculate derivatives, we are calculating rates of change**. Results and answers we obtain for derivatives translate directly into results and answers about rates of change. Let us look at some examples where more than one variable is involved, and where our job is to analyze and exploit relations between the rates of change of these variables. The mathematical step of relating the rates of change turns out to be largely an exercise in differentiation using the chain rule or implicit differentiation. This explains why some textbooks place this section shortly after the sections on the chain rule and implicit differentiation.

Suppose we have two variables x and y (in most problems the letters will be different, but for now let's use x and y) which are both changing with time. A "related rates" problem is a problem in which we know one of the rates of change at a given instant—say, $\dot{x} = dx/dt$—and we want to find the other rate $\dot{y} = dy/dt$ at that instant. (The use of $\dot{x}$ to mean dx/dt goes back to Newton and is still used for this purpose, especially by physicists.)

If y is written in terms of x, i.e., $y = f(x)$, then this is easy to do using the chain rule:

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx}\dot{x}.$$

That is, find the derivative of $f(x)$, plug in the value of x at the instant in question, and multiply by the given value of $\dot{x} = dx/dt$ to get $\dot{y} = dy/dt$.

Example 5.1: Speed at which a Coordinate is Changing

Suppose an object is moving along a path described by $y = x^2$, that is, it is moving on a parabolic path. At a particular time, say $t = 5$, the x coordinate is 6 and we measure the speed at which the x coordinate of the object is changing and find that $dx/dt = 3$.
At the same time, how fast is the y coordinate changing?

Solution. Using the chain rule, $dy/dt = 2x \cdot dx/dt$. At $t = 5$ we know that $x = 6$ and $dx/dt = 3$, so $dy/dt = 2 \cdot 6 \cdot 3 = 36$. ♣

In many cases, particularly interesting ones, x and y will be related in some other way, for example $x = f(y)$, or $F(x, y) = k$, or perhaps $F(x, y) = G(x, y)$, where $F(x, y)$ and $G(x, y)$ are expressions involving both variables. In all cases, you can solve the related rates problem by taking the derivative of both sides, plugging in all the known values (namely, x, y, and $\dot{x}$), and then solving for $\dot{y}$.

To summarize, here are the steps in doing a related rates problem.

Key Idea 5.1.0: Steps for Solving Related Rates Problems

1. Decide what the two variables are.
2. Find an equation relating them.
3. Take d/dt of both sides.
4. Plug in all known values at the instant in question.
5. Solve for the unknown rate.

Example 5.2: Receding Airplanes

A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

Solution. To see what's going on, we first draw a schematic representation of the situation, as in Figure 5.1. Because the plane is in level flight directly away from you, the rate at which x changes is the speed of the plane, $dx/dt = 500$. The distance between you and the plane is y; it is dy/dt that we wish to know. By the Pythagorean Theorem we know that $x^2 + 9 = y^2$. Taking the derivative:

$$2x\dot{x} = 2y\dot{y}.$$

We are interested in the time at which $x = 4$; at this time we know that $4^2 + 9 = y^2$, so $y = 5$. Putting together all the information we get

$$2(4)(500) = 2(5)\dot{y}.$$

Thus, $\dot{y} = 400$ mph. ♣

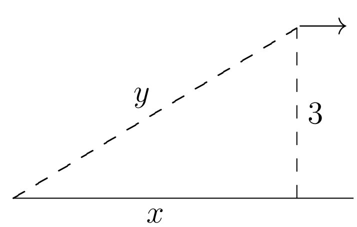


Figure 5.1: Receding airplane.

Example 5.3: Spherical Balloon

You are inflating a spherical balloon at the rate of 7 cm^3/sec. How fast is its radius increasing when the radius is 4 cm?

Solution. Here the variables are the radius r and the volume V. We know dV/dt, and we want dr/dt. The two variables are related by the equation $V = 4\pi r^3/3$. Taking the derivative of both sides gives $dV/dt = 4\pi r^2 \dot{r}$. We now substitute the values we know at the instant in question: $7 = 4\pi 4^2 \dot{r}$, so $\dot{r} = 7/(64\pi)$ cm/sec. ♣

Example 5.4: Conical Container

Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm; see Figure 5.2. How fast is the water level rising when the water is 4 cm deep (at its deepest point)?

Solution. The water forms a conical shape within the big cone; its height and base radius and volume are all increasing as water is poured into the container. This means that we actually have three things varying with time: the water level h (the height of the cone of water), the radius r of the circular top surface of water (the base radius of the cone of water), and the volume of water V. The volume of a cone is given by $V = \pi r^2 h/3$. We know dV/dt, and we want dh/dt. At first something seems to be wrong: we have a third variable, r, whose rate we don't know.

However, the dimensions of the cone of water must have the same proportions as those of the container. That is, because of similar triangles, $r/h = 10/30$ so $r = h/3$. Now we can eliminate r from the problem entirely: $V = \pi(h/3)^2 h/3 = \pi h^3/27$. We take the derivative of both sides and plug in $h = 4$ and $dV/dt = 10$, obtaining $10 = (3\pi \cdot 4^2/27)(dh/dt)$. Thus, $dh/dt = 90/(16\pi)$ cm/sec. ♣

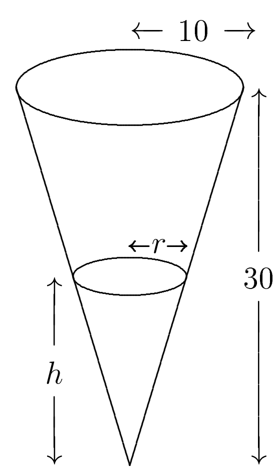


Figure 5.2: Conical water tank.

Example 5.5: Swing Set

A swing consists of a board at the end of a 10 ft long rope. Think of the board as a point P at the end of the rope, and let Q be the point of attachment at the other end. Suppose that the swing is directly below Q at time $t = 0$, and is being pushed by someone who walks at 6 ft/sec from left to right.
Find (a) how fast the swing is rising after 1 sec; (b) the angular speed of the rope in deg/sec after 1 sec.

Solution. We start out by asking: What is the geometric quantity whose rate of change we know, and what is the geometric quantity whose rate of change we're being asked about? Note that the person pushing the swing is moving horizontally at a rate we know. In other words, the horizontal coordinate of P is increasing at 6 ft/sec. In the xy-plane let us make the convenient choice of putting the origin at the location of P at time $t = 0$, i.e., a distance 10 directly below the point of attachment. Then the rate we know is dx/dt, and in part (a) the rate we want is dy/dt (the rate at which P is rising). In part (b) the rate we want is $\dot{\theta} = d\theta/dt$, where θ stands for the angle in radians through which the swing has swung from the vertical. (Actually, since we want our answer in deg/sec, at the end we must convert $d\theta/dt$ from rad/sec by multiplying by $180/\pi$.)

(a) From the diagram we see that we have a right triangle whose legs are x and $10-y$, and whose hypotenuse is 10. Hence $x^2+(10-y)^2 = 100$. Taking the derivative of both sides we obtain: $2x\dot{x}+2(10-y)(0-\dot{y}) = 0$. We now look at what we know after 1 second, namely $x = 6$ (because x started at 0 and has been increasing at the rate of 6 ft/sec for 1 sec), thus $y = 2$ (because we get $10-y = 8$ from the Pythagorean theorem applied to the triangle with hypotenuse 10 and leg 6), and $\dot{x} = 6$. Putting in these values gives us $2\cdot 6\cdot 6-2\cdot 8\dot{y} = 0$, from which we can easily solve for $\dot{y}$: $\dot{y} = 4.5$ ft/sec.

(b) Here our two variables are x and θ, so we want to use the same right triangle as in part (a), but this time relate θ to x. Since the hypotenuse is constant (equal to 10), the best way to do this is to use the sine: $\sin\theta = x/10$. Taking derivatives we obtain $(\cos\theta)\dot{\theta} = 0.1\dot{x}$. At the instant in question ($t = 1$ sec), when we have a right triangle with sides 6–8–10, $\cos\theta = 8/10$ and $\dot{x} = 6$. Thus $(8/10)\dot{\theta} = 6/10$, i.e., $\dot{\theta} = 6/8 = 3/4$ rad/sec, or approximately 43 deg/sec. ♣

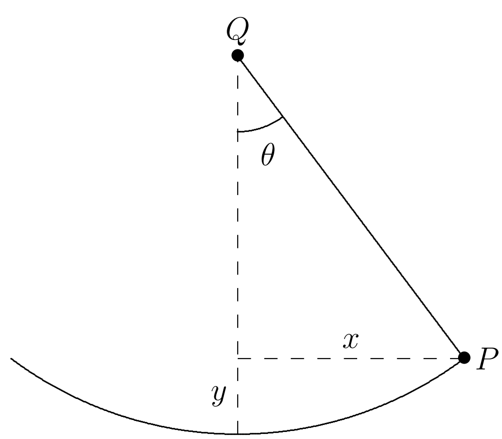


Figure 5.3: Swing.

We have seen that sometimes there are apparently more than two variables that change with time, but in reality there are just two, as the others can be expressed in terms of just two. However sometimes there really are several variables that change with time; as long as you know the rates of change of all but one of them you can find the rate of change of the remaining one. As in the case when there are just two variables, take the derivative of both sides of the equation relating all of the variables, and then substitute all of the known values and solve for the unknown rate.

Example 5.6: Distance Changing Rate

A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometers to the north of P and traveling at 80 km/hr, while car B is 15 kilometers to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

Solution. Let $a(t)$ be the distance of car A north of P at time t, and $b(t)$ the distance of car B east of P at time t, and let $c(t)$ be the distance from car A to car B at time t. By the Pythagorean Theorem, $c(t)^2 = a(t)^2 + b(t)^2$. Taking derivatives we get $2c(t)c'(t) = 2a(t)a'(t) + 2b(t)b'(t)$, so

$$\dot{c} = \frac{a\dot{a} + b\dot{b}}{c} = \frac{a\dot{a} + b\dot{b}}{\sqrt{a^2+b^2}}.$$

Substituting known values we get:

$$\dot{c} = \frac{10 \cdot 80 + 15 \cdot 100}{\sqrt{10^2+15^2}} = \frac{460}{\sqrt{13}} \approx 127.6\text{km/hr}$$

at the time of interest. ♣

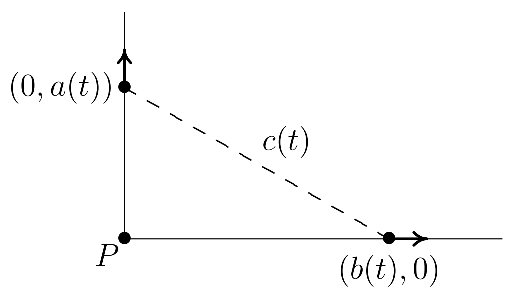


Figure 5.4: Cars moving apart.

Notice how this problem differs from Example 5.2. In both cases we started with the Pythagorean Theorem and took derivatives on both sides. However, in Example 5.2 one of the sides was a constant (the altitude of the plane), and so the derivative of the square of that side of the triangle was simply zero. In this Example, on the other hand, all three sides of the right triangle are variables, even though we are interested in a specific value of each side of the triangle (namely, when the sides have lengths 10 and 15). Make sure that you understand at the start of the problem what are the variables and what are the constants.

Exercises for Section 5.1

5.1.1 *Air is being pumped into a spherical balloon at a constant rate of 3 cm^3/s. How fast is the radius of the balloon increasing when the radius reaches 5cm?*

5.1.2 *A cylindrical tank standing upright (with one circular base on the ground) has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 25* cm^3*/sec?*

5.1.3 *A cylindrical tank standing upright (with one circular base on the ground) has radius 1 meter. How fast does the water level in the tank drop when the water is being drained at 3 liters per second?*

5.1.4 *A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall?*

5.1.5 *A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The top of the ladder is being pulled up the wall at* 0.1 *meters per second. How fast is the foot of the ladder approaching the wall when the foot of the ladder is 5 m from the wall?*

5.1.6 *A rotating beacon is located 2 miles out in the water. Let* A *be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point where the beam hits the shore moving at an instant when the beam is lighting up a point 2 miles along the shore from the point* A*?*

5.1.7 *A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 15 ft/sec. At what rate is the player's distance from third base decreasing when she is half way from first to second base?*

5.1.8 *Sand is poured onto a surface at 15* cm^3*/sec, forming a conical pile whose base diameter is always equal to its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high?*

5.1.9 *A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 5 ft higher than the front of the boat. The rope is being pulled through the ring at the rate of 0.6 ft/sec. How fast is the boat approaching the dock when 13 ft of rope are out?*

5.1.10 *A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. A bicyclist passes beneath it, traveling in a straight line at the constant speed of 10 m/sec. How fast is the distance between the bicyclist and the balloon increasing 2 seconds later?*

5.1.11 *A pyramid-shaped vat has square cross-section and stands on its tip. The dimensions at the top are 2 m* $\times$ *2 m, and the depth is 5 m. If water is flowing into the vat at 3* m^3*/min, how fast is the water level rising when the depth of water (at the deepest point) is 4 m? Note: the volume of any "conical" shape (including pyramids) is* (1/3)(*height*)(*area of base*).

5.1.12 *A woman 5 ft tall walks at the rate of 3.5 ft/sec away from a streetlight that is 12 ft above the ground. At what rate is the tip of her shadow moving? At what rate is her shadow lengthening?*

5.1.13 *A man 1.8 meters tall walks at the rate of 1 meter per second toward a streetlight that is 4 meters above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow shortening?*

5.1.14 *A police helicopter is flying at 150 mph at a constant altitude of 0.5 mile above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 mile from the helicopter, and that this distance is decreasing at 190 mph. Find the speed of the car.*

5.1.15 *A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 2 kilometers from the helicopter, and that this distance is decreasing at 250 kph. Find the speed of the car.*

5.1.16 *A light shines from the top of a pole 20 m high. An object is dropped from the same height from a point 10 m away, so that its height at time t seconds is* $h(t) = 20 - 9.8t^2/2$. *How fast is the object's shadow moving on the ground one second later?*

5.2 Extrema of a Function

In calculus, there is much emphasis placed on analyzing the behaviour of a function f on an interval I. Does f have a maximum value on I? Does it have a minimum value? How does the interval I impact our discussion of extrema?

5.2.1. Local Extrema

A **local maximum** point on a function is a point (x, y) on the graph of the function whose y coordinate is larger than all other y coordinates on the graph at points "close to" (x, y). More precisely, $(x, f(x))$ is a local maximum if there is an interval (a, b) with $a < x < b$ and $f(x) \geq f(z)$ for every z in (a, b). Similarly, (x, y) is a **local minimum** point if it has locally the smallest y coordinate. Again being more precise: $(x, f(x))$ is a local minimum if there is an interval (a, b) with $a < x < b$ and $f(x) \leq f(z)$ for every z in (a, b). A **local extremum** is either a local minimum or a local maximum.

Definition 5.1: Local Maxima and Minima

A real-valued function f *has a* **local maximum** *at* x_0 *if* $f(x_0)$ *is the largest value of* f *near* x_0*; in other words,* $f(x_0 \geq f(x)$ *when* x *is near* x_0.

A real-valued function f *has a* **local minimum** *at* x_0 *if* $f(x_0)$ *is the smallest value of* f *near* x_0*; in other words,* $f(x_0) \leq f(x)$ *when* x *is near* x_0.

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well. Some examples of local maximum and minimum points are shown in Figure 5.5.

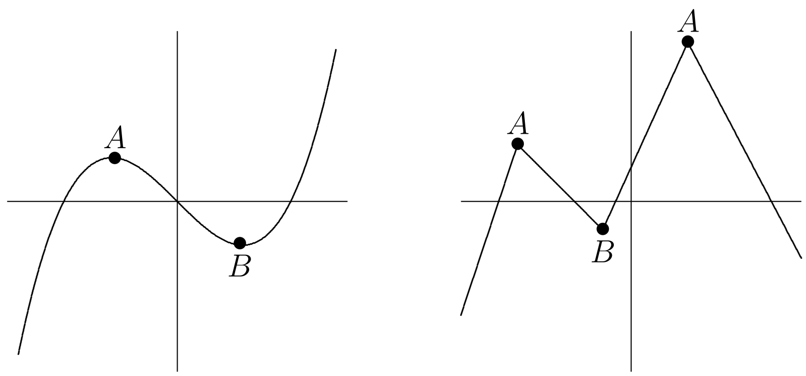


Figure 5.5: Some local maximum points (A) and minimum points (B).

If $(x, f(x))$ is a point where $f(x)$ reaches a local maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line *must* be horizontal. This is important enough to state as a theorem.

The proof is simple enough and we include it here, but you may accept Fermat's Theorem based on its strong intuitive appeal and come back to its proof at a later time.

Theorem 5.1: Fermat's Theorem

If $f(x)$ has a local extremum at $x = a$ and f is differentiable at a, then $f'(a) = 0$.

Proof. We shall give the proof for the case where $f(x)$ has a local maximum at $x = a$. The proof for the local minimum case is similar.

Since $f(x)$ has a local maximum at $x = a$, there is an open interval (c, d) with $c < a < d$ and $f(x) \leq f(a)$ for every x in (c, d). So, $f(x) - f(a) \leq 0$ for all such x. Let us now look at the sign of the difference quotient $\dfrac{f(x) - f(a)}{x - a}$. We consider two cases according as $x > a$ or $x < a$.

If $x > a$, then $x - a > 0$ and so, $\dfrac{f(x) - f(a)}{x - a} \leq 0$. Taking limit as x approach a from the right, we get

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leq 0.$$

On the other hand, if $x < a$, then $x - a < 0$ and so, $\dfrac{f(x) - f(a)}{x - a} \geq 0$. Taking limit as x approach a from the left, we get

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \geq 0.$$

Since f is differentiable at a, $f'(a) = \displaystyle\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$. Therefore, we have both $f'(a) \leq 0$ and $f'(a) \geq 0$. So, $f'(a) = 0$. ♣

Thus, the only points at which a function can have a local maximum or minimum are points at which the derivative is zero, as in the left hand graph in Figure 5.5, or the derivative is undefined, as in the right hand graph. Any value of x in the domain of f for which $f'(x)$ is zero or undefined is called a **critical point** for f. When looking for local maximum and minimum points, you are likely to make two sorts of mistakes: You may forget that a maximum or minimum can occur where the derivative does not exist, and so forget to check whether the derivative exists everywhere. You might also assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true. A portion of the graph of $f(x) = x^3$ is shown in Figure 5.6. The derivative of f is $f'(x) = 3x^2$, and $f'(0) = 0$, but there is neither a maximum nor minimum at $(0, 0)$.

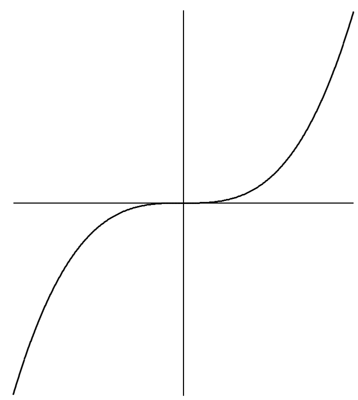


Figure 5.6: No maximum or minimum even though the derivative is zero.

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach, but one that is often tedious or difficult, is to test directly whether the y coordinates "near" the potential maximum or minimum are above or below the y coordinate at the point of interest. Of course, there are too many points "near" the point to test, but a little thought shows we need only test two provided we know that f is continuous (recall that this means that the graph of f has no jumps or gaps).

Suppose, for example, that we have identified three points at which f' is zero or nonexistent: (x_1, y_1), (x_2, y_2), (x_3, y_3), and $x_1 < x_2 < x_3$ (see Figure 5.7). Suppose that we compute the value of $f(a)$ for $x_1 < a < x_2$, and that $f(a) < f(x_2)$. What can we say about the graph between a and x_2? Could there be a point $(b, f(b))$, $a < b < x_2$ with $f(b) > f(x_2)$? No: if there were, the graph would go up from $(a, f(a))$ to $(b, f(b))$ then down to $(x_2, f(x_2))$ and somewhere in between would have a local maximum point. (This is not obvious; it is a result of the Extreme Value Theorem.) But at that local maximum point the derivative of f would be zero or nonexistent, yet we already know that the derivative is zero or nonexistent only at x_1, x_2, and x_3. The upshot is that one computation tells us that $(x_2, f(x_2))$ has the largest y coordinate of any point on the graph near x_2 and to the left of x_2. We can perform the same test on the right. If we find that on both sides of x_2 the values are smaller, then there must be a local maximum at $(x_2, f(x_2))$; if we find that on both sides of x_2 the values are larger, then there must be a local minimum at $(x_2, f(x_2))$; if we find one of each, then there is neither a local maximum or minimum at x_2.

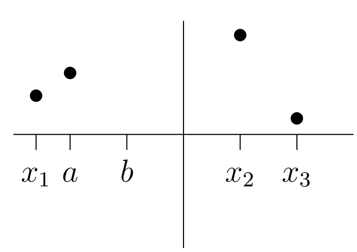


Figure 5.7: Testing for a maximum or minimum.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks—they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.

Example 5.7: Simple Cubic

Find all local maximum and minimum points for the function $f(x) = x^3 - x$.

Solution. The derivative is $f'(x) = 3x^2 - 1$. This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. Looking first at $x = \sqrt{3}/3$, we see that $f(\sqrt{3}/3) = -2\sqrt{3}/9$. Now we test two points on either side of $x = \sqrt{3}/3$, choosing one point in the interval $(-\sqrt{3}/3, \sqrt{3}/3)$ and one point in the interval $(\sqrt{3}/3, \infty)$. Since $f(0) = 0 > -2\sqrt{3}/9$ and $f(1) = 0 > -2\sqrt{3}/9$, there must be a local minimum at $x = \sqrt{3}/3$. For $x = -\sqrt{3}/3$, we see that $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. This time we can use $x = 0$ and $x = -1$, and we find that $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, so there must be a local maximum at $x = -\sqrt{3}/3$. ♣

Of course this example is made very simple by our choice of points to test, namely $x = -1$, 0, 1. We could have used other values, say $-5/4$, $1/3$, and $3/4$, but this would have made the calculations considerably more tedious, and we should always choose very simple points to test if we can.

Example 5.8: Max and Min

Find all local maximum and minimum points for $f(x) = \sin x + \cos x$.

Solution. The derivative is $f'(x) = \cos x - \sin x$. This is always defined and is zero whenever $\cos x = \sin x$. Recalling that the $\cos x$ and $\sin x$ are the x and y coordinates of points on a unit circle, we see that $\cos x = \sin x$ when x is $\pi/4$, $\pi/4 \pm \pi$, $\pi/4 \pm 2\pi$, $\pi/4 \pm 3\pi$, etc. Since both sine and cosine have a period of 2π, we need only determine the status of $x = \pi/4$ and $x = 5\pi/4$. We can use 0 and $\pi/2$ to test the critical value $x = \pi/4$. We find that $f(\pi/4) = \sqrt{2}$, $f(0) = 1 < \sqrt{2}$ and $f(\pi/2) = 1$, so there is a local maximum when $x = \pi/4$ and also when $x = \pi/4 \pm 2\pi$, $\pi/4 \pm 4\pi$, etc. We can summarize this more neatly by saying that there are local maxima at $\pi/4 \pm 2k\pi$ for every integer k.

We use π and 2π to test the critical value $x = 5\pi/4$. The relevant values are $f(5\pi/4) = -\sqrt{2}$, $f(\pi) = -1 > -\sqrt{2}$, $f(2\pi) = 1 > -\sqrt{2}$, so there is a local minimum at $x = 5\pi/4$, $5\pi/4 \pm 2\pi$, $5\pi/4 \pm 4\pi$, etc. More succinctly, there are local minima at $5\pi/4 \pm 2k\pi$ for every integer k. ♣

Example 5.9: Max and Min

Find all local maximum and minimum points for $g(x) = x^{2/3}$.

Solution. The derivative is $g'(x) = \frac{2}{3}x^{-1/3}$. This is undefined when $x = 0$ and is not equal to zero for any x in the domain of g'. Now we test two points on either side of $x = 0$. We use $x = -1$ and $x = 1$. Since $g(0) = 0$, $g(-1) = 1 > 0$ and $g(1) = 1 > 0$, there must be a local minimum at $x = 0$. ♣

Exercises for 5.2.1

Find all local maximum and minimum points (x, y) by the method of this section.

5.2.1 $y = x^2 - x$

5.2.2 $y = 2 + 3x - x^3$

5.2.3 $y = x^3 - 9x^2 + 24x$

5.2.4 $y = x^4 - 2x^2 + 3$

5.2.5 $y = 3x^4 - 4x^3$

5.2.6 $y = (x^2 - 1)/x$

5.2.7 $y = 3x^2 - (1/x^2)$

5.2.8 $y = \cos(2x) - x$

5.2.9 $f(x) = x^2 - 98x + 4$

5.2.10 *For any real number x there is a unique integer n such that $n \leq x < n+1$, and the greatest integer function is defined as $\lfloor x \rfloor = n$. Where are the critical values of the greatest integer function? Which are local maxima and which are local minima?*

5.2.11 *Explain why the function $f(x) = 1/x$ has no local maxima or minima.*

5.2.12 *How many critical points can a quadratic polynomial function have?*

5.2.13 *Show that a cubic polynomial can have at most two critical points. Give examples to show that a cubic polynomial can have zero, one, or two critical points.*

5.2.14 *Explore the family of functions $f(x) = x^3 + cx + 1$ where c is a constant. How many and what types of local extremes are there? Your answer should depend on the value of c, that is, different values of c will give different answers.*

5.2.15 *We generalize the preceding two questions. Let n be a positive integer and let f be a polynomial of degree n. How many critical points can f have? (Hint: Recall the* **Fundamental Theorem of Algebra**, *which says that a polynomial of degree n has at most n roots.)*

5.2.2. Absolute Extrema

Absolute extrema are also commonly referred to as **global extrema**. Unlike local extrema, which are only “extreme" relative to points “close to” them, an absolute (or global) extrema is “extreme" relative to *all* other points in the interval under consideration.

Definition 5.2: Absolute Maxima and Minima

A real-valued function f has an **absolute maximum** *on an interval I at x_0 if $f(x_0)$ is the largest value of f on I; in other words, $f(x_0) \geq f(x)$ for all x in the domain of f that are in I.*

A real-valued function f has an **absolute minimum** *on an interval I at x_0 if $f(x_0)$ is the smallest value of f on I; in other words, $f(x_0) \leq f(x)$ for all x in the domain of f that are in I.*

Example 5.10: Absolute Extrema

Consider the function $f(x) = x^2$ on the interval $(-\infty, \infty)$. This parabola has an absolute minimum at $x = 0$. However, it does not have an absolute maximum.

Consider the function $f(x) = |x|$ on the interval $[-1, 2]$. This graph looks like a check mark. It has an absolute minimum at $x = 0$ and an absolute maximum at $x = 2$.

Consider the function $f(x) = \cos x$ on the interval $[0, \pi]$. It has an absolute minimum at $x = \pi$ and an absolute maximum at $x = 0$.

Consider the function $f(x) = e^x$ on any interval $[a, b]$, where $a < b$. Since this exponential function is increasing, it has an absolute minimum at $x = a$ and an absolute maximum at $x = b$.

Like Fermat's Theorem, the following theorem has an intuitive appeal. However, unlike Fermat's Theorem, the proof relies on a more advanced concept called **compactness**, which will only be covered in a course typically entitled Analysis. So, we will be content with understanding the statement of the theorem.

Theorem 5.2: Extreme-Value Theorem

If a function f is continuous on a closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

Although this theorem tells us that an absolute extremum exists, it does not tell us what it is or how to find it.

Note that if an absolute extremum is inside the interval (i.e. not an endpoint), then it must also be a local extremum. This immediately tells us that to find the absolute extrema of a function on an interval, we need only examine the local extrema inside the interval, and the endpoints of the interval.

We can devise a method for finding absolute extrema for a function f on a closed interval $[a, b]$:

1. Verify the function is continuous on $[a, b]$.
2. Find the derivative and determine all critical values of f that are in $[a, b]$.
3. Evaluate the function at the critical values found in Step 2 and the end points of the interval.
4. Identify the absolute extrema.

Why must a function be continuous on a closed interval in order to use this theorem? Consider the following example.

Example 5.11: Absolute Extrema of a $1/x$

Find any absolute extrema for $f(x) = 1/x$ on the interval $[-1, 1]$.

Solution. The function f is not continuous at $x = 0$. Since $0 \in [-1, 1]$, f is not continous on the closed interval:

$$\lim_{x \to 0^+} f(x) = +\infty$$
$$\lim_{x \to 0^-} f(x) = -\infty\,,$$

so we are *unable* to apply the Extreme-Value Theorem. Therefore, $f(x) = 1/x$ does not have an absolute maximum or an absolute minimum on $[-1, 1]$. ♣

However, if we consider the same function on an interval where it is continuous, the theorem will apply. This is illustrated in the following example.

Example 5.12: Absolute Extrema of a $1/x$

Find any absolute extrema for $f(x) = 1/x$ on the interval $[1, 2]$.

Solution. The function f is continous on the interval, so we can apply the Extreme-Value Theorem. We begin with taking the derivative to be $f'(x) = -1/x^2$ which has a critical value at $x = 0$, but since this critical value is not in $[1, 2]$ we ignore it. The only points where an extrema can occur are the endpoints of the interval. To find the maximum or minimum we can simply evaluate the function: $f(1) = 1$ and $f(2) = 1/2$, so the absolute maximum is at $x = 1$ and the absolute minimum is at $x = 2$. ♣

Why must an interval be closed in order to use the above theorem? Recall the difference between open and closed intervals. Consider a function f on the open interval $(0, 1)$. If we choose successive values of x moving closer and closer to 1, what happens? Since 1 is not included in the interval we will not attain exactly the value of 1. Suppose we reach a value of 0.9999 — is it possible to get closer to 1? Yes: There are infinitely many real numbers between 0.9999 and 1. In fact, any conceivable real number close to 1 will have infinitely many real numbers between itself and 1. Now, suppose f is decreasing on $(0, 1)$: As we approach 1, f will continue to decrease, even if the difference between successive values of f is slight. Similarly if f is increasing on $(0, 1)$.

Consider a few more examples:

Example 5.13: Determining Absolute Extrema

Determine the absolute extrema of $f(x) = x^3 - x^2 + 1$ on the interval $[-1, 2]$.

Solution. First, notice f is continuous on the closed interval $[-1, 2]$, so we're able to use Theorem 5.2 to determine the absolute extrema. The derivative is $f'(x) = 3x^2 - 2x$, and the critical values are $x = 0, 2/3$ which are both in the interval $[-1, 2]$. In order to find the absolute extrema, we must consider all critical values that lie within the interval (that is, in $(-1, 2)$) *and* the endpoints of the interval.

$$\begin{aligned}
f(-1) &= (-1)^3 - (-1)^2 + 1 = -1 \\
f(0) &= (0)^3 - (0)^2 + 1 = 1 \\
f(2/3) &= (2/3)^3 - (2/3)^2 + 1 = 23/27 \\
f(2) &= (2)^3 - (2)^2 + 1 = 5
\end{aligned}$$

The absolute maximum is at (2,5) and the absolute minimum is at (-1,-1). ♣

Example 5.14: Determining Absolute Extrema

Determine the absolute extrema of $f(x) = -9/x - x + 10$ *on the interval* $[2,6]$.

Solution. First, notice f is continuous on the closed interval $[2,6]$, so we're able to use Theorem 5.2 to determine the absolute extrema. The function is not continuous at $x = 0$, but we can ignore this fact since 0 is not in $[2,6]$. The derivative is $f'(x) = 9/x^2 - 1$, and the critical values are $x = \pm 3$, but only $x = +3$ is in the interval. In order to find the absolute extrema, we must consider all critical values that lie within the interval *and* the endpoints of the interval.

$$\begin{aligned}
f(2) &= -9/(2) - (2) + 10 = 7/2 = 3.5 \\
f(3) &= -9/(3) - (3) + 10 = 4 \\
f(6) &= -9/(6) - (6) + 10 = 5/2 = 2.5
\end{aligned}$$

The absolute maximum is at (3,4) and the absolute minimum is at (6,2.5). ♣

When we are trying to find the absolute extrema of a function on an open interval, we cannot use the Extreme Value Theorem. However, if the function is continuous on the interval, many of the same ideas apply. In particular, if an absolute extremum exists, it must also be a local extremum. In addition to checking values at the local extrema, we must check the behaviour of the function as it approaches the ends of the interval.

Some examples to illustrate this method.

Example 5.15: Extrema of Secant

Find the extrema of $\sec(x)$ *on* $(-\pi/2, \pi/2)$.

Solution. Notice $\sec(x)$ is continuous on $(-\pi/2, \pi/2)$ and has one local minimum at 0. Also

$$\lim_{x \to (-\pi/2)^+} \sec(x) = \lim_{x \to (\pi/2)^-} \sec(x) = +\infty\,,$$

so $\sec(x)$ has no absolute maximum, but the point $(0,1)$ is the absolute minimum. ♣

A similar approach can be used for infinite intervals.

Example 5.16: Extrema of $\frac{x^2}{x^2+1}$

Find the extrema of $\dfrac{x^2}{x^2+1}$ *on* $(-\infty, \infty)$.

Solution. Since $x^2 + 1 \neq 0$ for all x in $(-\infty, \infty)$ the function is continuous on this interval. This function has only one critical value at $x = 0$, which is the local minimum and also the absolute minimum. Now, $\lim_{x \to \pm\infty} \frac{x^2}{x^2+1} = 1$, so the function does not have an absolute maximum: It continues to increase towards 1, but does not attain this exact value. ♣

Exercises for 5.2.2

5.2.16 *Find the absolute extrema for* $f(x) = -\frac{x+4}{x-4}$ *on* $[0,3]$.

5.2.17 *Find the absolute extrema for* $f(x) = -\frac{x+4}{x-4}$ *on* $[0,3]$.

5.2.18 *Find the absolute extrema for* $f(x) = \csc(x)$ *on* $[0,\pi]$.

5.2.19 *Find the absolute extrema for* $f(x) = \ln(x)/x^2$ *on* $[1,4]$.

5.2.20 *Find the absolute extrema for* $f(x) = x\sqrt{1-x^2}$ *on* $[-1,1]$.

5.2.21 *Find the absolute extrema for* $f(x) = xe^{-x^2/32}$ *on* $[0,2]$.

5.2.22 *Find the absolute extrema for* $f(x) = x - \tan^{-1}(2x)$ *on* $[0,2]$.

5.2.23 *Find the absolute extrema for* $f(x) = \frac{x}{x^2+1}$.

For the following exercises, sketch a potential graph of a continuous function on the closed interval $[0,4]$ with the given properties.

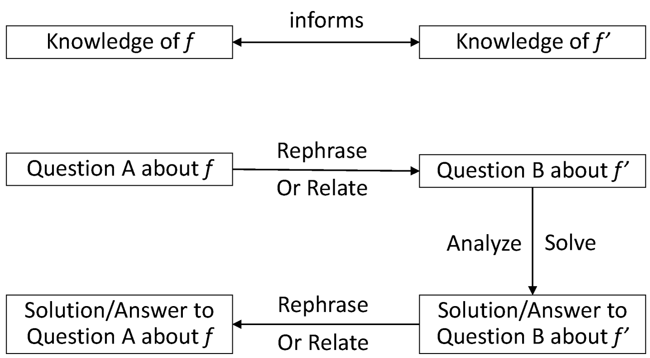
5.2.24 *Absolute minimum at 0, absolute maximum at 2, local minimum at 3.*

5.2.25 *Absolute maximum at 1, absolute minimum at 2, local maximum at 3.*

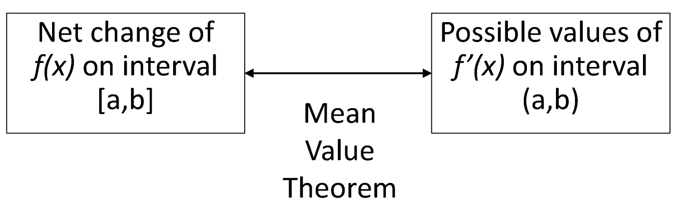
5.2.26 *Absolute minimum at 4, absolute maximum at 1, local minimum at 2, local maxima at 1 and 3.*

5.3 The Mean Value Theorem

There are numerous applications of the derivative through its **definition** as rate of change and as the slope of the tangent line. In this section we shall look at some deeper reasons why the derivative turns out to be so useful. The simple answer is that **the derivative of a function tells us a lot about the function**. More important, "hard" questions about a function can sometimes be answered by solving a relatively simple problem about the derivative of the function.



The Mean Value Theore tells us that there is an intimate connection between the net change of the value of any "sufficiently nice" function over an interval and the possible values of its derivative on that interval. Because of this connection, we can draw conclusions about the possible values of the derivative based on information about the values of the function, and conversely, we can draw conclusions about the values of the function based on information about the values of its derivative.



Let us illustrate the idea through the following two interesting questions involving derivatives:

1. Suppose two different functions have the same derivative; what can you say about the relationship between the two functions?

2. Suppose you drive a car from toll booth on a toll road to another toll booth at an average speed of 70 miles per hour. What can be concluded about your actual speed during the trip? In particular, did you exceed the 65 mile per hour speed limit?

While these sound very different, it turns out that the two problems are very closely related. We know that "speed" is really the derivative by a different name; let's start by translating the second question into something that may be easier to visualize. Suppose that the function $f(t)$ gives the position of your car on the toll road at time t. Your change in position between one toll booth and the next is given by $f(t_1)-f(t_0)$, assuming that at time t_0 you were at the first booth and at time t_1 you arrived at the second booth. Your average speed for the trip is $(f(t_1)-f(t_0))/(t_1-t_0)$. If we think about the graph of $f(t)$, the average speed is the slope of the line that connects the two points $(t_0, f(t_0))$ and $(t_1, f(t_1))$. Your speed at any particular time t between t_0 and t_1 is $f'(t)$, the slope of the curve. Now question (2) becomes a question about slope. In particular, if the slope between endpoints is 70, what can be said of the slopes at points between the endpoints?

As a general rule, when faced with a new problem it is often a good idea to examine one or more simplified versions of the problem, in the hope that this will lead to an understanding of the original problem. In this case, the problem in its "slope" form is somewhat easier to simplify than the original, but equivalent, problem.

Here is a special instance of the problem. Suppose that $f(t_0) = f(t_1)$. Then the two endpoints have the same height and the slope of the line connecting the endpoints is zero. What can we say about the slope between the endpoints? It shouldn't take much experimentation before you are convinced of the truth of this statement: Somewhere between t_0 and t_1 the slope is exactly zero, that is, somewhere between t_0 and t_1 the slope is equal to the slope of the line between the endpoints. This suggests that perhaps the same is true even if the endpoints are at different heights, and again a bit of experimentation will probably convince you that this is so. But we can do better than "experimentation"—we can prove that this is so.

We start with the simplified version:

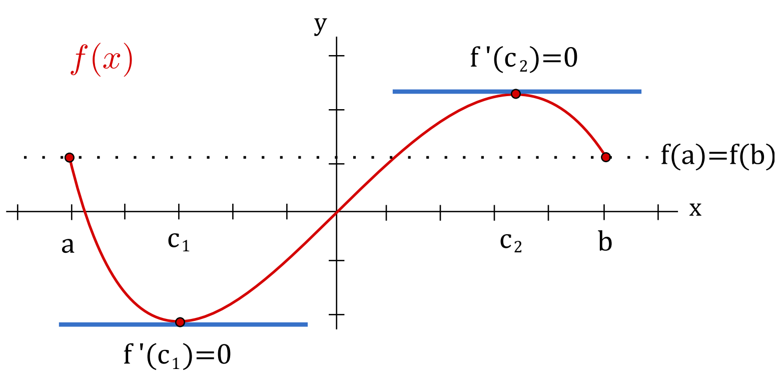
Theorem 5.3: Rolle's Theorem

(Rolle's Theorem) Suppose that $f(x)$ has a derivative on the interval (a, b), is continuous on the interval $[a, b]$, and $f(a) = f(b)$. Then at some value $c \in (a, b)$, $f'(c) = 0$.

Proof. We know that $f(x)$ has a maximum and minimum value on $[a, b]$ (because it is continuous), and we also know that the maximum and minimum must occur at an endpoint, at a point at which the derivative is zero, or at a point where the derivative is undefined. Since the derivative is never undefined, that possibility is removed.

If the maximum or minimum occurs at a point c, other than an endpoint, where $f'(c) = 0$, then we have found the point we seek. Otherwise, the maximum and minimum both occur at an endpoint, and since the endpoints have the same height, the maximum and minimum are the same. This means that $f(x) = f(a) = f(b)$ at every $x \in [a, b]$, so the function is a horizontal line, and it has derivative zero everywhere in (a, b). Then we may choose any c at all to get $f'(c) = 0$. ♣

Rolle's Theorem is illustrated below for a function $f(x)$ where $f'(x) = 0$ holds for two values of $x = c_1$ and $x = c_2$:



Perhaps remarkably, this special case is all we need to prove the more general one as well.

Theorem 5.4: Mean Value Theorem

Suppose that $f(x)$ has a derivative on the interval (a,b) and is continuous on the interval $[a,b]$. Then at some value $c \in (a,b)$, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Proof. Let $m = \dfrac{f(b)-f(a)}{b-a}$, and consider a new function $g(x) = f(x) - m(x-a) - f(a)$. We know that $g(x)$ has a derivative everywhere, since $g'(x) = f'(x) - m$. We can compute $g(a) = f(a) - m(a-a) - f(a) = 0$ and

$$\begin{aligned} g(b) = f(b) - m(b-a) - f(a) \quad &= f(b) - \tfrac{f(b)-f(a)}{b-a}(b-a) - f(a) \\ &= f(b) - (f(b) - f(a)) - f(a) = 0. \end{aligned}$$

So the height of $g(x)$ is the same at both endpoints. This means, by Rolle's Theorem, that at some c, $g'(c) = 0$. But we know that $g'(c) = f'(c) - m$, so

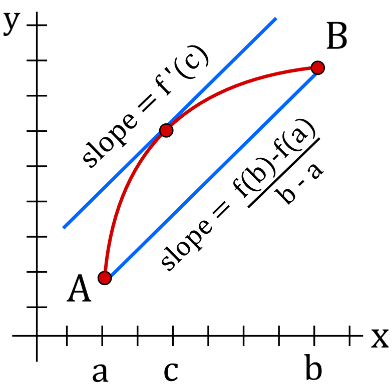
$$0 = f'(c) - m = f'(c) - \frac{f(b)-f(a)}{b-a},$$

which turns into

$$f'(c) = \frac{f(b)-f(a)}{b-a},$$

exactly what we want. ♣

The Mean Value Theorem is illustrated below showing the existence of a point $x = c$ for a function $f(x)$ where the tangent line at $x = c$ (with slope $f'(c)$) is parallel to the secant line connecting $A(a, f(a))$ and $B(b, f(b))$ (with slope $\frac{f(b)-f(a)}{b-a}$):



Returning to the original formulation of question (2), we see that if $f(t)$ gives the position of your car at time t, then the Mean Value Theorem says that at some time c, $f'(c) = 70$, that is, at some time you must have been traveling at exactly your average speed for the trip, and that indeed you exceeded the speed limit.

Now let's return to question (1). Suppose, for example, that two functions are known to have derivative equal to 5 everywhere, $f'(x) = g'(x) = 5$. It is easy to find such functions: $5x$, $5x + 47$, $5x - 132$, etc. Are there other, more complicated, examples? No—the only functions that work are the "obvious" ones, namely, $5x$ plus some constant. How can we see that this is true?

Although "5" is a very simple derivative, let's look at an even simpler one. Suppose that $f'(x) = g'(x) = 0$. Again we can find examples: $f(x) = 0$, $f(x) = 47$, $f(x) = -511$ all have $f'(x) = 0$. Are there non-constant functions f with derivative 0? No, and here's why: Suppose that $f(x)$ is not a constant function. This means that there are two points on the function with different heights, say $f(a) \neq f(b)$. The Mean Value Theorem tells us that at some point c, $f'(c) = (f(b) - f(a))/(b - a) \neq 0$. So any non-constant function does not have a derivative that is zero everywhere; this is the same as saying that the only functions with zero derivative are the constant functions.

Let's go back to the slightly less easy example: suppose that $f'(x) = g'(x) = 5$. Then $(f(x)-g(x))' = f'(x)-g'(x) = 5 - 5 = 0$. So using what we discovered in the previous paragraph, we know that $f(x) - g(x) = k$, for some constant k. So any two functions with derivative 5 must differ by a constant; since $5x$ is known to work, the only other examples must look like $5x + k$.

Now we can extend this to more complicated functions, without any extra work. Suppose that $f'(x) = g'(x)$. Then as before $(f(x) - g(x))' = f'(x) - g'(x) = 0$, so $f(x) - g(x) = k$. Again this means that if we find just a single function $g(x)$ with a certain derivative, then every other function with the same derivative must be of the form $g(x) + k$.

Example 5.17: Given Derivative

Describe all functions that have derivative $5x - 3$.

Solution. It's easy to find one: $g(x) = (5/2)x^2 - 3x$ has $g'(x) = 5x - 3$. The only other functions with the same derivative are therefore of the form $f(x) = (5/2)x^2 - 3x + k$.

Alternately, though not obviously, you might have first noticed that $g(x) = (5/2)x^2 - 3x + 47$ has $g'(x) = 5x-3$. Then every other function with the same derivative must have the form $f(x) = (5/2)x^2-3x+47+k$. This looks different, but it really isn't. The functions of the form $f(x) = (5/2)x^2 - 3x + k$ are exactly the same as the ones of the form $f(x) = (5/2)x^2 - 3x + 47 + k$. For example, $(5/2)x^2 - 3x + 10$ is the same as $(5/2)x^2 - 3x + 47 + (-37)$, and the first is of the first form while the second has the second form. ♣

This is worth calling a theorem:

Theorem 5.5: Functions with the Same Derivative

If $f'(x) = g'(x)$ for every $x \in (a, b)$, then for some constant k, $f(x) = g(x) + k$ on the interval (a, b).

Example 5.18: Same Derivative

Describe all functions with derivative $\sin x + e^x$. One such function is $-\cos x + e^x$, so all such functions have the form $-\cos x + e^x + k$.

Theorem 5.5 and the above example illustrate what the Mean Value Theorem allows us to say about $f(x)$ when we have perfect information about $f'(x)$. Specifically, $f(x)$ is determined up to a constant. Our next example illustrates almost the opposite extreme situation, one where we have much less information about $f'(x)$ beyond the fact that $f'(x)$ exists. Specifically, assuming that we know an upper bound on the values of $f'(x)$, what can we say about the values of $f(x)$?

Example 5.19: Conclusion Regarding Function Value Based on Derivative Information

Suppose that f is a differentiable function such that $f'(x) \leq 2$ for all x. What is the largest possible value of $f(7)$ if $f(3) = 5$?

Solution. We are interested in the values of $f(x)$ at $x = 3$ and $x = 7$. It makes sense to focus our attention on the interval between 3 and 7. It is given that $f(x)$ is differentiable for all x. So, $f(x)$ is also continuous at all x. In particular, $f(x)$ is continuous on the interval $[3, 7]$ and differentiable on the interval $(3, 7)$. By the Mean Value Theorem, we know that there is some c in $(3, 7)$ such that

$$f'(c) = \frac{f(7) - f(3)}{7 - 3}.$$

Simplifying and using the given information $f(3) = 5$, we get

$$f'(c) = \frac{f(7) - 5}{4},$$

or, after re-arranging the terms,

$$f(7) = 4f'(c) + 5.$$

We do not know the exact value of c, but we do know that $f'(x) \leq 2$ for all x. This implies that $f'(c) \leq 2$. Therefore,

$$f(7) \leq 4 \cdot 2 + 5 = 13.$$

That is, the value of $f(7)$ cannot exceed 13. To convince ourselves that 13 (as opposed to some smaller number) is the largest possible value of $f(7)$, we still need to show that it is possible for the value of $f(7)$ to reach 13. If we review our proof, we notice that the inequality will be an equality if $f'(c) = 2$. One way to guarantee this without knowing anything about c is to require $f'(x) = 2$ for all x. This means that $f(x) = 2x + k$ for some constant k. From the condition $f(3) = 5$, we see that $k = -1$. We can easily verify that indeed $f(x) = 2x - 1$ meets all our requirements and $f(7) = 13$. ♣

Exercises for Section 5.3

5.3.1 *Let $f(x) = x^2$. Find a value $c \in (-1, 2)$ so that $f'(c)$ equals the slope between the endpoints of $f(x)$ on $[-1, 2]$.*

5.3.2 *Verify that $f(x) = x/(x + 2)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 4]$ and then find all of the values, c, that satisfy the conclusion of the theorem.*

5.3.3 *Verify that $f(x) = 3x/(x + 7)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-2, 6]$ and then find all of the values, c, that satisfy the conclusion of the theorem.*

5.3.4 *Let $f(x) = \tan x$. Show that $f(\pi) = f(2\pi) = 0$ but there is no number $c \in (\pi, 2\pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?*

5.3.5 *Let $f(x) = (x - 3)^{-2}$. Show that there is no value $c \in (1, 4)$ such that $f'(c) = (f(4) - f(1))/(4 - 1)$. Why is this not a contradiction of the Mean Value Theorem?*

5.3.6 *Describe all functions with derivative* $x^2 + 47x - 5$.

5.3.7 *Describe all functions with derivative* $\dfrac{1}{1+x^2}$.

5.3.8 *Describe all functions with derivative* $x^3 - \dfrac{1}{x}$.

5.3.9 *Describe all functions with derivative* $\sin(2x)$.

5.3.10 *Find* $f(x)$ *if* $f'(x) = e^{-x}$ *and* $f(0) = 2$.

5.3.11 *Suppose that* f *is a differentiable function such that* $f'(x) \geq -3$ *for all* x. *What is the smallest possible value of* $f(4)$ *if* $f(-1) = 2$?

5.3.12 *Show that the equation* $6x^4 - 7x + 1 = 0$ *does not have more than two distinct real roots.*

5.3.13 *Let* f *be differentiable on* $\mathbb{R}$. *Suppose that* $f'(x) \neq 0$ *for every* x. *Prove that* f *has at most one real root.*

5.3.14 *Prove that for all real* x *and* y $|\cos x - \cos y| \leq |x - y|$. *State and prove an analogous result involving sine.*

5.3.15 *Show that* $\sqrt{1+x} \leq 1 + (x/2)$ *if* $-1 < x < 1$.

5.3.16 *Suppose that* $f(a) = g(a)$ *and that* $f'(x) \leq g'(x)$ *for all* $x \geq a$.

(a) *Prove that* $f(x) \leq g(x)$ *for all* $x \geq a$.

(b) *Use part (a) to prove that* $e^x \geq 1 + x$ *for all* $x \geq 0$.

(c) *Use parts (a) and (b) to prove that* $e^x \geq 1 + x + \dfrac{x^2}{2}$ *for all* $x \geq 0$.

(d) *Can you generalize these results?*

5.4 Linear and Higher Order Approximations

When we define the derivative $f'(x)$ as the rate of change of $f(x)$ with respect to x, we notice that in relation to the graph of f, the derivative is the slope of the tangent line, which (loosely speaking) is the line that just grazes the graph. But what precisely do we mean by this? In short, **the tangent line approximates the graph near the point of contact**. The definition of the derivative $f'(a)$ guarantees this when it exists: By taking x sufficiently close to a but not equal to a,

$$\frac{f(x) - f(a)}{x - a} \approx f'(a),$$

and consequently,

$$f(x) \approx f'(a)(x-a) + f(a).$$

The left hand side gives us the y-value of the function $y = f(x)$ and the right hand side gives us the y-value $y = f'(a)(x-a) + f(a)$ for the tangent line to the graph of f at the point $(a, f(a))$.

In this section we will explore how to apply this idea to approximate some values of f, some changes in the values of f, and also the roots of f.

5.4.1. Linear Approximations

We begin by the first derivative as an application of the tangent line to approximate f.

Recall that the tangent line to $f(x)$ at a point $x = a$ is given by

$$L(x) = f'(a)(x-a) + f(a).$$

The tangent line in this context is also called the **linear approximation** to f at a.

If f is differentiable at a then L is a good approximation of f so long as x is "not too far" from a. Put another way, if f is differentiable at a then under a microscope f will look very much like a straight line, and thus will look very much like L; since $L(x)$ is often much easier to compute than $f(x)$, then it makes sense to use L as an approximation. Figure 5.8 shows a tangent line to $y = x^2$ at three different magnifications.

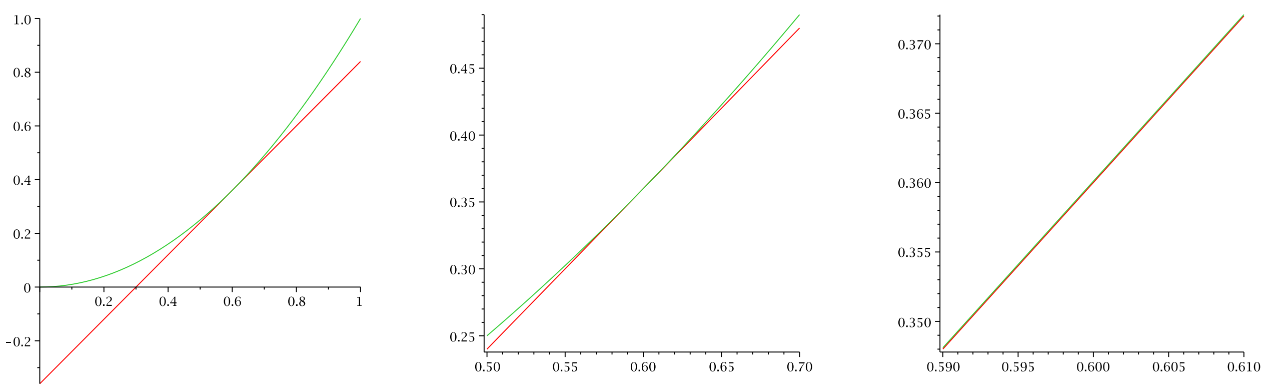


Figure 5.8: The linear approximation to $y = x^2$.

Thus in practice if we want to approximate a difficult value of $f(b)$, then we may be able to approximate this value using a linear approximation, provided that we can compute the tangent line at some point a close to b. Here is an example.

Example 5.20: Linear Approximation

Let $f(x) = \sqrt{x+4}$, what is $f(6)$?

Solution. We are asked to calculate $f(6) = \sqrt{6+4} = \sqrt{10}$ which is not easy to do without a calculator. However 9 is (relatively) close to 10 and of course $f(5) = \sqrt{9}$ is easy to compute, and we use this to approximate $\sqrt{10}$.

To do so we have $f'(x) = 1/(2\sqrt{x+4})$, and thus the linear approximation to f at $x = 5$ is

$$L(x) = \left(\frac{1}{2\sqrt{5+4}}\right)(x-5) + \sqrt{5+4} = \frac{x-5}{6} + 3.$$

Now to estimate $\sqrt{10}$, we substitute 6 into the linear approximation $L(x)$ instead of $f(x)$, to obtain

$$\sqrt{6+4} \approx \frac{6-5}{6} + 3 = \frac{19}{6} = 3^1/_6 = 3.1\bar{6} \approx 3.17$$

It turns out the exact value of $\sqrt{10}$ is actually 3.16227766... but our estimate of 3.17 was very easy to obtain and is relatively accurate. This estimate is only accurate to one decimal place. ♣

With modern calculators and computing software it may not appear necessary to use linear approximations, but in fact they are quite useful. For example in cases requiring an explicit numerical approximation, they allow us to get a quick estimate which can be used as a "reality check" on a more complex calculation. Further in some complex calculations involving functions, the linear approximation makes an otherwise intractable calculation possible without serious loss of accuracy.

Example 5.21: Linear Approximation of Sine

Find the linear approximation of $\sin x$ *at* $x = 0$, *and use it to compute small values of* $\sin x$.

Solution. If $f(x) = \sin x$, then $f'(x) = \cos x$, and thus the linear approximation of $\sin x$ at $x = 0$ is:

$$L(x) = \cos(0)(x-0) + \sin(0) = x.$$

Thus when x is small this is quite a good approximation and is used frequently by engineers and scientists to simplify some calculations.

For example you can use your calculator (in radian mode since the derivative of $\sin x$ is $\cos x$ only in radian) to see that

$$\sin(0.1) = 0.099833416\ldots$$

and thus $L(0.1) = 0.1$ is a very good and quick approximation without any calculator! ♣

Exercises for 5.4.1

5.4.1 *Find the linearization* $L(x)$ *of* $f(x) = \ln(1+x)$ *at* $a = 0$. *Use this linearization to approximate* $f(0.1)$.

5.4.2 *Use linear approximation to estimate* $(1.9)^3$.

5.4.3 *Show in detail that the linear approximation of* $\sin x$ *at* $x = 0$ *is* $L(x) = x$ *and the linear approximation of* $\cos x$ *at* $x = 0$ *is* $L(x) = 1$.

5.4.4 *Use* $f(x) = \sqrt[3]{x+1}$ *to approximate* $\sqrt[3]{9}$ *by choosing an appropriate point* $x = a$. *Are we over- or under-estimating the value of* $\sqrt[3]{9}$? *Explain.*

5.4.2. Differentials

Very much related to linear approximations are the *differentials* dx and dy, used not to approximate values of f, but instead the change (or rise) in the values of f.

Definition 5.3: Differentials dx and dy

Let $y = f(x)$ be a differentiable function. We define a new independent variable dx, and a new dependent variable $dy = f'(x)\,dx$. Notice that dy is a function both of x (since $f'(x)$ is a function of x) and of dx. We call both dx and dy **differentials**.

Now fix a point a and let $\Delta x = x - a$ and $\Delta y = f(x) - f(a)$. If x is near a then Δx is clearly small. If we set $dx = \Delta x$ then we obtain

$$dy = f'(a)\,dx \approx \frac{\Delta y}{\Delta x}\Delta x = \Delta y.$$

Thus, dy can be used to approximate Δy, the actual change in the function f between a and x. This is exactly the approximation given by the tangent line:

$$dy = f'(a)(x-a) = f'(a)(x-a) + f(a) - f(a) = L(x) - f(a).$$

While $L(x)$ approximates $f(x)$, dy approximates how $f(x)$ has changed from $f(a)$. Figure 5.9 illustrates the relationships.

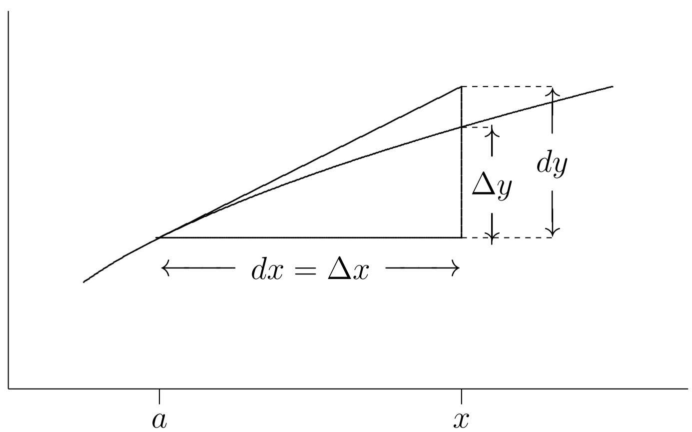


Figure 5.9: Differentials.

Here is a concrete example.

Example 5.22: Rise of Natural Logarithm

Approximate the rise of $f(x) = \ln x$ from $x = 1$ to $x = 1.1$, using linear approximation.

Solution. Note that $\ln(1.1)$ is not readily calculated (without a calculator) hence why we wish to use linear approximation to approximate $f(1.1) - f(1)$.

We fix $a = 1$ and as above we have $\Delta x = x - 1$ and $\Delta y = f(x) - f(1) = \ln x$, and obtain

$$dy = f'(1)dx \approx \frac{\Delta y}{\Delta x}\Delta x = \Delta y.$$

But $f'(x) = 1/x$ and thus $f'(1) = 1/1 = 1$, we obtain in this case

$$dy = dx \approx \Delta y.$$

Finally for $x = 1.1$, we can easily approximate the rise of f as

$$f(1.1) - f(1) = \Delta y \approx dy = 1.1 - 1 = 0.1.$$

The correct value of $\ln(1.1) = \ln 1$ is $0.0953\ldots$ and thus we were relatively close. ♣

Exercises for 5.4.2

5.4.5 *Let $f(x) = x^4$. If $a = 1$ and $dx = \Delta x = 1/2$, what are Δy and dy?*

5.4.6 *Let $f(x) = \sqrt{x}$. If $a = 1$ and $dx = \Delta x = 1/10$, what are Δy and dy?*

5.4.7 *Let $f(x) = \sin(2x)$. If $a = \pi$ and $dx = \Delta x = \pi/100$, what are Δy and dy?*

5.4.8 *Use differentials to estimate the amount of paint needed to apply a coat of paint 0.02 cm thick to a sphere with diameter 40 meters. (Recall that the volume of a sphere of radius r is $V = (4/3)\pi r^3$. Notice that you are given that $dr = 0.02$.)*

5.4.3. Taylor Polynomials

We can go beyond first order derivatives to create polynomials approximating a function as closely as we wish, these are called *Taylor Polynomials*.

While our linear approximation $L(x) = f'(a)(x-a) + f(a)$ at a point a was a polynomial of degree 1 such that both $L(a) = f(a)$ and $L'(a) = f'(a)$, we can now form a polynomial

$$T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n$$

which has the same first n derivatives at $x = a$ as the function f.

By successively computing the derivatives of T_n, we obtain:

$$\begin{array}{l} a_0 = f(a) = \frac{f(a)}{0!} \\ a_1 = \frac{f'(a)}{1!} \\ a_2 = \frac{f''(a)}{2!} \\ \cdots \\ a_k = \frac{f^{(k)}(a)}{k!} \\ \cdots a_n = \frac{f^{(n)}(a)}{n!} \end{array}$$

where $f^{(k)}(x)$ is the k^{th} derivative of $f(x)$, and $n! = n(n-1)(n-2)\ldots(2)(1)$, referred to as *factorial* notation.

Here is an example.

Example 5.23: Approximate e using Taylor Polynomials

Approximate e^x using Taylor polynomials at $a = 0$, and use this to approximate e.

Solution. In this case we use the function $f(x) = e^x$ at $a = 0$, and therefore

$$T_n(x) = a_0 + a_1 x + a_x x^2 + a_3 x^3 + \ldots + a_n x^n$$

Since all derivatives $f^{(k)}(x) = e^x$, we get:

$$\begin{aligned} a_0 &= f(0) = 1 \\ a_1 &= \tfrac{f'(0)}{1!} = 1 \\ a_2 &= \tfrac{f''(0)}{2!} = \tfrac{1}{2!} \\ a_3 &= \tfrac{f'''(0)}{3!} = \tfrac{1}{3!} \\ &\ldots \\ a_k &= \tfrac{f^{(k)}(0)}{k!} = \tfrac{1}{k!} \\ &\ldots \\ a_n &= \tfrac{f^{(n)}(0)}{n!} = \tfrac{1}{n!} \end{aligned}$$

Thus

$$\begin{aligned} T_1(x) &= 1 + x = L(x) \\ T_2(x) &= 1 + x + \tfrac{x^2}{2!} \\ T_3(x) &= 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} \end{aligned}$$

and in general

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Finally we can approximate $e = f(1)$ by simply calculating $T_n(1)$. A few values are:

$$\begin{aligned} T_1(1) &= 1 + 1 = 2 \\ T_2(1) &= 1 + 1 + \tfrac{1^2}{2!} = 2.5 \\ T_4(1) &= 1 + 1 + \tfrac{1^2}{2!} + \tfrac{1^3}{3!} = 2.\overline{6} \\ T_8(1) &= 2.71825396825 \\ T_{20}(1) &= 2.71828182845 \end{aligned}$$

We can continue this way for larger values of n, but $T_{20}(1)$ is already a pretty good approximation of e, and we took only 20 terms! ♣

Exercises for 5.4.3

5.4.9 *Find the 5^{th} degree Taylor polynomial for $f(x) = \sin x$ around $a = 0$.*

(a) Use this Taylor polynomial to approximate $\sin(0.1)$.

(b) Use a calculator to find $\sin(0.1)$. *How does this compare to our approximation in part (a)?*

5.4.10 *Suppose that f'' exists and is continuous on $[1,2]$. Suppose also that $|f''(x)| \leq \frac{1}{4}$ for all x in $(1,2)$. Prove that if we use the linearization $y = L(x)$ of $y = f(x)$ at $x = 1$ as an approximation of $y = f(x)$ near $x = 1$, then our estimated value of $f(1.2)$ is guaranteed to have an accuracy of at least 0.01, i.e., our estimate will lie within 0.01 units of the true value.*

5.4.11 *Find the 3^{rd} degree Taylor polynomial for $f(x) = \frac{1}{1-x} - 1$ around $a = 0$. Explain why this approximation would not be useful for calculating $f(5)$.*

5.4.12 *Consider $f(x) = \ln x$ around $a = 1$.*

(a) *Find a general formula for $f^{(n)}(x)$ for $n \geq 1$.*

(b) *Find a general formula for the Taylor Polynomial, $T_n(x)$.*

5.4.4. Newton's Method

A well known numeric method is *Newton's Method* (also sometimes referred to as *Newton-Raphson's Method*), named after Isaac Newton and Joseph Raphson. This method is used to find roots, or x-intercepts, of a function. While we may be able to find the roots of a polynomial which we can easily factor, we saw in the previous chapter on **Limits**, that for example the function $e^x + x = 0$ has a solution (*i.e.* root, or x-intercept) at $x \approx -0.56714$. By the *Intermediate Value Theorem* we know that the function $e^x + x = 0$ does have a solution. We cannot here simply solve for such a root algebraically, but we can use a numerical method such as $Newton's$. Such a process is typically classified as an *iterative* method, a name given to a technique which involves repeating similar steps until the desired accuracy is obtained. Many computer *algorithms* are coded with a for-loop, repeating an iterative step to converge to a solution.

The idea is to start with an initial value x_0 (approximating the root), and use linear approximation to create values x_1, x_2, $\cdots$ getting closer and closer to a root.

The first value x_1 corresponds to the intercept of the tangent line of $f(x_0)$ with the x-axis, which is:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

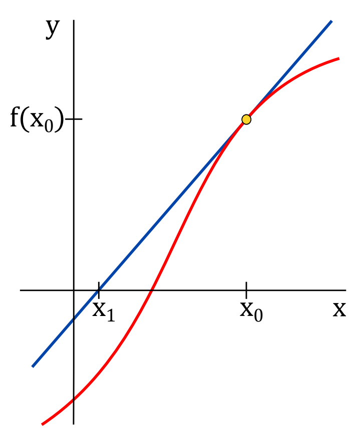


Figure 5.10: First iteration of Newton's Method.

We can see in Figure 5.10, that if we compare the point $(x_0, 0)$ to $(x_1, 0)$, we would likely come to the conclusion that $(x_1, 0)$ is closer to the actual root of $f(x)$ than our original guess, $(x_0, 0)$. As will be discussed, the choice of x_0 must be done correctly, and it may occur that x_1 does not yield a better estimate of the root.

Newton's method is simply to repeat this process again and again in an effort to obtain a more accurate solution. Thus at the next step we obtain:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

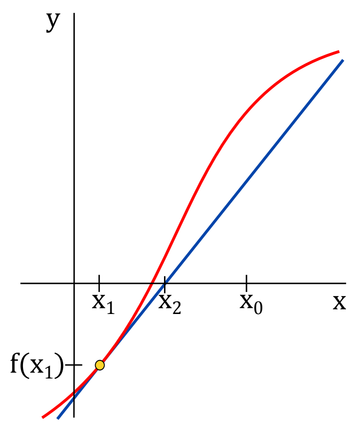


Figure 5.11: Second iteration of Newton's Method.

We can now clearly see how $(x_2, 0)$ is a better estimate of the root of $f(x)$, rather than any of the previous points. Moving forward, we will get:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Rest assured, $(x_3, 0)$ will be an even better estimate of the root! We express the general iterative step as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The idea is to iterate these steps to obtain the desired accuracy. Here is an example.

Example 5.24: Newton method to approximate roots

Use Newton's method to approximate the roots of $f(x) = x^3 - x + 1$.

Solution. You can try to find solve the equation algebraically to see that this is a difficult task, and thus it make sense to try a numerical method such as Newton's.

To find an initial value x_0, note that $f(-1) = -5$ and $f(0) = 1$, and by the Intermediate Value Theorem this f has a root between these two values, and we decide to start with $x_0 = -1$ (you can try other values to see what happens).

Note that $f'(x) = 3x^2 - 1$, and thus we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x + n - \frac{x_n^3 - x_n + 1}{3x_n^2 - 1}$$

Thus we can produce the following values (try it):

$$
\begin{aligned}
x_0 &= -1 \\
x_1 &= -1.5000 \\
x_2 &= -1.347826.. \\
x_3 &= -1.325200.. \\
x_4 &= -1.324718.. \\
x_5 &= -1.324717.. \\
x_6 &= -1.324717.. \\
&\dots
\end{aligned}
$$

and we can now approximate the root as -1.324717. ♣

As with any numerical method, we need to be aware of the weaknesses of any technique we are using.

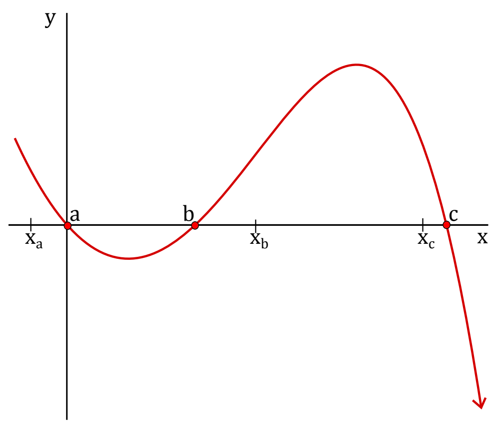


Figure 5.12: Function with three distinct solutions.

If we know our root is somewhere near a, we would make our guess $x_0 = a$. Generally speaking, a good practice is to make our guess as close to the actual root as possible. In some cases we may have no idea where the root is, so it would be prudent to perform the algorithm several times on several different initial guesses and analyze the results.

For example we can see in Figure 5.12 that $f(x)$ in fact has three roots, and depending on our initial guess, we may get the algorithm to converge to different roots. If we did not know where the roots were, we would try the technique several times. In one instance, if our initial guess was x_a, we'd likely converge to $(a, 0)$. Then if we were to choose another guess, x_b, then we'd likely converge to $(b, 0)$. Eventually, using various initial guesses we'd get one of three roots: a, b, or c. Under these circumstances we can clearly see the effectiveness of this numeric method.

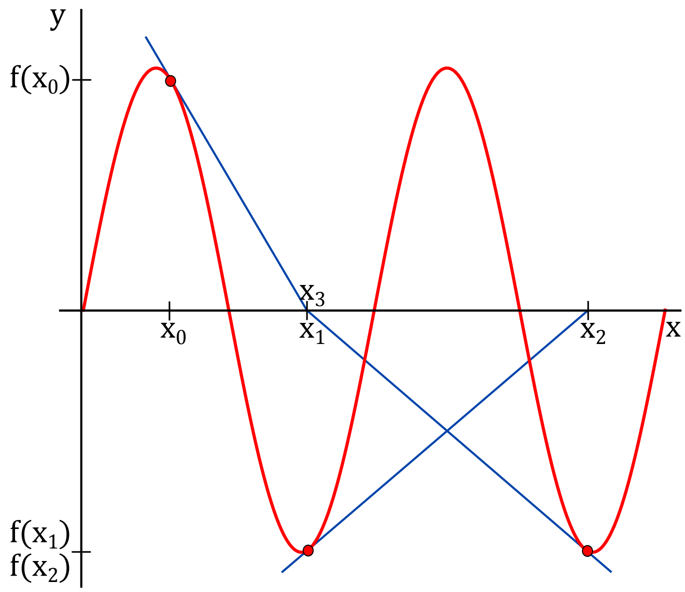


Figure 5.13: Newton's Method applied to $\sin x$**.**

As another example if we attempt to use $Newton's\ Method$ on $f(x) = \sin x$ using $x_0 = \pi/2$, then $f'(x_0) = 0$ so x_1 is undefined and we cannot proceed. Even in general x_{n+1} is typically nowhere near x_n, and in general not converging to the root nearest to our initial guess of x_0. In effect, the algorithm keeps "bouncing around". An example of which is depicted in Figure 5.13. Based on our initial guess for such a function, the algorithm may or may not converge to a root, or it may or may not converge to the root **closest** to the initial guess. This gives rise to the more common issue: Selection of the initial guess, x_0.

Here is a summary.

Key Points in using Newton's method to approximate a root of $f(x)$

1. Choosing x_0 as close as possible to the root we wish to find.
2. A guess for x_0 which makes the algorithm "bounce around" is considered *unstable*.
3. Even the smallest changes to x_0 can have drastic effects: We may converge to another root, we may converge very slowly (requiring many more iterations), or we may encounter an unstable point.
4. We may encounter a *stationary point* if we choose x_0 such that $f'(x) = 0$ (*i.e.* at a critical point!) in which case the algorithm fails.

This is all to say that your initial guess for x_0 can be extremely important.

Exercises for 5.4.4

5.4.13 *Use Newton's Method to find all roots of* $f(x) = 3x^2 - 9x - 11$*. (Hint: use Intermediate Value Theorem to choose an appropriate* x_0*)*

5.4.14 *Consider* $f(x) = x^3 - x^2 + x - 1$*.*

(a) *Using initial approximation* $x_0 = 2$*, find* x_4*.*

(b) *What is the exact value of the root of* f*? How does this compare to our approximation* x_4 *in part (a)?*

(c) *What would happen if we chose* $x_0 = 0$ *as our initial approximation?*

5.4.15 *Consider* $f(x) = \sin x$*. What happens when we choose* $x_0 = \pi/2$*? Explain.*

5.5 L'Hôpital's Rule

This section is concerned with a technique of evaluating certain limits that will be useful for determining asymptotes, and also in the following chapters, where integration is discussed.

Our treatment of limits exposed us to "0/0", an indeterminate form. If $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, we do not conclude that $\lim\limits_{x\to c} f(x)/g(x)$ is $0/0$; rather, we use $0/0$ as notation to describe the fact that both the numerator and denominator approach 0. The expression $0/0$ has no numeric value; other work must be done to evaluate the limit.

Other indeterminate forms exist; they are: ∞/∞, $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞ and ∞^0. Just as "0/0" does not mean "divide 0 by 0," the expression "∞/∞" does not mean "divide infinity by infinity." Instead, it means "a quantity is growing without bound and is being divided by another quantity that is growing without bound." We cannot determine from such a statement what value, if any, results in the limit. Likewise, "$0 \cdot \infty$" does not mean "multiply zero by infinity." Instead, it means "one quantity is shrinking to zero, and is being multiplied by a quantity that is growing without bound." We cannot determine from such a description what the result of such a limit will be.

This section introduces l'Hôpital's Rule, a method of resolving limits that produce the indeterminate forms $0/0$ and ∞/∞. We'll also show how algebraic manipulation can be used to convert other indeterminate expressions into one of these two forms so that our new rule can be applied.

Theorem 5.6: L'Hôpital's Rule, Part 1

Let $\lim\limits_{x\to c} f(x) = 0$ *and* $\lim\limits_{x\to c} g(x) = 0$*, where* f *and* g *are differentiable functions on an open interval* I *containing* c*, and* $g'(x) \neq 0$ *on* I *except possibly at* c*. Then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

We demonstrate the use of l'Hôpital's Rule in the following examples; we will often use "LHR" as an abbreviation of "l'Hôpital's Rule."

Example 5.25: Using l'Hôpital's Rule

Evaluate the following limits, using l'Hôpital's Rule as needed.

1. $\lim_{x\to 0} \frac{\sin x}{x}$
2. $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x}$
3. $\lim_{x\to 0} \frac{x^2}{1-\cos x}$
4. $\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2}$

Solution.

1. We could solve this using the Squeeze Theorem, but l'Hôpital's Rule is much simpler to apply:
$$\lim_{x\to 0} \frac{\sin x}{x} \quad \left(\to \frac{0}{0}\right) \quad \stackrel{\text{by LHR}}{=} \quad \lim_{x\to 0} \frac{\cos x}{1} = 1.$$
2. $$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x} \quad \left(\to \frac{0}{0}\right) \quad \stackrel{\text{by LHR}}{=} \quad \lim_{x\to 1} \frac{\frac{1}{2}(x+3)^{-1/2}}{-1} = -\frac{1}{4}.$$
3. $$\lim_{x\to 0} \frac{x^2}{1-\cos x} \quad \left(\to \frac{0}{0}\right) \quad \stackrel{\text{by LHR}}{=} \quad \lim_{x\to 0} \frac{2x}{\sin x}.$$
This latter limit also evaluates to the $0/0$ indeterminate form. To evaluate it, we apply l'Hôpital's Rule again.
$$\lim_{x\to 0} \frac{2x}{\sin x} \quad \stackrel{\text{by LHR}}{=} \quad \frac{2}{\cos x} = 2.$$
Thus $\lim_{x\to 0} \frac{x^2}{1-\cos x} = 2.$
4. We already know how to evaluate this limit; first factor the numerator and denominator. We then have:
$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x+3}{x-1} = 5.$$
We now show how to solve this using l'Hôpital's Rule.
$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2} \quad \left(\to \frac{0}{0}\right) \quad \stackrel{\text{by LHR}}{=} \quad \lim_{x\to 2} \frac{2x+1}{2x-3} = 5.$$

♣

Note that at each step where l'Hôpital's Rule was applied, it was *needed*: the initial limit returned the indeterminate form of "0/0." If the initial limit returns, for example, 1/2, then l'Hôpital's Rule does not apply.

The following theorem extends our initial version of l'Hôpital's Rule in two ways. It allows the technique to be applied to the indeterminate form ∞/∞ and to limits where x approaches $\pm\infty$.

Theorem 5.7: L'Hôpital's Rule, Part 2

1. *Let* $\lim_{x\to a} f(x) = \pm\infty$ *and* $\lim_{x\to a} g(x) = \pm\infty$, *where* f *and* g *are differentiable on an open interval* I *containing* a. *Then*
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$
2. *Let* f *and* g *be differentiable functions on the open interval* (a,∞) *for some value* a, *where* $g'(x) \neq 0$ *on* (a,∞) *and* $\lim_{x\to\infty} f(x)/g(x)$ *returns either 0/0 or* ∞/∞. *Then*
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$
A similar statement can be made for limits where x *approaches* $-\infty$.

Example 5.26: Using l'Hôpital's Rule with limits involving ∞

Evaluate the following limits.

1. $\lim_{x\to\infty} \dfrac{3x^2-100x+2}{4x^2+5x-1000}$ 2. $\lim_{x\to\infty} \dfrac{e^x}{x^3}$.

Solution.

1. We can evaluate this limit already using other techniques; the answer is 3/4. We apply l'Hôpital's Rule to demonstrate its applicability.
$$\lim_{x\to\infty} \frac{3x^2-100x+2}{4x^2+5x-1000} \quad \left(\to \frac{\infty}{\infty}\right) \quad \overset{\text{by LHR}}{=} \quad \lim_{x\to\infty} \frac{6x-100}{8x+5} \quad \left(\to \frac{\infty}{\infty}\right) \quad \overset{\text{by LHR}}{=} \quad \lim_{x\to\infty} \frac{6}{8} = \frac{3}{4}.$$
2. $\lim_{x\to\infty} \dfrac{e^x}{x^3} \quad \left(\to \dfrac{\infty}{\infty}\right) \quad \overset{\text{by LHR}}{=} \quad \lim_{x\to\infty} \dfrac{e^x}{3x^2} \quad \left(\to \dfrac{\infty}{\infty}\right) \quad \overset{\text{by LHR}}{=} \quad \lim_{x\to\infty} \dfrac{e^x}{6x} \quad \left(\to \dfrac{\infty}{\infty}\right) \quad \overset{\text{by LHR}}{=} \quad \lim_{x\to\infty} \dfrac{e^x}{6} = \infty.$

   Recall that this means that the limit does not exist; as x approaches ∞, the expression e^x/x^3 grows without bound. We can infer from this that e^x grows "faster" than x^3; as x gets large, e^x is far larger than x^3. (This has important implications in computing when considering efficiency of algorithms.)

♣

Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$

L'Hôpital's Rule can only be applied to ratios of functions. When faced with an indeterminate form such as $0 \cdot \infty$ or $\infty - \infty$, we can sometimes apply algebra to rewrite the limit so that l'Hôpital's Rule can be applied. We demonstrate the general idea in the next example.

Example 5.27: L'Hôpital's Rule

Compute the following limits:

1. $\lim_{x\to 0^+} x \cdot e^{1/x}$
2. $\lim_{x\to 0^-} x \cdot e^{1/x}$
3. $\lim_{x\to 0^+} x \ln x$
4. $\lim_{x\to\infty} \ln(x+1) - \ln x$
5. $\lim_{x\to\infty} x^2 - e^x$

Solution.

1. As $x \to 0^+$, $x \to 0$ and $e^{1/x} \to \infty$. Thus we have the indeterminate form $0 \cdot \infty$. We rewrite the expression $x \cdot e^{1/x}$ as $\dfrac{e^{1/x}}{1/x}$; now, as $x \to 0^+$, we get the indeterminate form ∞/∞ to which l'Hôpital's Rule can be applied.
$$\lim_{x\to 0^+} x \cdot e^{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} \overset{\text{by LHR}}{=} \lim_{x\to 0^+} \frac{(-1/x^2)e^{1/x}}{-1/x^2} = \lim_{x\to 0^+} e^{1/x} = \infty.$$
Interpretation: $e^{1/x}$ grows "faster" than x shrinks to zero, meaning their product grows without bound.

2. As $x \to 0^-$, $x \to 0$ and $e^{1/x} \to e^{-\infty} \to 0$. The the limit evaluates to $0 \cdot 0$ which is not an indeterminate form. We conclude then that
$$\lim_{x\to 0^-} x \cdot e^{1/x} = 0.$$

3. As $x \to 0^+$, $\ln x \to -\infty$, so the product is indeterminate of the form $\pm 0 \cdot \infty$. So we can apply L'Hôpital's Rule after re-writing it in the form $\frac{\infty}{\infty}$:
$$x \ln x = \frac{\ln x}{1/x} = \frac{\ln x}{x^{-1}}.$$
Now as x approaches zero, both the numerator and denominator approach infinity (one $-\infty$ and one $+\infty$, but only the size is important). Using L'Hôpital's Rule:
$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \overset{\text{by LHR}}{=} \lim_{x\to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} -x = 0.$$
Interpretation: x approaches zero much faster than the $\ln x$ approaches $-\infty$.

4. This limit initially evaluates to the indeterminate form $\infty - \infty$. By applying a logarithmic rule, we can rewrite the limit as
$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right).$$
As $x \to \infty$, the argument of the ln term approaches ∞/∞, to which we can apply l'Hôpital's Rule.
$$\lim_{x\to\infty} \frac{x+1}{x} \overset{\text{by LHR}}{=} \frac{1}{1} = 1.$$

Since $x \to \infty$ implies $\dfrac{x+1}{x} \to 1$, it follows that

$$x \to \infty \quad \text{implies} \quad \ln\left(\frac{x+1}{x}\right) \to \ln 1 = 0.$$

Thus

$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = 0.$$

Interpretation: since this limit evaluates to 0, it means that for large x, there is essentially no difference between $\ln(x+1)$ and $\ln x$; their difference is essentially 0.

5. The limit $\lim\limits_{x\to\infty} x^2 - e^x$ initially returns the indeterminate form $\infty - \infty$. We can rewrite the expression by factoring out x^2; $x^2 - e^x = x^2\left(1 - \dfrac{e^x}{x^2}\right)$. We need to evaluate how e^x/x^2 behaves as $x \to \infty$:

$$\lim_{x\to\infty} \frac{e^x}{x^2} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Thus $\lim_{x\to\infty} x^2(1 - e^x/x^2)$ evaluates to $\infty \cdot (-\infty)$, which is not an indeterminate form; rather, $\infty \cdot (-\infty)$ evaluates to $-\infty$. We conclude that $\lim\limits_{x\to\infty} x^2 - e^x = -\infty$.

Interpretation: as x gets large, the difference between x^2 and e^x grows very large.

♣

Indeterminate Forms 0^0, 1^∞ and ∞^0

When faced with an indeterminate form that involves a power, it often helps to employ the natural logarithmic function. The following Key Idea expresses the concept, which is followed by an example that demonstrates its use.

Key Idea 5.5.0: Limits Involving Indeterminate Powers

If $\lim\limits_{x\to c} \ln\big(f(x)\big) = L$, then $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} e^{\ln(f(x))} = e^{\,L}$.

Example 5.28: Using l'Hôpital's Rule with indeterminate forms involving exponents

Evaluate the following limits.

1. $\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^x$
2. $\lim\limits_{x\to 0^+} x^x$

3. $\lim_{x\to 1^+} x^{1/(x-1)}$.

Solution.

1. This limit has important applications within mathematics and finance. Note that the exponent approaches ∞ while the base approaches 1, leading to the indeterminate form 1^∞. Let $f(x) = (1+1/x)^x$; the problem asks to evaluate $\lim_{x\to\infty} f(x)$. Let's first evaluate $\lim_{x\to\infty} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to\infty} \ln\big(f(x)\big) &= \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x \\ &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\ &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\end{aligned}$$

This produces the indeterminate form $0/0$, so we apply l'Hôpital's Rule.

$$\begin{aligned}&\stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cdot(-1/x^2)}{(-1/x^2)} \\ &= \lim_{x\to\infty} \frac{1}{1+1/x} \\ &= 1.\end{aligned}$$

Thus $\lim_{x\to\infty} \ln\big(f(x)\big) = 1$. We return to the original limit and apply Key Idea 5.5.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln(f(x))} = e^1 = e.$$

2. This limit leads to the indeterminate form 0^0. Let $f(x) = x^x$ and consider first $\lim_{x\to 0^+} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to 0^+} \ln\big(f(x)\big) &= \lim_{x\to 0^+} \ln\left(x^x\right) \\ &= \lim_{x\to 0^+} x\ln x \\ &= \lim_{x\to 0^+} \frac{\ln x}{1/x}.\end{aligned}$$

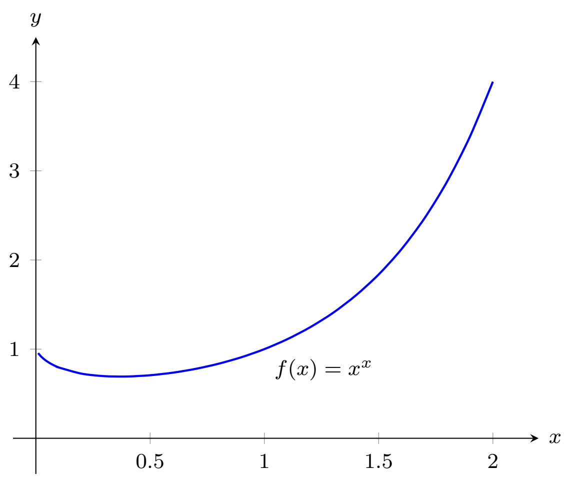
This produces the indeterminate form $-\infty/\infty$ so we apply l'Hôpital's Rule.

$$\begin{aligned}&\stackrel{\text{by LHR}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \\ &= \lim_{x\to 0^+} -x \\ &= 0.\end{aligned}$$

Thus $\lim_{x\to 0^+} \ln\big(f(x)\big) = 0$. We return to the original limit and apply Key Idea 5.5.

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln(f(x))} = e^0 = 1.$$

This result is supported by the graph of $f(x) = x^x$ given in Figure 5.14.



3. This limit is of the type "1^∞". To deal with this type of limit we will again use logarithms. Let

$$L = \lim_{x \to 1^+} x^{1/(x-1)}.$$

Now, take the natural log of both sides:

$$\ln L = \lim_{x \to 1^+} \ln \left(x^{1/(x-1)} \right).$$

Using log properties we have:

$$\ln L = \lim_{x \to 1^+} \frac{\ln x}{x-1}.$$

The right side limit is now of the type $0/0$, therefore, we can apply L'Hôpital's Rule:

$$\ln L = \lim_{x \to 1^+} \frac{\ln x}{x-1} \overset{\text{by LHR}}{=} \lim_{x \to 1^+} \frac{1/x}{1} = 1$$

Thus, $\ln L = 1$ and hence, our original limit (denoted by L) is: $L = e^1 = e$. That is,

$$L = \lim_{x \to 1^+} x^{1/(x-1)} = e.$$

In this case, even though our limit had a type of "1^∞", it actually had a value of e.

Figure 5.14: A graph of $f(x) = x^x$ supporting the fact that as $x \to 0^+$, $f(x) \to 1$.

♣

Exercises for 5.5

Compute the following limits.

5.5.1 $\displaystyle\lim_{x\to 0}\frac{\cos x - 1}{\sin x}$

5.5.2 $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^3}$

5.5.3 $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x}$

5.5.4 $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$

5.5.5 $\displaystyle\lim_{x\to 0}\frac{\sqrt{9+x}-3}{x}$

5.5.6 $\displaystyle\lim_{x\to 2}\frac{2-\sqrt{x+2}}{4-x^2}$

5.5.7 $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$

5.5.8 $\displaystyle\lim_{x\to 0}\frac{(1-x)^{1/4}-1}{x}$

5.5.9 $\displaystyle\lim_{t\to 0}\left(t+\frac{1}{t}\right)((4-t)^{3/2}-8)$

5.5.10 $\displaystyle\lim_{t\to 0^+}\left(\frac{1}{t}+\frac{1}{\sqrt{t}}\right)(\sqrt{t+1}-1)$

5.5.11 $\displaystyle\lim_{x\to 0}\frac{x^2}{\sqrt{2x+1}-1}$

5.5.12 $\displaystyle\lim_{u\to 1}\frac{(u-1)^3}{(1/u)-u^2+3/u-3}$

5.5.13 $\displaystyle\lim_{x\to 0}\frac{2+(1/x)}{3-(2/x)}$

5.5.14 $\displaystyle\lim_{x\to 0^+}\frac{1+5/\sqrt{x}}{2+1/\sqrt{x}}$

5.5.15 $\displaystyle\lim_{x\to\pi/2}\frac{\cos x}{(\pi/2)-x}$

5.5.16 $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}$

5.5.17 $\displaystyle\lim_{x\to 0}\frac{x^2}{e^x-x-1}$

5.5.18 $\displaystyle\lim_{x\to 1}\frac{\ln x}{x-1}$

5.5.19 $\displaystyle\lim_{x\to 0}\frac{\ln(x^2+1)}{x}$

5.5.20 $\displaystyle\lim_{x\to 1}\frac{x\ln x}{x^2-1}$

5.5.21 $\displaystyle\lim_{x\to 0}\frac{\sin(2x)}{\ln(x+1)}$

5.5.22 $\displaystyle\lim_{x\to 1}\frac{x^{1/4}-1}{x}$

5.5.23 $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$

5.5.24 $\displaystyle\lim_{x\to 0}\frac{3x^2+x+2}{x-4}$

5.5.25 $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\sqrt{x+4}-2}$

5.5.26 $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\sqrt{x+2}-2}$

5.5.27 $\displaystyle\lim_{x\to 0^+}\frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}$

5.5.28 $\displaystyle\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x+1}-1}$

5.5.29 $\displaystyle\lim_{x\to 1}(x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)$

5.5.30 $\displaystyle\lim_{x\to 2}\frac{x^3-6x-2}{x^3+4}$

5.5.31 $\lim\limits_{x\to\pi} \dfrac{\sin x}{x-\pi}$

5.5.32 $\lim\limits_{x\to\pi/4} \dfrac{\sin x - \cos x}{\cos(2x)}$

5.5.33 $\lim\limits_{x\to 0} \dfrac{\sin(5x)}{x}$

5.5.34 $\lim\limits_{x\to 0^+} \dfrac{e^x - 1}{x^2}$

5.5.35 $\lim\limits_{x\to 0^+} \dfrac{e^x - x - 1}{x^2}$

5.5.36 $\lim\limits_{x\to 0^+} \dfrac{x - \sin x}{x^3 - x^2}$

5.5.37 $\lim\limits_{x\to\infty} \dfrac{\left(\ln x\right)^2}{x}$

5.5.38 $\lim\limits_{x\to\infty} x^3 - x^2$

5.5.39 $\lim\limits_{x\to\infty} \sqrt{x} - \ln x$

5.5.40 $\lim\limits_{x\to-\infty} xe^x$

5.5.41 $\lim\limits_{x\to 0^+} (\sin x)^x$

Hint: use the Squeeze Theorem.

5.5.42 $\lim\limits_{x\to 1^+} (1-x)^{1-x}$

5.5.43 $\lim\limits_{x\to\infty} (x)^{1/x}$

5.5.44 $\lim\limits_{x\to 1^+} (\ln x)^{1-x}$

5.5.45 $\lim\limits_{x\to\infty} (1+x)^{1/x}$

5.5.46 $\lim\limits_{x\to\infty} (1+x^2)^{1/x}$

5.5.47 $\lim\limits_{x\to\pi/2} \tan x \cos x$

5.5.48 $\lim\limits_{x\to\pi/2} \tan x \sin(2x)$

5.5.49 $\lim\limits_{x\to 1^+} \dfrac{1}{\ln x} - \dfrac{1}{x-1}$

5.5.50 $\lim\limits_{x\to 3^+} \dfrac{5}{x^2-9} - \dfrac{x}{x-3}$

5.5.51 $\lim\limits_{x\to\infty} x\tan(1/x)$

5.5.52 $\lim\limits_{x\to\infty} \dfrac{(\ln x)^3}{x}$

5.5.53 $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{\ln x}$

5.5.54 $\lim\limits_{x\to 0^+} xe^{1/x}$

5.5.55 *Discuss what happens if we try to use L'Hôpital's rule to find the limit* $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{x+1}$.

5.6 Curve Sketching

In this section, we discuss how we can tell what the graph of a function looks like by performing simple tests on its derivatives.

5.6.1. Intervals of Increase/Decrease, and the First Derivative Test

The method of Section 5.2.1 for deciding whether there is a local maximum or minimum at a critical value is not always convenient. We can instead use information about the derivative $f'(x)$ to decide; since we have already had to compute the derivative to find the critical values, there is often relatively little extra work involved in this method.

How can the derivative tell us whether there is a maximum, minimum, or neither at a point? Suppose that f is differentiable at and around $x = a$, and suppose further that a is a critical point of f. Then we have several possibilities:

1. There is a local maximum at $x = a$. This happens if $f'(x) > 0$ as we approach $x = a$ from the left (i.e. when x is in the vicinity of a, and $x < a$) and $f'(x) < 0$ as we move to the right of $x = a$ (i.e. when x is in the vicinity of a, and $x > a$).
2. There is a local minimum at $x = a$. This happens if $f'(x) < 0$ as we approach $x = a$ from the left (i.e. when x is in the vicinity of a, and $x < a$) and $f'(x) > 0$ as we move to the right of $x = a$ (i.e. when x is in the vicinity of a, and $x > a$).
3. There is neither a local maximum or local minimum at $x = a$. If $f'(x)$ does not change from negative to positive, or from positive to negative, as we move from the left of $x = a$ to the right of $x = a$ (that is, $f'(x)$ is positive on both sides of $x = a$, or negative on both sides of $x = a$) then there is neither a maximum nor minimum when $x = a$.

See the first graph in Figure 5.5 and the graph in Figure 5.6 for examples.

Example 5.29: Local Maximum and Minimum

Find all local maximum and minimum points for $f(x) = \sin x + \cos x$ using the first derivative test.

Solution. The derivative is $f'(x) = \cos x - \sin x$ and from Example 5.8 the critical values we need to consider are $\pi/4$ and $5\pi/4$.

We analyze the graphs of $\sin x$ and $\cos x$. Just to the left of $\pi/4$ the cosine is larger than the sine, so $f'(x)$ is positive; just to the right the cosine is smaller than the sine, so $f'(x)$ is negative. This means there is a local maximum at $\pi/4$. Just to the left of $5\pi/4$ the cosine is smaller than the sine, and to the right the cosine is larger than the sine. This means that the derivative $f'(x)$ is negative to the left and positive to the right, so f has a local minimum at $5\pi/4$. ♣

The above observations have obvious intuitive appeal as you examine the graphs in Figures 5.5 and 5.6. We can extend these ideas further and then formulate and prove a theorem: If the graph of f is increasing before (i.e., to the left of) $x = a$ and decreasing after (i.e., to the right of) $x = a$, then there is a local maximum at $x = a$. If the graph of f is decreasing before $x = a$ and increasing after $x = a$, then there is a local minimum at $x = a$. If the graph of f is consistently increasing on either side of $x = a$ or consistently decreasing on either side of $x = a$, then there is neither a local maximum nor a local minimum at $x = a$. We can prove the following theorem using the Mean Value Theorem.

Theorem 5.8: Intervals of Increase and Decrease

If $f'(x) > 0$ for every x in an interval, then f is increasing on that interval.
If $f'(x) < 0$ for every x in an interval, then f is decreasing on that interval.

Proof. We will prove the increasing case. The proof of the decreasing case is similar. Suppose that $f'(x) > 0$ on an interval I. Then f is differentiable, and hence also, continuous on I. If x_1 and x_2 are any two numbers in I and $x_1 < x_2$, then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Mean Value Theorem, there is some c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But c must be in I, and thus, since $f'(x) > 0$ for every x in I, $f'(c) > 0$. Also, since $x_1 < x_2$, we have $x_2 - x_1 > 0$. Therefore, both the left hand side and the denominator of the right hand side are positive. It follows that the numerator of the right hand must be positive. That is, $f(x_2) - f(x_1) > 0$, or in other words, $f(x_1) < f(x_2)$. This shows that between x_1 and x_2 in I, the larger one, x_2, necessarily has the larger function value, $f(x_2)$, and the smaller one, x_1, necessarily have the smaller function value, $f(x_1)$. This means that f is increasing on I. ♣

Example 5.30: Local Minimum and Maximum

Consider the function $f(x) = x^4 - 2x^2$. Find where f is increasing and where f is decreasing. Use this information to find the local maximum and minimum points of f.

Solution. We compute $f'(x)$ and analyze its sign.

$$f'(x) = 4x^3 - 4x = 4x\left(x^2 - 1\right) = 4x(x-1)(x+1).$$

The solution of the inequality $f'(x) > 0$ is $(-1, 0) \cup (1, \infty)$. So, f is increasing on the interval $(-1, 0)$ and on the interval $(1, \infty)$. The solution of the inequality $f'(x) < 0$ is $(-\infty, -1) \cup (0, 1)$. So, f is decreasing on the interval $(-\infty, -1)$ and on the interval $(0, 1)$. Therefore, at the critical points -1, 0 and 1, respectively, f has a local minimum, a local maximum and a local minimum. ♣

Exercises for 5.6.1

Find all critical points and identify them as local maximum points, local minimum points, or neither.

5.6.1 $y = x^2 - x$

5.6.2 $y = 2 + 3x - x^3$

5.6.3 $y = x^3 - 9x^2 + 24x$

5.6.4 $y = x^4 - 2x^2 + 3$

5.6.5 $y = 3x^4 - 4x^3$

5.6.6 $y = (x^2 - 1)/x$

5.6.7 $y = 3x^2 - (1/x^2)$

5.6.8 $y = \cos(2x) - x$

5.6.9 $f(x) = (5 - x)/(x + 2)$

5.6.10 $f(x) = |x^2 - 121|$

5.6.11 $f(x) = x^3/(x + 1)$

5.6.12 $f(x) = \sin^2 x$

5.6.13 *Find the maxima and minima of* $f(x) = \sec x$.

5.6.14 *Let* $f(\theta) = \cos^2(\theta) - 2\sin(\theta)$. *Find the intervals where* f *is increasing and the intervals where* f *is decreasing in* $[0, 2\pi]$. *Use this information to classify the critical points of* f *as either local maximums, local minimums, or neither.*

5.6.15 *Let* $r > 0$. *Find the local maxima and minima of the function* $f(x) = \sqrt{r^2 - x^2}$ *on its domain* $[-r, r]$.

5.6.16 *Let* $f(x) = ax^2 + bx + c$ *with* $a \neq 0$. *Show that* f *has exactly one critical point using the first derivative test. Give conditions on* a *and* b *which guarantee that the critical point will be a maximum. It is possible to see this without using calculus at all; explain.*

5.6.2. The Second Derivative Test

The basis of the first derivative test is that if the derivative changes from positive to negative at a point at which the derivative is zero then there is a local maximum at the point, and similarly for a local minimum. If f' changes from positive to negative it is decreasing; this means that the derivative of f', f'', might be negative, and if in fact f'' is negative then f' is definitely decreasing. From this we determine that there is a local maximum at the point in question. Note that f' might change from positive to negative while f'' is zero, in which case f'' gives us no information about the critical value. Similarly, if f' changes from negative to positive there is a local minimum at the point, and f' is increasing. If $f'' > 0$ at the point, this tells us that f' is increasing, and so there is a local minimum.

Example 5.31: Second Derivative

Consider again $f(x) = \sin x + \cos x$, with $f'(x) = \cos x - \sin x$ and $f''(x) = -\sin x - \cos x$. Use the second derivative test to determine which critical points are local maximum or minima.

Solution. Since $f''(\pi/4) = -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2} < 0$, we know there is a local maximum at $\pi/4$. Since $f''(5\pi/4) = -(-\sqrt{2}/2) - (-\sqrt{2}/2) = \sqrt{2} > 0$, there is a local minimum at $5\pi/4$. ♣

When it works, the second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails, and sometimes the second derivative is quite difficult to evaluate; in such cases we must fall back on one of the previous tests.

Example 5.32: Second Derivative

Let $f(x) = x^4$ and $g(x) = -x^4$. Classify the critical points of $f(x)$ and $g(x)$ as either maximum or minimum.

Solution. The derivatives for $f(x)$ are $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Zero is the only critical value, but $f''(0) = 0$, so the second derivative test tells us nothing. However, $f(x)$ is positive everywhere except at zero, so clearly $f(x)$ has a local minimum at zero.

On the other hand, for $g(x) = -x^4$, $g'(x) = -4x^3$ and $g''(x) = -12x^2$. So $g(x)$ also has zero as its only only critical value, and the second derivative is again zero, but $-x^4$ has a local maximum at zero. ♣

Exercises for 5.6.2

Find all local maximum and minimum points by the second derivative test.

5.6.17 $y = x^2 - x$

5.6.18 $y = 2 + 3x - x^3$

5.6.19 $y = x^3 - 9x^2 + 24x$

5.6.20 $y = x^4 - 2x^2 + 3$

5.6.21 $y = 3x^4 - 4x^3$

5.6.22 $y = (x^2 - 1)/x$

5.6.23 $y = 3x^2 - (1/x^2)$

5.6.24 $y = \cos(2x) - x$

5.6.25 $y = 4x + \sqrt{1 - x}$

5.6.26 $y = (x+1)/\sqrt{5x^2+35}$

5.6.27 $y = x^5 - x$

5.6.28 $y = 6x + \sin 3x$

5.6.29 $y = x + 1/x$

5.6.30 $y = x^2 + 1/x$

5.6.31 $y = (x+5)^{1/4}$

5.6.32 $y = \tan^2 x$

5.6.33 $y = \cos^2 x - \sin^2 x$

5.6.34 $y = \sin^3 x$

5.6.3. Concavity and Inflection Points

We know that the sign of the derivative tells us whether a function is increasing or decreasing; for example, when $f'(x) > 0$, $f(x)$ is increasing. The sign of the second derivative $f''(x)$ tells us whether f' is increasing or decreasing; we have seen that if f' is zero and increasing at a point then there is a local minimum at the point. If f' is zero and decreasing at a point then there is a local maximum at the point. Thus, we extracted information about f from information about f''.

We can get information from the sign of f'' even when f' is not zero. Suppose that $f''(a) > 0$. This means that near $x = a$, f' is increasing. If $f'(a) > 0$, this means that f slopes up and is getting steeper; if $f'(a) < 0$, this means that f slopes down and is getting *less* steep. The two situations are shown in figure 5.15. A curve that is shaped like this is called **concave up**.

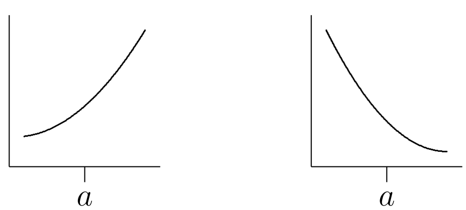


Figure 5.15: $f''(a) > 0$**:** $f'(a)$ **positive and increasing,** $f'(a)$ **negative and increasing.**

Now suppose that $f''(a) < 0$. This means that near $x = a$, f' is decreasing. If $f'(a) > 0$, this means that f slopes up and is getting less steep; if $f'(a) < 0$, this means that f slopes down and is getting steeper. The two situations are shown in figure 5.16. A curve that is shaped like this is called **concave down**.

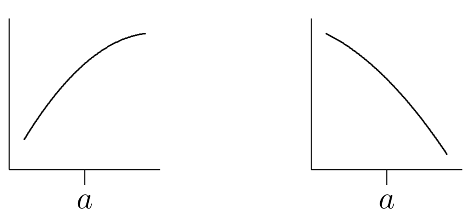


Figure 5.16: $f''(a) < 0$**:** $f'(a)$ **positive and decreasing,** $f'(a)$ **negative and decreasing.**

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. Of particular interest are points at which the concavity changes from up to down or down to up; such points are called **inflection points**. If the concavity changes from up to down at $x = a$, f'' changes from positive to the left of a to negative to the right of a, and usually $f''(a) = 0$. We can identify such points by first finding where $f''(x)$ is zero and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points. Note that it is possible that $f''(a) = 0$ but the concavity is the same on both sides; $f(x) = x^4$ at $x = 0$ is an example.

Example 5.33: Concavity

Describe the concavity of $f(x) = x^3 - x$.

Solution. The derivatives are $f'(x) = 3x^2 - 1$ and $f''(x) = 6x$. Since $f''(0) = 0$, there is potentially an inflection point at zero. Since $f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$ the concavity does change from concave down to concave up at zero, and the curve is concave down for all $x < 0$ and concave up for all $x > 0$. ♣

Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.

Exercises for 5.6.3

Describe the concavity of the functions below.

5.6.35 $y = x^2 - x$

5.6.36 $y = 2 + 3x - x^3$

5.6.37 $y = x^3 - 9x^2 + 24x$

5.6.38 $y = x^4 - 2x^2 + 3$

5.6.39 $y = 3x^4 - 4x^3$

5.6.40 $y = (x^2 - 1)/x$

5.6.41 $y = 3x^2 - (1/x^2)$

5.6.42 $y = \sin x + \cos x$

5.6.43 $y = 4x + \sqrt{1 - x}$

5.6.44 $y = (x + 1)/\sqrt{5x^2 + 35}$

5.6.45 $y = x^5 - x$

5.6.46 $y = 6x + \sin 3x$

5.6.47 $y = x + 1/x$

5.6.48 $y = x^2 + 1/x$

5.6.49 $y = (x + 5)^{1/4}$

5.6.50 $y = \tan^2 x$

5.6.51 $y = \cos^2 x - \sin^2 x$

5.6.52 $y = \sin^3 x$

5.6.53 *Identify the intervals on which the graph of the function* $f(x) = x^4 - 4x^3 + 10$ *is of one of these four shapes: concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing.*

5.6.54 *Describe the concavity of* $y = x^3 + bx^2 + cx + d$. *You will need to consider different cases, depending on the values of the coefficients.*

5.6.55 *Let* n *be an integer greater than or equal to two, and suppose* f *is a polynomial of degree* n. *How many inflection points can* f *have? Hint: Use the second derivative test and the fundamental theorem of algebra.*

5.6.4. Asymptotes and Other Things to Look For

A vertical asymptote is a place where the function becomes infinite, typically because the formula for the function has a denominator that becomes zero. For example, the reciprocal function $f(x) = 1/x$ has a vertical asymptote at $x = 0$, and the function $\tan x$ has a vertical asymptote at $x = \pi/2$ (and also at $x = -\pi/2$, $x = 3\pi/2$, etc.). Whenever the formula for a function contains a denominator it is worth looking for a vertical asymptote by checking to see if the denominator can ever be zero, and then checking the limit at such points. Note that there is not always a vertical asymptote where the derivative is zero: $f(x) = (\sin x)/x$ has a zero denominator at $x = 0$, but since $\lim_{x\to 0}(\sin x)/x = 1$ there is no asymptote there.

A horizontal asymptote is a horizontal line to which $f(x)$ gets closer and closer as x approaches ∞ (or as x approaches $-\infty$). For example, the reciprocal function has the x-axis for a horizontal asymptote. Horizontal asymptotes can be identified by computing the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Since $\lim_{x\to\infty} 1/x = \lim_{x\to-\infty} 1/x = 0$, the line $y = 0$ (that is, the x-axis) is a horizontal asymptote in both directions.

Some functions have asymptotes that are neither horizontal nor vertical, but some other line. Such asymptotes are somewhat more difficult to identify and we will ignore them.

If the domain of the function does not extend out to infinity, we should also ask what happens as x approaches the boundary of the domain. For example, the function $y = f(x) = 1/\sqrt{r^2 - x^2}$ has domain $-r < x < r$, and y becomes infinite as x approaches either r or $-r$. In this case we might also identify this behavior because when $x = \pm r$ the denominator of the function is zero.

If there are any points where the derivative fails to exist (a cusp or corner), then we should take special note of what the function does at such a point.

Finally, it is worthwhile to notice any symmetry. A function $f(x)$ that has the same value for $-x$ as for x, i.e., $f(-x) = f(x)$, is called an "even function." Its graph is symmetric with respect to the y-axis. Some examples of even functions are: x^n when n is an even number, $\cos x$, and $\sin^2 x$. On the other hand, a function that satisfies the property $f(-x) = -f(x)$ is called an "odd function." Its graph is symmetric with respect to the origin. Some examples of odd functions are: x^n when n is an odd number, $\sin x$, and $\tan x$. Of course, most functions are neither even nor odd, and do not have any particular symmetry.

5.6.5. Summary of Curve Sketching

The following is a guideline for sketching a curve $y = f(x)$ by hand. Each item may not be relevant to the function in question, but utilizing this guideline will provide all information needed to make a detailed sketch of the function.

Key Idea 5.6.0: Guideline for Curve Sketching

1. Domain of the function
2. x- and y-Intercepts
3. Symmetry
4. Vertical and Horizontal Asymptotes
5. Intervals of Increase/Decrease, and Local Extrema
6. Concavity and Points of Inflection
7. Sketch the Graph

Example 5.34: Graph Sketching

Sketch the graph of $y = f(x)$ *where* $f(x) = \dfrac{2x^2}{x^2-1}$

Solution.

1. The domain is $\{x : x^2 - 1 \neq 0\} = \{x : x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
2. There is an x-intercept at $x = 0$. The y intercept is $y = 0$.
3. $f(-x) = f(x)$, so f is an even function (symmetric about y-axis)
4. $\displaystyle\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{2}{1-1/x^2} = 2$, so $y = 2$ is a horizontal asymptote.

   Now the denominator is 0 at $x = \pm 1$, so we compute:

   $$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = +\infty, \ \lim_{x\to 1^-} \frac{2x^2}{x^2-1} = -\infty, \ \lim_{x\to -1^+} \frac{2x^2}{x^2-1} = -\infty, \ \lim_{x\to -1^-} \frac{2x^2}{x^2-1} = +\infty.$$

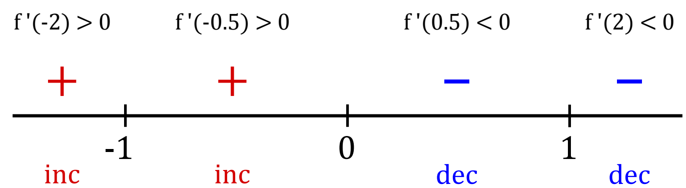
   So the lines $x = 1$ and $x = -1$ are vertical asymptotes.

5. For critical values we take the derivative:

$$f'(x) = \frac{4x(x^2-1) - 2x^2 \cdot 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

Note that $f'(x) = 0$ when $x = 0$ (the top is zero). Also, $f'(x) = DNE$ when $x = \pm 1$ (the bottom is zero). As $x = \pm 1$ is *not* in the domain of $f(x)$, the only critical number is $x = 0$ (recall that to be a critical number we need it to be in the domain of the original function).

Drawing a number line and including *all* of the split points of $f'(x)$ we have:



Thus f is increasing on $(-\infty, -1) \cup (-1, 0)$ and decreasing on $(0, 1) \cup (1, \infty)$.

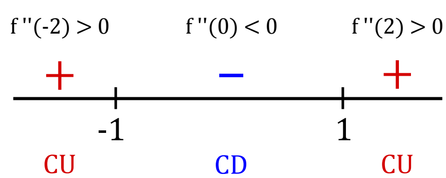
By the first derivative test, $x = 0$ is a local max.

6. For possible inflection points we take the second derivative:

$$f''(x) = \frac{12x^2 + 4}{(x^2-1)^3}$$

The top is never zero. Also, the bottom is only zero when $x = \pm 1$ (neither of which are in the domain of $f(x)$). Thus, there are no possible inflection points to consider.

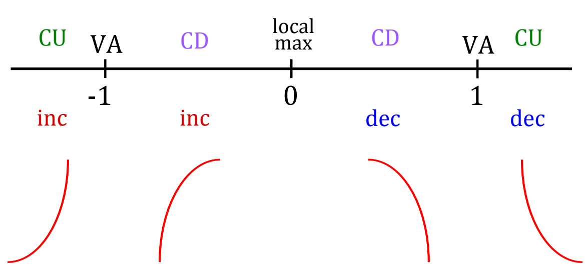
Drawing a number line and including *all* of the split points of $f''(x)$ we have:



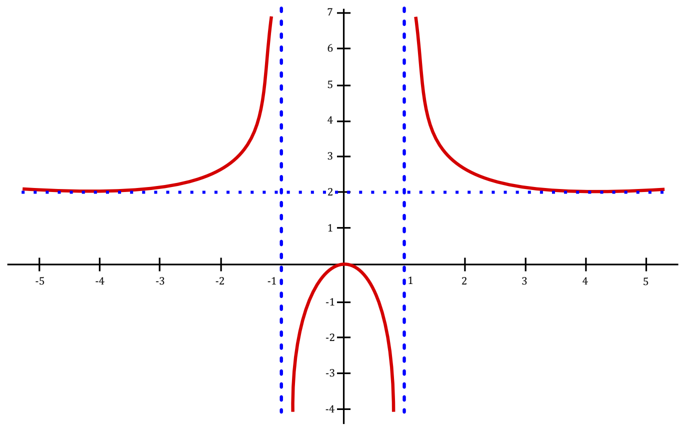
Hence f is concave up on $(-\infty, -1) \cup (1, \infty)$, concave down on $(-1, 1)$.

7. We put this information together and sketch the graph.

We combine some of this information on a single number line to see what *shape* the graph has on certain intervals:



Note that there is a horizontal asymptote at $y = 2$ and that the curve has x-int of $x = 0$ and y-int of $y = 0$. Therefore, a sketch of $f(x)$ is as follows:



♣

Exercises for 5.6

Sketch the curves. Identify clearly any interesting features, including local maximum and minimum points, inflection points, asymptotes, and intercepts.

5.6.56 $y = x^5 - 5x^4 + 5x^3$

5.6.57 $y = x^3 - 3x^2 - 9x + 5$

5.6.58 $y = (x-1)^2(x+3)^{2/3}$

5.6.59 $x^2 + x^2y^2 = a^2y^2$, $a > 0$.

5.6.60 $y = xe^x$

5.6.61 $y = (e^x + e^{-x})/2$

5.6.62 $y = e^{-x}\cos x$

5.6.63 $y = e^x - \sin x$

5.6.64 $y = e^x/x$

5.6.65 $y = 4x + \sqrt{1-x}$

5.6.66 $y = (x+1)/\sqrt{5x^2+35}$

5.6.67 $y = x^5 - x$

5.6.68 $y = 6x + \sin 3x$

5.6.69 $y = x + 1/x$

5.6.70 $y = x^2 + 1/x$

5.6.71 $y = (x+5)^{1/4}$

5.6.72 $y = \tan^2 x$

5.6.73 $y = \cos^2 x - \sin^2 x$

5.6.74 $y = \sin^3 x$

5.6.75 $y = x(x^2 + 1)$

5.6.76 $y = x^3 + 6x^2 + 9x$

5.6.77 $y = x/(x^2 - 9)$

5.6.78 $y = x^2/(x^2 + 9)$

5.6.79 $y = 2\sqrt{x} - x$

5.6.80 $y = 3\sin(x) - \sin^3(x)$, *for* $x \in [0, 2\pi]$

5.6.81 $y = (x-1)/(x^2)$

5.7 Optimization Problems

Many important applied problems involve finding the best way to accomplish some task. Often this involves finding the maximum or minimum value of some function: the minimum time to make a certain journey, the minimum cost for doing a task, the maximum power that can be generated by a device, and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

Generally such a problem will have the following mathematical form: Find the largest (or smallest) value of $f(x)$ when $a \le x \le b$. Sometimes a or b are infinite, but frequently the real world imposes some constraint on the values that x may have.

Such a problem differs in two ways from the local maximum and minimum problems we encountered when graphing functions: We are interested only in the function between a and b, and we want to know the largest or smallest value that $f(x)$ takes on, not merely values that are the largest or smallest in a small interval. That is, we seek not a local maximum or minimum but a *global* (or *absolute*) maximum or minimum.

Key Idea 5.7.0: Guidelines to solving an optimization problem.

1. Understand clearly what is to be maximized or minimized and what the contraints are.
2. Draw a diagram (if appropriate) and label it.
3. Decide what the variables are. For example, A for area, r for radius, C for cost.
4. Write a formula for the function for which you wish to find the maximum or minimum.
5. Express that formula in terms of only one variable, that is, in the form $f(x)$. Usually this is accomplished by using the given constraints.
6. Set $f'(x) = 0$ and solve. Check all critical values and endpoints to determine the extreme value(s) of $f(x)$.

Example 5.35: Largest Rectangle

Find the largest rectangle (that is, the rectangle with largest area) that fits inside the graph of the parabola $y = x^2$ below the line $y = a$ (a is an unspecified constant value), with the top side of the rectangle on the horizontal line $y = a$; see Figure 5.17.)

Solution. We want to find the maximum value of some function $A(x)$ representing area. Perhaps the hardest part of this problem is deciding what x should represent. The lower right corner of the rectangle is at (x, x^2), and once this is chosen the rectangle is completely determined. So we can let the x in $A(x)$ be the x of the parabola $f(x) = x^2$. Then the area is

$$A(x) = (2x)(a - x^2) = -2x^3 + 2ax.$$

We want the maximum value of $A(x)$ when x is in $[0, \sqrt{a}]$. (You might object to allowing $x = 0$ or $x = \sqrt{a}$, since then the "rectangle" has either no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area.)

Setting $0 = A'(x) = 6x^2 + 2a$ we get $x = \sqrt{a/3}$ as the only critical value. Testing this and the two endpoints, we have $A(0) = A(\sqrt{a}) = 0$ and $A(\sqrt{a/3}) = (4/9)\sqrt{3}a^{3/2}$. The maximum area thus occurs when the rectangle has dimensions $2\sqrt{a/3} \times (2/3)a$. ♣

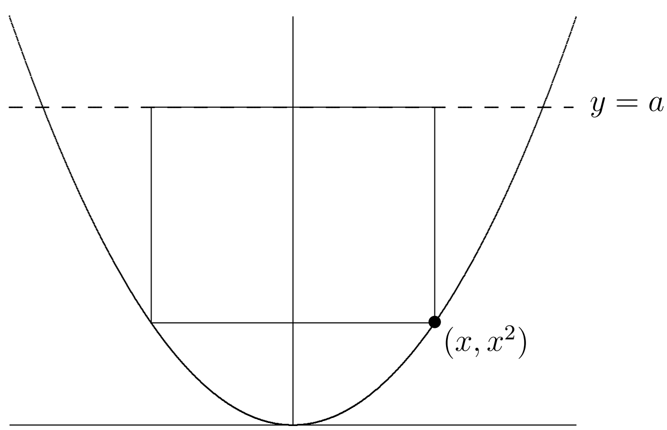


Figure 5.17: Rectangle in a parabola.

Example 5.36: Largest Cone

If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)

Solution. Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: $\pi r^2 h/3$. Here R is a fixed value, but r and h can vary. Namely, we could choose r to be as large as possible—equal to R—by taking the height equal to R; or we could make the cone's height h larger at the expense of making r a little less than R. See the cross-section depicted in Figure 5.18. We have situated the picture in a convenient way relative to the x and y axes, namely, with the center of the sphere at the origin and the vertex of the cone at the far left on the x-axis.

Notice that the function we want to maximize, $\pi r^2 h/3$, depends on *two* variables. This is frequently the case, but often the two variables are related in some way so that "really" there is only one variable. So our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure: the upper corner of the triangle, whose coordinates are $(h-R, r)$, must be on the circle of radius R. That is,

$$(h-R)^2 + r^2 = R^2.$$

We can solve for h in terms of r or for r in terms of h. Either involves taking a square root, but we notice that the volume function contains r^2, not r by itself, so it is easiest to solve for r^2 directly: $r^2 = R^2 - (h-R)^2$. Then we substitute the result into $\pi r^2 h/3$:

$$\begin{aligned} V(h) &= \pi(R^2 - (h-R)^2)h/3 \\ &= -\frac{\pi}{3}h^3 + \frac{2}{3}\pi h^2 R \end{aligned}$$

We want to maximize $V(h)$ when h is between 0 and $2R$. Now we solve $0 = f'(h) = -\pi h^2 + (4/3)\pi h R$, getting $h = 0$ or $h = 4R/3$. We compute $V(0) = V(2R) = 0$ and $V(4R/3) = (32/81)\pi R^3$. The maximum is the latter; since the volume of the sphere is $(4/3)\pi R^3$, the fraction of the sphere occupied by the cone is

$$\frac{(32/81)\pi R^3}{(4/3)\pi R^3} = \frac{8}{27} \approx 30\%.$$

♣

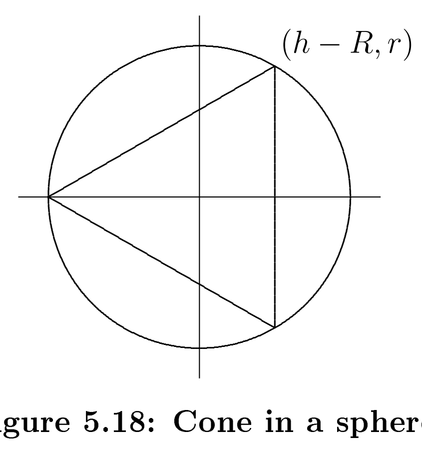


Figure 5.18: Cone in a sphere.

Example 5.37: Containers of Given Volume

You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder.
Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.

Solution. Let us first choose letters to represent various things: h for the height, r for the base radius, V for the volume of the cylinder, and c for the cost per unit area of the lateral side of the cylinder; V and c are constants, h and r are variables. Now we can write the cost of materials:

$$c(2\pi rh) + Nc(2\pi r^2).$$

Again we have two variables; the relationship is provided by the fixed volume of the cylinder: $V = \pi r^2 h$. We use this relationship to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). The result is

$$f(r) = 2c\pi r \frac{V}{\pi r^2} + 2Nc\pi r^2 = \frac{2cV}{r} + 2Nc\pi r^2.$$

We want to know the minimum value of this function when r is in $(0, \infty)$. We now set $0 = f'(r) = -2cV/r^2 + 4Nc\pi r$, giving $r = \sqrt[3]{V/(2N\pi)}$. Since $f''(r) = 4cV/r^3 + 4Nc\pi$ is positive when r is positive, there is a local minimum at the critical value, and hence a global minimum since there is only one critical value.

Finally, since $h = V/(\pi r^2)$,

$$\frac{h}{r} = \frac{V}{\pi r^3} = \frac{V}{\pi(V/(2N\pi))} = 2N,$$

so the minimum cost occurs when the height h is $2N$ times the radius. If, for example, there is no difference in the cost of materials, the height is twice the radius (or the height is equal to the diameter). ♣

Example 5.38: Rectangles of Given Area

Of all rectangles of area 100, which has the smallest perimeter?

Solution. First we must translate this into a purely mathematical problem in which we want to find the minimum value of a function. If x denotes one of the sides of the rectangle, then the adjacent side must be $100/x$ (in order that the area be 100). So the function we want to minimize is

$$f(x) = 2x + 2\frac{100}{x}$$

since the perimeter is twice the length plus twice the width of the rectangle. Not all values of x make sense in this problem: lengths of sides of rectangles must be positive, so $x > 0$. If $x > 0$ then so is $100/x$, so we need no second condition on x.

We next find $f'(x)$ and set it equal to zero: $0 = f'(x) = 2 - 200/x^2$. Solving $f'(x) = 0$ for x gives us $x = \pm 10$. We are interested only in $x > 0$, so only the value $x = 10$ is of interest. Since $f'(x)$ is defined everywhere on the interval $(0, \infty)$, there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at $x = 10$? The second derivative is $f''(x) = 400/x^3$, and $f''(10) > 0$, so there is a local minimum. Since there is only one critical value, this is also the global minimum, so the rectangle with smallest perimeter is the 10×10 square. ♣

Example 5.39: Maximize your Profit

You want to sell a certain number n of items in order to maximize your profit. Market research tells you that if you set the price at \$1.50, you will be able to sell 5000 items, and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total \$2000, and the per item cost of production ("marginal cost") is \$0.50.
Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

Solution. The first step is to convert the problem into a function maximization problem. Since we want to maximize profit by setting the price per item, we should look for a function $P(x)$ representing the profit when the price per item is x. Profit is revenue minus costs, and revenue is number of items sold times the price per item, so we get $P = nx - 2000 - 0.50n$. The number of items sold is itself a function of x, $n = 5000 + 1000(1.5 - x)/0.10$, because $(1.5 - x)/0.10$ is the number of multiples of 10 cents that the price is below \$1.50. Now we substitute for n in the profit function:

$$\begin{aligned} P(x) &= (5000 + 1000(1.5 - x)/0.10)x - 2000 - 0.5(5000 + 1000(1.5 - x)/0.10) \\ &= -10000x^2 + 25000x - 12000 \end{aligned}$$

We want to know the maximum value of this function when x is between 0 and 1.5. The derivative is $P'(x) = -20000x + 25000$, which is zero when $x = 1.25$. Since $P''(x) = -20000 < 0$, there must be a local maximum at $x = 1.25$, and since this is the only critical value it must be a global maximum as well. (Alternately, we could compute $P(0) = -12000$, $P(1.25) = 3625$, and $P(1.5) = 3000$ and note that $P(1.25)$ is the maximum of these.) Thus the maximum profit is \$3625, attained when we set the price at \$1.25 and sell 7500 items. ♣

Example 5.40: Minimize Travel Time

Suppose you want to reach a point A that is located across the sand from a nearby road (see Figure 5.19). Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Solution. Let x be the distance short of C where you turn off, i.e., the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is distance divided by velocity.

You travel the distance $\overline{DB}$ at speed v, and then the distance $\overline{BA}$ at speed w. Since $\overline{DB} = a - x$ and, by the Pythagorean theorem, $\overline{BA} = \sqrt{x^2 + b^2}$, the total time for the trip is

$$f(x) = \frac{a - x}{v} + \frac{\sqrt{x^2 + b^2}}{w}.$$

We want to find the minimum value of f when x is between 0 and a. As usual we set $f'(x) = 0$ and solve for x:

$$\begin{aligned} 0 = f'(x) &= -\frac{1}{v} + \frac{x}{w\sqrt{x^2 + b^2}} \\ w\sqrt{x^2 + b^2} &= vx \end{aligned}$$

$$w^2(x^2+b^2) = v^2x^2$$
$$w^2b^2 = (v^2-w^2)x^2$$
$$x = \frac{wb}{\sqrt{v^2-w^2}}$$

Notice that a does not appear in the last expression, but a is not irrelevant, since we are interested only in critical values that are in $[0,a]$, and $wb/\sqrt{v^2-w^2}$ is either in this interval or not. If it is, we can use the second derivative to test it:

$$f''(x) = \frac{b^2}{(x^2+b^2)^{3/2}w}.$$

Since this is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0,a]$ it is larger than a. In this case the minimum must occur at one of the endpoints. We can compute

$$f(0) = \frac{a}{v} + \frac{b}{w}$$

$$f(a) = \frac{\sqrt{a^2+b^2}}{w}$$

but it is difficult to determine which of these is smaller by direct comparison. If, as is likely in practice, we know the values of v, w, a, and b, then it is easy to determine this. With a little cleverness, however, we can determine the minimum in general. We have seen that $f''(x)$ is always positive, so the derivative $f'(x)$ is always increasing. We know that at $wb/\sqrt{v^2-w^2}$ the derivative is zero, so for values of x less than that critical value, the derivative is negative. This means that $f(0) > f(a)$, so the minimum occurs when $x = a$.

So the upshot is this: If you start farther away from C than $wb/\sqrt{v^2-w^2}$ then you always want to cut across the sand when you are a distance $wb/\sqrt{v^2-w^2}$ from point C. If you start closer than this to C, you should cut directly across the sand. ♣

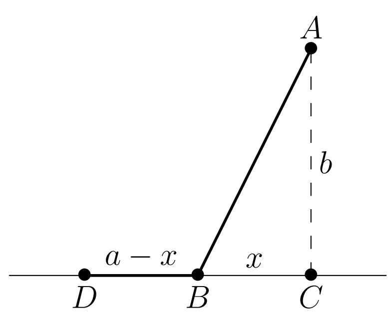


Figure 5.19: Minimizing travel time.

Exercises for Section 5.7

5.7.1 *Find the dimensions of the rectangle of largest area having fixed perimeter* 100.

5.7.2 *Find the dimensions of the rectangle of largest area having fixed perimeter* P.

5.7.3 *A box with square base and no top is to hold a volume* 100. *Find the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base.*

5.7.4 *A box with square base is to hold a volume* 200. *The bottom and top are formed by folding in flaps from all four sides, so that the bottom and top consist of two layers of cardboard. Find the dimensions of the box that requires the least material. Also find the ratio of height to side of the base.*

5.7.5 *A box with square base and no top is to hold a volume* V. *Find (in terms of* V*) the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. (This ratio will not involve* V*.)*

5.7.6 *You have* 100 *feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area?*

5.7.7 *You have* l *feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area?*

5.7.8 *Marketing tells you that if you set the price of an item at $10 then you will be unable to sell it, but that you can sell 500 items for each dollar below $10 that you set the price. Suppose your fixed costs total $3000, and your marginal cost is $2 per item. What is the most profit you can make?*

5.7.9 *Find the area of the largest rectangle that fits inside a semicircle of radius* 10 *(one side of the rectangle is along the diameter of the semicircle).*

5.7.10 *Find the area of the largest rectangle that fits inside a semicircle of radius* r *(one side of the rectangle is along the diameter of the semicircle).*

5.7.11 *For a cylinder with surface area* 50, *including the top and the bottom, find the ratio of height to base radius that maximizes the volume.*

5.7.12 *For a cylinder with given surface area* S, *including the top and the bottom, find the ratio of height to base radius that maximizes the volume.*

5.7.13 *You want to make cylindrical containers to hold 1 liter using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side* $2r$, *so that* $2(2r)^2 = 8r^2$ *of material is needed (rather than* $2\pi r^2$, *which is the total area of the top and bottom). Find the dimensions of the container using the least amount of material, and also find the ratio of height to radius for this container.*

5.7.14 *You want to make cylindrical containers of a given volume* V *using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side* $2r$, *so that* $2(2r)^2 = 8r^2$ *of material is needed (rather than* $2\pi r^2$, *which is the total area of the top and bottom). Find the optimal ratio of height to radius.*

5.7.15 *Given a right circular cone, you put an upside-down cone inside it so that its vertex is at the center of the base of the larger cone and its base is parallel to the base of the larger cone. If you choose the* *upside-down cone to have the largest possible volume, what fraction of the volume of the larger cone does it occupy? (Let H and R be the height and base radius of the larger cone, and let h and r be the height and base radius of the smaller cone. Hint: Use similar triangles to get an equation relating h and r.)*

5.7.16 *A container holding a fixed volume is being made in the shape of a cylinder with a hemispherical top. (The hemispherical top has the same radius as the cylinder.) Find the ratio of height to radius of the cylinder which minimizes the cost of the container if (a) the cost per unit area of the top is twice as great as the cost per unit area of the side, and the container is made with no bottom; (b) the same as in (a), except that the container is made with a circular bottom, for which the cost per unit area is 1.5 times the cost per unit area of the side.*

5.7.17 *A piece of cardboard is 1 meter by $1/2$ meter. A square is to be cut from each corner and the sides folded up to make an open-top box. What are the dimensions of the box with maximum possible volume?*

5.7.18 *(a) A square piece of cardboard of side a is used to make an open-top box by cutting out a small square from each corner and bending up the sides. How large a square should be cut from each corner in order that the box have maximum volume? (b) What if the piece of cardboard used to make the box is a rectangle of sides a and b?*

5.7.19 *A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top; the colored glass transmits only $1/2$ as much light per unit area as the the clear glass. If the distance from top to bottom (across both the rectangle and the semicircle) is 2 meters and the window may be no more than 1.5 meters wide, find the dimensions of the rectangular portion of the window that lets through the most light.*

5.7.20 *A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top. Suppose that the colored glass transmits only k times as much light per unit area as the clear glass (k is between 0 and 1). If the distance from top to bottom (across both the rectangle and the semicircle) is a fixed distance H, find (in terms of k) the ratio of vertical side to horizontal side of the rectangle for which the window lets through the most light.*

5.7.21 *You are designing a poster to contain a fixed amount A of printing (measured in square centimeters) and have margins of a centimeters at the top and bottom and b centimeters at the sides. Find the ratio of vertical dimension to horizontal dimension of the printed area on the poster if you want to minimize the amount of posterboard needed.*

5.7.22 *What fraction of the volume of a sphere is taken up by the largest cylinder that can be fit inside the sphere?*

5.7.23 *The U.S. post office will accept a box for shipment only if the sum of the length and girth (distance around) is at most 108 in. Find the dimensions of the largest acceptable box with square front and back.*

5.7.24 *Find the dimensions of the lightest cylindrical can containing 0.25 liter (=250 cm^3) if the top and bottom are made of a material that is twice as heavy (per unit area) as the material used for the side.*

5.7.25 *A conical paper cup is to hold $1/4$ of a liter. Find the height and radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula $\pi r\sqrt{r^2+h^2}$ for the area of the side of a cone.*

5.7.26 *A conical paper cup is to hold a fixed volume of water. Find the ratio of height to base radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula* $\pi r\sqrt{r^2+h^2}$ *for the area of the side of a cone, called the* **lateral area** *of the cone.*

5.7.27 *Find the fraction of the area of a triangle that is occupied by the largest rectangle that can be drawn in the triangle (with one of its sides along a side of the triangle). Show that this fraction does not depend on the dimensions of the given triangle.*

5.7.28 *How are your answers to Problem 5.7.8 affected if the cost per item for the* x *items, instead of being simply \$2, decreases below \$2 in proportion to* x *(because of economy of scale and volume discounts) by 1 cent for each 25 items produced?*

5.8 Antiderivatives and Indefinite Integration

We have spent considerable time considering the derivatives of a function and their applications. In the following chapters, we are going to starting thinking in "the other direction." That is, given a function $f(x)$, we are going to consider functions $F(x)$ such that $F'(x) = f(x)$. There are numerous reasons this will prove to be useful: these functions will help us compute areas, volumes, mass, force, pressure, work, and much more.

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another?

And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. "Finding another" may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2 + 123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

Definition 5.4: Antiderivatives and Indefinite Integrals

Let a function $f(x)$ *be given. An* ***antiderivative*** *of* $f(x)$ *on an open interval* I *is a function* $F(x)$ *such that* $F'(x) = f(x)$ *for all* $x \in I$.

The set of all antiderivatives of $f(x)$ *is referred to as the General Antiderivative, ot the* ***(indefinite) integral of*** f*, denoted by*

$$\int f(x)\, dx.$$

The process of finding the indefinite integral is called **integration** (or **integrating** $f(x)$). Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them. We often use upper-case letters to denote antiderivatives.

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem 5.9: Antiderivative Forms

Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then there exists a constant C such that

$$G(x) = F(x) + C.$$

To see why this is true, observe that if F and G are both antiderivatives of f, then $F' = G' = f$, so $\frac{d}{dx}(G - F) = G' - F' = f - f = 0$, so $G - F$ is a constant function giving $G - F = C$, so $G = F + C$.

So, given a function f and one of its antiderivatives F, we know *all* antiderivatives of f have the form $F(x) + C$ for some constant C. Using Definition 6.1, we can say that

$$\int f(x)\ dx = F(x) + C.$$

Let's analyze this indefinite integral notation.

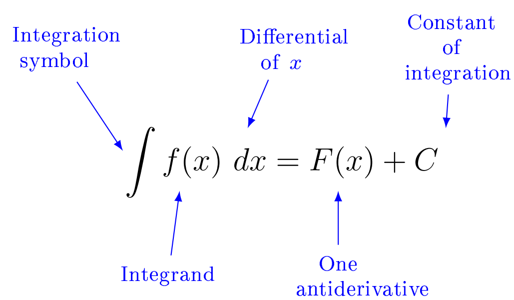


Figure 5.20: Understanding the indefinite integral notation.

Figure 6.1 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related.

The function we want to find an antiderivative of is called the *integrand*. It contains the differential of the variable we are integrating with respect to. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Let's practice using this notation.

Example 5.41: Evaluating indefinite integrals

Evaluate $\int \sin x\ dx.$

Solution. We are asked to find all functions $F(x)$ such that $F'(x) = \sin x$. Some thought will lead us to one solution: $F(x) = -\cos x$, because $\frac{d}{dx}(-\cos x) = \sin x$.

The indefinite integral of $\sin x$ is thus $-\cos x$, plus a constant of integration. So:

$$\int \sin x \, dx = -\cos x + C.$$

♣

Common mistakes: One habit students make with integrals is to *drop the dx* at the end of the integral. This is required! Think of the integral as a set of parenthesis. Both are required so it is clear where the integrand ends and what variable you are integrating with respect to.

Another common mistake is to *forget the $+C$* for indefinite integrals.

A commonly asked question is "What happened to the dx?" The unenlightened response is "Don't worry about it. It just goes away." A full understanding includes the following.

This process of *antidifferentiation* is really solving a *differential* question. The integral

$$\int \sin x \, dx$$

presents us with a differential, $dy = \sin x \, dx$. It is asking: "What is y?" We found lots of solutions, all of the form $y = -\cos x + C$.

Letting $dy = \sin x \, dx$, rewrite

$$\int \sin x \, dx \quad \text{as} \quad \int dy.$$

This is asking: "What functions have a differential of the form dy?" The answer is "Functions of the form $y + C$, where C is a constant." What is y? We have lots of choices, all differing by a constant; the simplest choice is $y = -\cos x$.

Understanding all of this is more important later as we try to find antiderivatives of more complicated functions. In this section, we will simply explore the rules of indefinite integration, and one can succeed for now with answering "What happened to the dx?" with "It went away."

Let's practice once more before stating integration rules.

Example 5.42: Evaluating indefinite integrals

Evaluate $\int (3x^2 + 4x + 5) \, dx$.

Solution. We seek a function $F(x)$ whose derivative is $3x^2 + 4x + 5$. When taking derivatives, we can consider functions term–by–term, so we can likely do that here.

What functions have a derivative of $3x^2$? Some thought will lead us to a cubic, specifically $x^3 + C_1$, where C_1 is a constant.

What functions have a derivative of $4x$? Here the x term is raised to the first power, so we likely seek a quadratic. Some thought should lead us to $2x^2 + C_2$, where C_2 is a constant.

Finally, what functions have a derivative of 5? Functions of the form $5x + C_3$, where C_3 is a constant.

Our answer appears to be

$$\int (3x^2 + 4x + 5) \, dx = x^3 + C_1 + 2x^2 + C_2 + 5x + C_3.$$

We do not need three separate constants of integration; combine them as one constant, giving the final answer of

$$\int (3x^2 + 4x + 5) \, dx = x^3 + 2x^2 + 5x + C.$$

It is easy to verify our answer; take the derivative of $x^3 + 2x^3 + 5x + C$ and see we indeed get $3x^2 + 4x + 5$. ♣

This final step of "verifying our answer" is important both practically and theoretically. In general, taking derivatives is easier than finding antiderivatives so checking our work is easy and vital as we learn.

We also see that taking the derivative of our answer returns the function in the integrand. Thus we can say that:

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x).$$

Differentiation "undoes" the work done by antidifferentiation.

The table at the end of Chapter 4 gave a list of the derivatives of common functions we had learned at that point. We restate part of that list here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn.

Theorem 5.10: Derivatives and Antiderivatives

Common Differentiation Rules

1. $\frac{d}{dx}\big(cf(x)\big) = c \cdot f'(x)$
2. $\frac{d}{dx}\big(f(x) \pm g(x)\big) =$
   $f'(x) \pm g'(x)$
3. $\frac{d}{dx}\big(C\big) = 0$
4. $\frac{d}{dx}\big(x\big) = 1$
5. $\frac{d}{dx}\big(x^n\big) = n \cdot x^{n-1}$
6. $\frac{d}{dx}\big(\sin x\big) = \cos x$
7. $\frac{d}{dx}\big(\cos x\big) = -\sin x$
8. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$
9. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$
10. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$
11. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$
12. $\frac{d}{dx}\big(e^x\big) = e^x$
13. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$
14. $\frac{d}{dx}\big(\ln x\big) = \frac{d}{dx}\big(\ln |x|\big) = \frac{1}{x}$
15. $\frac{d}{dx}\big(\cosh x\big) = \sinh x$
16. $\frac{d}{dx}\big(\sinh x\big) = \cosh x$
17. $\frac{d}{dx}\big(\ln(\cosh x)\big) = \tanh x$
18. $\frac{d}{dx}\big(\ln(\sinh x)\big) = \coth x$
19. $\dfrac{d}{dx}\big(\sin^{-1}(x)\big) = \dfrac{1}{\sqrt{1-x^2}}$
20. $\dfrac{d}{dx}\big(\tan^{-1}(x)\big) = \dfrac{1}{1+x^2}$

Common Indefinite Integral Rules

1. $\int c \cdot f(x)\ dx = c \cdot \int f(x)\ dx$
2. $\int \big(f(x) \pm g(x)\big)\ dx =$
   $\int f(x)\ dx \pm \int g(x)\ dx$
3. $\int 0\ dx = C$
4. $\int 1\ dx = \int dx = x + C$
5. $\int x^n\ dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
6. $\int \cos x\ dx = \sin x + C$
7. $\int \sin x\ dx = -\cos x + C$
8. $\int \sec^2 x\ dx = \tan x + C$
9. $\int \csc x \cot x\ dx = -\csc x + C$
10. $\int \sec x \tan x\ dx = \sec x + C$
11. $\int \csc^2 x\ dx = -\cot x + C$
12. $\int e^x\ dx = e^x + C$
13. $\int a^x\ dx = \frac{1}{\ln a} \cdot a^x + C$
14. $\int \frac{1}{x}\ dx = \ln|x| + C$
15. $\displaystyle\int \cosh x\ dx = \sinh x + C$
16. $\displaystyle\int \sinh x\ dx = \cosh x + C$
17. $\displaystyle\int \tanh x\ dx = \ln(\cosh x) + C$
18. $\displaystyle\int \coth x\ dx = \ln|\sinh x| + C$
19. $\displaystyle\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$
20. $\displaystyle\int \frac{1}{1+x^2}\ dx = \tan^{-1}(x) + C$

We highlight a few important points from Theorem 6.2:

- Rule #1 states $\int c \cdot f(x)\ dx = c \cdot \int f(x)\ dx$. This is the Constant Multiple Rule: we can temporarily ignore constants when finding antiderivatives, just as we did when computing derivatives (i.e., $\frac{d}{dx}\big(3x^2\big)$

is just as easy to compute as $\frac{d}{dx}\left(x^2\right)$). An example:

$$\int 5\cos x\ dx = 5\cdot\int \cos x\ dx = 5\cdot(\sin x + C) = 5\sin x + C.$$

In the last step we can consider the constant as also being multiplied by 5, but "5 times a constant" is still a constant, so we just write "C".

- Rule #2 is the Sum/Difference Rule: we can split integrals apart when the integrand contains terms that are added/subtracted, as we did in Example **??**. So:

$$\begin{aligned}\int (3x^2+4x+5)\ dx &= \int 3x^2\ dx + \int 4x\ dx + \int 5\ dx\\ &= 3\int x^2\ dx + 4\int x\ dx + \int 5\ dx\\ &= 3\cdot\frac{1}{3}x^3 + 4\cdot\frac{1}{2}x^2 + 5x + C\\ &= x^3 + 2x^2 + 5x + C\end{aligned}$$

  In practice we generally do not write out all these steps, but we demonstrate them here for completeness.

- Rule #5 is the Power Rule of indefinite integration. There are two important things to keep in mind:

  1. Notice the restriction that $n \neq -1$. This is important: $\int \frac{1}{x}\ dx \neq$ "$\frac{1}{0}x^0 + C$"; rather, see Rule #14.
  2. We are presenting antidifferentiation as the "inverse operation" of differentiation. Here is a useful quote to remember:

     "Inverse operations do the opposite things in the opposite order."

     When taking a derivative using the Power Rule, we **first** *multiply* by the power, then **second** *subtract* 1 from the power. To find the antiderivative, do the opposite things in the opposite order: **first** *add* one to the power, then **second** *divide* by the power.

- Note that Rule #14 incorporates the absolute value of x. The exercises will work the reader through why this is the case; for now, know the absolute value is important and cannot be ignored.

Note that we don't have properties to deal with products or quotients of functions, that is, in general

$$\int f(x)\cdot g(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx.$$

$$\int \frac{f(x)}{g(x)}\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}.$$

With derivatives, we had the product and quotient rules to deal with these cases. For integrals, we have no such rules, but we will learn a variety of different techniques to deal with these cases.

Example 5.43: Indefinite Integral

If $f'(x) = x^4 + 2x - 8\sin x$ then what is $f(x)$?

Solution. The answer is:

$$\begin{aligned} f(x) = \int f'(x)\,dx &= \int \left(x^4 + 2x - 8\sin x\right)\,dx \\ &= \int x^4\,dx + 2\int x\,dx - 8\int \sin x\,dx \\ &= \frac{x^5}{5} + x^2 + 8\cos x + C, \end{aligned}$$

where C is a constant. ♣

Example 5.44: Indefinite Integral

Find the indefinite integral: $\int 3x^2\,dx$.

Solution.

$$\begin{aligned} \int 3x^2\,dx &= 3\int x^2\,dx \\ &= 3\frac{x^3}{3} + C \\ &= x^3 + C \end{aligned}$$

♣

Example 5.45: Indefinite Integral

Find the general antiderivative of $\frac{2}{\sqrt{x}}\,dx$.

Solution.

$$\begin{aligned} \int \frac{2}{\sqrt{x}}\,dx &= 2\int x^{-\frac{1}{2}}\,dx \\ &= 2\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\ &= 4\sqrt{x} + C \end{aligned}$$

♣

Example 5.46: Indefinite Integral

Find: $\displaystyle\int \left(\frac{1}{x} + e^{7x} + x^{\pi} + 7\right) dx.$

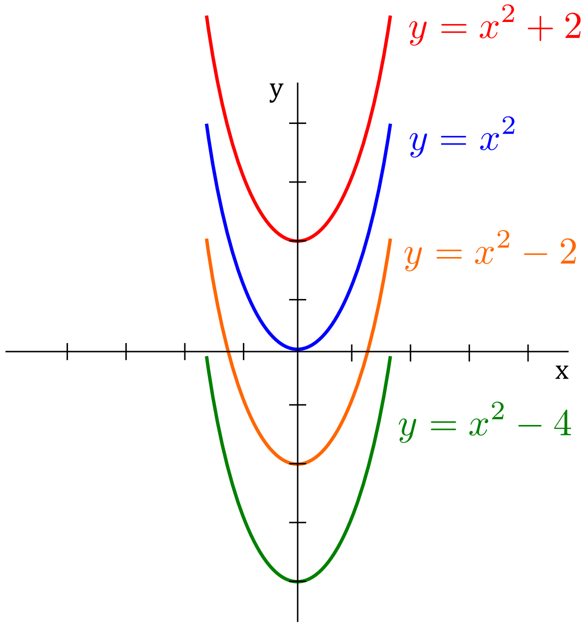
Solution.

$$\begin{aligned}\int \left(\frac{1}{x} + e^{7x} + x^{\pi} + 7\right) dx &= \int \frac{1}{x}\,dx + \int e^{7x}\,dx + \int x^{\pi}\,dx + \int 7\,dx \\ &= \ln|x| + \frac{1}{7}e^{7x} + \frac{x^{\pi+1}}{\pi+1} + 7x + C\end{aligned}$$

♣

Differential Equations and Initial Value Problems

An equation involving derivatives where we want to solve for the original function is called a **differential equation**. For example, $f'(x) = 2x$ is a differential equation with general solution $f(x) = x^2 + C$. Some solutions (i.e., particular values of C) are shown below.



As seen with integral curves, we may have an infinite family of solutions satisfying the differential equation. However, if we were given a point (called an *initial value*) on the curve then we could determine $f(x)$ completely. Such a problem is known as an *initial value problem*.

Example 5.47: Initial Value Problem

If $f'(x) = 2x$ and $f(0) = 2$ then determine $f(x)$.

Solution. As previously stated, we have a solution of:

$$f(x) = x^2 + C.$$

But $f(0) = 2$ implies:

$$2 = 0^2 + C \quad \to \quad C = 2.$$

Therefore, $f(x) = x^2 + 2$ is the solution to the initial value problem. ♣

We have seen that the derivative of a position function is a velocity function, and the derivative of a velocity function describes acceleration. We can now go "the other way:" the antiderivative of an acceleration function gives a velocity function, etc. While there is just one derivative of a given function, there are infinite antiderivatives. Therefore we cannot ask "What is *the* velocity of an object whose acceleration is -32ft/s^2?", since there is more than one answer.

As above, we can find *the* answer if we provide more information with the question, in the form of an initial value, a value of the function that one knows beforehand.

Example 5.48: Solving initial value problems

The acceleration due to gravity of a falling object is -32 ft/s^2. At time $t = 3$, a falling object had a velocity of -10 ft/s. Find the equation of the object's velocity.

Solution. We want to know a velocity function, $v(t)$. We know two things:

- The acceleration, i.e., $v'(t) = -32$, and
- the velocity at a specific time, i.e., $v(3) = -10$.

Using the first piece of information, we know that $v(t)$ is an antiderivative of $v'(t) = -32$. So we begin by finding the indefinite integral of -32:

$$\int (-32)\; dt = -32t + C = v(t).$$

Now we use the fact that $v(3) = -10$ to find C:

$$\begin{aligned} v(t) &= -32t + C \\ v(3) &= -10 \\ -32(3) + C &= -10 \\ C &= 86 \end{aligned}$$

Thus $v(t) = -32t + 86$. We can use this equation to understand the motion of the object: when $t = 0$, the object had a velocity of $v(0) = 86$ ft/s. Since the velocity is positive, the object was moving upward.

When did the object begin moving down? Immediately after $v(t) = 0$:

$$-32t + 86 = 0 \quad \Rightarrow \quad t = \frac{43}{16} \approx 2.69\text{s}.$$

Recognize that we are able to determine quite a bit about the path of the object knowing just its acceleration and its velocity at a single point in time. ♣

Example 5.49: Solving initial value problems

Find $f(t)$, given that $f''(t) = \cos t$, $f'(0) = 3$ and $f(0) = 5$.

Solution. We start by finding $f'(t)$, which is an antiderivative of $f''(t)$:

$$\int f''(t)\ dt = \int \cos t\ dt = \sin t + C = f'(t).$$

So $f'(t) = \sin t + C$ for the correct value of C. We are given that $f'(0) = 3$, so:

$$f'(0) = 3 \quad \Rightarrow \quad \sin 0 + C = 3 \quad \Rightarrow \quad C = 3.$$

Using the initial value, we have found $f'(t) = \sin t + 3$.

We now find $f(t)$ by integrating again.

$$f(t) = \int f'(t)\ dt = \int (\sin t + 3)\ dt = -\cos t + 3t + C.$$

We are given that $f(0) = 5$, so

$$\begin{aligned} -\cos 0 + 3(0) + C &= 5 \\ -1 + C &= 5 \\ C &= 6 \end{aligned}$$

Thus $f(t) = -\cos t + 3t + 6$. ♣

This section introduced antiderivatives and the indefinite integral. We found they are needed when finding a function given information about its derivative(s). For instance, we found a position function given a velocity function.

In the next section, we will see how position and velocity are unexpectedly related by the areas of certain regions on a graph of the velocity function. Then, in Section 6.3, we will see how areas and antiderivatives are closely tied together.

6. Integration

6.1 Antiderivatives and Indefinite Integration

We have spent considerable time considering the derivatives of a function and their applications. In the following chapters, we are going to starting thinking in "the other direction." That is, given a function $f(x)$, we are going to consider functions $F(x)$ such that $F'(x) = f(x)$. There are numerous reasons this will prove to be useful: these functions will help us compute areas, volumes, mass, force, pressure, work, and much more.

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another?

And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. "Finding another" may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2 + 123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

Definition 6.1: Antiderivatives and Indefinite Integrals

Let a function $f(x)$ be given. An ***antiderivative*** *of $f(x)$ on an open interval I is a function $F(x)$ such that $F'(x) = f(x)$ for all $x \in I$.*

The set of all antiderivatives of $f(x)$ is referred to as the General Antiderivative, ot the ***(indefinite) integral of*** *f, denoted by*

$$\int f(x)\, dx.$$

The process of finding the indefinite integral is called **integration** (or **integrating** $f(x)$). Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them. We often use upper-case letters to denote antiderivatives.

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem 6.1: Antiderivative Forms

Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then there exists a constant C such that

$$G(x) = F(x) + C.$$

To see why this is true, observe that if F and G are both antiderivatives of f, then $F' = G' = f$, so $\frac{d}{dx}(G - F) = G' - F' = f - f = 0$, so $G - F$ is a constant function giving $G - F = C$, so $G = F + C$.

So, given a function f and one of its antiderivatives F, we know *all* antiderivatives of f have the form $F(x) + C$ for some constant C. Using Definition 6.1, we can say that

$$\int f(x)\ dx = F(x) + C.$$

Let's analyze this indefinite integral notation.

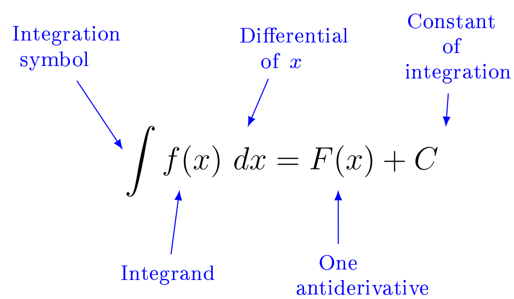


Figure 6.1: Understanding the indefinite integral notation.

Figure 6.1 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related.

The function we want to find an antiderivative of is called the *integrand*. It contains the differential of the variable we are integrating with respect to. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Let's practice using this notation.

Example 6.1: Evaluating indefinite integrals

Evaluate $\displaystyle\int \sin x\ dx$.

Solution. We are asked to find all functions $F(x)$ such that $F'(x) = \sin x$. Some thought will lead us to one solution: $F(x) = -\cos x$, because $\frac{d}{dx}(-\cos x) = \sin x$.

The indefinite integral of $\sin x$ is thus $-\cos x$, plus a constant of integration. So:

$$\int \sin x\ dx = -\cos x + C.$$

♣

Common mistakes: One habit students make with integrals is to *drop the dx* at the end of the integral. This is required! Think of the integral as a set of parenthesis. Both are required so it is clear where the integrand ends and what variable you are integrating with respect to.

Another common mistake is to *forget the $+C$* for indefinite integrals.

A commonly asked question is "What happened to the dx?" The unenlightened response is "Don't worry about it. It just goes away." A full understanding includes the following.

This process of *antidifferentiation* is really solving a *differential* question. The integral

$$\int \sin x \; dx$$

presents us with a differential, $dy = \sin x \; dx$. It is asking: "What is y?" We found lots of solutions, all of the form $y = -\cos x + C$.

Letting $dy = \sin x \; dx$, rewrite

$$\int \sin x \; dx \quad \text{as} \quad \int dy.$$

This is asking: "What functions have a differential of the form dy?" The answer is "Functions of the form $y + C$, where C is a constant." What is y? We have lots of choices, all differing by a constant; the simplest choice is $y = -\cos x$.

Understanding all of this is more important later as we try to find antiderivatives of more complicated functions. In this section, we will simply explore the rules of indefinite integration, and one can succeed for now with answering "What happened to the dx?" with "It went away."

Let's practice once more before stating integration rules.

Example 6.2: Evaluating indefinite integrals

Evaluate $\int (3x^2 + 4x + 5) \; dx$.

Solution. We seek a function $F(x)$ whose derivative is $3x^2 + 4x + 5$. When taking derivatives, we can consider functions term–by–term, so we can likely do that here.

What functions have a derivative of $3x^2$? Some thought will lead us to a cubic, specifically $x^3 + C_1$, where C_1 is a constant.

What functions have a derivative of $4x$? Here the x term is raised to the first power, so we likely seek a quadratic. Some thought should lead us to $2x^2 + C_2$, where C_2 is a constant.

Finally, what functions have a derivative of 5? Functions of the form $5x + C_3$, where C_3 is a constant.

Our answer appears to be

$$\int (3x^2 + 4x + 5) \; dx = x^3 + C_1 + 2x^2 + C_2 + 5x + C_3.$$

We do not need three separate constants of integration; combine them as one constant, giving the final answer of

$$\int (3x^2 + 4x + 5) \; dx = x^3 + 2x^2 + 5x + C.$$

It is easy to verify our answer; take the derivative of $x^3 + 2x^3 + 5x + C$ and see we indeed get $3x^2 + 4x + 5$.

♣

This final step of “verifying our answer” is important both practically and theoretically. In general, taking derivatives is easier than finding antiderivatives so checking our work is easy and vital as we learn.

We also see that taking the derivative of our answer returns the function in the integrand. Thus we can say that:

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x).$$

Differentiation “undoes” the work done by antidifferentiation.

The table at the end of Chapter 4 gave a list of the derivatives of common functions we had learned at that point. We restate part of that list here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn.

Theorem 6.2: Derivatives and Antiderivatives

Common Differentiation Rules

1. $\frac{d}{dx}\big(cf(x)\big) = c \cdot f'(x)$
2. $\frac{d}{dx}\big(f(x) \pm g(x)\big) =$ $f'(x) \pm g'(x)$
3. $\frac{d}{dx}\big(C\big) = 0$
4. $\frac{d}{dx}\big(x\big) = 1$
5. $\frac{d}{dx}\big(x^n\big) = n \cdot x^{n-1}$
6. $\frac{d}{dx}\big(\sin x\big) = \cos x$
7. $\frac{d}{dx}\big(\cos x\big) = -\sin x$
8. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$
9. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$
10. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$
11. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$
12. $\frac{d}{dx}\big(e^x\big) = e^x$
13. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$
14. $\frac{d}{dx}\big(\ln x\big) = \frac{d}{dx}\big(\ln |x|\big) = \frac{1}{x}$
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19. $\dfrac{d}{dx}\big(\sin^{-1}(x)\big) = \dfrac{1}{\sqrt{1-x^2}}$
20. $\dfrac{d}{dx}\big(\tan^{-1}(x)\big) = \dfrac{1}{1+x^2}$

Common Indefinite Integral Rules

1. $\int c \cdot f(x)\ dx = c \cdot \int f(x)\ dx$
2. $\int \big(f(x) \pm g(x)\big)\ dx =$ $\int f(x)\ dx \pm \int g(x)\ dx$
3. $\int 0\ dx = C$
4. $\int 1\ dx = \int dx = x + C$
5. $\int x^n\ dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
6. $\int \cos x\ dx = \sin x + C$
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We highlight a few important points from Theorem 6.2:

- Rule #1 states $\int c \cdot f(x)\ dx = c \cdot \int f(x)\ dx$. This is the Constant Multiple Rule: we can temporarily ignore constants when finding antiderivatives, just as we did when computing derivatives (i.e., $\frac{d}{dx}\big(3x^2\big)$

is just as easy to compute as $\frac{d}{dx}(x^2)$). An example:

$$\int 5\cos x\ dx = 5\cdot\int \cos x\ dx = 5\cdot(\sin x + C) = 5\sin x + C.$$

In the last step we can consider the constant as also being multiplied by 5, but "5 times a constant" is still a constant, so we just write "C".

- Rule #2 is the Sum/Difference Rule: we can split integrals apart when the integrand contains terms that are added/subtracted, as we did in Example **??**. So:

$$\begin{aligned}\int (3x^2+4x+5)\ dx &= \int 3x^2\ dx + \int 4x\ dx + \int 5\ dx\\ &= 3\int x^2\ dx + 4\int x\ dx + \int 5\ dx\\ &= 3\cdot\frac{1}{3}x^3 + 4\cdot\frac{1}{2}x^2 + 5x + C\\ &= x^3 + 2x^2 + 5x + C\end{aligned}$$

  In practice we generally do not write out all these steps, but we demonstrate them here for completeness.

- Rule #5 is the Power Rule of indefinite integration. There are two important things to keep in mind:

  1. Notice the restriction that $n \neq -1$. This is important: $\int \frac{1}{x}\ dx \neq$ "$\frac{1}{0}x^0 + C$"; rather, see Rule #14.
  2. We are presenting antidifferentiation as the "inverse operation" of differentiation. Here is a useful quote to remember:

     "Inverse operations do the opposite things in the opposite order."

     When taking a derivative using the Power Rule, we **first** *multiply* by the power, then **second** *subtract* 1 from the power. To find the antiderivative, do the opposite things in the opposite order: **first** *add* one to the power, then **second** *divide* by the power.

- Note that Rule #14 incorporates the absolute value of x. The exercises will work the reader through why this is the case; for now, know the absolute value is important and cannot be ignored.

Note that we don't have properties to deal with products or quotients of functions, that is, in general

$$\int f(x)\cdot g(x)\,dx \neq \int f(x)\,dx \int g(x)\,dx.$$

$$\int \frac{f(x)}{g(x)}\,dx \neq \frac{\int f(x)\,dx}{\int g(x)\,dx}.$$

With derivatives, we had the product and quotient rules to deal with these cases. For integrals, we have no such rules, but we will learn a variety of different techniques to deal with these cases.

Example 6.3: Indefinite Integral

If $f'(x) = x^4 + 2x - 8\sin x$ then what is $f(x)$?

Solution. The answer is:

$$\begin{aligned} f(x) = \int f'(x)\,dx &= \int \left(x^4 + 2x - 8\sin x\right)\,dx \\ &= \int x^4\,dx + 2\int x\,dx - 8\int \sin x\,dx \\ &= \frac{x^5}{5} + x^2 + 8\cos x + C, \end{aligned}$$

where C is a constant. ♣

Example 6.4: Indefinite Integral

Find the indefinite integral: $\int 3x^2\,dx$.

Solution.

$$\begin{aligned} \int 3x^2\,dx &= 3\int x^2\,dx \\ &= 3\frac{x^3}{3} + C \\ &= x^3 + C \end{aligned}$$

♣

Example 6.5: Indefinite Integral

Find the general antiderivative of $\frac{2}{\sqrt{x}}\,dx$.

Solution.

$$\begin{aligned} \int \frac{2}{\sqrt{x}}\,dx &= 2\int x^{-\frac{1}{2}}\,dx \\ &= 2\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\ &= 4\sqrt{x} + C \end{aligned}$$

♣

Example 6.6: Indefinite Integral

Find: $\int \left(\frac{1}{x} + e^{7x} + x^{\pi} + 7 \right) dx.$

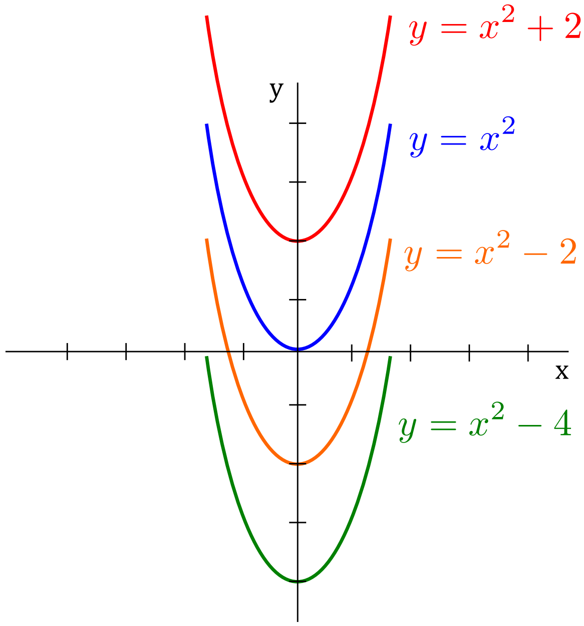
Solution.

$$\begin{aligned} \int \left(\frac{1}{x} + e^{7x} + x^{\pi} + 7 \right) dx &= \int \frac{1}{x}\, dx + \int e^{7x}\, dx + \int x^{\pi}\, dx + \int 7\, dx \\ &= \ln|x| + \frac{1}{7} e^{7x} + \frac{x^{\pi+1}}{\pi+1} + 7x + C \end{aligned}$$

♣

Differential Equations and Initial Value Problems

An equation involving derivatives where we want to solve for the original function is called a **differential equation**. For example, $f'(x) = 2x$ is a differential equation with general solution $f(x) = x^2 + C$. Some solutions (i.e., particular values of C) are shown below.



As seen with integral curves, we may have an infinite family of solutions satisfying the differential equation. However, if we were given a point (called an *initial value*) on the curve then we could determine $f(x)$ completely. Such a problem is known as an *initial value problem.*

Example 6.7: Initial Value Problem

If $f'(x) = 2x$ *and* $f(0) = 2$ *then determine* $f(x)$*.*

Solution. As previously stated, we have a solution of:

$$f(x) = x^2 + C.$$

But $f(0) = 2$ implies:

$$2 = 0^2 + C \quad \to \quad C = 2.$$

Therefore, $f(x) = x^2 + 2$ is the solution to the initial value problem. ♣

We have seen that the derivative of a position function is a velocity function, and the derivative of a velocity function describes acceleration. We can now go "the other way:" the antiderivative of an acceleration function gives a velocity function, etc. While there is just one derivative of a given function, there are infinite antiderivatives. Therefore we cannot ask "What is *the* velocity of an object whose acceleration is -32ft/s^2?", since there is more than one answer.

As above, we can find *the* answer if we provide more information with the question, in the form of an initial value, a value of the function that one knows beforehand.

Example 6.8: Solving initial value problems

The acceleration due to gravity of a falling object is -32 ft/s^2. At time $t = 3$, a falling object had a velocity of -10 ft/s. Find the equation of the object's velocity.

Solution. We want to know a velocity function, $v(t)$. We know two things:

- The acceleration, i.e., $v'(t) = -32$, and
- the velocity at a specific time, i.e., $v(3) = -10$.

Using the first piece of information, we know that $v(t)$ is an antiderivative of $v'(t) = -32$. So we begin by finding the indefinite integral of -32:

$$\int (-32)\ dt = -32t + C = v(t).$$

Now we use the fact that $v(3) = -10$ to find C:

$$\begin{aligned} v(t) &= -32t + C \\ v(3) &= -10 \\ -32(3) + C &= -10 \\ C &= 86 \end{aligned}$$

Thus $v(t) = -32t + 86$. We can use this equation to understand the motion of the object: when $t = 0$, the object had a velocity of $v(0) = 86$ ft/s. Since the velocity is positive, the object was moving upward.

When did the object begin moving down? Immediately after $v(t) = 0$:

$$-32t + 86 = 0 \quad \Rightarrow \quad t = \frac{43}{16} \approx 2.69\text{s}.$$

Recognize that we are able to determine quite a bit about the path of the object knowing just its acceleration and its velocity at a single point in time. ♣

Example 6.9: Solving initial value problems

Find $f(t)$, given that $f''(t) = \cos t$, $f'(0) = 3$ and $f(0) = 5$.

Solution. We start by finding $f'(t)$, which is an antiderivative of $f''(t)$:

$$\int f''(t)\ dt = \int \cos t\ dt = \sin t + C = f'(t).$$

So $f'(t) = \sin t + C$ for the correct value of C. We are given that $f'(0) = 3$, so:

$$f'(0) = 3 \quad \Rightarrow \quad \sin 0 + C = 3 \quad \Rightarrow \quad C = 3.$$

Using the initial value, we have found $f'(t) = \sin t + 3$.

We now find $f(t)$ by integrating again.

$$f(t) = \int f'(t)\ dt = \int (\sin t + 3)\ dt = -\cos t + 3t + C.$$

We are given that $f(0) = 5$, so

$$\begin{aligned} -\cos 0 + 3(0) + C &= 5 \\ -1 + C &= 5 \\ C &= 6 \end{aligned}$$

Thus $f(t) = -\cos t + 3t + 6$. ♣

This section introduced antiderivatives and the indefinite integral. We found they are needed when finding a function given information about its derivative(s). For instance, we found a position function given a velocity function.

In the next section, we will see how position and velocity are unexpectedly related by the areas of certain regions on a graph of the velocity function. Then, in Section 6.3, we will see how areas and antiderivatives are closely tied together.

6.2 Displacement and Area

Example 6.10: Object Moving in a Straight Line

An object moves in a straight line so that its speed at time t is given by $v(t) = 3t$ in, say, cm/sec. If the object is at position 10 on the straight line when $t = 0$, where is the object at any time t?

Solution. There are two reasonable ways to approach this problem. If $s(t)$ is the position of the object at time t, we know that $s'(t) = v(t)$. Based on our knowledge of derivatives, we therefore know that $s(t) = 3t^2/2 + k$, and because $s(0) = 10$ we easily discover that $k = 10$, so $s(t) = 3t^2/2 + 10$. For example, at $t = 1$ the object is at position $3/2 + 10 = 11.5$. This is certainly the easiest way to deal with this problem. Not all similar problems are so easy, as we will see; the second approach to the problem is more difficult but also more general.

We start by considering how we might approximate a solution. We know that at $t = 0$ the object is at position 10. How might we approximate its position at, say, $t = 1$? We know that the speed of the object at time $t = 0$ is 0; if its speed were constant then in the first second the object would not move and its position would still be 10 when $t = 1$. In fact, the object will not be too far from 10 at $t = 1$, but certainly we can do better. Let's look at the times 0.1, 0.2, 0.3, ..., 1.0, and try approximating the location of the object at each, by supposing that during each tenth of a second the object is going at a constant speed. Since the object initially has speed 0, we again suppose it maintains this speed, but only for a tenth of second; during that time the object would not move. During the tenth of a second from $t = 0.1$ to $t = 0.2$, we suppose that the object is traveling at 0.3 cm/sec, namely, its actual speed at $t = 0.1$. In this case the object would travel $(0.3)(0.1) = 0.03$ centimeters: 0.3 cm/sec times 0.1 seconds. Similarly, between $t = 0.2$ and $t = 0.3$ the object would travel $(0.6)(0.1) = 0.06$ centimeters. Continuing, we get as an approximation that the object travels

$$(0.0)(0.1) + (0.3)(0.1) + (0.6)(0.1) + \cdots + (2.7)(0.1) = 1.35$$

centimeters, ending up at position 11.35. This is a better approximation than 10, certainly, but is still just an approximation. (We know in fact that the object ends up at position 11.5, because we've already done the problem using the first approach.) Presumably, we will get a better approximation if we divide the time into one hundred intervals of a hundredth of a second each, and repeat the process:

$$(0.0)(0.01) + (0.03)(0.01) + (0.06)(0.01) + \cdots + (2.97)(0.01) = 1.485.$$

We thus approximate the position as 11.485. Since we know the exact answer, we can see that this is much closer, but if we did not already know the answer, we wouldn't really know how close.

We can keep this up, but we'll never really know the exact answer if we simply compute more and more examples. Let's instead look at a "typical" approximation. Suppose we divide the time into n equal intervals, and imagine that on each of these the object travels at a constant speed. Over the first time interval we approximate the distance traveled as $(0.0)(1/n) = 0$, as before. During the second time interval, from $t = 1/n$ to $t = 2/n$, the object travels approximately $3(1/n)(1/n) = 3/n^2$ centimeters. During time interval number i, the object travels approximately $(3(i-1)/n)(1/n) = 3(i-1)/n^2$ centimeters, that is, its speed at time $(i-1)/n$, $3(i-1)/n$, times the length of time interval number i, $1/n$. Adding these up as before, we approximate the distance traveled as

$$(0)\frac{1}{n} + 3\frac{1}{n^2} + 3(2)\frac{1}{n^2} + 3(3)\frac{1}{n^2} + \cdots + 3(n-1)\frac{1}{n^2}$$

centimeters. What can we say about this? At first it looks rather less useful than the concrete calculations we've already done, but in fact a bit of algebra reveals it to be much more useful. We can factor out a 3 and $1/n^2$ to get

$$\frac{3}{n^2}(0 + 1 + 2 + 3 + \cdots + (n-1)),$$

that is, $3/n^2$ times the sum of the first $n-1$ positive integers. Now we make use of a fact you may have run across before, Gauss's Equation:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

In our case we're interested in $k = n - 1$, so

$$1 + 2 + 3 + \cdots + (n-1) = \frac{(n-1)(n)}{2} = \frac{n^2 - n}{2}.$$

This simplifies the approximate distance traveled to

$$\frac{3}{n^2}\frac{n^2-n}{2}=\frac{3}{2}\frac{n^2-n}{n^2}=\frac{3}{2}\left(\frac{n^2}{n^2}-\frac{n}{n^2}\right)=\frac{3}{2}\left(1-\frac{1}{n}\right).$$

Now this is quite easy to understand: as n gets larger and larger this approximation gets closer and closer to $(3/2)(1-0)=3/2$, so that $3/2$ is the exact distance traveled during one second, and the final position is 11.5.

So for $t=1$, at least, this rather cumbersome approach gives the same answer as the first approach. But really there's nothing special about $t=1$; let's just call it t instead. In this case the approximate distance traveled during time interval number i is $3(i-1)(t/n)(t/n)=3(i-1)t^2/n^2$, that is, speed $3(i-1)(t/n)$ times time t/n, and the total distance traveled is approximately

$$(0)\frac{t}{n}+3(1)\frac{t^2}{n^2}+3(2)\frac{t^2}{n^2}+3(3)\frac{t^2}{n^2}+\cdots+3(n-1)\frac{t^2}{n^2}.$$

As before we can simplify this to

$$\frac{3t^2}{n^2}(0+1+2+\cdots+(n-1))=\frac{3t^2}{n^2}\frac{n^2-n}{2}=\frac{3}{2}t^2\left(1-\frac{1}{n}\right).$$

In the limit, as n gets larger, this gets closer and closer to $(3/2)t^2$ and the approximated position of the object gets closer and closer to $(3/2)t^2+10$, so the actual position is $(3/2)t^2+10$, exactly the answer given by the first approach to the problem. ♣

Example 6.11: Area under the Line

Find the area under the curve $y=3x$ between $x=0$ and any positive value x.

Solution. There is here no obvious analogue to the first approach in the previous example, but the second approach works fine. (Since the function $y=3x$ is so simple, there is another approach that works here, but it is even more limited in potential application than is approach number one.) How might we approximate the desired area? We know how to compute areas of rectangles, so we approximate the area by rectangles. Jumping straight to the general case, suppose we divide the interval between 0 and x into n equal subintervals, and use a rectangle above each subinterval to approximate the area under the curve. There are many ways we might do this, but let's use the height of the curve at the left endpoint of the subinterval as the height of the rectangle, as in figure 6.2. The height of rectangle number i is then $3(i-1)(x/n)$, the width is x/n, and the area is $3(i-1)(x^2/n^2)$. The total area of the rectangles is

$$(0)\frac{x}{n}+3(1)\frac{x^2}{n^2}+3(2)\frac{x^2}{n^2}+3(3)\frac{x^2}{n^2}+\cdots+3(n-1)\frac{x^2}{n^2}.$$

By factoring out $3x^2/n^2$ this simplifies to

$$\frac{3x^2}{n^2}(0+1+2+\cdots+(n-1))=\frac{3x^2}{n^2}\frac{n^2-n}{2}=\frac{3}{2}x^2\left(1-\frac{1}{n}\right).$$

As n gets larger this gets closer and closer to $3x^2/2$, which must therefore be the true area under the curve. ♣

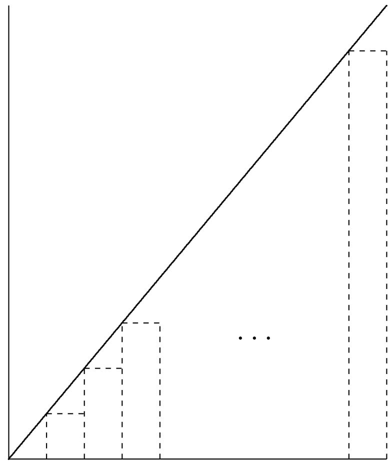


Figure 6.2: Approximating the area under $y = 3x$ with rectangles.

What you will have noticed, of course, is that while the problem in the second example appears to be much different than the problem in the first example, and while the easy approach to problem one does not appear to apply to problem two, the "approximation" approach works in both, and moreover the *calculations are identical.* As we will see, there are many, many problems that appear much different on the surface but turn out to be the same as these problems, in the sense that when we try to approximate solutions we end up with mathematics that looks like the two examples, though of course the function involved will not always be so simple.

Even better, we now see that while the second problem did not appear to be amenable to approach one, it can in fact be solved in the same way. The reasoning is this: we know that problem one can be solved easily by finding a function whose derivative is $3t$. We also know that mathematically the two problems are the same, because both can be solved by taking a limit of a sum, and the sums are identical. Therefore, we don't really need to compute the limit of either sum because we know that we will get the same answer by computing a function with the derivative $3t$ or, which is the same thing, $3x$.

It's true that the first problem had the added complication of the "10", and we certainly need to be able to deal with such minor variations, but that turns out to be quite simple. The lesson then is this: whenever we can solve a problem by taking the limit of a sum of a certain form, instead of computing the (often nasty) limit we can find a new function with a certain derivative.

6.2.1. Sigma Notation

To refine the area approximations we use more rectangles. The notation can become unwieldy, as we add up longer and longer lists of numbers. For this reason we introduce **sigma notation**.

Suppose we wish to add up a list of numbers a_1, a_2, a_3, …, a_9. Instead of writing

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9,$$

we use sigma notation and write

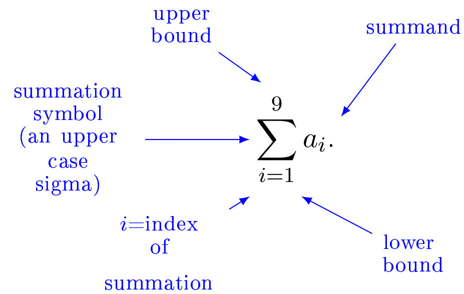


Figure 6.3: Understanding sigma notation.

The upper case sigma represents the term "sum." The index of summation in this example is i; any symbol can be used. By convention, the index takes on only the integer values between (and including) the lower and upper bounds.

Let's practice using this notation.

Example 6.12: Using sigma notation

Let the numbers $\{a_i\}$ be defined as $a_i = 2i - 1$ for integers i, where $i \geq 1$. So $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, etc. (The output is the positive odd integers). Evaluate the following summations:

1. $\displaystyle\sum_{i=1}^{6} a_i$ 2. $\displaystyle\sum_{i=3}^{7} (3a_i - 4)$ 3. $\displaystyle\sum_{i=1}^{4} (a_i)^2$

Solution.

1.
$$\begin{aligned}\sum_{i=1}^{6} a_i &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\ &= 1 + 3 + 5 + 7 + 9 + 11 \\ &= 36.\end{aligned}$$

2. Note the starting value is different than 1:
$$\begin{aligned}\sum_{i=3}^{7} a_i &= (3a_3 - 4) + (3a_4 - 4) + (3a_5 - 4) + (3a_6 - 4) + (3a_7 - 4) \\ &= 11 + 17 + 23 + 29 + 35 \\ &= 115.\end{aligned}$$

3.
$$\begin{aligned}\sum_{i=1}^{4} (a_i)^2 &= (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_4)^2 \\ &= 1^2 + 3^2 + 5^2 + 7^2 \\ &= 84\end{aligned}$$

♣

It might seem odd to stress a new, concise way of writing summations only to write each term out as we add them up. It is. The following theorem gives some of the properties of summations that allow us to work with them without writing individual terms. The first three properties are typically referred to as the *linearity properties*. Examples will follow.

Theorem 6.3: Properties of Summations

1. $\displaystyle\sum_{i=1}^{n} c = c \cdot n$, *where c is a constant.*
2. $\displaystyle\sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$
3. $\displaystyle\sum_{i=m}^{n} c \cdot a_i = c \cdot \sum_{i=m}^{n} a_i$
4. $\displaystyle\sum_{i=m}^{j} a_i + \sum_{i=j+1}^{n} a_i = \sum_{i=m}^{n} a_i$
5. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
6. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
7. $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 6.13: Evaluating summations using Theorem 6.3

Revisit Example 6.12 and, using Theorem 6.3, evaluate

$$\sum_{i=1}^{6} a_i = \sum_{i=1}^{6} (2i-1).$$

Solution.

$$\begin{aligned}
\sum_{i=1}^{6} (2i-1) &= \sum_{i=1}^{6} 2i - \sum_{i=1}^{6} (1) \\
&= \left(2\sum_{i=1}^{6} i\right) - 6 \\
&= 2\frac{6(6+1)}{2} - 6 \\
&= 42 - 6 = 36
\end{aligned}$$

We obtained the same answer without writing out all six terms. When dealing with small sizes of n, it may be faster to write the terms out by hand. However, Theorem 6.3 is incredibly important when dealing with large sums as we'll soon see. ♣

6.2.2. Approximating the Area of a Plane Region

As we have observed above, if $f(t)$ is a positive velocity function, then the area under the graph of $f(x)$ over the interval $[t_1, t_2]$ is the distance travelled over the same time interval. Note that if $f(t)$ is allowed to be negative, then the area provides the displacement over the interval.

For the rest of this section, we assume that $f(x)$ is **continuous and positive**, so that the graph lies above the x-axis. Our goal is to compute the area "under the graph", that is, the area between the graph and the x-axis. As a first step, we approximate the area using rectangles.

There are three common ways to determine the height of these rectangles: the **Right Hand Rule** (LHR), the **Left Hand Rule** (RHR), and the **Midpoint Rule** (MPR). The **Riight Hand Rule** says to evaluate the function at the right–hand endpoint of the subinterval and make the rectangle that height.

The **Left Hand Rule** says the opposite: on each subinterval, evaluate the function at the left endpoint and make the rectangle that height.

The **Midpoint Rule** says that on each subinterval, evaluate the function at the midpoint and make the rectangle that height.

Example 6.14: Using the Left Hand, Right Hand and Midpoint Rules

Approximate the value of $\int_0^4 (4x - x^2)\, dx$ using the Left Hand Rule, the Right Hand Rule, and the Midpoint Rule, using 4 equally spaced subintervals.

Solution. We break the interval $[0, 4]$ into four subintervals. In Figure 6.4 we see 4 rectangles drawn on $f(x) = 4x - x^2$ using the Left Hand Rule.

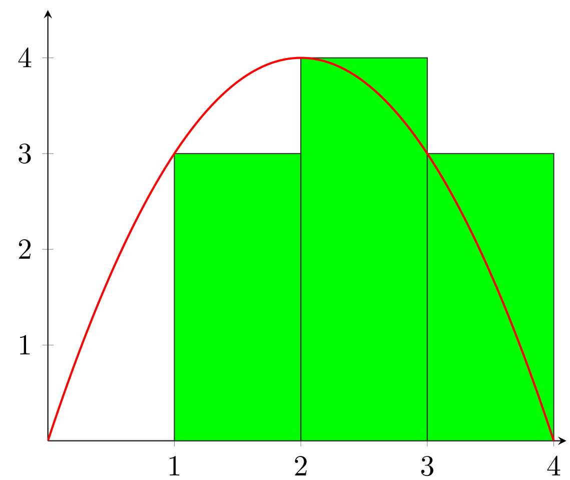


Figure 6.4: Approximating the area under $f(x) = 4x - x^2$ on $[0, 4]$ using the Left Hand Rule

Note how in the first subinterval, $[0, 1]$, the rectangle has height $f(0) = 0$. We add up the areas of each rectangle (height× width) for our Left Hand Rule approximation:

$$f(0) \cdot 1 + f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 =$$
$$0 + 3 + 4 + 3 = 10.$$

Figure 6.5 shows 4 rectangles drawn under f using the Right Hand Rule; note how the $[3, 4]$ subinterval has a rectangle of height 0.

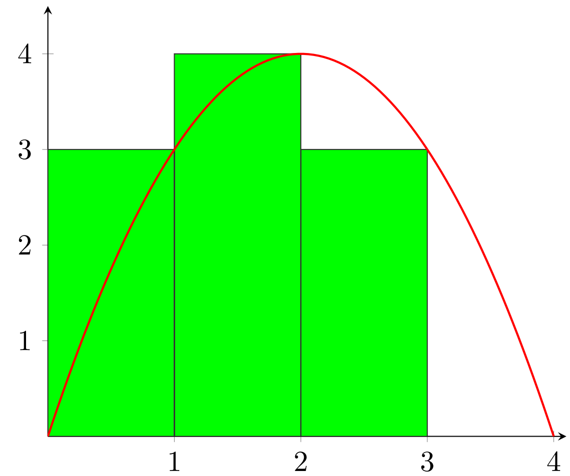


Figure 6.5: Approximating the area under $f(x) = 4x - x^2$ on $[0, 4]$ using the Right Hand Rule

In this example, these rectangle seem to be the mirror image of those found in Figure 6.4. (This is because of the symmetry of our shaded region.) Our approximation gives the same answer as before, though calculated a different way:

$$f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 = \\ 3 + 4 + 3 + 0 = 10.$$

Figure 6.6 shows 4 rectangles drawn under f using the Midpoint Rule.

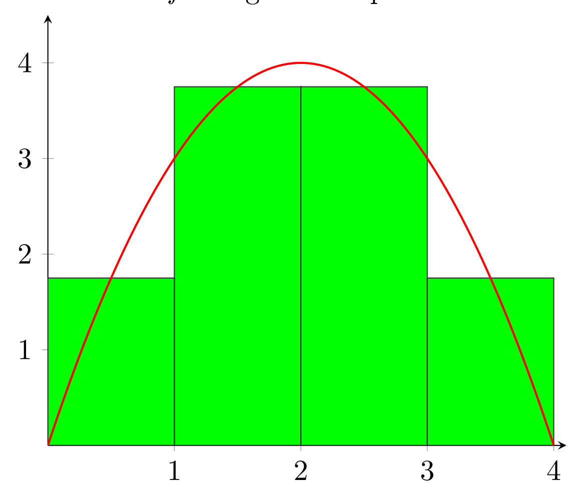


Figure 6.6: Approximating the area under $f(x) = 4x - x^2$ on $[0, 4]$ using the Midpoint Rule

This gives an approximation of

$$f(0.5) \cdot 1 + f(1.5) \cdot 1 + f(2.5) \cdot 1 + f(3.5) \cdot 1 = \\ 1.75 + 3.75 + 3.75 + 1.75 = 11.$$

Our three methods provide two approximations of the area under $f(x) = 4x - x^2$ on [0.4]: 10 and 11. ♣

6.2.3. Riemann Sums

For now, we continue to focus on determining an accurate estimate of area through the use of a sum of the areas of rectangles, doing so in the setting where $f(x) \geq 0$ on $[a, b]$. Throughout, unless otherwise indicated, we also assume that f is continuous on $[a, b]$.

The first choice we make in any such approximation is the number of rectangles.

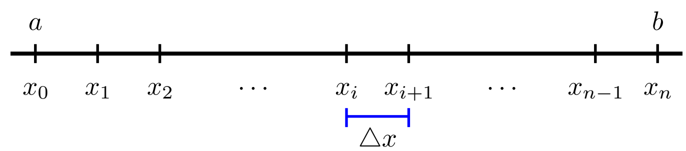


Figure 6.7: Subdividing the interval $[a, b]$ into n subintervals of equal length $\triangle x$.

If we say that the total number of rectangles is n, and we desire n rectangles of equal width to subdivide the interval $[a, b]$, then each rectangle must have width $\triangle x = \frac{b-a}{n}$. We observe further that $x_1 = x_0 + \triangle x$, $x_2 = x_0 + 2\triangle x$, and thus in general $x_i = a + i\triangle x$, as pictured in Figure 6.7.

We use each subinterval $[x_i, x_{i+1}]$ as the base of a rectangle, and next must choose how to decide the height of the rectangle that will be used to approximate the area under $y = f(x)$ on the subinterval. The three standard choices are the left endpoint, right endpoint, or the midpoint of each. These are precisely the options encountered in the previous section. We next explore how these choices can be reflected in sigma notation.

If we now consider an arbitrary positive function f on $[a, b]$ with the interval subdivided as shown in Figure 6.7, and choose to use left endpoints, then on each interval of the form $[x_i, x_{i+1}]$, the area of the rectangle formed is given by

$$A_{i+1} = f(x_i) \cdot \triangle x,$$

as seen in Figure 6.8.

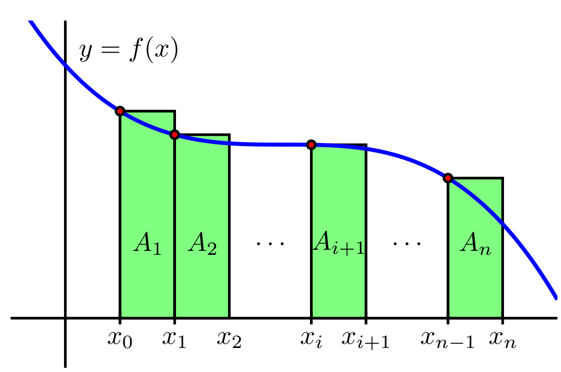


Figure 6.8: Subdividing the interval $[a, b]$ into n subintervals of equal length $\triangle x$ and approximating the area under $y = f(x)$ over $[a, b]$ using left rectangles.

If we let L_n denote the sum of the areas of rectangles whose heights are given by the function value at each respective left endpoint, then we see that

$$\begin{aligned} L_n &= A_1 + A_2 + \cdots + A_{i+1} + \cdots + A_n \\ &= f(x_0)\cdot\triangle x + f(x_1)\cdot\triangle x + \cdots + f(x_i)\cdot\triangle x + \cdots + f(x_{n-1})\cdot\triangle x. \end{aligned}$$

In the more compact sigma notation, we have

$$L_n = \sum_{i=0}^{n-1} f(x_i)\triangle x.$$

Note particularly that since the index of summation begins at 0 and ends at $n-1$, there are indeed n terms in this sum. We call L_n the *left Riemann sum* for the function f on the interval $[a, b]$.

There are now two fundamental issues to explore: the number of rectangles we choose to use and the selection of the pattern by which we identify the height of each rectangle. It is best to explore these choices dynamically, and the applet[1] found at http://gvsu.edu/s/a9 is a particularly useful one. There we see

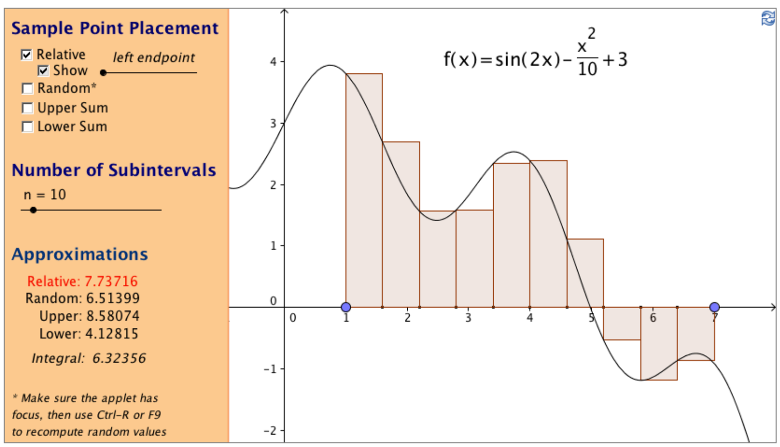


Figure 6.9: A snapshot of the applet found at http://gvsu.edu/s/a9.

the image shown in Figure 6.9, but with the opportunity to adjust the slider bars for the left endpoint and the number of subintervals. By moving the sliders, we can see how the heights of the rectangles change as we consider left endpoints, midpoints, and right endpoints, as well as the impact that a larger number of narrower rectangles has on the approximation of the exact area bounded by the function and the horizontal axis.

To see how the Riemann sums for right endpoints and midpoints are constructed, we consider Figure 6.10.

[1]Marc Renault, Geogebra Calculus Applets.

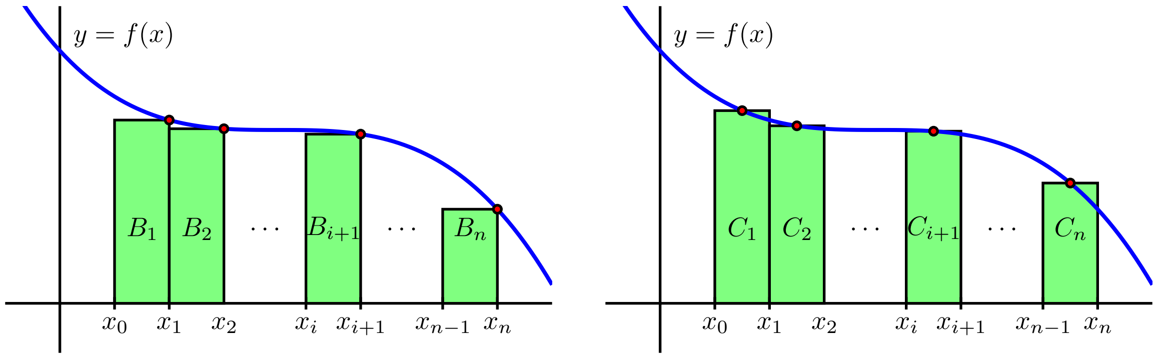


Figure 6.10: Riemann sums using right endpoints and midpoints.

For the sum with right endpoints, we see that the area of the rectangle on an arbitrary interval $[x_i, x_{i+1}]$ is given by

$$B_{i+1} = f(x_{i+1}) \cdot \triangle x,$$

so that the sum of all such areas of rectangles is given by

$$\begin{aligned} R_n &= B_1 + B_2 + \cdots + B_{i+1} + \cdots + B_n \\ &= f(x_1) \cdot \triangle x + f(x_2) \cdot \triangle x + \cdots + f(x_{i+1}) \cdot \triangle x + \cdots + f(x_n) \cdot \triangle x \\ &= \sum_{i=1}^{n} f(x_i) \triangle x. \end{aligned}$$

We call R_n the *right Riemann sum* for the function f on the interval $[a, b]$. For the sum that uses midpoints, we introduce the notation

$$\overline{x}_{i+1} = \frac{x_i + x_{i+1}}{2}$$

so that $\overline{x}_{i+1}$ is the midpoint of the interval $[x_i, x_{i+1}]$. For instance, for the rectangle with area C_1 in Figure 6.10, we now have

$$C_1 = f(\overline{x}_1) \cdot \triangle x.$$

Hence, the sum of all the areas of rectangles that use midpoints is

$$\begin{aligned} M_n &= C_1 + C_2 + \cdots + C_{i+1} + \cdots + C_n \\ &= f(\overline{x_1}) \cdot \triangle x + f(\overline{x_2}) \cdot \triangle x + \cdots + f(\overline{x}_{i+1}) \cdot \triangle x + \cdots + f(\overline{x}_n) \cdot \triangle x \\ &= \sum_{i=1}^{n} f(\overline{x}_i) \triangle x, \end{aligned}$$

and we say that M_n is the *middle (or midpoint) Riemann sum* for f on $[a, b]$.

When $f(x) \geq 0$ on $[a, b]$, each of the Riemann sums L_n, R_n, and M_n provides an estimate of the area under the curve $y = f(x)$ over the interval $[a, b]$. As n increases, each of the approximations get better and better:

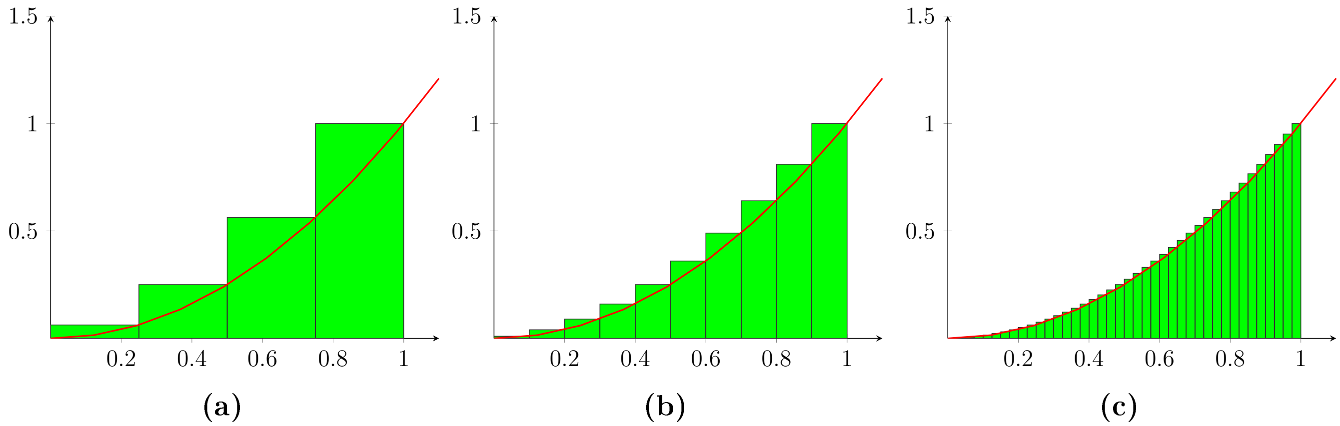


Figure 6.11: The approximations get better and better as the number of strips (n) increases.

The following example lets us practice using the Right Hand Rule and the summation formulas introduced in Theorem 6.3.

Example 6.15: Approximating definite integrals using sums

Approximate the area A under $f(x) = 4x - x^2\ dx$ on $[0, 4]$ using the Right Hand Rule with 16 and 1000 equally spaced intervals.

Solution. Using 16 equally spaced intervals and the Right Hand Rule, we can approximate the area as

$$\sum_{i=1}^{16} f(x_{i+1})\Delta x.$$

We have $\Delta x = 4/16 = 0.25$. Since $x_i = 0 + (i-1)\Delta x$, we have

$$\begin{aligned} x_{i+1} &= 0 + \big((i+1) - 1\big)\Delta x \\ &= i\Delta x \end{aligned}$$

This gives:

$$
\begin{aligned}
\text{Area } = A &\approx \sum_{i=1}^{16} f(x_{i+1})\Delta x \\
&= \sum_{i=1}^{16} f(i\Delta x)\Delta x \\
&= \sum_{i=1}^{16} \big(4i\Delta x - (i\Delta x)^2\big)\Delta x \\
&= \sum_{i=1}^{16}(4i\Delta x^2 - i^2\Delta x^3) \\
&= (4\Delta x^2)\sum_{i=1}^{16} i - \Delta x^3 \sum_{i=1}^{16} i^2 \qquad (6.1) \\
&= (4\Delta x^2)\frac{16\cdot 17}{2} - \Delta x^3 \frac{16(17)(33)}{6} \\
&= 4\cdot 0.25^2 \cdot 136 - 0.25^3 \cdot 1496 \\
&= 10.625
\end{aligned}
$$

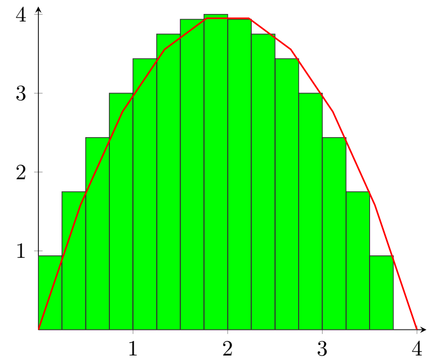


Figure 6.12

We were able to sum up the areas of 16 rectangles with very little computation. In Figure 6.12 the function and the 16 rectangles are graphed.

While some rectangles over–approximate the area, others under–approximate the area (by about the same amount). Thus our approximate area of 10.625 is likely a fairly good approximation.

Notice Equation (6.1); by changing the 16's to 1,000's (and appropriately changing the value of Δx), we can use that equation to sum up 1000 rectangles!

We do so here, skipping from the original summand to the equivalent of Equation (6.1) to save space. Note that $\Delta x = 4/1000 = 0.004$.

$$
\begin{aligned}
A &\approx \sum_{i=1}^{1000} f(x_{i+1})\Delta x \\
&= (4\Delta x^2)\sum_{i=1}^{1000} i - \Delta x^3 \sum_{i=1}^{1000} i^2 \\
&= (4\Delta x^2)\frac{1000\cdot 1001}{2} - \Delta x^3 \frac{1000(1001)(2001)}{6} \\
&= 4\cdot 0.004^2 \cdot 500500 - 0.004^3 \cdot 333,833,500 \\
&= 10.666656
\end{aligned}
$$

Using many, many rectangles, we have a likely good approximation of the area. That is,

$$A \approx 10.666656.$$

♣

The previous example motivates the following definition.

Definition 6.2: Area

The area A of the region that lies under the graph of a continuous nonnegative function f over the interval $[a, b]$ is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$, and $x_i = a + 1\Delta x$.

Example 6.16: Approximating definite integrals with a formula, using sums

Revisit Example 6.15. Use Definition 6.2 to determine the area under $f(x) = 4x - x^2\, dx$ on $[0, 4]$.

Solution. We have $\Delta x = \frac{4-0}{n} = 4/n$, and $x_i = 0 + \Delta x(i) = 4i/n$.

We construct the Riemann sum as follows. Be sure to follow each step carefully. If you get stuck, and do not understand how one line proceeds to the next, you may skip to the result and consider how this result is used. You should come back, though, and work through each step for full understanding.

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(x_{i+1})\Delta x \\
&= \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\Delta x \\
&= \sum_{i=1}^{n} \left[4\frac{4i}{n} - \left(\frac{4i}{n}\right)^2\right]\Delta x \\
&= \sum_{i=1}^{n} \left(\frac{16\Delta x}{n}\right) i - \sum_{i=1}^{n} \left(\frac{16\Delta x}{n^2}\right) i^2 \\
&= \left(\frac{16\Delta x}{n}\right)\sum_{i=1}^{n} i - \left(\frac{16\Delta x}{n^2}\right)\sum_{i=1}^{n} i^2 \\
&= \left(\frac{16\Delta x}{n}\right)\cdot\frac{n(n+1)}{2} - \left(\frac{16\Delta x}{n^2}\right)\frac{n(n+1)(2n+1)}{6} \qquad \left(\begin{matrix}\text{recall} \\ \Delta x = \\ 4/n\end{matrix}\right) \\
&= \frac{32(n+1)}{n} - \frac{32(n+1)(2n+1)}{3n^2} \qquad \text{(now simplify)} \\
&= \frac{32}{3}\left(1 - \frac{1}{n^2}\right)
\end{aligned}$$

Therefore, we have

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{32}{3}\left(1 - \frac{1}{n^2}\right) = \frac{32}{3}(1-0) = \frac{32}{3}$$

Note that our approximation in Example 6.15 using 16 subintervals was actually quite a good estimate. ♣

It can be proved that the limit in Definition 6.2 always exists (as f) is assumed to be continuous). Moreover, we get the same value of we take the limit using left endpoints, or midpoints.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} M_n.$$

Momentarily, we will discuss the meaning of Riemann sums in the setting when f is sometimes negative. We also recall that in the context of a nonnegative velocity function $y = v(t)$, the corresponding Riemann sums are approximating the distance traveled on $[a, b]$ by the moving object with velocity function v.

There is a more general way to think of Riemann sums, and that is to not restrict the choice of where the function is evaluated to determine the respective rectangle heights. That is, rather than saying we'll always choose left endpoints, or always choose midpoints, we simply say that a point x_{i+1}^* will be selected at random in the interval $[x_i, x_{i+1}]$ (so that $x_i \leq x_{i+1}^* \leq x_{i+1}$), which makes the Riemann sum given by

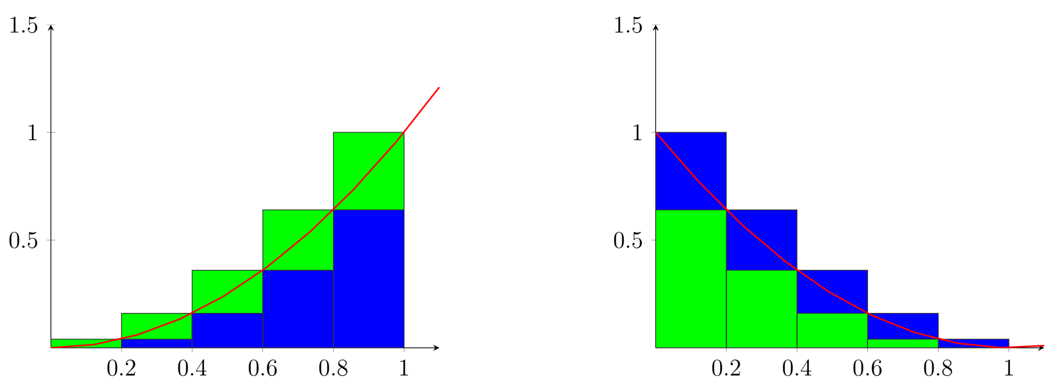
$$f(x_1^*) \cdot \triangle x + f(x_2^*) \cdot \triangle x + \cdots + f(x_{i+1}^*) \cdot \triangle x + \cdots + f(x_n^*) \cdot \triangle x = \sum_{i=1}^{n} f(x_i^*) \triangle x.$$

Definition 6.2 could also be made using this more general Riemann sum:

Key Idea 6.2.0: Area: another definition

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \triangle x.$$

In general, the **lower** (resp. **upper**) **sums** are formed by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (resp. maximum) value of on the ith subinterval. In the special case that f is monotone increasing (resp. decreasing), the left and right endpoint approximations correspond to the lower and upper (resp. upper and lower) sums.



(a) f **monotone increasing:** $L_n \leq$ **Area** $\leq R_n$

(b) f **monotone decreasing:** $R_n \leq$ **Area** $\leq L_n$

Figure 6.13: For monttonic functions lower and upper sums are given by L_n or R_n.

At http://gvsu.edu/s/a9, the applet noted earlier and referenced in Figure 6.9, by unchecking the "relative" box at the top left, and instead checking "random," we can easily explore the effect of using random point locations in subintervals on a given Riemann sum. In computational practice, we most often use L_n, R_n, or M_n, while the random Riemann sum is useful in theoretical discussions.

When the function is sometimes negative

For a Riemann sum such as

$$L_n = \sum_{i=0}^{n-1} f(x_i)\triangle x,$$

we can of course compute the sum even when f takes on negative values. We know that when f is positive on $[a, b]$, the corresponding left Riemann sum L_n estimates the area bounded by f and the horizontal axis over the interval.

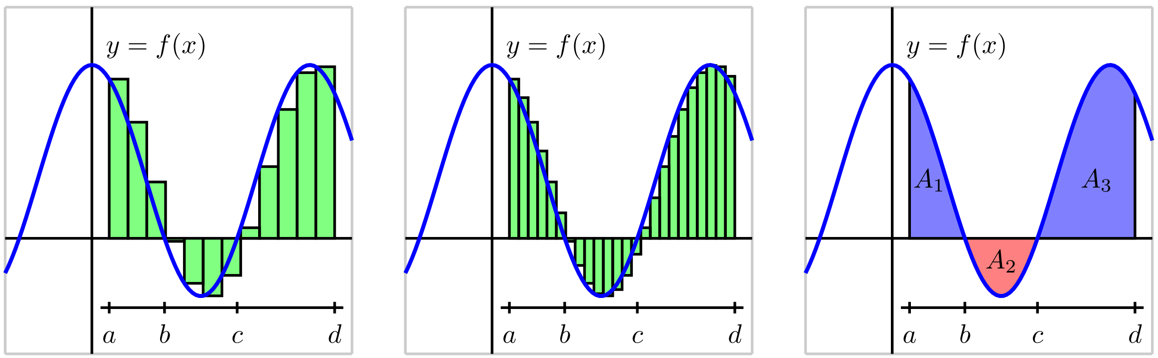


Figure 6.14: At left and center, two left Riemann sums for a function f that is sometimes negative; at right, the areas bounded by f on the interval $[a, d]$.

For a function such as the one pictured in Figure 6.14, where in the first figure a left Riemann sum is being taken with 12 subintervals over $[a, d]$, we observe that the function is negative on the interval $b \leq x \leq c$, and so for the four left endpoints that fall in $[b, c]$, the terms $f(x_i)\triangle x$ have negative function values. This means that those four terms in the Riemann sum produce an estimate of the *opposite* of the area bounded by $y = f(x)$ and the x-axis on $[b, c]$.

In Figure 6.14, we also see evidence that by increasing the number of rectangles used in a Riemann sum, it appears that the approximation of the area (or the opposite of the area) bounded by a curve appears to improve. For instance, in the middle graph, we use 24 left rectangles, and from the shaded areas, it appears that we have decreased the error from the approximation that uses 12. When we proceed to the next section, we will discuss the natural idea of letting the number of rectangles in the sum increase without bound.

For now, it is most important for us to observe that, in general, any Riemann sum of a continuous function f on an interval $[a, b]$ approximates the difference between the area that lies above the horizontal axis on $[a, b]$ and under f and the area that lies below the horizontal axis on $[a, b]$ and above f. In the notation of Figure 6.14, we may say that

$$L_{24} \approx A_1 - A_2 + A_3,$$

where L_{24} is the left Riemann sum using 24 subintervals shown in the middle graph, and A_1 and A_3 are the areas of the regions where f is positive on the interval of interest, while A_2 is the area of the region where f is negative. We will also call the quantity $A_1 - A_2 + A_3$ the *net signed area* bounded by f over the interval $[a, d]$, where by the phrase "signed area" we indicate that we are attaching a minus sign to the areas of regions that fall below the horizontal axis.

Finally, we recall that in the context where the function f represents the velocity of a moving object, the total sum of the areas bounded by the curve tells us the total distance traveled over the relevant time interval, while the total net signed area bounded by the curve computes the object's change in position on the interval.

Summary:

- A Riemann sum is simply a sum of products of the form $f(x_i^*)\triangle x$ that estimates the area between a positive function and the horizontal axis over a given interval. If the function is sometimes negative on the interval, the Riemann sum estimates the difference between the areas that lie above the horizontal axis and those that lie below the axis.
- The three most common types of Riemann sums are left, right, and middle sums, plus we can also work with a more general, random Riemann sum. The only difference among these sums is the location of the point at which the function is evaluated to determine the height of the rectangle whose area is being computed in the sum. For a left Riemann sum, we evaluate the function at the left endpoint of each subinterval, while for right and middle sums, we use right endpoints and midpoints, respectively.
- The left, right, and middle Riemann sums are denoted L_n, R_n, and M_n, with formulas

$$L_n = f(x_0)\triangle x + f(x_1)\triangle x + \cdots + f(x_{n-1})\triangle x = \sum_{i=0}^{n-1} f(x_i)\triangle x,$$

$$R_n = f(x_1)\triangle x + f(x_2)\triangle x + \cdots + f(x_n)\triangle x = \sum_{i=1}^{n} f(x_i)\triangle x,$$

$$M_n = f(\overline{x}_1)\triangle x + f(\overline{x}_2)\triangle x + \cdots + f(\overline{x}_n)\triangle x = \sum_{i=1}^{n} f(\overline{x}_i)\triangle x,$$

  where $x_0 = a$, $x_i = a + i\triangle x$, and $x_n = b$, using $\triangle x = \frac{b-a}{n}$. For the midpoint sum, $\overline{x}_i = (x_{i-1} + x_i)/2$.

6.2.4. The Definite Integral

In the previous examples it appears that as the number of rectangles got larger and larger, the values of L_n, M_n, and R_n all grew closer and closer to the same value. It turns out that this occurs for any continuous function on an interval $[a, b]$, and even more generally for a Riemann sum using any point x_{i+1}^* in the interval $[x_i, x_{i+1}]$. Said differently, as we let $n \to \infty$, it doesn't really matter where we choose to evaluate the function within a given subinterval, because

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\triangle x.$$

That these limits always exist (and share the same value) for a continuous[2] function f allows us to make the following definition.

[2] It turns out that a function need not be continuous in order to have a definite integral. For our purposes, we assume that the functions we consider are continuous on the interval(s) of interest. It is straightforward to see that any function that is piecewise continuous on an interval of interest will also have a well-defined definite integral.

Definition 6.3:

The definite integral *of a continuous function f on the interval $[a,b]$, denoted $\int_a^b f(x)\,dx$, is the real number given by*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\triangle x,$$

where $\triangle x = \frac{b-a}{n}$, $x_i = a + i\triangle x$ (for $i = 0,\dots,n$), and x_i^* satisfies $x_{i-1} \le x_i^* \le x_i$ (for $i = 1,\dots,n$).

We call the values a and b the *lower and upper limits of integration* respectively. The process of determining the real number $\int_a^b f(x)\,dx$ is called *evaluating the definite integral.*

Example 6.17: Finding definite integrals with Riemann sums

Find $\int_{-1}^5 x^3\,dx$ using the limit definition of the definite integral..

Solution. As we may choose x_i^* freely in Definition 6.3, we will use the right hand rule, so we have $\Delta x = (b-a)/n = \frac{5-(-1)}{n} = 6/n$, and $x_i = a + i\Delta x = (-1) + i\Delta x$. The Riemann sum corresponding to the Right Hand Rule is (followed by simplifications):

$$\begin{aligned}
\int_{-1}^5 x^3\,dx &\approx \sum_{i=1}^n f(x_{i+1})\Delta x\\
&= \sum_{i=1}^n f(-1+i\Delta x)\Delta x\\
&= \sum_{i=1}^n (-1+i\Delta x)^3\Delta x\\
&= \sum_{i=1}^n \big((i\Delta x)^3 - 3(i\Delta x)^2 + 3i\Delta x - 1\big)\Delta x \quad \text{(now distribute } \Delta x\text{)}\\
&= \sum_{i=1}^n \left(i^3\Delta x^4 - 3i^2\Delta x^3 + 3i\Delta x^2 - \Delta x\right) \quad \text{(now split up summation)}\\
&= \Delta x^4\sum_{i=1}^n i^3 - 3\Delta x^3\sum_{i=1}^n i^2 + 3\Delta x^2\sum_{i=1}^n i - \sum_{i=1}^n \Delta x\\
&= \Delta x^4\left(\frac{n(n+1)}{2}\right)^2 - 3\Delta x^3\frac{n(n+1)(2n+1)}{6} + 3\Delta x^2\frac{n(n+1)}{2} - n\Delta x
\end{aligned}$$

(use $\Delta x = 6/n$)

$$\begin{aligned}
&= \frac{1296}{n^4}\cdot\frac{n^2(n+1)^2}{4} - 3\frac{216}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + 3\frac{36}{n^2}\frac{n(n+1)}{2} - 6\\
&= 324\cdot\left(\frac{n+1}{n}\right)^2 - 108\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} + 54\cdot\frac{n+1}{n} - 6\\
&= 324\cdot\left(1+\frac{1}{n}\right)^2 - 108\cdot\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right) + 54\cdot\left(1+\frac{1}{n}\right) - 6
\end{aligned}$$

Now find the exact answer using a limit:

$$\int_{-1}^{5} x^3\, dx = \lim_{n\to\infty} \left(324 \cdot \left(1 + \frac{1}{n}\right)^2 - 108 \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right) + 54 \cdot \left(1 + \frac{1}{n}\right) - 6 \right) = 156.$$

♣

If we wish to compute the value of a definite integral using the definition, we have to take the limit of a sum. While this is possible to do in select circumstances, it is also tedious and time-consuming; moreover, computing these limits does not offer much additional insight into the meaning or interpretation of the definite integral. Instead, in the next section, we will learn the Fundamental Theorem of Calculus, a result that provides a shortcut for evaluating a large class of definite integrals. This will enable us to determine the exact net signed area bounded by a continuous function and the x-axis in many circumstances.

While we will come to understand that there are several different interpretations of the value of the definite integral, for now the most important is that $\int_a^b f(x)\, dx$ measures the net signed area bounded by $y = f(x)$ and the x-axis on the interval $[a, b]$.

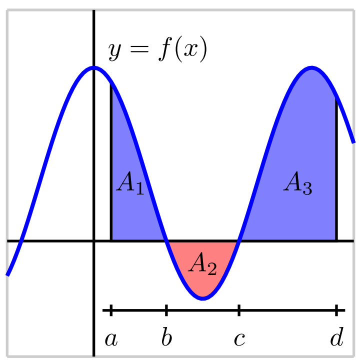


Figure 6.15: A continuous function f on the interval $[a, d]$.

For example, in the notation of the definite integral, if f is the function pictured in Figure 6.15 and A_1, A_2, and A_3 are the exact areas bounded by f and the x-axis on the respective intervals $[a, b]$, $[b, c]$, and $[c, d]$, then

$$\int_a^b f(x)\, dx = A_1, \quad \int_b^c f(x)\, dx = -A_2, \quad \int_c^d f(x)\, dx = A_3,$$

and

$$\int_a^d f(x)\, dx = A_1 - A_2 + A_3.$$

If a given curve produces regions whose areas we can compute exactly through known area formulas, we can thus compute the exact value of the integral. Let's look at a few more examples.

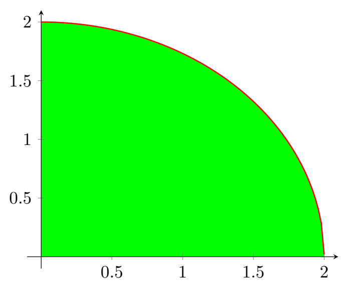
Example 6.18: Finding definite integrals as signed areas

Evaluate the following integrals by interpreting each in terms of signed areas:

1. $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$
2. $\displaystyle\int_0^3 2-x\,dx$
3. $\displaystyle\int_1^4 2x+2\,dx$
4. $\displaystyle\int_0^3 |2-x|\,dx$

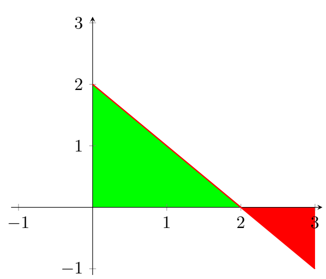
Solution.

1. Since $f(x)=\sqrt{4-x^2}\geq 0$, the integral corresponds to the area under the curve $y=f(x)$ between $x=0$ and $x=2$. Since $y=\sqrt{4-x^2}\Rightarrow y^2=4-x^2\Rightarrow x^2+y^2=4$, the region we are interested in is that of a quarter circle with radius 2. So we have



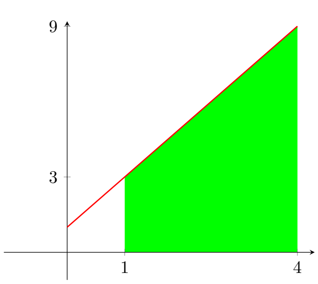
$$\int_0^2 \sqrt{4-x^2}\,dx=\frac{1}{4}\pi(2)^2=\pi.$$

2. The region between the x-axis and $y=2-x$ over the interval $[0,3]$ consists of two triangular regions of areas 2 and $\frac{1}{2}$. Since the second triangle is below the x-axis, it has a signed area of $-\frac{1}{2}$. so we have



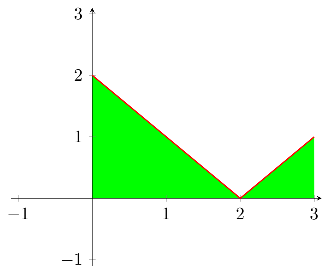
$$\int_0^3 2-x\,dx=2-\frac{1}{2}=\frac{3}{2}.$$

3. Observe that the region bounded by this function and the x-axis is the trapezoid, and by the known formula for the area of a trapezoid, its area is $A=\frac{1}{2}(3+9)\cdot 3=18$, so



$$\int_1^4 (2x+1)\,dx=18.$$

4. The region between the x-axis and $y = |2-x|$ over the interval $[0,3]$ consists of two triangular regions of areas 2 and $\frac{1}{2}$. Both are above the x-axis, so have a positive signed area, and we have



$$\int_0^3 2 - x \, dx = 2 + \frac{1}{2} = \frac{5}{2}.$$

♣

We can also use definite integrals to express the change in position and distance traveled by a moving object. In the setting of a velocity function v on an interval $[a,b]$, it follows from our work above and in preceding sections that the change in position, $s(b) - s(a)$, is given by

$$s(b) - s(a) = \int_a^b v(t)\, dt.$$

If the velocity function is nonnegative on $[a,b]$, then $\int_a^b v(t)\, dt$ tells us the distance the object travelled. When velocity is sometimes negative on $[a,b]$, the areas bounded by the function on intervals where v does not change sign can be found using integrals, and the sum of these values will tell us the distance the object travelled.

Example 6.19: Understanding motion given velocity

Consider the graph of a velocity function of an object moving in a straight line, given in Figure 6.16, where the numbers in the given regions gives the area of that region. Find the maximum speed of the object and its maximum displacement from its starting position.

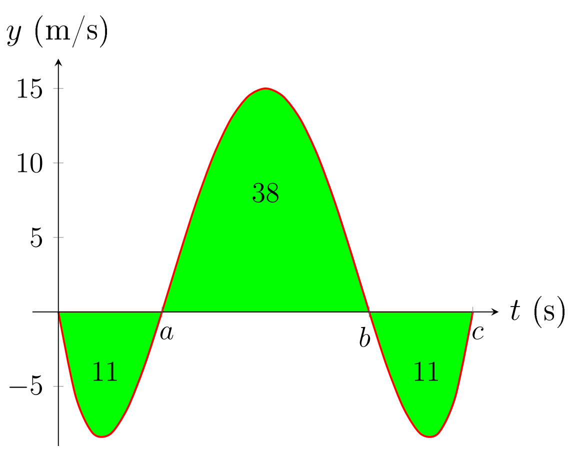


Figure 6.16: A graph of a velocity in Example 6.19.

Solution. Since the graph gives velocity, finding the maximum speed is simple: it looks to be 15m/s.

At time $t = 0$, the displacement is 0; the object is at its starting position. At time $t = a$, the object has moved backward 11 meters. Between times $t = a$ and $t = b$, the object moves forward 38 meters, bringing it into a position 27 meters forward of its starting position. From $t = b$ to $t = c$ the object is moving backwards again, hence its total displacement is 27 meters from its starting position.

$$\text{Total Displacement} = \int_0^c v(t)\ dt = -11 + 38 - 11 = 16m.$$

$$\text{Maximum Displacement} = \int_0^b v(t)\ dt = -11 + 38 = 27m.$$

♣

Some properties of the definite integral

With the perspective that the definite integral of a function f over an interval $[a, b]$ measures the net signed area bounded by f and the x-axis over the interval, we naturally arrive at several different standard properties of the definite integral. In addition, it is helpful to remember that the definite integral is defined in terms of Riemann sums that fundamentally consist of the areas of rectangles.

If we consider the definite integral $\int_a^a f(x)\,dx$ for any real number a, it is evident that no area is being bounded because the interval begins and ends with the same point. Hence,

> If f is a continuous function and a is a real number, then $\displaystyle\int_a^a f(x)\,dx = 0$.

Next, we consider the results of subdividing a given interval. In Figure 6.17, we see that

$$\int_a^b f(x)\,dx = A_1, \quad \int_b^c f(x)\,dx = A_2, \text{ and } \int_a^c f(x)\,dx = A_1 + A_2,$$

which is indicative of the following general rule.

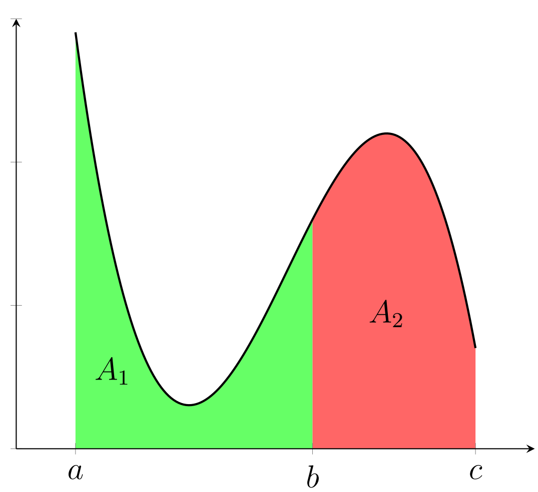


Figure 6.17: The area bounded by $y = f(x)$ on the interval $[a, c]$.

If f is a continuous function and a, b, and c are real numbers, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

While this rule is most apparent in the situation where $a < b < c$, it in fact holds in general for any values of a, b, and c. This result is connected to another property of the definite integral, which states that if we reverse the order of the limits of integration, we change the sign of the integral's value.

If f is a continuous function and a and b are real numbers, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

This result makes sense because if we integrate from a to b, then in the defining Riemann sum $\triangle x = \frac{b-a}{n}$, while if we integrate from b to a, $\triangle x = \frac{a-b}{n} = -\frac{b-a}{n}$, and this is the only change in the sum used to define the integral.

There are two additional properties of the definite integral that we need to understand. Recall that when we worked with derivative rules in Chapter **??**, we found that both the Constant Multiple Rule and the Sum Rule held. The Constant Multiple Rule tells us that if f is a differentiable function and k is a constant, then

$$\frac{d}{dx}[kf(x)] = kf'(x),$$

and the Sum Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x).$$

These rules are useful because they enable us to deal individually with the simplest parts of certain functions and take advantage of the elementary operations of addition and multiplying by a constant. They also tell us that the process of taking the derivative respects addition and multiplying by constants in the simplest possible way.

It turns out that similar rules hold for the definite integral. First, let's consider the situation pictured in Figure 6.18,

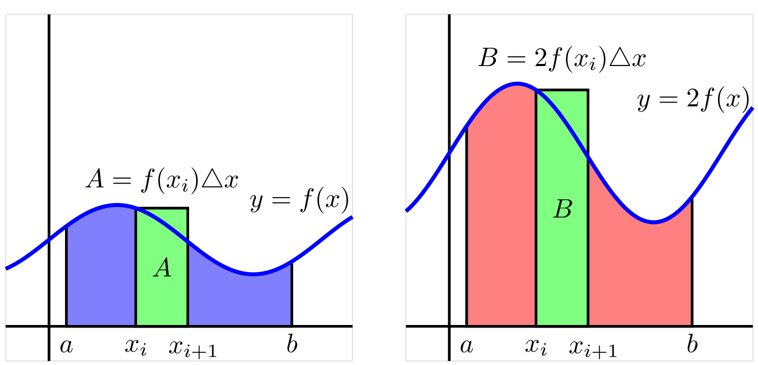


Figure 6.18: The areas bounded by $y = f(x)$ and $y = 2f(x)$ on $[a, b]$.

where we examine the effect of multiplying a function by a factor of 2 on the area it bounds with the x-axis. Because multiplying the function by 2 doubles its height at every x-value, we see that if we consider a typical rectangle from a Riemann sum, the difference in area comes from the changed height of the rectangle: $f(x_i)$ for the original function, versus $2f(x_i)$ in the doubled function, in the case of left sum. Hence, in Figure 6.18, we see that for the pictured rectangles with areas A and B, it follows $B = 2A$. As this will happen in every such rectangle, regardless of the value of n and the type of sum we use, we see that in the limit, the area of the red region bounded by $y = 2f(x)$ will be twice that of the area of the blue region bounded by $y = f(x)$. As there is nothing special about the value 2 compared to an arbitrary constant k, it turns out that the following general principle holds.

Constant Multiple Rule: If f is a continuous function and k is any real number then

$$\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx.$$

Finally, we see a similar situation geometrically with the sum of two functions f and g.

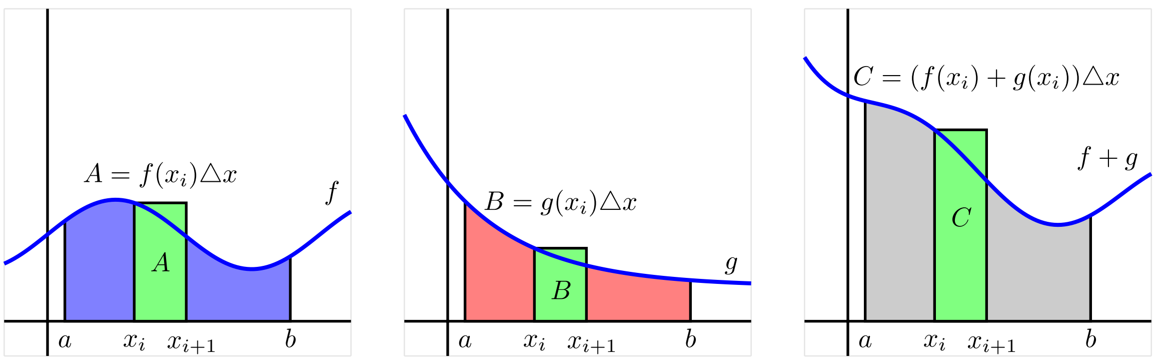


Figure 6.19: The areas bounded by $y = f(x)$ and $y = g(x)$ on $[a, b]$, as well as the area bounded by $y = f(x) + g(x)$.

In particular, as shown in Figure 6.19, if we take the sum of two functions f and g, at every point in the interval, the height of the function $f + g$ is given by $(f + g)(x_i) = f(x_i) + g(x_i)$, which is the sum of the individual function values of f and g (taken at left endpoints). Hence, for the pictured rectangles with areas A, B, and C, it follows that $C = A + B$, and because this will occur for every such rectangle, in the limit the area of the gray region will be the sum of the areas of the blue and red regions. Stated in terms of definite integrals, we have the following general rule.

Sum Rule: If f and g are continuous functions, then

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

More generally, the Constant Multiple and Sum Rules can be combined to make the observation that for any continuous functions f and g and any constants c and k,

$$\int_a^b [cf(x) \pm kg(x)]\,dx = c\int_a^b f(x)\,dx \pm k\int_a^b g(x)\,dx.$$

In summary we have the following:

Key Idea 6.2.0: Properties of Definite Integrals

Some properties are as follows:

Order of limits matters: $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

If interval is empty, integral is zero: $$\int_a^a f(x)\,dx = 0$$

Constant Multiple Rule: $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

Sum/Difference Rule: $$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Can split up interval $[a,b] = [a,c] \cup [c,b]$: $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The variable does not matter!: $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

The reason for the last property is that a definite integral is a *number*, not a function, so the variable is just a placeholder that won't appear in the final answer.

Some additional properties are *comparison* types of properties.

Key Idea 6.2.0: Comparison Properties of Definite Integrals

If $f(x) \geq 0$ for $x \in [a,b]$, then: $$\int_a^b f(x)\,dx \geq 0.$$

If $f(x) \geq g(x)$ for $x \in [a,b]$, then: $$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

If $m \leq f(x) \leq M$ for $x \in [a,b]$, then: $$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Example 6.20: Properties of Definite Integrals

Suppose $\int_a^b f(x)\ dx = 7$ *and* $\int_a^b g(x)\ dx = 3$. *Find:*

1. $\int_a^b 2f(x) - 3g(x)\,dx$.
2. $\int_b^a 2g(x)\,dx$.
3. $\int_a^a f(x)\cdot g(x)\,dx$.
4. $\int_a^c f(x)\ dx + \int_c^b f(x)\,dx$.

Solution.

1. $\int_a^b 2f(x) - 3g(x)\,dx = 2\int_a^b f(x)\,dx - 3\int_a^b g(x)\,dx = 2(7) - 3(3) = 5.$

2. $\displaystyle\int_b^a 2g(x)\,dx = -2\int_a^b g(x)\,dx = -2(3) = -6.$

3. $\displaystyle\int_a^a f(x)\cdot g(x)\,dx = 0.$

4. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx = 7.$

♣

Summary

- Any Riemann sum of a continuous function f on an interval $[a,b]$ provides an estimate of the net signed area bounded by the function and the horizontal axis on the interval. Increasing the number of subintervals in the Riemann sum improves the accuracy of this estimate, and letting the number of subintervals increase without bound results in the values of the corresponding Riemann sums approaching the exact value of the enclosed net signed area.

- When we take the just described limit of Riemann sums, we arrive at what we call the definite integral of f over the interval $[a,b]$. In particular, the symbol $\int_a^b f(x)\,dx$ denotes the definite integral of f over $[a,b]$, and this quantity is defined by the equation

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\triangle x,$$

where $\triangle x = \frac{b-a}{n}$, $x_i = a + i\triangle x$ (for $i = 0,\ldots,n$), and x_i^* satisfies $x_{i-1} \le x_i^* \le x_i$ (for $i = 1,\ldots,n$).

- The definite integral $\int_a^b f(x)\,dx$ measures the exact net signed area bounded by f and the horizontal axis on $[a,b]$.In the setting where we consider the integral of a velocity function v, $\int_a^b v(t)\,dt$ measures the displacement of the moving object on $[a,b]$; when v is nonnegative, $\int_a^b v(t)\,dt$ is the object's distance traveled on $[a,b]$.

- The definite integral is a sophisticated sum, and thus has some of the same natural properties that finite sums have. Perhaps most important of these is how the definite integral respects sums and constant multiples of functions, which can be summarized by the rule

$$\int_a^b [cf(x) \pm kg(x)]\,dx = c\int_a^b f(x)\,dx \pm k\int_a^b g(x)\,dx$$

where f and g are continuous functions on $[a,b]$ and c and k are arbitrary constants.

Exercises for Section 6.2

6.2.1 *Suppose an object moves in a straight line so that its speed at time t is given by $v(t) = 2t + 2$, and that at $t = 1$ the object is at position 5. Find the position of the object at $t = 2$.*

6.2.2 *Suppose an object moves in a straight line so that its speed at time t is given by $v(t) = t^2 + 2$, and that at $t = 0$ the object is at position 5. Find the position of the object at $t = 2$.*

6.2.3 *Find the area under $y = 2x$ between $x = 0$ and any positive value for x.*

6.2.4 *Find the area under $y = 4x$ between $x = 0$ and any positive value for x.*

6.2.5 *Find the area under $y = 4x$ between $x = 2$ and any positive value for x bigger than 2.*

6.2.6 *Find the area under $y = 4x$ between any two positive values for x, say $a < b$.*

6.2.7 *Let $f(x) = x^2 + 3x + 2$. Approximate the area under the curve between $x = 0$ and $x = 2$ using 4 rectangles and also using 8 rectangles.*

6.2.8 *Let $f(x) = x^2 - 2x + 3$. Approximate the area under the curve between $x = 1$ and $x = 3$ using 4 rectangles.*

6.3 The Fundamental Theorem of Calculus

Consider the setting where we know the position function $s(t)$ of an object moving along an axis, as well as its corresponding velocity function $v(t)$, and for the moment let us assume that $v(t)$ is positive on $[a, b]$. Then, as shown in Figure 6.20,

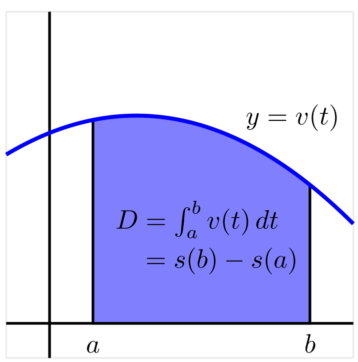


Figure 6.20: Finding the distance traveled when we know an object's velocity function v.

we know two different perspectives on the distance, D, the object travels: one is that $D = s(b) - s(a)$, which is the object's change in position. The other is that the distance traveled is the area under the velocity curve, which is given by the definite integral, so $D = \int_a^b v(t)\,dt$.

Of course, since both of these expressions tell us the distance traveled, it follows that they are equal, so

$$s(b) - s(a) = \int_a^b v(t)\,dt. \tag{6.2}$$

Furthermore, we know that Equation (6.2) holds even when velocity is sometimes negative, since $s(b) - s(a)$ is the object's change in position over $[a, b]$, which is simultaneously measured by the total net signed area on $[a, b]$ given by $\int_a^b v(t)\,dt$.

Perhaps the most powerful part of Equation (6.2) lies in the fact that we can compute the integral's value if we can find a formula for s. Remember, s and v are related by the fact that v is the derivative of s, or equivalently that s is an antiderivative of v. For example, if we have an object whose velocity is $v(t) = 3t^2 + 40$ feet per second (which is always nonnegative), and wish to know the distance traveled on the interval $[1, 5]$, we have that

$$\begin{aligned} D &= \int_1^5 v(t)\,dt \\ &= \int_1^5 (3t^2 + 40)\,dt \\ &= s(5) - s(1), \end{aligned}$$

where s is an antiderivative of v. We know that the derivative of t^3 is $3t^2$ and that the derivative of $40t$ is 40, so it follows that if $s(t) = t^3 + 40t$, then s is a function whose derivative is $v(t) = s'(t) = 3t^2 + 40$, and thus we have found an antiderivative of v. Therefore,

$$\begin{aligned} D &= \int_1^5 3t^2 + 40\,dt \\ &= s(5) - s(1) \\ &= (5^3 + 40 \cdot 5) - (1^3 + 40 \cdot 1) \\ &= 284 \text{ feet.} \end{aligned}$$

Note the key lesson of this example: to find the distance traveled, we needed to compute the area under a curve, which is given by the definite integral. But to evaluate the integral, we found an antiderivative, s, of the velocity function, and then computed the net change in s on the interval. In particular, observe that we have found the exact area of the region shown in Figure 6.21, and done so without a familiar formula (such as those for the area of a triangle or circle) and without directly computing the limit of a Riemann sum.

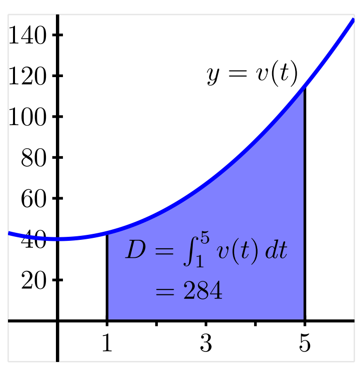


Figure 6.21: The exact area of the region enclosed by $v(t) = 3t^2 + 40$ on $[1, 5]$.

As we proceed to thinking about contexts other than just velocity and position, it turns out to be advantageous to have a shorthand symbol for a function's antiderivative. In the general setting of a continuous function f, we will often denote an antiderivative of f by F, so that the relationship between F and f is that $F'(x) = f(x)$ for all relevant x. Using the notation V in place of s (so that V is an antiderivative of v) in Equation (6.2), we find it is equivalent to write that

$$V(b) - V(a) = \int_a^b v(t)\,dt. \tag{6.3}$$

Now, in the general setting of wanting to evaluate the definite integral $\int_a^b f(x)\,dx$ for an arbitrary continuous function f, we could certainly think of f as representing the velocity of some moving object, and x as the variable that represents time. And again, Equations (6.2) and (6.3) hold for any continuous velocity function, even when v is sometimes negative. This leads us to see that Equation (6.3) tells us something even more important than the change in position of a moving object: it offers a shortcut route to evaluating any definite integral, provided that we can find an antiderivative of the integrand. The Fundamental Theorem of Calculus (FTC) summarizes these observations.

Theorem 6.4: Fundamental Theorem of Calculus

Suppose that $f(x)$ is continuous on the interval $[a, b]$. If $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Note that we will prove Theorem 6.4 in Section 6.3.3.

A common alternate notation for $F(b) - F(a)$ is

$$F(b) - F(a) = F(x)\big|_a^b,$$

where we read the righthand side as “the function F evaluated from a to b.” In this notation, the FTC says that

$$\int_a^b f(x)\,dx = F(x)\big|_a^b.$$

The FTC opens the door to evaluating exactly a wide range of integrals. In particular, if we are interested in a definite integral for which we can find an antiderivative F for the integrand f, then we can evaluate the integral exactly. For instance since $\frac{d}{dx}[\frac{1}{3}x^3] = x^2$, the FTC tells us that

$$\begin{aligned}\int_0^1 x^2\,dx &= \frac{1}{3}x^3\Big|_0^1 \\ &= \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 \\ &= \frac{1}{3}.\end{aligned}$$

Example 6.21: Fundamental Theorem of Calculus

Evaluate $\displaystyle\int_1^4 x^3 + \sqrt{x} + \frac{1}{x^2}\,dx.$

Solution.

$$\begin{aligned}
\int_1^4 x^3 + \sqrt{x} + \frac{1}{x^2}\,dx &= \left.\frac{x^4}{4} + \frac{2x^{3/2}}{3} - x^{-1}\right|_1^4 \\
&= \left(\frac{(4)^4}{4} + \frac{2(4)^{3/2}}{3} - 4^{-1}\right) \\
&\quad - \left(\frac{(1)^4}{4} + \frac{2(1)^{3/2}}{3} - 1^{-1}\right) \\
&= \frac{415}{6}
\end{aligned}$$

♣

Finding an antiderivative can be far from simple; in fact, often finding a formula for an antiderivative is very hard or even impossible. While we can differentiate just about any function, even some relatively simple ones don't have an elementary antiderivative. A significant portion of integral calculus (which is the main focus of second semester calculus) is devoted to understanding the problem of finding antiderivatives.

Example 6.22: Three Different Techniques

Evaluate $\displaystyle\int_0^2 x + 1\,dx$ *by*

1. *Using FTC (the shortcut)*
2. *Using the definition of a definite integral (the limit sum definition)*
3. *Interpreting the problem in terms of areas (graphically)*

Solution. 1. The shortcut (FTC) is the method of choice as it is the fastest. Integrating and using the '*top minus bottom*' rule we have:

$$\begin{aligned}
\int_0^2 x + 1\,dx &= \left.\frac{x^2}{2} + x\right|_0^2 \\
&= \left[\frac{2^2}{2} + 2\right] - \left[\frac{0^2}{2} + 0\right] = 4.
\end{aligned}$$

2. We now use the definition of a definite integral. We divide the interval $[0, 2]$ into n subintervals of equal width Δx, and from each interval choose a point x_i^*. Using the formulas

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\Delta x,$$

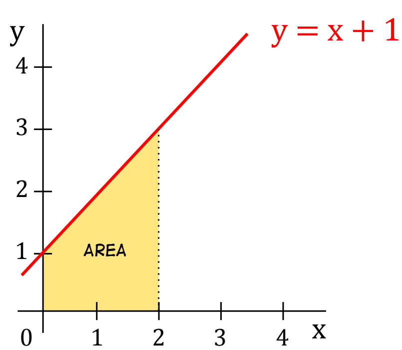
we have

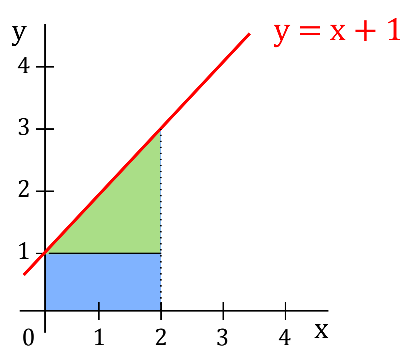
$$\Delta x = \frac{2}{n} \qquad \text{and} \qquad x_i = 0 + i\Delta x = \frac{2i}{n}.$$

Then taking x_i^*'s as right endpoints for convenience (so that $x_i^* = x_i$), we have:

$$\begin{aligned}
\int_0^2 x + 1\ dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n} + 1\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n^2} + \frac{2}{n}\right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{4i}{n^2} + \sum_{i=1}^{n} \frac{2}{n}\right) \\
&= \lim_{n\to\infty} \left(\frac{4}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}\sum_{i=1}^{n} 1\right) \\
&= \lim_{n\to\infty} \left(\frac{4}{n^2}\frac{n(n+1)}{2} + \frac{2}{n}n\right) \\
&= \lim_{n\to\infty} \left(2 + \frac{2}{n} + 2\right) \\
&= 4.
\end{aligned}$$

3. Finally, let's evaluate the net area under $x+1$ from 0 to 2.





Thus, the area is the sum of the areas of a rectangle and a triangle. Hence,

$$\begin{aligned}
\int_0^2 x + 1\ dx &= \text{Net Area} \\
&= \text{Area of rectangle} + \text{Area of triangle} \\
&= (2)(1) + \frac{1}{2}(2)(2) \\
&= 4.
\end{aligned}$$

♣

6.3.1. The net change theorem

As we use the Fundamental Theorem of Calculus to evaluate definite integrals, it is essential that we remember and understand the meaning of the numbers we find. We briefly summarize three key interpretations to date.

- For a moving object with instantaneous velocity $v(t)$, the object's change in position on the time interval $[a, b]$ is given by $\int_a^b v(t)\,dt$, and whenever $v(t) \geq 0$ on $[a, b]$, $\int_a^b v(t)\,dt$ tells us the total distance traveled by the object on $[a, b]$.
- For any continuous function f, its definite integral $\int_a^b f(x)\,dx$ represents the total net signed area bounded by $y = f(x)$ and the x-axis on $[a, b]$, where regions that lie below the x-axis have a minus sign associated with their area.

The Fundamental Theorem of Calculus now enables us to evaluate exactly (without taking a limit of Riemann sums) any definite integral for which we are able to find an antiderivative of the integrand.

A slight change in notational perspective allows us to gain even more insight into the meaning of the definite integral. To begin, recall Equation (6.3), where we wrote the Fundamental Theorem of Calculus for a velocity function v with antiderivative V as

$$V(b) - V(a) = \int_a^b v(t)\,dt.$$

If we instead replace V with s (which represents position) and replace v with s' (since velocity is the derivative of position), Equation (6.3) equivalently reads

$$s(b) - s(a) = \int_a^b s'(t)\,dt. \tag{6.4}$$

In words, this version of the FTC tells us that the net change in the object's position function on a particular interval is given by the definite integral of the position function's derivative over that interval.

Of course, this result is not limited to only the setting of position and velocity. Writing the result in terms of a more general function f, we have the Net Change Theorem.

Theorem 6.5: Net Change Theorem

If f is a continuously differentiable function on $[a, b]$ with derivative f', then

$$f(b) - f(a) = \int_a^b f'(x)\,dx. \tag{6.5}$$

That is, the integral of the rate of change (derivative) of a function on $[a, b]$ is the net change of the function itself on $[a, b]$.

The Net Change Theorem tells us more about the relationship between the graph of a function and that of its derivative. Recall from Chapter 4 that heights (or values) on the graph of the derivative function correspond to slopes on the graph of the function itself. That observation occurred in the context where we knew f and were seeking f'; if now instead we think about knowing f' and seeking information about f, we can instead say the following:

differences in heights on f correspond to net signed areas bounded by f'.

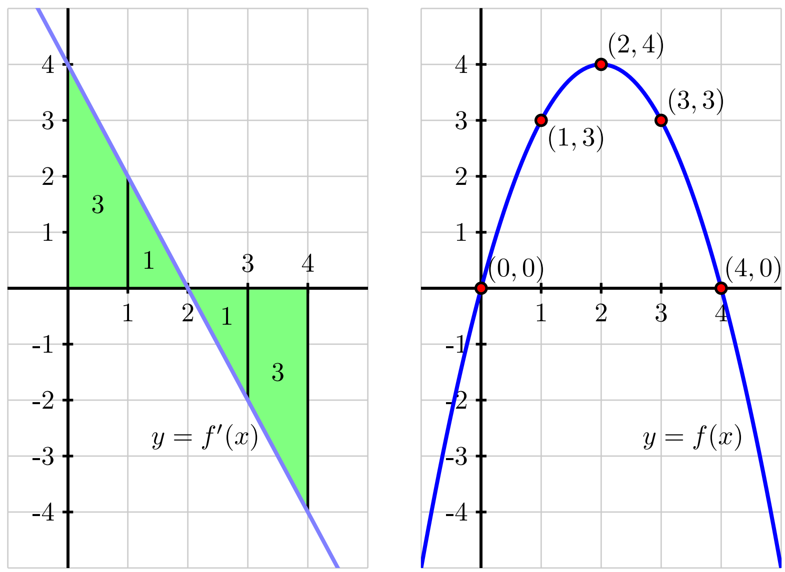


Figure 6.22: The graphs of $f'(x) = 4-2x$ (at left) and an antiderivative $f(x) = 4x - x^2$ at right. Differences in heights on f correspond to net signed areas bounded by f'.

To see why this is so, say we consider the difference $f(1) - f(0)$. Note that this value is 3, in part because $f(1) = 3$ and $f(0) = 0$, but also because the net signed area bounded by $y = f'(x)$ on $[0,1]$ is 3. That is, $f(1) - f(0) = \int_0^1 f'(x)\,dx$. A similar pattern holds throughout, including the fact that since the total net signed area bounded by f' on $[0,4]$ is 0, $\int_0^4 f'(x)\,dx = 0$, so it must be that $f(4) - f(0) = 0$, so $f(4) = f(0)$. Beyond this general observation about area, the Net Change Theorem enables us to consider interesting and important problems where we know the rate of change, and answer key questions about the function whose rate of change we know.

Example 6.23:

Suppose that pollutants are leaking out of an underground storage tank at a rate of $r(t)$ gallons/day, where t is measured in days. It is conjectured that $r(t)$ is given by the formula $r(t) = 0.0069t^3 - 0.125t^2 + 11.079$ over a certain 12-day period. The graph of $y = r(t)$ is given in Figure 6.23. What is the meaning of $\int_4^{10} r(t)\,dt$ and what is its value?

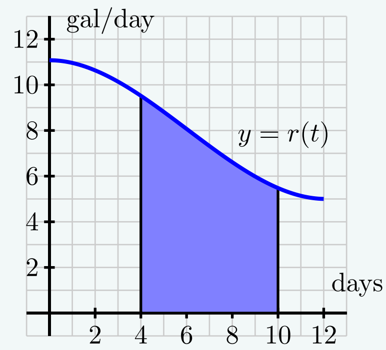


Figure 6.23: The rate $r(t)$ of pollution leaking from a tank, measured in gallons per day.

Solution. We know that since $r(t) \geq 0$, the value of $\int_4^{10} r(t)\,dt$ is the area under the curve on the interval $[4, 10]$. If we think about this area from the perspective of a Riemann sum, the rectangles will have heights measured in gallons per day and widths measured in days, thus the area of each rectangle will have units of

$$\frac{\text{gallons}}{\text{day}} \cdot \text{days} = \text{gallons}.$$

Thus, the definite integral tells us the total number of gallons of pollutant that leak from the tank from day 4 to day 10. The Net Change Theorem tells us the same thing: if we let $R(t)$ denote the function that measures the total number of gallons of pollutant that have leaked from the tank up to day t, then $R'(t) = r(t)$, and

$$\int_4^{10} r(t)\,dt = R(10) - R(4),$$

which is the net change in the function that measures total gallons leaked over time, thus the number of gallons that have leaked from day 4 to day 10.

To compute the exact value, we use the Fundamental Theorem of Calculus. Antidifferentiating $r(t) = 0.0069t^3 - 0.125t^2 + 11.079$, we find that

$$\begin{aligned}
\int_4^{10} (0.0069t^3 - 0.125t^2 + 11.079)\,dt &= \left(0.0069 \cdot \frac{1}{4}t^4 - 0.125 \cdot \frac{1}{3}t^3 + 11.079t\right)\Bigg|_4^{10} \\
&= \left(0.0069 \cdot \frac{1}{4}(10)^4 - 0.125 \cdot \frac{1}{3}(10)^3 + 11.079(10)\right) - \\
&\quad \left(0.0069 \cdot \frac{1}{4}(4)^4 - 0.125 \cdot \frac{1}{3}(4)^3 + 11.079(4)\right) \\
&\approx 44.282.
\end{aligned}$$

Thus, approximately 44.282 gallons of pollutant leaked over the six day time period.

♣

Summary

- We can find the exact value of a definite integral without taking the limit of a Riemann sum or using a familiar area formula by finding the antiderivative of the integrand, and hence applying the Fundamental Theorem of Calculus.

- The Fundamental Theorem of Calculus says that if f is a continuous function on $[a, b]$ and F is an antiderivative of f, then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$
Hence, if we can find an antiderivative for the integrand f, evaluating the definite integral comes from simply computing the change in F on $[a, b]$.

- A slightly different perspective on the FTC allows us to restate it as the Net Change Theorem, which says that
$$\int_a^b f'(x)\,dx = f(b) - f(a),$$
for any continuously differentiable function f. This means that the definite integral of the instantaneous rate of change of a function f on an interval $[a, b]$ is equal to the net change in the function f on $[a, b]$.

6.3.2. Functions defined by integrals

The FTC enables us to compute the value of the antiderivative F at a point b, provided that we know $F(a)$ and can evaluate the definite integral from a to b of f:

$$F(b) = F(a) + \int_a^b f(x)\,dx.$$

We may think of b, the upper limit of integration, as a variable itself. To that end, we introduce the idea of an *integral function*, a function whose formula involves a definite integral.

Given a continuous function f, we define the corresponding integral function A according to the rule

$$A(x) = \int_a^x f(t)\,dt. \tag{6.6}$$

Note particularly that because we are using the variable x as the independent variable in the function A, and x determines the other endpoint of the interval over which we integrate (starting from a), we need to use a variable other than x as the variable of integration. A standard choice is t, but any variable other than x is acceptable.

One way to think of the function A is as the "net-signed area from a up to x" function, where we consider the region bounded by $y = f(t)$ on the relevant interval. For example, in Figure 6.24, we see a given function f pictured at left, and its corresponding area function (choosing $a = 0$), $A(x) = \int_0^x f(t)\,dt$ shown at right.

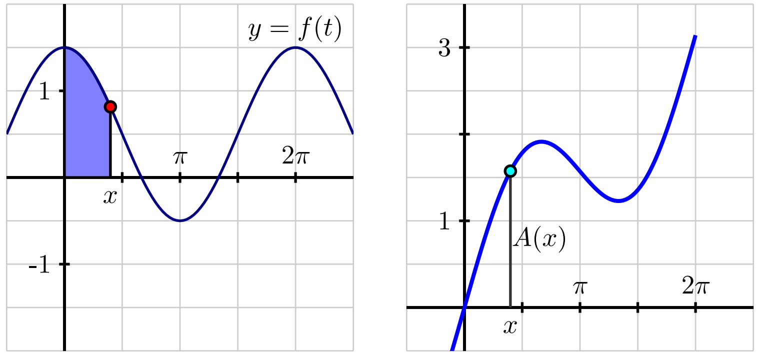


Figure 6.24: At left, the graph of the given function f. At right, the area function $A(x) = \int_0^x f(t)\,dt$.

Note particularly that the function A measures the net-signed area from $t = 0$ to $t = x$ bounded by the curve $y = f(t)$; this value is then reported as the corresponding height on the graph of $y = A(x)$. It is even more natural to think of this relationship between f and A dynamically. At http://gvsu.edu/s/cz, we find a java applet[3] that brings the static picture in Figure 6.24 to life. There, the user can move the red point on the function f and see how the corresponding height changes at the light blue point on the graph of A.

6.3.3. FTC 2

In the previous section we learned the Fundamental Theorem of Calculus (FTC), which from here forward will be referred to as the *First* Fundamental Theorem of Calculus, as in this section we develop a corresponding result that follows it.

We begin by way of example. If we let $f(t) = \cos(t) - t$ and set $A(x) = \int_2^x f(t)\,dt$, then we can determine a formula for A without integrals by the First FTC. Specifically,

$$\begin{aligned} A(x) &= \int_2^x (\cos(t) - t)\,dt \\ &= \sin(t) - \frac{1}{2}t^2 \bigg|_2^x \\ &= \sin(x) - \frac{1}{2}x^2 - (\sin(2) - 2)\,. \end{aligned}$$

Differentiating $A(x)$, since $(\sin(2) - 2)$ is constant, it follows that

$$A'(x) = \cos(x) - x,$$

and thus we see that $A'(x) = f(x)$. This tells us that for this particular choice of f, A is an antiderivative of f. More specifically, since $A(2) = \int_2^2 f(t)\,dt = 0$, A is the only antiderivative of f for which $A(2) = 0$.

In general, if f is any continuous function, and we define the function A by the rule

$$A(x) = \int_c^x f(t)\,dt,$$

[3] David Austin, Grand Valley State University

where c is an arbitrary constant, then we can show that A is an antiderivative of f. To see why, let's demonstrate that $A'(x) = f(x)$ by using the limit definition of the derivative. Doing so, we observe that

$$\begin{aligned} A'(x) &= \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \\ &= \lim_{h \to 0} \frac{\int_c^{x+h} f(t)\,dt - \int_c^x f(t)\,dt}{h} \\ &= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}, \end{aligned} \tag{6.7}$$

where Equation (6.7) in the preceding chain follows from the fact that $\int_c^x f(t)\,dt + \int_x^{x+h} f(t)\,dt = \int_c^{x+h} f(t)\,dt$. Now, observe that for small values of h,

$$\int_x^{x+h} f(t)\,dt \approx f(x) \cdot h,$$

by a simple left-hand approximation of the integral. Thus, as we take the limit in Equation (6.7), it follows that

$$A'(x) = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h \to 0} \frac{f(x) \cdot h}{h} = f(x).$$

Hence, A is indeed an antiderivative of f. In addition, $A(c) = \int_c^c f(t)\,dt = 0$. The preceding argument demonstrates the truth of the Second Fundamental Theorem of Calculus, which we state as follows.

Theorem 6.6: FTC 2

If f is a continuous function and a is any constant, then f has a unique antiderivative A that satisfies $A(a) = 0$, and that antiderivative is given by the rule $A(x) = \int_a^x f(t)\,dt$. That is to say

$$A'(x) = \frac{d}{dx}\left[\int_c^x f(t)\,dt\right] = f(x) \text{ and } A(c) = 0.$$

We can prove the first version of the FTC using the second:

Proof. Proof of Theorem 6.4.

We know from Theorem 6.6 that

$$A(x) = \int_a^x f(t)\,dt$$

is an antiderivative of $f(x)$, and therefore any antiderivative $F(x)$ of $f(x)$ is of the form $F(x) = A(x) + k$. Then

$$\begin{aligned} F(b) - F(a) = A(b) + k - (A(a) + k) &= A(b) - A(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt. \end{aligned}$$

It is not hard to see that $\int_a^a f(t)\,dt = 0$, so this means that

$$F(b) - F(a) = \int_a^b f(t)\,dt,$$

which is exactly what Theorem 6.4 says. ♣

Example 6.24: Using FTC

Differentiate *the following function:*

$$g(x) = \int_{-2}^{x} \cos(1+5t)\sin t\,dt.$$

Solution. We simply apply the Fundamental Theorem of Calculus directly to get:

$$g'(x) = \cos(1+5x)\sin x.$$

♣

Using the Chain Rule we can derive a formula for some more complicated problems. If F is an antiderivative of f, then we have:

$$\frac{d}{dx}\int_{a}^{v(x)} f(t)\,dt = \frac{d}{dx}\left(F(v(x)) - F(a)\right) = f(v(x))\cdot v'(x) - 0 = f(v(x))\cdot v'(x).$$

Now what if the upper limit is constant and the lower limit is a function of x? Then we interchange the limits and add a minus sign to get:

$$\frac{d}{dx}\int_{u(x)}^{a} f(t)\,dt = -\frac{d}{dx}\int_{a}^{u(x)} f(t)\,dt = -f(u(x))\cdot u'(x).$$

Combining these two we can get a formula where both limits are a function of x. We break up the integral as follows:

$$\int_{u(x)}^{v(x)} f(t)\,dt = \int_{u(x)}^{a} f(t)\,dt + \int_{a}^{v(x)} f(t)\,dt.$$

We just need to make sure $f(a)$ exists after we break up the integral. Then differentiating and using the above two formulas gives:

Key Idea 6.3.0: FTC I + Chain Rule:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))v'(x) - f(u(x))u'(x)$$

Many textbooks do not show this formula and instead to solve these types of problems will use FTC I along with the tricks we used to derive the formula above. Either method is perfectly fine.

Example 6.25: FTC I + Chain Rule

Differentiate the following integral:

$$\int_{10x}^{x^2} t^3 \sin(1+t)\,dt.$$

Solution. We will use the formula above. We have $f(t) = t^3 \sin(1+t)$, $u(x) = 10x$ and $v(x) = x^2$. Then $u'(x) = 10$ and $v'(x) = 2x$. Thus,

$$\begin{aligned}\frac{d}{dx}\int_{10x}^{x^2} t^3 \sin(1+t)\,dt &= (x^2)^3 \sin(1+(x^2))(2x) - (10x)^3 \sin(1+(10x))(10) \\ &= 2x^7 \sin(1+x^2) - 10000x^3 \sin(1+10x)\end{aligned}$$

♣

Example 6.26: FTC I + Chain Rule

Differentiate the following integral with respect to x:

$$\int_{x^3}^{2x} 1+\cos t\,dt$$

Solution. Using the formula we have:

$$\frac{d}{dx}\int_{x^3}^{2x} 1+\cos t\,dt = (1+\cos(2x))(2) - (1+\cos(x^3))(3x^2).$$

♣

6.3.4. More on Differentiating Integral Functions

The Second FTC provides us with a means to construct an antiderivative of any continuous function. In particular, if we are given a continuous function g and wish to find an antiderivative of g, we can now say that

$$G(x) = \int_a^x g(t)\,dt$$

provides the rule for such an antiderivative, and moreover that $G(a) = 0$. Note especially that we know that $G'(x) = g(x)$. We sometimes want to write this relationship between G and g from a different notational perspective. In particular, observe that

$$\frac{d}{dx}\left[\int_a^x g(t)\,dt\right] = g(x). \tag{6.8}$$

This result can be particularly useful when we're given an integral function such as G and wish to understand properties of its graph by recognizing that $G'(x) = g(x)$, while not necessarily being able to exactly evaluate the definite integral $\int_c^x g(t)\,dt$.

This shows that integral functions, while perhaps having the most complicated formulas of any functions we have encountered, are nonetheless particularly simple to differentiate. For instance, if

$$F(x) = \int_\pi^x \sin(t^2)\,dt,$$

then by the Second FTC, we know immediately that

$$F'(x) = \sin(x^2).$$

Stating this result more generally for an arbitrary function f, we know by the Second FTC that

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

In words, the last equation essentially says that "the derivative of the integral function whose integrand is f, is f." In this sense, we see that if we first integrate the function f from $t = a$ to $t = x$, and then differentiate with respect to x, these two processes "undo" one another.

Taking a different approach, say we begin with a function $f(t)$ and differentiate with respect to t. What happens if we follow this by integrating the result from $t = a$ to $t = x$? That is, what can we say about the quantity

$$\int_a^x \frac{d}{dt}\left[f(t)\right]\,dt?$$

Here, we use the First FTC and note that $f(t)$ is an antiderivative of $\frac{d}{dt}\left[f(t)\right]$. Applying this result and evaluating the antiderivative function, we see that

$$\begin{aligned} \int_a^x \frac{d}{dt}\left[f(t)\right]\,dt &= f(t)\Big|_a^x \\ &= f(x) - f(a). \end{aligned}$$

Thus, we see that if we apply the processes of first differentiating f and then integrating the result from a to x, we return to the function f, minus the constant value $f(a)$. So in this situation, the two processes almost undo one another, up to the constant $f(a)$.

The observations made in the preceding two paragraphs demonstrate that differentiating and integrating (where we integrate from a constant up to a variable) are almost inverse processes. In one sense, this should not be surprising: integrating involves antidifferentiating, which reverses the process of differentiating. On the other hand, we see that there is some subtlety involved, as integrating the derivative of a function does not quite produce the function itself. This is connected to a key fact that any function has an infinite family of antiderivatives, and any two of those antiderivatives differ only by a constant.

This section has laid the groundwork for a lot of great mathematics to follow. The most important lesson is this: definite integrals can be evaluated using antiderivatives. Since the previous section established that definite integrals are the limit of Riemann sums, we can later create Riemann sums to approximate values other than "area under the curve," convert the sums to definite integrals, then evaluate these using the Fundamental Theorem of Calculus. This will allow us to compute the work done by a variable force, the volume of certain solids, the arc length of curves, and more.

The downside is this: generally speaking, computing antiderivatives is much more difficult than computing derivatives. The next chapter is devoted to techniques of finding antiderivatives so that a wide variety of definite integrals can be evaluated. Before that, the next section explores techniques of approximating the value of definite integrals beyond using the Left Hand, Right Hand and Midpoint Rules.

Exercises for Section 6.3

6.3.1 *Evaluate* $\displaystyle\int_1^4 t^2 + 3t\,dt$

6.3.2 *Evaluate* $\displaystyle\int_0^\pi \sin t\,dt$

6.3.3 *Evaluate* $\displaystyle\int_1^{10} \frac{1}{x}\,dx$

6.3.4 *Evaluate* $\displaystyle\int_0^5 e^x\,dx$

6.3.5 *Evaluate* $\displaystyle\int_0^3 x^3\,dx$

6.3.6 *Evaluate* $\displaystyle\int_1^2 x^5\,dx$

6.3.7 *Find the derivative of* $\displaystyle G(x) = \int_1^x t^2 - 3t\,dt$

6.3.8 *Find the derivative of* $\displaystyle G(x) = \int_1^{x^2} t^2 - 3t\,dt$

6.3.9 *Find the derivative of* $\displaystyle G(x) = \int_1^x e^{t^2}\,dt$

6.3.10 *Find the derivative of* $\displaystyle G(x) = \int_1^{x^2} e^{t^2}\,dt$

6.3.11 *Find the derivative of* $\displaystyle G(x) = \int_1^x \tan(t^2)\,dt$

6.3.12 *Find the derivative of* $\displaystyle G(x) = \int_{10x}^{x^2} \tan(t^2)\,dt$

6.3.13 *Suppose* $\int_1^4 f(x)\,dx = 2$ *and* $\int_1^4 g(x)\,dx = 7$. *Find* $\int_1^4 (5f(x) + 3g(x))\,dx$ *and* $\int_1^4 (6 - 2f(x))\,dx$.

6.3.14 *Suppose* $\int_{-2}^5 f(x)\,dx = 3$ *and* $\int_1^5 f(x)\,dx = -2$. *Find* $\int_{-2}^1 f(x)\,dx$.

6.3.15 *If* f *is continuous on* $[a,b]$, *we define the average of* $f(x)$ *on* $[a,b]$ *to be*

$$avg_{[a,b]}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

(a) What is the average of $\sqrt{x}$ *on the interval* $[0,1]$?

(b) If the average of $f(x)$ *on* $[0,2]$ *and on* $[2,5]$ *are 6 and 4 respectively, then what is its average on* $[0,5]$?

7. Techniques of Integration

Over the next few sections we examine some techniques that are frequently successful when seeking antiderivatives of functions.

7.1 Substitution Rule

Antiderivatives play a key role in the the process of evaluating definite integrals exactly. In particular, the Fundamental Theorem of Calculus tells us that if F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Furthermore, we realized that each elementary derivative rule leads to a corresponding elementary antiderivative (or indefinite integral), as summarized in Table **??**. Thus, if we wish to evaluate an integral such as

$$\int_0^1 \left(x^3 - \sqrt{x} + 5^x\right)\,dx,$$

it is straightforward to do so, since we can easily integrate $f(x) = x^3 - \sqrt{x} + 5^x$. In particular, since a function F whose derivative is f is given by $F(x) = \frac{1}{4}x^4 - \frac{2}{3}x^{3/2} + \frac{1}{\ln(5)}5^x$, the Fundamental Theorem of Calculus tells us that

$$\begin{aligned}
\int_0^1 \left(x^3 - \sqrt{x} + 5^x\right)\,dx &= \left.\frac{1}{4}x^4 - \frac{2}{3}x^{3/2} + \frac{1}{\ln(5)}5^x\right|_0^1 \\
&= \left(\frac{1}{4}(1)^4 - \frac{2}{3}(1)^{3/2} + \frac{1}{\ln(5)}5^1\right) - \left(\frac{1}{4}(0)^4 - \frac{2}{3}(0)^{3/2} + \frac{1}{\ln(5)}5^0\right) \\
&= \frac{1}{4} - \frac{2}{3} + \frac{5}{\ln(5)} - \frac{1}{\ln(5)} \\
&= -\frac{5}{12} + \frac{4}{\ln(5)}.
\end{aligned}$$

Because an algebraic formula for an antiderivative of f enables us to evaluate the definite integral $\int_a^b f(x)\,dx$ exactly, we see that we have a natural interest in being able to find such algebraic antiderivatives. Note that we emphasize *algebraic* antiderivatives, as opposed to any antiderivative, since we know by the Second Fundamental Theorem of Calculus that $G(x) = \int_a^x f(t)\,dt$ is indeed an antiderivative of the given function f, but one that still involves a definite integral. One of our main goals in this section is to develop understanding, in select circumstances, of how to "undo" the process of differentiation in order to find an algebraic antiderivative for a given function.

Reversing the Chain Rule: Substitution

It is usually straightforward to integrate a function of the form

$$h(x) = f(u(x)),$$

whenever f is a familiar function whose antiderivative is known and $u(x)$ is a linear function. For example, if we consider

$$h(x) = (5x-3)^6,$$

in this context the *outer function* f is $f(u) = u^6$, while the *inner function* is $u(x) = 5x - 3$. Since the antiderivative of f is $F(u) = \frac{1}{7}u^7 + C$, we see that the antiderivative of h is

$$H(x) = \frac{1}{7}(5x-3)^7 \cdot \frac{1}{5} + C = \frac{1}{35}(5x-3)^7 + C.$$

The inclusion of the constant $\frac{1}{5}$ is essential precisely because the derivative of the inner function is $u'(x) = 5$. Indeed, if we now compute $H'(x)$, we find by the Chain Rule (and Constant Multiple Rule) that

$$H'(x) = \frac{1}{35} \cdot 7(5x-3)^6 \cdot 5 = (5x-3)^6 = h(x),$$

and thus H is indeed the general antiderivative of h.

Hence, in the special case where the outer function is familiar and the inner function is linear, we can antidifferentiate composite functions according to the following rule.

Key Idea 7.1.0:

If $h(x) = f(ax+b)$ and F is a known algebraic antiderivative of f, then the general antiderivative of h is given by

$$H(x) = \frac{1}{a}F(ax+b) + C.$$

Of course, a natural question arises: what happens when the inner function is not a linear function? For example, can we find antiderivatives of such functions as

$$g(x) = xe^{x^2} \text{ and } h(x) = e^{x^2}?$$

It is important to explicitly remember that differentiation and antidifferentiation are essentially inverse processes; that they are not quite inverse processes is due to the $+C$ that arises when antidifferentiating. This close relationship enables us to take any known derivative rule and translate it to a corresponding rule for an indefinite integral. For example, since

$$\frac{d}{dx}\left[x^5\right] = 5x^4,$$

we can equivalently write

$$\int 5x^4\,dx = x^5 + C.$$

Recall that the Chain Rule states that

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x).$$

Restating this relationship in terms of an indefinite integral,

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + C. \tag{7.1}$$

Hence, Equation (7.1) tells us that if we can take a given function and view its algebraic structure as $f'(g(x))g'(x)$ for some appropriate choices of f and g, then we can antidifferentiate the function by reversing the Chain Rule. It is especially notable that both $g(x)$ and $g'(x)$ appear in the form of $f'(g(x))g'(x)$; we will sometimes say that we seek to *identify a function-derivative pair* when trying to apply the rule in Equation (7.1).

For example: Find

$$\int 2x\cos(x^2)\,dx.$$

This is not a "simple" derivative, but a little thought reveals that it must have come from an application of the chain rule. Multiplied on the "outside" is $2x$, which is the derivative of the "inside" function x^2. Checking:

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)\frac{d}{dx}x^2 = 2x\cos(x^2),$$

so

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

In the situation where we can identify a function-derivative pair, we will introduce a new variable u to represent the function $g(x)$. Observing that with $u = g(x)$, it follows in Leibniz notation that $\frac{du}{dx} = g'(x)$, so that in terms of differentials[1], $du = g'(x)\,dx$. Now converting the indefinite integral of interest to a new one in terms of u, we have

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du.$$

Provided that f' is an elementary function whose antiderivative is known, we can now easily evaluate the indefinite integral in u, and then go on to determine the desired overall antiderivative of $f'(g(x))g'(x)$. We call this process *u-substitution*.

To summarize: If we suspect that a given function is the derivative of another via the chain rule, we let u denote a likely candidate for the inner function, then translate the given function so that it is written entirely in terms of u, with no x remaining in the expression. If we can integrate this new function of u, then the antiderivative of the original function is obtained by replacing u by the equivalent expression in x.

Theorem 7.1: u-Substitution Rule for Indefinite Integrals

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Even in simple cases you may prefer to use this mechanical procedure, since it often helps to avoid silly mistakes. For example, consider again this simple problem:

$$\int 2x\cos(x^2)\,dx.$$

Let $u = x^2$, then $du/dx = 2x$ or $du = 2x\,dx$. Since we have exactly $2x\,dx$ in the original integral, we can replace it by du:

$$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

[1]If we recall from the definition of the derivative that $\frac{du}{dx} \approx \frac{\triangle u}{\triangle x}$ and use the fact that $\frac{du}{dx} = g'(x)$, then we see that $g'(x) \approx \frac{\triangle u}{\triangle x}$. Solving for $\triangle u$, $\triangle u \approx g'(x)\triangle x$. It is this last relationship that, when expressed in "differential" notation enables us to write $du = g'(x)\,dx$ in the change of variable formula.

This is not the only way to do the algebra, and typically there are many paths to the correct answer. Another possibility, for example, is: Since $du/dx = 2x$, $dx = du/2x$, and then the integral becomes

$$\int 2x\cos(x^2)\,dx = \int 2x\cos u\,\frac{du}{2x} = \int \cos u\,du.$$

The important thing to remember is that you must eliminate all instances of the original variable x.

Example 7.1: Substitution Rule

Evaluate the indefinite integral

$$\int x^3 \cdot \sin(7x^4+3)\,dx$$

and check the result by differentiating.

Solution. We can make two key algebraic observations stand regarding the integrand, $x^3 \cdot \sin(7x^4+3)$. First, $\sin(7x^4+3)$ is a composite function; as such, we know we'll need a more sophisticated approach to antidifferentiating. Second, x^3 is almost the derivative of $(7x^4+3)$; the only issue is a missing constant. Thus, x^3 and $(7x^4+3)$ are nearly a function-derivative pair. Furthermore, we know the antiderivative of $f(u) = \sin(u)$. The combination of these observations suggests that we can evaluate the given indefinite integral by reversing the chain rule through u-substitution.

Letting u represent the inner function of the composite function $\sin(7x^4+3)$, we have

$$u = 7x^4+3,$$

and thus $\frac{du}{dx} = 28x^3$. In differential notation, it follows that $du = 28x^3\,dx$, and thus $x^3\,dx = \frac{1}{28}\,du$. We make this last observation because the original indefinite integral may now be written

$$\int \sin(7x^4+3)\cdot x^3\,dx,$$

and so by substituting the expressions in u for x (specifically u for $7x^4+3$ and $\frac{1}{28}\,du$ for $x^3\,dx$), it follows that

$$\int \sin(7x^4+3)\cdot x^3\,dx = \int \sin(u)\cdot\frac{1}{28}\,du.$$

Now we may evaluate the original integral by first evaluating the easier integral in u, followed by replacing u by the expression $7x^4+3$. Doing so, we find

$$\begin{aligned}
\int \sin(7x^4+3)\cdot x^3\,dx &= \int \sin(u)\cdot\frac{1}{28}\,du \\
&= \frac{1}{28}\int \sin(u)\,du \\
&= \frac{1}{28}(-\cos(u)) + C \\
&= -\frac{1}{28}\cos(7x^4+3) + C.
\end{aligned}$$

To check our work, we observe by the Chain Rule that

$$\frac{d}{dx}\left[-\frac{1}{28}\cos(7x^4+3)+C\right] = -\frac{1}{28}\cdot(-1)\sin(7x^4+3)\cdot 28x^3 = \sin(7x^4+3)\cdot x^3,$$

which is indeed the original integrand. ♣

An essential observation about our work in Example 7.1 is that the u-substitution only worked because the function multiplying $\sin(7x^4+3)$ was x^3. If instead that function was x^2 or x^4, the substitution process may not (and likely would not) have worked. This is one of the primary challenges of antidifferentiation: slight changes in the integrand make tremendous differences. For instance, we can use u-substitution with $u = x^2$ and $du = 2xdx$ to find that

$$\begin{aligned}\int xe^{x^2}\,dx &= \int e^u \cdot \frac{1}{2}\,du \\ &= \frac{1}{2}\int e^u\,du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{x^2} + C.\end{aligned}$$

If, however, we consider the similar indefinite integral

$$\int e^{x^2}\,dx,$$

the missing x to multiply e^{x^2} makes the u-substitution $u = x^2$ no longer possible. Hence, part of the lesson of u-substitution is just how specialized the process is: it only applies to situations where, up to a missing constant, the integrand that is present is the result of applying the Chain Rule to a different, related function.

Example 7.2: Substitution Rule

Evaluate $\int (ax+b)^n\,dx$*, assuming* a, b *are constants,* $a \neq 0$*, and* n *is a positive integer.*

Solution. We let $u = ax + b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int (ax+b)^n\,dx = \int \frac{1}{a}u^n\,du = \frac{1}{a(n+1)}u^{n+1} + C = \frac{1}{a(n+1)}(ax+b)^{n+1} + C.$$

♣

Example 7.3: Substitution Rule

Evaluate $\int \sin(ax+b)\,dx$*, assuming that* a *and* b *are constants and* $a \neq 0$*.*

Solution. Again we let $u = ax + b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int \sin(ax+b)\,dx = \int \frac{1}{a}\sin u\,du = \frac{1}{a}(-\cos u) + C = -\frac{1}{a}\cos(ax+b) + C.$$

♣

Key Idea 7.1.0: Strategy for Substitution Rule

A general strategy to follow is:

1. Choose a possible $u = u(x)$. **Tip:** Choose a substitution u so that its derivate also appears in the integral (up to a constant).
2. Calculate $du = u'(x)\ dx$.
3. Either replace $u'(x)\ dx$ by du, or replace dx by $\dfrac{du}{u'(x)}$, and cancel.
4. Write the rest of the integrand in terms of u. If this is not possible, the substitution will not work: You must go back to step 1.
5. Find the indefinite integral. (Again, if this is not possible, try a different substitution, or a different method).
6. Rewrite the result in terms of x.

Example 7.4: Substitution

Evaluate the following integral: $\displaystyle\int \frac{2x}{\sqrt{1-4x^2}}\,dx$.

Solution. We try the substitution:

$$u = 1 - 4x^2.$$

Then,

$$du = -8x\ dx$$

In the numerator we have $2x\ dx$, so rewriting the differential gives:

$$-\frac{1}{4}du = 2x\ dx.$$

Then the integral is:

$$\begin{aligned}
\int \frac{2x}{\sqrt{1-4x^2}}\,dx &= \int \left(1-4x^2\right)^{-1/2}(2x\ dx) \\
&= \int u^{-1/2}\left(-\frac{1}{4}du\right) \\
&= \left(\frac{-1}{4}\right)\frac{u^{1/2}}{1/2} + C \\
&= -\frac{\sqrt{1-4x^2}}{2} + C
\end{aligned}$$

♣

Example 7.5: Substitution

Evaluate the following integral: $\int \operatorname{sech}^2(7t-3)\ dt$

Solution. We employ substitution, with $u = 7t - 3$ and $du = 7dt$. We have:

$$\int \operatorname{sech}^2(7t-3)\ dt = \frac{1}{7}\int \operatorname{sech}^2(u)\ du = \frac{1}{7}\tanh(u) + C = \frac{1}{7}\tanh(7t-3) + C.$$

♣

Example 7.6: Substitution

Evaluate the following integral: $\int \cos x(\sin x)^5\, dx.$

Solution. In this question we will let $u = \sin x$. Then,

$$du = \cos x\ dx.$$

Thus, the integral becomes:

$$\begin{aligned}\int \cos x(\sin x)^5\, dx &= \int u^5\, du\\ &= \frac{u^6}{6} + C\\ &= \frac{(\sin x)^6}{6} + C\end{aligned}$$

♣

Example 7.7: Substitution

Evaluate the following integral: $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\ dx.$

Solution. We use the substitution:

$$u = x^{1/2}.$$

Then,

$$du = \frac{1}{2}x^{-1/2}dx.$$

Rewriting the differential we get:

$$2\ du = \frac{1}{\sqrt{x}}\ dx.$$

The integral becomes:

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\, dx \quad = \quad 2\int \cos u\, du$$

$$= \quad 2\sin u + C$$

$$= \quad 2\sin(\sqrt{x}) + C$$

♣

Example 7.8: Integrating by substitution

Evaluate $\displaystyle\int \sin x \cos x \ dx.$

Solution. There is not a composition of function here to exploit; rather, just a product of functions. Do not be afraid to experiment; when given an integral to evaluate, it is often beneficial to think "If I let u be *this*, then du must be *that* ..." and see if this helps simplify the integral at all.

In this example, let's set $u = \sin x$. Then $du = \cos x \ dx$, which we have as part of the integrand! The substitution becomes very straightforward:

$$\begin{aligned}\int \sin x \cos x \ dx &= \int u \ du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\sin^2 x + C.\end{aligned}$$

One would do well to ask "What would happen if we let $u = \cos x$?" The result is just as easy to find, yet looks very different. The challenge to the reader is to evaluate the integral letting $u = \cos x$ and discover why the answer is the same, yet looks different. ♣

Our examples so far have required "basic substitution." The next example demonstrates how substitutions can be made that often strike the new learner as being "nonstandard."

Example 7.9: Substitution

Evaluate the following integral: $\displaystyle\int 2x^3\sqrt{x^2+1}\,dx.$

Solution. This problem is a little bit different than the previous ones. It makes sense to let:

$$u = x^2 + 1,$$

then

$$du = 2x \ dx.$$

Making this substitution gives:

$$\int 2x^3\sqrt{x^2+1}\,dx \quad = \quad \int x^2\sqrt{x^2+1}(2x)\,dx$$

$$= \int x^2 u^{1/2}\, du$$

This is a problem because our integrals can't have a mixture of two variables in them. Usually this means we chose our u incorrectly. However, in this case we can eliminate the remaining x's from our integral by using:

$$u = x^2 + 1 \quad \rightarrow \quad x^2 = u - 1.$$

We get:

$$\begin{aligned}
\int x^2 u^{1/2}\, du &= \int (u-1)u^{1/2}\, du \\
&= \int u^{3/2} - u^{1/2}\, du \\
&= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C \\
&= \frac{2}{5}(x^2+1)^{5/2} - \frac{2}{3}(x^2+1)^{3/2} + C
\end{aligned}$$

♣

Example 7.10: Integrating by substitution

Evaluate $\int x\sqrt{x+3}\; dx$.

Solution. Recognizing the composition of functions, set $u = x + 3$. Then $du = dx$, giving what seems initially to be a simple substitution. But at this stage, we have:

$$\int x\sqrt{x+3}\; dx = \int x\sqrt{u}\; du.$$

We cannot evaluate an integral that has both an x and an u in it. We need to convert the x to an expression involving just u.

Since we set $u = x + 3$, we can also state that $u - 3 = x$. Thus we can replace x in the integrand with $u - 3$. It will also be helpful to rewrite $\sqrt{u}$ as $u^{\frac{1}{2}}$.

$$\begin{aligned}
\int x\sqrt{x+3}\; dx &= \int (u-3)u^{\frac{1}{2}}\; du \\
&= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right)\; du \\
&= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C \\
&= \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.
\end{aligned}$$

Checking your work is always a good idea. In this particular case, some algebra will be needed to make one's answer match the integrand in the original problem. ♣

Example 7.11: Integrating by substitution

Evaluate $\displaystyle\int \frac{1}{x \ln x}\, dx$.

Solution. This is another example where there does not seem to be an obvious composition of functions. The line of thinking used in Example 7.10 is useful here: choose something for u and consider what this implies du must be. If u can be chosen such that du also appears in the integrand, then we have chosen well.

Choosing $u = 1/x$ makes $du = -1/x^2\ dx$; that does not seem helpful. However, setting $u = \ln x$ makes $du = 1/x\ dx$, which is part of the integrand. Thus:

$$\begin{aligned}
\int \frac{1}{x \ln x}\ dx &= \int \underbrace{\frac{1}{\ln x}}_{1/u}\underbrace{\frac{1}{x}\ dx}_{du} \\
&= \int \frac{1}{u}\ du \\
&= \ln |u| + C \\
&= \ln |\ln x| + C.
\end{aligned}$$

The final answer is interesting; the natural log of the natural log. Take the derivative to confirm this answer is indeed correct. ♣

7.1.1. Integrals Involving Trigonometric Functions

Section 7.2 delves deeper into integrals of a variety of trigonometric functions; here we use substitution to establish a foundation that we will build upon.

The next three examples will help fill in some missing pieces of our antiderivative knowledge. We know the antiderivatives of the sine and cosine functions; what about the other standard functions tangent, cotangent, secant and cosecant? We discover these next.

Example 7.12: Integration by substitution: antiderivatives of $\tan x$

Evaluate $\displaystyle\int \tan x\ dx$.

Solution. The previous paragraph established that we did not know the antiderivatives of tangent, hence we must assume that we have learned something in this section that can help us evaluate this indefinite integral.

Rewrite $\tan x$ as $\sin x/\cos x$. While the presence of a composition of functions may not be immediately obvious, recognize that $\cos x$ is "inside" the $1/x$ function. Therefore, we see if setting $u = \cos x$ returns usable results. We have that $du = -\sin x\ dx$, hence $-du = \sin x\ dx$. We can integrate:

$$\int \tan x\ dx = \int \frac{\sin x}{\cos x}\ dx$$

$$
\begin{aligned}
&= \int \frac{1}{\underbrace{\cos x}_{u}} \underbrace{\sin x \; dx}_{-du} \\
&= \int \frac{-1}{u} \; du \\
&= -\ln|u| + C \\
&= -\ln|\cos x| + C.
\end{aligned}
$$

Some texts prefer to bring the -1 inside the logarithm as a power of $\cos x$, as in:

$$
\begin{aligned}
-\ln|\cos x| + C &= \ln|(\cos x)^{-1}| + C \\
&= \ln\left|\frac{1}{\cos x}\right| + C \\
&= \ln|\sec x| + C.
\end{aligned}
$$

Thus the result they give is $\int \tan x \; dx = \ln|\sec x| + C$. These two answers are equivalent.

♣

Example 7.13: Integrating by substitution: antiderivatives of $\sec x$

Evaluate $\displaystyle\int \sec x \; dx$.

Solution. This example employs a wonderful trick: multiply the integrand by "1" so that we see how to integrate more clearly. In this case, we write "1" as

$$
1 = \frac{\sec x + \tan x}{\sec x + \tan x}.
$$

This may seem like it came out of left field, but it works beautifully. Consider:

$$
\begin{aligned}
\int \sec x \; dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \; dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx.
\end{aligned}
$$

Now let $u = \sec x + \tan x$; this means $du = (\sec x \tan x + \sec^2 x) \; dx$, which is our numerator. Thus:

$$
\begin{aligned}
&= \int \frac{du}{u} \\
&= \ln|u| + C \\
&= \ln|\sec x + \tan x| + C.
\end{aligned}
$$

♣

We can use similar techniques to those used in Examples **??** and **??** to find antiderivatives of $\cot x$ and $\csc x$ (which the reader can explore in the exercises.) We summarize our results here.

Theorem 7.2: Antiderivatives of Trigonometric Functions

1. $\int \sin x \; dx = -\cos x + C$
2. $\int \cos x \; dx = \sin x + C$
3. $\int \tan x \; dx = -\ln|\cos x| + C$
4. $\int \csc x \; dx = -\ln|\csc x + \cot x| + C$
5. $\int \sec x \; dx = \ln|\sec x + \tan x| + C$
6. $\int \cot x \; dx = \ln|\sin x| + C$

We explore one more common trigonometric integral.

Example 7.14: Integration by substitution: powers of $\cos x$ and $\sin x$

Evaluate $\int \cos^2 x \; dx$.

Solution. We have a composition of functions as $\cos^2 x = \big(\cos x\big)^2$. However, setting $u = \cos x$ means $du = -\sin x \; dx$, which we do not have in the integral. Another technique is needed.

The process we'll employ is to use a Power Reducing formula for $\cos^2 x$ (perhaps consult the back of this text for this formula), which states

$$\cos^2 x = \frac{1+\cos(2x)}{2}.$$

The right hand side of this equation is not difficult to integrate. We have:

$$\begin{aligned}\int \cos^2 x \; dx &= \int \frac{1+\cos(2x)}{2} \; dx \\ &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) \; dx.\end{aligned}$$

Now use Key Idea **??**:

$$\begin{aligned}&= \frac{1}{2}x + \frac{1}{2}\frac{\sin(2x)}{2} + C \\ &= \frac{1}{2}x + \frac{\sin(2x)}{4} + C.\end{aligned}$$

We'll make significant use of this power–reducing technique in future sections.

♣

7.1.2. Simplifying the Integrand

It is common to be reluctant to manipulate the integrand of an integral; at first, our grasp of integration is tenuous and one may think that working with the integrand will improperly change the results. Integration by substitution works using a different logic: as long as *equality* is maintained, the integrand can be manipulated so that its *form* is easier to deal with. The next two examples demonstrate common ways in which using algebra first makes the integration easier to perform.

Example 7.15: Integration by substitution: simplifying first

Evaluate $\displaystyle\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\,dx.$

Solution. One may try to start by setting u equal to either the numerator or denominator; in each instance, the result is not workable.

When dealing with rational functions (i.e., quotients made up of polynomial functions), it is an almost universal rule that everything works better when the degree of the numerator is less than the degree of the denominator. Hence we use polynomial division.

We skip the specifics of the steps, but note that when x^2+2x+1 is divided into x^3+4x^2+8x+5, it goes in $x+2$ times with a remainder of $3x+3$. Thus

$$\frac{x^3+4x^2+8x+5}{x^2+2x+1} = x+2+\frac{3x+3}{x^2+2x+1}.$$

Integrating $x+2$ is simple. The fraction can be integrated by setting $u=x^2+2x+1$, giving $du=(2x+2)\,dx$. This is very similar to the numerator. Note that $du/2=(x+1)\,dx$ and then consider the following:

$$\begin{aligned}
\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\,dx &= \int\left(x+2+\frac{3x+3}{x^2+2x+1}\right)\,dx \\
&= \int (x+2)\,dx + \int \frac{3(x+1)}{x^2+2x+1}\,dx \\
&= \frac{1}{2}x^2+2x+C_1+\int \frac{3}{u}\frac{du}{2} \\
&= \frac{1}{2}x^2+2x+C_1+\frac{3}{2}\ln|u|+C_2 \\
&= \frac{1}{2}x^2+2x+\frac{3}{2}\ln|x^2+2x+1|+C.
\end{aligned}$$

In some ways, we "lucked out" in that after dividing, substitution was able to be done. In later sections we'll develop techniques for handling rational functions where substitution is not directly feasible.

♣

Example 7.16: Integration by alternate methods

Evaluate $\displaystyle\int \frac{x^2+2x+3}{\sqrt{x}}\,dx$ *with, and without, substitution.*

Solution. We already know how to integrate this particular example. Rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$ and simplify the fraction:

$$\frac{x^2+2x+3}{x^{1/2}} = x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}}.$$

We can now integrate using the Power Rule:

$$\begin{aligned}
\int \frac{x^2+2x+3}{x^{1/2}}\,dx &= \int\left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}}\right)\,dx \\
&= \frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+6x^{\frac{1}{2}}+C
\end{aligned}$$

This is a perfectly fine approach. We demonstrate how this can also be solved using substitution as its implementation is rather clever.
Let $u = \sqrt{x} = x^{\frac{1}{2}}$; therefore

$$du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}\ dx \quad \Rightarrow \quad 2du = \frac{1}{\sqrt{x}}\ dx.$$

This gives us $\int \frac{x^2+2x+3}{\sqrt{x}}\ dx = \int (x^2+2x+3)\cdot 2\ du$. What are we to do with the other x terms? Since $u = x^{\frac{1}{2}}$, $u^2 = x$, etc. We can then replace x^2 and x with appropriate powers of u. We thus have

$$\begin{aligned} \int \frac{x^2+2x+3}{\sqrt{x}}\ dx &= \int (x^2+2x+3)\cdot 2\ du \\ &= \int 2(u^4+2u^2+3)\ du \\ &= \frac{2}{5}u^5 + \frac{4}{3}u^3 + 6u + C \\ &= \frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + C, \end{aligned}$$

which is obviously the same answer we obtained before. In this situation, substitution is arguably more work than our other method. The fantastic thing is that it works. It demonstrates how flexible integration is. ♣

7.1.3. Substitution and Inverse Trigonometric Functions

When studying derivatives of inverse functions, we learned that

$$\frac{d}{dx}\big(\tan^{-1}x\big) = \frac{1}{1+x^2}.$$

Applying the Chain Rule to this is not difficult; for instance,

$$\frac{d}{dx}\big(\tan^{-1}5x\big) = \frac{5}{1+25x^2}.$$

We now explore how Substitution can be used to "undo" certain derivatives that are the result of the Chain Rule applied to Inverse Trigonometric functions. We begin with an example.

Example 7.17: Integrating by substitution: inverse trigonometric functions

Evaluate $\int \frac{1}{25+x^2}\ dx$.

Solution. The integrand looks similar to the derivative of the arctangent function. Note:

$$\begin{aligned} \frac{1}{25+x^2} &= \frac{1}{25(1+\frac{x^2}{25})} \\ &= \frac{1}{25(1+\left(\frac{x}{5}\right)^2)} \end{aligned}$$

$$= \frac{1}{25} \frac{1}{1 + \left(\frac{x}{5}\right)^2}.$$

Thus

$$\int \frac{1}{25 + x^2}\, dx = \frac{1}{25} \int \frac{1}{1 + \left(\frac{x}{5}\right)^2}\, dx.$$

This can be integrated using Substitution. Set $u = x/5$, hence $du = dx/5$ or $dx = 5du$. Thus

$$\begin{aligned} \int \frac{1}{25 + x^2}\, dx &= \frac{1}{25} \int \frac{1}{1 + \left(\frac{x}{5}\right)^2}\, dx \\ &= \frac{1}{5} \int \frac{1}{1 + u^2}\, du \\ &= \frac{1}{5} \tan^{-1} u + C \\ &= \frac{1}{5} \tan^{-1}\left(\frac{x}{5}\right) + C \end{aligned}$$

♣

Example 7.17 demonstrates a general technique that can be applied to other integrands that result in inverse trigonometric functions. The results are summarized here.

Theorem 7.3: Integrals Involving Inverse Trigonomentric Functions

Let $a > 0$.

1. $\displaystyle\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$
2. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a} \sec^{-1}\left(\frac{|x|}{a}\right) + C$

Let's practice using Theorem 7.3.

Example 7.18: Integrating by substitution: inverse trigonometric functions

Evaluate the given indefinite integrals.

$$\int \frac{1}{9 + x^2}\, dx, \quad \int \frac{1}{x\sqrt{x^2 - \frac{1}{100}}}\, dx \quad \text{and} \quad \int \frac{1}{\sqrt{5 - x^2}}\, dx.$$

Solution. Each can be answered using a straightforward application of Theorem 7.3.

$\displaystyle\int \frac{1}{9 + x^2}\, dx = \frac{1}{3} \tan^{-1} \frac{x}{3} + C$, as $a = 3$.

$$\int \frac{1}{x\sqrt{x^2 - \frac{1}{100}}}\,dx = 10\sec^{-1} 10x + C, \text{ as } a = \tfrac{1}{10}.$$

$$\int \frac{1}{\sqrt{5 - x^2}} = \sin^{-1} \frac{x}{\sqrt{5}} + C, \text{ as } a = \sqrt{5}.$$

♣

Most applications of Theorem 7.3 are not as straightforward. The next examples show some common integrals that can still be approached with this theorem.

Example 7.19: Integrating by substitution: completing the square

Evaluate $\displaystyle\int \frac{1}{x^2 - 4x + 13}\,dx$.

Solution. Initially, this integral seems to have nothing in common with the integrals in Theorem 7.3. As it lacks a square root, it almost certainly is not related to arcsine or arcsecant. It is, however, related to the arctangent function.

We see this by *completing the square* in the denominator. We give a brief reminder of the process here.

Start with a quadratic with a leading coefficient of 1. It will have the form of $x^2 + bx + c$. Take $1/2$ of b, square it, and add/subtract it back into the expression. I.e.,

$$\begin{aligned} x^2 + bx + c &= \underbrace{x^2 + bx + \frac{b^2}{4}}_{(x+b/2)^2} - \frac{b^2}{4} + c \\ &= \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4} \end{aligned}$$

In our example, we take half of -4 and square it, getting 4. We add/subtract it into the denominator as follows:

$$\begin{aligned} \frac{1}{x^2 - 4x + 13} &= \frac{1}{\underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4 + 13} \\ &= \frac{1}{(x-2)^2 + 9} \end{aligned}$$

We can now integrate this using the arctangent rule. Technically, we need to substitute first with $u = x - 2$, but we omit this step here. Thus we have

$$\int \frac{1}{x^2 - 4x + 13}\,dx = \int \frac{1}{(x-2)^2 + 9}\,dx = \frac{1}{3}\tan^{-1}\frac{x-2}{3} + C.$$

♣

Example 7.20: Integrals requiring multiple methods

Evaluate $\displaystyle\int \frac{4-x}{\sqrt{16-x^2}}\,dx.$

Solution. This integral requires two different methods to evaluate it. We get to those methods by splitting up the integral:

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = \int \frac{4}{\sqrt{16-x^2}}\,dx - \int \frac{x}{\sqrt{16-x^2}}\,dx.$$

The first integral is handled using a straightforward application of Theorem 7.3; the second integral is handled by substitution, with $u = 16-x^2$. We handle each separately.

$\displaystyle\int \frac{4}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + C.$

$\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$: Set $u = 16-x^2$, so $du = -2xdx$ and $xdx = -du/2$. We have

$$\begin{aligned}\int \frac{x}{\sqrt{16-x^2}}\,dx &= \int \frac{-du/2}{\sqrt{u}}\\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du\\ &= -\sqrt{u}+C\\ &= -\sqrt{16-x^2}+C.\end{aligned}$$

Combining these together, we have

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + \sqrt{16-x^2} + C.$$

♣

This section has focused on evaluating indefinite integrals as we are learning a new technique for finding antiderivatives. However, much of the time integration is used in the context of a definite integral. Definite integrals that require substitution can be calculated using the following workflow:

1. Start with a definite integral $\displaystyle\int_a^b f(x)\,dx$ that requires substitution.

2. Ignore the bounds; use substitution to evaluate $\displaystyle\int f(x)\,dx$ and find an antiderivative $F(x)$.

3. Evaluate $F(x)$ at the bounds; that is, evaluate $F(x)\Big|_a^b = F(b) - F(a)$.

This workflow works fine, but substitution offers an alternative that is powerful and amazing (and a little time saving). The next example shows how to use the Substitution Rule when dealing with definite integrals.

Example 7.21: Substitution Rule

Evaluate $\displaystyle\int_2^4 x\sin(x^2)\,dx$.

Solution. First we compute the antiderivative, then evaluate the integral. Let $u = x^2$, so $du = 2x\,dx$ or $x\,dx = du/2$. Then

$$\int x\sin(x^2)\,dx = \int \frac{1}{2}\sin u\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2) + C.$$

Now

$$\int_2^4 x\sin(x^2)\,dx = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

A somewhat neater alternative to this method is to change the original limits to match the variable u. Since $u = x^2$, when $x = 2$, $u = 4$, and when $x = 4$, $u = 16$. So we can do this:

$$\int_2^4 x\sin(x^2)\,dx = \int_4^{16} \frac{1}{2}\sin u\,du = -\frac{1}{2}(\cos u)\Big|_4^{16} = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

An incorrect, and dangerous, alternative is something like this:

$$\int_2^4 x\sin(x^2)\,dx = \int_2^4 \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos(u)\Big|_2^4 = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

This is incorrect because $\displaystyle\int_2^4 \frac{1}{2}\sin u\,du$ means that u takes on values between 2 and 4, which is wrong. It is dangerous, because it is very easy to get to the point $\displaystyle -\frac{1}{2}\cos(u)\Big|_2^4$ and forget to substitute x^2 back in for u, thus getting the incorrect answer $\displaystyle -\frac{1}{2}\cos(4) + \frac{1}{2}\cos(2)$. An acceptable alternative is something like:

$$\int_2^4 x\sin(x^2)\,dx = \int_{x=2}^{x=4} \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos(u)\Big|_{x=2}^{x=4} = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{\cos(16)}{2} + \frac{\cos(4)}{2}.$$

♣

7.1.4. Substitution and Definite Integrals

The following theorem states how the bounds of a definite integral can be changed as the substitution is performed.

Theorem 7.4: Substitution Rule for Definite Integrals

If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

In effect, Theorem 7.4 states that once you convert to integrating with respect to u, you do not need to switch back to evaluating with respect to x. A few examples will help one understand.

Example 7.22: Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^2 \cos(3x-1)\ dx$ *using Theorem 7.4.*

Solution. Observing the composition of functions, let $u = 3x-1$, hence $du = 3dx$. As $3dx$ does not appear in the integrand, divide the latter equation by 3 to get $du/3 = dx$.

By setting $u = 3x-1$, we are implicitly stating that $g(x) = 3x-1$. Theorem 7.4 states that the new lower bound is $g(0) = -1$; the new upper bound is $g(2) = 5$. We now evaluate the definite integral:

$$\begin{aligned}\int_1^2 \cos(3x-1)\ dx &= \int_{-1}^5 \cos u \frac{du}{3}\\ &= \frac{1}{3}\sin u\Big|_{-1}^5\\ &= \frac{1}{3}\big(\sin 5 - \sin(-1)\big) \approx -0.039.\end{aligned}$$

Notice how once we converted the integral to be in terms of u, we never went back to using x.

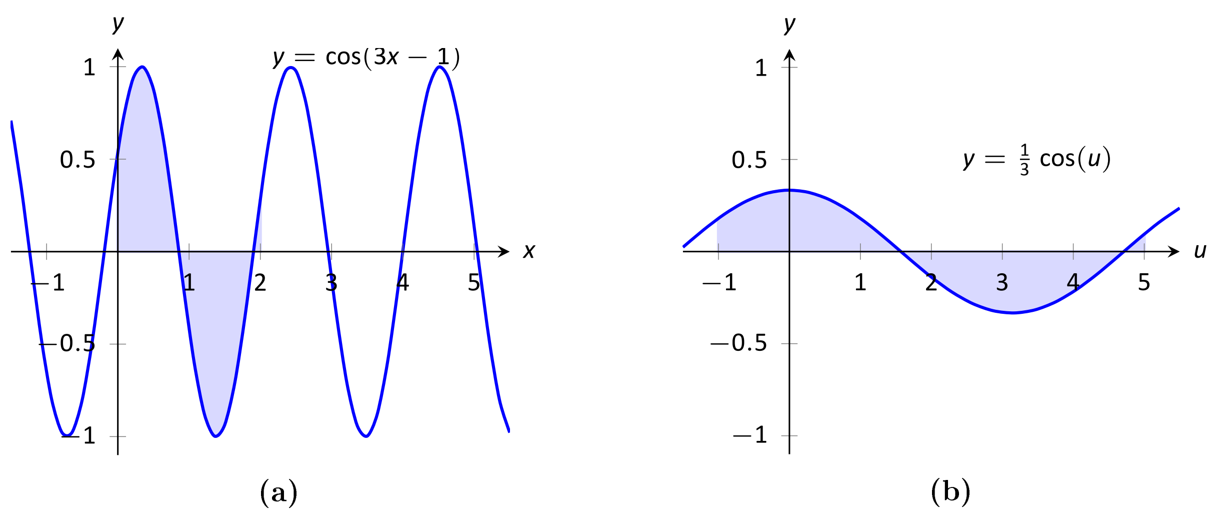


Figure 7.1: Graphing the areas defined by the definite integrals of Example 7.22.

The graphs in Figure 7.1 tell more of the story. In (a) the area defined by the original integrand is shaded, whereas in (b) the area defined by the new integrand is shaded. In this particular situation, the areas look very similar; the new region is "shorter" but "wider," giving the same area.

♣

Example 7.23: Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^{\pi/2} \sin x \cos x\ dx$ *using Theorem 7.4.*

Solution. We saw the corresponding indefinite integral in Example 7.8. In that example we set $u = \sin x$ but stated that we could have let $u = \cos x$. For variety, we do the latter here.

Let $u = g(x) = \cos x$, giving $du = -\sin x\ dx$ and hence $\sin x\ dx = -du$. The new upper bound is $g(\pi/2) = 0$; the new lower bound is $g(0) = 1$. Note how the lower bound is actually larger than the upper bound now. We have

$$\begin{aligned} \int_0^{\pi/2} \sin x \cos x\ dx &= \int_1^0 -u\ du \quad \text{(switch bounds \& change sign)} \\ &= \int_0^1 u\ du \\ &= \frac{1}{2}u^2\Big|_0^1 = 1/2. \end{aligned}$$

In Figure 7.11 we have again graphed the two regions defined by our definite integrals. Unlike the previous example, they bear no resemblance to each other. However, Theorem 7.4 guarantees that they have the same area.

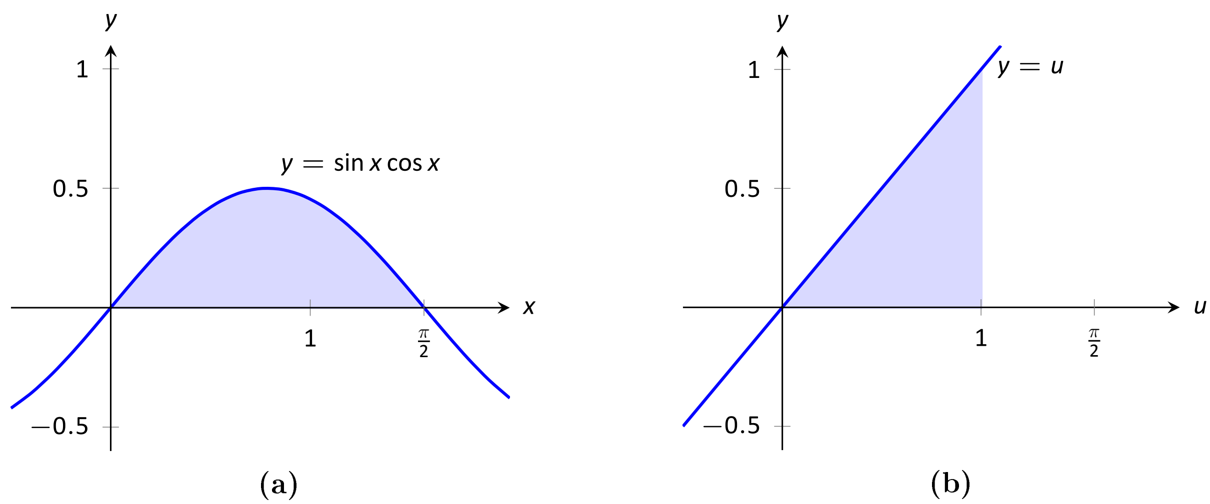


Figure 7.2: Graphing the areas defined by the definite integrals of Example 7.23.

♣

Example 7.24: Substitution Rule

Evaluate $\displaystyle\int_{1/4}^{1/2} \frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt.$

Solution. Let $u = \sin(\pi t)$ so $du = \pi\cos(\pi t)\,dt$ or $du/\pi = \cos(\pi t)\,dt$. We change the limits to $\sin(\pi/4) = \sqrt{2}/2$ and $\sin(\pi/2) = 1$. Then

$$\int_{1/4}^{1/2} \frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt = \int_{\sqrt{2}/2}^{1} \frac{1}{\pi}\frac{1}{u^2}\,du = \int_{\sqrt{2}/2}^{1} \frac{1}{\pi}u^{-2}\,du = \frac{1}{\pi}\frac{u^{-1}}{-1}\Big|_{\sqrt{2}/2}^{1} = -\frac{1}{\pi} + \frac{\sqrt{2}}{\pi}.$$

♣

The following theorem sometimes allows us to greatly simplify the calculations of integrals, by exploiting their symmetry.

Theorem 7.5: Integrals of Symmetric Functions

Suppose f is continuous on $[-a, a]$.

1. *If f is even (that is, $f(-x) = f(x)$), then* $\displaystyle\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$.

2. *If f is odd (that is, $f(-x) = -f(x)$), then* $\displaystyle\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} f(x)\, dx + \int_{-a}^{0} f(x)\, dx = 0$.

Proof. Since the definite integral is additive with respect to the interval of integration, we have

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx = \star$$

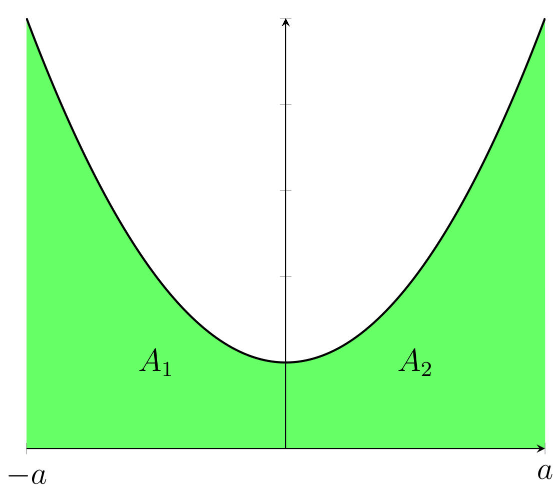
Substitute $u = -x$ in the first integral to get

$$\star = \int_{a}^{0} f(-u)\ -du + \int_{0}^{a} f(x)\, dx = \int_{0}^{a} f(-u)\, du + \int_{0}^{a} f(x)\, dx$$

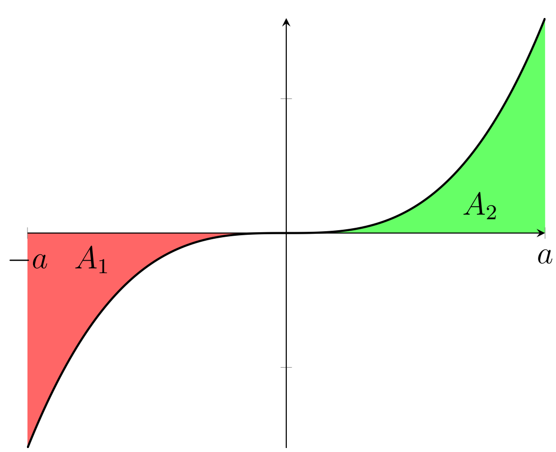
Since u is simply a "dummy variable" we can replace it with x to get

$$\star = \int_{0}^{a} f(-x)\, dx + \int_{0}^{a} f(x)\, dx = \int_{0}^{a} f(x) + f(-x)\, dx$$

If f is even, then $f(-x) = f(x)$, and f is odd then $f(x) + f(-x) = 0$ giving the equations in the theorem.
♣



(a) Even symmetry, $A_1 = A_2$



(b) Odd symmetry, $A_1 = -A_2$**.**

Example 7.25: Odd Symmetry and Integrals

Use properties of integrals to evaluate $\int_{-3}^{3} 1 - x^4 dx$

Solution. Since $1-x^4$ is an odd function, that is, if $f(x) = 1-x^4$, then $f(-x) = 1-(-x)^4 = 1-x^4 = f(x)$. So by symmetry (Theorem 7.5) we have

$$\int_{-3}^{3} 1 - x^4 dx = 2\int_{0}^{3} 1 - x^4 dx = 2\left(x - \frac{x^5}{5}\right|+0^3 = 2\left(3 - \frac{243}{5}\right) = -\frac{456}{5}$$

♣

Example 7.26: Even Symmetry and Integrals

Use properties of integrals to evaluate $\int_{-5}^{5} \left(\frac{\sin(x)}{x^2+1}\right) dx$

Solution. Since $sin(x)$ is an odd function, and x^2+1 is an even function, the quotient of the two is odd. That is, if $f(x) = \frac{\sin(x)}{x^2+1}$, then $f(-x) = \frac{\sin(-x)}{(-x)^2+1} = \frac{-\sin(x)}{x^2+1} = -f(x)$. So by symmetry (Theorem 7.5) we have

$$\int_{-5}^{5} \left(\frac{\sin(x)}{x^2+1}\right) dx = 0$$

♣

Integration by substitution is a powerful and useful integration technique. Section 7.4 introduces another technique, called Integration by Parts. As substitution "undoes" the Chain Rule, integration by parts "undoes" the Product Rule. Together, these two techniques provide a strong foundation on which most other integration techniques are based.

Exercises for Section 7.1

Find the following indefinite and definite integrals.

7.1.1 $\int (1-t)^9 \, dt$

7.1.2 $\int (x^2+1)^2 \, dx$

7.1.3 $\int x(x^2+1)^{100} \, dx$

7.1.4 $\int \frac{1}{\sqrt[3]{1-5t}} \, dt$

7.1.5 $\displaystyle\int \sin^3 x \cos x \, dx$

7.1.6 $\displaystyle\int x\sqrt{100 - x^2} \, dx$

7.1.7 $\displaystyle\int \frac{x^2}{\sqrt{1 - x^3}} \, dx$

7.1.8 $\displaystyle\int \cos(\pi t) \cos\big(\sin(\pi t)\big) \, dt$

7.1.9 $\displaystyle\int \frac{\sin x}{\cos^3 x} \, dx$

7.1.10 $\displaystyle\int \tan x \, dx$

7.1.11 $\displaystyle\int_0^{\pi} \sin^5(3x) \cos(3x) \, dx$

7.1.12 $\displaystyle\int \sec^2 x \tan x \, dx$

7.1.13 $\displaystyle\int_0^{\sqrt{\pi}/2} x \sec^2(x^2) \tan(x^2) \, dx$

7.1.14 $\displaystyle\int \frac{\sin(\tan x)}{\cos^2 x} \, dx$

7.1.15 $\displaystyle\int_3^4 \frac{1}{(3x - 7)^2} \, dx$

7.1.16 $\displaystyle\int_0^{\pi/6} (\cos^2 x - \sin^2 x) \, dx$

7.1.17 $\displaystyle\int \frac{6x}{(x^2 - 7)^{1/9}} \, dx$

7.1.18 $\displaystyle\int_{-1}^{1} (2x^3 - 1)(x^4 - 2x)^6 \, dx$

7.1.19 $\displaystyle\int_{-1}^{1} \sin^7 x \, dx$

7.1.20 $\displaystyle\int f(x) f'(x) \, dx$

7.2 Powers of Trigonometric Functions

Functions involving trigonometric functions are useful as they are good at describing periodic behavior. This section describes several techniques for finding antiderivatives of certain combinations of trigonometric functions.

Functions consisting of powers of the sine and cosine can be integrated by using substitution and trigonometric identities. These can sometimes be tedious, but the technique is straightforward. A similar technique is applicable to powers of secant and tangent (and also cosecant and cotangent, not discussed here).

Integrals of the form $\displaystyle\int \sin^m x \cos^n x \ dx$

Using the technique of Substitution, we see the integral $\int \sin x \cos x \ dx$ could easily be evaluated by letting $u = \sin x$ or by letting $u = \cos x$. This integral is easy since the power of both sine and cosine is 1.

We generalize this integral and consider integrals of the form $\int \sin^m x \cos^n x \ dx$, where m, n are nonnegative integers. Our strategy for evaluating these integrals is to use the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ to convert high powers of one trigonometric function into the other, leaving a single sine or cosine term in the integrand. We summarize the general technique in the following

Key Idea 7.2.0: Integrals Involving Powers of Sine and Cosine

Consider $\displaystyle\int \sin^m x \cos^n x \ dx$, where m, n are nonnegative integers.

1. If m is odd, then $m = 2k+1$ for some integer k. Rewrite

$$\sin^m x = \sin^{2k+1} x = \sin^{2k} x \sin x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x.$$

   Then

$$\int \sin^m x \cos^n x \ dx = \int (1-\cos^2 x)^k \sin x \cos^n x \ dx = -\int (1-u^2)^k u^n \ du,$$

   where $u = \cos x$ and $du = -\sin x \ dx$.

2. If n is odd, then using substitutions similar to that outlined above we have

$$\int \sin^m x \cos^n x \ dx = \int u^m (1-u^2)^k \ du,$$

   where $u = \sin x$ and $du = \cos x \ dx$.

3. If both m and n are even, use the power–reducing identities

$$\cos^2 x = \frac{1+\cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1-\cos(2x)}{2}$$

   to reduce the degree of the integrand. Expand the result and try again.

Now a few examples to practice the approach.

Example 7.27: Integrating powers of sine and cosine

Evaluate $\int \sin^6 x \cos^5 x \, dx$.

Solution. Since the power of cosine is odd, we use the substitution $u = \sin x$ and $du = \cos x \, dx$, that is, $\boxed{dx} = \boxed{\dfrac{du}{\cos x}}$. Then $\int \sin^6 x \cos^5 x \, dx$ is equal to:

$$
\begin{aligned}
&= \int u^6 \cos^5 x \boxed{\frac{du}{\cos x}} && \text{Using the substitution} \\
&= \int u^6 \left(\cos^2 x\right)^2 \, du && \text{Canceling a } \cos x \text{ and rewriting } \cos^4 x \\
&= \int u^6 (1 - \sin^2 x)^2 \, du && \text{Using trig identity } \cos^2 x = 1 - \sin^2 x \\
&= \int u^6 (1 - u^2)^2 \, du && \text{Writing integral in terms of } u\text{'s} \\
&= \int u^6 - 2u^8 + u^{10} \, du && \text{Expand and collect like terms} \\
&= \frac{u^7}{7} - \frac{2u^9}{9} + \frac{u^{11}}{11} + C && \text{Integrating} \\
&= \frac{\sin^7 x}{7} - \frac{2\sin^9 x}{9} + \frac{\sin^{11} x}{11} + C && \text{Replacing } u \text{ back in terms of } x
\end{aligned}
$$

♣

Example 7.28: Odd Power of Cosine

Evaluate $\int \cos^3 x \, dx$.

Solution. Since the power of cosine is odd, we use the substitution $u = \sin x$ and $du = \cos x \, dx$. This may seem strange at first since we don't have $\sin x$ in the question, but it does work!

$$
\begin{aligned}
\int \cos^3 x \boxed{dx} &= \int \cos^3 x \boxed{\frac{du}{\cos x}} && \text{Using the substitution} \\
&= \int \cos^2 x \, du && \text{Canceling a } \cos x \\
&= \int (1 - \sin^2 x) \, du && \text{Using trig identity } \cos^2 x = 1 - \sin^2 x \\
&= \int (1 - u^2) \, du && \text{Writing integral in terms of } u\text{'s} \\
&= u - \frac{u^3}{3} + C && \text{Integrating} \\
&= \sin x - \frac{\sin^3 x}{3} + C && \text{Replacing } u \text{ back in terms of } x
\end{aligned}
$$

♣

Example 7.29: Integrating powers of sine and cosine

Evaluate $\displaystyle\int \sin^5 x\,dx$.

Solution. Since the power of sine is odd, we factor out one $\sin(x)$, and exploit the Pythagorean identity: $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned}
\int \sin^5 x\,dx &= \int \sin x \sin^4 x\,dx \\
&= \int \sin x(\sin^2 x)^2\,dx \\
&= \int \sin x(1-\cos^2 x)^2\,dx.
\end{aligned}$$

Now use $u = \cos x$, $du = -\sin x\,dx$:

$$\begin{aligned}
\int \sin x(1-\cos^2 x)^2\,dx &= \int -(1-u^2)^2\,du \\
&= \int -(1-2u^2+u^4)\,du \\
&= \int 1+2u^2-u^4\,du \\
&= -u+\frac{2}{3}u^3-\frac{1}{5}u^5+C \\
&= -\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+C.
\end{aligned}$$

♣

Example 7.30: Integrating powers of sine and cosine

Evaluate $\displaystyle\int \sin^5 x\cos^8 x\;dx$.

Solution. The power of the sine term is odd, so we rewrite $\sin^5 x$ as

$$\sin^5 x = \sin^4 x \sin x = (\sin^2 x)^2 \sin x = (1-\cos^2 x)^2 \sin x.$$

Our integral is now $\displaystyle\int (1-\cos^2 x)^2 \cos^8 x \sin x\;dx$. Let $u=\cos x$, hence $du = -\sin x\;dx$. Making the substitution and expanding the integrand gives

$$\int (1-\cos^2)^2 \cos^8 x \sin x\;dx = -\int (1-u^2)^2 u^8\;du = -\int \left(1-2u^2+u^4\right)u^8\;du = -\int \left(u^8-2u^{10}+u^{12}\right)\,du.$$

This final integral is not difficult to evaluate, giving

$$\begin{aligned}
-\int \left(u^8-2u^{10}+u^{12}\right)\,du &= -\frac{1}{9}u^9+\frac{2}{11}u^{11}-\frac{1}{13}u^{13}+C \\
&= -\frac{1}{9}\cos^9 x+\frac{2}{11}\cos^{11} x-\frac{1}{13}\cos^{13} x+C.
\end{aligned}$$

♣

Example 7.31: Product of Even Powers of Sine and Cosine

Evaluate $\int \sin^2 x \cos^2 x \, dx$.

Solution. Use the formulas $\sin^2 x = (1 - \cos(2x))/2$ and $\cos^2 x = (1 + \cos(2x))/2$ to get:

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1 - \cos(2x)}{2} \cdot \frac{1 + \cos(2x)}{2} \, dx.$$

We then have

$$\begin{aligned}
\int \sin^2 x \cos^2 x \, dx &= \int \frac{1 - \cos(2x)}{2} \cdot \frac{1 + \cos(2x)}{2} \, dx \\
&= \frac{1}{4} \int 1 - \cos^2 2x \, dx \\
&= \frac{1}{4} \left(x - \int \cos^2 2x \, dx \right) \\
&= \frac{1}{4} \left(x - \frac{1}{2} \int 1 + \cos 4x \, dx \right) \\
&= \frac{1}{4} \left(x - \frac{1}{2} \left(x + \frac{\sin 4x}{4} \right) \right) \\
&= \frac{1}{4} \left(x - \frac{x}{2} - \frac{\sin 4x}{8} \right) + C
\end{aligned}$$

♣

Example 7.32: Even Power of Sine

Evaluate $\int \sin^6 x \, dx$.

Solution. Use $\sin^2 x = (1 - \cos(2x))/2$ to rewrite the function:

$$\begin{aligned}
\int \sin^6 x \, dx &= \int (\sin^2 x)^3 \, dx \\
&= \int \frac{(1 - \cos 2x)^3}{8} \, dx \\
&= \frac{1}{8} \int 1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x \, dx.
\end{aligned}$$

Now we have four integrals to evaluate:

$$\int 1 \, dx = x + C$$

and

$$\int -3\cos 2x \, dx = -\frac{3}{2} \sin 2x + C$$

The $\cos^3 2x$ integral is like the previous example:

$$\int -\cos^3 2x \, dx = \int -\cos 2x \cos^2 2x \, dx$$

$$\begin{aligned}
&= \int -\cos 2x(1-\sin^2 2x)\,dx \\
&= \int -\frac{1}{2}(1-u^2)\,du \\
&= -\frac{1}{2}\left(u-\frac{u^3}{3}\right)+C \\
&= -\frac{1}{2}\left(\sin 2x-\frac{\sin^3 2x}{3}\right)+C.
\end{aligned}$$

And finally we use another trigonometric identity, $\cos^2 x = (1+\cos(2x))/2$:

$$\int 3\cos^2 2x\,dx = 3\int \frac{1+\cos 4x}{2}\,dx = \frac{3}{2}\left(x+\frac{\sin 4x}{4}\right)+C.$$

So at long last, gathering and combining the arbitrary constants we get

$$\int \sin^6 x\,dx = \frac{x}{8}-\frac{3}{16}\sin 2x-\frac{1}{16}\left(\sin 2x-\frac{\sin^3 2x}{3}\right)+\frac{3}{16}\left(x+\frac{\sin 4x}{4}\right)+C.$$

♣

Example 7.33: Integrating powers of sine and cosine

Evaluate $\int \sin^5 x\cos^9 x\; dx$.

Solution. The powers of both the sine and cosine terms are odd, therefore we can apply our techniques to either power. We choose to work with the power of the cosine term since the previous example used the sine term's power.

We rewrite $\cos^9 x$ as

$$\begin{aligned}
\cos^9 x &= \cos^8 x\cos x \\
&= (\cos^2 x)^4\cos x \\
&= (1-\sin^2 x)^4\cos x \\
&= (1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x)\cos x.
\end{aligned}$$

We rewrite the integral as

$$\int \sin^5 x\cos^9 x\; dx = \int \sin^5 x\big(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\big)\cos x\; dx.$$

Now substitute and integrate, using $u=\sin x$ and $du=\cos x\; dx$.

$$\int \sin^5 x\big(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\big)\cos x\; dx =$$

$$\begin{aligned}
\int u^5(1-4u^2+6u^4-4u^6+u^8)\; du &= \int \big(u^5-4u^7+6u^9-4u^{11}+u^{13}\big)\; du \\
&= \frac{1}{6}u^6-\frac{1}{2}u^8+\frac{3}{5}u^{10}-\frac{1}{3}u^{12}+\frac{1}{14}u^{14}+C \\
&= \frac{1}{6}\sin^6 x-\frac{1}{2}\sin^8 x+\frac{3}{5}\sin^{10} x+\dots
\end{aligned}$$

$$-\frac{1}{3}\sin^{12}x + \frac{1}{14}\sin^{14}x + C.$$

♣

Technology Note: The work we are doing here can be a bit tedious, but the skills developed (problem solving, algebraic manipulation, etc.) are important. Nowadays problems of this sort are often solved using a computer algebra system. The powerful program *Mathematica*®, or *Wolfram Alpha* integrates $\int \sin^5 x \cos^9 x \, dx$ as

$$f(x) = -\frac{45\cos(2x)}{16384} - \frac{5\cos(4x)}{8192} + \frac{19\cos(6x)}{49152} + \frac{\cos(8x)}{4096} - \frac{\cos(10x)}{81920} - \frac{\cos(12x)}{24576} - \frac{\cos(14x)}{114688},$$

which clearly has a different form than our answer in Example 7.33, which is

$$g(x) = \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x - \frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x.$$

Figure 7.4 shows a graph of f and g; they are clearly not equal, but they differ *only by a constant*. That is $g(x) = f(x) + C$ for some constant C. So we have two different antiderivatives of the same function, meaning both answers are correct.

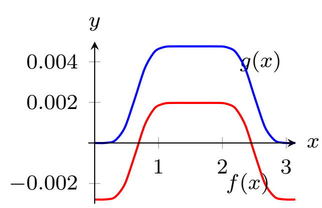


Figure 7.4: A plot of $f(x)$ and $g(x)$ from Example 7.33 and the Technology Note.

Example 7.34: Integrating powers of sine and cosine

Evaluate $\int \cos^4 x \sin^2 x \, dx$.

Solution. The powers of sine and cosine are both even, so we employ the power–reducing formulas and algebra as follows.

$$\begin{aligned}\int \cos^4 x \sin^2 x \, dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2 \left(\frac{1-\cos(2x)}{2}\right) \, dx \\ &= \int \frac{1+2\cos(2x)+\cos^2(2x)}{4} \cdot \frac{1-\cos(2x)}{2} \, dx \\ &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big) \, dx\end{aligned}$$

The $\cos(2x)$ term is easy to integrate, especially with Key Idea **??**. The $\cos^2(2x)$ term is another trigonometric integral with an even power, requiring the power–reducing formula again. The

$\cos^3(2x)$ term is a cosine function with an odd power, requiring a substitution as done before. We integrate each in turn below.

$$\int \cos(2x)\ dx = \frac{1}{2}\sin(2x) + C.$$

$$\int \cos^2(2x)\ dx = \int \frac{1+\cos(4x)}{2}\ dx = \frac{1}{2}\big(x + \frac{1}{4}\sin(4x)\big) + C.$$

Finally, we rewrite $\cos^3(2x)$ as

$$\cos^3(2x) = \cos^2(2x)\cos(2x) = \big(1-\sin^2(2x)\big)\cos(2x).$$

Letting $u = \sin(2x)$, we have $du = 2\cos(2x)\ dx$, hence

$$\begin{aligned}
\int \cos^3(2x)\ dx &= \int \big(1-\sin^2(2x)\big)\cos(2x)\ dx \\
&= \int \frac{1}{2}(1-u^2)\ du \\
&= \frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C \\
&= \frac{1}{2}\left(\sin(2x) - \frac{1}{3}\sin^3(2x)\right) + C
\end{aligned}$$

Putting all the pieces together, we have

$$\begin{aligned}
\int \cos^4 x \sin^2 x\ dx &= \int \frac{1}{8}\big(1+\cos(2x) - \cos^2(2x) - \cos^3(2x)\big)\ dx \\
&= \frac{1}{8}\Big[x + \frac{1}{2}\sin(2x) - \frac{1}{2}\big(x + \frac{1}{4}\sin(4x)\big) - \frac{1}{2}\Big(\sin(2x) - \frac{1}{3}\sin^3(2x)\Big)\Big] + C \\
&= \frac{1}{8}\Big[\frac{1}{2}x - \frac{1}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\Big] + C
\end{aligned}$$

♣

Integrals of the form $\int \sin(mx)\sin(nx)\ dx$, $\int \cos(mx)\cos(nx)\ dx$, and $\int \sin(mx)\cos(nx)\ dx$.

Functions that contain products of sines and cosines of differing periods are important in many applications including the analysis of sound waves. Integrals of the form

$$\int \sin(mx)\sin(nx)\ dx, \quad \int \cos(mx)\cos(nx)\ dx \quad \text{and} \quad \int \sin(mx)\cos(nx)\ dx$$

are best approached by first applying the Product to Sum Formulas found in the back cover of this text, namely

$$\begin{aligned}
\sin(mx)\sin(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) - \cos\big((m+n)x\big)\Big] \\
\cos(mx)\cos(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) + \cos\big((m+n)x\big)\Big] \\
\sin(mx)\cos(nx) &= \frac{1}{2}\Big[\sin\big((m-n)x\big) + \sin\big((m+n)x\big)\Big]
\end{aligned}$$

Example 7.35: Integrating products of $\sin(mx)$ and $\cos(nx)$

Evaluate $\displaystyle\int \sin(5x)\cos(2x)\;dx.$

Solution. The application of the formula and subsequent integration are straightforward:

$$\begin{aligned}\int \sin(5x)\cos(2x)\;dx &= \int \frac{1}{2}\Big[\sin(3x)+\sin(7x)\Big]\;dx\\ &= -\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C\end{aligned}$$

♣

Integrals of the form $\displaystyle\int \tan^m x \sec^n x\;dx$.

When evaluating integrals of the form $\int \sin^m x \cos^n x\;dx$, the Pythagorean identity allowed us to convert even powers of sine into even powers of cosine, and vise–versa. If, for instance, the power of sine was odd, we pulled out one $\sin x$ and converted the remaining even power of $\sin x$ into a function using powers of $\cos x$, leading to an easy substitution.

The same basic strategy applies to integrals of the form $\int \tan^m x \sec^n x\,dx$, albeit a bit more nuanced. The following three facts will prove useful:

- $\frac{d}{dx}(\tan x) = \sec^2 x,$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$, and
- $1 + \tan^2 x = \sec^2 x$ (the Pythagorean Theorem).

If the integrand can be manipulated to separate a $\sec^2 x$ term with the remaining secant power even, or if a $\sec x \tan x$ term can be separated with the remaining $\tan x$ power even, the Pythagorean Theorem can be employed, leading to a simple substitution. This strategy is outlined in the following.

Key Idea 7.2.0: Integrals Involving Powers of Tangent and Secant

Consider $\displaystyle\int \tan^m x \sec^n x \; dx$, where m, n are nonnegative integers.

1. If n is even, then $n = 2k$ for some integer k. Rewrite $\sec^n x$ as

$$\sec^n x = \sec^{2k} x = \sec^{2k-2} x \sec^2 x = (1 + \tan^2 x)^{k-1} \sec^2 x.$$

Then

$$\int \tan^m x \sec^n x \; dx = \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \; dx = \int u^m (1 + u^2)^{k-1} \; du,$$

where $u = \tan x$ and $du = \sec^2 x \; dx$.

2. If m is odd, then $m = 2k + 1$ for some integer k. Rewrite $\tan^m x \sec^n x$ as

$$\tan^m x \sec^n x = \tan^{2k+1} x \sec^n x = \tan^{2k} x \sec^{n-1} x \sec x \tan x = (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x.$$

Then

$$\int \tan^m x \sec^n x \; dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \; dx = \int (u^2 - 1)^k u^{n-1} \; du,$$

where $u = \sec x$ and $du = \sec x \tan x \; dx$.

3. If n is odd and m is even, then $m = 2k$ for some integer k. Convert $\tan^m x$ to $(\sec^2 x - 1)^k$. Expand the new integrand and use Integration By Parts, with $dv = \sec^2 x \; dx$.

4. If m is even and $n = 0$, rewrite $\tan^m x$ as

$$\tan^m x = \tan^{m-2} x \tan^2 x = \tan^{m-2} x (\sec^2 x - 1) = \tan^{m-2} \sec^2 x - \tan^{m-2} x.$$

So

$$\int \tan^m x \; dx = \underbrace{\int \tan^{m-2} \sec^2 x \; dx}_{\text{apply rule \#1}} - \underbrace{\int \tan^{m-2} x \; dx}_{\text{apply rule \#4 again}}.$$

The techniques described in items 1 and 2 are relatively straightforward, but the techniques in items 3 and 4 can be rather tedious. A few examples will help with these methods.

We can integrate $\tan x$ quite easily using substitution.

Example 7.36: Integrating Tangent

Evaluate $\displaystyle\int \tan x \, dx$.

Solution. Note that $\tan x = \dfrac{\sin x}{\cos x}$ and let $u = \cos x$, so that $du = -\sin x\,dx$.

$$\begin{array}{rcll}
\int \tan x\,dx & = & \int \dfrac{\sin x}{\cos x} \boxed{dx} & \text{Rewriting } \tan x \\
& = & \int \dfrac{\sin x}{u} \boxed{\dfrac{du}{-\sin x}} & \text{Using the substitution} \\
& = & -\int \dfrac{1}{u}\,du & \text{Cancelling and pulling the } -1 \text{ out} \\
& = & -\ln|u| + C & \text{Using formula } \int \dfrac{1}{u}\,dx = \ln|u| + C \\
& = & -\ln|\cos x| + C & \text{Replacing } u \text{ back in terms of } x \\
& = & \ln|\sec x| + C & \text{Using log properties and } \sec x = 1/\cos x
\end{array}$$

♣

Higher poers of $\tan(x)$ require the methods outlined above.

Example 7.37: Integrating Tangent Squared

Evaluate $\int \tan^2 x\,dx$.

Solution. The power on tangent is even, so we are in the fourth case outlined in the method. Ao we exploit the fact that $\tan^2 x = \sec^2 x - 1$.

$$\begin{array}{rcll}
\int \tan^2 x\,dx & = & \int \sec^2 x - 1\,dx & \text{Rewriting } \tan x \\
& = & \tan x - x + C & \text{Since } \int \sec^2 x\,dx = \tan x + C
\end{array}$$

♣

In problems with tangent and secant, two integrals come up frequently:

$$\int \sec^3 x\,dx \qquad \text{and} \qquad \int \sec x\,dx.$$

Both have relatively nice expressions but they are a bit tricky to discover.

First we do $\int \sec x\,dx$, which we will need to compute $\int \sec^3 x\,dx$.

Example 7.38: Integral of Secant

Evaluate $\int \sec x \, dx$.

Solution.

$$\begin{aligned} \int \sec x \, dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx. \end{aligned}$$

Now let $u = \sec x + \tan x$, $du = \sec x \tan x + \sec^2 x \, dx$, exactly the numerator of the function we are integrating. Thus

$$\begin{aligned} \int \sec x \, dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{1}{u} \, du = \ln |u| + C \\ &= \ln |\sec x + \tan x| + C. \end{aligned}$$

♣

Now we compute the integral $\int \sec^3 x \, dx$.

Example 7.39: Integral of Secant Cubed

Evaluate $\int \sec^3 x \, dx$.

Solution.

$$\begin{aligned} \sec^3 x &= \frac{\sec^3 x}{2} + \frac{\sec^3 x}{2} = \frac{\sec^3 x}{2} + \frac{(\tan^2 x + 1) \sec x}{2} \\ &= \frac{\sec^3 x}{2} + \frac{\sec x \tan^2 x}{2} + \frac{\sec x}{2} \\ &= \frac{\sec^3 x + \sec x \tan^2 x}{2} + \frac{\sec x}{2}. \end{aligned}$$

We already know how to integrate $\sec x$, so we just need the first quotient. This is "simply" a matter of recognizing the product rule in action:

$$\int \sec^3 x + \sec x \tan^2 x \, dx = \sec x \tan x.$$

So putting these together we get

$$\int \sec^3 x \, dx = \frac{\sec x \tan x}{2} + \frac{\ln |\sec x + \tan x|}{2} + C,$$

Note: Once we learn a technique called Integration by Parts, we will see another way to solve this integral. ♣

Example 7.40: Even Power of Secant

Evaluate $\displaystyle\int \tan^2 x \sec^6 x\, dx$.

Solution. Since the power of secant is even, we use rule #1 and pull out a $\sec^2 x$ in the integrand. We convert the remaining powers of secant into powers of tangent.

$$\begin{aligned}\int \tan^2 x \sec^6 x\, dx &= \int \tan^2 x \sec^4 x \sec^2 x\, dx \\ &= \int \tan^2 x \left(1+\tan^2 x\right)^2 \sec^2 x\, dx\end{aligned}$$

Now substitute, with $u = \tan x$, with $du = \sec^2 x\, dx$.

$$= \int u^2 \left(1+u^2\right)^2\, du$$

We leave the integration and subsequent substitution to the reader. The final answer is

$$= \frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C.$$

♣

Example 7.41: Odd Power of Tangent

Evaluate $\displaystyle\int \sec^5 x \tan x\, dx$.

Solution. Since the power of tangent is odd, we use rule # 2, and factor out $\sec x \tan x$ and substitute $u = \sec(x)$. Then we have:

$$\begin{aligned}\int \sec^5 x \tan x\, dx &= \int \sec^4 x \sec x \tan x\, dx && \text{Substituting } dx \text{ first} \\ &= \int u^4\, du && \text{Using the substitution} \\ &= \frac{u^5}{5} + C && \text{Integrating} \\ &= \frac{\sec^5 x}{5} + C && \text{Rewriting in terms of } x\end{aligned}$$

♣

Example 7.42: Odd Power of Secant and Even Power of Tangent

Evaluate $\displaystyle\int \sec x \tan^2 x \, dx$.

Solution. The guidelines don't help us in this scenario. However, since $\tan^2 x = \sec^2 x - 1$, we have

$$\begin{aligned}
\int \sec x \tan^2 x \, dx &= \int \sec x(\sec^2 x - 1)\, dx \\
&= \int (\sec^3 x - \sec x)\, dx \\
&= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C \\
&= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C \\
&= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C
\end{aligned}$$

♣

Example 7.43: Integrating powers of tangent and secant

Evaluate $\displaystyle\int \tan^6 x \; dx$.

Solution. We employ rule #4.

$$\begin{aligned}
\int \tan^6 x \; dx &= \int \tan^4 x \tan^2 x \; dx \\
&= \int \tan^4 x \left(\sec^2 x - 1\right) \; dx \\
&= \int \tan^4 x \sec^2 x \; dx - \int \tan^4 x \; dx
\end{aligned}$$

Integrate the first integral with substitution, $u = \tan x$; integrate the second by employing rule #4 again.

$$\begin{aligned}
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \tan^2 x \; dx \\
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \left(\sec^2 x - 1\right) \; dx \\
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \sec^2 x \; dx + \int \tan^2 x \; dx
\end{aligned}$$

Again, use substitution for the first integral and rule #4 for the second.

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int \left(\sec^2 x - 1\right) \; dx$$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.$$

♣

Some of these examples were admittedly long, with repeated applications of the same rule. Try to not be overwhelmed by the length of the problem, but rather admire how robust this solution method is. A trigonometric function of a high power can be systematically reduced to trigonometric functions of lower powers until all antiderivatives can be computed.

The next section introduces an integration technique known as Trigonometric Substitution, a clever combination of Substitution and the Pythagorean Theorem.

Exercises for 7.2

Find the antiderivatives.

7.2.1 $\displaystyle\int \sin^2 x\,dx$

7.2.2 $\displaystyle\int \sin^3 x\,dx$

7.2.3 $\displaystyle\int \sin^4 x\,dx$

7.2.4 $\displaystyle\int \cos^2 x \sin^3 x\,dx$

7.2.5 $\displaystyle\int \cos^3 x\,dx$

7.2.6 $\displaystyle\int \cos^3 x \sin^2 x\,dx$

7.2.7 $\displaystyle\int \sin x (\cos x)^{3/2}\,dx$

7.2.8 $\displaystyle\int \sec^2 x \csc^2 x\,dx$

7.2.9 $\displaystyle\int \tan^3 x \sec x\,dx$

7.2.10 $\displaystyle\int \left(\frac{1}{\csc x} + \frac{1}{\sec x}\right) dx$

7.2.11 $\displaystyle\int \frac{\cos^2 x + \cos x + 1}{\cos^3 x}\,dx$

7.2.12 $\displaystyle\int x\sec^2(x^2)\tan^4(x^2)\,dx$

7.3 Trigonometric Substitutions

So far we have seen that it sometimes helps to replace a subexpression of a function by a single variable. Occasionally it can help to replace the original variable by something more complicated. This seems like a "reverse" substitution, but it is really no different in principle than ordinary substitution.

Example 7.44: Sine Subsitution

Evaluate $\displaystyle\int \sqrt{1-x^2}\,dx$.

Solution. Let $x = \sin u$ so $dx = \cos u\,du$. Then

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2 u}\cos u\,du = \int \sqrt{\cos^2 u}\cos u\,du.$$

We would like to replace $\sqrt{\cos^2 u}$ by $\cos u$, but this is valid only if $\cos u$ is positive, since $\sqrt{\cos^2 u}$ is positive. Consider again the substitution $x = \sin u$. We could just as well think of this as $u = \arcsin x$. If we do, then by the definition of the arcsine, $-\pi/2 \le u \le \pi/2$, so $\cos u \ge 0$. Then we continue:

$$\begin{aligned}
\int \sqrt{\cos^2 u}\cos u\,du &= \int \cos^2 u\,du \\
&= \int \frac{1+\cos 2u}{2}\,du = \frac{u}{2} + \frac{\sin 2u}{4} + C \\
&= \frac{\arcsin x}{2} + \frac{\sin(2\arcsin x)}{4} + C.
\end{aligned}$$

This is a perfectly good answer, though the term $\sin(2\arcsin x)$ is a bit unpleasant. It is possible to simplify this. Using the identity $\sin 2x = 2\sin x\cos x$, we can write $\sin 2u = 2\sin u\cos u = 2\sin(\arcsin x)\sqrt{1-\sin^2 u} = 2x\sqrt{1-\sin^2(\arcsin x)} = 2x\sqrt{1-x^2}$. Then the full antiderivative is

$$\frac{\arcsin x}{2} + \frac{2x\sqrt{1-x^2}}{4} = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C.$$

♣

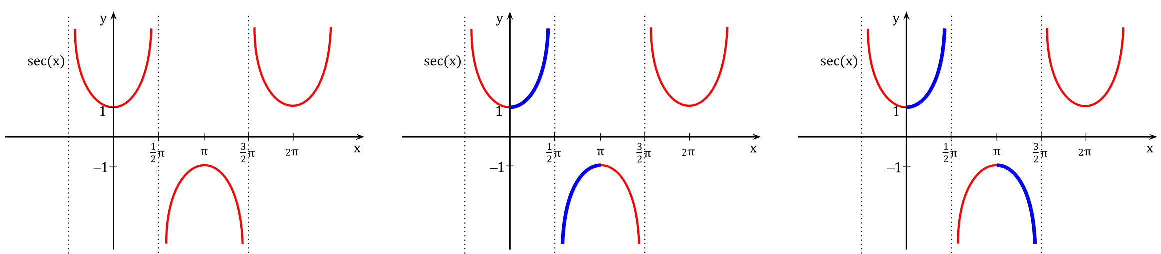
This type of substitution is usually indicated when the function you wish to integrate contains a polynomial expression that might allow you to use the fundamental identity $\sin^2 x + \cos^2 x = 1$ in one of three forms:

$$\cos^2 x = 1 - \sin^2 x \qquad \sec^2 x = 1 + \tan^2 x \qquad \tan^2 x = \sec^2 x - 1.$$

If your function contains $1 - x^2$, as in the example above, try $x = \sin u$; if it contains $1 + x^2$ try $x = \tan u$; and if it contains $x^2 - 1$, try $x = \sec u$. Sometimes you will need to try something a bit different to handle constants other than one which we will describe below. First we discuss inverse substitutions.

In a **traditional** substitution we let $u = u(x)$, i.e., our new variable is defined in terms of x. In an **inverse** substitution we let $x = g(u)$, i.e., we assume x can be written in terms of u. We cannot do this arbitrarily since we do **NOT** get to "choose" x. For example, an inverse substitution of $x = 1$ will give an obviously wrong answer. However, when $x = g(u)$ is an invertible function, then we are really doing a u-substitution with $u = g^{-1}(x)$. Now the substitution rule applies.

Sometimes with inverse substitutions involving trig functions we use θ instead of u. Thus, we would take $x = \sin\theta$ instead of $x = \sin u$. However, as we discussed above, we would like our inverse substitution $x = g(u)$ to be a one-to-one function, and $x = \sin u$ is not one-to-one. We can overcome this issue by using the restricted trigonometric functions. The three common trigonometric substitutions are the restricted sine, restricted tangent and restricted secant. Thus, for sine we use the domain $[-\pi/2,\ \pi/2]$ and for tangent we use $(-\pi/2,\ \pi/2)$. Depending on the convention chosen, the restricted secant function is usually defined in one of two ways.



One convention is to restrict secant to the region $[0,\ \pi/2) \cup (\pi/2, \pi]$ as shown in the middle graph. The other convention is to use $[0,\ \pi/2) \cup [\pi,\ 3\pi/2)$ as shown in the right graph. Both choices give a one-to-one restricted secant function and no universal convention has been adopted. To make the analysis in this section less cumbersome, we will use the domain $[0,\ \pi/2) \cup [\pi,\ 3\pi/2)$ for the restricted secant function. Then $\sec^{-1} x$ is defined to be the inverse of this restricted secant function.

Typically trigonometric substitions are used for problems that involve radical expressions. The table below outlines when each substitution is typically used along with their intervals of validity.

Key Idea 7.3.0: Trigonometric Substitution

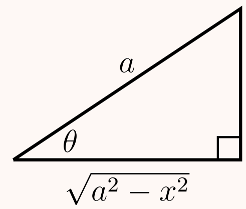
(a) For integrands containing $\sqrt{a^2 - x^2}$:

Let $x = a \sin\theta, \qquad dx = a\cos\theta\ d\theta$

Thus $\theta = \sin^{-1}(x/a)$, for $-\pi/2 \le \theta \le \pi/2$.

On this interval, $\cos\theta \ge 0$, so

$\sqrt{a^2 - x^2} = a\cos\theta$



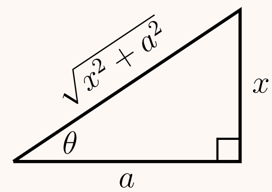
(b) For integrands containing $\sqrt{x^2 + a^2}$:

Let $x = a\tan\theta, \qquad dx = a\sec^2\theta\ d\theta$

Thus $\theta = \tan^{-1}(x/a)$, for $-\pi/2 < \theta < \pi/2$.

On this interval, $\sec\theta > 0$, so

$\sqrt{x^2 + a^2} = a\sec\theta$



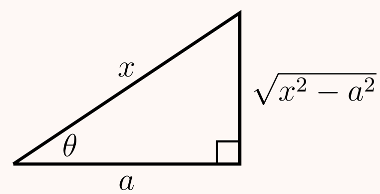
(c) For integrands containing $\sqrt{x^2 - a^2}$:

Let $x = a\sec\theta, \qquad dx = a\sec\theta\tan\theta\ d\theta$

Thus $\theta = \sec^{-1}(x/a)$. If $x/a \ge 1$, then $0 \le \theta < \pi/2$; if $x/a \le -1$, then $\pi \le \theta < 3\pi/2$.

We restrict our work to where $x \ge a$, so $x/a \ge 1$, and $0 \le \theta < \pi/2$. On this interval, $\tan\theta \ge 0$, so

$\sqrt{x^2 - a^2} = a\tan\theta$



All three substitutions are one-to-one on the listed intervals. When dealing with radicals we often end up with absolute values since

$$\sqrt{z^2} = |z|.$$

For each of the three trigonometric substitions above we will verify that we can ignore the absolute value in each case when encountering a radical.

For $x = a\sin\theta$, the expression $\sqrt{a^2 - x^2}$ becomes

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = a\sqrt{\cos^2\theta} = a|\cos\theta| = a\cos\theta$$

This is because $\cos\theta \ge 0$ when $\theta \in [-\pi/2,\ \pi/2]$. For $x = a\tan\theta$, the expression $\sqrt{a^2 + x^2}$ becomes

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1 + \tan^2\theta)} = a\sqrt{\sec^2\theta} = a|\sec\theta| = a\sec\theta$$

This is because $\sec\theta > 0$ when $\theta \in (-\pi/2,\ \pi/2)$.

Finally, for $x = a\sec\theta$, the expression $\sqrt{x^2 - a^2}$ becomes

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = a\sqrt{\tan^2\theta} = a|\tan\theta| = a\tan\theta$$

This is because $\tan\theta \ge 0$ when $\theta \in [0,\ \pi/2) \cup [\pi,\ 3\pi/2)$.

Thus, when using an appropriate trigonometric substitution we can usually ignore the absolute value. After integrating, we typically get an answer in terms of θ (or u) and need to convert back to x's. To do so, we use the two guidelines below:

- For trig functions containing θ, use a triangle to convert to x's.
- For θ by itself, use the inverse trig function.

All pieces needed for such a trigonometric substitution can be summarized as follows:

Expression	Substitution	Differential	Identity	Inverse of Substitution
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$dx = a\cos\theta d\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$\theta = \sin^{-1}\left(\frac{x}{a}\right)$
$\sqrt{a^2+x^2}$ or a^2+x^2	$x = a\tan\theta$	$dx = a\sec^2\theta d\theta$	$\sqrt{a^2+x^2} = a\sec\theta$	$\theta = \tan^{-1}\left(\frac{x}{a}\right)$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta d\theta$	$\sqrt{x^2-a^2} = a\tan\theta$	$\theta = \sec^{-1}\left(\frac{x}{a}\right)$

To emphasize the technique, we redo the computation for $\displaystyle\int \sqrt{1-x^2}\,dx$.

Example 7.45: Sine Subsitution

Evaluate $\displaystyle\int \sqrt{1-x^2}\,dx$.

Solution. Since $\sqrt{1-x^2}$ appears in the integrand we try the trigonometric substitution $x = \sin\theta$. (Here we are using the restricted sine function with $\theta \in [-\pi/2,\ \pi/2]$ but typically omit this detail when writing out the solution.) Then $\boxed{dx} = \boxed{\cos\theta\, d\theta}$.

$$
\begin{aligned}
\int \sqrt{1-x^2}\,\boxed{dx} &= \int \sqrt{1-\sin^2\theta}\,\boxed{\cos\theta\,d\theta} && \text{Using our (inverse) substitution}\\
&= \int \sqrt{\cos^2\theta}\cos\theta\,d\theta && \text{Since } \sin^2\theta+\cos^2\theta = 1\\
&= \int |\cos\theta|\cdot\cos\theta\,d\theta && \text{Since } \sqrt{\cos^2\theta} = |\cos\theta|\\
&= \int \cos^2\theta\,d\theta && \text{Since for } \theta\in[-\tfrac{\pi}{2},\tfrac{\pi}{2}] \text{ we have } \cos\theta \geq 0.
\end{aligned}
$$

Often we omit the step containing the absolute value by our discussion above. Now, to integrate a power of cosine we use the guidelines for products of sine and cosine and make use of the identity

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta)).$$

Our integral then becomes

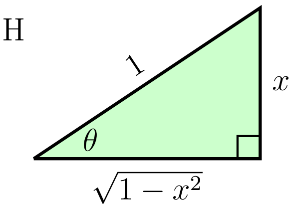
$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\int (1+\cos(2\theta))\,d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C$$

To write the answer back in terms of x we first use the identity $\sin(2\theta) = 2\sin\theta\cos\theta$. So

$$\int \sqrt{1-x^2}\,dx = \frac{\theta}{2} + \frac{\sin(\theta)\cos(\theta)}{2} + C$$

Since $x = \sin\theta$, we can build the corresponding triangle and use SOH CAH TOA we get $\cos\theta = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. This gives

$$\int \sqrt{1-x^2}\,dx = \frac{\sin^{-1} x}{2} + \frac{x\sqrt{1-x^2}}{2} + C$$



♣

Example 7.46: Secant Subsitution

Evaluate $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}\,dx$.

Solution. We do not have $\sqrt{x^2-a^2}$ because of the 25, but if we factor 25 out we get:

$$\int \frac{\sqrt{25(x^2-(4/25))}}{x}\,dx = \int 5\frac{\sqrt{x^2-(4/25)}}{x}\,dx.$$

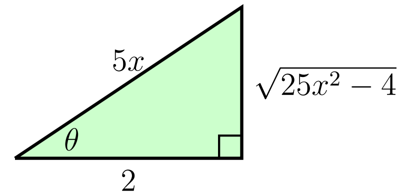
Now, $a = 2/5$, so let $x = \frac{2}{5}\sec\theta$. Alternatively, we can think of the integral as being:

$$\int \frac{\sqrt{(5x)^2-4}}{x}\,dx$$

Then we could let $u = 5x$ followed by $u = 2\sec\theta$, etc. Or equivalently, we can avoid a u-substitution by letting $5x = 2\sec\theta$. In either case we are using the trigonometric substitution $x = \frac{2}{5}\sec\theta$, but do use the method that makes the most sense to you! As $x = \frac{2}{5}\sec\theta$ we have $\boxed{dx} = \boxed{\frac{2}{5}\sec\theta\tan\theta\,d\theta}$.

$$\begin{aligned}
\int \frac{\sqrt{25x^2-4}}{x}\boxed{dx} &= \int \frac{\sqrt{25\frac{4\sec^2\theta}{25}-4}}{\frac{2}{5}\sec\theta}\boxed{\frac{2}{5}\sec\theta\tan\theta\,d\theta} && \text{Using the substitution}\\
&= \int \sqrt{4(\sec^2\theta-1)}\cdot\tan\theta\,d\theta && \text{Cancelling}\\
&= 2\int \sqrt{\tan^2\theta}\cdot\tan\theta\,d\theta && \text{Using } \tan^2\theta+1=\sec^2\theta\\
&= 2\int \tan^2\theta\,d\theta && \text{Simplifying}\\
&= 2\int (\sec^2\theta-1)\,d\theta && \text{Using } \tan^2\theta+1=\sec^2\theta\\
&= 2(\tan\theta-\theta)+C && \text{Since } \int \sec^2\theta\,d\theta = \tan\theta + C
\end{aligned}$$

To solve for $\tan\theta$, we build a triangle where $\sec\theta = \frac{5x}{2}$, or equivalently, $\cos\theta = \frac{2}{5x}$. Using SOH CAH TOA, the triangle is as shown at the right:



From the triangle, we get $\tan\theta = \dfrac{\sqrt{25x^2-4}}{2}$. As $\theta = \sec^{-1}(5x/2)$, we get

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx = 2\left(\frac{\sqrt{25x^2-4}}{2} - \sec^{-1}\left(\frac{5x}{2}\right)\right) + C$$

♣

In the context of the previous example, some resources give alternate guidelines when choosing a trigonometric substitution.

$$\sqrt{a^2-b^2x^2} \quad \to \quad x = \frac{a}{b}\sin\theta$$

$$\sqrt{b^2x^2+a^2} \text{ or } (b^2x^2+a^2) \quad \to \quad x = \frac{a}{b}\tan\theta$$

$$\sqrt{b^2x^2-a^2} \quad \to \quad x = \frac{a}{b}\sec\theta$$

We next look at a tangent substitution.

Example 7.47: Tangent Substitution

Evaluate $\displaystyle\int \frac{1}{\sqrt{5+x^2}}\,dx.$

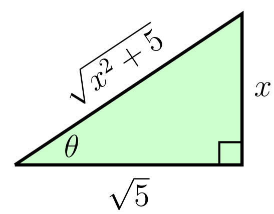
Solution. Using part (b), we recognize $a=\sqrt{5}$ and set $x=\sqrt{5}\tan\theta$. This makes $dx = \sqrt{5}\sec^2\theta\,d\theta$. We will use the fact that $\sqrt{5+x^2} = \sqrt{5+5\tan^2\theta} = \sqrt{5\sec^2\theta} = \sqrt{5}\sec\theta$. Substituting, we have:

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \int \frac{1}{\sqrt{5+5\tan^2\theta}}\sqrt{5}\sec^2\theta\;d\theta \\ &= \int \frac{\sqrt{5}\sec^2\theta}{\sqrt{5}\sec\theta}\;d\theta \\ &= \int \sec\theta\;d\theta \\ &= \ln\big|\sec\theta+\tan\theta\big| + C.\end{aligned}$$

While the integration steps are over, we are not yet done. The original problem was stated in terms of x, whereas our answer is given in terms of θ. We must convert back to x.

With $x=\sqrt{5}\tan\theta$, we have

$$\tan\theta = \frac{x}{\sqrt{5}} \quad \text{and} \quad \sec\theta = \frac{\sqrt{x^2+5}}{\sqrt{5}}.$$



This gives

$$\int \frac{1}{\sqrt{5+x^2}}\,dx = \ln\left|\sec\theta+\tan\theta\right| + C$$
$$= \ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}} + \frac{x}{\sqrt{5}}\right| + C.$$

We can leave this answer as is, or we can use a logarithmic identity to simplify it. Note:

$$\ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}} + \frac{x}{\sqrt{5}}\right| + C = \ln\left|\frac{1}{\sqrt{5}}\left(\sqrt{x^2+5}+x\right)\right| + C$$
$$= \ln\left|\frac{1}{\sqrt{5}}\right| + \ln\left|\sqrt{x^2+5}+x\right| + C$$
$$= \ln\left|\sqrt{x^2+5}+x\right| + C,$$

where the $\ln\left(1/\sqrt{5}\right)$ term is absorbed into the constant C. ♣

In the next example, we will use the technique of completing the square in order to rewrite the integrand.

Example 7.48: Completing the Square

Evaluate $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx.$

Solution. First, complete the square to write

$$3-2x-x^2 = 4-(x+1)^2$$

Now, we may let $u = x+1$ so that $du = dx$ (note that $x = u-1$) to get:

$$\int \frac{x}{\sqrt{4-(x+1)^2}}\,dx = \int \frac{u-1}{\sqrt{4-u^2}}\,du$$

Let $u = 2\sin\theta$ giving $du = 2\cos\theta\,d\theta$:

$$\int \frac{u-1}{\sqrt{4-u^2}}\,du = \int \frac{2\sin\theta-1}{2\cos\theta}\cdot 2\cos\theta\,d\theta = \int (2\sin\theta-1)\,d\theta$$

Integrating and using a triangle we get:

$$\begin{aligned}\int \frac{x}{\sqrt{3-2x-x^2}} &= -2\cos\theta-\theta+C \\ &= -\sqrt{4-u^2}-\sin^{-1}\left(\frac{u}{2}\right)+C \\ &= -\sqrt{3-2x-x^2}-\sin^{-1}\left(\frac{x+1}{2}\right)+C\end{aligned}$$

Note that in this problem we could have skipped the u-substitution if instead we let $x+1 = 2\sin\theta$. (For the triangle we would then use $\sin\theta = \dfrac{x+1}{2}$.) ♣

Exercises for 7.3

7.3.1 $\displaystyle\int \sqrt{x^2-1}\,dx$

7.3.2 $\displaystyle\int \sqrt{9+4x^2}\,dx$

7.3.3 $\displaystyle\int x\sqrt{1-x^2}\,dx$

7.3.4 $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

7.3.5 $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$

7.3.6 $\displaystyle\int \sqrt{x^2+2x}\,dx$

7.3.7 $\displaystyle\int \frac{1}{x^2(1+x^2)}\,dx$

7.3.8 $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

7.3.9 $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$

7.3.10 $\displaystyle\int \frac{x^3}{\sqrt{4x^2-1}}\,dx$

7.4 Integration by Parts

Here's a simple integral that we can't yet evaluate:

$$\int x\cos x\,dx.$$

It's a simple matter to take the derivative of the integrand using the Product Rule, but there is no Product Rule for integrals. However, this section introduces *Integration by Parts*, a method of integration that is based on the Product Rule for derivatives. It will enable us to evaluate this integral.

The Product Rule says that if u and v are functions of x, then $(uv)' = u'v + uv'$. For simplicity, we've written u for $u(x)$ and v for $v(x)$. Suppose we integrate both sides with respect to x. This gives

$$\int (uv)'\,dx = \int (u'v + uv')\,dx.$$

By the Fundamental Theorem of Calculus, the left side integrates to uv. The right side can be broken up into two integrals, and we have

$$\int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx.$$

rearranging we get

$$\int uv'\,dx = \int (uv)'\,dx - \int u'v\,dx.$$

which gives

$$\int uv'\,dx = uv - \int u'v\,dx.$$

Using differential notation, we can write $du = u'(x)dx$ and $dv = v'(x)dx$ and the expression above can be written as follows:

$$\int u\,dv = uv - \int v\,du.$$

This is the Integration by Parts formula. For reference purposes, we state this in a theorem.

Theorem 7.6: Integration by Parts

Let u and v be differentiable functions of x on an interval I containing a and b. Then

$$\int u\ dv = uv - \int v\ du,$$

and

$$\int_{x=a}^{x=b} u\ dv = uv\Big|_a^b - \int_{x=a}^{x=b} v\ du.$$

Let's try an example to understand our new technique.

Example 7.49: Integrating using Integration by Parts

Evaluate $\displaystyle\int x\cos x\ dx.$

Solution. The key to Integration by Parts is to identify part of the integrand as "u" and part as "dv." Regular practice will help one make good identifications, and later we will introduce some principles that help. For now, let $u = x$ and $dv = \cos x\ dx$.

It is generally useful to make a small table of these values as done below. Right now we only know u and dv as shown on the left of Figure 7.5; on the right we fill in the rest of what we need. If

$u = x$, then $du = dx$. Since $dv = \cos x\ dx$, v is an antiderivative of $\cos x$. We choose $v = \sin x$.

$$\begin{array}{cc} u = x & v = ? \\ du = ? & dv = \cos x\ dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x & v = \sin x \\ du = dx & dv = \cos x\ dx \end{array}$$

Figure 7.5: Setting up Integration by Parts.

Now substitute all of this into the Integration by Parts formula, giving

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx.$$

We can then integrate $\sin x$ to get $-\cos x + C$ and overall our answer is

$$\int x \cos x\ dx = x \sin x + \cos x + C.$$

Note how the antiderivative contains a product, $x \sin x$. This product is what makes Integration by Parts necessary. ♣

The example above demonstrates how Integration by Parts works in general. We try to identify u and dv in the integral we are given, and the key is that we usually want to choose u and dv so that du is simpler than u and v is hopefully not too much more complicated than dv. This will mean that the integral on the right side of the Integration by Parts formula, $\int v \, du$ will be simpler to integrate than the original integral $\int u \, dv$.

In the example above, we chose $u = x$ and $dv = \cos x \, dx$. Then $du = dx$ was simpler than u and $v = \sin x$ is no more complicated than dv. Therefore, instead of integrating $x \cos x \, dx$, we could integrate $\sin x \, dx$, which we knew how to do.

A useful mnemonic for helping to determine u is "LIPET," where

L = **L**ogarithmic, I = **I**nverse Trig., P = **P**olynomial (algebraic), E = **E**xponential, and T = **T**rigonometric.

If the integrand contains both a logarithmic and an polynomial term, in general letting u be the logarithmic term works best, as indicated by L coming before P in LIPET.

Note: Some texts us "LIATE," where A = **A**lgebraic. This method works just as well as LIPET.

We now consider another example.

Example 7.50: Integrating using Integration by Parts

Evaluate $\displaystyle\int xe^x \, dx$.

Solution. Using the LIPET rule, we see that the integrand contains a **P**olynomial term (x) and an **E**xponential term (e^x). Our mnemonic suggests letting u be the polynomial term, so we choose $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$ as indicated by the tables below.

$$\begin{array}{cc} u = x & v = ? \\ du = ? & dv = e^x\ dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x & v = e^x \\ du = dx & dv = e^x\ dx \end{array}$$

Figure 7.6: Setting up Integration by Parts.

We see du is simpler than u, while there is no change in going from dv to v. This is good. The Integration by Parts formula gives

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

The integral on the right is simple; our final answer is

$$\int xe^x\ dx = xe^x - e^x + C.$$

Note again how the antiderivatives contain a product term. ♣

Example 7.51: Product of a Linear Function and Logarithm

Evaluate $\int x \ln x\,dx$.

Solution. Using the LIPET rule, we let $u = \ln x$ so $du = 1/x\,dx$. Then we must let $dv = x\,dx$ so $v = x^2/2$ and

$$\int x \ln x\,dx = \frac{x^2 \ln x}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

♣

Example 7.52: Secant Cubed (again)

Evaluate $\int \sec^3 x\,dx$.

Solution. Of course we already know the answer to this, but we needed to be clever to discover it. Here we'll use the new technique to discover the antiderivative. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x$ and $v = \tan x$ and

$$\int \sec^3 x\,dx \quad = \quad \sec x \tan x - \int \tan^2 x \sec x\,dx$$

$$\begin{aligned} &= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx \\ &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx. \end{aligned}$$

At first this looks useless—we're right back to $\int \sec^3 x \, dx$. But looking more closely:

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\ \int \sec^3 x \, dx + \int \sec^3 x \, dx &= \sec x \tan x + \int \sec x \, dx \\ 2 \int \sec^3 x \, dx &= \sec x \tan x + \int \sec x \, dx \\ \int \sec^3 x \, dx &= \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x \, dx \\ &= \frac{\sec x \tan x}{2} + \frac{\ln |\sec x + \tan x|}{2} + C. \end{aligned}$$

♣

Tabular Method

Example 7.53: Product of a Polynomial and Trigonometric Function

Evaluate $\int x^2 \sin x \, dx$.

Solution. Let $u = x^2$, $dv = \sin x \, dx$; then $du = 2x \, dx$ and $v = -\cos x$. Now

$$\int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx.$$

This is better than the original integral, but we need to do integration by parts again. Let $u = 2x$, $dv = \cos x \, dx$; then $du = 2$ and $v = \sin x$, and

$$\begin{aligned} \int x^2 \sin x \, dx &= -x^2 \cos x + \int 2x \cos x \, dx \\ &= -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C. \end{aligned}$$

♣

Such repeated use of integration by parts is fairly common, but it can be a bit tedious to accomplish, and it is easy to make errors, especially sign errors involving the subtraction in the formula. There is a nice tabular method to accomplish the calculation that minimizes the chance for error and speeds up the whole process. We illustrate with the previous example. Here is the table:

sign	u	dv
$+$	x^2	$\sin x$
-	$2x$	$-\cos x$
$+$	2	$-\sin x$
-	0	$\cos x$

To form this table, we start with u at the top of the second column and repeatedly compute the derivative; starting with dv at the top of the third column, we repeatedly compute the antiderivative. In the first column, we place a "$-$" in every second row. To form the second table we combine the first and second columns by ignoring the boundary; if you do this by hand, you may simply start with two columns and add a "$-$" to every second row.

Alternatively, we can use the following table:

u	dv
x^2	$\sin x$
$-2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

To compute with this second table we begin at the top. Multiply the first entry in column u by the second entry in column dv to get $-x^2\cos x$, and add this to the integral of the product of the second entry in column u and second entry in column dv. This gives:

$$-x^2\cos x + \int 2x\cos x\,dx,$$

or exactly the result of the first application of integration by parts. Since this integral is not yet easy, we return to the table. Now we multiply twice on the diagonal, $(x^2)(-\cos x)$ and $(-2x)(-\sin x)$ and then once straight across, $(2)(-\sin x)$, and combine these as

$$-x^2\cos x + 2x\sin x - \int 2\sin x\,dx,$$

giving the same result as the second application of integration by parts. While this integral is easy, we may return yet once more to the table. Now multiply three times on the diagonal to get $(x^2)(-\cos x)$, $(-2x)(-\sin x)$, and $(2)(\cos x)$, and once straight across, $(0)(\cos x)$. We combine these as before to get

$$-x^2\cos x + 2x\sin x + 2\cos x + \int 0\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

Typically we would fill in the table one line at a time, until the "straight across" multiplication gives an easy integral. If we can see that the u column will eventually become zero, we can instead fill in the whole table; computing the products as indicated will then give the entire integral, including the "$+C$", as above.

Example 7.54:

Do Example 7.50 again, using the tabular method.

Solution. Recognising that x has a derivative that vanishes, and e^x is easy to repeatedly integrate, we construct the table with $u = x$:

u	dv
x	e^x
-1	e^x
$+0$	e^x

This gives

$$\int xe^x \, dx = xe^x - e^x + C.$$

which agrees with the result in Example 7.50. ♣

Some Classic Examples of IBP

Example 7.55: Solving for the unknown integral

Evaluate $\int e^x \cos x \, dx$.

Solution. This is a classic problem. Our mnemonic (LIPET) suggests letting u be the exponential factor, sowe choose $u = e^x$ and hence $dv = \cos x \, dx$. Then $du = e^x \, dx$ and $v = \sin x$ as shown below.

$$\begin{array}{llcll} u = e^x & v = ? & \Rightarrow & u = e^x & v = \sin x \\ du = ? & dv = \cos x \ dx & & du = e^x \ dx & dv = \cos x \ dx \end{array}$$

Figure 7.7: Setting up Integration by Parts.

Notice that du is no simpler than u, going against our general rule (but bear with us). The Integration by Parts formula yields

$$\int e^x \cos x \ dx = e^x \sin x - \int e^x \sin x \, dx.$$

The integral on the right is not much different than the one we started with, so it seems like we have gotten nowhere. Let's keep working and apply Integration by Parts to the new integral, using $u = e^x$ and $dv = \sin x \, dx$. This leads us to the following:

$$\begin{array}{llcll} u = e^x & v = ? & \Rightarrow & u = e^x & v = -\cos x \\ du = ? & dv = \sin x \ dx & & du = e^x \ dx & dv = \sin x \ dx \end{array}$$

Figure 7.8: Setting up Integration by Parts (again).

The Integration by Parts formula then gives:

$$\int e^x \cos x\, dx = e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x\, dx \right)$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x\ dx.$$

It seems we are back right where we started, as the right hand side contains $\int e^x \cos x\, dx$. But this is actually a good thing.

Add $\displaystyle\int e^x \cos x\ dx$ to both sides. This gives

$$2 \int e^x \cos x\ dx = e^x \sin x + e^x \cos x$$

Now divide both sides by 2:

$$\int e^x \cos x\ dx = \frac{1}{2}\big(e^x \sin x + e^x \cos x\big).$$

Simplifying a little and adding the constant of integration, our answer is thus

$$\int e^x \cos x\ dx = \frac{1}{2} e^x \left(\sin x + \cos x\right) + C.$$

♣

Example 7.56: IBP: the antiderivative of $\ln x$

Evaluate $\displaystyle\int \ln x\, dx$.

Solution. One may have noticed that we have rules for integrating the familiar trigonometric functions and e^x, but we have not yet given a rule for integrating $\ln x$. That is because $\ln x$ can't easily be integrated with any of the rules we have learned up to this point. But we can find its antiderivative by a clever application of Integration by Parts. Set $u = \ln x$ and $dv = dx$. This is a good, sneaky trick to learn as it can help in other situations. This determines $du = (1/x)\, dx$ and $v = x$ as shown below.

$$\begin{array}{cc} u = \ln x & v = ? \\ du = ? & dv = dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = \ln x & v = x \\ du = 1/x\ dx & dv = dx \end{array}$$

Figure 7.9: Setting up Integration by Parts.

Putting this all together in the Integration by Parts formula, things work out very nicely:

$$\int \ln x\, dx = x \ln x - \int x\, \frac{1}{x}\, dx.$$

The new integral simplifies to $\int 1\,dx$, which is about as simple as things get. Its integral is $x+C$ and our answer is

$$\int \ln x\ dx = x\ln x - x + C.$$

♣

Example 7.57: Integrating using Int. by Parts: antiderivative of $\arctan x$

Evaluate $\int \arctan x\,dx$.

Solution. The same sneaky trick we used above works here. Let $u = \arctan x$ and $dv = dx$. Then $du = 1/(1+x^2)\,dx$ and $v = x$. The Integration by Parts formula gives

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

The integral on the right can be solved by substitution. Taking $u = 1+x^2$, we get $du = 2x\,dx$. The integral then becomes

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\int \frac{1}{u}\,du.$$

The integral on the right evaluates to $\ln|u| + C$, which becomes $\ln(1+x^2)+C$. Therefore, the answer is

$$\int \arctan x\ dx = x\arctan x - \ln(1+x^2) + C.$$

♣

Substitution Before Integration

When taking derivatives, it was common to employ multiple rules (such as using both the Quotient and the Chain Rules). It should then come as no surprise that some integrals are best evaluated by combining integration techniques. In particular, here we illustrate making an “unusual” substitution first before using Integration by Parts.

Example 7.58: Integration by Parts after substitution

Evaluate $\int \cos(\ln x)\ dx$.

Solution. The integrand contains a composition of functions, leading us to think Substitution would be beneficial. Letting $u = \ln x$, we have $du = 1/x\ dx$. This seems problematic, as we do not have a $1/x$ in the integrand. But consider:

$$du = \frac{1}{x}\ dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln x$, we can use inverse functions and conclude that $x = e^u$. Therefore we have that

$$\begin{aligned} dx &= x \cdot du \\ &= e^u\ du. \end{aligned}$$

We can thus replace $\ln x$ with u and dx with $e^u\ du$. Thus we rewrite our integral as

$$\int \cos(\ln x)\ dx = \int e^u \cos u\ du.$$

We evaluated this integral in Example 7.55. Using the result there, we have:

$$\begin{aligned} \int \cos(\ln x)\ dx &= \int e^u \cos u\ du \\ &= \frac{1}{2} e^u \big(\sin u + \cos u\big) + C \\ &= \frac{1}{2} e^{\ln x} \big(\sin(\ln x) + \cos(\ln x)\big) + C \\ &= \frac{1}{2} x \big(\sin(\ln x) + \cos(\ln x)\big) + C. \end{aligned}$$

♣

Definite Integrals and Integration By Parts

So far we have focused only on evaluating indefinite integrals. Of course, we can use Integration by Parts to evaluate definite integrals as well, as Theorem 7.6 states. We do so in the next example.

Example 7.59: Definite integration using Integration by Parts

Evaluate $\displaystyle\int_1^2 x^2 \ln x\, dx$.

Solution. Our mnemonic suggests letting $u = \ln x$, hence $dv = x^2\, dx$. We then get $du = (1/x)\, dx$ and $v = x^3/3$ as shown below.

$$\begin{array}{cccccc} u = \ln x & v = ? & & & u = \ln x & v = x^3/3 \\ du = ? & dv = x^2\ dx & & \Rightarrow & du = 1/x\ dx & dv = x^2\ dx \end{array}$$

Figure 7.10: Setting up Integration by Parts.

The Integration by Parts formula then gives

$$\int_1^2 x^2 \ln x\, dx = \frac{x^3}{3} \ln x \bigg|_1^2 - \int_1^2 \frac{x^3}{3} \frac{1}{x}\, dx$$

$$\begin{aligned}
&= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{3}\,dx \\
&= \frac{x^3}{3}\ln x\Big|_1^2 - \frac{x^3}{9}\Big|_1^2 \\
&= \left(\frac{x^3}{3}\ln x - \frac{x^3}{9}\right)\Big|_1^2 \\
&= \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) - \left(\frac{1}{3}\ln 1 - \frac{1}{9}\right) \\
&= \frac{8}{3}\ln 2 - \frac{7}{9} \\
&\approx 1.07.
\end{aligned}$$

♣

In general, Integration by Parts is useful for integrating certain products of functions, like $\int xe^x\,dx$ or $\int x^3 \sin x\,dx$. It is also useful for integrals involving logarithms and inverse trigonometric functions.

As stated before, integration is generally more difficult than derivation. We are developing tools for handling a large array of integrals, and experience will tell us when one tool is preferable/necessary over another. For instance, consider the three similar–looking integrals

$$\int xe^x\,dx, \qquad \int xe^{x^2}\,dx \qquad \text{and} \qquad \int xe^{x^3}\,dx.$$

While the first is calculated easily with Integration by Parts, the second is best approached with Substitution. Taking things one step further, the third integral has no answer in terms of elementary functions, so none of the methods we learn in calculus will get us the exact answer.

Integration by Parts is a very useful method, second only to substitution. In the following sections of this chapter, we continue to learn other integration techniques.

Exercises for 7.4

Find the antiderivatives.

7.4.1 $\displaystyle\int x\cos x\,dx$

7.4.2 $\displaystyle\int x^2\cos x\,dx$

7.4.3 $\displaystyle\int xe^x\,dx$

7.4.4 $\displaystyle\int xe^{x^2}\,dx$

7.4.5 $\displaystyle\int \sin^2 x\,dx$

7.4.6 $\displaystyle\int \ln x\,dx$

7.4.7 $\displaystyle\int x\arctan x\,dx$

7.4.8 $\displaystyle\int x^3\sin x\,dx$

7.4.9 $\displaystyle\int x^3\cos x\,dx$

7.4.10 $\displaystyle\int x\sin^2 x\,dx$

7.4.11 $\displaystyle\int x\sin x\cos x\,dx$

7.4.12 $\displaystyle\int \arctan(\sqrt{x})\,dx$

7.4.13 $\displaystyle\int \sin(\sqrt{x})\,dx$

7.4.14 $\displaystyle\int \sec^2 x\csc^2 x\,dx$

7.5 Rational Functions and Partial Fractions

A **rational function** is a fraction with polynomials in the numerator and denominator. For example,

$$\frac{x^3}{x^2+x-6}, \qquad \frac{1}{(x-3)^2}, \qquad \frac{x^2+1}{x^2-1},$$

are all rational functions of x. We should mention a special type of rational function that we already know how to integrate: If the denominator has the form $(ax+b)^n$, the substitution $u = ax+b$ will always work. The denominator becomes u^n, and each x in the numerator is replaced by $(u-b)/a$, and $dx = du/a$. While it may be tedious to complete the integration if the numerator has high degree, it is merely a matter of algebra.

Example 7.60: Substitution and Splitting Up a Fraction

Find $\displaystyle\int \frac{x^3}{(3-2x)^5}\,dx$.

Solution. Using the substitution $u = 3-2x$ we get

$$\begin{aligned}
\int \frac{x^3}{(3-2x)^5}\,dx &= \frac{1}{-2}\int \frac{\left(\frac{u-3}{-2}\right)^3}{u^5}\,du = \frac{1}{16}\int \frac{u^3-9u^2+27u-27}{u^5}\,du \\
&= \frac{1}{16}\int u^{-2}-9u^{-3}+27u^{-4}-27u^{-5}\,du \\
&= \frac{1}{16}\left(\frac{u^{-1}}{-1}-\frac{9u^{-2}}{-2}+\frac{27u^{-3}}{-3}-\frac{27u^{-4}}{-4}\right)+C \\
&= \frac{1}{16}\left(\frac{(3-2x)^{-1}}{-1}-\frac{9(3-2x)^{-2}}{-2}+\frac{27(3-2x)^{-3}}{-3}-\frac{27(3-2x)^{-4}}{-4}\right)+C \\
&= -\frac{1}{16(3-2x)}+\frac{9}{32(3-2x)^2}-\frac{9}{16(3-2x)^3}+\frac{27}{64(3-2x)^4}+C
\end{aligned}$$

♣

Of course there are other situation in which we can apply known techniques, perhaps after some clever manipulation. The following example demonstrates one such case.

Example 7.61: Denominator Does Not Factor

Evaluate $\displaystyle\int \frac{x+1}{x^2+4x+8}\,dx$.

Solution. The quadratic denominator does not factor. We could complete the square and use a trigonometric substitution, but it is simpler to rearrange the integrand:

$$\int \frac{x+1}{x^2+4x+8}\,dx = \int \frac{x+2}{x^2+4x+8}\,dx - \int \frac{1}{x^2+4x+8}\,dx.$$

The first integral is an easy substitution problem, using $u = x^2+4x+8$:

$$\int \frac{x+2}{x^2+4x+8}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|x^2+4x+8|.$$

For the second integral we complete the square:

$$x^2+4x+8 = (x+2)^2+4 = 4\left(\left(\frac{x+2}{2}\right)^2+1\right),$$

making the integral

$$\frac{1}{4}\int \frac{1}{\left(\frac{x+2}{2}\right)^2+1}\,dx.$$

Using $u = \dfrac{x+2}{2}$ we get

$$\frac{1}{4}\int \frac{1}{\left(\frac{x+2}{2}\right)^2 + 1}\, dx = \frac{1}{4}\int \frac{2}{u^2+1}\, dx = \frac{1}{2}\arctan\left(\frac{x+2}{2}\right).$$

The final answer is now

$$\int \frac{x+1}{x^2+4x+8}\, dx = \frac{1}{2}\ln|x^2+4x+8| - \frac{1}{2}\arctan\left(\frac{x+2}{2}\right) + C.$$

♣

More generally, we can not rely on a rational function having such "nice" forms as those above. There is a general technique called "partial fractions" that, in principle, allows us to integrate any rational function.

7.5.1. Partial Fraction Decomposition

Consider the integral $\displaystyle\int \frac{1}{x^2-1}\, dx$. We do not have a simple formula for this (if the denominator were x^2+1, we would recognize the antiderivative as being the arctangent function). It can be solved using Trigonometric Substitution, but note how the integral is easy to evaluate once we realize:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\, dx &= \int \frac{1/2}{x-1}\, dx - \int \frac{1/2}{x+1}\, dx \\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

This section teaches how to *decompose*

$$\frac{1}{x^2-1} \quad \text{into} \quad \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

We start with a rational function $f(x) = \frac{p(x)}{q(x)}$, where p and q do not have any common factors and the degree of p is less than the degree of q. It can be shown that any polynomial, and hence q, can be factored into a product of linear and irreducible quadratic terms. The following Key Idea states how to decompose a rational function into a sum of rational functions whose denominators are all of lower degree than q.

If we start with a rational function $f(x) = \frac{p(x)}{q(x)}$, we can assume that perhaps after long division the the fraction is reduced, and is "proper", in that p and q do not have any common factors and the degree of p is less than the degree of q. Any polynomial, and hence q, can be factored into a product of linear and irreducible quadratic terms. The following outlines how to decompose a rational function into a sum of rational functions whose denominators are all of lower degree than q.

Key Idea 7.5.0: Partial Fraction Decomposition

Let $\dfrac{p(x)}{q(x)}$ be a rational function, where the degree of p is less than the degree of q.

1. **Linear Terms:** Let $(x-a)$ divide $q(x)$, where $(x-a)^n$ is the highest power of $(x-a)$ that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum
$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_n}{(x-a)^n}.$$

2. **Quadratic Terms:** Let x^2+bx+c divide $q(x)$, where $(x^2+bx+c)^n$ is the highest power of x^2+bx+c that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum
$$\frac{B_1x+C_1}{x^2+bx+c}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

To find the coefficients A_i, B_i and C_i:

1. Multiply all fractions by $q(x)$, clearing the denominators. Collect like terms.
2. Equate the resulting coefficients of the powers of x and solve the resulting system of linear equations.

To find the coefficients A_i, B_i and C_i, it is helpful to have a process:

Key Idea 7.5.0: Partial Fractions: Solving for Coefficients

Let $\dfrac{p(x)}{q(x)}$ be a rational function, set up an equation with $\dfrac{p(x)}{q(x)}$ set equal to it's partial fraction form.

1. Multiply by the denominator $q(x)$ to clear all fractions and obtain the "**Basic Equation**".

2. Solve the Basic Equation for the unknowns using the following guidelines:

   (a) Expand the Basic Equation, collect terms according to powers of x and equate coefficients of like powers of x. This will give a system of linear equations to be solved.

   (b) Alternatively, for distinct linear factors, you may substitute the roots of the distinct linear factors to determine the constants.

   (c) Another alternative: For repeated linear factors, you may also first substitute the roots of the linear factors, then rewrite the Basic Equation and use other âĂIJconvenient" choices for x to solve for the remaining coefficients (or use the method of equating coefficients).

The following examples will demonstrate how to put this into practice. Example 7.62 stresses the decomposition aspect of the Key Idea.

Example 7.62: Decomposing into partial fractions

Decompose $f(x) = \dfrac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2}$ *without solving for the resulting coefficients.*

Solution. The denominator is already factored, as both x^2+x+2 and x^2+x+7 cannot be factored further. We need to decompose $f(x)$ properly. Since $(x+5)$ is a linear term that divides the denominator, there will be a

$$\frac{A}{x+5}$$

term in the decomposition.

As $(x-2)^3$ divides the denominator, we will have the following terms in the decomposition:

$$\frac{B}{x-2}, \quad \frac{C}{(x-2)^2} \quad \text{and} \quad \frac{D}{(x-2)^3}.$$

The x^2+x+2 term in the denominator results in a $\dfrac{Ex+F}{x^2+x+2}$ term.

Finally, the $(x^2+x+7)^2$ term results in the terms

$$\frac{Gx+H}{x^2+x+7} \quad \text{and} \quad \frac{Ix+J}{(x^2+x+7)^2}.$$

All together, we have

$$\frac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2} = \frac{A}{x+5} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3} + \frac{Ex+F}{x^2+x+2} + \frac{Gx+H}{x^2+x+7} + \frac{Ix+J}{(x^2+x+7)^2}$$

Solving for the coefficients A, $B \dots J$ would be a bit tedious but not "hard." ♣

Example 7.63: Decomposing into partial fractions

Perform the partial fraction decomposition of $\frac{1}{x^2-1}$.

Solution. The denominator factors into two linear terms: $x^2-1=(x-1)(x+1)$. Thus

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

To solve for A and B, first multiply through by $x^2-1=(x-1)(x+1)$ to obtain the Basic Equation:

$$1 = \frac{A(x-1)(x+1)}{x-1} + \frac{B(x-1)(x+1)}{x+1} \tag{7.2}$$

$$= A(x+1) + B(x-1) \tag{7.3}$$

$$= Ax + A + Bx - B \tag{7.4}$$

Now collect like terms.

$$= (A+B)x + (A-B). \tag{7.5}$$

The next step is key. Note the equality we have:

$$1 = (A+B)x + (A-B).$$

For clarity's sake, rewrite the left hand side as

$$0x + 1 = (A+B)x + (A-B).$$

On the left, the coefficient of the x term is 0; on the right, it is $(A+B)$. Since both sides are equal, we must have that $0=A+B$.

Likewise, on the left, we have a constant term of 1; on the right, the constant term is $(A-B)$. Therefore we have $1=A-B$.

We have two linear equations with two unknowns. This one is easy to solve by hand, leading to

$$\begin{array}{c} A+B=0 \\ A-B=1 \end{array} \Rightarrow \begin{array}{c} A=1/2 \\ B=-1/2 \end{array}.$$

Note, that alternatively, we could have substituted the two roots $(usedx = \pm 1)$ into the basic equation 7.3 to solve for the coefficients. $x=1$ gives $1=2A$, and $x=-1$ gives $1=-2B$.

Thus we arrive at the partial fraction decomposition

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

♣

Example 7.64: Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\,dx$.

Solution. We decompose the integrand as follows, as described in the process 7.5.1:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

To solve for A, B and C, we multiply both sides by $(x-1)(x+2)^2$ to obtain the Basic Equation and collect like terms:

$$\begin{aligned} 1 &= A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad (7.6)\\ &= Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C \\ &= (A+B)x^2 + (4A+B+C)x + (4A-2B-C) \end{aligned}$$

We have

$$0x^2 + 0x + 1 = (A+B)x^2 + (4A+B+C)x + (4A-2B-C)$$

leading to the equations

$$A+B=0, \quad 4A+B+C=0 \quad \text{and} \quad 4A-2B-C=1.$$

These three equations of three unknowns lead to a unique solution:

$$A = 1/9, \quad B = -1/9 \quad \text{and} \quad C = -1/3.$$

Thus

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \int \frac{1/9}{x-1}\,dx + \int \frac{-1/9}{x+2}\,dx + \int \frac{-1/3}{(x+2)^2}\,dx.$$

Each can be integrated with a simple substitution with $u = x-1$ or $u = x+2$ as the denominators are linear functions). The end result is

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + C.$$

Note: The Basic Equation 7.6 offers a direct route to finding the values of A, B and C. When $x=1$, the right hand side simplifies to $A(1+2)^2 = 9A$. Since the left hand side is still 1, we have $1 = 9A$. Hence $A = 1/9$. Likewise, when $x = -2$we obtain $1 = -3C$, so $C = -1/3$. Knowing A and C, we can find the value of B by choosing yet another value of x, such as $x = 0$, and solving for B, or by equating coefficients.

♣

Example 7.65: Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{x^3}{(x-5)(x+3)}\,dx$.

Solution. Our method presumes that the degree of the numerator is less than the degree of the denominator. Since this is not the case here, we begin by using polynomial division to reduce the degree of the numerator. We omit the steps, but encourage the reader to verify that

$$\frac{x^3}{(x-5)(x+3)} = x+2+\frac{19x+30}{(x-5)(x+3)}.$$

We can rewrite the new rational function in partial fraction form:

$$\frac{19x+30}{(x-5)(x+3)} = \frac{A}{x-5}+\frac{B}{x+3}$$

for appropriate values of A and B. Clearing denominators, we have the basic equation:

$$19x+30 = A(x+3)+B(x-5)$$

Setting $x=-3$ gives $-27=-8B$, so $B=27/8$.
Setting $x=5$ gives $125=8A$, so $A=125/8$.

We can now integrate.

$$\begin{aligned}\int \frac{x^3}{(x-5)(x+3)}\,dx &= \int\left(x+2+\frac{125/8}{x-5}+\frac{27/8}{x+3}\right)\,dx\\ &= \frac{x^2}{2}+2x+\frac{125}{8}\ln|x-5|+\frac{27}{8}\ln|x+3|+C.\end{aligned}$$

Note: Alternatively, we could have used the method of equating coefficients to solve for A and B.

$$\begin{aligned}19x+30 &= A(x+3)+B(x-5)\\ &= (A+B)x+(3A-5B).\end{aligned}$$

This implies that:

$$\begin{aligned}19 &= A+B\\ 30 &= 3A-5B.\end{aligned}$$

Solving this system of linear equations gives

$$\begin{aligned}125/8 &= A\\ 27/8 &= B.\end{aligned}$$

♣

Example 7.66: Integrating using partial fractions

Use partial fraction decomposition to evaluate $\displaystyle\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx.$

Solution. The degree of the numerator is less than the degree of the denominator so we begin by setting up the partial fraction form. We have:

$$\frac{7x^2+31x+54}{(x+1)(x^2+6x+11)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+6x+11}.$$

Now clear the denominators to get the Basic Equation.

$$7x^2+31x+54 = A(x^2+6x+11) + (Bx+C)(x+1)$$

Now collect terms to equate coefficients.

$$= (A+B)x^2 + (6A+B+C)x + (11A+C).$$

This implies that:

$$\begin{aligned} 7 &= A+B \\ 31 &= 6A+B+C \\ 54 &= 11A+C. \end{aligned}$$

Solving this system of linear equations gives the nice result of $A=5$, $B=2$ and $C=-1$. Thus

$$\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx = \int \left(\frac{5}{x+1} + \frac{2x-1}{x^2+6x+11}\right)\,dx.$$

The first term of this new integrand is easy to evaluate; it leads to a $5\ln|x+1|$ term. The second term is not hard, but takes several steps and uses substitution techniques.

The integrand $\dfrac{2x-1}{x^2+6x+11}$ has a quadratic in the denominator and a linear term in the numerator. This leads us to try substitution. Let $u = x^2+6x+11$, so $du = (2x+6)\,dx$. The numerator is $2x-1$, not $2x+6$, but we can get a $2x+6$ term in the numerator by adding 0 in the form of "$7-7$."

$$\begin{aligned} \frac{2x-1}{x^2+6x+11} &= \frac{2x-1+7-7}{x^2+6x+11} \\ &= \frac{2x+6}{x^2+6x+11} - \frac{7}{x^2+6x+11}. \end{aligned}$$

We can now integrate the first term with substitution, leading to a $\ln|x^2+6x+11|$ term. The final term can be integrated using arctangent. First, complete the square in the denominator:

$$\frac{7}{x^2+6x+11} = \frac{7}{(x+3)^2+2}.$$

An antiderivative of the latter term can be found using Theorem 7.3 and substitution:

$$\int \frac{7}{x^2+6x+11}\,dx = \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)+C.$$

Let's start at the beginning and put all of the steps together.

$$\begin{aligned}\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx &= \int\left(\frac{5}{x+1}+\frac{2x-1}{x^2+6x+11}\right)\,dx\\ &= \int \frac{5}{x+1}\,dx + \int \frac{2x+6}{x^2+6x+11}\,dx - \int \frac{7}{x^2+6x+11}\,dx\\ &= 5\ln|x+1| + \ln|x^2+6x+11| - \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)+C.\end{aligned}$$

As with many other problems in calculus, it is important to remember that one is not expected to "see" the final answer immediately after seeing the problem. Rather, given the initial problem, we break it down into smaller problems that are easier to solve. The final answer is a combination of the answers of the smaller problems.

♣

Partial Fraction Decomposition is an important tool when dealing with rational functions. Note that at its heart, it is a technique of algebra, not calculus, as we are rewriting a fraction in a new form. Regardless, it is very useful in the realm of calculus as it lets us evaluate a certain set of "complicated" integrals.

Exercises for 7.5

7.5.1 $\displaystyle\int \frac{1}{4-x^2}\,dx$

7.5.2 $\displaystyle\int \frac{x^4}{4-x^2}\,dx$

7.5.3 $\displaystyle\int \frac{1}{x^2+10x+25}\,dx$

7.5.4 $\displaystyle\int \frac{x^2}{4-x^2}\,dx$

7.5.5 $\displaystyle\int \frac{x^4}{4+x^2}\,dx$

7.5.6 $\displaystyle\int \frac{1}{x^2+10x+29}\,dx$

7.5.7 $\displaystyle\int \frac{x^3}{4+x^2}\,dx$

7.5.8 $\displaystyle\int \frac{1}{x^2+10x+21}\,dx$

7.5.9 $\displaystyle\int \frac{1}{2x^2-x-3}\,dx$

7.5.10 $\displaystyle\int \frac{1}{x^2+3x}\,dx$

7.6 Numerical Integration

The Fundamental Theorem of Calculus gives a concrete technique for finding the exact value of a definite integral. That technique is based on computing antiderivatives. Despite the power of this theorem, there are still situations where we must *approximate* the value of the definite integral instead of finding its exact value. The first situation we explore is where we *cannot* compute the antiderivative of the integrand. The second case is when we actually do not know the integrand, but only its value when evaluated at certain points.

An **elementary function** is any function that is a combination of polynomials, n^{th} roots, rational, exponential, logarithmic and trigonometric functions. We can compute the derivative of any elementary function, but there are many elementary functions of which we cannot compute an antiderivative. For example, the following functions do not have antiderivatives that we can express with elementary functions:

$$e^{-x^2}, \quad \sin(x^3) \quad \text{and} \quad \frac{\sin x}{x}.$$

The simplest way to refer to the antiderivatives of e^{-x^2} is to simply write $\int e^{-x^2}\,dx$.

How, then, can we solve problems involving definite integrals of such functions? We *approximate.* This section outlines three common methods of approximating the value of definite integrals. We describe each as a systematic method of approximating area under a curve. By approximating this area accurately, we find an accurate approximation of the corresponding definite integral.

We will apply the methods we learn in this section to the following definite integrals:

$$\int_0^1 e^{-x^2}\,dx, \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx, \quad \text{and} \quad \int_{0.5}^{4\pi} \frac{\sin(x)}{x}\,dx,$$

as pictured in Figure 7.11.

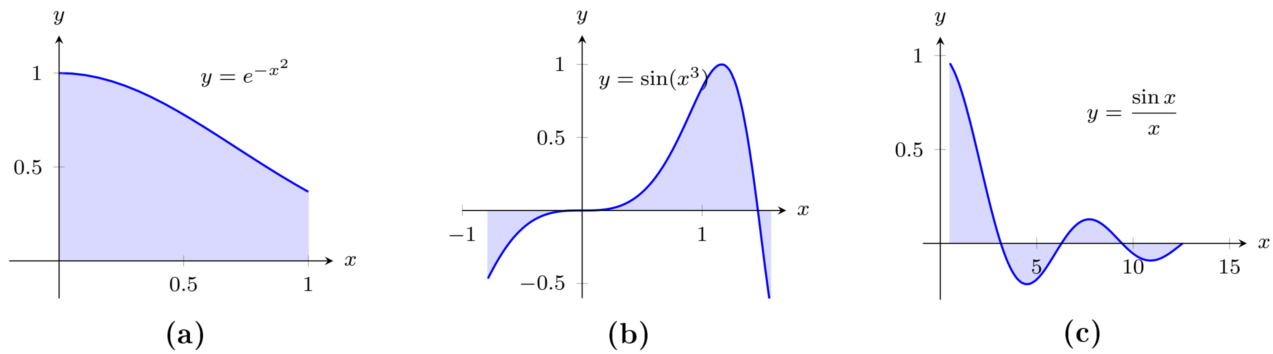


Figure 7.11: Graphing the areas defined by the definite integrals of Example 7.23.

The Left Hand, Right Hand, and Midpoint Rule Methods

In Section 6.2 we addressed the problem of evaluating definite integrals by approximating the area under the curve using rectangles. We revisit those ideas here before introducing other methods of approximating definite integrals.

We start with a review of notation. Let f be a continuous function on the interval $[a, b]$. We wish to approximate $\int_a^b f(x)\, dx$. We partition $[a, b]$ into n equally spaced subintervals, each of length $\Delta x = \dfrac{b-a}{n}$. The endpoints of these subintervals are labelled as

$$x_0 = a,\ x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ \ldots,\ x_i = a + (i)\Delta x,\ \ldots,\ x_n = b.$$

To use the Left Hand Rule we use the summation $\sum_{i=1}^{n} f(x_{i-1})\Delta x$, to use the Right Hand Rule we use $\sum_{i=1}^{n} f(x_i)\Delta x$, and for the Midpoint rule we used $\sum_{i=1}^{n} f(\overline{x}_i)\triangle x$, where $\overline{x}_i = (x_{i-1} + x_i)/2$. We review the use of the Left and Right Hand rules in the context of examples.

Example 7.67: Approximating definite integrals with rectangles

Approximate $\int_0^1 e^{-x^2}\, dx$ *using the Left and Right Hand Rules with* 5 *equally spaced subintervals.*

Solution. We begin by partitioning the interval $[0, 1]$ into 5 equally spaced intervals. We have $\Delta x = \frac{1-0}{5} = 1/5 = 0.2$, so

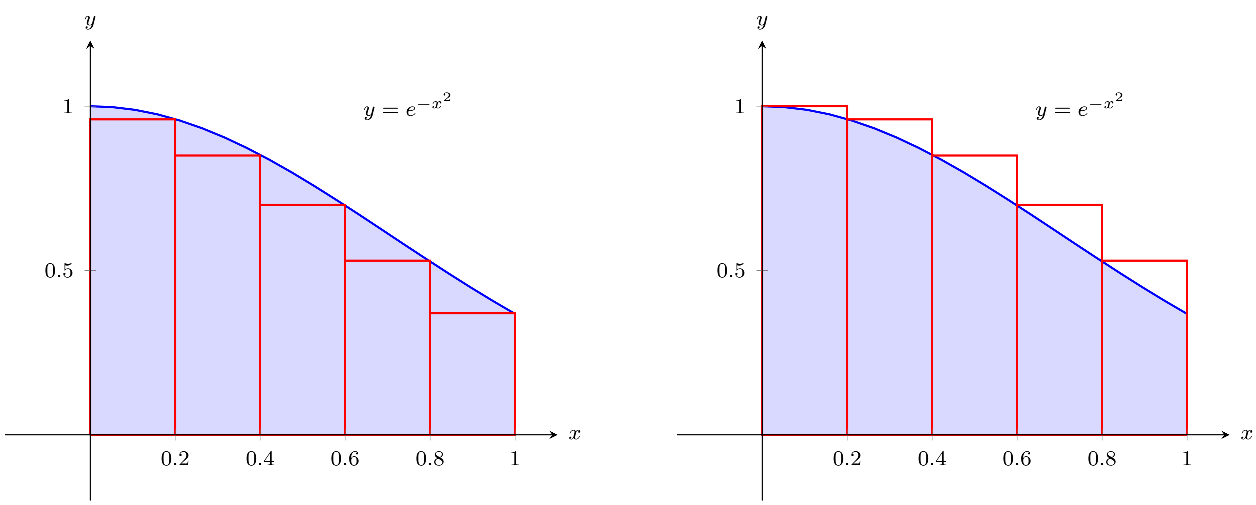
$$x_1 = 0,\ x_2 = 0.2,\ x_3 = 0.4,\ x_4 = 0.6,\ x_5 = 0.8,\ \text{and } x_6 = 1.$$

Using the Left Hand Rule, we have:

$$\begin{aligned}\sum_{i=1}^{n} f(x_{i-1})\Delta x &= \big(f(x_0)+f(x_1)+f(x_2)+f(x_3)+f(x_4)\big)\Delta x\\ &= \big(f(0)+f(0.2)+f(0.4)+f(0.6)+f(0.8)\big)\Delta x\\ &\approx \big(1+0.961+0.852+0.698+0.527)(0.2)\\ &\approx 0.808.\end{aligned}$$

Using the Right Hand Rule, we have:

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \big(f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)\big)\Delta x\\ &= \big(f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1)\big)\Delta x\\ &\approx \big(0.961+0.852+0.698+0.527+0.368)(0.2)\\ &\approx 0.681.\end{aligned}$$



(a) **(b)**

Figure 7.12: Approximating $\int_0^1 e^{-x^2}\,dx$ in Example 7.67.

Figure 7.12 shows the rectangles used in each method to approximate the definite integral. These graphs show that in this particular case, the Left Hand Rule is an over approximation and the Right Hand Rule is an under approximation. To get a better approximation, we could use more rectangles. We could also average the Left and Right Hand Rule results together, giving

$$\frac{0.808+0.681}{2} = 0.7445.$$

The actual answer, accurate to 4 places after the decimal, is 0.7468, showing our average is a good approximation.

♣

Example 7.68: Approximating definite integrals with rectangles

Approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ *using the Left and Right Hand Rules with* 10 *equally spaced subintervals.*

Solution. We begin by finding Δx:

$$\frac{b-a}{n} = \frac{\pi/2-(-\pi/4)}{10} = \frac{3\pi}{40} \approx 0.236.$$

It is useful to write out the endpoints of the subintervals in a table; in Figure 7.13, we give the exact values of the endpoints, their decimal approximations, and decimal approximations of $\sin(x^3)$ evaluated at these points.

x_i	Exact	Approx.	$\sin(x_i^3)$
x_0	$-\pi/4$	-0.785	-0.466
x_1	$-7\pi/40$	-0.550	-0.165
x_2	$-\pi/10$	-0.314	-0.031
x_3	$-\pi/40$	-0.0785	0
x_4	$\pi/20$	0.157	0.004
x_5	$\pi/8$	0.393	0.061
x_6	$\pi/5$	0.628	0.246
x_7	$11\pi/40$	0.864	0.601
x_8	$7\pi/20$	1.10	0.971
x_9	$17\pi/40$	1.34	0.690
x_{10}	$\pi/2$	1.57	-0.670

Figure 7.13: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ in Example 7.68.

Once this table is created, it is straightforward to approximate the definite integral using the Left and Right Hand Rules. (Note: the table itself is easy to create, especially with a standard spreadsheet program on a computer. The last two columns are all that are needed.) The Left Hand Rule sums the first 10 values of $\sin(x_i^3)$ and multiplies the sum by Δx; the Right Hand Rule sums the last 10 values of $\sin(x_i^3)$ and multiplies by Δx. Therefore we have:

Left Hand Rule: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx \approx (1.91)(0.236) = 0.451.$

Right Hand Rule: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx \approx (1.71)(0.236) = 0.404.$

Average of the Left and Right Hand Rules: 0.4275.

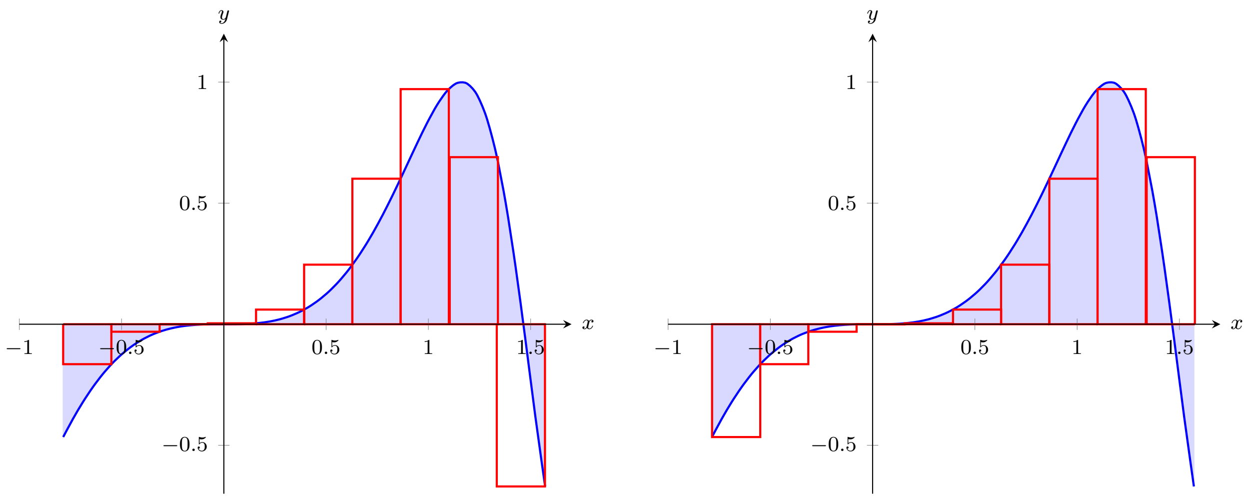


Figure 7.14: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ in Example ??.

The actual answer, accurate to 3 places after the decimal, is 0.460. Our approximations were once again fairly good. The rectangles used in each approximation are shown in Figure 7.14. It is clear from the graphs that using more rectangles (and hence, narrower rectangles) should result in a more accurate approximation. ♣

7.6.1. The Trapezoidal Rule

In Example **??** we approximated the value of $\displaystyle\int_0^1 e^{-x^2}\ dx$ with 5 rectangles of equal width. Figure 7.12 shows the rectangles used in the Left and Right Hand Rules. These graphs clearly show that rectangles do not match the shape of the graph all that well, and that accurate approximations will only come by using lots of rectangles.

Instead of using rectangles to approximate the area, we can instead use *trapezoids.* In Figure 7.15, we show the region under $f(x) = e^{-x^2}$ on $[0, 1]$ approximated with 5 trapezoids of equal width; the top "corners" of each trapezoid lies on the graph of $f(x)$. It is clear from this figure that these trapezoids more accurately approximate the area under f and hence should give a better approximation of $\int_0^1 e^{-x^2}\ dx$. (In fact, these trapezoids seem to give a *great* approximation of the area!)

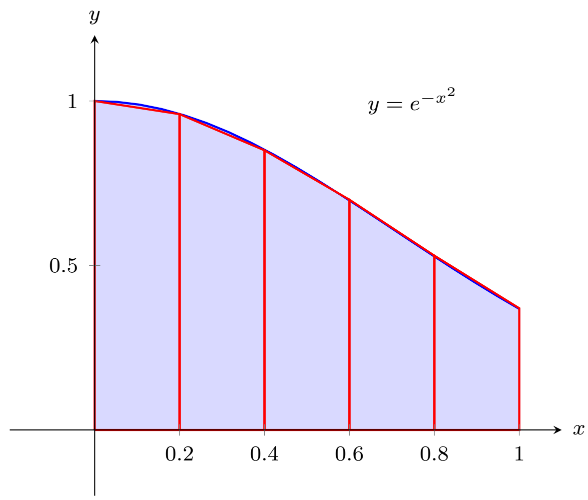
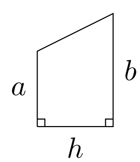


Figure 7.15: Approximating $\int_0^1 e^{-x^2}\,dx$ using 5 trapezoids of equal widths.

The formula for the area of a trapezoid is given in Figure 7.16.



$$\text{Area} = \frac{a+b}{2}h$$

Figure 7.16: The area of a trapezoid.

We approximate $\int_0^1 e^{-x^2}\,dx$ with these trapezoids in the following example.

Example 7.69: Approximating definite integrals using trapezoids

Use 5 trapezoids of equal width to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution. To compute the areas of the 5 trapezoids in Figure 7.15, it will again be useful to create a table of values as shown in Figure 7.17.

x_i	$e^{-x_i^2}$
0	1
0.2	0.961
0.4	0.852
0.6	0.698
0.8	0.527
1	0.368

Figure 7.17: A table of values of e^{-x^2}.

The leftmost trapezoid has legs of length 1 and 0.961 and a height of 0.2. Thus, by our formula, the area of the leftmost trapezoid is:

$$\frac{1+0.961}{2}(0.2) = 0.1961.$$

Moving right, the next trapezoid has legs of length 0.961 and 0.852 and a height of 0.2. Thus its area is:

$$\frac{0.961+0.852}{2}(0.2) = 0.1813.$$

The sum of the areas of all 5 trapezoids is:

$$\begin{aligned}\frac{1+0.961}{2}(0.2) + \frac{0.961+0.852}{2}(0.2) + \frac{0.852+0.698}{2}(0.2) &+ \\ \frac{0.698+0.527}{2}(0.2) + \frac{0.527+0.368}{2}(0.2) &= 0.7445.\end{aligned}$$

We approximate $\displaystyle\int_0^1 e^{-x^2}\,dx \approx 0.7445.$ ♣

There are many things to observe in this example. Note how each term in the final summation was multiplied by both 1/2 and by $\Delta x = 0.2$. We can factor these coefficients out, leaving a more concise summation as:

$$\frac{1}{2}(0.2)\Big[(1+0.961)+(0.961+0.852)+(0.852+0.698)+(0.698+0.527)+(0.527+0.368)\Big].$$

Now notice that all numbers except for the first and the last are added twice. Therefore we can write the summation even more concisely as

$$\frac{0.2}{2}\Big[1+2(0.961+0.852+0.698+0.527)+0.368\Big].$$

This is the heart of the **Trapezoidal Rule**, wherein a definite integral $\displaystyle\int_a^b f(x)\,dx$ is approximated by using trapezoids of equal widths to approximate the corresponding area under f. Using n equally spaced subintervals with endpoints $x_1, x_2, \ldots, x_{n+1}$, we again have $\Delta x = \dfrac{b-a}{n}$. Thus:

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \sum_{i=1}^{n} \frac{f(x_i)+f(x_{i+1})}{2}\Delta x \\ &= \frac{\Delta x}{2}\sum_{i=1}^{n}\big(f(x_i)+f(x_{i+1})\big) \\ &= \frac{\Delta x}{2}\Big[f(x_1) + 2\sum_{i=2}^{n} f(x_i) + f(x_{n+1})\Big].\end{aligned}$$

Example 7.70: Using the Trapezoidal Rule

Revisit Example 7.68 and approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ *using the Trapezoidal Rule and 10 equally spaced subintervals.*

Solution. We refer back to Figure 7.13 for the table of values of $\sin(x^3)$. Recall that $\Delta x = 3\pi/40 \approx 0.236$. Thus we have:

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx &\approx \frac{0.236}{2}\Big[-0.466 + 2\Big(-0.165 + (-0.031) + \ldots + 0.69\Big) + (-0.67)\Big] \\ &= 0.4275.\end{aligned}$$

♣

Notice how "quickly" the Trapezoidal Rule can be implemented once the table of values is created. This is true for all the methods explored in this section; the real work is creating a table of x_i and $f(x_i)$ values. Once this is completed, approximating the definite integral is not difficult. Again, using technology is wise. Spreadsheets can make quick work of these computations and make using lots of subintervals easy.

Also notice the approximations the Trapezoidal Rule gives. It is the average of the approximations given by the Left and Right Hand Rules! This effectively renders the Left and Right Hand Rules obsolete. They are useful when first learning about definite integrals, but if a real approximation is needed, one is generally better off using the Trapezoidal Rule instead of either the Left or Right Hand Rule.

How can we improve on the Trapezoidal Rule, apart from using more and more trapezoids? The answer is clear once we look back and consider what we have *really* done so far. The Left Hand Rule is not *really* about using rectangles to approximate area. Instead, it approximates a function f with constant functions on small subintervals and then computes the definite integral of these constant functions. The Trapezoidal Rule is really approximating a function f with a linear function on a small subinterval, then computes the definite integral of this linear function. In both of these cases the definite integrals are easy to compute in geometric terms.

So we have a progression: we start by approximating f with a constant function and then with a linear function. What is next? A quadratic function. By approximating the curve of a function with lots of parabolas, we generally get an even better approximation of the definite integral. We call this process **Simpson's Rule**, named after Thomas Simpson (1710-1761), even though others had used this rule as much as 100 years prior.

7.6.2. Simpson's Rule

Given one point, we can create a constant function that goes through that point. Given two points, we can create a linear function that goes through those points. Given three points, we can create a quadratic function that goes through those three points (given that no two have the same x–value).

Consider three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) whose x–values are equally spaced and $x_1 < x_2 < x_3$. Let f be the quadratic function that goes through these three points. It is not hard to show that

$$\int_{x_1}^{x_3} f(x)\ dx = \frac{x_3 - x_1}{6}\big(y_1 + 4y_2 + y_3\big). \tag{7.7}$$

Consider Figure 7.18. A function f goes through the 3 points shown and the parabola g that also goes through those points is graphed with a dashed line. Using our equation from above, we know exactly that

$$\int_1^3 g(x)\ dx = \frac{3-1}{6}\big(3 + 4(1) + 2\big) = 3.$$

Since g is a good approximation for f on $[1, 3]$, we can state that

$$\int_1^3 f(x)\ dx \approx 3.$$

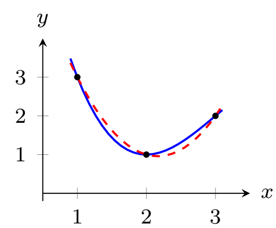


Figure 7.18: A graph of a function f and a parabola that approximates it well on $[1, 3]$.

Notice how the interval $[1, 3]$ was split into two subintervals as we needed 3 points. Because of this, whenever we use Simpson's Rule, we need to break the interval into an even number of subintervals.

In general, to approximate $\displaystyle\int_a^b f(x)\ dx$ using Simpson's Rule, subdivide $[a, b]$ into n subintervals, where n is even and each subinterval has width $\Delta x = (b - a)/n$. We approximate f with $n/2$ parabolic curves, using Equation (7.7) to compute the area under these parabolas. Adding up these areas gives the formula:

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big].$$

Note how the coefficients of the terms in the summation have the pattern $1, 4, 2, 4, 2, 4, \ldots, 2, 4, 1$.

Let's demonstrate Simpson's Rule with a concrete example.

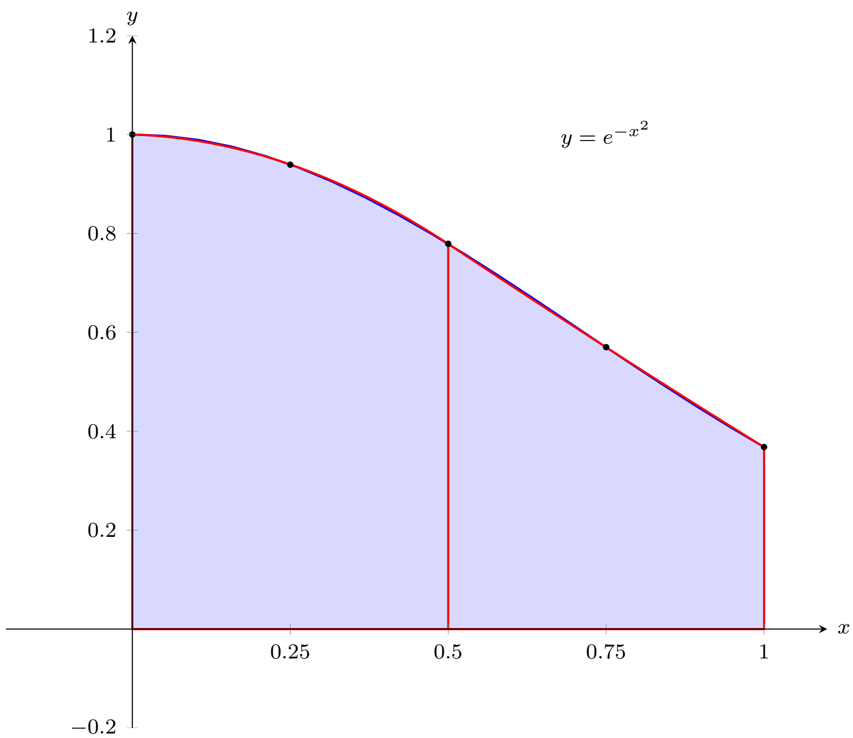
Example 7.71: Using Simpson's Rule

Approximate $\int_0^1 e^{-x^2}\, dx$ *using Simpson's Rule and 4 equally spaced subintervals.*

Solution. We begin by making a table of values as we have in the past, as shown in Figure 7.19(a).

x_i	$e^{-x_i^2}$
0	1
0.25	0.939
0.5	0.779
0.75	0.570
1	0.368

(a)



(b)

Figure 7.19: A table of values to approximate $\int_0^1 e^{-x^2}\, dx$, along with a graph of the function.

Simpson's Rule states that

$$\int_0^1 e^{-x^2}\, dx \approx \frac{0.25}{3}\Big[1 + 4(0.939) + 2(0.779) + 4(0.570) + 0.368\Big] = 0.74683\overline{3}.$$

Recall in Example 7.67 we stated that the correct answer, accurate to 4 places after the decimal, was 0.7468. Our approximation with Simpson's Rule, with 4 subintervals, is better than our approximation with the Trapezoidal Rule using 5!

Figure 7.19(b) shows $f(x) = e^{-x^2}$ along with its approximating parabolas, demonstrating how good our approximation is. The approximating curves are nearly indistinguishable from the actual function. ♣

Example 7.72: Using Simpson's Rule

Approximate $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ *using Simpson's Rule and* 10 *equally spaced intervals.*

Solution. Figure 7.20 shows the table of values that we used in the past for this problem, shown here again for convenience. Again, $\Delta x = (\pi/2 + \pi/4)/10 \approx 0.236$.

x_i	$\sin(x_i^3)$
-0.785	-0.466
-0.550	-0.165
-0.314	-0.031
-0.0785	0
0.157	0.004
0.393	0.061
0.628	0.246
0.864	0.601
1.10	0.971
1.34	0.690
1.57	-0.670

Figure 7.20: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ in Example 7.72.

Simpson's Rule states that

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx \approx \frac{0.236}{3}\Big[&(-0.466) + 4(-0.165) + 2(-0.031) + \dots \\ &\dots + 2(0.971) + 4(0.69) + (-0.67)\Big] \\ &= 0.4701\end{aligned}$$

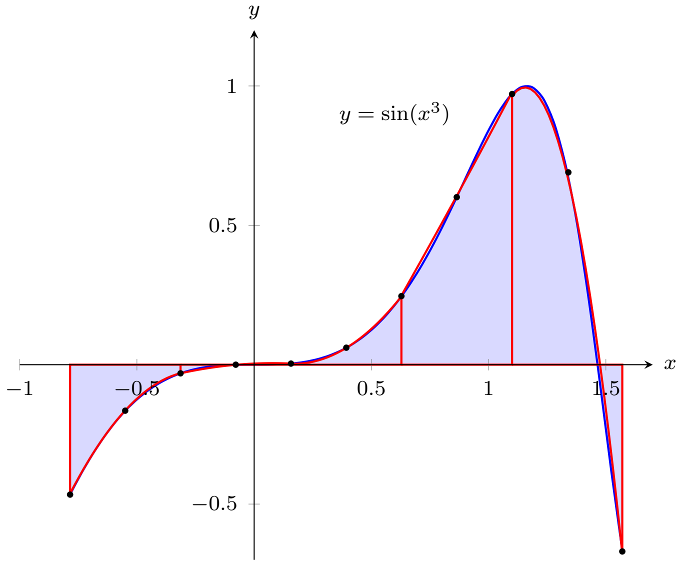


Figure 7.21: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 7.72 with Simpson's Rule and 10 equally spaced intervals.

Recall that the actual value, accurate to 3 decimal places, is 0.460. Our approximation is within one $1/100^{\text{th}}$ of the correct value. The graph in Figure 7.21 shows how closely the parabolas match the shape of the graph. ♣

7.6.3. Summary and Error Analysis

We summarize the key concepts of this section thus far in the following Key Idea.

Key Idea 7.6.0: Numerical Integration

Let f be a continuous function on $[a,b]$, let n be a positive integer, and let $\Delta x = \dfrac{b-a}{n}$.
Set $x_1 = a$, $x_2 = a + \Delta x$, ..., $x_i = a + (i-1)\Delta x$, $x_{n+1} = b$.

Consider $\displaystyle\int_a^b f(x)\,dx$.

Left Hand Rule:	$\displaystyle\int_a^b f(x)\,dx \approx \Delta x\Big[f(x_0) + f(x_1) + \ldots + f(x_{n-1})\Big]$.
Right Hand Rule:	$\displaystyle\int_a^b f(x)\,dx \approx \Delta x\Big[f(x_1) + f(x_2) + \ldots + f(x_n)\Big]$.
Trapezoidal Rule:	$\displaystyle\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\Big]$.
Simpson's Rule:	$\displaystyle\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_{n-1}) + f(x_n)\Big]$ (n even).

In our examples, we approximated the value of a definite integral using a given method then compared it to the “right” answer. This should have raised several questions in the reader's mind, such as:

1. How was the "right" answer computed?
2. If the right answer can be found, what is the point of approximating?
3. If there is value to approximating, how are we supposed to know if the approximation is any good?

These are good questions, and their answers are educational. In the examples, *the* right answer was never computed. Rather, an approximation accurate to a certain number of places after the decimal was given. In Example **??**, we do not know the *exact* answer, but we know it starts with 0.7468. These more accurate approximations were computed using numerical integration but with more precision (i.e., more subintervals and the help of a computer).

Since the exact answer cannot be found, approximation still has its place. How are we to tell if the approximation is any good?

"Trial and error" provides one way. Using technology, make an approximation with, say, 10, 100, and 200 subintervals. This likely will not take much time at all, and a trend should emerge. If a trend does not emerge, try using yet more subintervals. Keep in mind that trial and error is never foolproof; you might stumble upon a problem in which a trend will not emerge.

A second method is to use Error Analysis. While the details are beyond the scope of this text, there are some formulas that give *bounds* for how good your approximation will be. For instance, the formula might state that the approximation is within 0.1 of the correct answer. If the approximation is 1.58, then one knows that the correct answer is between 1.48 and 1.68. By using lots of subintervals, one can get an approximation as accurate as one likes. Theorem **??** states what these bounds are.

Theorem 7.7: Error Bounds in the Trapezoidal and Simpson's Rules

1. *Let E_T be the error in approximating $\displaystyle\int_a^b f(x)\,dx$ using the Trapezoidal Rule. If f has a continuous 2^{nd} derivative on $[a,b]$ and M is any upper bound of $\left|f''(x)\right|$ on $[a,b]$, then*

$$E_T \leq \frac{(b-a)^3}{12n^2}M.$$

2. *Let E_S be the error in approximating $\displaystyle\int_a^b f(x)\,dx$ using Simpson's Rule.*

   If f has a continuous 4^{th} derivative on $[a,b]$ and M is any upper bound of $\left|f^{(4)}\right|$ on $[a,b]$, then

$$E_S \leq \frac{(b-a)^5}{180n^4}M.$$

There are some key things to note about this theorem.

1. The larger the interval, the larger the error. This should make sense intuitively.

2. The error shrinks as more subintervals are used (i.e., as n gets larger).

3. The error in Simpson's Rule has a term relating to the 4^{th} derivative of f. Consider a cubic polynomial: it's 4^{th} derivative is 0. Therefore, the error in approximating the definite integral of a cubic polynomial with Simpson's Rule is 0 – Simpson's Rule computes the exact answer!

We revisit Examples 7.69 and 7.71 and compute the error bounds using Theorem 7.7 in the following example.

Example 7.73: Computing error bounds

Find the error bounds when approximating $\int_0^1 e^{-x^2}\,dx$ *using the Trapezoidal Rule and 5 subintervals, and using Simpson's Rule with 4 subintervals.*

Solution. **Trapezoidal Rule with $n = 5$:**

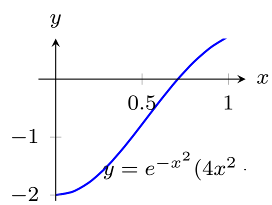


Figure 7.22: Graphing $f''(x)$ in Example ?? to help establish error bounds.

We start by computing the 2^{nd} derivative of $f(x) = e^{-x^2}$:

$$f''(x) = e^{-x^2}(4x^2 - 2).$$

Figure 7.22 shows a graph of $f''(x)$ on $[0, 1]$. It is clear that the largest value of f'', in absolute value, is 2. Thus we let $M = 2$ and apply the error formula from Theorem **??**.

$$E_T = \frac{(1-0)^3}{12 \cdot 5^2} \cdot 2 = 0.00\overline{6}.$$

Our error estimation formula states that our approximation of 0.7445 found in Example 7.69 is within 0.0067 of the correct answer, hence we know that

$$0.7445 - 0.0067 = .7378 \leq \int_0^1 e^{-x^2}\,dx \leq 0.7512 = 0.7445 + 0.0067.$$

We had earlier computed the exact answer, correct to 4 decimal places, to be 0.7468, affirming the validity of Theorem **??**.

Simpson's Rule with $n = 4$:

We start by computing the 4^{th} derivative of $f(x) = e^{-x^2}$:

$$f^{(4)}(x) = e^{-x^2}(16x^4 - 48x^2 + 12).$$

Figure 7.23 shows a graph of $f^{(4)}(x)$ on $[0,1]$. It is clear that the largest value of $f^{(4)}$, in absolute value, is 12. Thus we let $M = 12$ and apply the error formula from Theorem **??**.

$$E_s = \frac{(1-0)^5}{180 \cdot 4^4} \cdot 12 = 0.00026.$$

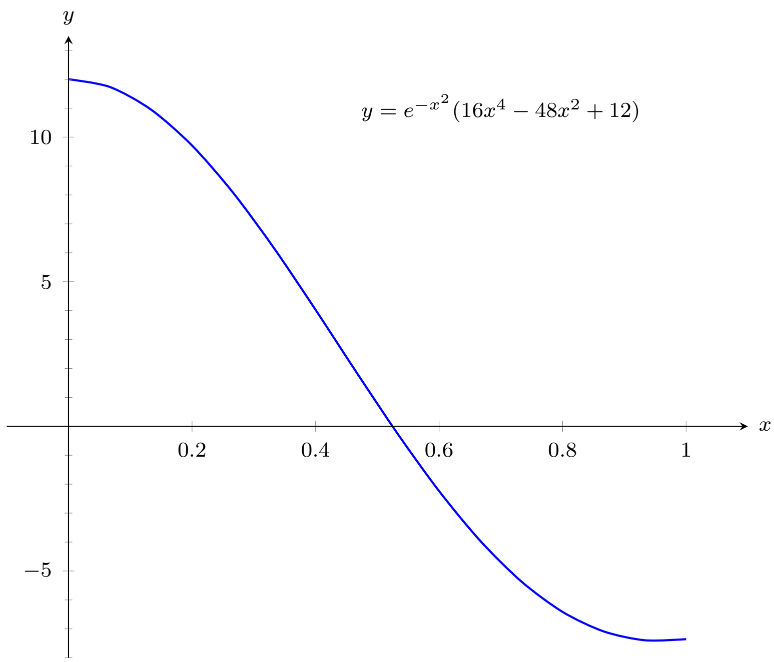


Figure 7.23: Graphing $f^{(4)}(x)$ in Example ?? to help establish error bounds.

Our error estimation formula states that our approximation of $0.7468\overline{3}$ found in Example 7.71 is within 0.00026 of the correct answer, hence we know that

$$0.74683 - 0.00026 = .74657 \le \int_0^1 e^{-x^2}\, dx \le 0.74709 = 0.74683 + 0.00026.$$

Once again we affirm the validity of Theorem **??**.

♣

At the beginning of this section we mentioned two main situations where numerical integration was desirable. We have considered the case where an antiderivative of the integrand cannot be computed. We now investigate the situation where the integrand is not known. This is, in fact, the most widely used application of Numerical Integration methods. “Most of the time” we observe behavior but do not know “the” function that describes it. We instead collect data about the behavior and make approximations based off of this data. We demonstrate this in an example.

Example 7.74: Approximating distance traveled

One of the authors drove his daughter home from school while she recorded their speed every 30 seconds. The data is given in Figure 7.24. Approximate the distance they travelled.

Solution. Recall that by integrating a speed function we get distance traveled. We have information about $v(t)$; we will use Simpson's Rule to approximate $\int_a^b v(t)\,dt$.

The most difficult aspect of this problem is converting the given data into the form we need it to be in. The speed is measured in miles per hour, whereas the time is measured in 30 second increments.

Time	Speed (mph)
0	0
1	25
2	22
3	19
4	39
5	0
6	43
7	59
8	54
9	51
10	43
11	35
12	40
13	43
14	30
15	0
16	0
17	28
18	40
19	42
20	40
21	39
22	40
23	23
24	0

Figure 7.24: Speed data collected at 30 second intervals for Example ??.

We need to compute $\Delta x = (b-a)/n$. Clearly, $n = 24$. What are a and b? Since we start at time $t = 0$, we have that $a = 0$. The final recorded time came after 24 periods of 30 seconds, which is 12 minutes or $1/5$ of an hour. Thus we have

$$\Delta x = \frac{b-a}{n} = \frac{1/5 - 0}{24} = \frac{1}{120}; \quad \frac{\Delta x}{3} = \frac{1}{360}.$$

Thus the distance traveled is approximately:

$$\begin{aligned}\int_0^{0.2} v(t)\ dt &\approx \frac{1}{360}\Big[f(x_1)+4f(x_2)+2f(x_3)+\cdots+4f(x_n)+f(x_{n+1})\Big]\\ &= \frac{1}{360}\Big[0+4\cdot 25+2\cdot 22+\cdots+2\cdot 40+4\cdot 23+0\Big]\\ &\approx 6.2167 \text{ miles.}\end{aligned}$$

We approximate the author drove 6.2 miles. (Because we are sure the reader wants to know, the author's odometer recorded the distance as about 6.05 miles.)

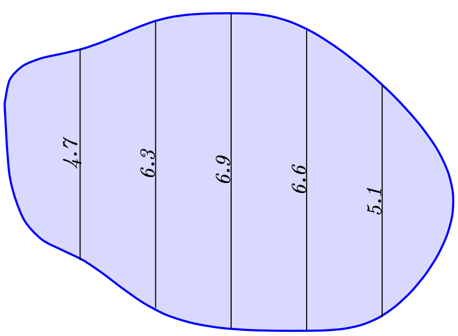
♣

Exercises for ??

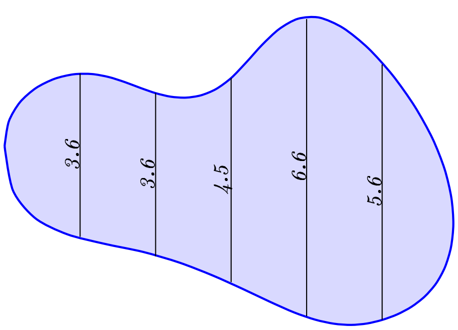
7.6.1 *For the region given, estimate the area using Simpson's Rule:*

(a) where the measurements are in centimetres, taken in 1 *cm increments, and*

(b) where the measurements are in hundreds of meters, taken in 100 *m increments.*



(a)



(b)

7.6.2 *For each definite integral:*

(a) Approximate the definite integral with the Trapezoidal Rule and $n = 4$.

(b) Approximate the definite integral with Simpson's Rule and $n = 4$.

(c) *Find the exact value of the integral.*

(a) $\displaystyle\int_{-1}^{1} x^2 \; dx$

(b) $\displaystyle\int_{0}^{10} 5x \; dx$

(c) $\displaystyle\int_{0}^{\pi} \sin x \; dx$

(d) $\displaystyle\int_{0}^{4} \sqrt{x} \; dx$

(e) $\displaystyle\int_{0}^{3} (x^3 + 2x^2 - 5x + 7) \; dx$

(f) $\displaystyle\int_{0}^{1} x^4 \; dx$

(g) $\displaystyle\int_{0}^{2\pi} \cos x \; dx$

(h) $\displaystyle\int_{-3}^{3} \sqrt{9 - x^2} \; dx$

7.6.3 *Approximate the definite integral with the Trapezoidal Rule and Simpson's Rule, with $n = 6$.*

(a) $\displaystyle\int_{0}^{1} \cos\left(x^2\right) \; dx$

(b) $\displaystyle\int_{-1}^{1} e^{x^2} \; dx$

(c) $\displaystyle\int_{0}^{5} \sqrt{x^2 + 1} \; dx$

(d) $\displaystyle\int_{0}^{\pi} x \sin x \; dx$

(e) $\displaystyle\int_{0}^{\pi/2} \sqrt{\cos x} \; dx$

(f) $\displaystyle\int_{1}^{4} \ln x \; dx$

(g) $\displaystyle\int_{-1}^{1} \frac{1}{\sin x + 2} \; dx$

(h) $\displaystyle\int_0^6 \frac{1}{\sin x + 2}\, dx$

7.6.4 *Find n such that the error in approximating the given definite integral is less than* 0.0001 *when using:*

(a) *the Trapezoidal Rule*

(b) *Simpson's Rule*

(a) $\displaystyle\int_0^\pi \sin x \ dx$

(b) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\, dx$

(c) $\displaystyle\int_0^\pi \cos\left(x^2\right)\, dx$

(d) $\displaystyle\int_0^5 x^4 \ dx$

In the following problems, compute the trapezoid and Simpson approximations using 4 subintervals, and compute the error bound for each. (Finding the maximum values of the second and fourth derivatives can be challenging for some of these; you may use a graphing calculator or computer software to estimate the maximum values.)

7.6.5 $\displaystyle\int_1^3 x\, dx$

7.6.6 $\displaystyle\int_0^3 x^2\, dx$

7.6.7 $\displaystyle\int_2^4 x^3\, dx$

7.6.8 $\displaystyle\int_1^3 \frac{1}{x}\, dx$

7.6.9 $\displaystyle\int_1^2 \frac{1}{1+x^2}\, dx$

7.6.10 $\displaystyle\int_0^1 x\sqrt{1+x}\, dx$

7.6.11 $\displaystyle\int_1^5 \frac{x}{1+x}\, dx$

7.6.12 $\int_0^1 \sqrt{x^3+1}\,dx$

7.6.13 $\int_0^1 \sqrt{x^4+1}\,dx$

7.6.14 $\int_1^4 \sqrt{1+1/x}\,dx$

7.6.15 *Using Simpson's rule on a parabola $f(x)$, even with just two subintervals, gives the exact value of the integral, because the parabolas used to approximate f will be f itself. Remarkably, Simpson's rule also computes the integral of a cubic function $f(x) = ax^3 + bx^2 + cx + d$ exactly. Show this is true by showing that*

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{x_2 - x_0}{3 \cdot 2}(f(x_0) + 4f((x_0 + x_2)/2) + f(x_2)).$$

This does require a bit of messy algebra, so you may prefer to use Sage.

7.7 Improper Integrals

We begin this section by considering the following definite integrals:

- $\displaystyle\int_0^{100} \frac{1}{1+x^2}\,dx \approx 1.5608,$
- $\displaystyle\int_0^{1000} \frac{1}{1+x^2}\,dx \approx 1.5698,$
- $\displaystyle\int_0^{10{,}000} \frac{1}{1+x^2}\,dx \approx 1.5707.$

Notice how the integrand is $1/(1+x^2)$ in each integral (which is sketched in Figure 7.25). As the upper bound gets larger, one would expect the "area under the curve" would also grow. While the definite integrals do increase in value as the upper bound grows, they are not increasing by much. In fact, consider:

$$\int_0^b \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^b = \tan^{-1} b - \tan^{-1} 0 = \tan^{-1} b.$$

As $b \to \infty$, $\tan^{-1} b \to \pi/2$. Therefore it seems that as the upper bound b grows, the value of the definite integral $\displaystyle\int_0^b \frac{1}{1+x^2}\,dx$ approaches $\pi/2 \approx 1.5708$. This should strike the reader as being a bit amazing: even though the curve extends "to infinity," it has a finite amount of area underneath it.

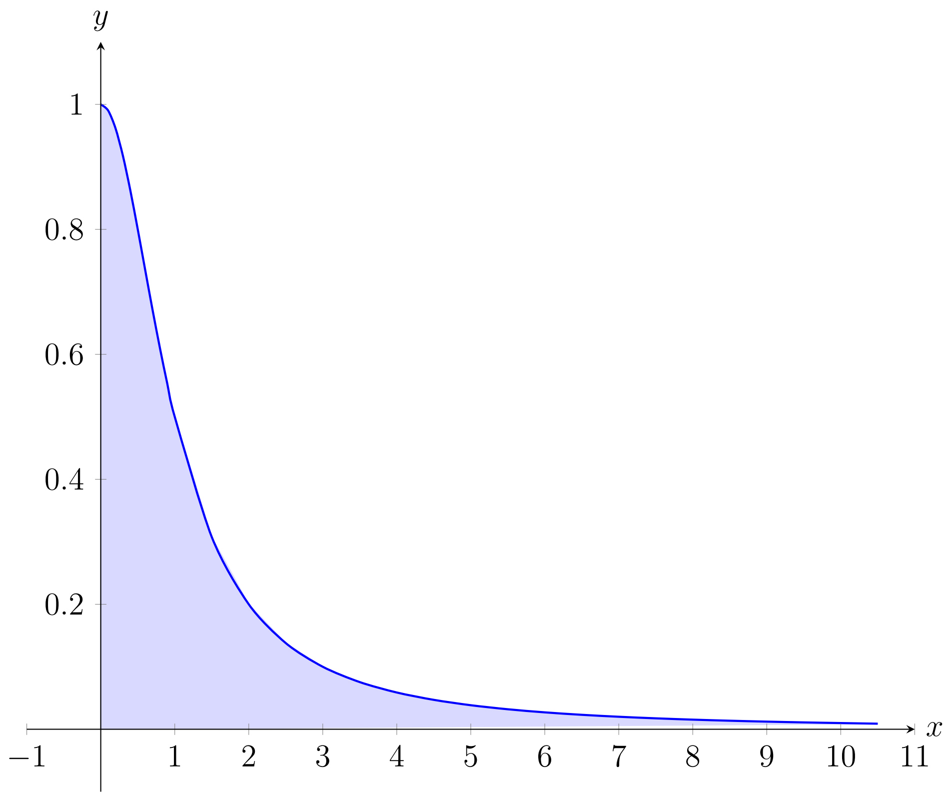


Figure 7.25: Graphing $f(x) = \dfrac{1}{1+x^2}$**.**

When we defined the definite integral $\displaystyle\int_a^b f(x)\,dx$, we made two stipulations:

1. The interval over which we integrated, $[a, b]$, was a finite interval, and
2. The function $f(x)$ was continuous on $[a, b]$ (ensuring that the range of f was finite).

In this section we consider integrals where one or both of the above conditions do not hold. Such integrals are called **improper integrals.**

Improper Integrals with Infinite Bounds

Definition 7.1: Improper Integrals with Infinite Bounds; Converge, Diverge

1. *Let f be a continuous function on $[a, \infty)$. Define*

$$\int_a^\infty f(x)\,dx \quad \text{to be} \quad \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. *Let f be a continuous function on $(-\infty, b]$. Define*

$$\int_{-\infty}^b f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. *Let f be a continuous function on $(-\infty, \infty)$. Let c be any real number; define*

$$\int_{-\infty}^\infty f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty} \int_a^c f(x)\,dx \;+\; \lim_{b\to\infty} \int_c^b f(x)\,dx.$$

An improper integral is said to ***converge*** *if its corresponding limit exists; otherwise, it* ***diverges****. The improper integral in part 3 converges if and only if both of its limits exist.*

Example 7.75: Evaluating improper integrals

Evaluate the following improper integrals.

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_{-\infty}^0 e^x\,dx$
4. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Solution.

1.
$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx &= \lim_{b\to\infty} \left.\frac{-1}{x}\right|_1^b \\ &= \lim_{b\to\infty} \frac{-1}{b} + 1 \\ &= 1.\end{aligned}$$

A graph of the area defined by this integral is given in Figure 7.26.

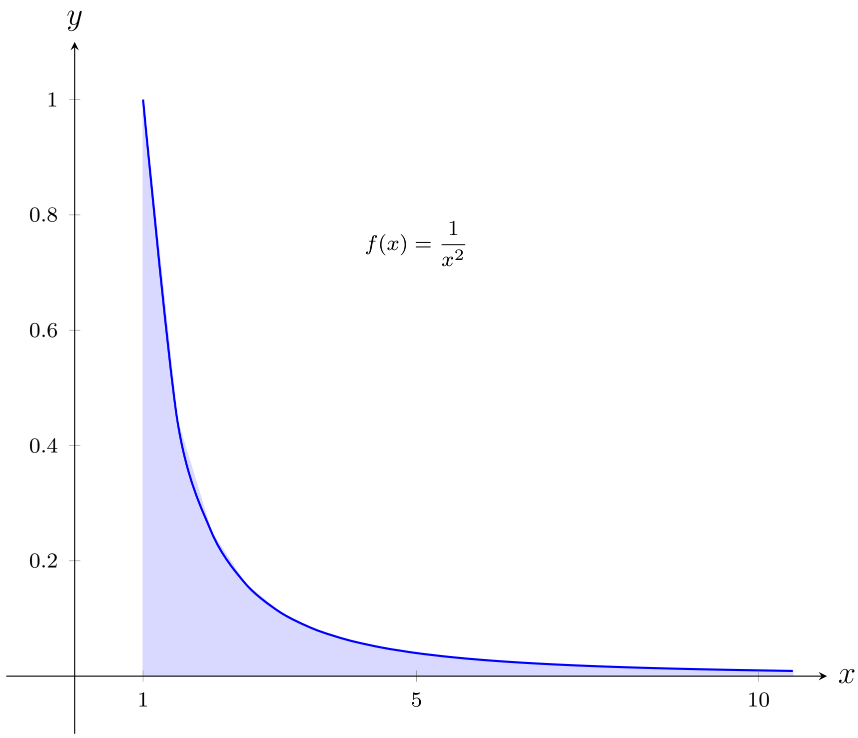


Figure 7.26: A graph of $f(x) = \frac{1}{x^2}$ in Example 7.75.

2.
$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx\\ &= \lim_{b\to\infty} \ln|x|\Big|_1^b\\ &= \lim_{b\to\infty} \ln(b)\\ &= \infty.\end{aligned}$$

The limit does not exist, hence the improper integral $\int_1^\infty \frac{1}{x}\,dx$ diverges. Compare the graphs in Figures 7.26 and 7.27; notice how the graph of $f(x) = 1/x$ is noticeably larger. This difference is enough to cause the improper integral to diverge.

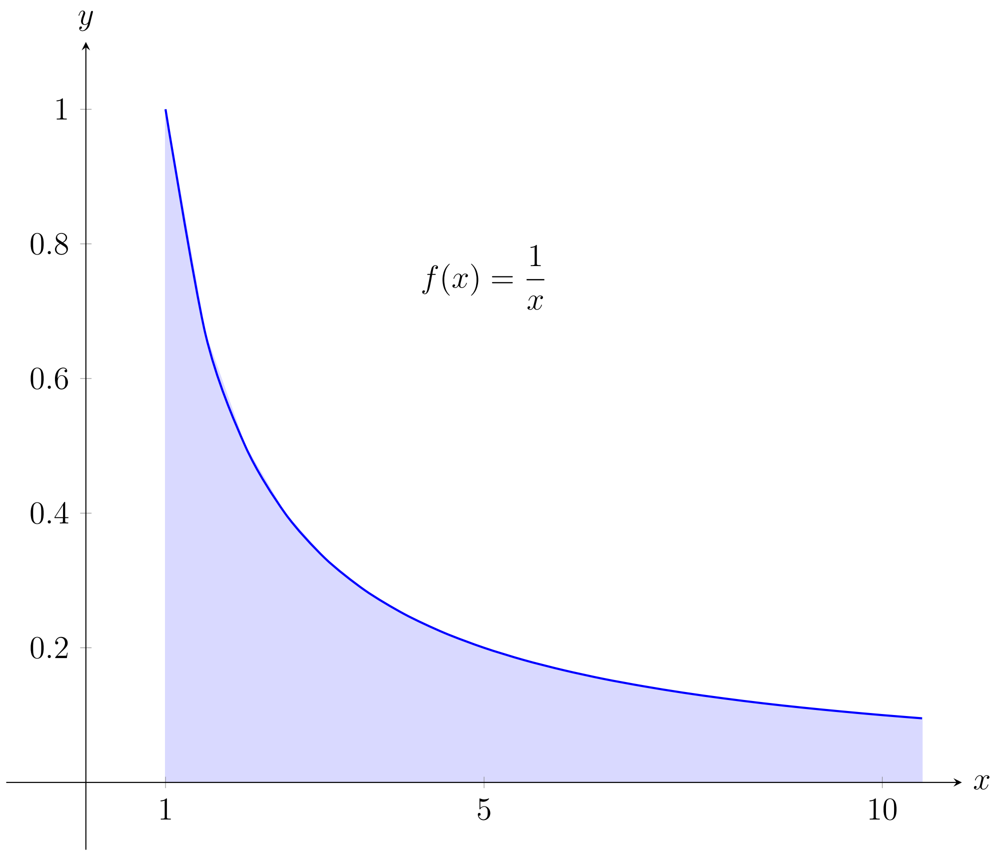


Figure 7.27: A graph of $f(x) = \frac{1}{x}$ in Example 7.75.

3.
$$\begin{aligned}\int_{-\infty}^{0} e^x \, dx &= \lim_{a\to-\infty} \int_a^0 e^x \, dx \\ &= \lim_{a\to-\infty} e^x\Big|_a^0 \\ &= \lim_{a\to-\infty} e^0 - e^a \\ &= 1.\end{aligned}$$

A graph of the area defined by this integral is given in Figure 7.28.

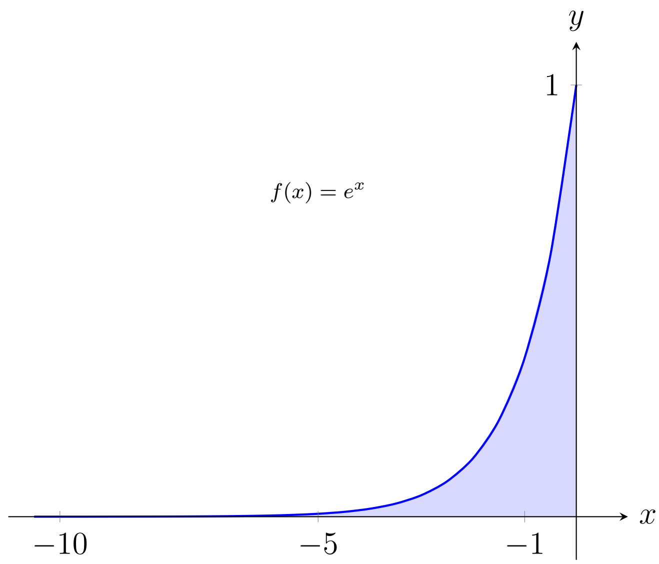


Figure 7.28: A graph of $f(x) = e^x$ in Example 7.75.

4. We will need to break this into two improper integrals and choose a value of c as in part 3 of Definition **??**. Any value of c is fine; we choose $c = 0$.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx\\ &= \lim_{a\to-\infty} \tan^{-1} x\Big|_a^0 + \lim_{b\to\infty} \tan^{-1} x\Big|_0^b\\ &= \lim_{a\to-\infty}\left(\tan^{-1} 0 - \tan^{-1} a\right) + \lim_{b\to\infty}\left(\tan^{-1} b - \tan^{-1} 0\right)\\ &= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right).\end{aligned}$$

Each limit exists, hence the original integral converges and has value:

$$= \pi.$$

A graph of the area defined by this integral is given in Figure 7.29.

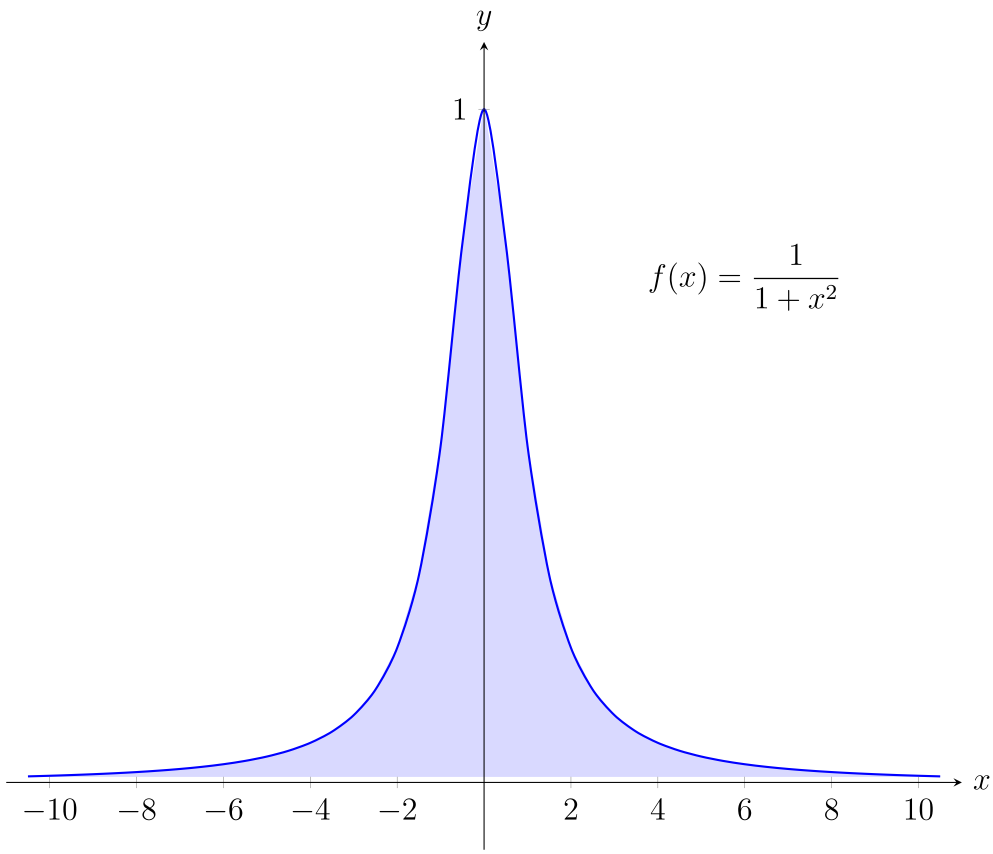


Figure 7.29: A graph of $f(x) = \frac{1}{1+x^2}$ in Example 7.75.

♣

The previous section introduced l'Hôpital's Rule, a method of evaluating limits that return indeterminate forms. It is not uncommon for the limits resulting from improper integrals to need this rule as demonstrated next.

Example 7.76: Improper integration and l'Hôpital's Rule

Evaluate the improper integral $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$.

Solution. This integral will require the use of Integration by Parts. Let $u = \ln x$ and $dv = 1/x^2\,dx$. Then

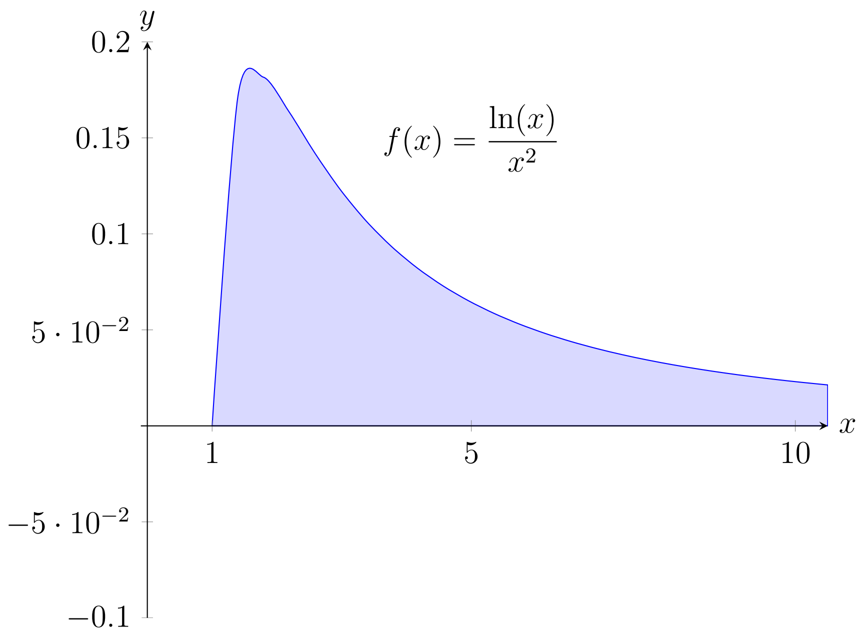


Figure 7.30: A graph of $f(x) = \frac{\ln x}{x^2}$ in Example 7.76.

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^2}\ dx &= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\ dx \\
&= \lim_{b\to\infty} \left(-\frac{\ln x}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\ dx \right) \\
&= \lim_{b\to\infty} \left(-\frac{\ln x}{x} - \frac{1}{x} \right)\Big|_1^b \\
&= \lim_{b\to\infty} \left(-\frac{\ln b}{b} - \frac{1}{b} - (-\ln 1 - 1) \right).
\end{aligned}$$

The $1/b$ and $\ln 1$ terms go to 0, leaving $\lim\limits_{b\to\infty} -\dfrac{\ln b}{b} + 1$. We need to evaluate $\lim\limits_{b\to\infty} \dfrac{\ln b}{b}$ with l'Hôpital's Rule. We have:

$$\begin{aligned}
\lim_{b\to\infty} \frac{\ln b}{b} &\overset{\text{by LHR}}{=} \lim_{b\to\infty} \frac{1/b}{1} \\
&= 0.
\end{aligned}$$

Thus the improper integral evaluates as:

$$\int_1^\infty \frac{\ln x}{x^2}\ dx = 1.$$

♣

Improper Integrals with Infinite Range

We have just considered definite integrals where the interval of integration was infinite. We now consider another type of improper integration, where the range of the integrand is infinite.

Definition 7.2: Improper Integration with Infinite Range

Let $f(x)$ be a continuous function on $[a,b]$ except at c, $a \leq c \leq b$, where $x = c$ is a vertical asymptote of f. Define

$$\int_a^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx.$$

Again, the integral converges if both limits exist and diverges otherwise.

Example 7.77: Improper integration of functions with infinite range

Evaluate the following improper integrals:

1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ 2. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx.$

Solution.

1. A graph of $f(x) = 1/\sqrt{x}$ is given in Figure 7.31. Notice that f has a vertical asymptote at $x = 0$; in some sense, we are trying to compute the area of a region that has no "top." Could this have a finite value?

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx\\ &= \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1\\ &= \lim_{a\to 0^+} 2\left(\sqrt{1}-\sqrt{a}\right)\\ &= 2.\end{aligned}$$

It turns out that the region does have a finite area even though it has no upper bound (strange things can occur in mathematics when considering the infinite).

Note: In Definition **??**, c can be one of the endpoints (a or b). In that case, there is only one limit to consider as part of the definition.

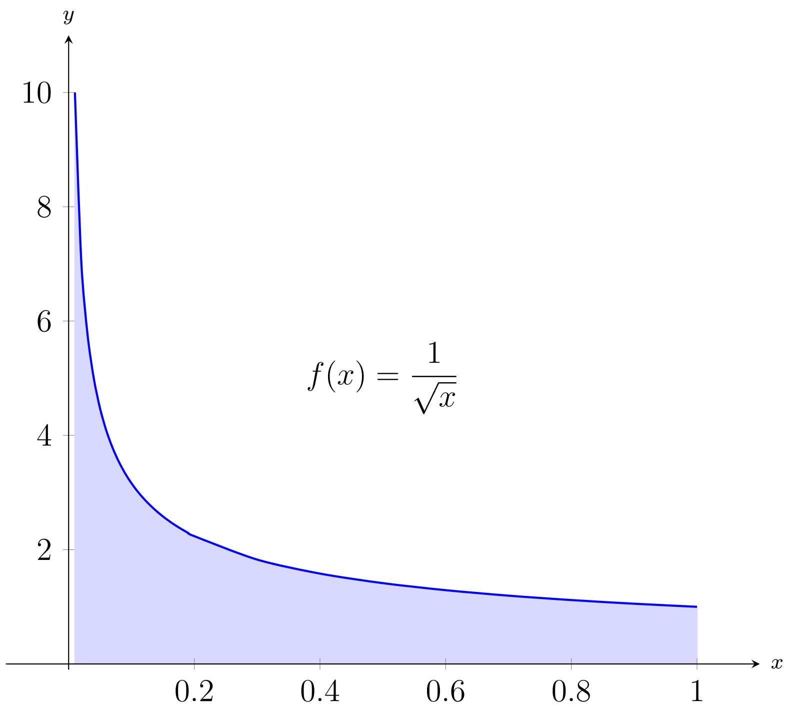


Figure 7.31: A graph of $f(x) = \frac{1}{\sqrt{x}}$ in Example 7.77.

2. The function $f(x) = 1/x^2$ has a vertical asymptote at $x = 0$, as shown in Figure 7.32, so this integral is an improper integral. Let's eschew using limits for a moment and proceed without recognizing the improper nature of the integral. This leads to:

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2}\, dx &= -\frac{1}{x}\Big|_{-1}^{1} \\ &= -1 - (1) \\ &= -2!\end{aligned}$$

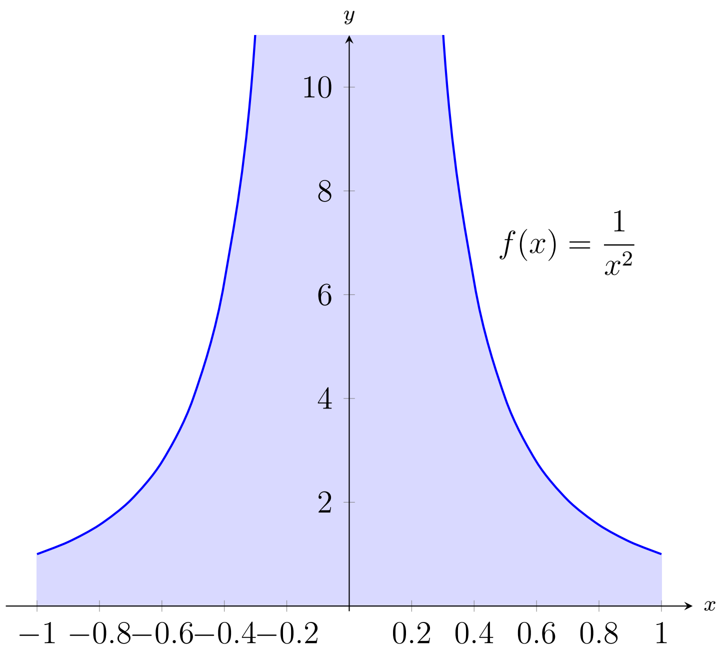


Figure 7.32: A graph of $f(x) = \frac{1}{x^2}$ in Example 7.77.

Clearly the area in question is above the x-axis, yet the area is supposedly negative! Why does our answer not match our intuition? To answer this, evaluate the integral using Definition **??**.

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2}\,dx &= \lim_{t\to 0^-}\int_{-1}^{t}\frac{1}{x^2}\,dx + \lim_{t\to 0^+}\int_{t}^{1}\frac{1}{x^2}\,dx\\ &= \lim_{t\to 0^-} -\frac{1}{x}\Big|_{-1}^{t} + \lim_{t\to 0^+} -\frac{1}{x}\Big|_{t}^{1}\\ &= \lim_{t\to 0^-} -\frac{1}{t} - 1 + \lim_{t\to 0^+} -1 + \frac{1}{t}\\ &\Rightarrow \Big(\infty - 1\Big) + \Big(-1 + \infty\Big).\end{aligned}$$

Neither limit converges hence the original improper integral diverges. The nonsensical answer we obtained by ignoring the improper nature of the integral is just that: nonsensical.

♣

Understanding Convergence and Divergence

Oftentimes we are interested in knowing simply whether or not an improper integral converges, and not necessarily the value of a convergent integral. We provide here several tools that help determine the convergence or divergence of improper integrals without integrating.

Our first tool is to understand the behavior of functions of the form $\dfrac{1}{x^p}$.

Example 7.78: Improper integration of $1/x^p$

Determine the values of p for which $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ *converges.*

Solution. We begin by integrating and then evaluating the limit.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx\\ &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \qquad (\text{assume } p\neq 1)\\ &= \lim_{b\to\infty}\frac{1}{-p+1}x^{-p+1}\Big|_1^b\\ &= \lim_{b\to\infty}\frac{1}{1-p}\big(b^{1-p}-1^{1-p}\big).\end{aligned}$$

When does this limit converge – i.e., when is this limit *not* ∞? This limit converges precisely when the power of b is less than 0: when $1-p<0 \Rightarrow 1<p$.

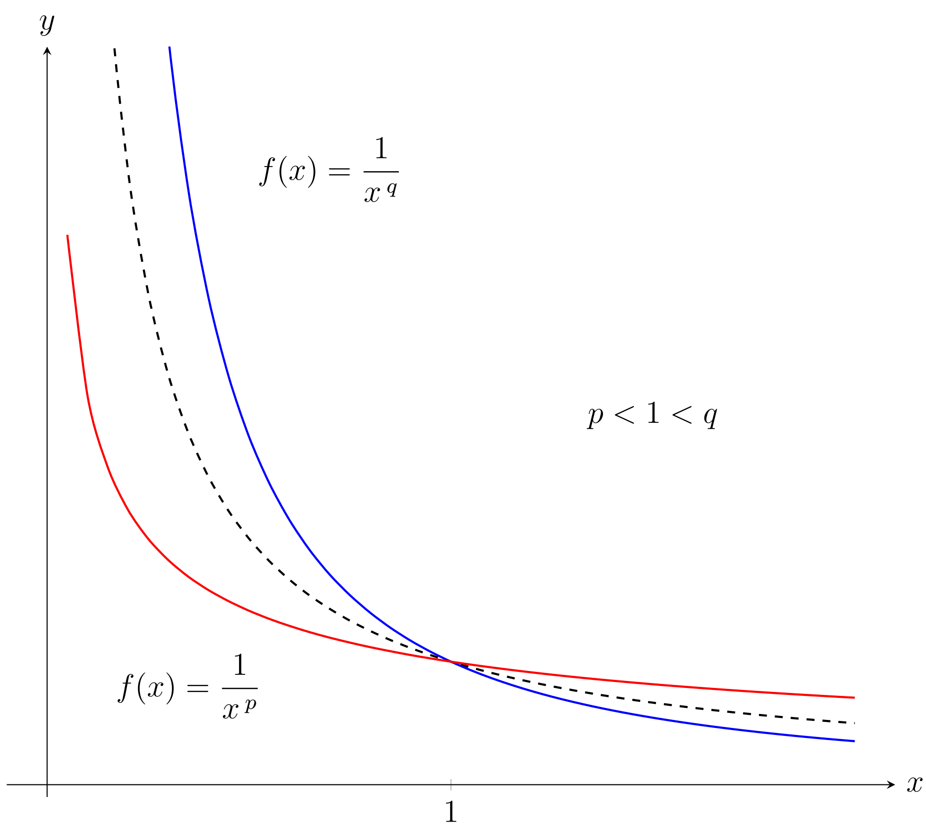


Figure 7.33: Plotting functions of the form $1/x^p$ in Example 7.78.

Our analysis shows that if $p>1$, then $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges. When $p<1$ the improper integral diverges; we showed in Example 7.75 that when $p=1$ the integral also diverges.

Figure 7.33 graphs $y = 1/x$ with a dashed line, along with graphs of $y = 1/x^p$, $p < 1$, and $y = 1/x^q$, $q > 1$. Somehow the dashed line forms a dividing line between convergence and divergence.

♣

The result of Example 7.78 provides an important tool in determining the convergence of other integrals. A similar result is proved in the exercises about improper integrals of the form $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$. These results are summarized in the following Theorem.

Theorem 7.8: Convergence of Improper Integrals $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ and $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$.

1. *The improper integral* $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ *converges when* $p > 1$ *and diverges when* $p \leq 1$.
2. *The improper integral* $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ *converges when* $p < 1$ *and diverges when* $p \geq 1$.

A basic technique in determining convergence of improper integrals is to compare an integrand whose convergence is unknown to an integrand whose convergence is known. We often use integrands of the form $1/x^p$ to compare to as their convergence on certain intervals is known. This is described in the following theorem.

Note: We used the upper and lower bound of "1" in Theorem 7.8 for convenience. It can be replaced by any a where $a > 0$.

Theorem 7.9: Direct Comparison Test for Improper Integrals

Let f *and* g *be continuous on* $[a, \infty)$ *where* $0 \leq f(x) \leq g(x)$ *for all* x *in* $[a, \infty)$.

1. *If* $\displaystyle\int_a^\infty g(x)\,dx$ *converges, then* $\displaystyle\int_a^\infty f(x)\,dx$ *converges.*
2. *If* $\displaystyle\int_a^\infty f(x)\,dx$ *diverges, then* $\displaystyle\int_a^\infty g(x)\,dx$ *diverges.*

Example 7.79: Determining convergence of improper integrals

Determine the convergence of the following improper integrals.

1. $\displaystyle\int_1^\infty e^{-x^2}\,dx$ 2. $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2 - x}}\,dx$

Solution.

1. The function $f(x) = e^{-x^2}$ does not have an antiderivative expressible in terms of elementary functions, so we cannot integrate directly. It is comparable to $g(x) = 1/x^2$, and as demon-

strated in Figure 7.34, $e^{-x^2} < 1/x^2$ on $[1,\infty)$. We know from Theorem 7.8 that $\int_1^\infty \frac{1}{x^2}\,dx$ converges, hence $\int_1^\infty e^{-x^2}\,dx$ also converges.

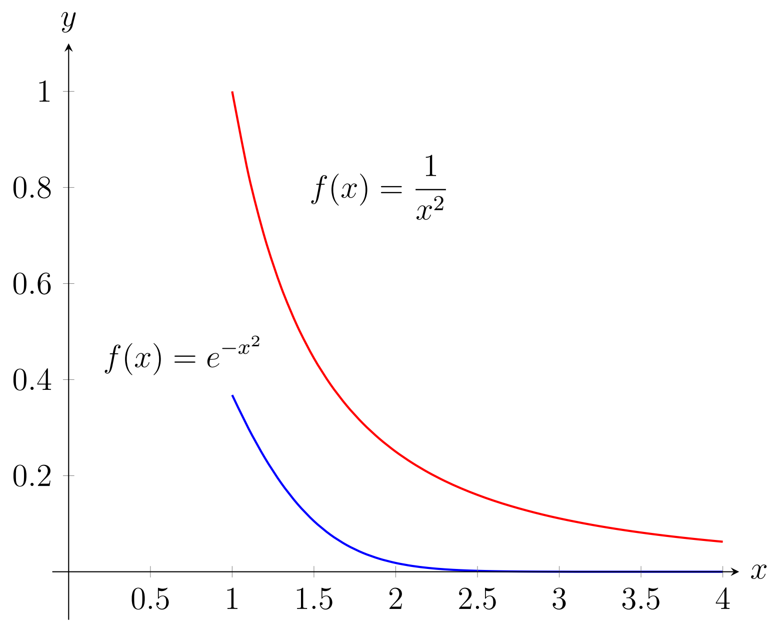


Figure 7.34: Graphs of $f(x) = e^{-x^2}$ and $f(x) = 1/x^2$ in Example 7.79.

2. Note that for large values of x, $\frac{1}{\sqrt{x^2-x}} \approx \frac{1}{\sqrt{x^2}} = \frac{1}{x}$. We know from Theorem 7.8 and the subsequent note that $\int_3^\infty \frac{1}{x}\,dx$ diverges, so we seek to compare the original integrand to $1/x$.

   It is easy to see that when $x > 0$, we have $x = \sqrt{x^2} > \sqrt{x^2-x}$. Taking reciprocals reverses the inequality, giving

$$\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}.$$

   Using Theorem 7.9, we conclude that since $\int_3^\infty \frac{1}{x}\,dx$ diverges, $\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$ diverges as well. Figure 7.35 illustrates this.

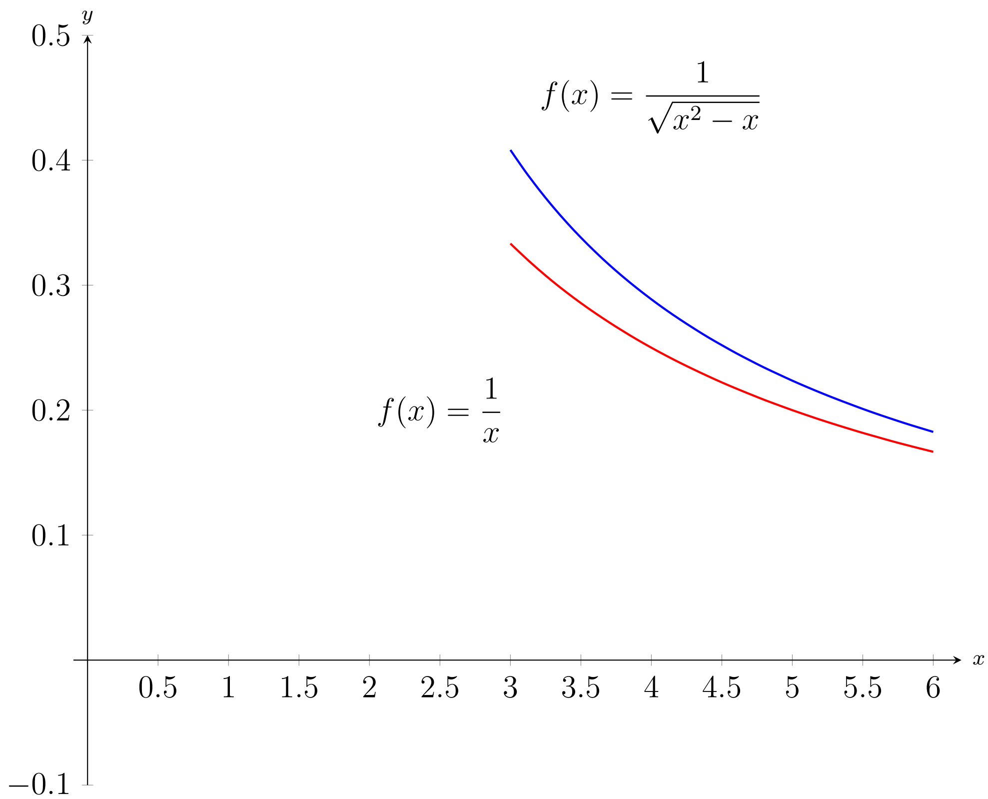


Figure 7.35: Graphs of $f(x) = 1/\sqrt{x^2 - x}$ and $f(x) = 1/x$ in Example 7.79.

♣

Being able to compare "unknown" integrals to "known" integrals is very useful in determining convergence. However, some of our examples were a little "too nice." For instance, it was convenient that $\dfrac{1}{x} < \dfrac{1}{\sqrt{x^2 - x}}$, but what if the "$-x$" were replaced with a "$+2x + 5$"? That is, what can we say about the convergence of $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2 + 2x + 5}}\, dx$? We have $\dfrac{1}{x} > \dfrac{1}{\sqrt{x^2 + 2x + 5}}$, so we cannot use Theorem 7.9.

In cases like this (and many more) it is useful to employ the following theorem.

Theorem 7.10: Limit Comparison Test for Improper Integrals

Let f and g be continuous functions on $[a, \infty)$ where $f(x) > 0$ and $g(x) > 0$ for all x. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\, dx \quad \text{and} \quad \int_a^\infty g(x)\, dx$$

either both converge or both diverge.

Example 7.80: Determining convergence of improper integrals

Determine the convergence of $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$.

Solution. As x gets large, the quadratic inside the square root function will begin to behave much like $y = x$. So we compare $\dfrac{1}{\sqrt{x^2+2x+5}}$ to $\dfrac{1}{x}$ with the Limit Comparison Test:

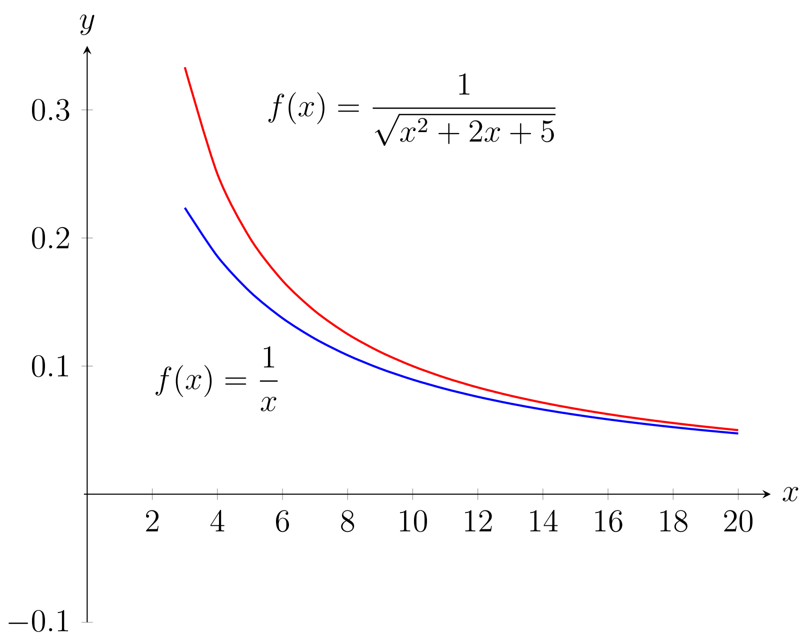
$$\lim_{x\to\infty} \frac{1/\sqrt{x^2+2x+5}}{1/x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+2x+5}}.$$

The immediate evaluation of this limit returns ∞/∞, an indeterminate form. Using l'Hôpital's Rule seems appropriate, but in this situation, it does not lead to useful results. (We encourage the reader to employ l'Hôpital's Rule at least once to verify this.)

The trouble is the square root function. To get rid of it, we employ the following fact: If $\lim\limits_{x\to c} f(x) = L$, then $\lim\limits_{x\to c} f(x)^2 = L^2$. (This is true when either c or L is ∞.) So we consider now the limit

$$\lim_{x\to\infty} \frac{x^2}{x^2+2x+5}.$$

This converges to 1, meaning the original limit also converged to 1. As x gets very large, the function $\dfrac{1}{\sqrt{x^2+2x+5}}$ looks very much like $\dfrac{1}{x}$. Since we know that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, by the Limit Comparison Test we know that $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$ also diverges. Figure 7.36 graphs $f(x) = 1/\sqrt{x^2+2x+5}$ and $f(x) = 1/x$, illustrating that as x gets large, the functions become indistinguishable.



♣

Figure 7.36: Graphing $f(x) = \frac{1}{\sqrt{x^2+2x+5}}$ and $f(x) = \frac{1}{x}$ in Example 7.80.

Both the Direct and Limit Comparison Tests were given in terms of integrals over an infinite interval. There are versions that apply to improper integrals with an infinite range, but as they are a bit wordy and a little more difficult to employ, they are omitted from this text.

This chapter has explored many integration techniques. We learned Substitution, which "undoes" the Chain Rule of differentiation, as well as Integration by Parts, which "undoes" the Product Rule. We also learned specialized techniques for handling trigonometric functions All techniques effectively have this goal in common: rewrite the integrand in a new way so that the integration step is easier to see and implement.

As stated before, integration is, in general, hard. It is easy to write a function whose antiderivative is impossible to write in terms of elementary functions, and even when a function does have an antiderivative expressible by elementary functions, it may be really hard to discover what it is. The powerful computer algebra system *Mathematica*® has approximately 1,000 pages of code dedicated to integration.

Do not let this difficulty discourage you. There is great value in learning integration techniques, as they allow one to manipulate an integral in ways that can illuminate a concept for greater understanding. There is also great value in understanding the need for good numerical techniques: the Trapezoidal and Simpson's Rules are just the beginning of powerful techniques for approximating the value of integration.

Chapter 8 stresses the uses of integration. We generally do not find antiderivatives for antiderivative's sake, but rather because they provide the solution to some type of problem. The following chapter introduces us to a number of different problems whose solution is provided by integration.

Exercises for Section 7.7

7.7.1 *Determine whether* $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ *is convergent or divergent.*

7.7.2 *Determine whether* $\displaystyle\int_e^\infty \frac{1}{x\sqrt{\ln x}}\,dx$ *is convergent or divergent.*

7.7.3 *Evaluate the improper integral* $\displaystyle\int_0^\infty e^{-3x}\,dx$*.*

7.7.4 *Determine if* $\displaystyle\int_1^e \frac{1}{x(\ln x)^2}\,dx$ *is convergent or divergent. Evaluate it if it is convergent.*

7.7.5 *Show that* $\displaystyle\int_0^\infty e^{-x}\sin^2\left(\frac{\pi x}{2}\right)dx$ *converges.*

7.7.6 *Evaluate* $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2+1}\,dx$ *and* $\displaystyle\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$*.*

7.7.7 *Determine whether the following improper integrals are convergent or divergent. Evaluate those that are convergent.*

(a) $\int_0^\infty \dfrac{1}{x^2+1}\,dx$

(b) $\int_0^\infty \dfrac{x}{x^2+1}\,dx$

(c) $\int_0^\infty e^{-x}(\cos x + \sin x)\,dx$. *[Hint: What is the derivative of $-e^{-x}\cos x$?]*

(d) $\int_0^{\pi/2} \sec^2 x\,dx$

(e) $\int_0^4 \dfrac{1}{(4-x)^{2/5}}\,dx$

7.7.8 *Prove that the integral* $\int_1^\infty \dfrac{1}{x^p}\,dx$ *is convergent if* $p > 1$ *and divergent if* $0 < p \leq 1$.

7.7.9 *Suppose that* $p > 0$. *Find all values of* p *for which* $\int_0^1 \dfrac{1}{x^p}\,dx$ *converges.*

7.7.10 *Show that* $\int_1^\infty \dfrac{\sin^2 x}{x(\sqrt{x}+1)}\,dx$ *converges.*

7.8 Hyperbolic Functions Revisited

In Section 2.7 we introduced the hyperbolic functions. From Definition 2.2 we recall their respective definitions.

1. $\cosh x = \dfrac{e^x + e^{-x}}{2}$
2. $\sinh x = \dfrac{e^x - e^{-x}}{2}$
3. $\tanh x = \dfrac{\sinh x}{\cosh x}$
4. $\text{sech} x = \dfrac{1}{\cosh x}$
5. $\text{csch} x = \dfrac{1}{\sinh x}$
6. $\coth x = \dfrac{\cosh x}{\sinh x}$

Example 7.81: Exploring properties of hyperbolic functions

Use the definitions of the hyperbolic functions to rewrite the following expressions.

1. $\frac{d}{dx}\big(\cosh x\big)$
2. $\frac{d}{dx}\big(\sinh x\big)$
3. $\frac{d}{dx}\big(\tanh x\big)$

Solution.

1. $$\begin{aligned}\frac{d}{dx}\big(\cosh x\big) &= \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right)\\ &= \frac{e^x-e^{-x}}{2}\\ &= \sinh x.\end{aligned}$$

   So $\frac{d}{dx}\big(\cosh x\big) = \sinh x$.

2. $$\begin{aligned}\frac{d}{dx}\big(\sinh x\big) &= \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right)\\ &= \frac{e^x+e^{-x}}{2}\\ &= \cosh x.\end{aligned}$$

   So $\frac{d}{dx}\big(\sinh x\big) = \cosh x$.

3. $$\begin{aligned}\frac{d}{dx}\big(\tanh x\big) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)\\ &= \frac{\cosh x\cosh x-\sinh x\sinh x}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x}\\ &= \operatorname{sech}^2 x.\end{aligned}$$

   So $\frac{d}{dx}\big(\tanh x\big) = \operatorname{sech}^2 x$.

♣

The following Key Idea summarizes many of the important identities relating to hyperbolic functions. Each can be verified by referring back to Definition 2.2.

Key Idea 7.8.0: Useful Hyperbolic Function Properties

Basic Identities

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\tanh^2 x + \text{sech}^2 x = 1$
3. $\coth^2 x - \text{csch}^2 x = 1$
4. $\cosh 2x = \cosh^2 x + \sinh^2 x$
5. $\sinh 2x = 2\sinh x \cosh x$
6. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
7. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Derivatives

1. $\frac{d}{dx}\big(\cosh x\big) = \sinh x$
2. $\frac{d}{dx}\big(\sinh x\big) = \cosh x$
3. $\frac{d}{dx}\big(\tanh x\big) = \text{sech}^2 x$
4. $\frac{d}{dx}\big(\text{sech} x\big) = -\text{sech} x \tanh x$
5. $\frac{d}{dx}\big(\text{csch} x\big) = -\text{csch} x \coth x$
6. $\frac{d}{dx}\big(\coth x\big) = -\text{csch}^2 x$

Integrals

1. $\displaystyle\int \cosh x\ dx = \sinh x + C$
2. $\displaystyle\int \sinh x\ dx = \cosh x + C$
3. $\displaystyle\int \tanh x\ dx = \ln(\cosh x) + C$
4. $\displaystyle\int \coth x\ dx = \ln|\sinh x| + C$

Next, we practice using these properties.

Example 7.82: Derivatives and integrals of hyperbolic functions

Evaluate the following derivatives and integrals.

1. $\dfrac{d}{dx}\big(\cosh 2x\big)$
2. $\displaystyle\int \text{sech}^2(7t-3)\ dt$
3. $\displaystyle\int_0^{\ln 2} \cosh x\ dx$

Solution.

1. Using the Chain Rule directly, we have $\frac{d}{dx}\big(\cosh 2x\big) = 2\sinh 2x$.

   Just to demonstrate that it works, let's also use the Basic Identity found in Key Idea 7.8: $\cosh 2x = \cosh^2 x + \sinh^2 x$.

$$\begin{aligned}\frac{d}{dx}\big(\cosh 2x\big) = \frac{d}{dx}\big(\cosh^2 x + \sinh^2 x\big) &= 2\cosh x \sinh x + 2\sinh x \cosh x \\ &= 4\cosh x \sinh x.\end{aligned}$$

   Using another Basic Identity, we can see that $4\cosh x \sinh x = 2\sinh 2x$. We get the same answer either way.

2. We employ substitution, with $u = 7t-3$ and $du = 7dt$ to get:

$$\int \text{sech}^2(7t-3)\ dt = \int \frac{1}{7}\text{sech}^2(u)\ du = \frac{1}{7}\tanh(u) + C = \frac{1}{7}\tanh(7t-3) + C.$$

3.

$$\int_0^{\ln 2} \cosh x\ dx = \sinh x\Big|_0^{\ln 2} = \sinh(\ln 2) - \sinh 0 = \sinh(\ln 2).$$

We can simplify this last expression as $\sinh x$ is based on exponentials:

$$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - 1/2}{2} = \frac{3}{4}.$$

♣

Inverse Hyperbolic Functions

Just as the inverse trigonometric functions are useful in certain integrations, the inverse hyperbolic functions are useful with others. Figure 7.39 shows the restrictions on the domains to make each function one-to-one and the resulting domains and ranges of their inverse functions. Their graphs are shown in Figure 7.40.

Because the hyperbolic functions are defined in terms of exponential functions, their inverses can be expressed in terms of logarithms as shown in Definition 7.3. It is often more convenient to refer to $\sinh^{-1} x$ than to $\ln\left(x + \sqrt{x^2+1}\right)$, especially when one is working on theory and does not need to compute actual values. On the other hand, when computations are needed, technology is often helpful but many hand-held calculators lack a *convenient* $\sinh^{-1} x$ button. (Often it can be accessed under a menu system, but not conveniently.) In such a situation, the logarithmic representation is useful. The reader is not encouraged to memorize these, but rather know they exist and know how to use them when needed.

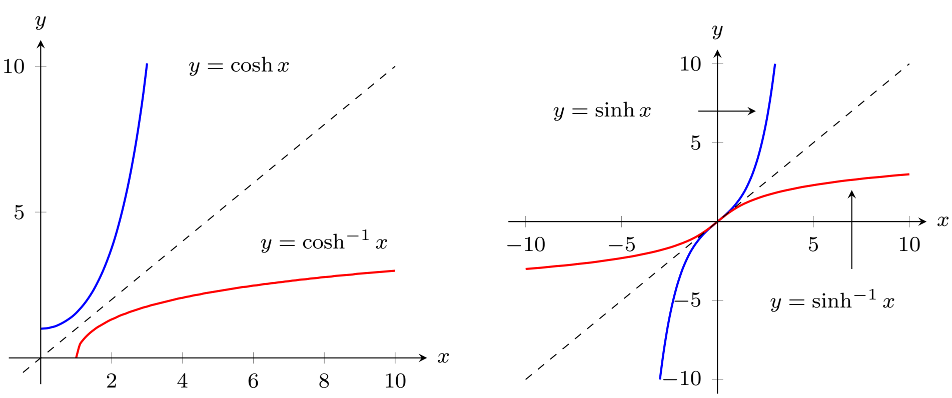


Figure 7.37: Graphs of $\cosh x$, $\sinh x$ and their inverses.

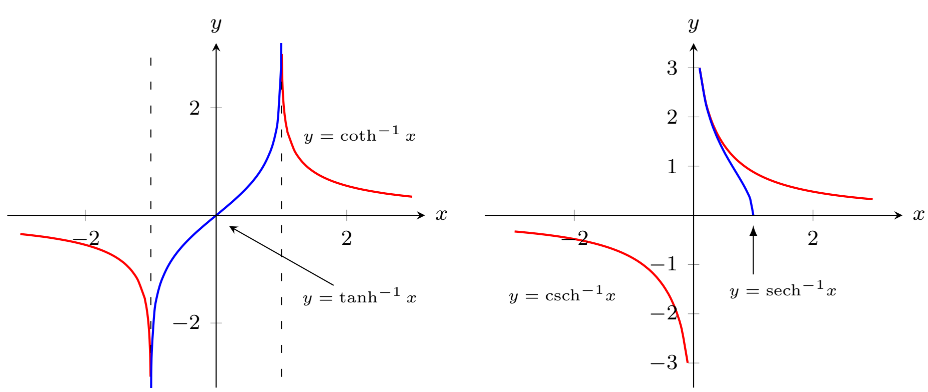


Figure 7.38: Graphs of $\tanh^{-1} x$, $\coth^{-1} x$, $\operatorname{sech}^{-1} x$ and $\operatorname{csch}^{-1} x$.

Function	Domain	Range
$\cosh x$	$[0, \infty)$	$[1, \infty)$
$\sinh x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\tanh x$	$(-\infty, \infty)$	$(-1, 1)$
$\text{sech} x$	$[0, \infty)$	$(0, 1]$
$\text{csch} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\coth x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$

Function	Domain	Range
$\cosh^{-1} x$	$[1, \infty)$	$[0, \infty)$
$\sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$
$\text{sech}^{-1} x$	$(0, 1]$	$[0, \infty)$
$\text{csch}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Figure 7.39: Domains and ranges of the hyperbolic and inverse hyperbolic functions.

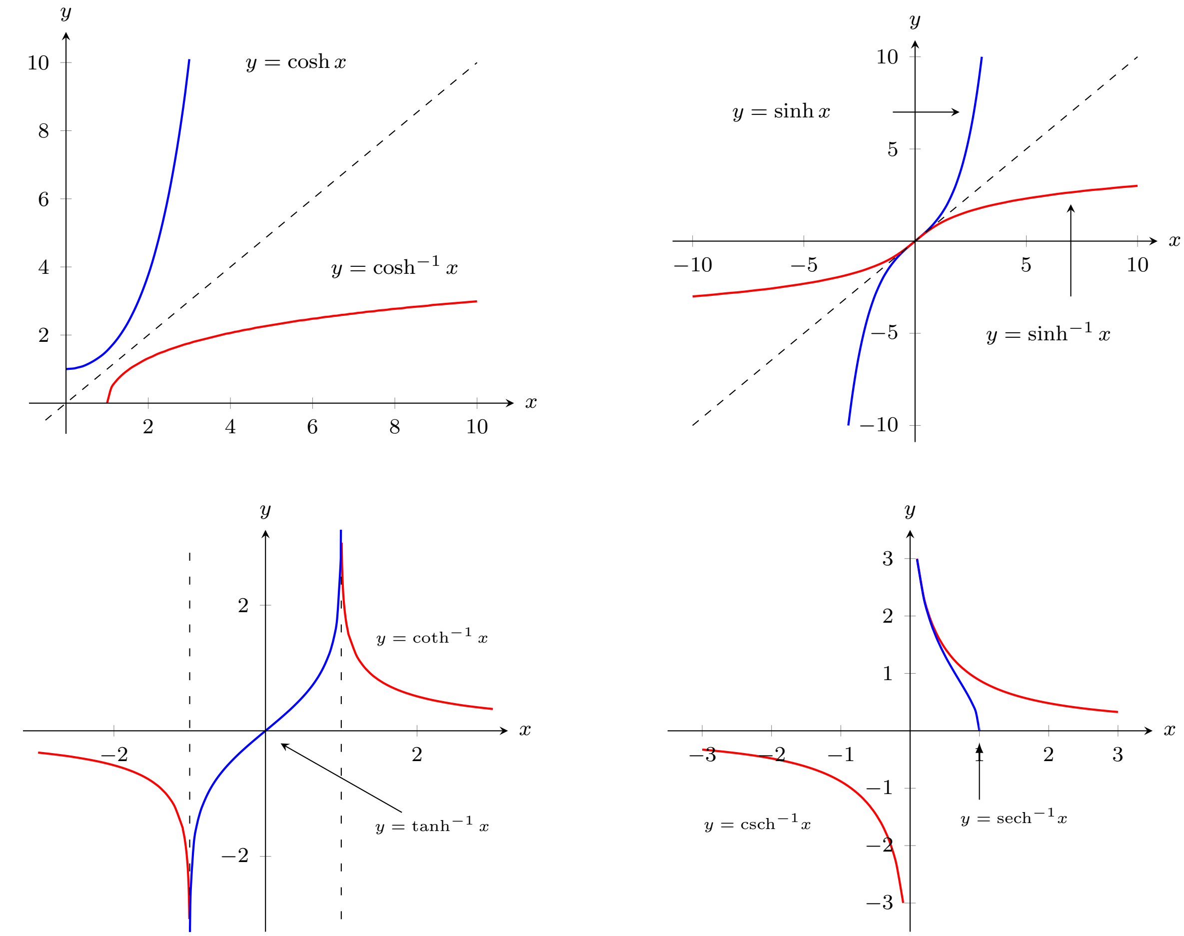


Figure 7.40: Graphs of the hyperbolic functions and their inverses.

Definition 7.3: Logarithmic definitions of Inverse Hyperbolic Functions

1. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right); \; x \geq 1$
2. $\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1 + x}{1 - x}\right); \; |x| < 1$
3. $\text{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right); \; 0 < x \leq 1$
4. $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$
5. $\coth^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{x + 1}{x - 1}\right); \; |x| > 1$
6. $\text{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1 + x^2}}{|x|}\right); \; x \neq 0$

The following gives the derivatives and integrals relating to the inverse hyperbolic functions. Both the inverse hyperbolic and logarithmic function representations of the antiderivative are given, based on Definition 7.3. Again, these latter functions are often more useful than the former. Note how inverse hyperbolic functions can be used to solve integrals we used Trigonometric Substitution to solve in Section **??**.

Key Idea 7.8.0: Derivatives Involving Inverse Hyperbolic Functions

1. $\dfrac{d}{dx}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{x^2-1}};\ x > 1$
2. $\dfrac{d}{dx}\left(\sinh^{-1} x\right) = \dfrac{1}{\sqrt{x^2+1}}$
3. $\dfrac{d}{dx}\left(\tanh^{-1} x\right) = \dfrac{1}{1-x^2};\ |x| < 1$
4. $\dfrac{d}{dx}\left(\text{sech}^{-1} x\right) = \dfrac{-1}{x\sqrt{1-x^2}}; 0 < x < 1$
5. $\dfrac{d}{dx}\left(\text{csch}^{-1} x\right) = \dfrac{-1}{|x|\sqrt{1+x^2}};\ x \neq 0$
6. $\dfrac{d}{dx}\left(\coth^{-1} x\right) = \dfrac{1}{1-x^2};\ |x| > 1$

Key Idea 7.8.0: Integrals Involving Inverse Hyperbolic Functions

1. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C;\ 0 < a < x = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
2. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C;\ a > 0 = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
3. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C & x^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C & a^2 < x^2 \end{cases} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$
4. $\displaystyle\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\text{sech}^{-1}\left(\frac{x}{a}\right) + C;\ 0 < x < a = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
5. $\displaystyle\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}\text{csch}^{-1}\left|\frac{x}{a}\right| + C;\ x \neq 0,\ a > 0 = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right| + C$

We practice using the derivative and integral formulas in the following example.

Example 7.83: Derivatives and integrals involving inverse hyperbolic functions

Evaluate the following.

1. $\dfrac{d}{dx}\left[\cosh^{-1}\left(\dfrac{3x-2}{5}\right)\right]$

2. $\displaystyle\int \frac{1}{x^2-1}\,dx$

3. $\displaystyle\int \frac{1}{\sqrt{9x^2+10}}\,dx$

Solution.

1. Applying the Chain Rule gives:
$$\frac{d}{dx}\left[\cosh^{-1}\left(\frac{3x-2}{5}\right)\right] = \frac{1}{\sqrt{\left(\frac{3x-2}{5}\right)^2-1}}\cdot\frac{3}{5}.$$

2. Multiplying the numerator and denominator by (-1) gives: $\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{-1}{1-x^2}\,dx$. The second integral can be solved with a direct application of item #3 from Integrals Involving Inverse Hyperbolic Functions, with $a=1$. Thus
$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= -\int \frac{1}{1-x^2}\,dx \\ &= \begin{cases} -\tanh^{-1}(x)+C & x^2<1 \\ -\coth^{-1}(x)+C & 1<x^2 \end{cases} \\ &= -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+C \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C. \end{aligned} \tag{7.8}$$
We should note that this exact problem was solved at the beginning of Section 7.5.1. In that example the answer was given as $\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C$. Note that this is equivalent to the answer given in Equation 7.8, as $\ln(a/b)=\ln a-\ln b$.

3. This requires a substitution, then item #2 of Integrals Involving Inverse Hyperbolic Functions can be applied.
Let $u=3x$, hence $du=3dx$. We have
$$\int \frac{1}{\sqrt{9x^2+10}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u^2+10}}\,du.$$
Note $a^2=10$, hence $a=\sqrt{10}$. Now apply the integral rule.
$$\begin{aligned} &= \frac{1}{3}\sinh^{-1}\left(\frac{3x}{\sqrt{10}}\right)+C \\ &= \frac{1}{3}\ln\left|3x+\sqrt{9x^2+10}\right|+C. \end{aligned}$$

♣

This section covers a lot of ground. New functions were introduced, along with some of their fundamental identities, their derivatives and antiderivatives, their inverses, and the derivatives and antiderivatives of these inverses.

Do not view this section as containing a source of information to be memorized, but rather as a reference for future problem solving.

Exercises for Section 7.8

7.8.1 *Verify the given identity.*

(a) $\dfrac{d}{dx}\left[\mathrm{sech} x\right] = -\mathrm{sech} x \tanh x$

(b) $\dfrac{d}{dx}\left[\coth x\right] = -\mathrm{csch}^2 x$

(c) $\displaystyle\int \tanh x \; dx = \ln(\cosh x) + C$

(d) $\displaystyle\int \coth x \; dx = \ln|\sinh x| + C$

7.8.2 *Find the derivative of the given function.*

(a) $f(x) = \cosh 2x$

(b) $f(x) = \tanh(x^2)$

(c) $f(x) = \ln(\sinh x)$

(d) $f(x) = \sinh x \cosh x$

(e) $f(x) = x \sinh x - \cosh x$

(f) $f(x) = \mathrm{sech}^{-1}(x^2)$

(g) $f(x) = \sinh^{-1}(3x)$

(h) $f(x) = \cosh^{-1}(2x^2)$

(i) $f(x) = \tanh^{-1}(x + 5)$

(j) $f(x) = \tanh^{-1}(\cos x)$

(k) $f(x) = \cosh^{-1}(\sec x)$

7.8.3 *Find the equation of the line tangent to the function at the given x-value.*

(a) $f(x) = \sinh x$ *at* $x = 0$

(b) $f(x) = \cosh x$ *at* $x = \ln 2$

(c) $f(x) = \text{sech}^2 x$ *at* $x = \ln 3$

(d) $f(x) = \sinh^{-1} x$ *at* $x = 0$

(e) $f(x) = \cosh^{-1} x$ *at* $x = \sqrt{2}$

7.8.4 *Evaluate the given indefinite integral.*

(a) $\displaystyle\int \tanh(2x)\ dx$

(b) $\displaystyle\int \cosh(3x - 7)\ dx$

(c) $\displaystyle\int \sinh x \cosh x\ dx$

(d) $\displaystyle\int x \cosh x\ dx$

(e) $\displaystyle\int x \sinh x\ dx$

(f) $\displaystyle\int \frac{1}{9 - x^2}\ dx$

(g) $\displaystyle\int \frac{2x}{\sqrt{x^4 - 4}}\ dx$

(h) $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1 + x^3}}\ dx$

(i) $\displaystyle\int \frac{1}{x^4 - 16}\ dx$

(j) $\displaystyle\int \frac{1}{x^2 + x}\ dx$

(k) $\displaystyle\int \frac{e^x}{e^{2x} + 1}\ dx$

(l) $\displaystyle\int \sinh^{-1} x\ dx$

(m) $\displaystyle\int \tanh^{-1} x\ dx$

(n) $\int \operatorname{sech} x \, dx$ *(Hint: mutiply by* $\frac{\cosh x}{\cosh x}$*; set* $u = \sinh x$*.)*

7.8.5 *Evaluate the given definite integral.*

(a) $\int_{-1}^{1} \sinh x \, dx$

(b) $\int_{-\ln 2}^{\ln 2} \cosh x \, dx$

(c) $\int_{0}^{1} \tanh^{-1} x \, dx$

7.9 Additional exercises

These problems require the techniques of this chapter, and are in no particular order. Some problems may be done in more than one way.

7.9.1 $\int (t+4)^3 \, dt$

7.9.2 $\int t(t^2-9)^{3/2} \, dt$

7.9.3 $\int (e^{t^2}+16)te^{t^2} \, dt$

7.9.4 $\int \sin t \cos 2t \, dt$

7.9.5 $\int \tan t \sec^2 t \, dt$

7.9.6 $\int \frac{2t+1}{t^2+t+3} \, dt$

7.9.7 $\int \frac{1}{t(t^2-4)} \, dt$

7.9.8 $\int \frac{1}{(25-t^2)^{3/2}} \, dt$

7.9.9 $\int \frac{\cos 3t}{\sqrt{\sin 3t}} \, dt$

7.9.10 $\displaystyle\int t\sec^2 t\,dt$

7.9.11 $\displaystyle\int \frac{e^t}{\sqrt{e^t+1}}\,dt$

7.9.12 $\displaystyle\int \cos^4 t\,dt$

7.9.13 $\displaystyle\int \frac{1}{t^2+3t}\,dt$

7.9.14 $\displaystyle\int \frac{1}{t^2\sqrt{1+t^2}}\,dt$

7.9.15 $\displaystyle\int \frac{\sec^2 t}{(1+\tan t)^3}\,dt$

7.9.16 $\displaystyle\int t^3\sqrt{t^2+1}\,dt$

7.9.17 $\displaystyle\int e^t\sin t\,dt$

7.9.18 $\displaystyle\int (t^{3/2}+47)^3\sqrt{t}\,dt$

7.9.19 $\displaystyle\int \frac{t^3}{(2-t^2)^{5/2}}\,dt$

7.9.20 $\displaystyle\int \frac{1}{t(9+4t^2)}\,dt$

7.9.21 $\displaystyle\int \frac{\arctan 2t}{1+4t^2}\,dt$

7.9.22 $\displaystyle\int \frac{t}{t^2+2t-3}\,dt$

7.9.23 $\displaystyle\int \sin^3 t\cos^4 t\,dt$

7.9.24 $\displaystyle\int \frac{1}{t^2-6t+9}\,dt$

7.9.25 $\displaystyle\int \frac{1}{t(\ln t)^2}\,dt$

7.9.26 $\displaystyle\int t(\ln t)^2\,dt$

7.9.27 $\displaystyle\int t^3 e^t\,dt$

7.9.28 $\displaystyle\int \frac{t+1}{t^2+t-1}\,dt$

8. Applications of Integration

8.1 Distance, Velocity, Acceleration

We recall a general principle that will later be applied to distance-velocity-acceleration problems, among other things. If $F(u)$ is an anti-derivative of $f(u)$, then $\int_a^b f(u)\,du = F(b) - F(a)$. Suppose that we want to let the upper limit of integration vary, i.e., we replace b by some variable x. We think of a as a fixed starting value x_0. In this new notation the last equation (after adding $F(a)$ to both sides) becomes:

$$F(x) = F(x_0) + \int_{x_0}^{x} f(u)\,du.$$

Here u is the variable of integration, called a "dummy variable," since it is not the variable in the function $F(x)$. In general, it is not a good idea to use the same letter as a variable of integration and as a limit of integration. That is, $\int_{x_0}^{x} f(x)dx$ is bad notation, and can lead to errors and confusion.

An important application of this principle occurs when we are interested in the position of an object at time t (say, on the x-axis) and we know its position at time t_0. Let $s(t)$ denote the position of the object at time t (its distance from a reference point, such as the origin on the x-axis). Then the net change in position between t_0 and t is $s(t) - s(t_0)$. Since $s(t)$ is an anti-derivative of the velocity function $v(t)$, we can write

$$s(t) = s(t_0) + \int_{t_0}^{t} v(u)du.$$

Similarly, since the velocity is an anti-derivative of the acceleration function $a(t)$, we have

$$v(t) = v(t_0) + \int_{t_0}^{t} a(u)du.$$

Example 8.1: Constant Force

Suppose an object is acted upon by a constant force F. Find $v(t)$ and $s(t)$.

Solution. By Newton's law $F = ma$, so the acceleration is F/m, where m is the mass of the object. Then we first have

$$v(t) = v(t_0) + \int_{t_0}^{t} \frac{F}{m}\,du = v_0 + \frac{F}{m}u\Big|_{t_0}^{t} = v_0 + \frac{F}{m}(t - t_0),$$

using the usual convention $v_0 = v(t_0)$. Then

$$\begin{aligned} s(t) &= s(t_0) + \int_{t_0}^t \left(v_0 + \tfrac{F}{m}(u - t_0)\right) du = s_0 + \left(v_0 u + \tfrac{F}{2m}(u - t_0)^2\right)\Big|_{t_0}^t \\ &= s_0 + v_0(t - t_0) + \tfrac{F}{2m}(t - t_0)^2. \end{aligned}$$

For instance, when $F/m = -g$ is the constant of gravitational acceleration, then this is the falling body formula (if we neglect air resistance) familiar from elementary physics:

$$s_0 + v_0(t - t_0) - \frac{g}{2}(t - t_0)^2,$$

or in the common case that $t_0 = 0$,

$$s_0 + v_0 t - \frac{g}{2}t^2.$$

♣

Recall that the integral of the velocity function gives the *net* distance traveled. If you want to know the *total* distance traveled, you must find out where the velocity function crosses the t-axis, integrate separately over the time intervals when $v(t)$ is positive and when $v(t)$ is negative, and add up the absolute values of the different integrals. For example, if an object is thrown straight upward at 19.6 m/sec, its velocity function is $v(t) = -9.8t + 19.6$, using $g = 9.8$ m/sec for the force of gravity. This is a straight line which is positive for $t < 2$ and negative for $t > 2$. The net distance traveled in the first 4 seconds is thus

$$\int_0^4 (-9.8t + 19.6)dt = 0,$$

while the total distance traveled in the first 4 seconds is

$$\int_0^2 (-9.8t + 19.6)dt + \left|\int_2^4 (-9.8t + 19.6)dt\right| = 19.6 + |-19.6| = 39.2$$

meters, 19.6 meters up and 19.6 meters down.

Example 8.2: Net and Total Distance

The acceleration of an object is given by $a(t) = \cos(\pi t)$, and its velocity at time $t = 0$ is $1/(2\pi)$. Find both the net and the total distance traveled in the first 1.5 seconds.

Solution. We compute

$$v(t) = v(0) + \int_0^t \cos(\pi u)du = \frac{1}{2\pi} + \frac{1}{\pi}\sin(\pi u)\Big|_0^t = \frac{1}{\pi}\big(\frac{1}{2} + \sin(\pi t)\big).$$

The *net* distance traveled is then

$$s(3/2) - s(0) \quad = \quad \int_0^{3/2} \frac{1}{\pi}\left(\frac{1}{2} + \sin(\pi t)\right)\, dt$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left(\frac{t}{2} - \frac{1}{\pi}\cos(\pi t)\right)\Big|_0^{3/2} \\
&= \frac{3}{4\pi} + \frac{1}{\pi^2} \approx 0.340 \text{ meters.}
\end{aligned}
$$

To find the *total* distance traveled, we need to know when $(0.5+\sin(\pi t))$ is positive and when it is negative. This function is 0 when $\sin(\pi t)$ is -0.5, i.e., when $\pi t = 7\pi/6$, $11\pi/6$, etc. The value $\pi t = 7\pi/6$, i.e., $t = 7/6$, is the only value in the range $0 \le t \le 1.5$. Since $v(t) > 0$ for $t < 7/6$ and $v(t) < 0$ for $t > 7/6$, the total distance traveled is

$$
\begin{aligned}
&\int_0^{7/6} \frac{1}{\pi}\left(\frac{1}{2} + \sin(\pi t)\right)\,dt + \left|\int_{7/6}^{3/2} \frac{1}{\pi}\left(\frac{1}{2} + \sin(\pi t)\right)\,dt\right| \\
&= \frac{1}{\pi}\left(\frac{7}{12} + \frac{1}{\pi}\cos(7\pi/6) + \frac{1}{\pi}\right) + \frac{1}{\pi}\left|\frac{3}{4} - \frac{7}{12} + \frac{1}{\pi}\cos(7\pi/6)\right| \\
&= \frac{1}{\pi}\left(\frac{7}{12} + \frac{1}{\pi}\frac{\sqrt{3}}{2} + \frac{1}{\pi}\right) + \frac{1}{\pi}\left|\frac{3}{4} - \frac{7}{12} + \frac{1}{\pi}\frac{\sqrt{3}}{2}.\right| \\
&\approx 0.409 \text{ meters.}
\end{aligned}
$$

♣

Exercises for Section 8.1

8.1.1 *An object moves so that its velocity at time t is $v(t) = -9.8t + 20$ m/s. Describe the motion of the object between $t = 0$ and $t = 5$, find the total distance traveled by the object during that time, and find the net distance traveled.*

8.1.2 *An object moves so that its velocity at time t is $v(t) = \sin t$. Set up and evaluate a single definite integral to compute the net distance traveled between $t = 0$ and $t = 2\pi$.*

8.1.3 *An object moves so that its velocity at time t is $v(t) = 1 + 2\sin t$ m/s. Find the net distance traveled by the object between $t = 0$ and $t = 2\pi$, and find the total distance traveled during the same period.*

8.1.4 *Consider the function $f(x) = (x+2)(x+1)(x-1)(x-2)$ on $[-2, 2]$. Find the total area between the curve and the x-axis (measuring all area as positive).*

8.1.5 *Consider the function $f(x) = x^2 - 3x + 2$ on $[0, 4]$. Find the total area between the curve and the x-axis (measuring all area as positive).*

8.1.6 *Evaluate the three integrals:*

$$
A = \int_0^3 (-x^2+9)\,dx \qquad B = \int_0^4 (-x^2+9)\,dx \qquad C = \int_4^3 (-x^2+9)\,dx,
$$

and verify that $A = B + C$.

8.2 Area Between Curves

We begin this chapter with a reminder of a few key concepts discussed thus far. Let f be a continuous function on $[a, b]$ which is partitioned into n equally spaced subintervals as

$$a < x_1 < x_2 < \cdots < x_n < x_{n+1} = b.$$

Let $\Delta x = (b - a)/n$ denote the length of the subintervals, and let c_i be any x-value in the i th subinterval. Definition **??** states that the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x$$

is a *Riemann Sum.* Riemann Sums are often used to approximate some quantity (area, volume, work, pressure, etc.). The *approximation* becomes *exact* by taking the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

Theorem **??** connects limits of Riemann Sums to definite integrals:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\ dx.$$

Finally, the Fundamental Theorem of Calculus states how definite integrals can be evaluated using antiderivatives.

This chapter employs the following technique to a variety of applications. Suppose the value Q of a quantity is to be calculated. We first approximate the value of Q using a Riemann Sum, then find the exact value via a definite integral. We spell out this technique in the following Key Idea.

Key Idea 8.2.0: Application of Definite Integrals Strategy

Let a quantity be given whose value Q is to be computed.

1. Divide the quantity into n smaller "subquantities" of value Q_i.
2. Identify a variable x and function $f(x)$ such that each subquantity can be approximated with the product $f(c_i)\Delta x$, where Δx represents a small change in x. Thus $Q_i \approx f(c_i)\Delta x$. A sample approximation $f(c_i)\Delta x$ of Q_i is called a *differential element.*
3. Recognize that $Q = \sum_{i=1}^{n} Q_i \approx \sum_{i=1}^{n} f(c_i)\Delta x$, which is a Riemann Sum.
4. Taking the appropriate limit gives $Q = \int_a^b f(x)\ dx$

This Key Idea will make more sense after we have had a chance to use it several times. We begin with Area Between Curves.

Area Between Curves

We are often interested in knowing the area of a region. Forget momentarily that we discussed this already in Section **??**.6.3 and approach it instead using the technique described in Key Idea 8.2.

Let Q be the area of a region bounded by continuous functions f and g. If we break the region into many subregions, we have an obvious equation:

$$\text{Total Area} = \text{sum of the areas of the subregions.}$$

The issue to address next is how to systematically break a region into subregions. A graph will help. Consider Figure 8.1 (a) where a region between two curves is shaded. While there are many ways to break this into subregions, one particularly efficient way is to "slice" it vertically, as shown in Figure 8.1 (b), into n equally spaced slices.

We now approximate the area of a slice. Again, we have many options, but using a rectangle seems simplest. Picking any x-value c_i in the i $^{\text{th}}$ slice, we set the height of the rectangle to be $f(c_i)-g(c_i)$, the difference of the corresponding y-values. The width of the rectangle is a small difference in x-values, which we represent with Δx. Figure 8.1 (c) shows sample points c_i chosen in each subinterval and appropriate rectangles drawn. (Each of these rectangles represents a differential element.) Each slice has an area approximately equal to $\big(f(c_i) - g(c_i)\big)\Delta x$; hence, the total area is approximately the Riemann Sum

$$Q = \sum_{i=1}^{n} \big(f(c_i) - g(c_i)\big)\Delta x.$$

Taking the limit as $n \to \infty$ gives the exact area as $\int_a^b \big(f(x) - g(x)\big)\ dx$.

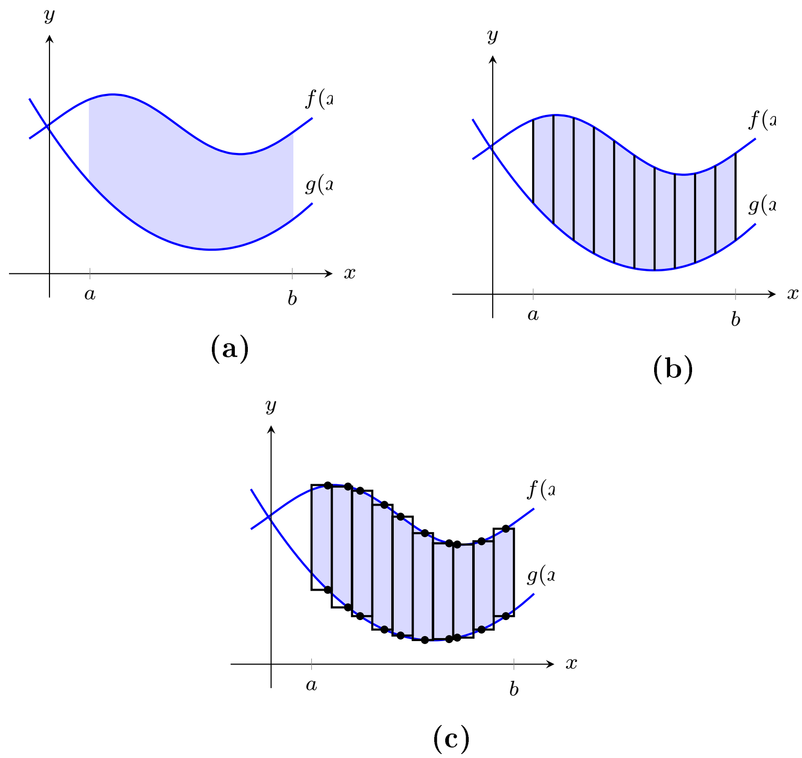


Figure 8.1: Subdividing a region into vertical slices and approximating the areas with rectangles.

Theorem 8.1: Area Between Curves

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$. The area, A, of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$A = \int_a^b \left| f(x) - g(x) \right| \, dx.$$

In particular, if $f(x) \geq g(x)$ everywhere on the interval $[a, b]$, then

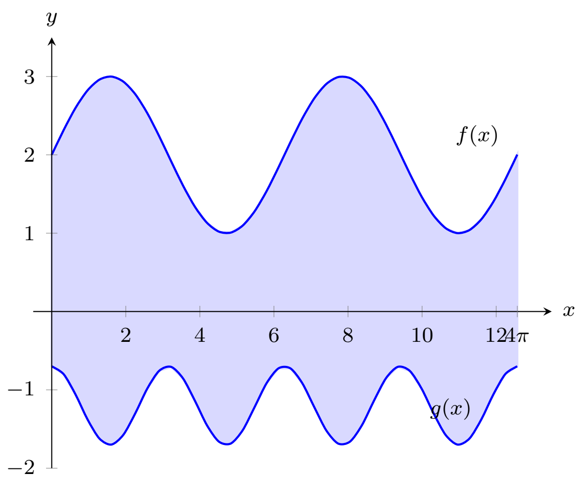
$$A = \int_a^b \big(f(x) - g(x) \big) \, dx.$$

Example 8.3: Finding area enclosed by curves

Find the area of the region bounded by $f(x) = \sin x + 2$, $g(x) = \frac{1}{2}\cos(2x) - 1$, $x = 0$ and $x = 4\pi$, as shown in Figure 8.2.

Solution. The graph verifies that the upper boundary of the region is given by f and the lower bound is given by g. Therefore the area of the region is the value of the integral

$$\begin{aligned}
\textbf{Area} &= \int_1^4 \left|f(x) - g(x)\right|\, dx \\
&= \int_0^{4\pi} \big(f(x) - g(x)\big)\, dx \qquad = \int_0^{4\pi} \left(\sin x + 2 - \big(\frac{1}{2}\cos(2x) - 1\big)\right) dx \\
&= -\cos x - \frac{1}{4}\sin(2x) + 3x\Big|_0^{4\pi} \\
&= 12\pi \approx 37.7 \text{ units}^2.
\end{aligned}$$



♣

Figure 8.2: Graphing an enclosed region in Example ??.

Example 8.4: Finding total area enclosed by curves

Find the total area of the region enclosed by the functions $f(x) = -2x + 5$ *and* $g(x) = x^3 - 7x^2 + 12x - 3$ *as shown in Figure 8.3.*

Solution.

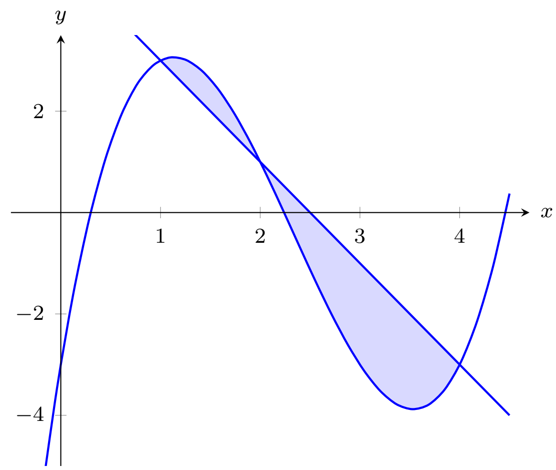


Figure 8.3: Graphing a region enclosed by two functions in Example ??.

A quick calculation shows that $f = g$ at $x = 1, 2$ and 4. One can proceed thoughtlessly by computing $\displaystyle\int_1^4 \big(f(x) - g(x)\big)\, dx$, but this ignores the fact that on $[1, 2]$, $g(x) > f(x)$. (In fact, the thoughtless integration returns $-9/4$, hardly the expected value of an *area*.) Thus we compute the total area by breaking the interval $[1, 4]$ into two subintervals, $[1, 2]$ and $[2, 4]$ and using the proper integrand in each.

$$\begin{aligned}
\textbf{Area} &= \int_1^4 \big|f(x) - g(x)\big|\, dx \\
&= \int_1^2 \big(g(x) - f(x)\big)\, dx + \int_2^4 \big(f(x) - g(x)\big)\, dx \\
&= \int_1^2 \big(x^3 - 7x^2 + 14x - 8\big)\, dx + \int_2^4 \big(-x^3 + 7x^2 - 14x + 8\big)\, dx \\
&= 5/12 + 8/3 \\
&= 37/12 = 3.083 \text{ units}^2.
\end{aligned}$$

♣

The previous example makes note that we are expecting area to be *positive*. When first learning about the definite integral, we interpreted it as “signed area under the curve,” allowing for “negative area.” That doesn’t apply here; area is to be positive.

The previous example also demonstrates that we often have to break a given region into subregions before applying Theorem 8.1. The following example shows another situation where this is applicable, along with an alternate view of applying the Theorem.

Example 8.5: Finding area: integrating with respect to y

Find the area of the region enclosed by the functions $y = \sqrt{x} + 2$, $y = -(x-1)^2 + 3$ and $y = 2$, as shown in Figure 8.4.

Solution. We give two approaches to this problem. In the first approach, we notice that the region's "top" is defined by two different curves. On $[0, 1]$, the top function is $y = \sqrt{x} + 2$; on $[1, 2]$, the top function is $y = -(x-1)^2 + 3$.

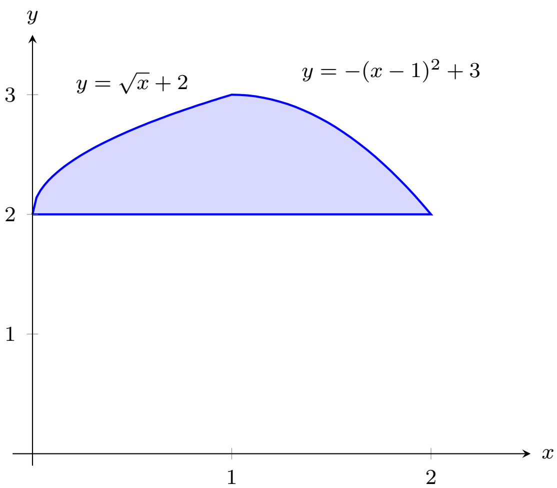


Figure 8.4: Graphing a region for Example 8.5.

Thus we compute the area as the sum of two integrals:

$$\begin{aligned}\textbf{Area} &= \int_0^1 \Big(\big(\sqrt{x}+2\big) - 2\Big)\,dx + \int_1^2 \Big(\big(-(x-1)^2+3\big) - 2\Big)\,dx \\ &= 2/3 + 2/3 \\ &= 4/3.\end{aligned}$$

The second approach is clever and very useful in certain situations. We are used to viewing curves as functions of x; we input an x-value and a y-value is returned. Some curves can also be described as functions of y: input a y-value and an x-value is returned. We can rewrite the equations describing the boundary by solving for x:

$$y = \sqrt{x} + 2 \quad \Rightarrow \quad x = (y-2)^2$$

$$y = -(x-1)^2 + 3 \quad \Rightarrow \quad x = \sqrt{3-y} + 1.$$

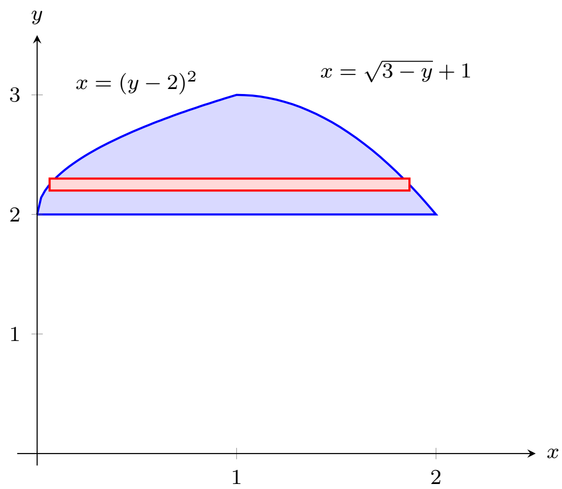


Figure 8.5: The region used in Example ?? with boundaries relabeled as functions of y.

Figure 8.5 shows the region with the boundaries relabeled. A differential element, a horizontal rectangle, is also pictured. The width of the rectangle is a small change in y: Δy. The height of the rectangle is a difference in x-values. The "top" x-value is the largest value, i.e., the rightmost. The "bottom" x-value is the smaller, i.e., the leftmost. Therefore the height of the rectangle is

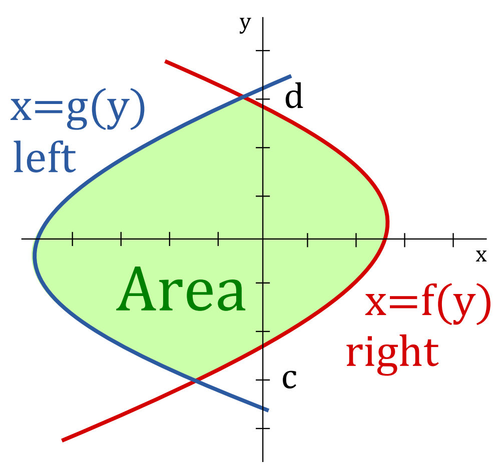
$$\big(\sqrt{3-y}+1\big) - (y-2)^2.$$

The area is found by integrating the above function with respect to y with the appropriate bounds. We determine these by considering the y-values the region occupies. It is bounded below by $y = 2$, and bounded above by $y = 3$. That is, both the "top" and "bottom" functions exist on the y interval $[2, 3]$. Thus

$$\begin{aligned}\textbf{Area} &= \int_2^3 \big(\sqrt{3-y}+1-(y-2)^2\big)\,dy \\ &= \Big(-\frac{2}{3}(3-y)^{3/2}+y-\frac{1}{3}(y-2)^3\Big)\Big|_2^3 \\ &= 4/3.\end{aligned}$$

♣

Sometimes the given curves are not given as functions of x, but rather functions of y. In this instances, it may be more useful to use the "horizontal rectangle" approach outlined above.



The area A of the region bounded by the curves $x = f(y)$ and $x = g(y)$ and the lines $y = c$ and $y = d$ is:

$$A = \int_c^d |f(y) - g(y)|\, dy.$$

Informally this can be thought of as follows:

Key Idea 8.2.0: Area Between Two Curves

$$Area = \int_c^d (\text{right curve}) - (\text{left curve})\, dy, \qquad c \le y \le d.$$

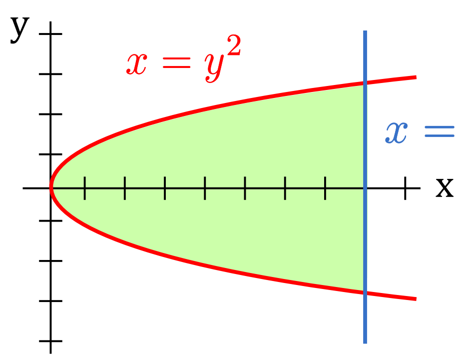
Example 8.6: Area Between Two Curves

Determine the area enclosed by $x = y^2$ and $x = 8$.

Solution. Note that $x = y^2$ and $x = 8$ intersect when:

$$y^2 = 8 \qquad \to \qquad y = \pm\sqrt{8} \qquad \to \qquad y = \pm 2\sqrt{2}$$

Sketching the two curves gives:



From the sketch $c = -2\sqrt{2}$, $d = 2\sqrt{2}$, the right curve is $x = 8$ and the left curve is $x = y^2$.

$$\textbf{Area} = \int_c^d [\text{right} - \text{left}]\, dy = \int_{-2\sqrt{2}}^{2\sqrt{2}} (8 - y^2)\, dy = \left(8y - \frac{1}{3}y^3\right)\Bigg|_{-2\sqrt{2}}^{2\sqrt{2}}$$

$$= \left[8(2\sqrt{2}) - \frac{1}{3}(2\sqrt{2})^3\right] - \left[8(-2\sqrt{2}) - \frac{1}{3}(-2\sqrt{2})^3\right] = \frac{64\sqrt{2}}{3}$$

♣

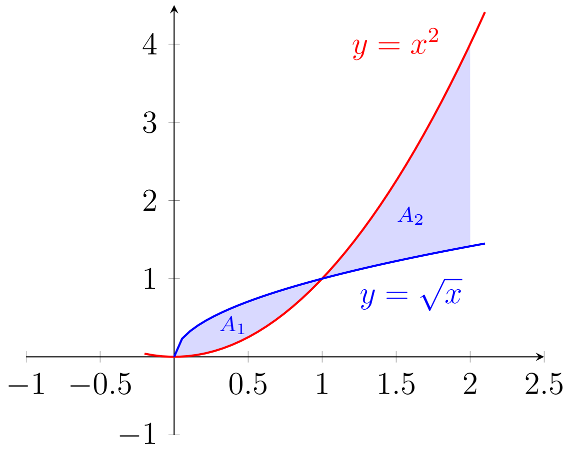
Example 8.7: Area Between Two Curves

Determine the area enclosed by $y = x^2$, $y = \sqrt{x}$, $x = 0$ *and* $x = 2$.

Solution. The points of intersection of $y = x^2$ and $y = \sqrt{x}$ are

$$x^2 = \sqrt{x} \quad \to \quad x^4 = x \quad \to \quad x^4 - x = 0 \quad \to \quad x(x^3 - 1) = 0.$$

Thus, either $x = 0$ or $x = 1$. Sketching the curves gives:



The area we want to compute is the shaded region. Since the top curve changes at $x = 1$, we need to use the formula twice. For A_1 we have $a = 0$, $b = 1$, the top curve is $y = \sqrt{x}$ and the bottom curve is $y = x^2$. For A_2 we have $a = 1$, $b = 2$, the top curve is $y = x^2$ and the bottom curve is $y = \sqrt{x}$.

$$\textbf{Area} \;=\; \mathbf{A}_1 + \mathbf{A}_2 \;=\; \int_0^1 (\sqrt{x} - x^2)\,dx + \int_1^2 (x^2 - \sqrt{x})\,dx$$

For the first integral we have:

$$\int_0^1 (\sqrt{x} - x^2)\,dx \;=\; \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Bigg|_0^1 \;=\; \frac{1}{3}$$

Thus,

$$\textbf{Area} \;=\; \frac{1}{3} + \left(\frac{1}{3}x^3 - \frac{2}{3}x^{3/2}\right)\Bigg|_1^2 \;=\; \frac{1}{3} + \left[\left(\frac{8}{3} - \frac{2(\sqrt{2})^3}{3}\right) - \left(\frac{1}{3} - \frac{2}{3}\right)\right] \;=\; \frac{10 - 4\sqrt{2}}{3}$$

♣

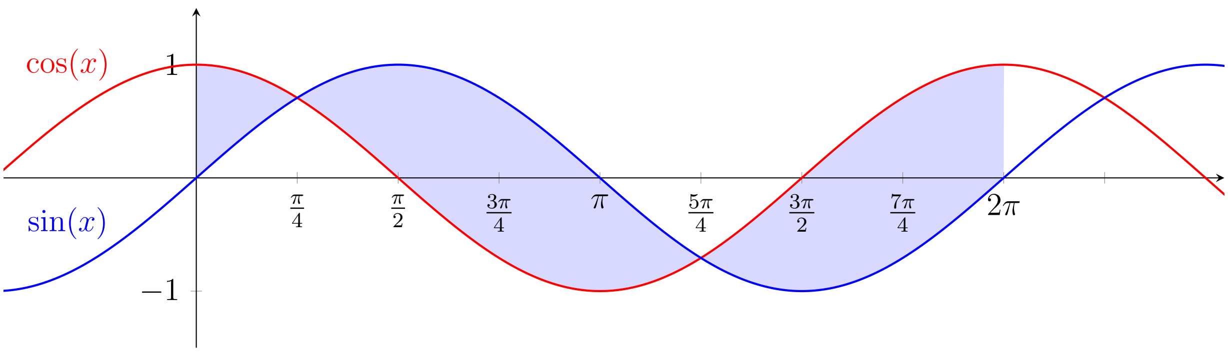
Example 8.8: Area Between Sine and Cosine

Determine the area enclosed by $y = \sin x$ and $y = \cos x$ on the interval $[0,\ 2\pi]$.

Solution. The curves $y = \sin x$ and $y = \cos x$ intersect when:

$$\sin x = \cos x \qquad \to \qquad \tan x = 1 \qquad \to \qquad x = \frac{\pi}{4} + \pi k,\ k \text{ an integer.}$$

We have the following sketch:



The area we want to compute is the shaded region. The top curve changes at $x = \pi/4$ and $x = 5\pi/4$, thus, we need to split the area up into three regions: from 0 to $\pi/4$; from $\pi/4$ to $5\pi/4$; and from $5\pi/4$ to 2π.

$$\begin{aligned}
\textbf{Area} &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\, dx + \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)\, dx \\
&= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4} + (\sin x + \cos x)\Big|_{5\pi/4}^{2\pi} \\
&= \left(\sqrt{2} - 1\right) + \left(\sqrt{2} + \sqrt{2}\right) + \left(1 + \sqrt{2}\right) \\
&= 4\sqrt{2}
\end{aligned}$$

♣

This calculus–based technique of finding area can be useful even with shapes that we normally think of as “easy.” Example **??** computes the area of a triangle. While the formula “$\frac{1}{2} \times \text{base} \times \text{height}$” is well known, in arbitrary triangles it can be nontrivial to compute the height. Calculus makes the problem simple.

Example 8.9: Finding the area of a triangle

Compute the area of the regions bounded by the lines $y = x+1$, $y = -2x+7$ and $y = -\frac{1}{2}x + \frac{5}{2}$, as shown in Figure 8.6.

Solution. Recognize that there are two "top" functions to this region, causing us to use two definite integrals.

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big((x+1) - (-\frac{1}{2}x + \frac{5}{2})\big)\ dx + \int_2^3 \big((-2x+7) - (-\frac{1}{2}x + \frac{5}{2})\big)\ dx \\ &= 3/4 + 3/4 \\ &= 3/2.\end{aligned}$$

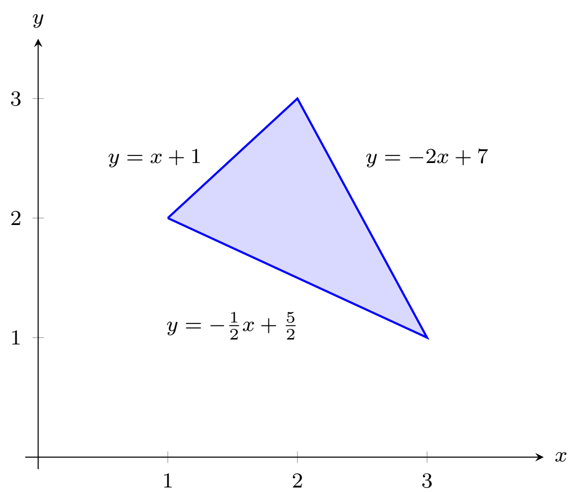


Figure 8.6: Graphing a triangular region in Example ??.

We can also approach this by converting each function into a function of y. This also requires 2 integrals, so there isn't really any advantage to doing so. We do it here for demonstration purposes. The "top" function is always $x = \frac{7-y}{2}$ while there are two "bottom" functions. Being mindful of the proper integration bounds, we have

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big(\frac{7-y}{2} - (5-2y)\big)\ dy + \int_2^3 \big(\frac{7-y}{2} - (y-1)\big)\ dy \\ &= 3/4 + 3/4 \\ &= 3/2.\end{aligned}$$

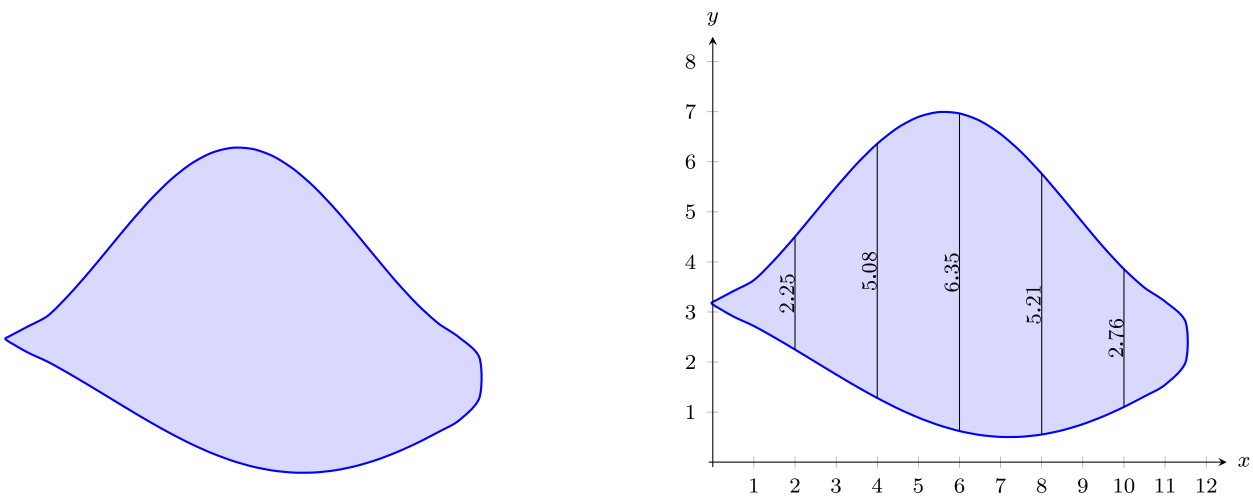
Of course, the final answer is the same. (It is interesting to note that the area of all 4 subregions used is 3/4. This is coincidental.) ♣

While we have focused on producing exact answers, we are also able to make approximations using the principle of Theorem 8.1. The integrand in the theorem is a distance ("top minus bottom"); integrating this distance function gives an area. By taking discrete measurements of distance, we can approximate an area using numerical integration techniques developed in Section 7.6. The following example demonstrates this.

Example 8.10: Numerically approximating area

To approximate the area of a lake, shown in Figure 8.7 (a), the "length" of the lake is measured at 200-meter increments as shown in Figure 8.7 (b), where the lengths are given in hundreds of meters. Approximate the area of the lake.

Solution. The measurements of length can be viewed as measuring "top minus bottom" of two functions. The exact answer is found by integrating $\int_0^{12} \big(f(x) - g(x)\big)\,dx$, but of course we don't know the functions f and g. Our discrete measurements instead allow us to approximate.



(a) A sketch of a lake. **(b) The lake with length measurements.**

Figure 8.7

We have the following data points:

$$(0, 0),\ (2, 2.25),\ (4, 5.08),\ (6, 6.35),\ (8, 5.21),\ (10, 2.76),\ (12, 0).$$

We also have that $\Delta x = \frac{b-a}{n} = 2$, so Simpson's Rule gives

$$\begin{aligned} \text{Area} &\approx \frac{2}{3}\Big(1 \cdot 0 + 4 \cdot 2.25 + 2 \cdot 5.08 + 4 \cdot 6.35 + 2 \cdot 5.21 + 4 \cdot 2.76 + 1 \cdot 0\Big) \\ &= 44.01\overline{3}\ 100\text{m.}^2. \end{aligned}$$

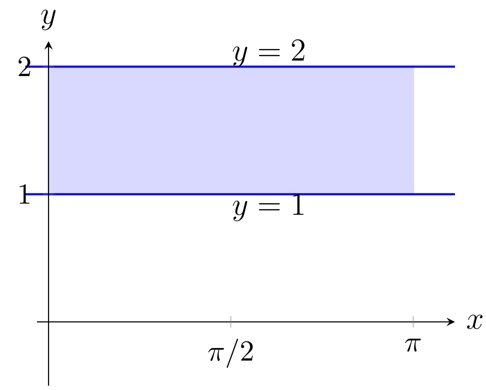
Since the measurements are in hundreds of meters, units^2 = $(100 \text{ m})^2 = 10{,}000 \text{ m}^2$, giving a total area of $440{,}133 \text{ m}^2$. (Since we are approximating, we'd likely say the area was about $440{,}000 \text{ m}^2$, which is 44 hectares.) ♣

In the next section we apply our applications–of–integration techniques to finding the volumes of certain solids.

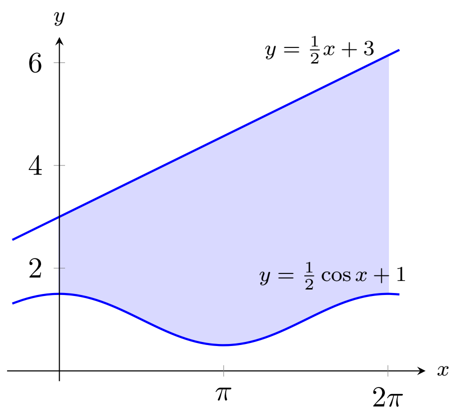
Exercises for Section 8.2

8.2.1 *Find the area of the shaded region in the given graph.*

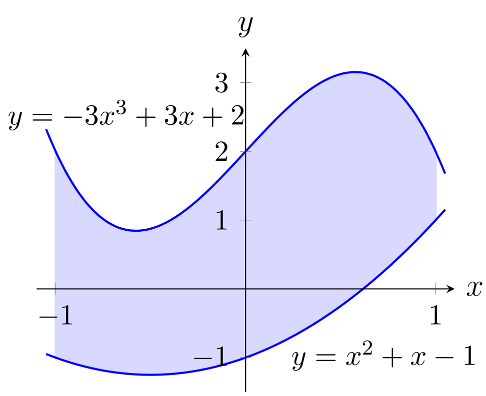
(a)



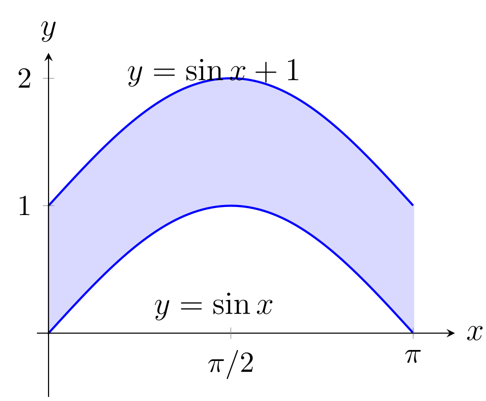
(b)



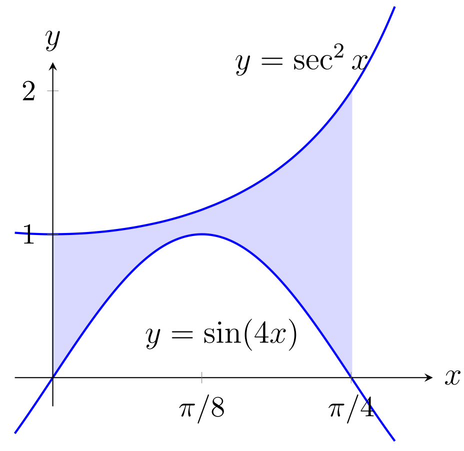
(c)



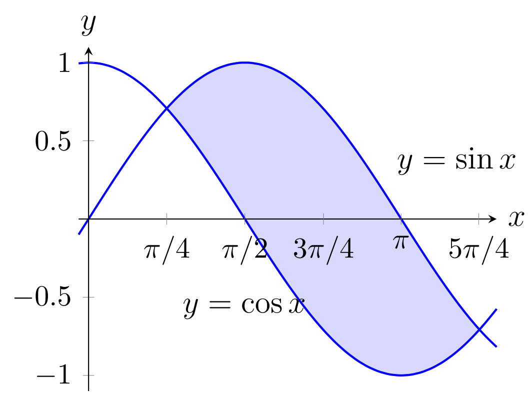
(d)



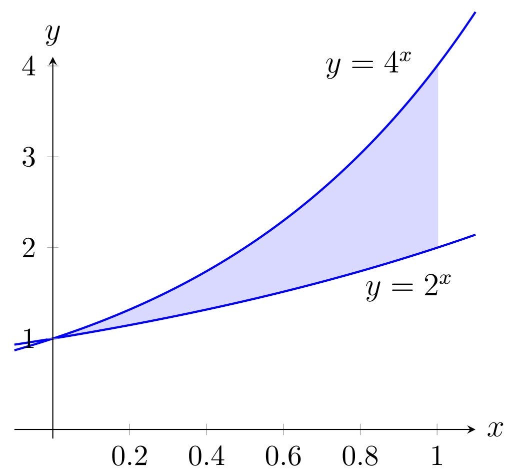
(e)



(f)

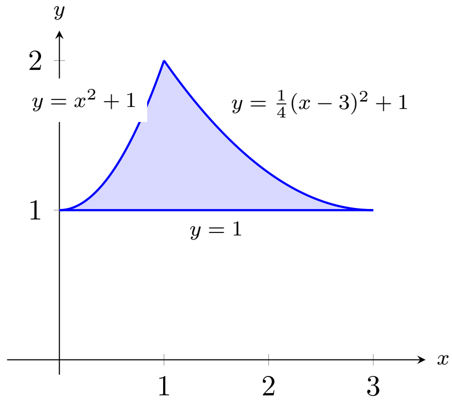


(g)

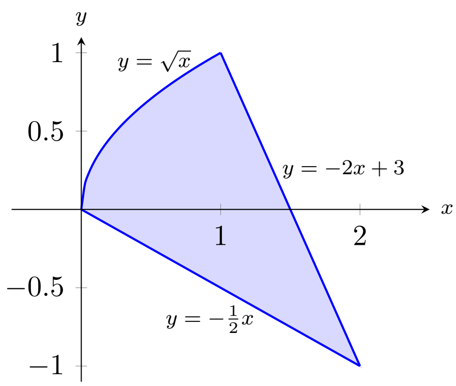


8.2.2 *Find the area of the enclosed region in two ways:*
(1) by treating the boundaries as functions of x, and
(2) by treating the boundaries as functions of y.

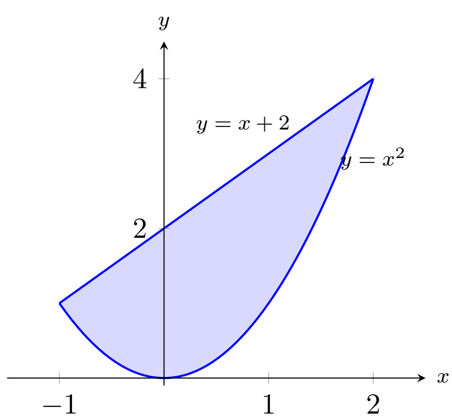
(a)



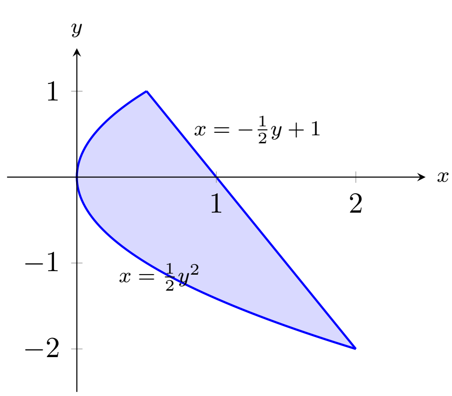
(b)



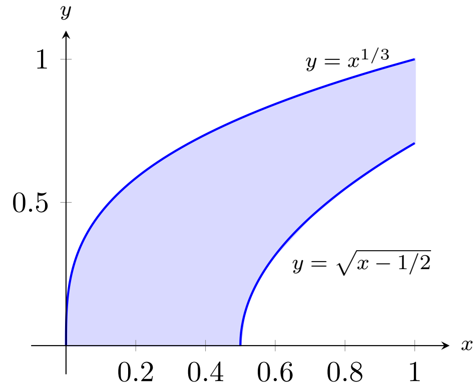
(c)



(d)



(e)



8.2.3

(a) $(1,1)$, $(2,3)$, *and* $(3,3)$

(b) $(-1,1)$, $(1,3)$, *and* $(2,-1)$

(c) $(1,1)$, $(3,3)$, *and* $(3,3)$

(d) $(0,0)$, $(2,5)$, *and* $(5,2)$

Find the area bounded by the curves.

8.2.4 $y = x^4 - x^2$ *and* $y = x^2$ *(the part to the right of the y-axis)*

8.2.5 $x = y^3$ *and* $x = y^2$

8.2.6 $x = 1 - y^2$ *and* $y = -x - 1$

8.2.7 $x = 3y - y^2$ *and* $x + y = 3$

8.2.8 $y = \cos(\pi x/2)$ *and* $y = 1 - x^2$ *(in the first quadrant)*

8.2.9 $y = \sin(\pi x/3)$ *and* $y = x$ *(in the first quadrant)*

8.2.10 $y = \sqrt{x}$ *and* $y = x^2$

8.2.11 $y = \sqrt{x}$ *and* $y = \sqrt{x+1}$, $0 \le x \le 4$

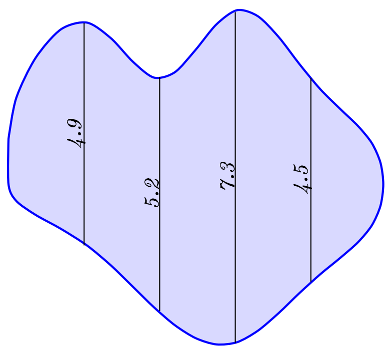
8.2.12 $x = 0$ *and* $x = 25 - y^2$

8.2.13 $y = \sin x \cos x$ *and* $y = \sin x$, $0 \le x \le \pi$

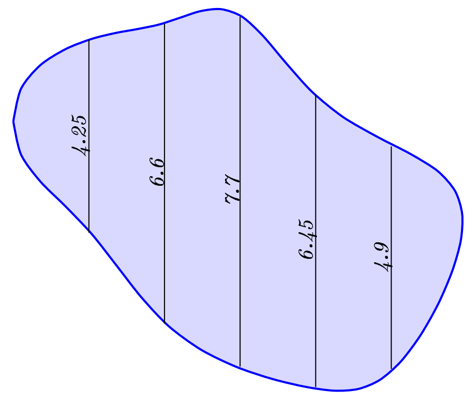
8.2.14 $y = x^{3/2}$ *and* $y = x^{2/3}$

8.2.15 $y = x^2 - 2x$ *and* $y = x - 2$

8.2.16 *Use the Trapezoidal Rule to approximate the area of the pictured lake whose lengths, in hundreds of meters, are measured in* 100-*m. increments.*



8.2.17 *Use Simpson's Rule to approximate the area of the pictured lake whose lengths, in hundreds of meters, are measured in 200-meter increments.*



8.3 Volume

Now that we have seen how to compute certain areas by using integration; we will now look into how some volumes may also be computed by evaluating an integral. Generally, the volumes that we can compute this way have cross-sections that are easy to describe.

The volume of a general right cylinder, as shown in Figure 8.8 has a simple calculation, it is equal to base $\times$ height.

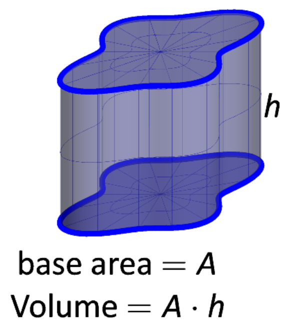


Figure 8.8: The volume of a general right cylinder

We can use this fact as the building block in finding volumes of a variety of shapes.

Given an arbitrary solid, we can *approximate* its volume by cutting it into n thin slices. When the slices are thin, each slice can be approximated well by a general right cylinder. Thus the volume of each slice is approximately its cross-sectional area $\times$ thickness. (These slices are the differential elements.)

By orienting a solid along the x-axis, we can let $A(x_i)$ represent the cross-sectional area of the i^{th} slice, and let Δx_i represent the thickness of this slice (the thickness is a small change in x). The total volume of the solid is approximately:

$$\begin{aligned} \text{Volume} &\approx \sum_{i=1}^{n} \Big[\text{Area} \ \times \ \text{thickness}\Big] \\ &= \sum_{i=1}^{n} A(x_i)\Delta x_i. \end{aligned}$$

Recognize that this is a Riemann Sum. By taking a limit (as the thickness of the slices goes to 0) we can find the volume exactly.

Theorem 8.2: Volume By Cross-Sectional Area

The volume V of a solid, oriented along the x-axis with cross-sectional area $A(x)$ from $x = a$ to $x = b$, is

$$V = \int_a^b A(x)\ dx.$$

Example 8.11: Finding the volume of a solid

Find the volume of a pyramid with a square base of side length 10 cm and a height of 5 cm.

Solution. There are many ways to "orient" the pyramid along the x-axis; Figure 8.9 gives one such way, with the pointed top of the pyramid at the origin and the x-axis going through the center of the base.

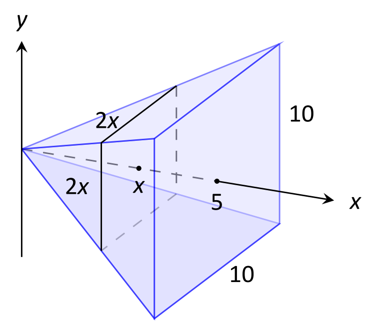


Figure 8.9: Orienting a pyramid along the x-axis in Example ??.

Each cross section of the pyramid is a square; this is a sample differential element. To determine its area $A(x)$, we need to determine the side lengths of the square.

When $x = 5$, the square has side length 10; when $x = 0$, the square has side length 0. Since the edges of the pyramid are lines, it is easy to figure that each cross-sectional square sitting above x, has side length $2x$, giving $A(x) = (2x)^2 = 4x^2$.

If one were to cut a slice out of the pyramid at $x = 3$, as shown in Figure 8.10, one would have a shape with square bottom and top with sloped sides. If the slice were thin, both the bottom and top squares would have sides lengths of about 6, and thus the cross–sectional area of the bottom and top would be about 36in^2. Letting Δx_i represent the thickness of the slice, the volume of this slice would then be about $36\Delta x_i\text{in}^3$.

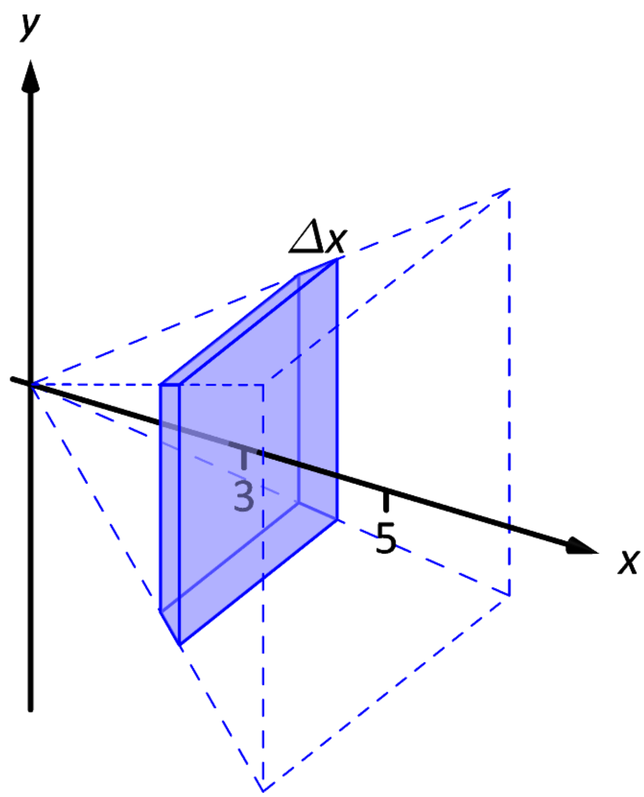


Figure 8.10: Cutting a slice in the pyramid in Example ?? at $x = 3$.

Cutting the pyramid into n slices divides the total volume into n equally–spaced smaller pieces, each with volume $(2x_i)^2\Delta x$, where x_i is the approximate location of the slice along the x-axis and Δx represents the thickness of each slice. One can approximate total volume of the pyramid by summing up the volumes of these slices:

$$\text{Approximate volume } = \sum_{i=1}^{n} (2x_i)^2 \Delta x.$$

Taking the limit as $n \to \infty$ gives the actual volume of the pyramid; recognizing this sum as a Riemann Sum allows us to find the exact answer using a definite integral, matching the definite integral given by Theorem 8.2.

We have

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} (2x_i)^2 \Delta x \\ &= \int_0^5 4x^2 \, dx \\ &= \frac{4}{3}x^3 \Big|_0^5 \\ &= \frac{500}{3} \text{ cm}^3 \approx 166.67 \text{ cm}^3. \end{aligned}$$

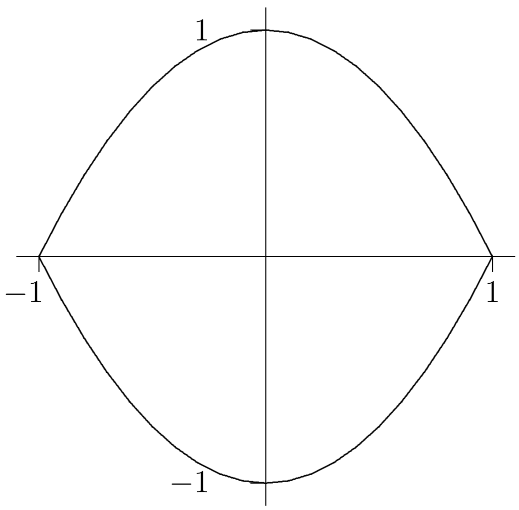
We can check our work by consulting the general equation for the volume of a pyramid (see the back cover under “Volume of A General Cone”):

$$\tfrac{1}{3} \times \text{area of base} \times \text{height}.$$

Certainly, using this formula from geometry is faster than our new method, but the calculus–based method can be applied to much more than just cones. ♣

Example 8.12: Volume of an Object

The base of a solid is the region between $f(x) = x^2 - 1$ and $g(x) = -x^2 + 1$, and its cross-sections perpendicular to the x-axis are equilateral triangles, as indicated in Figure 8.11. The solid has been truncated to show a triangular cross-section above $x = 1/2$. Find the volume of the solid.



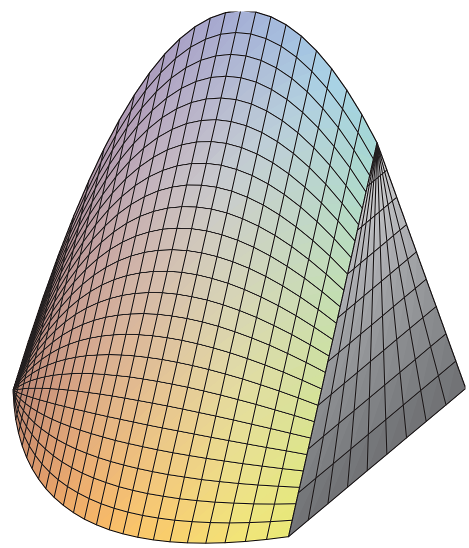


Figure 8.11: Solid with equilateral triangles as cross-sections.

Solution. A cross-section taken at say x_i on the x-axis is a triangle with base $2(1 - x_i^2)$ and height $\sqrt{3}(1 - x_i^2)$, so the area of the cross-section is

$$A(x_i)\frac{1}{2}(\text{base})(\text{height}) = (1 - x_i^2)\sqrt{3}(1 - x_i^2),$$

and the volume of a thin "slab" is then

$$(1 - x_i^2)\sqrt{3}(1 - x_i^2)\Delta x.$$

Thus the total volume is

$$\int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}(1 - x^2)^2\,dx = \frac{16}{15}\sqrt{3}.$$

♣

An important special case of Theorem 8.2 is when the solid is a **solid of revolution**, that is, when the solid is formed by rotating a shape around some axis of rotation.

Start with a function $y = f(x)$ from $x = a$ to $x = b$. Revolving this curve about a horizontal axis creates a three-dimensional solid whose cross sections are disks (thin circles). Let $R(x)$ represent the radius of the cross-sectional disk at x; the area of this disk is $\pi R(x)^2$. Applying Theorem 8.2 gives the Disk Method.

For example, in Figure 8.12 we see a plane region under a curve and between two vertical lines; then the result of rotating this around the x-axis, and a typical circular cross-section is a circle.

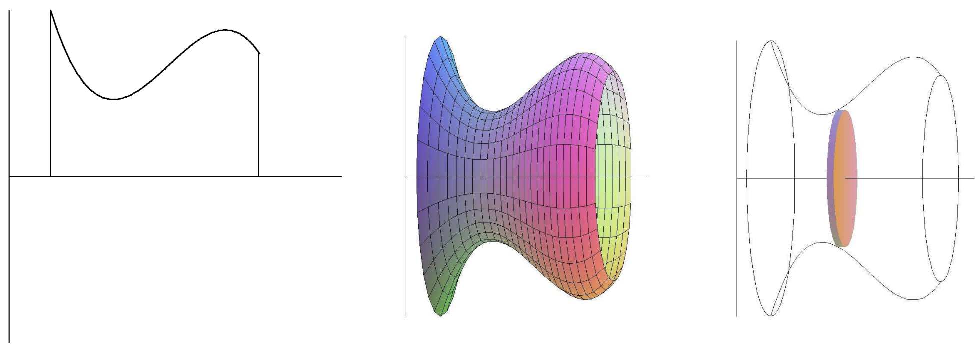


Figure 8.12: A solid of rotation.

Key Idea 8.3.0: The Disk Method

Let a solid be formed by revolving the curve $y = f(x)$ from $x = a$ to $x = b$ around a horizontal axis, and let $R(x)$ be the radius of the cross-sectional disk at x. The volume of the solid is

$$V = \pi \int_a^b R(x)^2 \, dx.$$

Of course a real "slice" of this figure will not have straight sides, but we can approximate the volume of the slice by a cylinder or disk with circular top and bottom and straight sides; the volume of this disk will have the form $\pi r^2 \Delta x$. As long as we can write r in terms of x we can compute the volume by an integral.

Example 8.13: Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, around the x-axis.

Solution. A sketch can help us understand this problem. In Figure **??**(a) the curve $y = 1/x$ is sketched along with the differential element – a disk – at x with radius $R(x) = 1/x$. In Figure **??** (b) the whole solid is pictured, along with the differential element.

The volume of the differential element shown in part (a) of the figure is approximately $\pi R(x_i)^2 \Delta x$, where $R(x_i)$ is the radius of the disk shown and Δx is the thickness of that slice. The radius $R(x_i)$ is the distance from the x-axis to the curve, hence $R(x_i) = 1/x_i$.

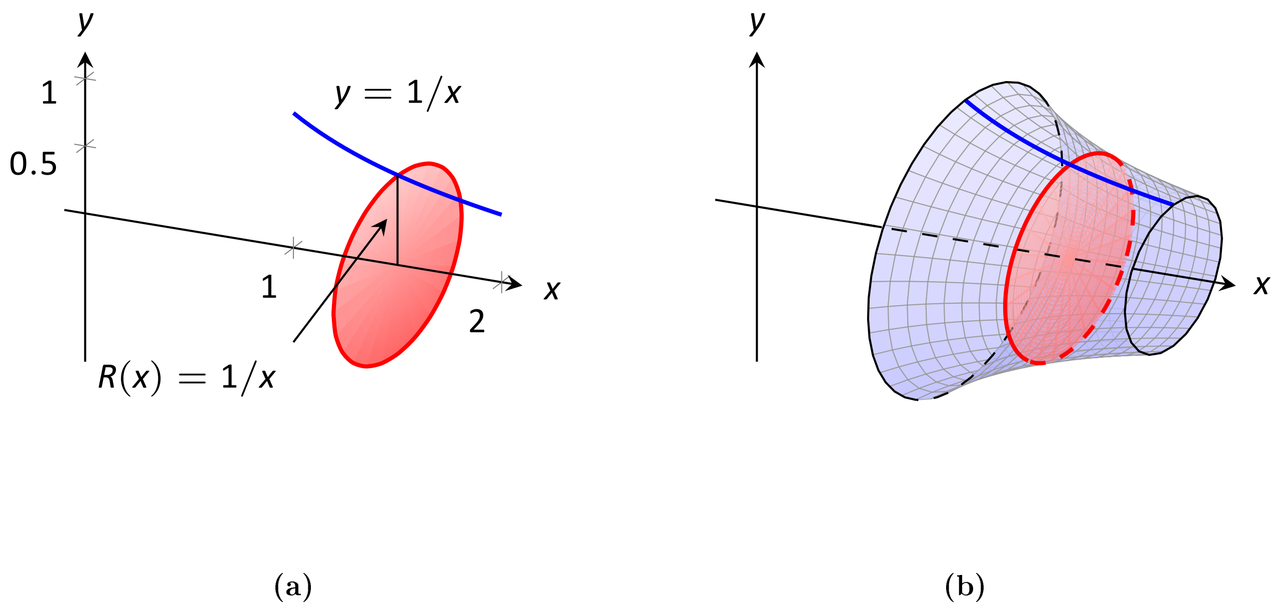


Figure 8.13: Sketching a solid in Example 8.13

Slicing the solid into n equally–spaced slices, we can approximate the total volume by adding up the approximate volume of each slice:

$$\text{Approximate volume } = \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x.$$

Taking the limit of the above sum as $n \to \infty$ gives the actual volume; recognizing this sum as a Riemann sum allows us to evaluate the limit with a definite integral, which matches the formula given in Key Idea 8.3:

$$\begin{aligned} V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x = \pi \int_1^2 \left(\frac{1}{x}\right)^2\, dx = \pi \int_1^2 \frac{1}{x^2}\, dx &= \pi \left[-\frac{1}{x}\right]\Big|_1^2 \\ &= \pi \left[-\frac{1}{2} - (-1)\right] \\ &= \frac{\pi}{2} \text{ units}^3. \end{aligned}$$

♣

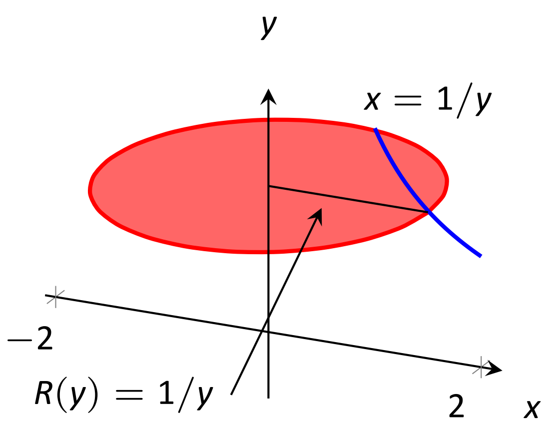
Note: While Key Idea 8.3 is given in terms of functions of x, the principle involved can be applied to functions of y when the axis of rotation is vertical, not horizontal. We demonstrate this in the next example.

Example 8.14: Finding volume using the Disk Method

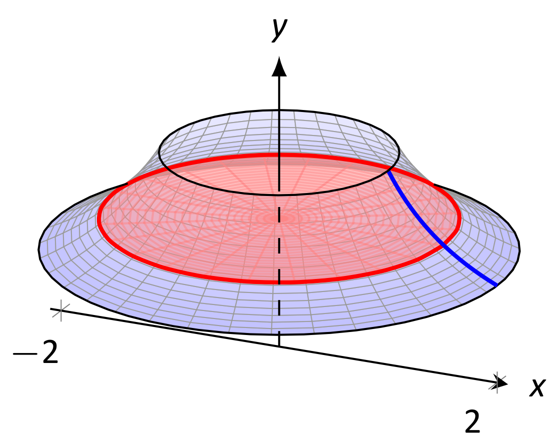
Find the volume of the solid formed by revolving the curve $y = \dfrac{1}{x}$, from $x = 1$ to $x = 2$, about the y-axis.

Solution. Since the axis of rotation is vertical, we need to convert the function into a function of y and convert the x-bounds to y-bounds. Since $y = 1/x$ defines the curve, we rewrite it as $x = 1/y$. The bound $x = 1$ corresponds to the y-bound $y = 1$, and the bound $x = 2$ corresponds to the y-bound $y = 1/2$.

Thus we are rotating the curve $x = 1/y$, from $y = 1/2$ to $y = 1$ about the y-axis to form a solid. The curve and sample differential element are sketched in Figure 8.14 (a), with a full sketch of the solid in Figure 8.14 (b).



(a)



(b)

Figure 8.14: Sketching a solid in Example 8.14.

We integrate to find the volume:

$$V = \pi \int_{1/2}^{1} \frac{1}{y^2}\, dy = -\frac{\pi}{y}\bigg|_{1/2}^{1} = \pi \text{ units}^3.$$

♣

Example 8.15: Volume of a Right Circular Cone

Find the volume of a right circular cone with base radius 10 and height 20. (A right circular cone is one with a circular base and with the tip of the cone directly over the center of the base.)

Solution. We can view this cone as produced by the rotation of the line $y = x/2$ between $x = 0$ and $x = 20$ rotated about the x-axis, as indicated in figure 8.15.

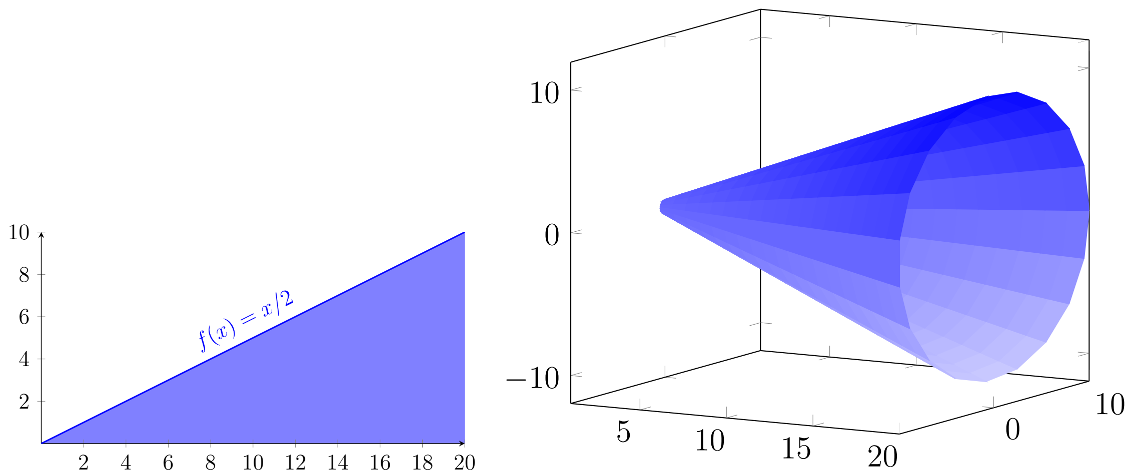


Figure 8.15: A region that generates a cone; approximating the volume by circular disks.

At a particular point on the x-axis, say x_i, the radius of the resulting cone is the y-coordinate of the corresponding point on the line, namely $y_i = x_i/2$. Thus the total volume is approximately

$$\sum_{i=0}^{n-1} \pi(x_i/2)^2 \, dx$$

and the exact volume is

$$\int_0^{20} \pi \frac{x^2}{4} \, dx = \frac{\pi}{4} \frac{20^3}{3} = \frac{2000\pi}{3}.$$

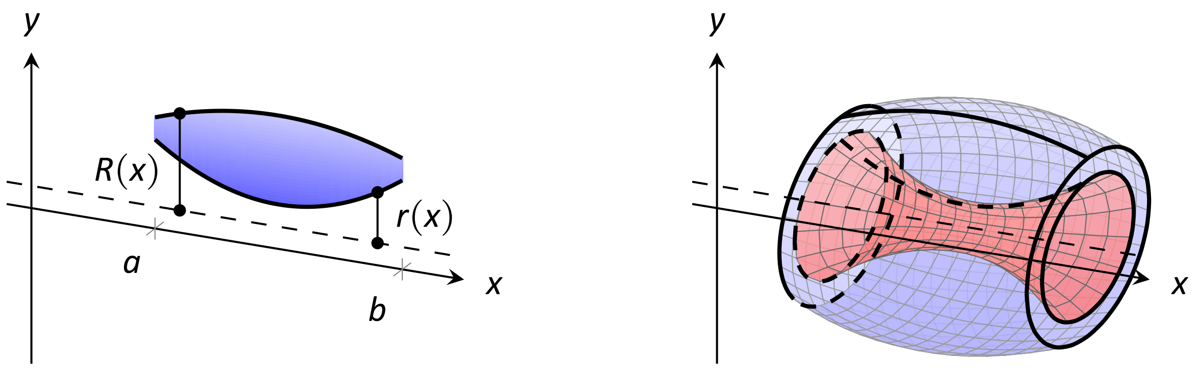
Note that we can instead do the calculation with a generic height and radius:

$$\int_0^h \pi \frac{r^2}{h^2} x^2 \, dx = \frac{\pi r^2}{h^2} \frac{h^3}{3} = \frac{\pi r^2 h}{3},$$

giving us the usual formula for the volume of a cone. ♣

We can also compute the volume of solids of revolution that have a hole in the centre. The general principle is simple: compute the volume of the solid irrespective of the hole, then subtract the volume of the hole. If the outside radius of the solid is $R(x)$ and the inside radius (defining the hole) is $r(x)$, then the volume is

$$V = \pi \int_a^b R(x)^2 \, dx - \pi \int_a^b r(x)^2 \, dx = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) \, dx.$$



(a) (b)

Figure 8.16: Establishing the Washer Method; see also Figure 8.17.

One can generate a solid of revolution with a hole in the middle by revolving a region about an axis. Consider Figure 8.16(a), where a region is sketched along with a dashed, horizontal axis of rotation. By rotating the region about the axis, a solid is formed as sketched in Figure 8.16(b). The outside of the solid has radius $R(x)$, whereas the inside has radius $r(x)$. Each cross section of this solid will be a washer (a disk with a hole in the center) as sketched in Figure 8.17. This leads us to the Washer Method.

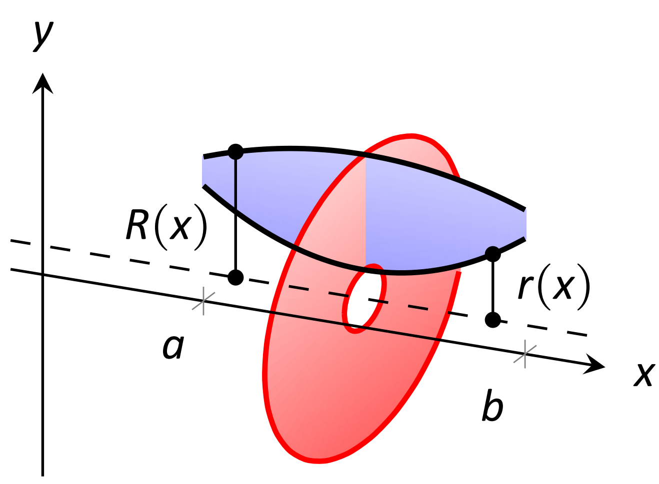


Figure 8.17: Establishing the Washer Method; see also Figure 8.16.

Key Idea 8.3.0: The Washer Method

Let a region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ be rotated about a horizontal axis that does not intersect the region, forming a solid. Each cross section at x will be a washer with outside radius $R(x)$ and inside radius $r(x)$. The volume of the solid is

$$V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) \, dx.$$

Even though we introduced it first, the Disk Method is just a special case of the Washer Method with an inside radius of $r(x) = 0$.

Example 8.16: Volumes with the Washer Method

Find the volume of the object generated when the area between $y = x^2$ and $y = x$ is rotated around the x-axis.

Solution. We begin with a sketch. In figure 8.18 we show the region that is rotated, the resulting solid with the front half cut away, the cone that forms the outer surface, the horn-shaped hole, and a cross-section perpendicular to the x-axis.

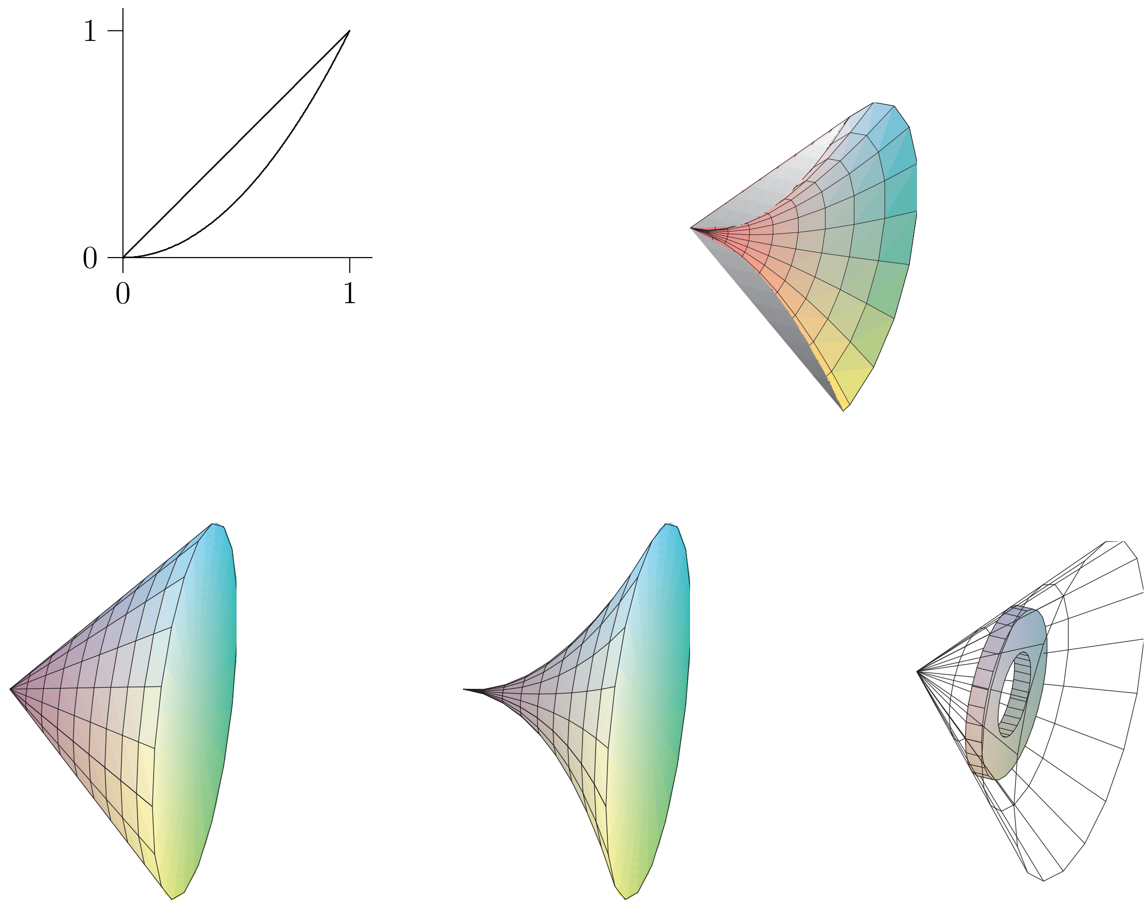


Figure 8.18: Solid with a hole, showing the outer cone and the shape to be removed to form the hole.

We can approximate the volume of a slice of the solid with a washer-shaped volume, as indicated in Figure 8.18.

The thickness is dx, while the area of the face is the area of the outer circle minus the area of the inner circle, say $\pi R(x)^2 - \pi r(x)^2$, or $\pi(\text{TOP})^2 - \pi(\text{BOTTOM})^2$. In the present example, $R(x) = x$ and $r(x) = x^2$. Hence, the whole volume is

$$\int_0^1 \pi\left(R(x)^2 - r(x)^2\right)\,dx = \int_0^1 \pi x^2 - \pi x^4\,dx = \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}.$$

♣

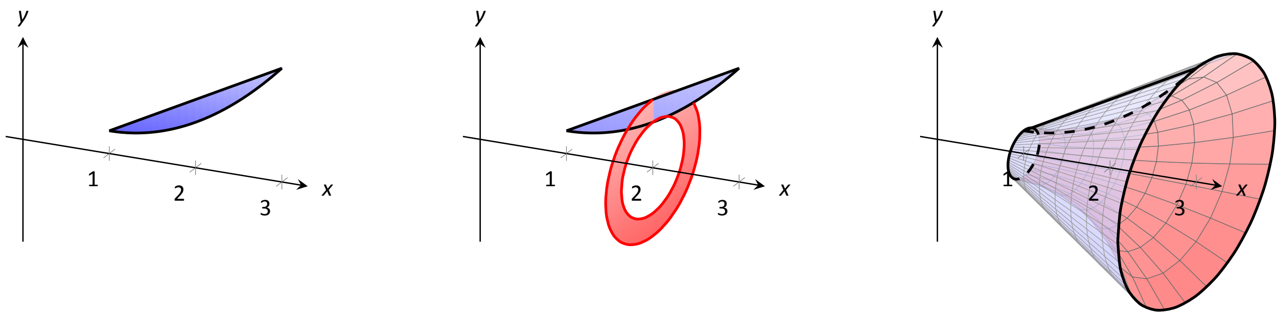
Example 8.17: Finding volume with the Washer Method

Find the volume of the solid formed by rotating the region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$ about the x-axis.

Solution. A sketch of the region will help, as given in Figure 8.19(a). Rotating about the x-axis will produce cross sections in the shape of washers, as shown in Figure 8.19(b); the complete solid is shown in part (c). The outside radius of this washer is $R(x) = 2x + 1$; the inside radius is

$r(x) = x^2 - 2x + 2$. As the region is bounded from $x = 1$ to $x = 3$, we integrate as follows to compute the volume.

$$\begin{aligned} V &= \pi \int_1^3 \Big((2x-1)^2 - (x^2-2x+2)^2 \Big)\, dx \\ &= \pi \int_1^3 \big(-x^4 + 4x^3 - 4x^2 + 4x - 3 \big)\, dx \\ &= \pi \Big[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 + 2x^2 - 3x \Big]\Big|_1^3 \\ &= \frac{104}{15}\pi \approx 21.78 \text{ units}^3. \end{aligned}$$



(a) (b) (c)

Figure 8.19: Sketching the differential element and solid in Example 8.17.

♣

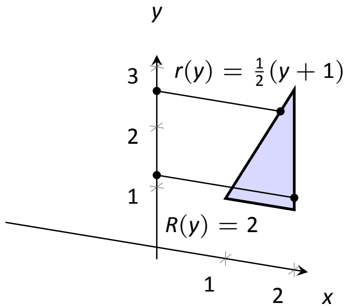
When rotating about a vertical axis, the outside and inside radius functions must be functions of y.

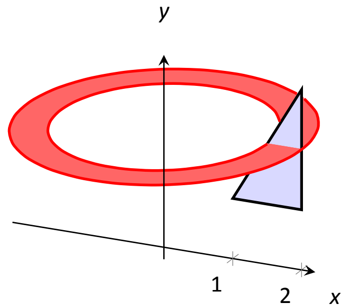
Example 8.18: Finding volume with the Washer Method

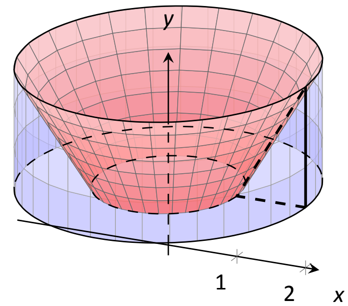
Find the volume of the solid formed by rotating the triangular region with vertices at $(1,1)$, $(2,1)$ *and* $(2,3)$ *about the* y*-axis.*

Solution. The triangular region is sketched in Figure 8.20(a); the differential element is sketched in (b) and the full solid is drawn in (c). They help us establish the outside and inside radii. Since the axis of rotation is vertical, each radius is a function of y.

The outside radius $R(y)$ is formed by the line connecting $(2,1)$ and $(2,3)$; it is a constant function, as regardless of the y-value the distance from the line to the axis of rotation is 2. Thus $R(y) = 2$.







(a) (b) (c)

Figure 8.20: Sketching the solid in Example ??.

The inside radius is formed by the line connecting $(1,1)$ and $(2,3)$. The equation of this line is $y = 2x - 1$, but we need to refer to it as a function of y. Solving for x gives $r(y) = \frac{1}{2}(y+1)$.

We integrate over the y-bounds of $y = 1$ to $y = 3$. Thus the volume is

$$\begin{aligned} V &= \pi \int_1^3 \left(2^2 - \left(\frac{1}{2}(y+1)\right)^2\right)\, dy \\ &= \pi \int_1^3 \left(-\frac{1}{4}y^2 - \frac{1}{2}y + \frac{15}{4}\right)\, dy \\ &= \pi\left[-\frac{1}{12}y^3 - \frac{1}{4}y^2 + \frac{15}{4}y\right]\Big|_1^3 \\ &= \frac{10}{3}\pi \approx 10.47 \text{ units}^3. \end{aligned}$$

♣

This section introduced a new application of the definite integral. Our default view of the definite integral is that it gives "the area under the curve." However, we can establish definite integrals that represent other quantities; in this section, we computed volume.

The ultimate goal of this section is not to compute volumes of solids. That can be useful, but what is more useful is the understanding of this basic principle of integral calculus: to find the exact value of some quantity,

- we start with an approximation (in this section, slice the solid and approximate the volume of each slice),
- then make the approximation better by refining our original approximation (i.e., use more slices),
- then use limits to establish a definite integral which gives the exact value.

Exercises for 8.3

8.3.1 *Verify that* $\pi\int_0^1 (1+\sqrt{y})^2-(1-\sqrt{y})^2\,dy+\pi\int_1^4 (1+\sqrt{y})^2-(y-1)^2=\frac{8}{3}\pi+\frac{65}{6}\pi=\frac{27}{2}\pi.$

8.3.2 *Verify that* $\int_0^3 2\pi x(x+1-(x-1)^2)\,dx=\frac{27}{2}\pi.$

8.3.3 *Verify that* $\int_0^1 \pi(1-x^2)^2\,dx=\frac{8}{15}\pi.$

8.3.4 *Verify that* $\int_0^1 2\pi y\sqrt{1-y}\,dy=\frac{8}{15}\pi.$

8.3.5 *Use integration to find the volume of the solid obtained by revolving the region bounded by* $x+y=2$ *and the* x *and* y *axes around the* x*-axis.*

8.3.6 *Find the volume of the solid obtained by revolving the region bounded by* $y=x-x^2$ *and the* x*-axis around the* x*-axis.*

8.3.7 *Find the volume of the solid obtained by revolving the region bounded by* $y=\sqrt{\sin x}$ *between* $x=0$ *and* $x=\pi/2$*, the* y*-axis, and the line* $y=1$ *around the* x*-axis.*

8.3.8 *Let* S *be the region of the* xy*-plane bounded above by the curve* $x^3y=64$*, below by the line* $y=1$*, on the left by the line* $x=2$*, and on the right by the line* $x=4$*. Find the volume of the solid obtained by rotating* S *around:*

(a) *the* x*-axis;*

(b) *the line* $y=1$*;*

(c) *the* y*-axis; and*

(d) *the line* $x=2$*.*

8.3.9 *The equation* $x^2/9+y^2/4=1$ *describes an ellipse. Find the volume of the solid obtained by rotating the ellipse around the* x*-axis and also around the* y*-axis. These solids are called* **ellipsoids***; one is vaguely rugby-ball shaped, one is sort of flying-saucer shaped, or perhaps squished-beach-ball-shaped.*

Figure 8.21: Ellipsoids.

8.3.10 *Use integration to compute the volume of a sphere of radius r. You should of course get the well-known formula $4\pi r^3/3$.*

8.3.11 *A hemispheric bowl of radius r contains water to a depth h. Find the volume of water in the bowl.*

8.3.12 *The base of a tetrahedron (a triangular pyramid) of height h is an equilateral triangle of side s. Its cross-sections perpendicular to an altitude are equilateral triangles. Express its volume V as an integral, and find a formula for V in terms of h and s. Verify that your answer is $(1/3)$(area of base)(height).*

8.3.13 *The base of a solid is the region between $f(x) = \cos x$ and $g(x) = -\cos x$, $-\pi/2 \le x \le \pi/2$, and its cross-sections perpendicular to the x-axis are squares. Find the volume of the solid.*

8.4 The Mean Value Theorem for Integrals and Average Value

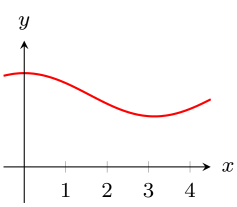
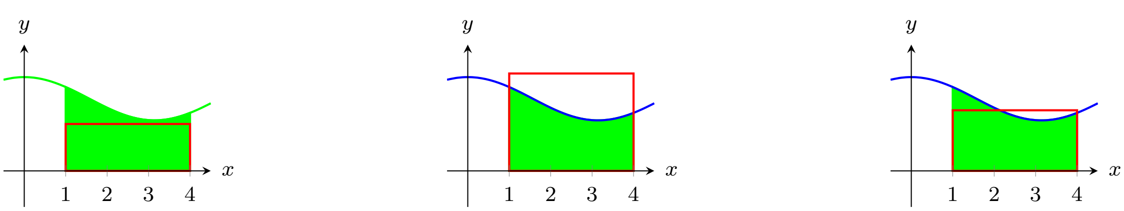


Figure 8.22: A graph of a function f to introduce the Mean Value Theorem.

Consider the graph of a function f in Figure 8.23(a) and the area defined by $\int_1^4 f(x)\ dx$. Three rectangles are drawn in Figure 8.23(b); in (a), the height of the rectangle is greater than f on $[1, 4]$, hence the area of this rectangle is is greater than $\int_0^4 f(x)\ dx$.
In (b), the height of the rectangle is smaller than f on $[1, 4]$, hence the area of this rectangle is less than $\int_1^4 f(x)\ dx$.



(a) (b) (c)

Figure 8.23: Differently sized rectangles give upper and lower bounds on $\int_1^4 f(x)\ dx$; the last rectangle matches the area exactly.

Finally, in (c) the height of the rectangle is such that the area of the rectangle is *exactly* that of $\int_0^4 f(x)\ dx$. Since rectangles that are "too big", as in (a), and rectangles that are "too little," as in (b), give areas greater/lesser than $\int_1^4 f(x)\ dx$, it makes sense that there is a rectangle, whose top intersects $f(x)$ somewhere on $[1,4]$, whose area is *exactly* that of the definite integral.

We state this idea formally in a theorem.

Theorem 8.3: The Mean Value Theorem of Integration

Let f be continuous on $[a,b]$. There exists a value c in $[a,b]$ such that

$$\int_a^b f(x)\ dx = f(c)(b-a).$$

This is an *existential* statement; c exists, but we do not provide a method of finding it. Theorem 8.3 is directly connected to the Mean Value Theorem of Differentiation, given as Theorem 5.4; we leave it to the reader to see how.

We demonstrate the principles involved in this version of the Mean Value Theorem in the following example.

Example 8.19: Using the Mean Value Theorem

Consider $\int_0^\pi \sin x\ dx$. Find a value c guaranteed by the Mean Value Theorem.

Solution. We first need to evaluate $\int_0^\pi \sin x\ dx$. (This was previously done in Example **??**.)

$$\int_0^\pi \sin x\ dx = -\cos x\Big|_0^\pi = 2.$$

Thus we seek a value c in $[0,\pi]$ such that $\pi \sin c = 2$.

$$\pi \sin c = 2 \quad \Rightarrow \quad \sin c = 2/\pi \quad \Rightarrow \quad c = \arcsin(2/\pi) \approx 0.69.$$

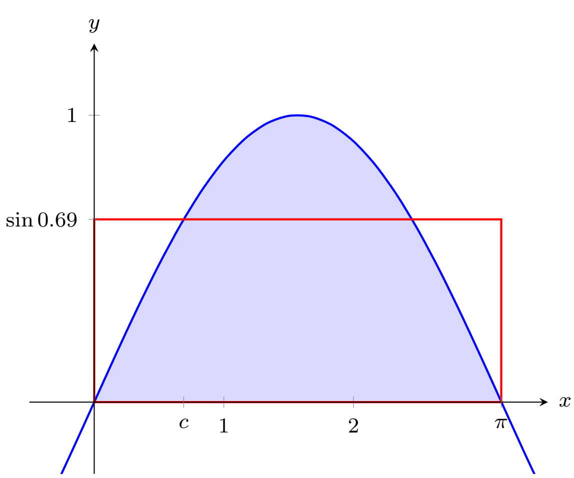


Figure 8.24: A graph of $y = \sin x$ on $[0,\pi]$ and the rectangle guaranteed by the Mean Value Theorem.

In Figure 8.24 $\sin x$ is sketched along with a rectangle with height $\sin(0.69)$. The area of the rectangle is the same as the area under $\sin x$ on $[0, \pi]$.

♣

Let f be a function on $[a, b]$ with c such that $f(c)(b-a) = \int_a^b f(x)\,dx$. Consider $\int_a^b \big(f(x) - f(c)\big)\,dx$:

$$\begin{aligned}\int_a^b \big(f(x) - f(c)\big)\,dx &= \int_a^b f(x) - \int_a^b f(c)\,dx \\ &= f(c)(b-a) - f(c)(b-a) \\ &= 0.\end{aligned}$$

When $f(x)$ is shifted by $-f(c)$, the amount of area under f above the x–axis on $[a, b]$ is the same as the amount of area below the x–axis above f; see Figure 8.25 for an illustration of this. In this sense, we can say that $f(c)$ is the *average value* of f on $[a, b]$.

(a) **(b)**

Figure 8.25: On the left, a graph of $y = f(x)$ and the rectangle guaranteed by the Mean Value Theorem. On the right, $y = f(x)$ is shifted down by $f(c)$; the resulting "area under the curve" is 0.

The value $f(c)$ is the average value in another sense. First, recognize that the Mean Value Theorem can be rewritten as

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

for some value of c in $[a, b]$. Next, partition the interval $[a, b]$ into n equally spaced subintervals, $a = x_1 < x_2 < \ldots < x_{n+1} = b$ and choose any c_i in $[x_i, x_{i+1}]$. The average of the numbers $f(c_1)$, $f(c_2)$, ..., $f(c_n)$ is:

$$\frac{1}{n}\Big(f(c_1) + f(c_2) + \ldots + f(c_n)\Big) = \frac{1}{n}\sum_{i=1}^n f(c_i).$$

Multiply this last expression by 1 in the form of $\frac{(b-a)}{(b-a)}$:

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n f(c_i) &= \sum_{i=1}^n f(c_i)\frac{1}{n} \\ &= \sum_{i=1}^n f(c_i)\frac{1}{n}\frac{(b-a)}{(b-a)} \\ &= \frac{1}{b-a}\sum_{i=1}^n f(c_i)\frac{b-a}{n}\end{aligned}$$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(c_i)\Delta x \quad \text{(where } \Delta x = (b-a)/n\text{)}$$

Now take the limit as $n \to \infty$:

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(c_i)\Delta x \quad = \quad \frac{1}{b-a}\int_a^b f(x)\, dx \quad = \quad f(c).$$

This tells us this: when we evaluate f at n (somewhat) equally spaced points in $[a,b]$, the average value of these samples is $f(c)$ as $n \to \infty$.

This leads us to a definition.

Definition 8.1: The Average Value of f on $[a,b]$

Let f be continuous on $[a,b]$. The ***average value of*** f ***on*** $[a,b]$ *is $f(c)$, where c is a value in $[a,b]$ guaranteed by the Mean Value Theorem. I.e.,*

$$\textit{Average Value of } f \textit{ on } [a,b] = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

An application of this definition is given in the following example.

Example 8.20: Finding the average value of a function

An object moves back and forth along a straight line with a velocity given by $v(t) = (t-1)^2$ on $[0,3]$, where t is measured in seconds and $v(t)$ is measured in m/s.
What is the average velocity of the object?

Solution. By our definition, the average velocity is:

$$\frac{1}{3-0}\int_0^3 (t-1)^2\, dt = \frac{1}{3}\int_0^3 \left(t^2 - 2t + 1\right)\, dt = \frac{1}{3}\left(\frac{1}{3}t^3 - t^2 + t\right)\Bigg|_0^3 = 1 \text{ m/s}.$$

♣

We can understand the above example through a simpler situation. Suppose you drove 100 kilometres in 2 hours. What was your average speed? The answer is simple: displacement/time = 100 km/2 hours = 50 kmph.

What was the displacement of the object in Example 8.20? We calculate this by integrating its velocity function: $\int_0^3 (t-1)^2\, dt = 3$ m. Its final position was 3 feet from its initial position after 3 seconds: its average velocity was 1 m/s.

Exercises for 8.4

8.4.1 *Find the average height of* $\cos x$ *over the intervals* $[0, \pi/2]$, $[-\pi/2, \pi/2]$, *and* $[0, 2\pi]$.

8.4.2 *Find the average height of* x^2 *over the interval* $[-2, 2]$.

8.4.3 *Find the average height of* $1/x^2$ *over the interval* $[1, A]$.

8.4.4 *Find the average height of* $\sqrt{1-x^2}$ *over the interval* $[-1, 1]$.

8.4.5 *An object moves with velocity* $v(t) = -t^2 + 1$ *feet per second between* $t = 0$ *and* $t = 2$. *Find the average velocity and the average speed of the object between* $t = 0$ *and* $t = 2$.

8.4.6 *The observation deck on the 102nd floor of the Empire State Building is 1,224 feet above the ground. If a steel ball is dropped from the observation deck its velocity at time* t *is approximately* $v(t) = -32t$ *feet per second. Find the average speed between the time it is dropped and the time it hits the ground, and find its speed when it hits the ground.*

8.5 Work

A fundamental concept in classical physics is **work**: If an object is moved in a straight line against a force F for a distance d the work done is $W = Fd$.

Example 8.21: Constant Force

How much work is done in lifting a 10 pound weight vertically a distance of 5 feet?

Solution. The force due to gravity on a 10 pound weight is 10 pounds at the surface of the earth, and it does not change appreciably over 5 feet. The work done is $W = 10 \cdot 5 = 50$ foot-pounds. ♣

In reality few situations are so simple. The force might not be constant over the range of motion, as in the next example.

Example 8.22: Lifting a Weight

How much work is done in lifting a 10 pound weight from the surface of the earth to an orbit 100 miles above the surface?

Solution. Over 100 miles the force due to gravity does change significantly, so we need to take this into account. The force exerted on a 10 pound weight at a distance r from the center of the earth is $F = k/r^2$ and by definition it is 10 when r is the radius of the earth (we assume the earth is a sphere). How can we approximate the work done? We divide the path from the surface to orbit into n small subpaths. On each subpath the force due to gravity is roughly constant, with value k/r_i^2 at distance r_i. The work to raise the object from r_i to r_{i+1} is thus approximately $k/r_i^2 \Delta r$ and the total work is approximately

$$\sum_{i=0}^{n-1} \frac{k}{r_i^2}\Delta r,$$

or in the limit

$$W = \int_{r_0}^{r_1} \frac{k}{r^2}\, dr,$$

where r_0 is the radius of the earth and r_1 is r_0 plus 100 miles. The work is

$$W = \int_{r_0}^{r_1} \frac{k}{r^2}\, dr = -\left.\frac{k}{r}\right|_{r_0}^{r_1} = -\frac{k}{r_1} + \frac{k}{r_0}.$$

Using $r_0 = 20925525$ feet we have $r_1 = 21453525$. The force on the 10 pound weight at the surface of the earth is 10 pounds, so $10 = k/20925525^2$, giving $k = 4378775965256250$. Then

$$-\frac{k}{r_1} + \frac{k}{r_0} = \frac{491052320000}{95349} \approx 5150052 \quad \text{foot-pounds.}$$

Note that if we assume the force due to gravity is 10 pounds over the whole distance we would calculate the work as $10(r_1 - r_0) = 10 \cdot 100 \cdot 5280 = 5280000$, somewhat higher since we don't account for the weakening of the gravitational force. ♣

Example 8.23: Lifting an Object

How much work is done in lifting a 10 kilogram object from the surface of the earth to a distance D from the center of the earth?

Solution. This is the same problem as before in different units, and we are not specifying a value for D. As before

$$W = \int_{r_0}^{D} \frac{k}{r^2}\, dr = -\left.\frac{k}{r}\right|_{r_0}^{D} = -\frac{k}{D} + \frac{k}{r_0}.$$

While "weight in pounds" is a measure of force, "weight in kilograms" is a measure of mass. To convert to force we need to use Newton's law $F = ma$. At the surface of the earth the acceleration due to gravity is approximately 9.8 meters per second squared, so the force is $F = 10 \cdot 9.8 = 98$. The units here are "kilogram-meters per second squared" or "kg m/s^2", also known as a Newton (N), so $F = 98$ N. The radius of the earth is approximately 6378.1 kilometers or 6378100 meters. Now the problem proceeds as before. From $F = k/r^2$ we compute k: $98 = k/6378100^2$, $k = 3.986655642 \cdot 10^{15}$. Then the work is:

$$W = -\frac{k}{D} + 6.250538000 \cdot 10^8 \quad \text{Newton-meters.}$$

As D increases W of course gets larger, since the quantity being subtracted, $-k/D$, gets smaller. But note that the work W will never exceed $6.250538000 \cdot 10^8$, and in fact will approach this value as D gets larger. In short, with a finite amount of work, namely $6.250538000 \cdot 10^8$ N-m, we can lift the 10 kilogram object as far as we wish from earth. ♣

Next is an example in which the force is constant, but there are many objects moving different distances.

Example 8.24: Multiple Objects Moving

Suppose that a water tank is shaped like a right circular cone with the tip at the bottom, and has height 10 meters and radius 2 meters at the top. If the tank is full, how much work is required to pump all the water out over the top?

Solution. Here we have a large number of atoms of water that must be lifted different distances to get to the top of the tank. Fortunately, we don't really have to deal with individual atoms—we can consider all the atoms at a given depth together.

To approximate the work, we can divide the water in the tank into horizontal sections, approximate the volume of water in a section by a thin disk, and compute the amount of work required to lift each disk to the top of the tank. As usual, we take the limit as the sections get thinner and thinner to get the total work.

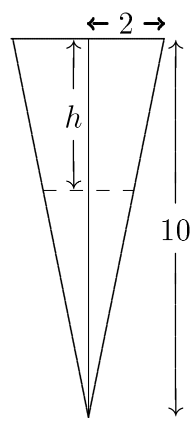


Figure 8.26: Cross-section of a conical water tank.

At depth h the circular cross-section through the tank has radius $r = (10-h)/5$, by similar triangles, and area $\pi(10-h)^2/25$. A section of the tank at depth h thus has volume approximately $\pi(10-h)^2/25\Delta h$ and so contains $\sigma\pi(10-h)^2/25\Delta h$ kilograms of water, where σ is the density of water in kilograms per cubic meter; $\sigma \approx 1000$. The force due to gravity on this much water is $9.8\sigma\pi(10-h)^2/25\Delta h$, and finally, this section of water must be lifted a distance h, which requires $h 9.8\sigma\pi(10-h)^2/25\Delta h$ Newton-meters of work. The total work is therefore

$$W = \frac{9.8\sigma\pi}{25}\int_0^{10} h(10-h)^2\,dh = \frac{980000}{3}\pi \approx 1026254 \quad \text{Newton-meters.}$$

♣

A spring has a "natural length," its length if nothing is stretching or compressing it. If the spring is either stretched or compressed the spring provides an opposing force; according to **Hooke's Law** the magnitude of this force is proportional to the distance the spring has been stretched or compressed: $F = kx$. The constant of proportionality, k, of course depends on the spring. Note that x here represents the *change* in length from the natural length.

Example 8.25: Compressing a Spring

Suppose $k = 5$ for a given spring that has a natural length of 0.1 meters. Suppose a force is applied that compresses the spring to length 0.08. What is the magnitude of the force?

Solution. Assuming that the constant k has appropriate dimensions (namely, kg/s^2), the force is $5(0.1 - 0.08) = 5(0.02) = 0.1$ Newtons. ♣

Example 8.26: Compressing a Spring (continued)

How much work is done in compressing the spring in the previous example from its natural length to 0.08 meters? From 0.08 meters to 0.05 meters? How much work is done to stretch the spring from 0.1 meters to 0.15 meters?

Solution. We can approximate the work by dividing the distance that the spring is compressed (or stretched) into small subintervals. Then the force exerted by the spring is approximately constant over the subinterval, so the work required to compress the spring from x_i to x_{i+1} is approximately $5(x_i - 0.1)\Delta x$. The total work is approximately

$$\sum_{i=0}^{n-1} 5(x_i - 0.1)\Delta x$$

and in the limit

$$W = \int_{0.1}^{0.08} 5(x - 0.1)\,dx = \left.\frac{5(x-0.1)^2}{2}\right|_{0.1}^{0.08} = \frac{5(0.08-0.1)^2}{2} - \frac{5(0.1-0.1)^2}{2} = \frac{1}{1000} \text{ N-m.}$$

The other values we seek simply use different limits. To compress the spring from 0.08 meters to 0.05 meters takes

$$W = \int_{0.08}^{0.05} 5(x - 0.1)\,dx = \left.\frac{5(x-0.1)^2}{2}\right|_{0.08}^{0.05} = \frac{5(0.05-0.1)^2}{2} - \frac{5(0.08-0.1)^2}{2} = \frac{21}{4000} \quad \text{N-m}$$

and to stretch the spring from 0.1 meters to 0.15 meters requires

$$W = \int_{0.1}^{0.15} 5(x - 0.1)\,dx = \left.\frac{5(x-0.1)^2}{2}\right|_{0.1}^{0.15} = \frac{5(0.15-0.1)^2}{2} - \frac{5(0.1-0.1)^2}{2} = \frac{1}{160} \quad \text{N-m.}$$

♣

Exercises for 8.5

8.5.1 *How much work is done in lifting a 100 kilogram weight from the surface of the earth to an orbit 35,786 kilometers above the surface of the earth?*

8.5.2 *How much work is done in lifting a 100 kilogram weight from an orbit 1000 kilometers above the surface of the earth to an orbit 35,786 kilometers above the surface of the earth?*

8.5.3 *A water tank has the shape of an upright cylinder with radius* $r = 1$ *meter and height 10 meters. If the depth of the water is 5 meters, how much work is required to pump all the water out the top of the tank?*

8.5.4 *Suppose the tank of the previous problem is lying on its side, so that the circular ends are vertical, and that it has the same amount of water as before. How much work is required to pump the water out the top of the tank (which is now 2 meters above the bottom of the tank)?*

8.5.5 *A water tank has the shape of the bottom half of a sphere with radius* $r = 1$ *meter. If the tank is full, how much work is required to pump all the water out the top of the tank?*

8.5.6 *A spring has constant* $k = 10$ *kg/s*^{2}*. How much work is done in compressing it* $1/10$ *meter from its natural length?*

8.5.7 *A force of 2 Newtons will compress a spring from 1 meter (its natural length) to 0.8 meters. How much work is required to stretch the spring from 1.1 meters to 1.5 meters?*

8.5.8 *A 20 meter long steel cable has density 2 kilograms per meter, and is hanging straight down. How much work is required to lift the entire cable to the height of its top end?*

8.5.9 *The cable in the previous problem has a 100 kilogram bucket of concrete attached to its lower end. How much work is required to lift the entire cable and bucket to the height of its top end?*

8.5.10 *Consider again the cable and bucket of the previous problem. How much work is required to lift the bucket 10 meters by raising the cable 10 meters? (The top half of the cable ends up at the height of the top end of the cable, while the bottom half of the cable is lifted 10 meters.)*

8.6 Center of Mass

Suppose a beam is 10 meters long, and that there are three weights on the beam: a 10 kilogram weight 3 meters from the left end, a 5 kilogram weight 6 meters from the left end, and a 4 kilogram weight 8 meters from the left end. Where should a fulcrum be placed so that the beam balances? Let's assign a scale to the beam, from 0 at the left end to 10 at the right, so that we can denote locations on the beam simply as x coordinates; the weights are at $x = 3$, $x = 6$, and $x = 8$, as in Figure 8.27.

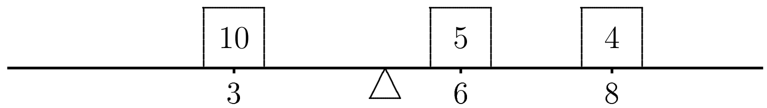


Figure 8.27: A beam with three masses.

Suppose to begin with that the fulcrum is placed at $x = 5$. What will happen? Each weight applies a force to the beam that tends to rotate it around the fulcrum; this effect is measured by a quantity called **torque**, proportional to the mass times the distance from the fulcrum. Of course, weights on different sides of the fulcrum rotate the beam in opposite directions. We can distinguish this by using a signed distance in the formula for torque. So with the fulcrum at 5, the torques induced by the three weights will be proportional to $(3-5)10 = -20$, $(6-5)5 = 5$, and $(8-5)4 = 12$. For the beam to balance, the sum of the torques must be zero; since the sum is $-20+5+12 = -3$, the beam rotates counter-clockwise, and to get the beam to balance we need to move the fulcrum to the left. To calculate exactly where the fulcrum should be, we let $\bar{x}$ denote the location of the fulcrum when the beam is in balance. The total torque on the beam is then $(3-\bar{x})10+(6-\bar{x})5+(8-\bar{x})4 = 92-19\bar{x}$. Since the beam balances at $\bar{x}$ it must be that $92-19\bar{x} = 0$ or $\bar{x} = 92/19 \approx 4.84$, that is, the fulcrum should be placed at $x = 92/19$ to balance the beam.

Now suppose that we have a beam with varying density—some portions of the beam contain more mass than other portions of the same size. We want to figure out where to put the fulcrum so that the beam balances.

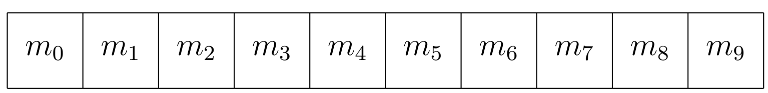


Figure 8.28: A solid beam.

Example 8.27: Balance Point of a Beam

Find the balance point of a solid beam, illustrated in Figure 8.28, assuming the beam is 10 meters long and that the density is $1+x$ kilograms per meter at location x on the beam.

Solution. To approximate the solution, we can think of the beam as a sequence of weights "on" a beam. For example, we can think of the portion of the beam between $x = 0$ and $x = 1$ as a weight sitting at $x = 0$, the portion between $x = 1$ and $x = 2$ as a weight sitting at $x = 1$, and so on, as indicated in Figure 8.28. We then approximate the mass of the weights by assuming that each portion of the beam has constant density. So the mass of the first weight is approximately $m_0 = (1+0)1 = 1$ kilograms, namely, $(1+0)$ kilograms per meter times 1 meter. The second weight is $m_1 = (1+1)1 = 2$ kilograms, and so on to the tenth weight with $m_9 = (1+9)1 = 10$ kilograms. So in this case the total torque is

$$(0-\bar{x})m_0 + (1-\bar{x})m_1 + \cdots + (9-\bar{x})m_9 = (0-\bar{x})1 + (1-\bar{x})2 + \cdots + (9-\bar{x})10.$$

If we set this to zero and solve for $\bar{x}$ we get $\bar{x} = 6$. In general, if we divide the beam into n portions, the mass of weight number i will be $m_i = (1 + x_i)(x_{i+1} - x_i) = (1 + x_i)\Delta x$ and the torque induced by weight number i will be $(x_i - \bar{x})m_i = (x_i - \bar{x})(1 + x_i)\Delta x$. The total torque is then

$$\begin{aligned}
(x_0 - \bar{x})(1 + x_0)\Delta x + (x_1 - \bar{x})(1 + x_1)\Delta x + \cdots + (x_{n-1} - \bar{x})(1 + x_{n-1})\Delta x \\
= \sum_{i=0}^{n-1} x_i(1 + x_i)\Delta x - \sum_{i=0}^{n-1} \bar{x}(1 + x_i)\Delta x \\
= \sum_{i=0}^{n-1} x_i(1 + x_i)\Delta x - \bar{x} \sum_{i=0}^{n-1} (1 + x_i)\Delta x.
\end{aligned}$$

If we set this equal to zero and solve for $\bar{x}$ we get an approximation to the balance point of the beam:

$$\begin{aligned}
0 &= \sum_{i=0}^{n-1} x_i(1 + x_i)\Delta x - \bar{x} \sum_{i=0}^{n-1} (1 + x_i)\Delta x \\
\bar{x} \sum_{i=0}^{n-1} (1 + x_i)\Delta x &= \sum_{i=0}^{n-1} x_i(1 + x_i)\Delta x \\
\bar{x} &= \frac{\displaystyle\sum_{i=0}^{n-1} x_i(1 + x_i)\Delta x}{\displaystyle\sum_{i=0}^{n-1} (1 + x_i)\Delta x}.
\end{aligned}$$

The denominator of this fraction has a very familiar interpretation. Consider one term of the sum in the denominator: $(1 + x_i)\Delta x$. This is the density near x_i times a short length, Δx, which in other words is approximately the mass of the beam between x_i and x_{i+1}. When we add these up we get approximately the mass of the beam.

Now each of the sums in the fraction has the right form to turn into an integral, which in turn gives us the exact value of $\bar{x}$:

$$\bar{x} = \frac{\displaystyle\int_0^{10} x(1 + x)\,dx}{\displaystyle\int_0^{10} (1 + x)\,dx}.$$

The numerator of this fraction is called the **moment** of the system around zero:

$$\int_0^{10} x(1 + x)\,dx = \int_0^{10} x + x^2\,dx = \frac{1150}{3},$$

and the denominator is the mass of the beam:

$$\int_0^{10} (1 + x)\,dx = 60,$$

and the balance point, officially called the **center of mass** is

$$\bar{x} = \frac{1150}{3}\frac{1}{60} = \frac{115}{18} \approx 6.39.$$

♣

It should be apparent that there was nothing special about the density function $\sigma(x) = 1 + x$ or the length of the beam, or even that the left end of the beam is at the origin. In general, if the density of the beam is $\sigma(x)$ and the beam covers the interval $[a, b]$, the moment of the beam around zero is

$$M_0 = \int_a^b x\sigma(x)\,dx$$

and the total mass of the beam is

$$M = \int_a^b \sigma(x)\,dx$$

and the center of mass is at

$$\bar{x} = \frac{M_0}{M}.$$

Example 8.28: Center of Mass of a Beam

Suppose a beam lies on the x-axis between 20 and 30, and has density function $\sigma(x) = x - 19$. Find the center of mass.

Solution. This is the same as the previous example except that the beam has been moved. Note that the density at the left end is $20 - 19 = 1$ and at the right end is $30 - 19 = 11$, as before. Hence the center of mass must be at approximately $20 + 6.39 = 26.39$. Let's see how the calculation works out.

$$\begin{aligned}
M_0 &= \int_{20}^{30} x(x-19)\,dx = \int_{20}^{30} x^2 - 19x\,dx = \left.\frac{x^3}{3} - \frac{19x^2}{2}\right|_{20}^{30} = \frac{4750}{3} \\
M &= \int_{20}^{30} x - 19\,dx = \left.\frac{x^2}{2} - 19x\right|_{20}^{30} = 60 \\
\frac{M_0}{M} &= \frac{4750}{3}\frac{1}{60} = \frac{475}{18} \approx 26.39.
\end{aligned}$$

♣

Example 8.29: Centroid of a Flat Plate

Suppose a flat plate of uniform density has the shape contained by $y = x^2$, $y = 1$, and $x = 0$, in the first quadrant. Find the center of mass. (Since the density is constant, the center of mass depends only on the shape of the plate, not the density, or in other words, this is a purely geometric quantity. In such a case the center of mass is called the **centroid***.)*

Solution. This is a two dimensional problem, but it can be solved as if it were two one dimensional problems: we need to find the x and y coordinates of the center of mass, $\bar{x}$ and $\bar{y}$, and fortunately we can do these independently. Imagine looking at the plate edge on, from below the x-axis. The plate will appear to be a beam, and the mass of a short section of the "beam", say between x_i and

x_{i+1}, is the mass of a strip of the plate between x_i and x_{i+1}. See Figure 8.29 showing the plate from above and as it appears edge on.

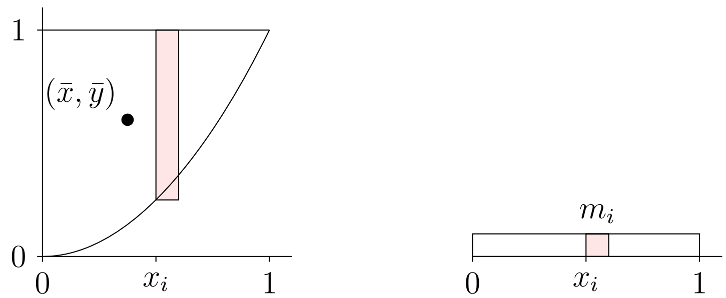


Figure 8.29: Center of mass for a two dimensional plate.

Since the plate has uniform density we may as well assume that $\sigma = 1$. Then the mass of the plate between x_i and x_{i+1} is approximately $m_i = \sigma(1 - x_i^2)\Delta x = (1 - x_i^2)\Delta x$. Now we can compute the moment around the y-axis:

$$M_y = \int_0^1 x(1 - x^2)\,dx = \frac{1}{4}$$

and the total mass

$$M = \int_0^1 (1 - x^2)\,dx = \frac{2}{3}$$

and finally

$$\bar{x} = \frac{1}{4}\frac{3}{2} = \frac{3}{8}.$$

Next we do the same thing to find $\bar{y}$. The mass of the plate between y_i and y_{i+1} is approximately $n_i = \sqrt{y}\Delta y$, so

$$M_x = \int_0^1 y\sqrt{y}\,dy = \frac{2}{5}$$

and

$$\bar{y} = \frac{2}{5}\frac{3}{2} = \frac{3}{5},$$

since the total mass M is the same. The center of mass is shown in Figure 8.29. ♣

Example 8.30: Center of Mass under Cosine

Find the center of mass of a thin, uniform plate whose shape is the region between $y = \cos x$ and the x-axis between $x = -\pi/2$ and $x = \pi/2$.

Solution. It is clear that $\bar{x} = 0$, but for practice let's compute it anyway. We will need the total mass, so we compute it first:

$$M = \int_{-\pi/2}^{\pi/2} \cos x\,dx = \sin x\Big|_{-\pi/2}^{\pi/2} = 2.$$

The moment around the y-axis is

$$M_y = \int_{-\pi/2}^{\pi/2} x\cos x\,dx = \cos x + x\sin x\Big|_{-\pi/2}^{\pi/2} = 0$$

and the moment around the x-axis is

$$M_x = \int_0^1 y\cdot 2\arccos y\,dy = y^2\arccos y - \frac{y\sqrt{1-y^2}}{2} + \frac{\arcsin y}{2}\Bigg|_0^1 = \frac{\pi}{4}.$$

Thus

$$\bar{x} = \frac{0}{2}, \quad \bar{y} = \frac{\pi}{8} \approx 0.393.$$

♣

Exercises for 8.6

8.6.1 *A beam 10 meters long has density $\sigma(x) = x^2$ at distance x from the left end of the beam. Find the center of mass $\bar{x}$.*

8.6.2 *A beam 10 meters long has density $\sigma(x) = \sin(\pi x/10)$ at distance x from the left end of the beam. Find the center of mass $\bar{x}$.*

8.6.3 *A beam 4 meters long has density $\sigma(x) = x^3$ at distance x from the left end of the beam. Find the center of mass $\bar{x}$.*

8.6.4 *Verify that* $\displaystyle\int 2x\arccos x\,dx = x^2\arccos x - \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin x}{2} + C.$

8.6.5 *A thin plate lies in the region between $y = x^2$ and the x-axis between $x = 1$ and $x = 2$. Find the centroid.*

8.6.6 *A thin plate fills the upper half of the unit circle $x^2 + y^2 = 1$. Find the centroid.*

8.6.7 *A thin plate lies in the region contained by $y = x$ and $y = x^2$. Find the centroid.*

8.6.8 *A thin plate lies in the region contained by $y = 4 - x^2$ and the x-axis. Find the centroid.*

8.6.9 *A thin plate lies in the region contained by $y = x^{1/3}$ and the x-axis between $x = 0$ and $x = 1$. Find the centroid.*

8.6.10 *A thin plate lies in the region contained by $\sqrt{x} + \sqrt{y} = 1$ and the axes in the first quadrant. Find the centroid.*

8.6.11 *A thin plate lies in the region between the circle $x^2 + y^2 = 4$ and the circle $x^2 + y^2 = 1$, above the x-axis. Find the centroid.*

8.6.12 *A thin plate lies in the region between the circle $x^2 + y^2 = 4$ and the circle $x^2 + y^2 = 1$ in the first quadrant. Find the centroid.*

8.6.13 *A thin plate lies in the region between the circle $x^2 + y^2 = 25$ and the circle $x^2 + y^2 = 16$ above the x-axis. Find the centroid.*

8.7 Arc Length

In previous sections we used integration to answer the following questions:

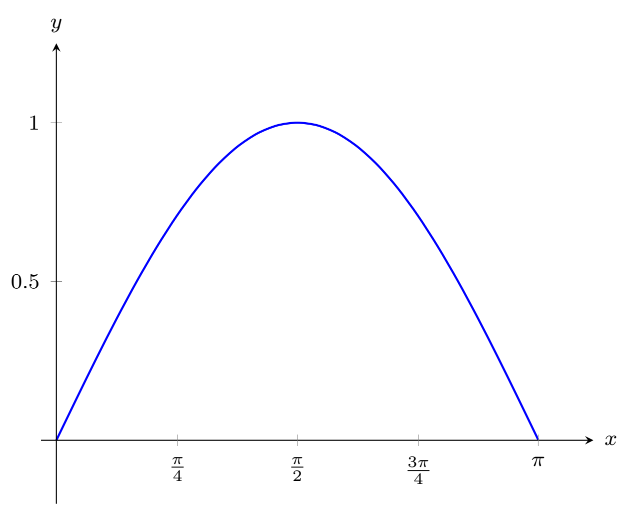
1. Given a region, what is its area?
2. Given a solid, what is its volume?

In this section, we address a related question: Given a curve, what is its length? This is often referred to as **arc length**.

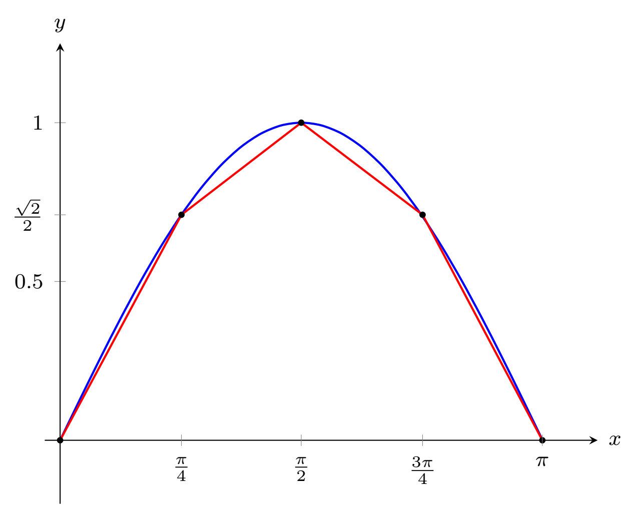
Consider the graph of $y = \sin x$ on $[0, \pi]$ given in Figure 8.30 (a). How long is this curve? That is, if we were to use a piece of string to exactly match the shape of this curve, how long would the string be?

As we have done in the past, we start by approximating; later, we will refine our answer using limits to get an exact solution.

The length of straight–line segments is easy to compute using the Distance Formula. We can approximate the length of the given curve by approximating the curve with straight lines and measuring their lengths.



(a)



(b)

Figure 8.30: Graphing $y = \sin x$ on $[0, \pi]$ and approximating the curve with line segments.

In Figure 8.30 (b), the curve $y = \sin x$ has been approximated with 4 line segments (the interval $[0, \pi]$ has been divided into 4 equally–lengthed subintervals). It is clear that these four line segments approximate $y = \sin x$ very well on the first and last subinterval, though not so well in the middle. Regardless, the sum of the lengths of the line segments is 3.79, so we approximate the arc length of $y = \sin x$ on $[0, \pi]$ to be 3.79.

In general, we can approximate the arc length of $y = f(x)$ on $[a, b]$ in the following manner. Let $a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b$ be a partition of $[a, b]$ into n subintervals. Let Δx_i represent the length of the i^{th} subinterval $[x_i, x_{i+1}]$.

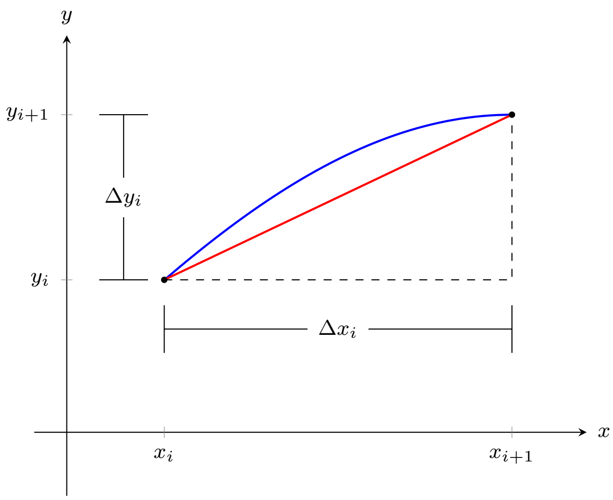


Figure 8.31: Zooming in on the i^{th} subinterval $[x_i, x_{i+1}]$ of a partition of $[a, b]$.

Figure ?? zooms in on the i^{th} subinterval where $y = f(x)$ is approximated by a straight line segment. The dashed lines show that we can view this line segment as they hypotenuse of a right triangle whose sides have length Δx_i and Δy_i. Using the Pythagorean Theorem, the length of this line segment is $\sqrt{\Delta x_i^2 + \Delta y_i^2}$. Summing over all subintervals gives an arc length approximation

$$L \approx \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

As shown here, this is *not* a Riemann Sum. While we could conclude that taking a limit as the subinterval length goes to zero gives the exact arc length, we would not be able to compute the answer with a definite integral. We need first to do a little algebra.

In the above expression factor out a Δx_i^2 term:

$$\sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{\Delta x_i^2 \left(1 + \frac{\Delta y_i^2}{\Delta x_i^2}\right)}.$$

Now pull the Δx_i^2 term out of the square root:

$$= \sum_{i=1}^{n} \sqrt{1 + \frac{\Delta y_i^2}{\Delta x_i^2}}\ \Delta x_i.$$

This is nearly a Riemann Sum. Consider the $\Delta y_i^2/\Delta x_i^2$ term. The expression $\Delta y_i/\Delta x_i$ measures the "change in y/change in x," that is, the "rise over run" of f on the i^{th} subinterval. The Mean Value Theorem of Differentiation (Theorem 5.4) states that there is a c_i in the i^{th} subinterval where $f'(c_i) = \Delta y_i/\Delta x_i$. Thus we can rewrite our above expression as:

$$= \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\ \Delta x_i.$$

This *is* a Riemann Sum. As long as f' is continuous, we can invoke Theorem **??** and conclude

$$= \int_a^b \sqrt{1 + f'(x)^2}\ dx.$$

Key Idea 8.7.0: Arc Length

Let f be differentiable on an open interval containing $[a, b]$, where f' is also continuous on $[a, b]$. Then the arc length of f from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ dx.$$

As the integrand contains a square root, it is often difficult to use the formula in Key Idea 8.7 to find the length exactly. When exact answers are difficult to come by, we resort to using numerical methods of approximating definite integrals. The following examples will demonstrate this.

Example 8.31: Finding arc length

Find the arc length of $f(x) = x^{3/2}$ from $x = 0$ to $x = 4$.

Solution. We begin by finding $f'(x) = \frac{3}{2}x^{1/2}$. Using the formula, we find the arc length L as

$$\begin{aligned}
L &= \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\ dx \\
&= \int_0^4 \sqrt{1 + \frac{9}{4}x}\ dx \\
&= \int_0^4 \left(1 + \frac{9}{4}x\right)^{1/2}\ dx \\
&= \frac{2}{3}\frac{4}{9}\left(1 + \frac{9}{4}x\right)^{3/2}\Big|_0^4 \\
&= \frac{8}{27}\left(10^{3/2} - 1\right) \approx 9.07\text{units}.
\end{aligned}$$

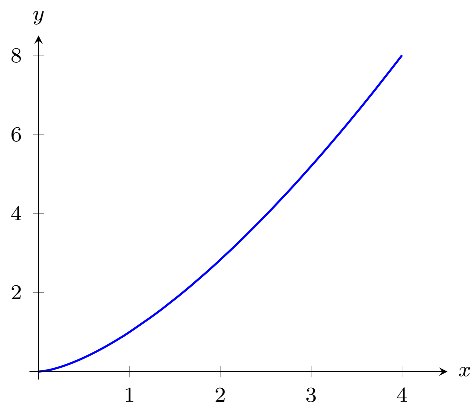


Figure 8.32: A graph of $f(x) = x^{3/2}$ from Example 8.31.

♣

Example 8.32: Finding arc length

Find the arc length of $f(x) = \frac{1}{8}x^2 - \ln x$ from $x = 1$ to $x = 2$.

Solution. This function was chosen specifically because the resulting integral can be evaluated exactly. We begin by finding $f'(x) = x/4 - 1/x$. The arc length is

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2}\, dx \\ &= \int_1^2 \sqrt{1 + \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2}}\, dx \\ &= \int_1^2 \sqrt{\frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2}}\, dx \\ &= \int_1^2 \sqrt{\left(\frac{x}{4} + \frac{1}{x}\right)^2}\, dx \end{aligned}$$

$$\begin{aligned} &= \int_1^2 \left(\frac{x}{4} + \frac{1}{x}\right)\, dx \\ &= \left(\frac{x^2}{8} + \ln x\right)\Big|_1^2 \\ &= \frac{3}{8} + \ln 2 \approx 1.07 \text{ units.} \end{aligned}$$

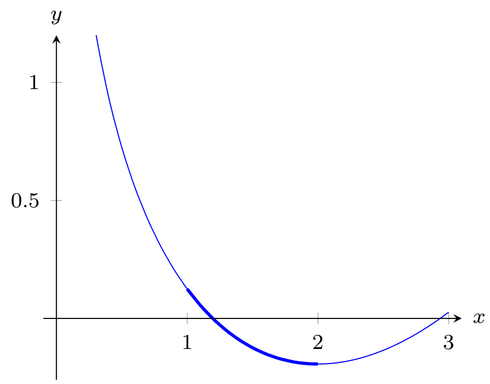


Figure 8.33: A graph of $f(x) = \frac{1}{8}x^2 - \ln x$ from Example ??.

A graph of f is given in Figure 8.33; the portion of the curve measured in this problem is in bold. ♣

Example 8.33: Circumference of a Circle

Let $f(x) = \sqrt{r^2 - x^2}$, the upper half circle of radius r. The length of this curve is half the circumference, namely πr. Compute this with the arc length formula.

Solution. The derivative f' is $-x/\sqrt{r^2 - x^2}$ so the integral is

$$\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = \int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\, dx = r \int_{-r}^{r} \sqrt{\frac{1}{r^2 - x^2}}\, dx.$$

Using a trigonometric substitution, we find the antiderivative, namely $\arcsin(x/r)$. Notice that the integral is improper at both endpoints, as the function $\sqrt{1/(r^2 - x^2)}$ is undefined when $x = \pm r$. So we need to compute

$$\lim_{D \to -r^+} \int_D^0 \sqrt{\frac{1}{r^2 - x^2}}\, dx + \lim_{D \to r^-} \int_0^D \sqrt{\frac{1}{r^2 - x^2}}\, dx.$$

This is not difficult, and has value π, so the original integral, with the extra r in front, has value πr as expected. ♣

The previous examples found the arc length exactly through careful choice of the functions. In general, exact answers are much more difficult to come by and numerical approximations are necessary.

Example 8.34: Approximating arc length numerically

Find the length of the sine curve from $x = 0$ to $x = \pi$.

Solution. This is somewhat of a mathematical curiosity; in Example **??** we found the area under one "hump" of the sine curve is 2 square units; now we are measuring its arc length.

The setup is straightforward: $f(x) = \sin x$ and $f'(x) = \cos x$. Thus

$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx.$$

This integral *cannot* be evaluated in terms of elementary functions so we will approximate it with Simpson's Method with $n = 4$.

x	$\sqrt{1+\cos^2 x}$
0	$\sqrt{2}$
$\pi/4$	$\sqrt{3/2}$
$\pi/2$	1
$3\pi/4$	$\sqrt{3/2}$
π	$\sqrt{2}$

Figure 8.34: A table of values of $y = \sqrt{1+\cos^2 x}$ to evaluate a definite integral in Example ??.

Figure 8.34 gives $\sqrt{1+\cos^2 x}$ evaluated at 5 evenly spaced points in $[0, \pi]$. Simpson's Rule then states that

$$\begin{aligned} \int_0^{\pi} \sqrt{1+\cos^2 x}\, dx &\approx \frac{\pi - 0}{4 \cdot 3}\left(\sqrt{2} + 4\sqrt{3/2} + 2(1) + 4\sqrt{3/2} + \sqrt{2}\right) \\ &= 3.82918. \end{aligned}$$

Using a computer with $n = 100$ the approximation is $L \approx 3.8202$; our approximation with $n = 4$ is quite good. ♣

Exercises for 8.7

8.7.1 *Find the arc length of the function on the given interval.*

(a) $f(x) = x$ *on* $[0, 1]$.

(b) $f(x) = \sqrt{8}x$ *on* $[-1, 1]$.

(c) $f(x) = \dfrac{1}{3}x^{3/2} - x^{1/2}$ *on* $[0, 1]$.

(d) $f(x) = \dfrac{1}{12}x^3 + \dfrac{1}{x}$ *on* $[1, 4]$.

(e) $f(x) = 2x^{3/2} - \dfrac{1}{6}\sqrt{x}$ *on* $[0, 9]$.

(f) $f(x) = \cosh x$ *on* $[-\ln 2, \ln 2]$.

(g) $f(x) = \dfrac{1}{2}\left(e^x + e^{-x}\right)$ *on* $[0, \ln 5]$.

(h) $f(x) = \dfrac{1}{12}x^5 + \dfrac{1}{5x^3}$ *on* $[.1, 1]$.

(i) $f(x) = \ln\big(\sin x\big)$ *on* $[\pi/6, \pi/2]$.

(j) $f(x) = \ln\big(\cos x\big)$ *on* $[0, \pi/4]$.

8.7.2 *Set up the integral to compute the arc length of the function on the given interval. Do **not** evaluate the integral.*

(a) $f(x) = x^2$ *on* $[0, 1]$.

(b) $f(x) = x^{10}$ *on* $[0, 1]$.

(c) $f(x) = \ln x$ *on* $[1, e]$.

(d) $f(x) = \sqrt{x}$ *on* $[0, 1]$.

(e) $f(x) = \sqrt{1 - x^2}$ *on* $[-1, 1]$. *(Note: this describes the top half of a circle with radius 1.)*

(f) $f(x) = \sqrt{1 - x^2/9}$ *on* $[-3, 3]$. *(Note: this describes the top half of an ellipse with a major axis of length* 6 *and a minor axis of length* 2*.)*

(g) $f(x) = \dfrac{1}{x}$ *on* $[1, 2]$.

(h) $f(x) = \sec x$ *on* $[-\pi/4, \pi/4]$.

8.7.3 *Use Simpson's Rule, with $n = 4$, to approximate the arc length of each the function on the given interval in Exercise 8.7.2.*

8.7.4 *Find the arc length of $f(x) = x^{3/2}$ on $[0, 2]$.*

8.7.5 *Find the arc length of $f(x) = x^2/8 - \ln x$ on $[1, 2]$.*

8.7.6

Find the arc length of $f(x) = (1/3)(x^2 + 2)^{3/2}$ on the interval $[0, a]$.

8.7.7 *Find the arc length of $f(x) = \ln(\sin x)$ on the interval $[\pi/4, \pi/3]$.*

8.7.8 *Let $a > 0$. Show that the length of $y = \cosh x$ on $[0, a]$ is equal to* $\displaystyle\int_0^a \cosh x \, dx$.

8.7.9 *Find the arc length of $f(x) = \cosh x$ on $[0, \ln 2]$.*

8.7.10 *Set up the integral to find the arc length of $\sin x$ on the interval $[0, \pi]$; do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral.*

8.7.11 *Set up the integral to find the arc length of $y = xe^{-x}$ on the interval $[2, 3]$; do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral.*

8.7.12 *Find the arc length of $y = e^x$ on the interval $[0, 1]$. (This can be done exactly; it is a bit tricky and a bit long.)*

8.8 Surface Area

Another geometric question that arises naturally is: "What is the surface area of a volume?" For example, what is the surface area of a sphere? More advanced techniques are required to approach this question in general, but we can compute the areas of some volumes generated by revolution.

As usual, the question is: How might we approximate the surface area? For a surface obtained by rotating a curve around an axis, we can take a polygonal approximation to the curve, as in the last section, and rotate it around the same axis. This gives a surface composed of many "truncated cones"; a truncated cone is called a **frustum** of a cone. Figure 8.35 illustrates this approximation.

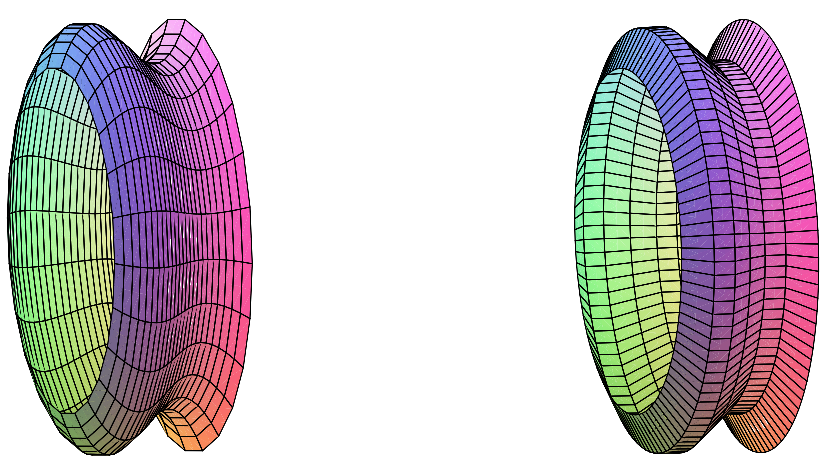


Figure 8.35: Approximating a surface (left) by portions of cones (right).

So we need to be able to compute the area of a frustum of a cone. Since the frustum can be formed by removing a small cone from the top of a larger one, we can compute the desired area if we know the surface area of a cone. Suppose a right circular cone has base radius r and slant height h. If we cut the cone from the vertex to the base circle and flatten it out, we obtain a sector of a circle with radius h and arc length $2\pi r$, as in Figure 8.36. The angle at the center, in radians, is then $2\pi r/h$, and the area of the cone is equal to the area of the sector of the circle. Let A be the area of the sector; since the area of the entire circle is πh^2, we have

$$\frac{A}{\pi h^2} = \frac{2\pi r/h}{2\pi}$$
$$A = \pi r h.$$

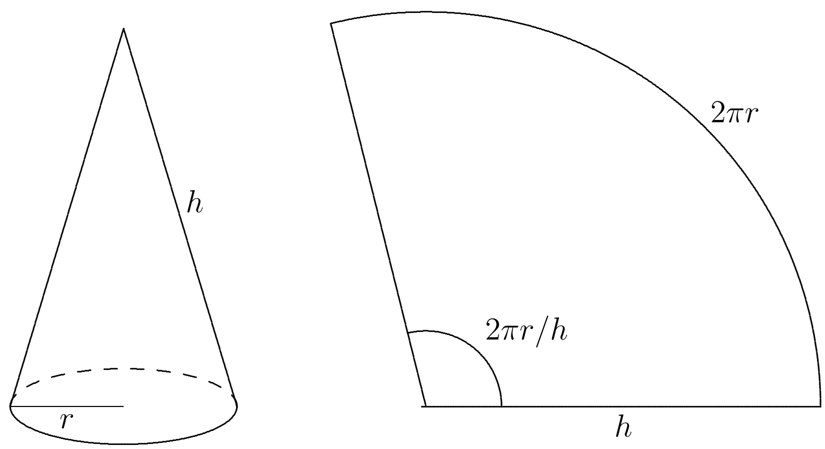


Figure 8.36: The area of a cone.

Now suppose we have a frustum of a cone with slant height h and radii r_0 and r_1, as in Figure 8.37. The area of the entire cone is $\pi r_1(h_0 + h)$, and the area of the small cone is $\pi r_0 h_0$; thus, the area of the frustum is $\pi r_1(h_0 + h) - \pi r_0 h_0 = \pi((r_1 - r_0)h_0 + r_1 h)$. By similar triangles,

$$\frac{h_0}{r_0} = \frac{h_0 + h}{r_1}.$$

With a bit of algebra this becomes $(r_1 - r_0)h_0 = r_0 h$; substitution into the area gives

$$\pi((r_1 - r_0)h_0 + r_1 h) = \pi(r_0 h + r_1 h) = \pi h(r_0 + r_1) = 2\pi\frac{r_0 + r_1}{2}h = 2\pi r h.$$

The final form is particularly easy to remember, with r equal to the average of r_0 and r_1, as it is also the formula for the area of a cylinder. (Think of a cylinder of radius r and height h as the frustum of a cone of infinite height.)

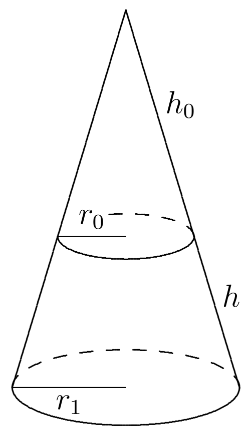


Figure 8.37: The area of a frustum.

Now we are ready to approximate the area of a surface of revolution. On one subinterval, the situation is as shown in Figure 8.38. When the line joining two points on the curve is rotated around the x-axis, it forms a frustum of a cone. The area is

$$2\pi rh = 2\pi\frac{f(x_i)+f(x_{i+1})}{2}\sqrt{1+(f'(t_i))^2}\,\Delta x.$$

Here $\sqrt{1+(f'(t_i))^2}\,\Delta x$ is the length of the line segment, as we found in the previous section. Assuming f is a continuous function, there must be some x_i^* in $[x_i, x_{i+1}]$ such that $(f(x_i)+f(x_{i+1}))/2 = f(x_i^*)$, so the approximation for the surface area is

$$\sum_{i=0}^{n-1} 2\pi f(x_i^*)\sqrt{1+(f'(t_i))^2}\,\Delta x.$$

This is not quite the sort of sum we have seen before, as it contains two different values in the interval $[x_i, x_{i+1}]$, namely x_i^* and t_i. Nevertheless, using more advanced techniques than we have available here, it turns out that

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} 2\pi f(x_i^*)\sqrt{1+(f'(t_i))^2}\,\Delta x = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

is the surface area we seek. (Roughly speaking, this is because while x_i^* and t_i are distinct values in $[x_i, x_{i+1}]$, they get closer and closer to each other as the length of the interval shrinks.)

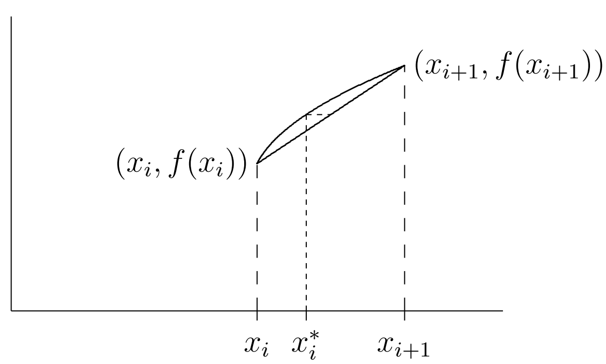


Figure 8.38: One subinterval.

Example 8.35: Surface Area of a Sphere

Compute the surface area of a sphere of radius r.

Solution. The sphere can be obtained by rotating the graph of $f(x) = \sqrt{r^2 - x^2}$ about the x-axis. The derivative f' is $-x/\sqrt{r^2 - x^2}$, so the surface area is given by

$$\begin{aligned} A &= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\ &= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \sqrt{\frac{r^2}{r^2 - x^2}}\, dx \\ &= 2\pi \int_{-r}^{r} r\, dx = 2\pi r \int_{-r}^{r} 1\, dx = 4\pi r^2 \end{aligned}$$

♣

If the curve is rotated around the y axis, the formula is nearly identical, because the length of the line segment we use to approximate a portion of the curve doesn't change. Instead of the radius $f(x_i^*)$, we use the new radius $\bar{x}_i = (x_i + x_{i+1})/2$, and the surface area integral becomes

$$\int_a^b 2\pi x \sqrt{1 + (f'(x))^2}\, dx.$$

Example 8.36: Surface Around y-axis

Compute the area of the surface formed when $f(x) = x^2$ between 0 and 2 is rotated around the y-axis.

Solution. We compute $f'(x) = 2x$, and then

$$2\pi \int_0^2 x\sqrt{1 + 4x^2}\, dx = \frac{\pi}{6}(17^{3/2} - 1),$$

by a simple substitution. ♣

Exercises for 8.8

8.8.1 *Compute the area of the surface formed when $f(x) = 2\sqrt{1 - x}$ between -1 and 0 is rotated around the x-axis.*

8.8.2 *Compute the surface area of example 8.36 by rotating $f(x) = \sqrt{x}$ around the x-axis.*

8.8.3 *Compute the area of the surface formed when $f(x) = x^3$ between 1 and 3 is rotated around the x-axis.*

8.8.4 *Compute the area of the surface formed when* $f(x) = 2 + \cosh(x)$ *between* 0 *and* 1 *is rotated around the* x*-axis.*

8.8.5 *Consider the surface obtained by rotating the graph of* $f(x) = 1/x$, $x \geq 1$, *around the* x*-axis. This surface is called* **Gabriel's horn** *or* **Toricelli's trumpet**. *Show that Gabriel's horn has infinite surface area.*

8.8.6 *Consider the circle* $(x-2)^2 + y^2 = 1$. *Sketch the surface obtained by rotating this circle about the* y*-axis. (The surface is called a* **torus**.*) What is the surface area?*

8.8.7 *Consider the ellipse with equation* $x^2/4 + y^2 = 1$. *If the ellipse is rotated around the* x*-axis it forms an* **ellipsoid**. *Compute the surface area.*

8.8.8 *Generalize the preceding result: rotate the ellipse given by* $x^2/a^2 + y^2/b^2 = 1$ *about the* x*-axis and find the surface area of the resulting ellipsoid. You should consider two cases, when* $a > b$ *and when* $a < b$. *Compare to the area of a sphere.*

9. Sequences and Series

Consider the following sum:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^i} + \cdots$$

The dots at the end indicate that the sum goes on forever. Does this make sense? Can we assign a numerical value to an infinite sum? While at first it may seem difficult or impossible, we have certainly done something similar when we talked about one quantity getting "closer and closer" to a fixed quantity. Here we could ask whether, as we add more and more terms, the sum gets closer and closer to some fixed value. That is, look at

$$\begin{aligned}
\frac{1}{2} &= \frac{1}{2} \\
\frac{3}{4} &= \frac{1}{2} + \frac{1}{4} \\
\frac{7}{8} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\
\frac{15}{16} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}
\end{aligned}$$

and so on, and consider whether these values have a limit. It seems likely that they do, namely 1. In fact, as we will see, it's not hard to show that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^i} = \frac{2^i - 1}{2^i} = 1 - \frac{1}{2^i}$$

and then

$$\lim_{i\to\infty} 1 - \frac{1}{2^i},$$

which gets closer and closer to 1 as i gets larger.

There is a context in which we already implicitly accept this notion of infinite sum without really thinking of it as a sum: The representation of a real number as an infinite decimal. For example,

$$0.3333\bar{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \frac{1}{3},$$

or likewise

$$3.14159\ldots = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000} + \cdots = \pi.$$

An infinite sum is called a **series**, and is usually written using the same sigma notation that we encountered in Chapter 6. In this case, however, we use ∞ to indicate that there is no 'last term'. The series we first examined can be written as

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^i} = \sum_{i=1}^{\infty} \frac{1}{2^i}$$

A related notion that will aid our investigations is that of a **sequence**. A sequence is just an ordered (possibly infinite) list of numbers. For example,

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$$

We will begin by learning some useful facts about sequences.

9.1 Sequences

While the idea of a sequence of numbers, $a_1, a_2, a_3, \ldots$ is straightforward, it is useful to think of a sequence as a function. We have dealt with functions whose domains are the real numbers, or a subset of the real numbers, like $f(x) = \sin x$. A sequence can be regarded as a function with domain as the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ or the non-negative integers, $\mathbb{Z}^{\geq 0} = \{0, 1, 2, 3, \ldots\}$. The range of the function is still allowed to be the set of all real numbers; we say that a sequence is a function $f\colon \mathbb{N} \to \mathbb{R}$. Sequences are commonly denoted in several different, but equally acceptable ways:

$$a_1, a_2, a_3, \ldots$$
$$\{a_n\}_{n=1}^{\infty}$$
$$\{f(n)\}_{n=1}^{\infty}$$

As with functions of the real numbers, we will most often encounter sequences that can be expressed by a formula. We have already seen the sequence $a_i = f(i) = 1 - 1/2^i$. Some other simple examples are:

$$\begin{aligned} f(i) &= \frac{i}{i+1} \\ f(n) &= \frac{1}{2^n} \\ f(n) &= \sin(n\pi/6) \\ f(i) &= \frac{(i-1)(i+2)}{2^i} \end{aligned}$$

Frequently these formulas will make sense if thought of either as functions with domain $\mathbb{R}$ or $\mathbb{N}$, though occasionally one will make sense for integer values only.

The main question of interest when dealing with sequences is what happens to the terms as we go further and further down the list. In particular, as i becomes extremely large, does a_i get closer to one specific value? This is reminiscent of a question we asked in Chapter 3, when looking at limits of functions. In fact, the problems are closely related and we define the limit of a sequence in a way similar to Definition 3.9.

Definition 9.1: Limit of a Sequence

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence. We say that $\lim\limits_{n\to\infty} a_n = L$ if for every $\epsilon > 0$ there is an $N > 0$ so that whenever $n > N$, $|a_n - L| < \epsilon$. If $\lim\limits_{n\to\infty} a_n = L$ we say that the sequence **converges** *to L, otherwise it* **diverges**.

Intuitively, $\lim_{n\to\infty} a_2 = L$ means that the further we go in the sequence, the closer the terms get to L.

Example 9.1: Exponential Sequence

Show that $\{2^{1/n}\}_{n=1}^{\infty}$ converges to 1.

Solution. Suppose $\epsilon > 0$. Then let $N = \frac{1}{\log_2(1+\epsilon)}$. Note that $N > 0$. Now if $n > N$, then

$$\begin{aligned} n &> \frac{1}{\log_2(1+\epsilon)} \\ \log_2(1+\epsilon) &> \frac{1}{n} \\ 1+\epsilon &> 2^{1/n} \\ \epsilon &> 2^{1/n} - 1 \\ \epsilon &> \left|2^{1/n} - 1\right| \end{aligned}$$

♣

Note that, as in Chapter 3, we generally need to work "backwards" from the last line of the proof to determine how to choose N. Having done so, we write the actual proof as we have done here to show that this value of N 'works'.

If a sequence is defined by a formula $\{f(i)\}_{i=1}^{\infty}$, we can often expand the domain of the function f to the set of all (or almost all) real numbers. For example, $f(i) = \frac{1}{i}$ is defined for all non-zero real numbers.

When this happens, we can sometimes find the limit of the sequence $\{f(i)\}_{i=1}^{\infty}$ more easily by finding the limit of the function $f(x)$, $x \in \mathbb{R}$, as x approaches infinity.

Theorem 9.1: Limit of a Sequence

If $\lim_{x\to\infty} f(x) = L$, where $f : \mathbb{R} \to \mathbb{R}$, then $\{f(i)\}_{i=1}^{\infty}$ converges to L.

Proof. This follows immediately from Definitions 3.9 and 9.1. ♣

Hereafter we will use the convention that x refers to a real-valued variable and i and n are integer-valued.

Example 9.2: Sequence of $1/n$

Show that $\{\frac{1}{n}\}_{n=0}^{\infty}$ converges to 0.

Solution. Since $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{n\to\infty} \frac{1}{n} = 0$. ♣

Note that the converse of Theorem 9.1 is not true.

Let $f(n) = \sin(n\pi)$. This is the sequence

$$\sin(0\pi), \sin(1\pi), \sin(2\pi), \sin(3\pi), \ldots = 0, 0, 0, 0, \ldots$$

since $\sin(n\pi) = 0$ when n is an integer. Thus $\lim\limits_{n\to\infty} f(n) = 0$. But $\lim\limits_{x\to\infty} f(x)$, when x is real, does not exist: as x gets bigger and bigger, the values $\sin(x\pi)$ do not get closer and closer to a single value, but take on all values between -1 and 1 over and over. In general, whenever you want to know $\lim\limits_{n\to\infty} f(n)$ you should first attempt to compute $\lim\limits_{x\to\infty} f(x)$, since if the latter exists it is also equal to the first limit. But if for some reason $\lim\limits_{x\to\infty} f(x)$ does not exist, it may still be true that $\lim\limits_{n\to\infty} f(n)$ exists, but you'll have to figure out another way to compute it.

It is occasionally useful to think of the graph of a sequence. Since the function is defined only for integer values, the graph is just a sequence of points. In Figure 9.1 we see the graphs of two sequences and the graphs of the corresponding real functions.

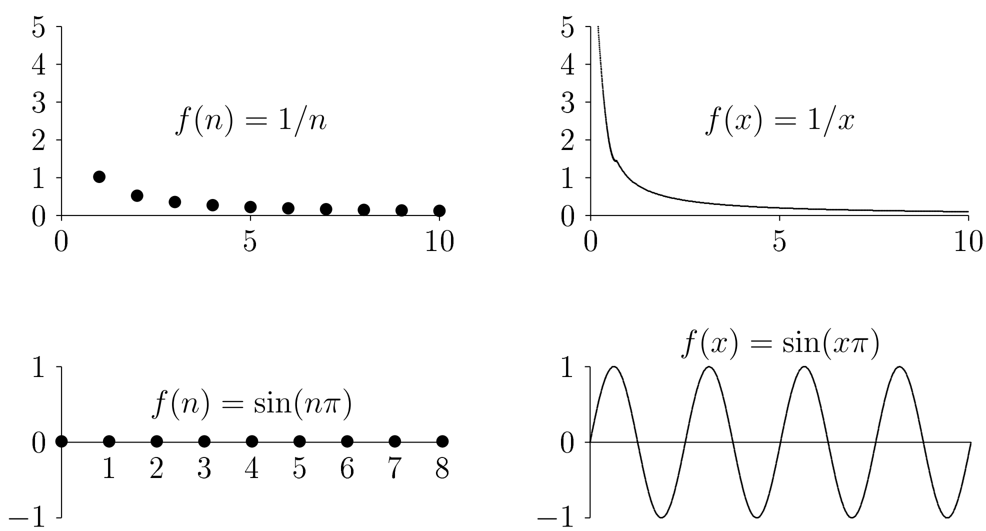


Figure 9.1: Graphs of sequences and their corresponding real functions.

Not surprisingly, the properties of limits of real functions translate into properties of sequences quite easily. Theorem **??** about limits becomes:

Theorem 9.2: Properties of Sequences

Suppose that $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$ *and* k *is some constant. Then*

$$\begin{aligned}
&\lim_{n\to\infty} ka_n = k\lim_{n\to\infty} a_n = kL\\
&\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M\\
&\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L - M\\
&\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = LM\\
&\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M},\ \textit{if } M \textit{ is not } 0
\end{aligned}$$

Likewise the Squeeze Theorem (**??**) becomes:

Theorem 9.3: Squeeze Theorem for Sequences

Suppose that $a_n \le b_n \le c_n$ *for all* $n > N$*, for some* N*. If* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$*, then* $\lim_{n\to\infty} b_n = L$*.*

And a final useful fact:

Theorem 9.4: Absolute Value Sequence

$\lim_{n\to\infty} |a_n| = 0$ *if and only if* $\lim_{n\to\infty} a_n = 0$.

This says simply that the size of $|a_n|$ gets close to zero if and only if a_n gets close to zero.

Example 9.3: Convergence of a Rational Fraction

Determine whether $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Solution. Defining $f(x) = \frac{x}{x+1}$ we obtain

$$\lim_{x\to\infty}\frac{x}{x+1} = \lim_{x\to\infty} 1 - \frac{1}{x+1} = 1 - 0 = 1.$$

Thus the sequence converges to 1. ♣

Example 9.4: Convergence of Ratio with Natural Logarithm

Determine whether $\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Solution. We compute

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0,$$

using L'Hôpital's Rule. Thus the sequence converges to 0. ♣

Example 9.5: Alternating Ones

Determine whether $\{(-1)^n\}_{n=0}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Solution. $f(x)=(-1)^x$ is undefined for irrational values of x so $\lim_{x\to\infty}(-1)^x$ does not exist. However, the sequence has a very simple pattern:

$$1,-1,1,-1,1\ldots$$

and clearly diverges. ♣

Example 9.6: Convergence of Exponential

Determine whether $\{(-1/2)^n\}_{n=0}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Solution. We consider the sequence $\{|(-1/2)^n|\}_{n=0}^{\infty}=\{(1/2)^n\}_{n=0}^{\infty}$. Then

$$\lim_{x\to\infty}\left(\frac{1}{2}\right)^x=\lim_{x\to\infty}\frac{1}{2^x}=0,$$

so by Theorem 9.4 the sequence converges to 0. ♣

Example 9.7: Using the Squeeze Theorem for Sequences

Determine whether $\{(\sin n)/\sqrt{n}\}_{n=1}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Solution. Since $|\sin n|\le 1$, $0\le|\sin n/\sqrt{n}|\le 1/\sqrt{n}$ and we can use Theorem 9.3 with $a_n=0$ and $c_n=1/\sqrt{n}$. Since $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}c_n=0$, $\lim\limits_{n\to\infty}\sin n/\sqrt{n}=0$ and the sequence converges to 0. ♣

Example 9.8: Geometric Sequence

Let r be a fixed real number. Determine when $\{r^n\}_{n=0}^{\infty}$ coverges.

Solution. A particularly common and useful sequence is $\{r^n\}_{n=0}^{\infty}$, for various values of r. Some are quite easy to understand: If $r = 1$ the sequence converges to 1 since every term is 1, and likewise if $r = 0$ the sequence converges to 0. If $r = -1$ this is the sequence of Example 9.5 and diverges. If $r > 1$ or $r < -1$ the terms r^n get large without limit, so the sequence diverges. If $0 < r < 1$ then the sequence converges to 0. If $-1 < r < 0$ then $|r^n| = |r|^n$ and $0 < |r| < 1$, so the sequence $\{|r|^n\}_{n=0}^{\infty}$ converges to 0, so also $\{r^n\}_{n=0}^{\infty}$ converges to 0. converges. In summary, $\{r^n\}$ converges precisely when $-1 < r \leq 1$ in which case

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

♣

Sequences of this form, or the more general form $\{kr^n\}_{n=0}^{\infty}$, are called **geometric sequences** or **geometric progressions**. They are encountered in a large variety of mathematical and real-world applications.

Sometimes we will not be able to determine the limit of a sequence, but we still would like to know whether it converges. In some cases we can determine this even without being able to compute the limit.

A sequence is called **increasing** or sometimes **strictly increasing** if $a_i < a_{i+1}$ for all i. It is called **non-decreasing** or sometimes (unfortunately) **increasing** if $a_i \leq a_{i+1}$ for all i. Similarly a sequence is **decreasing** if $a_i > a_{i+1}$ for all i and **non-increasing** if $a_i \geq a_{i+1}$ for all i. If a sequence has any of these properties it is called **monotonic**.

Example 9.9:

The sequence

$$\left\{\frac{2^i - 1}{2^i}\right\}_{i=1}^{\infty} = \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots,$$

is increasing, and

$$\left\{\frac{n+1}{n}\right\}_{i=1}^{\infty} = \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$$

is decreasing.

A sequence is **bounded above** if there is some number N such that $a_n \leq N$ for every n, and **bounded below** if there is some number N such that $a_n \geq N$ for every n. If a sequence is bounded above and bounded below it is **bounded**. If a sequence $\{a_n\}_{n=0}^{\infty}$ is increasing or non-decreasing it is bounded below (by a_0), and if it is decreasing or non-increasing it is bounded above (by a_0). Finally, with all this new terminology we can state an important theorem.

Theorem 9.5: Bounded Monotonic Sequence

If a sequence is bounded and monotonic then it converges.

We will not prove this, but the proof appears in many calculus books. It is not hard to believe: suppose that a sequence is increasing and bounded, so each term is larger than the one before, yet never larger than some fixed value N. The terms must then get closer and closer to some value between a_0 and N. It need not be N, since N may be a "too-generous" upper bound; the limit will be the smallest number that is above all of the terms a_i.

Example 9.10:

Determine whether $\left\{\dfrac{2^i-1}{2^i}\right\}_{i=1}^{\infty}$ *converges.*

Solution. For every $i \geq 1$ we have $0 < (2^i-1)/2^i < 1$, so the sequence is bounded, and we have already observed that it is necessary. Therefore, the sequence converges. ♣

We don't actually need to know that a sequence is monotonic to apply this theorem—it is enough to know that the sequence is "eventually" monotonic, that is, that at some point it becomes increasing or decreasing. For example, the sequence 10, 9, 8, 15, 3, 21, 4, 3/4, 7/8, 15/16, 31/32, ... is not increasing, because among the first few terms it is not. But starting with the term 3/4 it is increasing, so the theorem tells us that the sequence $3/4, 7/8, 15/16, 31/32, \ldots$ converges. Since convergence depends only on what happens as n gets large, adding a few terms at the beginning can't turn a convergent sequence into a divergent one.

Example 9.11:

Show that $\{n^{1/n}\}$ *converges.*

Solution. We first show that this sequence is decreasing, that is, that $n^{1/n} > (n+1)^{1/(n+1)}$. Consider the real function $f(x) = x^{1/x}$ when $x \geq 1$. We can compute the derivative, $f'(x) = x^{1/x}(1-\ln x)/x^2$, and note that when $x \geq 3$ this is negative. Since the function has negative slope, $n^{1/n} > (n+1)^{1/(n+1)}$ when $n \geq 3$. Since all terms of the sequence are positive, the sequence is decreasing and bounded when $n \geq 3$, and so the sequence converges. (As it happens, we can compute the limit in this case, but we know it converges even without knowing the limit; see Exercise 9.1.1.) ♣

Example 9.12:

Show that $\{n!/n^n\}$ *converges.*

Solution. If we look at the ratio of successive terms we see that:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} = \frac{n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^n < 1.$$

Therefore $a_{n+1} < a_n$, and so the sequence is decreasing. Since all terms are positive, it is also bounded, and so it must converge. (Again it is possible to compute the limit; see Exercise 9.1.2.)
♣

Exercises for 9.1

9.1.1 *Compute* $\lim_{x\to\infty} x^{1/x}$.

9.1.2 *Use the squeeze theorem to show that* $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

9.1.3 *Determine whether* $\{\sqrt{n+47} - \sqrt{n}\}_{n=0}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

9.1.4 *Determine whether* $\left\{\frac{n^2+1}{(n+1)^2}\right\}_{n=0}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

9.1.5 *Determine whether* $\left\{\frac{n+47}{\sqrt{n^2+3n}}\right\}_{n=1}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

9.1.6 *Determine whether* $\left\{\frac{2^n}{n!}\right\}_{n=0}^{\infty}$ *converges or diverges.*

9.2 Series

While much more can be said about sequences, we now turn to our principal interest, series. Recall that a series, roughly speaking, is the sum of a sequence: If $\{a_n\}_{n=0}^{\infty}$ is a sequence then the associated series is

$$\sum_{i=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

Associated with a series is a second sequence, called the **s**equence of partial sums $\{s_n\}_{n=0}^{\infty}$:

$$s_n = \sum_{i=0}^{n} a_i.$$

So

$$s_0 = a_0, \quad s_1 = a_0 + a_1, \quad s_2 = a_0 + a_1 + a_2, \quad \ldots$$

A series converges if the sequence of partial sums converges, and otherwise the series diverges.

If $\{kx^n\}_{n=0}^{\infty}$ is a geometric sequence, then the associated series $\sum_{i=0}^{\infty} kx^i$ is called a geometric series.

Theorem 9.6: Geometric Series Convergence

If $|x| < 1$*, the geometric series* $\sum_i kx^i$ *converges to* $\frac{k}{1-x}$*, otherwise the series diverges (unless* $k = 0$*).*

Proof. If $a_n = kx^n$, $\sum_{n=0}^{\infty} a_n$ is called a **geometric series**. A typical partial sum is

$$s_n = k + kx + kx^2 + kx^3 + \cdots + kx^n = k(1 + x + x^2 + x^3 + \cdots + x^n).$$

We note that

$$\begin{aligned} s_n(1-x) &= k(1 + x + x^2 + x^3 + \cdots + x^n)(1-x) \\ &= k(1 + x + x^2 + x^3 + \cdots + x^n)1 - k(1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n)x \\ &= k(1 + x + x^2 + x^3 + \cdots + x^n - x - x^2 - x^3 - \cdots - x^n - x^{n+1}) \\ &= k(1 - x^{n+1}) \end{aligned}$$

so

$$\begin{aligned} s_n(1-x) &= k(1 - x^{n+1}) \\ s_n &= k\frac{1 - x^{n+1}}{1-x}. \end{aligned}$$

If $|x| < 1$, $\lim_{n\to\infty} x^n = 0$ so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} k\frac{1 - x^{n+1}}{1-x} = k\frac{1}{1-x}.$$

Thus, when $|x| < 1$ the geometric series converges to $k/(1-x)$. ♣

When, for example, $k = 1$ and $x = 1/2$:

$$s_n = \frac{1 - (1/2)^{n+1}}{1 - 1/2} = \frac{2^{n+1} - 1}{2^n} = 2 - \frac{1}{2^n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - 1/2} = 2.$$

We began the chapter with the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n},$$

namely, the geometric series without the first term 1. Each partial sum of this series is 1 less than the corresponding partial sum for the geometric series, so of course the limit is also one less than the value of the geometric series, that is,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

It is not hard to see that the following theorem follows from Theorem 9.2.

Theorem 9.7: Series are Linear

Suppose that $\sum a_n$ *and* $\sum b_n$ *are convergent series, and* c *is a constant. Then*

1. $\sum ca_n$ *is convergent and* $\sum ca_n = c\sum a_n$
2. $\sum(a_n + b_n)$ *is convergent and* $\sum(a_n + b_n) = \sum a_n + \sum b_n$.

Note that when c is non-zero, the converse of the first part of this theorem is also true. That is, if $\sum ca_n$ is convergent, then $\sum a_n$ is also convergent; if $\sum ca_n$ converges then $\frac{1}{c}\sum ca_n$ must converge. On the other hand, the converse of the second part of the theorem is not true. For example, if $a_n = 1$ and $b_n = -1$, then $\sum a_n + \sum b_n = \sum 0 = 0$ converges, but each of $\sum a_n$ and $\sum b_n$ diverges. In general, the sequence of partial sums s_n is harder to understand and analyze than the sequence of terms a_n, and it is difficult to determine whether series converge and if so to what. The following result will let us deal with some simple cases easily.

Theorem 9.8: Divergence Test

If $\sum a_n$ *converges then* $\lim\limits_{n\to\infty} a_n = 0$.

Proof. Since $\sum a_n$ converges, $\lim\limits_{n\to\infty} s_n = L$ and $\lim\limits_{n\to\infty} s_{n-1} = L$, because this really says the same thing but “renumbers” the terms. By Theorem 9.2,

$$\lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0.$$

But

$$s_n - s_{n-1} = (a_0 + a_1 + a_2 + \cdots + a_n) - (a_0 + a_1 + a_2 + \cdots + a_{n-1}) = a_n,$$

so as desired $\lim\limits_{n\to\infty} a_n = 0$. ♣

This theorem presents an easy divergence test: Given a series $\sum a_n$, if the limit $\lim\limits_{n\to\infty} a_n$ does not exist or has a value other than zero, the series diverges. Note well that the converse is *not* true: If $\lim\limits_{n\to\infty} a_n = 0$ then the series does not necessarily converge.

Theorem 9.9: The n-th Term Test

If $\lim\limits_{n\to\infty} a_n \neq 0$ *or if the limit does not exist, then* $\sum a_n$ *diverges.*

Proof. Consider the statement of the theorem in contrapositive form:

$$\textbf{If } \sum_{n=1}^{\infty} a_n \text{ converges, then } \lim_{n\to\infty} a_n = 0.$$

If s_n are the partial sums of the series, then the assumption that the series converges gives us

$$\lim_{n\to\infty} s_n = s$$

for some number s. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

♣

Example 9.13:

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ *diverges.*

Solution. We compute the limit:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0.$$

Looking at the first few terms perhaps makes it clear that the series has no chance of converging:

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

will just get larger and larger; indeed, after a bit longer the series starts to look very much like $\cdots + 1 + 1 + 1 + 1 + \cdots$, and of course if we add up enough 1's we can make the sum as large as we desire. ♣

Example 9.14: Harmonic Series

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

Solution. Here the theorem does not apply: $\lim_{n\to\infty} 1/n = 0$, so it looks like perhaps the series converges. Indeed, if you have the fortitude (or the software) to add up the first 1000 terms you will find that

$$\sum_{n=1}^{1000} \frac{1}{n} \approx 7.49,$$

so it might be reasonable to speculate that the series converges to something in the neighborhood of 10. But in fact the partial sums do go to infinity; they just get big very, very slowly. Consider the following:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{16}>1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{16}=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

and so on. By swallowing up more and more terms we can always manage to add at least another 1/2 to the sum, and by adding enough of these we can make the partial sums as big as we like. In fact, it's not hard to see from this pattern that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n}>1+\frac{n}{2},$$

so to make sure the sum is over 100, for example, we'd add up terms until we get to around $1/2^{198}$, that is, about $4\cdot 10^{59}$ terms. This series, $\sum(1/n)$, is called the **harmonic series**. ♣

We will often make use of the fact that the first few (e.g. any finite number of) terms in a series are irrelevant when determining whether it will converge. In other words, $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges for some $N \geq 1$.

Exercises for 9.2

9.2.1 *Explain why* $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2n^2+1}$ *diverges.*

9.2.2 *Explain why* $\displaystyle\sum_{n=1}^{\infty}\frac{5}{2^{1/n}+14}$ *diverges.*

9.2.3 *Explain why* $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n}$ *diverges.*

9.2.4 *Compute* $\displaystyle\sum_{n=0}^{\infty}\frac{4}{(-3)^n}-\frac{3}{3^n}$.

9.2.5 *Compute* $\displaystyle\sum_{n=0}^{\infty}\frac{3}{2^n}+\frac{4}{5^n}$.

9.2.6 *Compute* $\displaystyle\sum_{n=0}^{\infty}\frac{4^{n+1}}{5^n}$.

9.2.7 *Compute* $\displaystyle\sum_{n=0}^{\infty}\frac{3^{n+1}}{7^{n+1}}$.

9.2.8 *Compute* $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n$.

9.2.9 *Compute* $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}}$.

9.3 The Integral Test

It is generally quite difficult, often impossible, to determine the value of a series exactly. In many cases it is possible at least to determine whether or not the series converges, and so we will spend most of our time on this problem.

If all of the terms a_n in a series are non-negative, then clearly the sequence of partial sums s_n is non-decreasing. This means that if we can show that the sequence of partial sums is bounded, the series must converge. Many useful and interesting series have this property, and they are among the easiest to understand. Let's look at an example.

Example 9.15:

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges.*

Solution. The terms $1/n^2$ are positive and decreasing, and since $\lim_{x\to\infty} 1/x^2 = 0$, the terms $1/n^2$ approach zero. We seek an upper bound for all the partial sums, that is, we want to find a number N so that $s_n \le N$ for every n. The upper bound is provided courtesy of integration, and is illustrated in figure 9.2.

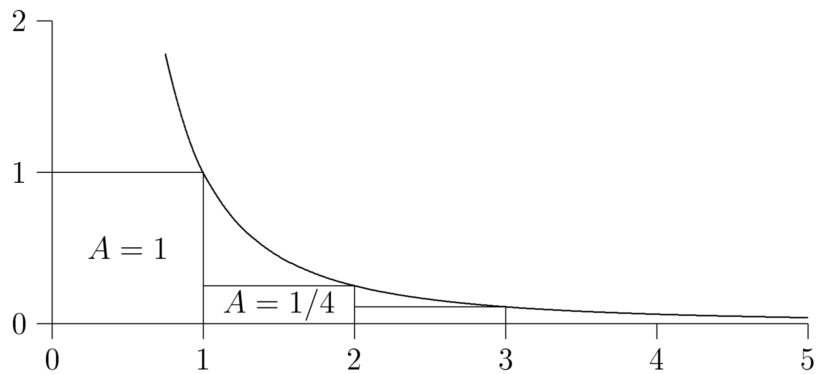


Figure 9.2: Graph of $y = 1/x^2$ with rectangles.

The figure shows the graph of $y = 1/x^2$ together with some rectangles that lie completely below the curve and that all have base length one. Because the heights of the rectangles are determined by the height of the curve, the areas of the rectangles are $1/1^2$, $1/2^2$, $1/3^2$, and so on—in other words, exactly the terms of the series. The partial sum s_n is simply the sum of the areas of the first n rectangles. Because the rectangles all lie between the curve and the x-axis, any sum of rectangle areas is less than the corresponding area under the curve, and so of course any sum of rectangle areas is less than the area under the entire curve. Unfortunately, because of the asymptote at $x = 0$, the integral $\int_0^\infty \frac{1}{x^2}$ is infinite, but we can deal with this by separating the first term from the series and integrating from 1:

$$s_n = \sum_{i=1}^{n} \frac{1}{i^2} = 1 + \sum_{i=2}^{n} \frac{1}{i^2} < 1 + \int_1^n \frac{1}{x^2}\,dx < 1 + \int_1^\infty \frac{1}{x^2}\,dx = 1 + 1 = 2$$

(Recalling that we computed this improper integral in section 7.7). Since the sequence of partial sums s_n is increasing and bounded above by 2, we know that $\lim_{n\to\infty} s_n = L < 2$, and so the series converges to some number less than 2. In fact, it is possible, though difficult, to show that $L = \pi^2/6 \approx 1.6$. ♣

We already know that $\sum 1/n$ diverges. What goes wrong if we try to apply this technique to it? Here's the calculation:

$$s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \int_1^n \frac{1}{x}\,dx < 1 + \int_1^\infty \frac{1}{x}\,dx = 1 + \infty.$$

The problem is that the improper integral doesn't converge. Note that this does *not* prove that $\sum 1/n$ diverges, just that this particular calculation fails to prove that it converges. A slight modification, however, allows us to prove in a second way that $\sum 1/n$ diverges.

Consider a slightly altered version of Figure 9.2, shown in Figure 9.3.

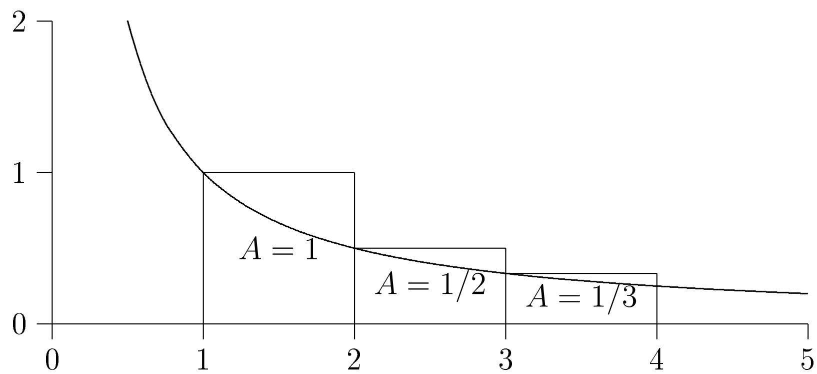


Figure 9.3: Graph of $y = 1/x$ with rectangles.

This time the rectangles are above the curve, that is, each rectangle completely contains the corresponding area under the curve. This means that

$$s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\,dx = \ln x\Big|_1^{n+1} = \ln(n+1).$$

As n gets bigger, $\ln(n+1)$ goes to infinity, so the sequence of partial sums s_n must also go to infinity, so the harmonic series diverges.

The key fact in this example is that

$$\lim_{n\to\infty} \int_1^{n+1} \frac{1}{x}\,dx = \int_1^\infty \frac{1}{x}\,dx = \infty$$

So these two examples taken together indicate that we can prove that a series converges or prove that it diverges with a single calculation of an improper integral. This is known as the **integral test**, which we state as a theorem.

Theorem 9.10: Integral Test

Suppose that $f(x) > 0$ and is decreasing on the infinite interval $[1, \infty)$ and that $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

The two examples we have seen are called ***p*-series**; a p-series is any series of the form $\sum 1/n^p$. If $p \le 0$, $\lim_{n\to\infty} 1/n^p \ne 0$, so the series diverges. For positive values of p we can determine precisely which series converge.

Theorem 9.11: p-Series Convergence

A p-series with $p > 0$ converges if and only if $p > 1$.

Proof. We use the integral test; we have already done $p = 1$, so assume that $p \ne 1$.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{D\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^D = \lim_{D\to\infty} \frac{D^{1-p}}{1-p} - \frac{1}{1-p}.$$

If $p > 1$ then $1 - p < 0$ and $\lim_{D\to\infty} D^{1-p} = 0$, so the integral converges. If $0 < p < 1$ then $1 - p > 0$ and $\lim_{D\to\infty} D^{1-p} = \infty$, so the integral diverges. ♣

Example 9.16: p-Series

Show that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Solution. We could of course use the integral test, but now that we have the theorem we may simply note that this is a p-series with $p > 1$. ♣

Example 9.17: p-Series

Show that $\sum_{n=1}^{\infty} \frac{5}{n^4}$ converges.

Solution. We know that if $\sum_{n=1}^{\infty} 1/n^4$ converges then $\sum_{n=1}^{\infty} 5/n^4$ also converges, by Theorem 9.7. Since $\sum_{n=1}^{\infty} 1/n^4$ is a convergent p-series, $\sum_{n=1}^{\infty} 5/n^4$ converges also. ♣

Example 9.18: p-Series

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$ *diverges.*

Solution. This also follows from Theorem 9.7: Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = 1/2 < 1$, it diverges, and so does $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$. ♣

Since it is typically difficult to compute the value of a series exactly, a good approximation is frequently required. In a real sense, a good approximation is only as good as we know it is, that is, while an approximation may in fact be good, it is only valuable in practice if we can guarantee its accuracy to some degree. This guarantee is usually easy to come by for series with decreasing positive terms.

Example 9.19:

Approximate $\sum 1/n^2$ *to within 0.01.*

Solution. Referring to Figure 9.2, if we approximate the sum by $\displaystyle\sum_{n=1}^{N} 1/n^2$, the size of the error we make is the total area of the remaining rectangles, all of which lie under the curve $1/x^2$ from $x = N$ to infinity. So we know the true value of the series is larger than the approximation, and no bigger than the approximation plus the area under the curve from N to infinity. Roughly, then, we need to find N so that

$$\int_N^\infty \frac{1}{x^2}\,dx < 1/100.$$

We can compute the integral:

$$\int_N^\infty \frac{1}{x^2}\,dx = \frac{1}{N},$$

so if we choose $N = 100$ the error will be less than 0.01. Adding up the first 100 terms gives approximately 1.634983900. In fact, we can do a bit better. Since we know that the correct value is between our approximation and our approximation plus the error (not minus), we can cut our error bound in half by taking the value midway between these two values. If we take $N = 50$, we get a sum of 1.6251327 with an error of at most 0.02, so the correct value is between 1.6251327 and 1.6451327, and therefore the value halfway between these, 1.6351327, is within 0.01 of the correct value. We have mentioned that the true value of this series can be shown to be $\pi^2/6 \approx 1.644934068$ which is 0.0098 more than our approximation, and so (just barely) within the required error. Frequently approximations will be even better than the "guaranteed" accuracy, but not always, as this example demonstrates. ♣

Exercises for 9.3

Determine whether each series converges or diverges.

9.3.1 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi/4}}$

9.3.2 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

9.3.3 $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

9.3.4 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

9.3.5 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$

9.3.6 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

9.3.7 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

9.3.8 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

9.3.9 *Find an* N *so that* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ *is between* $\displaystyle\sum_{n=1}^{N} \frac{1}{n^4}$ *and* $\displaystyle\sum_{n=1}^{N} \frac{1}{n^4} + 0.005$.

9.3.10 *Find an* N *so that* $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$ *is between* $\displaystyle\sum_{n=0}^{N} \frac{1}{e^n}$ *and* $\displaystyle\sum_{n=0}^{N} \frac{1}{e^n} + 10^{-4}$.

9.3.11 *Find an* N *so that* $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ *is between* $\displaystyle\sum_{n=1}^{N} \frac{\ln n}{n^2}$ *and* $\displaystyle\sum_{n=1}^{N} \frac{\ln n}{n^2} + 0.005$.

9.3.12 *Find an* N *so that* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ *is between* $\displaystyle\sum_{n=2}^{N} \frac{1}{n(\ln n)^2}$ *and* $\displaystyle\sum_{n=2}^{N} \frac{1}{n(\ln n)^2} + 0.005$.

9.4 Alternating Series

Next we consider series with both positive and negative terms, but in a regular pattern: they alternate, as in the **alternating harmonic series**:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \cdots = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

In this example the magnitude of the terms decrease, that is, $|a_n|$ forms a decreasing sequence, although this is not required in an alternating series. Recall that for a series with positive terms, if the limit of the terms is not zero, the series cannot converge; but even if the limit of the terms is zero, the series still may not converge. It turns out that for alternating series, the series converges exactly when the limit of the terms is zero. In Figure 9.4, we illustrate what happens to the partial sums of the alternating harmonic series. Because the sizes of the terms a_n are decreasing, the odd partial sums s_1, s_3, s_5, and so on, form a decreasing sequence that is bounded below by s_2, so this sequence must converge. Likewise, the even partial sums s_2, s_4, s_6, and so on, form an increasing sequence that is bounded above by s_1, so this sequence also converges. Since all the even numbered partial sums are less than all the odd numbered ones, and since the "jumps" (that is, the a_i terms) are getting smaller and smaller, the two sequences must converge to the same value, meaning the entire sequence of partial sums $s_1, s_2, s_3, \ldots$ converges as well.

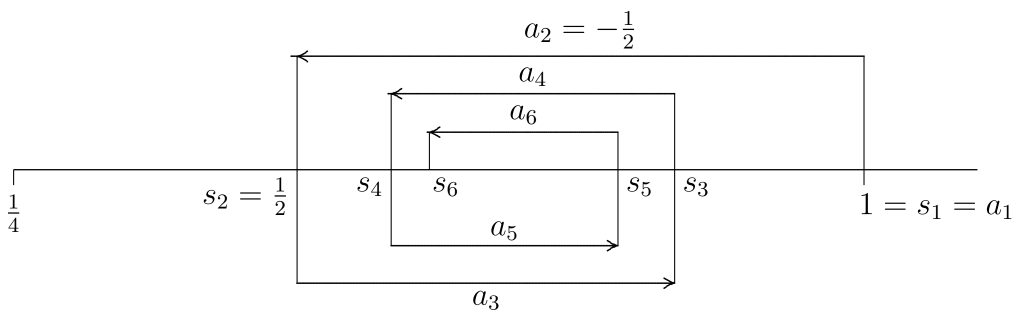


Figure 9.4: The alternating harmonic series.

The same argument works for any alternating sequence with terms that decrease in absolute value. The alternating series test is worth calling a theorem.

Theorem 9.12: Alternating Series Test

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a non-increasing sequence of positive numbers and $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.

Proof. The odd-numbered partial sums, $s_1, s_3, s_5, \ldots, s_{2k+1}, \ldots$, form a non-increasing sequence, because $s_{2k+3} = s_{2k+1} - a_{2k+2} + a_{2k+3} \le s_{2k+1}$, since $a_{2k+2} \ge a_{2k+3}$. This sequence is bounded below by s_2, so it must converge, to some value L. Likewise, the partial sums $s_2, s_4, s_6, \ldots, s_{2k}, \ldots$, form a non-decreasing sequence that is bounded above by s_1, so this sequence also converges, to some value M. Since $\lim_{n\to\infty} a_n = 0$ and $s_{2k+1} = s_{2k} + a_{2k+1}$,

$$L = \lim_{k\to\infty} s_{2k+1} = \lim_{k\to\infty}(s_{2k} + a_{2k+1}) = \lim_{k\to\infty} s_{2k} + \lim_{k\to\infty} a_{2k+1} = M + 0 = M,$$

so $L = M$; the two sequences of partial sums converge to the same limit, and this means the entire sequence of partial sums also converges to L. ♣

Another useful fact is implicit in this discussion. Suppose that

$$L = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

and that we approximate L by a finite part of this sum, say

$$L \approx \sum_{n=1}^{N} (-1)^{n-1} a_n.$$

Because the terms are decreasing in size, we know that the true value of L must be between this approximation and the next one, that is, between

$$\sum_{n=1}^{N} (-1)^{n-1} a_n \quad \text{and} \quad \sum_{n=1}^{N+1} (-1)^{n-1} a_n.$$

Depending on whether N is odd or even, the second will be larger or smaller than the first.

Example 9.20:

Approximate the sum of the alternating harmonic series to within 0.05.

Solution. We need to go to the point at which the next term to be added or subtracted is $1/10$. Adding up the first nine and the first ten terms we get approximately 0.746 and 0.646. These are $1/10$ apart, so the value halfway between them, 0.696, is within 0.05 of the correct value. ♣

We have considered alternating series with first index 1, and in which the first term is positive, but a little thought shows this is not crucial. The same test applies to any similar series, such as $\sum_{n=0}^{\infty} (-1)^n a_n$, $\sum_{n=1}^{\infty} (-1)^n a_n$, $\sum_{n=17}^{\infty} (-1)^n a_n$, etc.

Exercises for 9.4

Determine whether the following series converge or diverge.

9.4.1 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+5}$

9.4.2 $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n-3}}$

9.4.3 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{3n-2}$

9.4.4 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$

9.4.5 *Approximate* $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^3}$ *to within 0.005.*

9.4.6 *Approximate* $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^4}$ *to within 0.005.*

9.5 Comparison Tests

As we begin to compile a list of convergent and divergent series, new ones can sometimes be analyzed by comparing them to ones that we already understand.

Example 9.21:

Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ *converge?*

Solution. The obvious first approach, based on what we know, is the integral test. Unfortunately, we can't compute the required antiderivative. But looking at the series, it would appear that it must converge, because the terms we are adding are smaller than the terms of a p-series, that is,

$$\frac{1}{n^2 \ln n} < \frac{1}{n^2},$$

when $n \geq 3$. Since adding up the terms $1/n^2$ doesn't get "too big", the new series "should" also converge. Let's make this more precise.

The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ converges if and only if $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$ converges—all we've done is dropped the initial term. We know that $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges. Looking at two typical partial sums:

$$s_n = \frac{1}{3^2 \ln 3} + \frac{1}{4^2 \ln 4} + \frac{1}{5^2 \ln 5} + \cdots + \frac{1}{n^2 \ln n} < \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{n^2} = t_n.$$

Since the p-series converges, say to L, and since the terms are positive, $t_n < L$. Since the terms of the new series are positive, the s_n form an increasing sequence and $s_n < t_n < L$ for all n. Hence the sequence $\{s_n\}$ is bounded and so converges. ♣

Sometimes, even when the integral test applies, comparison to a known series is easier, so it's generally a good idea to think about doing a comparison before doing the integral test.

Example 9.22:

Does $\displaystyle\sum_{n=2}^{\infty} \frac{|\sin n|}{n^2}$ *converge?*

Solution. We can't apply the integral test here, because the terms of this series are not decreasing. Just as in the previous example, however,

$$\frac{|\sin n|}{n^2} \leq \frac{1}{n^2},$$

because $|\sin n| \le 1$. Once again the partial sums are non-decreasing and bounded above by $\sum 1/n^2 = L$, so the new series converges. ♣

Like the integral test, the comparison test can be used to show both convergence and divergence. In the case of the integral test, a single calculation will confirm whichever is the case. To use the comparison test we must first have a good idea as to convergence or divergence and pick the sequence for comparison accordingly.

Example 9.23:

Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-3}}$ *converge?*

Solution. We observe that the -3 should have little effect compared to the n^2 inside the square root, and therefore guess that the terms are enough like $1/\sqrt{n^2} = 1/n$ that the series should diverge. We attempt to show this by comparison to the harmonic series. We note that

$$\frac{1}{\sqrt{n^2-3}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n},$$

so that

$$s_n = \frac{1}{\sqrt{2^2-3}} + \frac{1}{\sqrt{3^2-3}} + \cdots + \frac{1}{\sqrt{n^2-3}} > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = t_n,$$

where t_n is 1 less than the corresponding partial sum of the harmonic series (because we start at $n = 2$ instead of $n = 1$). Since $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} s_n = \infty$ as well. ♣

So the general approach is this: If you believe that a new series is convergent, attempt to find a convergent series whose terms are larger than the terms of the new series; if you believe that a new series is divergent, attempt to find a divergent series whose terms are smaller than the terms of the new series.

Example 9.24:

Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$ *converge?*

Solution. Just as in the last example, we guess that this is very much like the harmonic series and so diverges. Unfortunately,

$$\frac{1}{\sqrt{n^2+3}} < \frac{1}{n},$$

so we can't compare the series directly to the harmonic series. A little thought leads us to

$$\frac{1}{\sqrt{n^2+3}} > \frac{1}{\sqrt{n^2+3n^2}} = \frac{1}{2n},$$

so if $\sum 1/(2n)$ diverges then the given series diverges. But since $\sum 1/(2n) = (1/2)\sum 1/n$, Theorem 9.7 implies that it does indeed diverge. ♣

For reference we summarize the comparison test in a theorem.

Theorem 9.13: Comparison Theorem

Suppose that a_n and b_n are non-negative for all n and that $a_n \le b_n$ when $n \ge N$, for some N.

- *If $\displaystyle\sum_{n=0}^{\infty} b_n$ converges, so does $\displaystyle\sum_{n=0}^{\infty} a_n$.*
- *If $\displaystyle\sum_{n=0}^{\infty} a_n$ diverges, so does $\displaystyle\sum_{n=0}^{\infty} b_n$.*

Exercises for 9.5

Determine whether the series converge or diverge.

9.5.1 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+3n+5}$

9.5.2 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2n^2+3n-5}$

9.5.3 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2-3n-5}$

9.5.4 $\displaystyle\sum_{n=1}^{\infty} \frac{3n+4}{2n^2+3n+5}$

9.5.5 $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4}{2n^2+3n+5}$

9.5.6 $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

9.5.7 $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

9.5.8 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

9.5.9 $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+5^n}$

9.5.10 $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+3^n}$

9.6 Absolute Convergence

Roughly speaking there are two ways for a series to converge: As in the case of $\sum 1/n^2$, the individual terms get small very quickly, so that the sum of all of them stays finite, or, as in the case of $\sum(-1)^{n-1}/n$, the terms don't get small fast enough ($\sum 1/n$ diverges), but a mixture of positive and negative terms provides enough cancellation to keep the sum finite. You might guess from what we've seen that if the terms get small fast enough that the sum of their absolute values converges, then the series will still converge regardless of which terms are actually positive or negative.

Theorem 9.14: Absolute Convergence

If $\sum_{n=0}^{\infty} |a_n|$ *converges, then* $\sum_{n=0}^{\infty} a_n$ *converges.*

Proof. Note that $0 \leq a_n + |a_n| \leq 2|a_n|$ so by the comparison test $\sum_{n=0}^{\infty}(a_n + |a_n|)$ converges. Now

$$\sum_{n=0}^{\infty}(a_n + |a_n|) - \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} a_n + |a_n| - |a_n| = \sum_{n=0}^{\infty} a_n$$

converges by Theorem 9.7. ♣

So given a series $\sum a_n$ with both positive and negative terms, you should first ask whether $\sum |a_n|$ converges. This may be an easier question to answer, because we have tests that apply specifically to terms with non-negative terms. If $\sum |a_n|$ converges then you know that $\sum a_n$ converges as well. If $\sum |a_n|$ diverges then it still may be true that $\sum a_n$ converges, but you will need to use other techniques to decide. Intuitively this results says that it is (potentially) easier for $\sum a_n$ to converge than for $\sum |a_n|$ to converge, because terms may partially cancel in the first series.

If $\sum |a_n|$ converges we say that $\sum a_n$ converges **absolutely**; to say that $\sum a_n$ converges absolutely is to say that the terms of the series get small (in absolute value) quickly enough to guarantee that the series converges, regardless of whether any of the terms cancel each other. If $\sum a_n$ converges but $\sum |a_n|$ does not, we say that $\sum a_n$ converges **conditionally**. For example $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ converges absolutely, while $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges conditionally.

Example 9.25:

Does $\sum_{n=2}^{\infty} \frac{\sin n}{n^2}$ *converge?*

Solution. In Example 9.22 we saw that $\sum_{n=2}^{\infty} \frac{|\sin n|}{n^2}$ converges, so the given series converges absolutely. ♣

Example 9.26:

Does $\sum_{n=0}^{\infty}(-1)^n \frac{3n+4}{2n^2+3n+5}$ *converge?*

Solution. Taking the absolute value, $\sum_{n=0}^{\infty} \frac{3n+4}{2n^2+3n+5}$ diverges by comparison to $\sum_{n=1}^{\infty} \frac{3}{10n}$, so if the series converges it does so conditionally. It is true that $\lim_{n\to\infty}(3n+4)/(2n^2+3n+5) = 0$, so to apply the alternating series test we need to know whether the terms are decreasing. If we let $f(x) = (3x+4)/(2x^2+3x+5)$ then $f'(x) = -(6x^2+16x-3)/(2x^2+3x+5)^2$, and it is not hard to see that this is negative for $x \geq 1$, so the series is decreasing and by the alternating series test it converges. ♣

Exercises for 9.6

Determine whether each series converges absolutely, converges conditionally, or diverges.

9.6.1 $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{2n^2+3n+5}$

9.6.2 $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{3n^2+4}{2n^2+3n+5}$

9.6.3 $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{\ln n}{n}$

9.6.4 $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{\ln n}{n^3}$

9.6.5 $\sum_{n=2}^{\infty}(-1)^{n} \frac{1}{\ln n}$

9.6.6 $\sum_{n=0}^{\infty}(-1)^{n} \frac{3^n}{2^n+5^n}$

9.6.7 $\sum_{n=0}^{\infty}(-1)^{n} \frac{3^n}{2^n+3^n}$

9.6.8 $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{\arctan n}{n}$

9.7 The Ratio and Root Tests

Does the series $\sum_{n=0}^{\infty} \frac{n^5}{5^n}$ converge? It is possible, but a bit unpleasant, to approach this with the integral test or the comparison test, but there is an easier way. Consider what happens as we move from one term to the next in this series:

$$\cdots + \frac{n^5}{5^n} + \frac{(n+1)^5}{5^{n+1}} + \cdots$$

The denominator goes up by a factor of 5, $5^{n+1} = 5 \cdot 5^n$, but the numerator goes up by much less: $(n+1)^5 = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1$, which is much less than $5n^5$ when n is large, because $5n^4$ is much less than n^5. So we might guess that in the long run it begins to look as if each term is $1/5$ of the previous term. We have seen series that behave like this: The geometric series.

$$\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{5}{4},$$

So we might try comparing the given series to some variation of this geometric series. This is possible, but a bit messy. We can in effect do the same thing, but bypass most of the unpleasant work.

The key is to notice that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^5}{5^{n+1}} \frac{5^n}{n^5} = \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \frac{1}{5} = 1 \cdot \frac{1}{5} = \frac{1}{5}.$$

This is really just what we noticed above, done a bit more formally: in the long run, each term is one fifth of the previous term. Now pick some number between $1/5$ and 1, say $1/2$. Because

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{5},$$

then when n is big enough, say $n \ge N$ for some N,

$$\frac{a_{n+1}}{a_n} < \frac{1}{2} \quad \text{so} \quad a_{n+1} < \frac{a_n}{2}.$$

So $a_{N+1} < a_N/2$, $a_{N+2} < a_{N+1}/2 < a_N/4$, $a_{N+3} < a_{N+2}/2 < a_N/8$, and so on. The general form is $a_{N+k} < a_N/2^k$. So if we look at the series

$$\sum_{k=0}^{\infty} a_{N+k} = a_N + a_{N+1} + a_{N+2} + a_{N+3} + \cdots + a_{N+k} + \cdots,$$

its terms are less than or equal to the terms of the sequence

$$a_N + \frac{a_N}{2} + \frac{a_N}{4} + \frac{a_N}{8} + \cdots + \frac{a_N}{2^k} + \cdots = \sum_{k=0}^{\infty} \frac{a_N}{2^k} = 2a_N.$$

So by the comparison test, $\sum_{k=0}^{\infty} a_{N+k}$ converges, and this means that $\sum_{n=0}^{\infty} a_n$ converges, since we've just added the fixed number $a_0 + a_1 + \cdots + a_{N-1}$.

Under what circumstances could we do this? What was crucial was that the limit of a_{n+1}/a_n, say L, was less than 1 so that we could pick a value r so that $L < r < 1$. The fact that $L < r$ ($1/5 < 1/2$ in our example) means that we can compare the series $\sum a_n$ to $\sum r^n$, and the fact that $r < 1$ guarantees that $\sum r^n$ converges. That's really all that is required to make the argument work. We also made use of the fact that the terms of the series were positive; in general we simply consider the absolute values of the terms and we end up testing for absolute convergence.

Theorem 9.15: The Ratio Test

Suppose that $\lim_{n\to\infty} |a_{n+1}/a_n| = L$. *If* $L < 1$ *the series* $\sum a_n$ *converges absolutely, if* $L > 1$ *the series diverges, and if* $L = 1$ *this test gives no information.*

Proof. The example above essentially proves the first part of this, if we simply replace $1/5$ by L and $1/2$ by r. Suppose that $L > 1$, and pick r so that $1 < r < L$. Then for $n \geq N$, for some N,

$$\frac{|a_{n+1}|}{|a_n|} > r \quad \text{and} \quad |a_{n+1}| > r|a_n|.$$

This implies that $|a_{N+k}| > r^k|a_N|$, but since $r > 1$ this means that $\lim_{k\to\infty} |a_{N+k}| \neq 0$, which means also that $\lim_{n\to\infty} a_n \neq 0$. By the divergence test, the series diverges.

To see that we get no information when $L = 1$, we need to exhibit two series with $L = 1$, one that converges and one that diverges. The series $\sum 1/n^2$ and $\sum 1/n$ provide a simple example. ♣

The ratio test is particularly useful for series involving the factorial function.

Example 9.27:

Analyze $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{n!}$.

Solution.

$$\lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!}\frac{n!}{5^n} = \lim_{n\to\infty} \frac{5^{n+1}}{5^n}\frac{n!}{(n+1)!} = \lim_{n\to\infty} 5\frac{1}{(n+1)} = 0.$$

Since $0 < 1$, the series converges. ♣

A similar argument justifies a similar test that is occasionally easier to apply.

Theorem 9.16: The Root Test

Suppose that $\lim_{n\to\infty} |a_n|^{1/n} = L$. *If* $L < 1$ *the series* $\sum a_n$ *converges absolutely, if* $L > 1$ *the series diverges, and if* $L = 1$ *this test gives no information.*

The proof of the root test is actually easier than that of the ratio test, and is left as an exercise.

Example 9.28:

Analyze $\sum_{n=0}^{\infty} \frac{5^n}{n^n}$.

Solution. The ratio test turns out to be a bit difficult on this series (try it). Using the root test:

$$\lim_{n\to\infty} \left(\frac{5^n}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{(5^n)^{1/n}}{(n^n)^{1/n}} = \lim_{n\to\infty} \frac{5}{n} = 0.$$

Since $0 < 1$, the series converges. ♣

The root test is frequently useful when n appears as an exponent in the general term of the series.

Exercises for 9.7

9.7.1 *Compute* $\lim_{n\to\infty} |a_{n+1}/a_n|$ *for the series* $\sum 1/n^2$.

9.7.2 *Compute* $\lim_{n\to\infty} |a_{n+1}/a_n|$ *for the series* $\sum 1/n$.

9.7.3 *Compute* $\lim_{n\to\infty} |a_n|^{1/n}$ *for the series* $\sum 1/n^2$.

9.7.4 *Compute* $\lim_{n\to\infty} |a_n|^{1/n}$ *for the series* $\sum 1/n$.

9.7.5 *Determine whether the series converge.*

(a) $\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{5^n}$

(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(c) $\sum_{n=1}^{\infty} \frac{n^5}{n^n}$

(d) $\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}$

9.7.6 *Prove Theorem 9.16, the root test.*

9.8 Power Series

Recall that the sum of a geometric series can be expressed using the simple formula:

$$\sum_{n=0}^{\infty} kx^n = \frac{k}{1-x},$$

if $|x| < 1$, and that the series diverges when $|x| \geq 1$. At the time, we thought of x as an unspecified constant, but we could just as well think of it as a variable, in which case the series

$$\sum_{n=0}^{\infty} kx^n$$

is a function, namely, the function $k/(1-x)$, as long as $|x| < 1$: Looking at this from the opposite perspective, this means that the function $k/(1-x)$ can be represented as the sum of an infinite series. Why would this be useful? While $k/(1-x)$ is a reasonably easy function to deal with, the more complicated representation $\sum kx^n$ does have some advantages: it appears to be an infinite version of one of the simplest function types—a polynomial. Later on we will investigate some of the ways we can take advantage of this 'infinite polynomial' representation, but first we should ask if other functions can even be represented this way.

The geometric series has a special feature that makes it unlike a typical polynomial—the coefficients of the powers of x are all the same, namely k. We will need to allow more general coefficients if we are to get anything other than the geometric series.

Definition 9.2: Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n,$$

where each a_n is a real number.

As we did in the section on sequences, we can think of the a_n as being a function $a(n)$ defined on the non-negative integers. Note, however, that the a_n do not depend on x.

Example 9.29:

Determine whether the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges.

Solution. We can investigate convergence using the ratio test:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \frac{n}{|x|^n} = \lim_{n\to\infty} |x| \frac{n}{n+1} = |x|.$$

Thus when $|x| < 1$ the series converges and when $|x| > 1$ it diverges, leaving only two values in doubt. When $x = 1$ the series is the harmonic series and diverges; when $x = -1$ it is the alternating harmonic series (actually the negative of the usual alternating harmonic series) and converges. Thus, we may think of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ as a function from the interval $[-1, 1)$ to the real numbers. ♣

A bit of thought reveals that the ratio test applied to a power series will always have the same nice form. In general, we will compute

$$\lim_{n\to\infty} \frac{|a_{n+1}||x|^{n+1}}{|a_n||x|^n} = \lim_{n\to\infty} |x| \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L|x|,$$

assuming that $\lim |a_{n+1}|/|a_n|$ exists. Then the series converges if $L|x| < 1$, that is, if $|x| < 1/L$, and diverges if $|x| > 1/L$. Only the two values $x = \pm 1/L$ require further investigation. Thus the series will always define a function on the interval $(-1/L, 1/L)$, that perhaps will extend to one or both endpoints as well. Two special cases deserve mention: if $L = 0$ the limit is 0 no matter what value x takes, so the series converges for all x and the function is defined for all real numbers. If $L = \infty$, then no matter what value x takes the limit is infinite and the series converges only when $x = 0$. The value $1/L$ is called the **radius of convergence** of the series, and the interval on which the series converges is the **interval of convergence**.

We can make these ideas a bit more general. Consider the series

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n}$$

This looks a lot like a power series, but with $(x+2)^n$ instead of x^n. Let's try to determine the values of x for which it converges. This is just a geometric series, so it converges when

$$\begin{aligned} |x+2|/3 &< 1 \\ |x+2| &< 3 \\ -3 < x+2 &< 3 \\ -5 < x &< 1. \end{aligned}$$

So the interval of convergence for this series is $(-5, 1)$. The center of this interval is at -2, which is at distance 3 from the endpoints, so the radius of convergence is 3, and we say that the series is centered at -2.

Interestingly, if we compute the sum of the series we get

$$\sum_{n=0}^{\infty} \left(\frac{x+2}{3}\right)^n = \frac{1}{1 - \frac{x+2}{3}} = \frac{3}{1-x}.$$

Multiplying both sides by $1/3$ we obtain

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}} = \frac{1}{1-x},$$

which we recognize as being equal to

$$\sum_{n=0}^{\infty} x^n,$$

so we have two series with the same sum but different intervals of convergence.

This leads to the following definition:

Definition 9.3: Power Series

A power series centered at c has the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

where c and each a_n are real numbers.

Exercises for 9.8

9.8.1 *Find the radius and interval of convergence for each series. In part c), do not attempt to determine whether the endpoints are in the interval of convergence.*

(a) $\displaystyle\sum_{n=0}^{\infty} nx^n$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}(x-2)^n$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}(x-2)^n$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)}$

9.8.2 *Find the radius of convergence for the series* $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}x^n$.

9.9 Calculus with Power Series

We now know that some functions can be expressed as power series, which look like infinite polynomials. Since it is easy to find derivatives and integrals of polynomials, we might hope that we can take derivatives and integrals of power series in an analogous way. In fact we can, as stated in the following theorem, which we will not prove here.

Theorem 9.17:

Suppose the power series $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *has radius of convergence* R. *Then*

$$f'(x) = \sum_{n=0}^{\infty} na_n(x-a)^{n-1},$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1},$$

and these two series have radius of convergence R.

Example 9.30:

Find a power series representation of $\ln|1-x|$.

Solution. Starting with the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\int \frac{1}{1-x}\,dx = -\ln|1-x| = \sum_{n=0}^{\infty} \frac{1}{n+1}x^{n+1}$$

$$\ln|1-x| = \sum_{n=0}^{\infty} -\frac{1}{n+1}x^{n+1}$$

when $|x| < 1$. The series does not converge when $x = 1$ but does converge when $x = -1$ or $1-x = 2$. The interval of convergence is $[-1, 1)$, or $0 < 1-x \leq 2$. We can use this series to express $\ln(a)$ as a series when $0 < a \leq 2$ by setting $x - 1 = a$. For example

$$\ln(3/2) = \ln(1-(-1/2)) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}\frac{1}{2^{n+1}}.$$

We can use this in turn to approximate $\ln(3/2)$:

$$\ln(3/2) \approx \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{160} - \frac{1}{384} + \frac{1}{896} = \frac{909}{2240} \approx 0.406.$$

Because this is an alternating series with decreasing terms, we know that the true value is between 909/2240 and $909/2240 - 1/2048 = 29053/71680 \approx .4053$, so $0.4053 \leq \ln(3/2) \leq 0.406$. ♣

With a bit of arithmetic, we can approximate values outside of the interval of convergence:

Example 9.31:

Find an approximation for $\ln(9/4)$.

Solution. We can use the approximation we just computed, plus some rules for logarithms:

$$\ln(9/4) = \ln((3/2)^2) = 2\ln(3/2) \approx 0.812,$$

and using our bounds above,

$$0.8106 \leq \ln(9/4) \leq 0.812.$$

♣

Exercises for 9.9

9.9.1 *Find a series representation for* $\ln 2$.

9.9.2 *Find a power series representation for* $1/(1-x)^2$.

9.9.3 *Find a power series representation for* $2/(1-x)^3$.

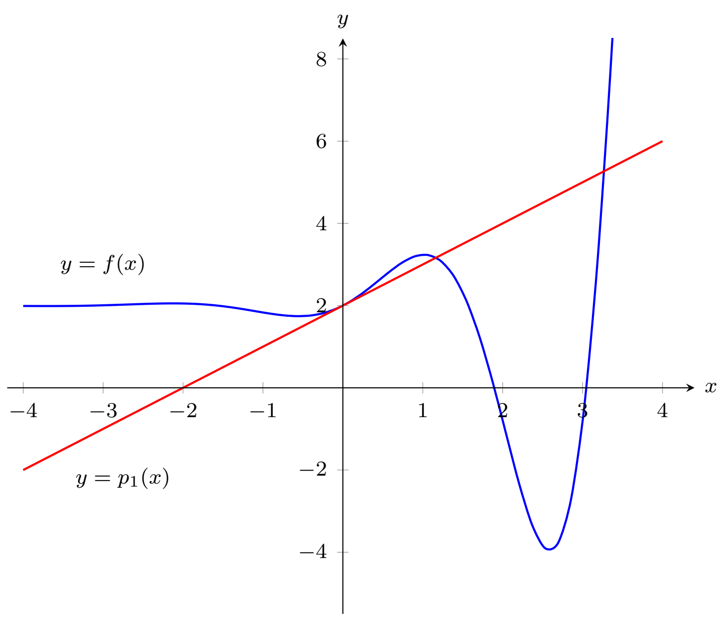
9.9.4 *Find a power series representation for* $1/(1-x)^3$. *What is the radius of convergence?*

9.9.5 *Find a power series representation for* $\int \ln(1-x)\,dx$.

9.10 Taylor Polynomials

Consider a function $y = f(x)$ and a point $\big(c, f(c)\big)$. The derivative, $f'(c)$, gives the instantaneous rate of change of f at $x = c$. Of all lines that pass through the point $\big(c, f(c)\big)$, the line that best approximates f at this point is the tangent line; that is, the line whose slope (rate of change) is $f'(c)$.

In Figure 9.5, we see a function $y = f(x)$ graphed. The table below the graph shows that $f(0) = 2$ and $f'(0) = 1$; therefore, the tangent line to f at $x = 0$ is $p_1(x) = 1(x - 0) + 2 = x + 2$. The tangent line is also given in the figure. Note that "near" $x = 0$, $p_1(x) \approx f(x)$; that is, the tangent line approximates f well.



$f(0) = 2$

$f'(0) = 1$

$f''(0) = 2$

$f'''(0) = -1$

$f^{(4)}(0) = -12$

$f^{(5)}(0) = -19$

Figure 9.5: Plotting $y = f(x)$ and a table of derivatives of f evaluated at 0.

One shortcoming of this approximation is that the tangent line only matches the slope of f; it does not, for instance, match the concavity of f. We can find a polynomial, $p_2(x)$, that does match the concavity without much difficulty, though. The table in Figure 9.5 gives the following information:

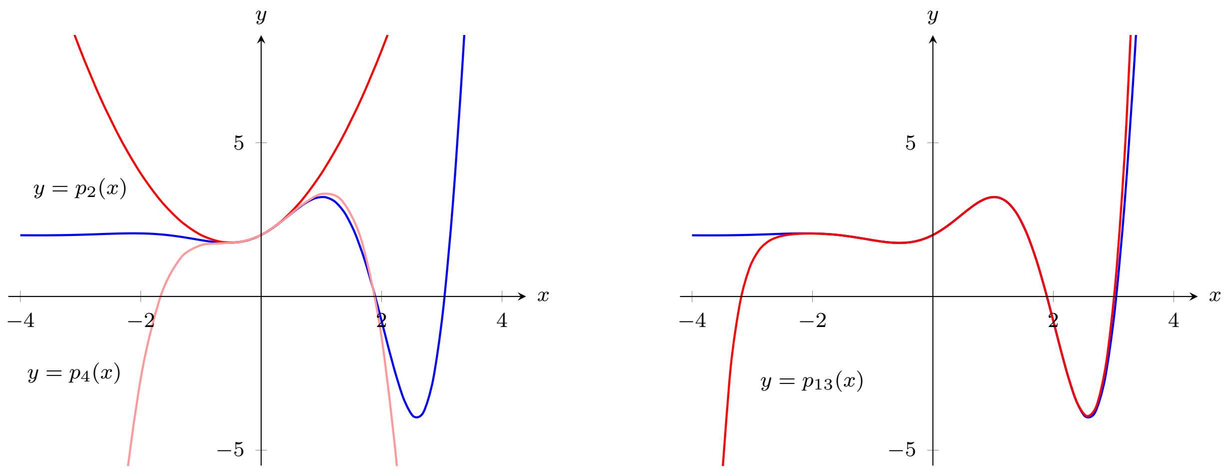
$$f(0) = 2 \qquad f'(0) = 1 \qquad f''(0) = 2.$$

Therefore, we want our polynomial $p_2(x)$ to have these same properties. That is, we need

$$p_2(0) = 2 \qquad p_2'(0) = 1 \qquad p_2''(0) = 2.$$

This is simply an initial–value problem. To keep $p_2(x)$ as simple as possible, we'll assume that not only $p_2''(0) = 2$, but that $p_2''(x) = 2$. That is, the second derivative of p_2 is constant.

If $p_2''(x) = 2$, then $p_2'(x) = 2x + C$ for some constant C. Since we have determined that $p_2'(0) = 1$, we find that $C = 1$ and so $p_2'(x) = 2x + 1$. Finally, we can compute $p_2(x) = x^2 + x + C$. Using our initial values, we know $p_2(0) = 2$ so $C = 2$. We conclude that $p_2(x) = x^2 + x + 2$. This function is plotted with f in Figure 9.6.



(a) Plotting f, p_2 and p_4. **(b) Plotting f and p_{13}.**

Figure 9.6

We can repeat this approximation process by creating polynomials of higher degree that match more of the derivatives of f at $x = 0$. In general, a polynomial of degree n can be created to match the first n derivatives of f. Figure **??** also shows $p_4(x) = -x^4/2 - x^3/6 + x^2 + x + 2$, whose first four derivatives at 0 match those of f. (Using the table in Figure 9.5, start with $p_4^{(4)}(x) = -12$ and solve the related initial–value problem.)

As we use more and more derivatives, our polynomial approximation to f gets better and better. In this example, the interval on which the approximation is "good" gets bigger and bigger. Figure **??** shows $p_{13}(x)$; we can visually affirm that this polynomial approximates f very well on $[-2, 3]$. (The polynomial $p_{13}(x)$ is not particularly "nice". It is

$$\frac{16901x^{13}}{6227020800} + \frac{13x^{12}}{1209600} - \frac{1321x^{11}}{39916800} - \frac{779x^{10}}{1814400} - \frac{359x^9}{362880} + \frac{x^8}{240} + \frac{139x^7}{5040} + \frac{11x^6}{360} - \frac{19x^5}{120} - \frac{x^4}{2} - \frac{x^3}{6} + x^2 + x + 2.)$$

The polynomials we have created are examples of *Taylor polynomials*, named after the British mathematician Brook Taylor who made important discoveries about such functions. While we created the above Taylor polynomials by solving initial–value problems, it can be shown that Taylor polynomials follow a general pattern that make their formation much more direct. This is described in the following definition.

Definition 9.4: Taylor Polynomial, Maclaurin Polynomial

Let f be a function whose first n derivatives exist at $x = c$.

1. *The **Taylor polynomial of degree n of f at $x = c$** is*

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. *A special case of the Taylor polynomial is the Maclaurin polynomial, where $c = 0$. That is, the **Maclaurin polynomial of degree n of f** is*

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

We will practice creating Taylor and Maclaurin polynomials in the following examples.

Example 9.32: Finding and using Maclaurin polynomials

1. *Find the n^{th} Maclaurin polynomial for $f(x) = e^x$.*
2. *Use $p_5(x)$ to approximate the value of e.*

Solution.

1. We start with creating a table of the derivatives of e^x evaluated at $x = 0$. In this particular case, this is relatively simple, as shown in Table 9.1.

$$\begin{array}{lcl} f(x) = e^x & \Rightarrow & f(0) = 1 \\ f'(x) = e^x & \Rightarrow & f'(0) = 1 \\ f''(x) = e^x & \Rightarrow & f''(0) = 1 \\ \vdots & & \vdots \\ f^{(n)}(x) = e^x & \Rightarrow & f^{(n)}(0) = 1 \end{array}$$

Table 9.1: The derivatives of $f(x) = e^x$ evaluated at $x = 0$.

By the definition of the Maclaurin polynomial, we have

$$\begin{aligned} p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{n}(0)}{n!}x^n \\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n. \end{aligned}$$

2. Using our answer from part 1, we have

$$p_5 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5.$$

To approximate the value of e, note that $e = e^1 = f(1) \approx p_5(1)$. It is very straightforward to evaluate $p_5(1)$:

$$p_5(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} \approx 2.71667.$$

A plot of $f(x) = e^x$ and $p_5(x)$ is given in Figure 9.7.

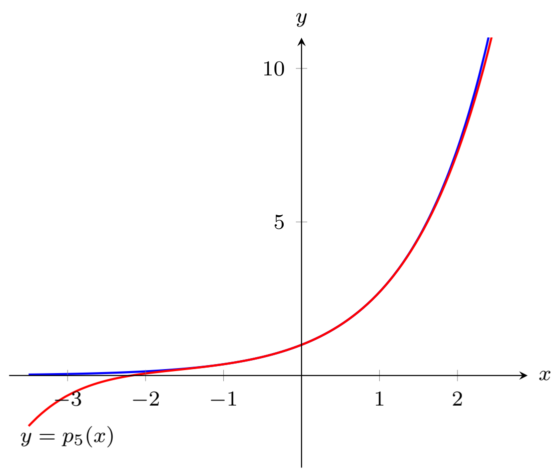


Figure 9.7: A plot of $f(x) = e^x$ and its 5^{th} degree Maclaurin polynomial $p_5(x)$.

♣

Example 9.33: Finding and using Taylor polynomials

1. *Find the n^{th} Taylor polynomial of $y = \ln x$ at $x = 1$.*
2. *Use $p_6(x)$ to approximate the value of $\ln 1.5$.*
3. *Use $p_6(x)$ to approximate the value of $\ln 2$.*

Solution.

1. We begin by creating a table of derivatives of $\ln x$ evaluated at $x = 1$. While this is not as straightforward as it was in the previous example, a pattern does emerge, as shown in Table

$f(x) = \ln x$	$\Rightarrow$	$f(1) = 0$
$f'(x) = 1/x$	$\Rightarrow$	$f'(1) = 1$
$f''(x) = -1/x^2$	$\Rightarrow$	$f''(1) = -1$
$f'''(x) = 2/x^3$	$\Rightarrow$	$f'''(1) = 2$
$f^{(4)}(x) = -6/x^4$	$\Rightarrow$	$f^{(4)}(1) = -6$
$\vdots$		$\vdots$
$f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{x^n}$	$\Rightarrow$	$f^{(n)}(1) = (-1)^{n+1}(n-1)!$

9.2.

Table 9.2: Derivatives of $\ln x$ evaluated at $x = 1$.

Using Definition **??**, we have

$$\begin{aligned} p_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{n}(c)}{n!}(x-c)^n \\ &= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n. \end{aligned}$$

Note how the coefficients of the $(x-1)$ terms turn out to be "nice."

2. We can compute $p_6(x)$ using our work above:

$$p_6(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6.$$

Since $p_6(x)$ approximates $\ln x$ well near $x = 1$, we approximate $\ln 1.5 \approx p_6(1.5)$:

$$\begin{aligned} p_6(1.5) &= (1.5-1) - \frac{1}{2}(1.5-1)^2 + \frac{1}{3}(1.5-1)^3 - \frac{1}{4}(1.5-1)^4 + \cdots \\ &\quad \cdots + \frac{1}{5}(1.5-1)^5 - \frac{1}{6}(1.5-1)^6 \\ &= \frac{259}{640} \\ &\approx 0.404688. \end{aligned}$$

This is a good approximation as a calculator shows that $\ln 1.5 \approx 0.4055$. Figure 9.8 plots $y = \ln x$ with $y = p_6(x)$. We can see that $\ln 1.5 \approx p_6(1.5)$.

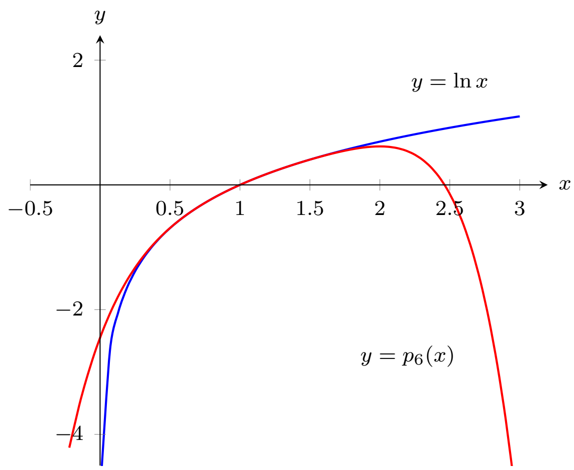


Figure 9.8: A plot of $y = \ln x$ and its 6$^{\text{th}}$ degree Taylor polynomial at $x = 1$.

3. We approximate $\ln 2$ with $p_6(2)$:

$$\begin{aligned} p_6(2) &= (2-1) - \frac{1}{2}(2-1)^2 + \frac{1}{3}(2-1)^3 - \frac{1}{4}(2-1)^4 + \cdots \\ &\cdots + \frac{1}{5}(2-1)^5 - \frac{1}{6}(2-1)^6 \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \\ &= \frac{37}{60} \\ &\approx 0.616667. \end{aligned}$$

This approximation is not terribly impressive: a hand held calculator shows that $\ln 2 \approx 0.693147$. The graph in Figure 9.8 shows that $p_6(x)$ provides less accurate approximations of $\ln x$ as x gets close to 0 or 2.

Surprisingly enough, even the 20$^{\text{th}}$ degree Taylor polynomial fails to approximate $\ln x$ for $x > 2$, as shown in Figure 9.9. We'll soon discuss why this is.

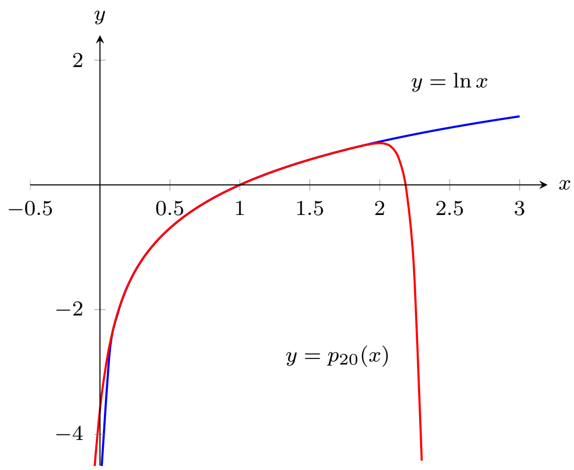


Figure 9.9: A plot of $y = \ln x$ and its 20$^{\text{th}}$ degree Taylor polynomial at $x = 1$.

♣

9.10.1. Taylor's Theorem

Taylor polynomials are used to approximate functions $f(x)$ in mainly two situations:

1. When $f(x)$ is known, but perhaps "hard" to compute directly. For instance, we can define $y = \cos x$ as either the ratio of sides of a right triangle ("adjacent over hypotenuse") or with the unit circle. However, neither of these provides a convenient way of computing $\cos 2$. A Taylor polynomial of sufficiently high degree can provide a reasonable method of computing such values using only operations usually hard–wired into a computer ($+$, $-$, $\times$ and $\div$).

2. When $f(x)$ is not known, but information about its derivatives is known. This occurs more often than one might think, especially in the study of differential equations.

Note: Even though Taylor polynomials *could* be used in calculators and computers to calculate values of trigonometric functions, in practice they generally aren't. Other more efficient and accurate methods have been developed, such as the CORDIC algorithm.

In both situations, a critical piece of information to have is "How good is my approximation?" If we use a Taylor polynomial to compute $\cos 2$, how do we know how accurate the approximation is?

We had the same problem when studying Numerical Integration. Theorem **??** provided bounds on the error when using, say, Simpson's Rule to approximate a definite integral. These bounds allowed us to determine that, for instance, using 10 subintervals provided an approximation within $\pm .01$ of the exact value. The following theorem gives similar bounds for Taylor (and hence Maclaurin) polynomials.

Theorem 9.18: Taylor's Theorem

1. *Let f be a function whose $n+1^{th}$ derivative exists on an interval I and let c be in I. Then, for each x in I, there exists z_x between x and c such that*

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x),$$

where $R_n(x) = \dfrac{f^{(n+1)}(z_x)}{(n+1)!}(x-c)^{(n+1)}$.

2. $\left|R_n(x)\right| \le \dfrac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-c)^{(n+1)}\right|$

Proof. The proof requires some cleverness to set up, but then the details are quite elementary. We define a function $F(t)$ as follows:

$$F(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k + B(x-t)^{n+1}.$$

(Once we have introduced Taylor series, you will notice that here we have replaced c by t in the first $n+1$ terms of the Taylor series, and added a carefully chosen term on the end, with B to be determined.) Note that we are temporarily keeping x fixed, so the only variable in this equation is t, and we will be interested only in t between c and x. Now substitute $t = c$:

$$F(c) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + B(x-c)^{n+1}.$$

Set this equal to $f(x)$:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + B(x-c)^{n+1}.$$

Since $x \neq c$, we can solve this for B, which is a "constant"—it depends on x and c but those are temporarily fixed. Now we have defined a function $F(t)$ with the property that $F(c) = f(x)$. Also, all terms with a positive power of $(x-t)$ become zero when we substitute x for t, so $F(x) = f^{(0)}(x)/0! = f(x)$. So $F(c) = F(x)$. By Rolle's theorem (5.3), we know that there is a value $z \in (c, x)$ such that $F'(z) = 0$. But what is F'? Each term in $F(t)$, except the first term and the extra term involving B, is a product, so to take the derivative we use the product rule on each of these terms.

$$\begin{aligned} F(t) = f(t) &+ \frac{f^{(1)}(t)}{1!}(x-t)^1 + \frac{f^{(2)}(t)}{2!}(x-t)^2 + \frac{f^{(3)}(t)}{3!}(x-t)^3 + \cdots \\ &+ \frac{f^{(n)}(t)}{n!}(x-t)^n + B(x-t)^{n+1}. \end{aligned}$$

So the derivative is

$$F'(t) = f'(t) + \left(\frac{f^{(1)}(t)}{1!}(x-t)^0(-1) + \frac{f^{(2)}(t)}{1!}(x-t)^1\right)$$

$$
\begin{aligned}
&+\left(\frac{f^{(2)}(t)}{1!}(x-t)^1(-1)+\frac{f^{(3)}(t)}{2!}(x-t)^2\right)\\
&+\left(\frac{f^{(3)}(t)}{2!}(x-t)^2(-1)+\frac{f^{(4)}(t)}{3!}(x-t)^3\right)+\cdots+\\
&+\left(\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}(-1)+\frac{f^{(n+1)}(t)}{n!}(x-t)^n\right)\\
&+B(n+1)(x-t)^n(-1).
\end{aligned}
$$

The second term in each parenthesis cancel with the first term in the next one, leaving just

$$F'(t)=\frac{f^{(n+1)}(t)}{n!}(x-t)^n+B(n+1)(x-t)^n(-1).$$

At some z, $F'(z)=0$ so

$$
\begin{aligned}
0&=\frac{f^{(n+1)}(z)}{n!}(x-z)^n+B(n+1)(x-z)^n(-1)\\
B(n+1)(x-z)^n&=\frac{f^{(n+1)}(z)}{n!}(x-z)^n\\
B&=\frac{f^{(n+1)}(z)}{(n+1)!}.
\end{aligned}
$$

Now we can write

$$F(t)=\sum_{k=0}^{n}\frac{f^{(k)}(t)}{k!}(x-t)^k+\frac{f^{(n+1)}(z)}{(n+1)!}(x-t)^{n+1}.$$

Recalling that $F(c)=f(x)$ we get

$$f(x)=F(c)=\sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k+\frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1},$$

which by taking $z_x=z$ is what we wanted to show. ♣

The first part of Taylor's Theorem states that $f(x)=p_n(x)+R_n(x)$, where $p_n(x)$ is the n^{th} order Taylor polynomial and $R_n(x)$ is the remainder, or error, in the Taylor approximation. The second part gives bounds on how big that error can be. If the $(n+1)^{\text{th}}$ derivative is large, the error may be large; if x is far from c, the error may also be large. However, the $(n+1)!$ term in the denominator tends to ensure that the error gets smaller as n increases.

The following example computes error estimates for the approximations of $\ln 1.5$ and $\ln 2$ made in Example 9.33.

Example 9.34: Finding error bounds of a Taylor polynomial

Use Theorem 9.18 to find error bounds when approximating $\ln 1.5$ *and* $\ln 2$ *with* $p_6(x)$*, the Taylor polynomial of degree 6 of* $f(x) = \ln x$ *at* $x = 1$*, as calculated in Example 9.33.*

Solution.

1. We start with the approximation of $\ln 1.5$ with $p_6(1.5)$. The theorem references an open interval I that contains both x and c. The smaller the interval we use the better; it will give us a more accurate (and smaller!) approximation of the error. We let $I = (0.9, 1.6)$, as this interval contains both $c = 1$ and $x = 1.5$.

   The theorem references $\max\left|f^{(n+1)}(z)\right|$. In our situation, this is asking "How big can the 7th derivative of $y = \ln x$ be on the interval $(0.9, 1.6)$?" The seventh derivative is $y = -6!/x^7$. The largest value it attains on I is about 1506. Thus we can bound the error as:

$$\begin{aligned} \left|R_6(1.5)\right| &\leq \frac{\max\left|f^{(7)}(z)\right|}{7!}\left|(1.5-1)^7\right| \\ &\leq \frac{1506}{5040} \cdot \frac{1}{2^7} \\ &\approx 0.0023. \end{aligned}$$

   We computed $p_6(1.5) = 0.404688$; using a calculator, we find $\ln 1.5 \approx 0.405465$, so the actual error is about 0.000778, which is less than our bound of 0.0023. This affirms Taylor's Theorem; the theorem states that our approximation would be within about 2 thousandths of the actual value, whereas the approximation was actually closer.

2. We again find an interval I that contains both $c = 1$ and $x = 2$; we choose $I = (0.9, 2.1)$. The maximum value of the seventh derivative of f on this interval is again about 1506 (as the largest values come near $x = 0.9$). Thus

$$\begin{aligned} \left|R_6(2)\right| &\leq \frac{\max\left|f^{(7)}(z)\right|}{7!}\left|(2-1)^7\right| \\ &\leq \frac{1506}{5040} \cdot 1^7 \\ &\approx 0.30. \end{aligned}$$

   This bound is not as nearly as good as before. Using the degree 6 Taylor polynomial at $x = 1$ will bring us within 0.3 of the correct answer. As $p_6(2) \approx 0.61667$, our error estimate guarantees that the actual value of $\ln 2$ is somewhere between 0.31667 and 0.91667. These bounds are not particularly useful.

   In reality, our approximation was only off by about 0.07. However, we are approximating ostensibly because we do not know the real answer. In order to be assured that we have a good approximation, we would have to resort to using a polynomial of higher degree.

♣

We practice again. This time, we use Taylor's theorem to find n that guarantees our approximation is within a certain amount.

Example 9.35: Finding sufficiently accurate Taylor polynomials

Find n such that the n^{th} Taylor polynomial of $f(x) = \cos x$ at $x = 0$ approximates $\cos 2$ to within 0.001 of the actual answer. What is $p_n(2)$?

Solution. Following Taylor's theorem, we need bounds on the size of the derivatives of $f(x) = \cos x$. In the case of this trigonometric function, this is easy. All derivatives of cosine are $\pm \sin x$ or $\pm \cos x$. In all cases, these functions are never greater than 1 in absolute value. We want the error to be less than 0.001. To find the appropriate n, consider the following inequalities:

$$\frac{\max \left| f^{(n+1)}(z) \right|}{(n+1)!} \left|(2-0)^{(n+1)}\right| \leq 0.001$$

$$\frac{1}{(n+1)!} \cdot 2^{(n+1)} \leq 0.001$$

We find an n that satisfies this last inequality with trial–and–error. When $n = 8$, we have $\dfrac{2^{8+1}}{(8+1)!} \approx 0.0014$; when $n = 9$, we have $\dfrac{2^{9+1}}{(9+1)!} \approx 0.000282 < 0.001$. Thus we want to approximate $\cos 2$ with $p_9(2)$.

We now set out to compute $p_9(x)$. We again need a table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$. A table of these values is given in Table 9.3.

$$\begin{array}{lcl}
f(x) = \cos x & \Rightarrow & f(0) = 1 \\
f'(x) = -\sin x & \Rightarrow & f'(0) = 0 \\
f''(x) = -\cos x & \Rightarrow & f''(0) = -1 \\
f'''(x) = \sin x & \Rightarrow & f'''(0) = 0 \\
f^{(4)}(x) = \cos x & \Rightarrow & f^{(4)}(0) = 1 \\
f^{(5)}(x) = -\sin x & \Rightarrow & f^{(5)}(0) = 0 \\
f^{(6)}(x) = -\cos x & \Rightarrow & f^{(6)}(0) = -1 \\
f^{(7)}(x) = \sin x & \Rightarrow & f^{(7)}(0) = 0 \\
f^{(8)}(x) = \cos x & \Rightarrow & f^{(8)}(0) = 1 \\
f^{(9)}(x) = -\sin x & \Rightarrow & f^{(9)}(0) = 0
\end{array}$$

Table 9.3: The derivatives of $f(x) = \cos x$ evaluat at $x = 0$.

Notice how the derivatives, evaluated at $x = 0$, follow a certain pattern. All the odd powers of x in the Taylor polynomial will disappear as their coefficient is 0. While our error bounds state that we need $p_9(x)$, our work shows that this will be the same as $p_8(x)$.

Since we are forming our polynomial at $x = 0$, we are creating a Maclaurin polynomial, and:

$$\begin{aligned}
p_8(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(8)}}{8!}x^8 \\
&= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8
\end{aligned}$$

We finally approximate $\cos 2$:

$$\cos 2 \approx p_8(2) = -\frac{131}{315} \approx -0.41587.$$

Our error bound guarantee that this approximation is within 0.001 of the correct answer. Technology shows us that our approximation is actually within about 0.0003 of the correct answer.

Figure 9.10 shows a graph of $y = p_8(x)$ and $y = \cos x$. Note how well the two functions agree on about $(-\pi, \pi)$.

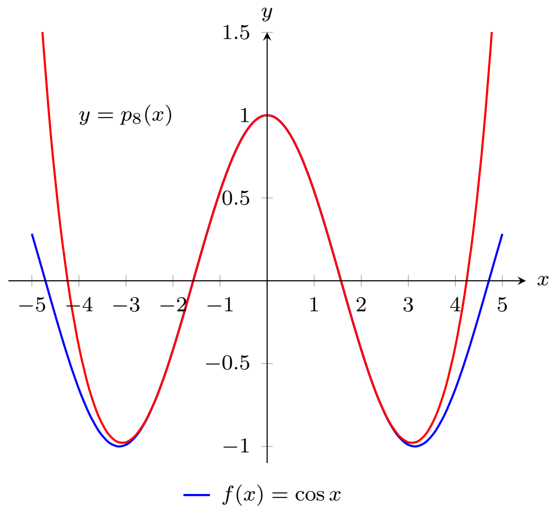


Figure 9.10: A graph of $f(x) = \cos x$ and its degree 8 Maclaurin polynomial.

♣

Example 9.36:

Find a polynomial approximation for $\sin x$ accurate to ± 0.005 for values of x in $[-\pi/2, \pi/2]$.

Solution. From Taylor's theorem with $a = 0$:

$$\sin x = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} x^n + \frac{f^{(N+1)}(z)}{(N+1)!} x^{N+1}.$$

What can we say about the size of the term

$$\frac{f^{(N+1)}(z)}{(N+1)!} x^{N+1}?$$

Every derivative of $\sin x$ is $\pm \sin x$ or $\pm \cos x$, so $|f^{(N+1)}(z)| \leq 1$.

So we need to pick N so that

$$\left| \frac{x^{N+1}}{(N+1)!} \right| < 0.005.$$

Since we have limited x to $[-\pi/2, \pi/2]$,

$$\left| \frac{x^{N+1}}{(N+1)!} \right| < \frac{2^{N+1}}{(N+1)!}.$$

The quantity on the right decreases with increasing N, so all we need to do is find an N so that

$$\frac{2^{N+1}}{(N+1)!} < 0.005.$$

A little trial and error shows that $N = 8$ works, and in fact $2^9/9! < 0.0015$, so

$$\begin{aligned}\sin x &= \sum_{n=0}^{8} \frac{f^{(n)}(0)}{n!} x^n \pm 0.0015 \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \pm 0.0015.\end{aligned}$$

Figure 9.11 shows the graphs of $\sin x$ and and the approximation on $[0, 3\pi/2]$. As x gets larger, the approximation heads to negative infinity very quickly, since it is essentially acting like $-x^7$. ♣

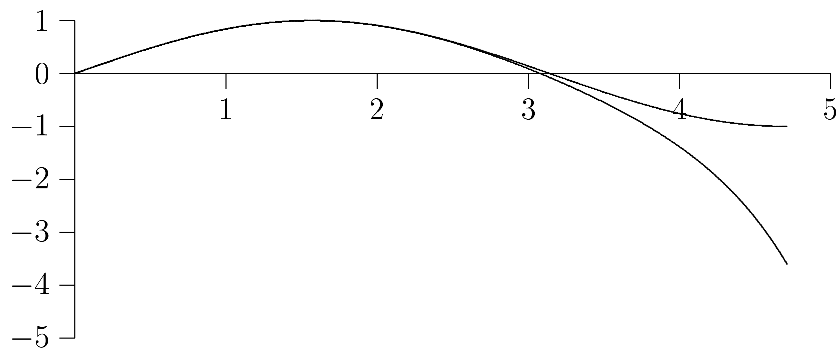


Figure 9.11: $\sin x$ **and a polynomial approximation.**

Note that we can now approximate the value of $\sin(x)$ to within 0.005 by using simple trigonometric identities to translate x into the interval $[-\pi/2, \pi/2]$.

Example 9.37:

Find a polynomial approximation for e^x near $x = 2$ accurate to ± 0.005.

Solution. From Taylor's theorem:

$$e^x = \sum_{n=0}^{N} \frac{e^2}{n!}(x-2)^n + \frac{e^z}{(N+1)!}(x-2)^{N+1},$$

since $f^{(n)}(x) = e^x$ for all n. We are interested in x near 2, and we need to keep $|(x-2)^{N+1}|$ in check, so we may as well specify that $|x-2| \le 1$, so $x \in [1, 3]$. Also

$$\left|\frac{e^z}{(N+1)!}\right| \le \frac{e^3}{(N+1)!},$$

so we need to find an N that makes $e^3/(N+1)! \le 0.005$. This time $N = 5$ makes $e^3/(N+1)! < 0.0015$, so the approximating polynomial is

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3 + \frac{e^2}{24}(x-2)^4 + \frac{e^2}{120}(x-2)^5 \pm 0.0015.$$

Note that our approximation requires that we already have a very accurate approximation of the value e^2, which we shouldn't assume we have in the context of trying to approximate e^x. For this reason we typically try to center our series on values for which the derivative of the function is easy to evaluate (e.g. $a = 0$). ♣

Note well that in these examples we found polynomials of a certain accuracy only on a small interval, even though the series for $\sin x$ and e^x converge for all x; this is typical. To get the same accuracy on a larger interval would require more terms.

Example 9.38: Finding and using Taylor polynomials

1. *Find the degree 4 Taylor polynomial, $p_4(x)$, for $f(x) = \sqrt{x}$ at $x = 4$.*
2. *Use $p_4(x)$ to approximate $\sqrt{3}$.*
3. *Find bounds on the error when approximating $\sqrt{3}$ with $p_4(3)$.*

Solution.

1. We begin by evaluating the derivatives of f at $x = 4$. This is done in Figure 9.4.

$f(x) =$
$f'(x) =$
$f''(x) =$
$f'''(x)$
$f^{(4)}(x)$

Table 9.4: The de

These values allow us to form the Taylor polynomial $p_4(x)$:

$$p_4(x) = 2 + \frac{1}{4}(x-4) + \frac{-1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3 + \frac{-15/2048}{4!}(x-4)^4.$$

2. As $p_4(x) \approx \sqrt{x}$ near $x = 4$, we approximate $\sqrt{3}$ with $p_4(3) = 1.73212$.

3. To find a bound on the error, we need an open interval that contains $x = 3$ and $x = 4$. We set $I = (2.9, 4.1)$. The largest value the fifth derivative of $f(x) = \sqrt{x}$ takes on this interval is near $x = 2.9$, at about 0.0273. Thus

$$\left|R_4(3)\right| \leq \frac{0.0273}{5!}\left|(3-4)^5\right| \approx 0.00023.$$

This shows our approximation is accurate to at least the first 2 places after the decimal. (It turns out that our approximation is actually accurate to 4 places after the decimal.) A graph of $f(x) = \sqrt{x}$ and $p_4(x)$ is given in Figure 9.12. Note how the two functions are nearly indistinguishable on $(2, 7)$.

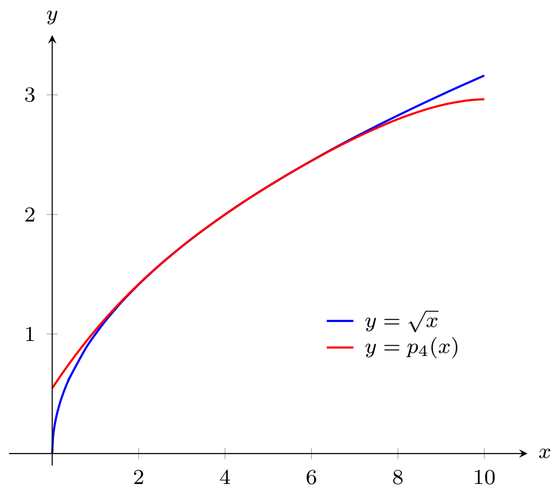


Figure 9.12: A graph of $f(x) = \sqrt{x}$ and its degree 4 Taylor polynomial at $x = 4$.

♣

Our final example gives a brief introduction to using Taylor polynomials to solve differential equations.

Example 9.39: Approximating an unknown function

A function $y = f(x)$ is unknown save for the following two facts.

1. $y(0) = f(0) = 1$, *and*
2. $y' = y^2$

(This second fact says that amazingly, the derivative of the function is actually the function squared!)
Find the degree 3 Maclaurin polynomial $p_3(x)$ of $y = f(x)$.

Solution. One might initially think that not enough information is given to find $p_3(x)$. However, note how the second fact above actually lets us know what $y'(0)$ is:

$$y' = y^2 \Rightarrow y'(0) = y^2(0).$$

Since $y(0) = 1$, we conclude that $y'(0) = 1$.

Now we find information about y''. Starting with $y' = y^2$, take derivatives of both sides, *with respect to x*. That means we must use implicit differentiation.

$$\begin{aligned} y' &= y^2 \\ \frac{d}{dx}\big(y'\big) &= \frac{d}{dx}\big(y^2\big) \\ y'' &= 2y \cdot y'. \end{aligned}$$

Now evaluate both sides at $x = 0$:

$$\begin{aligned} y''(0) &= 2y(0) \cdot y'(0) \\ y''(0) &= 2 \end{aligned}$$

We repeat this once more to find $y'''(0)$. We again use implicit differentiation; this time the Product Rule is also required.

$$\begin{aligned} \frac{d}{dx}\big(y''\big) &= \frac{d}{dx}\big(2yy'\big) \\ y''' &= 2y' \cdot y' + 2y \cdot y''. \end{aligned}$$

Now evaluate both sides at $x = 0$:

$$\begin{aligned} y'''(0) &= 2y'(0)^2 + 2y(0)y''(0) \\ y'''(0) &= 2 + 4 = 6 \end{aligned}$$

In summary, we have:

$$y(0) = 1 \qquad y'(0) = 1 \qquad y''(0) = 2 \qquad y'''(0) = 6.$$

We can now form $p_3(x)$:

$$\begin{aligned} p_3(x) &= 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 \\ &= 1 + x + x^2 + x^3. \end{aligned}$$

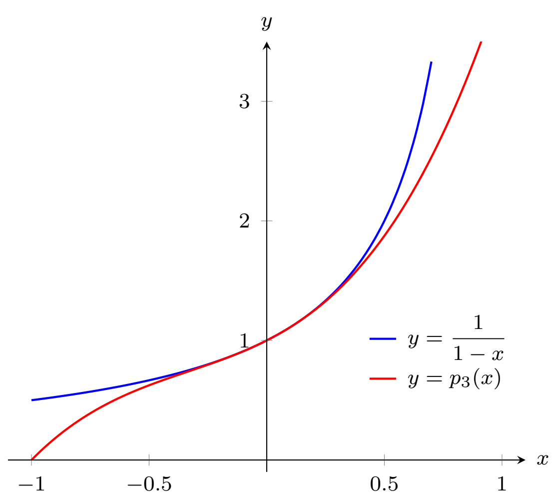


Figure 9.13: A graph of $y = -1/(x-1)$ and $y = p_3(x)$ from Example 9.39.

It turns out that the differential equation we started with, $y' = y^2$, where $y(0) = 1$, can be solved without too much difficulty: $y = \dfrac{1}{1-x}$. Figure 9.13 shows this function plotted with $p_3(x)$. Note how similar they are near $x = 0$. ♣

It is beyond the scope of this text to pursue error analysis when using Taylor polynomials to approximate solutions to differential equations. This topic is often broached in introductory Differential Equations courses and usually covered in depth in Numerical Analysis courses. Such an analysis is very important; one needs to know how good their approximation is. We explored this example simply to demonstrate the usefulness of Taylor polynomials.

Most of this chapter has been devoted to the study of infinite series. This section has taken a step back from this study, focusing instead on finite summation of terms. In the next section, we explore *Taylor Series*, where we represent a function with an infinite series.

Exercises for 9.10

9.10.1 *Find the Maclaurin polynomial of degree n for the given function.*

(a) $f(x) = e^{-x}, \quad n = 3$

(b) $f(x) = \sin x, \quad n = 8$

(c) $f(x) = x \cdot e^{x}, \quad n = 5$

(d) $f(x) = \tan x, \quad n = 6$

(e) $f(x) = e^{2x}, \quad n = 4$

(f) $f(x) = \dfrac{1}{1-x}, \quad n = 4$

(g) $f(x) = \dfrac{1}{1+x}, \quad n = 4$

(h) $f(x) = \dfrac{1}{1+x}, \quad n = 7$

9.10.2 *Find the Taylor polynomial of degree n, at $x = c$, for the given function.*

(a) $f(x) = \sqrt{x}, \quad n = 4, \quad c = 1$

(b) $f(x) = \ln(x+1), \quad n = 4, \quad c = 1$

(c) $f(x) = \cos x, \quad n = 6, \quad c = \pi/4$

(d) $f(x) = \sin x, \quad n = 5, \quad c = \pi/6$

(e) $f(x) = \dfrac{1}{x}, \quad n = 5, \quad c = 2$

(f) $f(x) = \dfrac{1}{x^2}, \quad n = 8, \quad c = 1$

(g) $f(x) = \dfrac{1}{x^2+1}, \quad n = 3, \quad c = -1$

(h) $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

9.10.3 *Approximate the function value with the indicated Taylor polynomial and give approximate bounds on the error.*

(a) *Approximate* $\sin 0.1$ *with the Maclaurin polynomial of degree 3.*

(b) *Approximate* $\cos 1$ *with the Maclaurin polynomial of degree 4.*

(c) *Approximate* $\sqrt{10}$ *with the Taylor polynomial of degree 2 centered at* $x = 9$.

(d) *Approximate* $\ln 1.5$ *with the Taylor polynomial of degree 3 centered at* $x = 1$.

9.10.4 *Find an* n *such that* $p_n(x)$ *approximates* $f(x)$ *within the specified bound of accuracy.*

(a) *Find* n *such that the Maclaurin polynomial of degree* n *of* $f(x) = e^x$ *approximates* e *within* 0.0001 *of the actual value.*

(b) *Find* n *such that the Taylor polynomial of degree* n *of* $f(x) = \sqrt{x}$, *centered at* $x = 4$, *approximates* $\sqrt{3}$ *within* 0.0001 *of the actual value.*

(c) *Find* n *such that the Maclaurin polynomial of degree* n *of* $f(x) = \cos x$ *approximates* $\cos \pi/3$ *within* 0.0001 *of the actual value.*

(d) *Find* n *such that the Maclaurin polynomial of degree* n *of* $f(x) = \sin x$ *approximates* $\cos \pi$ *within* 0.0001 *of the actual value.*

9.10.5 *Find the* n^{th} *term of the indicated Taylor polynomial.*

(a) *Find a formula for the* n^{th} *term of the Maclaurin polynomial for* $f(x) = e^x$.

(b) *Find a formula for the* n^{th} *term of the Maclaurin polynomial for* $f(x) = \cos x$.

(c) *Find a formula for the* n^{th} *term of the Maclaurin polynomial for* $f(x) = \dfrac{1}{1-x}$.

(d) *Find a formula for the* n^{th} *term of the Maclaurin polynomial for* $f(x) = \dfrac{1}{1+x}$.

(e) *Find a formula for the* n^{th} *term of the Taylor polynomial for* $f(x) = \ln x$.

9.10.6 *Approximate the solution to the given differential equation with a degree* 4 *Maclaurin polynomial.*

(a) $y' = y, \qquad y(0) = 1$

(b) $y' = 5y, \qquad y(0) = 3$

(c) $y' = \dfrac{2}{y}, \qquad y(0) = 1$

9.11 Taylor Series

In Section 9.8, we showed how certain functions can be represented by a power series function. In 9.10, we showed how we can approximate functions with polynomials, given that enough derivative information is available. In this section we combine these concepts: if a function $f(x)$ is infinitely differentiable, we show how to represent it with a power series function.

Definition 9.5: Taylor and Maclaurin Series

Let $f(x)$ have derivatives of all orders at $x = c$.

1. *The Taylor Series of $f(x)$, centered at c is*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. *Setting $c = 0$ gives the Maclaurin Series of $f(x)$:*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

The difference between a Taylor polynomial and a Taylor series is the former is a polynomial, containing only a finite number of terms, whereas the latter is a series, a summation of an infinite set of terms. When creating the Taylor polynomial of degree n for a function $f(x)$ at $x = c$, we needed to evaluate f, and the first n derivatives of f, at $x = c$. When creating the Taylor series of f, it helps to find a pattern that describes the n^{th} derivative of f at $x = c$. We demonstrate this in the next two examples.

Example 9.40: The Maclaurin series of $f(x) = \cos x$

Find the Maclaurin series of $f(x) = \cos x$.

Solution. In Example 9.35 we found the 8^{th} degree Maclaurin polynomial of $\cos x$. In doing so, we created the Table 9.5.

$$\begin{array}{lll}
f(x) = \cos x & \Rightarrow & f(0) = 1 \\
f'(x) = -\sin x & \Rightarrow & f'(0) = 0 \\
f''(x) = -\cos x & \Rightarrow & f''(0) = -1 \\
f'''(x) = \sin x & \Rightarrow & f'''(0) = 0 \\
f^{(4)}(x) = \cos x & \Rightarrow & f^{(4)}(0) = 1 \\
f^{(5)}(x) = -\sin x & \Rightarrow & f^{(5)}(0) = 0 \\
f^{(6)}(x) = -\cos x & \Rightarrow & f^{(6)}(0) = -1 \\
f^{(7)}(x) = \sin x & \Rightarrow & f^{(7)}(0) = 0 \\
f^{(8)}(x) = \cos x & \Rightarrow & f^{(8)}(0) = 1 \\
f^{(9)}(x) = -\sin x & \Rightarrow & f^{(9)}(0) = 0
\end{array}$$

Table 9.5: The derivatives of $f(x) = \cos x$ evaluated at $x = 0$.

Notice how $f^{(n)}(0) = 0$ when n is odd, $f^{(n)}(0) = 1$ when n is divisible by 4, and $f^{(n)}(0) = -1$ when n is even but not divisible by 4. Thus the Maclaurin series of $\cos x$ is

$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

We can go further and write this as a summation. Since we only need the terms where the power of x is even, we write the power series in terms of x^{2n}:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

♣

Example 9.41: The Taylor series of $f(x) = \ln x$ at $x = 1$

Find the Taylor series of $f(x) = \ln x$ centered at $x = 1$.

Solution. Table 9.6 shows the n^{th} derivative of $\ln x$ evaluated at $x = 1$ for $n = 0, \ldots, 5$, along with an expression for the n^{th} term:

$$f^{(n)}(1) = (-1)^{n+1}(n-1)! \quad \text{for } n \geq 1.$$

Remember that this is what distinguishes Taylor series from Taylor polynomials; we are very interested in finding a pattern for the n^{th} term, not just finding a finite set of coefficients for a polynomial.

$f(x) = \ln x$	$\Rightarrow$	$f(1) = 0$
$f'(x) = 1/x$	$\Rightarrow$	$f'(1) = 1$
$f''(x) = -1/x^2$	$\Rightarrow$	$f''(1) = -1$
$f'''(x) = 2/x^3$	$\Rightarrow$	$f'''(1) = 2$
$f^{(4)}(x) = -6/x^4$	$\Rightarrow$	$f^{(4)}(1) = -6$
$f^{(5)}(x) = 24/x^5$	$\Rightarrow$	$f^{(5)}(1) = 24$
$\vdots$		$\vdots$
$f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{x^n}$	$\Rightarrow$	$f^{(n)}(1) = (-1)^{n+1}(n-1)!$

Table 9.6: Derivatives of $\ln x$ evaluated at $x = 1$.

Since $f(1) = \ln 1 = 0$, we skip the first term and start the summation with $n = 1$, giving the Taylor series for $\ln x$, centered at $x = 1$, as

$$\sum_{n=1}^{\infty}(-1)^{n+1}(n-1)!\frac{1}{n!}(x-1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}.$$

♣

It is important to note that Definition **??** defines a Taylor series given a function $f(x)$; however, we *cannot* yet state that $f(x)$ *is equal* to its Taylor series. We will find that "most of the time" they are equal, but we need to consider the conditions that allow us to conclude this.

Theorem 9.18 states that the error between a function $f(x)$ and its n^{th}–degree Taylor polynomial $p_n(x)$ is $R_n(x)$, where

$$\left|R_n(x)\right| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-c)^{(n+1)}\right|.$$

If $R_n(x)$ goes to 0 for each x in an interval I as n approaches infinity, we conclude that the function is equal to its Taylor series expansion.

Theorem 9.19: Function and Taylor Series Equality

Let $f(x)$ have derivatives of all orders at $x = c$, let $R_n(x)$ be as stated in Theorem 9.18, and let I be an interval on which the Taylor series of $f(x)$ converges. If $\lim_{n\to\infty} R_n(x) = 0$ for all x in I, then

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n \quad \text{on } I.$$

We demonstrate the use of this theorem in an example.

Example 9.42: Establishing equality of a function and its Taylor series

*Show that $f(x) = \cos x$ is equal to its Maclaurin series, as found in Example **??**, for all x.*

Solution. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$\left|R_n(x)\right| \leq \frac{\max\left| f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{n+1}\right|.$$

Since all derivatives of $\cos x$ are $\pm\sin x$ or $\pm\cos x$, whose magnitudes are bounded by 1, we can state

$$\left|R_n(x)\right| \leq \frac{1}{(n+1)!}\left|x^{n+1}\right|$$

which implies

$$-\frac{\left|x^{n+1}\right|}{(n+1)!} \leq R_n(x) \leq \frac{\left|x^{n+1}\right|}{(n+1)!}. \tag{9.1}$$

For any x, $\displaystyle\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)!} = 0$. Applying the Squeeze Theorem to Equation (9.1), we conclude that $\displaystyle\lim_{n\to\infty} R_n(x) = 0$ for all x, and hence

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \quad \text{for all } x.$$

♣

It is natural to assume that a function is equal to its Taylor series on the series' interval of convergence, but this is not the case. In order to properly establish equality, one must use Theorem 9.19. This is a bit disappointing, as we developed beautiful techniques for determining the interval of convergence of a power series, and proving that $R_n(x) \to 0$ can be cumbersome as it deals with high order derivatives of the function.

There is good news. A function $f(x)$ that is equal to its Taylor series, centred at any point the domain of $f(x)$, is said to be an *analytic function*, and most, if not all, functions that we encounter within this course are analytic functions. Generally speaking, any function that one creates with elementary functions (polynomials, exponentials, trigonometric functions, etc.) that is not piecewise defined is probably analytic. For most functions, we assume the function is equal to its Taylor series on the series' interval of convergence and only use Theorem 9.19 when we suspect something may not work as expected.

We develop the Taylor series for one more important function, then give a table of the Taylor series for a number of common functions.

Example 9.43: The Binomial Series

Find the Maclaurin series of $f(x) = (1+x)^k$, $k \neq 0$.

Solution. When k is a positive integer, the Maclaurin series is finite. For instance, when $k = 4$, we have

$$f(x) = (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

The coefficients of x when k is a positive integer are known as the *binomial coefficients*, giving the series we are developing its name.

When $k = 1/2$, we have $f(x) = \sqrt{1+x}$. Knowing a series representation of this function would give a useful way of approximating $\sqrt{1.3}$, for instance.

To develop the Maclaurin series for $f(x) = (1+x)^k$ for any value of $k \neq 0$, we consider the derivatives of f evaluated at $x = 0$:

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1 \\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1) \\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots\big(k-(n-1)\big)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots\big(k-(n-1)\big)
\end{aligned}$$

Thus the Maclaurin series for $f(x) = (1+x)^k$ is

$$1 + k + \frac{k(k-1)}{2!} + \frac{k(k-1)(k-2)}{3!} + \ldots + \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!} + \ldots$$

It is important to determine the interval of convergence of this series. With

$$a_n = \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n,$$

we apply the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \left|\frac{k(k-1)\cdots(k-n)}{(n+1)!}x^{n+1}\right| \Bigg/ \left|\frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n\right| \\
&= \lim_{n\to\infty} \left|\frac{k-n}{n}x\right| \\
&= |x|.
\end{aligned}$$

The series converges absolutely when the limit of the Ratio Test is less than 1; therefore, we have absolute convergence when $|x| < 1$.

While outside the scope of this text, the interval of convergence depends on the value of k. When $k > 0$, the interval of convergence is $[-1, 1]$. When $-1 < k < 0$, the interval of convergence is $[-1, 1)$. If $k \le -1$, the interval of convergence is $(-1, 1)$. ♣

We learned that Taylor polynomials offer a way of approximating a "difficult to compute" function with a polynomial. Taylor series offer a way of exactly representing a function with a series. One probably can see the use of a good approximation; is there any use of representing a function exactly as a series?

While we should not overlook the mathematical beauty of Taylor series (which is reason enough to study them), there are practical uses as well. They provide a valuable tool for solving a variety of problems, including problems relating to integration and differential equations.

In Key Idea 9.11 (on the following page) we give a table of the Taylor series of a number of common functions. We then give a theorem about the "algebra of power series," that is, how we can combine power series to create power series of new functions. This allows us to find the Taylor series of functions like $f(x) = e^x \cos x$ by knowing the Taylor series of e^x and $\cos x$.

Before we investigate combining functions, consider the Taylor series for the arctangent function (see Key Idea 9.11). Knowing that $\tan^{-1}(1) = \pi/4$, we can use this series to approximate the value of π:

$$\begin{aligned} \frac{\pi}{4} &= \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \\ \pi &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right) \end{aligned}$$

Unfortunately, this particular expansion of π converges very slowly. The first 100 terms approximate π as 3.13159, which is not particularly good.

Key Idea 9.11.0: Important Taylor Series Expansions

Function and Series	First Few Terms	Interval of Convergence
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$	$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$	$(0, 2]$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!} x^n$	$1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots$	$(-1, 1)$[a]
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1, 1]$

[a] Convergence at $x = \pm 1$ depends on the value of k.

Theorem 9.20: Algebra of Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *converge absolutely for* $|x| < R$, *and let* $h(x)$ *be continuous.*

1. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n \quad$ *for* $|x| < R$.

2. $f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \big(a_0 b_n + a_1 b_{n-1} + \ldots a_n b_0\big) x^n$ *for* $|x| < R$.

3. $f\big(h(x)\big) = \sum_{n=0}^{\infty} a_n \big(h(x)\big)^n \quad$ *for* $|h(x)| < R$.

Example 9.44: Combining Taylor series

Write out the first 3 terms of the Taylor Series for $f(x) = e^x \cos x$ using Key Idea 9.11 and Theorem 9.20.

Solution. Key Idea 9.11 informs us that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots .$$

Applying Theorem 9.20, we find that

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right).$$

Distribute the right hand expression across the left:

$$\begin{aligned} &= 1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\ &\quad + \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^4}{4!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \cdots \end{aligned}$$

Distribute again and collect like terms.

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630} + \cdots$$

While this process is a bit tedious, it is much faster than evaluating all the necessary derivatives of $e^x \cos x$ and computing the Taylor series directly.

Because the series for e^x and $\cos x$ both converge on $(-\infty, \infty)$, so does the series expansion for $e^x \cos x$. ♣

Example 9.45: Creating new Taylor series

Use Theorem 9.20 to create series for $y = \sin(x^2)$ and $y = \ln(\sqrt{x})$.

Solution. Given that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ,$$

we simply substitute x^2 for x in the series, giving

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \cdots .$$

Since the Taylor series for $\sin x$ has an infinite radius of convergence, so does the Taylor series for $\sin(x^2)$.

The Taylor expansion for $\ln x$ given in Key Idea 9.11 is centred at $x = 1$, so we will centre the series for $\ln(\sqrt{x})$ at $x = 1$ as well. With

$$\ln x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots,$$

we substitute $\sqrt{x}$ for x to obtain

$$\ln(\sqrt{x}) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(\sqrt{x}-1)^n}{n} = (\sqrt{x}-1) - \frac{(\sqrt{x}-1)^2}{2} + \frac{(\sqrt{x}-1)^3}{3} - \cdots.$$

While this is not strictly a power series, it is a series that allows us to study the function $\ln(\sqrt{x})$. Since the interval of convergence of $\ln x$ is $(0, 2]$, and the range of $\sqrt{x}$ on $(0, 4]$ is $(0, 2]$, the interval of convergence of this series expansion of $\ln(\sqrt{x})$ is $(0, 4]$. **Note:** In Example **??**, one could create a series for $\ln(\sqrt{x})$ by simply recognizing that $\ln(\sqrt{x}) = \ln(x^{1/2}) = 1/2 \ln x$, and hence multiplying the Taylor series for $\ln x$ by $1/2$. This example was chosen to demonstrate other aspects of series, such as the fact that the interval of convergence changes. ♣

Example 9.46: Using Taylor series to evaluate definite integrals

Use the Taylor series of e^{-x^2} *to evaluate* $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution. We learned, when studying Numerical Integration, that e^{-x^2} does not have an antiderivative expressible in terms of elementary functions. This means any definite integral of this function must have its value approximated, and not computed exactly.

We can quickly write out the Taylor series for e^{-x^2} using the Taylor series of e^x:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and so

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!} \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots. \end{aligned}$$

We use Theorem **??** to integrate:

$$\int e^{-x^2}\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

This *is* the antiderivative of e^{-x^2}; while we can write it out as a series, we cannot write it out in terms of elementary functions. We can evaluate the definite integral $\displaystyle\int_0^1 e^{-x^2}\,dx$ using this antiderivative; substituting 1 and 0 for x and subtracting gives

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!}\cdots .$$

Summing the 5 terms shown above give the approximation of 0.74749. Since this is an alternating series, we can use the Alternating Series Approximation Theorem, (Theorem **??**), to determine how accurate this approximation is. The next term of the series is $1/(11\cdot 5!) \approx 0.00075758$. Thus we know our approximation is within 0.00075758 of the actual value of the integral. This is arguably much less work than using Simpson's Rule to approximate the value of the integral. ♣

Example 9.47: Using Taylor series to solve differential equations

Solve the differential equation $y' = 2y$ in terms of a power series, and use the theory of Taylor series to recognize the solution in terms of an elementary function.

Solution. We found the first 5 terms of the power series solution to this differential equation in Example **??** in Section **??**. These are:

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = \frac{4}{2} = 2, \quad a_3 = \frac{8}{2\cdot 3} = \frac{4}{3}, \quad a_4 = \frac{16}{2\cdot 3\cdot 4} = \frac{2}{3}.$$

We include the "unsimplified" expressions for the coefficients found in Example **??** as we are looking for a pattern. It can be shown that $a_n = 2^n/n!$. Thus the solution, written as a power series, is

$$y = \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

Using Key Idea 9.11 and Theorem 9.20, we recognize $f(x) = e^{2x}$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \Rightarrow \qquad e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

♣

Finding a pattern in the coefficients that match the series expansion of a known function, such as those shown in Key Idea 9.11, can be difficult. What if the coefficients in the previous example were given in their reduced form; how could we still recover the function $y = e^{2x}$?

Suppose that all we know is that

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 2, \quad a_3 = \frac{4}{3}, \quad a_4 = \frac{2}{3}.$$

Definition **??** states that each term of the Taylor expansion of a function includes an $n!$. This allows us to say that

$$a_2 = 2 = \frac{b_2}{2!}, \quad a_3 = \frac{4}{3} = \frac{b_3}{3!}, \quad \text{and} \quad a_4 = \frac{2}{3} = \frac{b_4}{4!}$$

for some values b_2, b_3 and b_4. Solving for these values, we see that $b_2 = 4$, $b_3 = 8$ and $b_4 = 16$. That is, we are recovering the pattern we had previously seen, allowing us to write

$$\begin{aligned} f(x) = \sum_{n=0}^{\infty} a_n x^n &= \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n \\ &= 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 + \cdots \end{aligned}$$

From here it is easier to recognize that the series is describing an exponential function.

There are simpler, more direct ways of solving the differential equation $y' = 2y$. We applied power series techniques to this equation to demonstrate its utility, and went on to show how *sometimes* we are able to recover the solution in terms of elementary functions using the theory of Taylor series. Most differential equations faced in real scientific and engineering situations are much more complicated than this one, but power series can offer a valuable tool in finding, or at least approximating, the solution.

This chapter introduced sequences, which are ordered lists of numbers, followed by series, wherein we add up the terms of a sequence. We quickly saw that such sums do not always add up to “infinity,” but rather converge. We studied tests for convergence, then ended the chapter with a formal way of defining functions based on series. Such “series–defined functions” are a valuable tool in solving a number of different problems throughout science and engineering.

Coming in the next chapters are new ways of defining curves in the plane apart from using functions of the form $y = f(x)$. Curves created by these new methods can be beautiful, useful, and important.

Exercises for 9.11

9.11.1 *Key Idea 9.11 gives the n^{th} term of the Taylor series of common functions. Verify the formula given in the Key Idea by finding the first few terms of the Taylor series of the given function and identifying a pattern.*

(a) $f(x) = e^x; \quad c = 0$

(b) $f(x) = \sin x$; $c = 0$

(c) $f(x) = 1/(1-x)$; $c = 0$

(d) $f(x) = \tan^{-1} x$; $c = 0$

9.11.2 *Find a formula for the n^{th} term of the Taylor series of $f(x)$, centered at c, by finding the coefficients of the first few powers of x and looking for a pattern. (The formulas for several of these are found in Key Idea 9.11; show work verifying these formula.)*

(a) $f(x) = \cos x$; $c = \pi/2$

(b) $f(x) = 1/x$; $c = 1$

(c) $f(x) = e^{-x}$; $c = 0$

(d) $f(x) = \ln(1+x)$; $c = 0$

(e) $f(x) = x/(x+1)$; $c = 1$

(f) $f(x) = \sin x$; $c = \pi/4$

9.11.3 *Show that the Taylor series for $f(x)$, as given in Key Idea 9.11, is equal to $f(x)$ by applying Theorem 9.19; that is, show $\lim_{n\to\infty} R_n(x) = 0$.*

(a) $f(x) = \sin x$

(b) $f(x) = e^x$

(c) $f(x) = \ln x$

(d) $f(x) = 1/(1-x)$ *(show equality only on $(-1, 0)$)*

9.11.4 *Use the Taylor series given in Key Idea 9.11 to verify the given identity.*

(a) $\cos(-x) = \cos x$

(b) $\sin(-x) = -\sin x$

(c) $\frac{d}{dx}\big(\sin x\big) = \cos x$

(d) $\frac{d}{dx}\big(\cos x\big) = -\sin x$

9.11.5 *Use the Taylor series given in Key Idea 9.11 to create the Taylor series of the given functions.*

(a) $f(x) = \cos\left(x^2\right)$

(b) $f(x) = e^{-x}$

(c) $f(x) = \sin\left(2x+3\right)$

(d) $f(x) = \tan^{-1}\left(x/2\right)$

(e) $f(x) = e^x \sin x$ *(only find the first 4 terms)*

(f) $f(x) = (1+x)^{1/2}\cos x$ *(only find the first 4 terms)*

9.11.6 *Write out the first* 5 *terms of the Binomial series with the given* k*-value.*

(a) $k = 1/2$

(b) $k = -1/2$

(c) $k = 1/3$

(d) $k = 4$

9.11.7 *Approximate the value of the given definite integral by using the first* 4 *nonzero terms of the integrand's Taylor series.*

(a) $\displaystyle\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\, dx$

(b) $\displaystyle\int_0^{\pi^2/4} \cos\left(\sqrt{x}\right)\, dx$

9.11.8 *For each function, find either the Maclaurin series, or Taylor series centred at* a *when* a *is specified, and the radius of convergence.*

(a) $\cos x$

(b) e^x

(c) $1/x$, $a = 5$

(d) $\ln x$, $a = 1$

(e) $\ln x$, $a = 2$

(f) $1/x^2$, $a = 1$

(g) $1/\sqrt{1-x}$

(h) *Find the first four terms of the Maclaurin series for* $\tan x$ *(up to and including the* x^3 *term).*

(i) *Use a combination of Maclaurin series and algebraic manipulation to find a series centered at zero for* $x\cos(x^2)$.

(j) *Use a combination of Maclaurin series and algebraic manipulation to find a series centered at zero for* xe^{-x}.

10. Differential Equations

Many physical phenomena can be modeled using the language of calculus. For example, observational evidence suggests that the temperature of a cup of tea (or some other liquid) in a room of constant temperature will cool over time at a rate proportional to the difference between the room temperature and the temperature of the tea.

In symbols, if t is the time, M is the room temperature, and $f(t)$ is the temperature of the tea at time t then $f'(t) = k(M - f(t))$ where $k > 0$ is a constant which will depend on the kind of tea (or more generally the kind of liquid) but not on the room temperature or the temperature of the tea. This is **Newton's law of cooling** and the equation that we just wrote down is an example of a **differential equation**. Ideally we would like to solve this equation, namely, find the function $f(t)$ that describes the temperature over time, though this often turns out to be impossible, in which case various approximation techniques must be used. The use and solution of differential equations is an important field of mathematics; here we see how to solve some simple but useful types of differential equation.

Informally, a differential equation is an equation in which one or more of the derivatives of some function appears. Typically, a scientific theory will produce a differential equation (or a system of differential equations) that describes or governs some physical process, but the theory will not produce the desired function or functions directly.

10.1 First Order Differential Equations

We start by considering equations in which only the first derivative of the function appears.

Definition 10.1: First Order Differential Equation

A **first order differential equation** *is an equation of the form $F(t, y, y') = 0$. A solution of a first order differential equation is a function $f(t)$ that makes $F(t, f(t), f'(t)) = 0$ for every value of t.*

Here, F is a function of three variables which we label t, y, and y'. It is understood that y' will explicitly appear in the equation although t and y need not. The term "first order" means that the first derivative of y appears, but no higher order derivatives do.

Example 10.1: Newton's Law of Cooling

The equation from Newton's law of cooling, $y' = k(y-T)$ is a first order differential equation; $F(t, y, y') = k(y-T) - y'$.

Example 10.2: A First Order Differential Equation

$y' = t^2 + 1$ is a first order differential equation; $F(t, y, y') = y' - t^2 - 1$. All solutions to this equation are of the form $t^3/3 + t + C$.

Definition 10.2: First Order Initial Value Problem

A **first order initial value problem** *is a system of equations of the form $F(t, y, y') = 0$, $y(t_0) = y_0$. Here t_0 is a fixed time and y_0 is a number. A solution of an initial value problem is a solution $f(t)$ of the differential equation that also satisfies the* **initial condition** *$f(t_0) = y_0$.*

Example 10.3: An Initial Value Problem

Verify that the initial value problem $y' = t^2 + 1$, $y(1) = 4$ has solution $f(t) = t^3/3 + t + 8/3$.

Solution. Observe that $f'(t) = t^2 + 1$ and $f(1) = 1^3/2 + 1 + 8/3 = 4$ as required. ♣

The general first order equation is too general, so we can't describe methods that will work on them all, or even a large portion of them. We can make progress with specific kinds of first order differential equations. For example, much can be said about equations of the form $y' = \phi(t, y)$ where ϕ is a function of the two variables t and y. Under reasonable conditions on ϕ, such an equation has a solution and the corresponding initial value problem has a unique solution. However, in general, these equations can be very difficult or impossible to solve explicitly.

A special case for which we do have a well defined method is that of separable differential equations.

10.1.1. Separable Differential Equations

Definition 10.3: Separable Differential Equations

A first order differential equation is **separable** *if it can be written in the form*

$$y' = f(t)g(y) \quad \text{or,} \quad \frac{dy}{dt} = f(t)g(y).$$

For example, the differential equation

$$\frac{dy}{dx} = \sin(x)\big(1 + y^2\big)$$

is separable, and one has $F(x) = \sin x$ and $G(y) = 1 + y^2$. On the other hand, the differential equation

$$\frac{dy}{dx} = x + y$$

is not separable.

The general approach to separable equations is as follows:

Suppose we wish to solve $y' = f(t)g(y)$ where f and g are continuous functions. If $g(a) = 0$ for some a then $y(t) = a$ is a constant solution of the equation, since in this case $y' = 0 = f(t)g(a)$. For example, $y' = y^2 - 1$ has constant solutions $y(t) = 1$ and $y(t) = -1$.

Such constant solutions to a differential equation are called *equilibrium solutions*. To find the nonconstant solutions, we divide by $g(y)$ to get

$$\frac{1}{g(y)} \frac{dy}{dt} = f(t). \tag{10.1}$$

Next find a function $H(y)$ whose derivative with respect to y is

$$H'(y) = \frac{1}{g(y)} \qquad \left(\text{solution: } H(y) = \int \frac{dy}{g(y)}.\right) \tag{10.2}$$

Then the chain rule implies that the left hand side in (10.1) can be written as

$$\frac{1}{g(y)} \frac{dy}{dt} = H'(y) \frac{dy}{dt} = \frac{dH(y)}{dt}.$$

Thus (10.1) is equivalent with

$$\frac{dH(y)}{dt} = f(t).$$

In words: $H(y)$ is an antiderivative of $f(t)$, which means we can find $H(y)$ by integrating $f(t)$:

$$H(y) = \int f(t)dt + C. \tag{10.3}$$

Once we have found the integral of $f(t)$ this gives us $y(t)$ in implicit form: the equation (10.3) gives us $y(t)$ as an *implicit function* of t. To get $y(t)$ itself we must solve the equation (10.3) for $y(t)$.

A quick way of organizing the calculation goes like this:

> To solve $\dfrac{dy}{dt} = f(t)g(y)$ we first *separate the variables*,
>
> $$\frac{dy}{g(y)} = f(t)\, dt,$$
>
> and then integrate,
>
> $$\int \frac{dy}{g(y)} = \int f(t)\, dt.$$
>
> The result is an implicit equation for the solution y with one undetermined integration constant.

This technique is called **separation of variables**.

As we have seen so far, a differential equation typically has an infinite number of solutions. Such a solution is called a **general solution**. A corresponding initial value problem will give rise to just one solution. Such a solution in which there are no unknown constants remaining is called a **particular solution**.

Example 10.4:

Find all functions y that are solutions to the differential equation

$$\frac{dy}{dt} = \frac{t}{y^2}.$$

Solution. We begin by separating the variables and writing

$$y^2 dy = t\ dt.$$

Integrating both sides of the equation with respect to the independent variable t shows that

$$\int y^2 \frac{dy}{dt}\ dt = \int t\ dt.$$

Next, we notice that the left-hand side allows us to change the variable of antidifferentiation[1] from t to y. In particular, $dy = \frac{dy}{dt}\ dt$, so we now have

$$\int y^2\ dy = \int t\ dt.$$

This most recent equation says that two families of antiderivatives are equal to one another. Therefore, when we find representative antiderivatives of both sides, we know they must differ by arbitrary constant C. Antidifferentiating and including the integration constant C on the right, we find that

$$\frac{y^3}{3} = \frac{t^2}{2} + C.$$

Again, note that it is not necessary to include an arbitrary constant on both sides of the equation; we know that $y^3/3$ and $t^2/2$ are in the same family of antiderivatives and must therefore differ by a single constant.

Finally, we may now solve the last equation above for y as a function of t, which gives

$$y(t) = \sqrt[3]{\frac{3}{2}\,t^2 + 3C}.$$

Of course, the term $3C$ on the right-hand side represents 3 times an unknown constant. It is, therefore, still an unknown constant, which we will rewrite as C. We thus conclude that the funtion

$$y(t) = \sqrt[3]{\frac{3}{2}\,t^2 + C}$$

[1]This is why we required that the left-hand side be written as a product in which dy/dt is one of the terms.

is a solution to the original differential equation for any value of C. ♣

Notice that because this solution depends on the arbitrary constant C, we have found an infinite family of solutions. This makes sense because we expect to find a unique solution that corresponds to any given initial value.

For example, if we want to solve the initial value problem

$$\frac{dy}{dt} = \frac{t}{y^2},\ y(0) = 2,$$

we know that the solution has the form $y(t) = \sqrt[3]{\frac{3}{2}t^2 + C}$ for some constant C. We therefore must find the appropriate value for C that gives the initial value $y(0) = 2$. Hence,

$$2 = y(0)\sqrt[3]{\frac{3}{2}0^2 + C} = \sqrt[3]{C},$$

which shows that $C = 2^3 = 8$. The solution to the initial value problem is then

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + 8}.$$

Example 10.5: Solving an IVP

Solve the IVP: $y' = 2t(25 - y)$, $y(0) = 20$.

Solution. We begin by finding the general solution to the differential equation. This is almost identical to the previous example. As before, $y(t) = 25$ is a solution. If $y \neq 25$,

$$\begin{aligned}
\int \frac{1}{25-y}\,dy &= \int 2t\,dt \\
(-1)\ln|25-y| &= t^2 + C_0 \\
\ln|25-y| &= -t^2 - C_0 = -t^2 + C \\
|25-y| &= e^{-t^2+C} = e^{-t^2}e^C \\
y - 25 &= \pm e^C e^{-t^2} \\
y &= 25 \pm e^C e^{-t^2} = 25 + Ae^{-t^2}.
\end{aligned} \tag{10.4}$$

As before, all solutions are represented by $y = 25 + Ae^{-t^2}$, allowing A to be zero.

To solve the IVP, we let $y = 20$, and $t = 0$ in Equation **??** to get

$$20 = 25 + A$$

which immediately gives $A = -20$. So the particular solution to the IVP is

$$y = 25 - 20e^{-t^2}$$

♣

One application often discussed when introducing Separable Equations is that of **mixing problems**. A typical mixing problems involves: A tank of fixed capacity; a completely mixed solution of some substance in the tank; a solution of a certain concentration entering the tank at a (usually) fixed rate; the solution immediately becomes completely stirred; and the mixture leaves at the other end at a (usually fixed) rate. We illustrate with an example.

Example 10.6: A

ank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after half an hour?

Solution. Let $y(t)$ denote the amount of salt (kg) in the tank after t minutes.
Given: $y(0) = 20$. We want to know: $y(30)$.

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$\text{rate in} = (\text{ concentration in})(\text{ rate of volume in}) = \left(0.03\ \frac{\text{kg}}{\text{L}}\right)\left(25\ \frac{\text{L}}{\text{min}}\right) = 0.75\ \frac{\text{kg}}{\text{min}}$$

$$\text{rate out} = (\text{ concentration out})(\text{ rate of volume out}) = \left(\frac{y(t)}{5000}\ \frac{\text{kg}}{\text{L}}\right)\left(25\ \frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\ \frac{\text{kg}}{\text{min}}$$

Therefore we have

$$\frac{dy}{dt} = \frac{150 - y(t)}{200}$$

Separating variables we get

$$\int \frac{1}{150 - y}\, dy = \int \frac{1}{200}\, dt$$

which gives

$$-\ln|150 - y| = t/200 + C$$

$y(0) = 20$, so $C = -\ln 130$. Also observe that since $y < 150 (= 0.3 \cdot 5000)$, so $|150 - y| = 150 - y$, so after simplification we get

$$y = 150 - 130e^{-t/200}$$

and therefore $y(30) = 150 - 130e^{-30/200} \approx 38.1$kg. ♣

Example 10.7:

Solve the differential equation

$$\frac{dy}{dt} = 3y.$$

Solution. Following the same strategy as in Example **??**, we have

$$\frac{1}{y}\frac{dy}{dt} = 3.$$

Integrating both sides with respect to t,

$$\int \frac{1}{y}\frac{dy}{dt}\ dt = \int 3\ dt,$$

and thus

$$\int \frac{1}{y}\ dy = \int 3\ dt.$$

Antidifferentiating and including the integration constant, we find that

$$\ln|y| = 3t + C_1$$

where C_1 is an arbitrary constant. Finally, we need to solve for y. Here, one point deserves careful attention. By the definition of the natural logarithm function, it follows that

$$|y| = e^{3t+C_1} = e^{3t}e^{C_1}.$$

Since C is an unknown constant, e^C is as well, though we do know that it is positive (because e^x is positive for any x). When we remove the absolute value in order to solve for y we obtain

$$y = \pm e^{C_1}e^{3t}.$$

As $\pm e^{C_1}$ may be either positive or negative, we will denote this by C to obtain

$$y(t) = Ce^{3t}.$$

There is one technical point to make here. Notice that $y = 0$ is an equilibrium solution to this differential equation. In solving the equation above, we begin by dividing both sides by y, which is not allowed if $y = 0$. To be perfectly careful, therefore, we will typically consider these equilibrium solutions separately. In this case, notice that the final form of our solution captures the equilibrium solution by allowing $C = 0$. ♣

10.1.2. Exponential Growth and Decay

The differential equation in the previous example ($y' = 3y$) describes a quantity y whose rate of change is directly proportional to the quantity itself. Such a differential equation is said to model exponential growth.

Example 10.8: Population Growth and Radioactive Decay

Analyze the differential equation $y' = ky$.

Solution. When $k > 0$, this describes certain simple cases of (exponential) population growth: It says that the change in the population y is proportional to the population. The underlying assumption is that each organism in the current population reproduces at a fixed rate, so the larger the population the more new organisms are produced. While this is too simple to model most real populations, it is useful in some cases over a limited time. The parameter k is called the *proportionality constant*. When $k < 0$, the differential equation describes a quantity that decreases in proportion to the current value (exponential decay); this can be used to model radioactive decay.

The constant solution is $y(t) = 0$; of course this will not be the solution to any interesting initial value problem. For the non-constant solutions, we proceed much as before:

$$\begin{aligned}
\int \frac{1}{y}\,dy &= \int k\,dt \\
\ln|y| &= kt + C \\
|y| &= e^{kt}e^{C} \\
y &= \pm\, e^{C} e^{kt} \\
y &= Ae^{kt}.
\end{aligned}$$

Again, if we allow $A = 0$ this includes the equilibrium solution, and we can simply say that $y = Ae^{kt}$ is the general solution. With an initial value we can easily solve for A to get the solution of the initial value problem. In particular, if the initial value is given for time $t = 0$, $y(0) = y_0$, then $A = y_0$ and the solution is $y = y_0e^{kt}$. ♣

In general, the work in the previous example shows the following to hold true.

Key Idea 10.1.0:

The solution of the initial value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

is $y = y_0e^{kt}$

Example 10.9: Global Population Growth

Assuming that the growth rate is proportional to population size, use the fact the world population in 1900 is 1650 million and the 1910 is 1750 million to estimate population in the year 2000.

Solution. Since growth rate is proportional to population, we know that the population $P(t)$ will be given by a function of the form:

$$P = P_0e^{kt}$$

taking t to be the number of years after 1900. We are asked to find the population in the year 2000, in other words, find $P(100)$. We know $P_0 = 1650$ (in millions), so

$$P = 1650e^{kt} \tag{10.5}$$

So we must solve for the growth constant k. In 1910, (when $t = 10$) the population was 1750 (million), so $P(10) = 1750$:

$$1750 = 1650e^{10k}$$

which gives

$$k = \frac{1}{10}\ln\left(\frac{175}{165}\right)$$

Substituting into equation 10.5 and simplifying gives

$$P = 1650\left(\frac{175}{165}\right)^{\frac{t}{10}}$$

Therefore, after 100 years, the population will be

$$P(100) = 1650\left(\frac{175}{165}\right)^{10} \approx 2972 \text{ million.}$$

♣

As mentioned previously, radioactive decay also follows an exponential model, $y = y_0e^{kt}$ (where $k < 0$). The *half-life* of a material is the time required for half of a given amount to decay. That is, the time for which $\frac{1}{2}y_0 = y_0e^{kt}$. Solving for t gives $t = -\frac{\ln(2)}{k}$.

Key Idea 10.1.0: Half Life

Radioactive decay of a material with decay constant k is modelled by $y = y_0e^{kt}$, and has a half-life of $-\dfrac{ln(2)}{k}$

Example 10.10:

The half-life of radium-226 is 1590 years. A sample of radium has a mass of 100 mg.

1. *Find a formula for the mass of radium after t years.*
2. *Find the mass after 1000 years.*

3. When will the mass be reduced to 30mg?

Solution.

1. As this model is one of exponential decay, we know the formula for the mass after t years will have the form:
$$y = y_0 e^{kt},$$
where $k < 0$. We are told 100mg are initially present, so $y_0 = 100$. To determine the decay constant, we use the given half-life with the formula in Key Idea 10.1.2:
$$k = -\frac{ln(2)}{1590}.$$
Therefore, after simplifications we have
$$y = 100e^{t\cdot -\frac{ln(2)}{1590}} = 100 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1590}}.$$

2. From part (a) we see that after 1000 years the amount remaining will be
$$y(1000) = 100 \cdot \left(\frac{1}{2}\right)^{\frac{1000}{1590}} \approx 64.67\text{mg}.$$

3. We wish to find t when $y = 30$, so we solve: $30 = 100 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1590}}$. Dividing by 100, taking the natural logarithm of both sides, and solving for t gives
$$\ln\left(\frac{3}{10}\right) = \frac{t}{1590}\ln\left(\frac{1}{2}\right) \to\ t = 1590 \cdot \frac{\ln\left(\frac{3}{10}\right)}{\ln\left(\frac{1}{2}\right)} \approx 2762 \text{ years}.$$

♣

More generally, a quantity y may grow (or shrink) with rate of change proportional to a difference $y - b$. Such is the case with Newton's Law of Cooling.

Key Idea 10.1.0: Newton's Law of Cooling

The rate of cooling of an object is directly proportional to the difference between the temperature $y(t)$ of the object and the ambient temperature T (i.e. the temperature T of its surroundings.)
$$\frac{dy}{dt} = k(y - T)$$
where k is called the cooling constant (in units of $(\text{time})^{-1}$), and depends on the physical properties of the materials involved. This differential equation may be solved in the same manner as in Example 10.8 to give
$$y = T + y_0 e^{kt}$$

More generally, if $\frac{dy}{dt} = k(y - b)$ for some constant b, then $y = b + Ce^{kt}$, where $C = y(0)$.

Example 10.11: IVP for Newton's Law of Cooling

Consider this specific example of an initial value problem for Newton's law of cooling: $y' = -2(y-25)$, $y(0) = 40$. *Discuss the solutions for this initial value problem.*

Solution. We first note the zero of the equation: If $y = 25$, the right hand side of the differential equation is zero, and so the constant function $y(t) = 25$ is a solution to the differential equation. It is not a solution to the initial value problem, since $y(0) \neq 25$. (The physical interpretation of this constant solution is that if a liquid is at the same temperature as its surroundings, then the liquid will stay at that temperature.)

At this point we may appeal to the key idea 10.1, taking $T = 25$, and $k = -2$ to solve the differential equation. However, just for practice we will derive the result directly.

Separating variables, so long as $y \neq 25$, we can rewrite the differential equation as

$$\begin{aligned} \frac{dy}{dt}\frac{1}{25-y} &= 2 \\ \frac{1}{25-y}\,dy &= 2\,dt, \end{aligned}$$

so

$$\int \frac{1}{25-y}\,dy = \int 2\,dt,$$

We can calculate these anti-derivatives and rearrange the results:

$$\begin{aligned} \int \frac{1}{25-y}\,dy &= \int 2\,dt \\ (-1)\ln|25-y| &= 2t + C_0 \\ \ln|25-y| &= -2t - C_0 = -2t + C_1 \\ |25-y| &= e^{-2t+C_1} = e^{-2t}e^{C_1} \\ y - 25 &= \pm\, e^{C_1}e^{-2t} \\ y &= 25 \pm e^{C_1}e^{-2t} = 25 + Ce^{-2t}. \end{aligned}$$

Here $C = \pm\, e^{C_1} = \pm\, e^{-C_0}$ is some non-zero constant. Note that this agrees with the solution we would have obtained directly from Key Idea 10.1.

Since we require $y(0) = 40$, we substitute and solve for C:

$$\begin{aligned} 40 &= 25 + Ce^0 \\ 15 &= C, \end{aligned}$$

and so $y = 25 + 15e^{-2t}$ is a solution to the initial value problem.

Note that y is never 25, so this makes sense for all values of t. However, if we allow $C = 0$ we get the solution $y = 25$ to the differential equation, which would be the solution to the initial value problem if we were to require $y(0) = 25$. Thus, $y = 25 + Ce^{-2t}$ describes all solutions to the differential equation $y' = 2(25 - y)$, and all solutions to the associated initial value problems. ♣

Example 10.12:

If an object takes 40 minutes to cool from 30 degrees to 24 degrees in a 20 degree room, how long will it take the object to cool to 21 degrees?

Solution. From the above discussion we know that the model for the temperature y, t minutes after the first temperature measurement is given by

$$y = 20 + 10e^{kt}.$$

To solve for k we use the fact that $y(40) = 24$. Substituting $t = 40$, and $y = 24$ into the last equation, and simplifying gives

$$24 = 2010e^{40k} \quad \rightarrow \quad k = \ln\left[\left(\frac{2}{5}\right)^{\frac{1}{40}}\right]$$

Therefore,

$$y = 10e^{t \cdot \ln\left[\left(\frac{2}{5}\right)^{\frac{1}{40}}\right]} + 20 = 10e^{\ln\left[\left(\frac{2}{5}\right)^{\frac{t}{40}}\right]} + 20 = 10\left(\frac{2}{5}\right)^{\frac{t}{40}} + 20$$

So when the temperature is 21 degrees, we have

$$21 = 10\left(\frac{2}{5}\right)^{\frac{t}{40}} + 20 \Rightarrow t = \frac{-40\ln(10)}{\ln\left(\frac{2}{5}\right)} \approx 100.52\text{min}.$$

Therefore, even though it took only 40 minutes to cool from 30 degrees to 24 degrees (a difference of 6 degrees), it will take over 100 minutes to cool to 21 degrees (the last 3 degrees add more than an hour to the time required!).

♣

Exercises for 10.1

10.1.1 *Which of the following equations are separable?*

(a) $y' = \sin(ty)$

(b) $y' = e^t e^y$

(c) $yy' = t$

(d) $y' = (t^3 - t)\arcsin(y)$

(e) $y' = t^2 \ln y + 4t^3 \ln y$

10.1.2 *Solve* $y' = 1/(1+t^2)$.

10.1.3 *Solve the initial value problem* $y' = t^n$ *with* $y(0) = 1$ *and* $n \geq 0$.

10.1.4 *Solve* $y' = \ln t$.

10.1.5 *Identify the constant solutions (if any) of* $y' = t \sin y$.

10.1.6 *Identify the constant solutions (if any) of* $y' = te^y$.

10.1.7 *Solve* $y' = t/y$.

10.1.8 *Solve* $y' = y^2 - 1$.

10.1.9 *Solve* $y' = t/(y^3 - 5)$. *You may leave your solution in implicit form: that is, you may stop once you have done the integration, without solving for* y.

10.1.10 *Find a non-constant solution of the initial value problem* $y' = y^{1/3}$, $y(0) = 0$, *using separation of variables. Note that the constant function* $y(t) = 0$ *also solves the initial value problem. This shows that an initial value problem can have more than one solution.*

10.1.11 *Solve the equation for Newton's law of cooling leaving* M *and* k *unknown.*

10.1.12 *After 10 minutes in Jean-Luc's room, his tea has cooled to* $40°$ *Celsius from* $100°$ *Celsius. The room temperature is* $25°$ *Celsius. How much longer will it take to cool to* $35°$*?*

10.1.13 *Solve the* **logistic equation** $y' = ky(M - y)$. *(This is a somewhat more reasonable population model in most cases than the simpler* $y' = ky$.*) Sketch the graph of the solution to this equation when* $M = 1000$, $k = 0.002$, $y(0) = 1$.

10.1.14 *Suppose that* $y' = ky$, $y(0) = 2$, *and* $y'(0) = 3$. *What is* y*?*

10.1.15 *A radioactive substance obeys the equation* $y' = ky$ *where* $k < 0$ *and* y *is the mass of the substance at time* t. *Suppose that initially, the mass of the substance is* $y(0) = M > 0$. *At what time does half of the mass remain? (This is known as the half life. Note that the half life depends on* k *but not on* M.*)*

10.1.16 *Bismuth-210 has a half life of five days. If there is initially 600 milligrams, how much is left after 6 days? When will there be only 2 milligrams left?*

10.1.17 *The half life of carbon-14 is 5730 years. If one starts with 100 milligrams of carbon-14, how much is left after 6000 years? How long do we have to wait before there is less than 2 milligrams?*

10.1.18 *A certain species of bacteria doubles its population (or its mass) every hour in the lab. The differential equation that models this phenomenon is* $y' = ky$, *where* $k > 0$ *and* y *is the population of bacteria at time* t. *What is* y*?*

10.1.19 *If a certain microbe doubles its population every 4 hours and after 5 hours the total population has mass 500 grams, what was the initial mass?*

10.2 First Order Homogeneous Linear Equations

A simple, but important and useful, type of separable equation is the **first order homogeneous linear equation**:

Definition 10.4: First Order Homogeneous Linear Equation

A first order homogeneous linear differential equation is one of the form $y' + p(t)y = 0$ or equivalently $y' = -p(t)y$.

"Homogeneous" refers to the zero on the right side of the equation, provided that y' and y are on the left. "Linear" in this definition indicates that both y' and y appear independently and explicitly; we don't see y' or y to any power greater than 1, or multiplied by each other (i.e. $y'y$).

Example 10.13: Linear Examples

The equation $y' = 2t(25-y)$ can be written $y'+2ty = 50t$. This is linear, but not homogeneous. The equation $y' = ky$, or $y' - ky = 0$ is linear and homogeneous, with a particularly simple $p(t) = -k$. The equation $y' + y^2 = 0$ is homogeneous, but not linear.

Since first order homogeneous linear equations are separable, we can solve them in the usual way:

$$\begin{aligned} y' &= -p(t)y \\ \int \frac{1}{y}\,dy &= \int -p(t)\,dt \\ \ln|y| &= P(t) + C \\ y &= \pm\, e^{P(t)} \\ y &= Ae^{P(t)}, \end{aligned}$$

where $P(t)$ is an anti-derivative of $-p(t)$. As in previous examples, if we allow $A = 0$ we get the constant solution $y = 0$.

Example 10.14: Solving an IVP

Solve the initial value problem

$$y' + y\cos t = 0,$$

subject to $y(0) = 1/2$ and $y(2) = 1/2$.

Solution. We start with

$$P(t) = \int -\cos t\, dt = -\sin t,$$

so the general solution to the differential equation is

$$y = Ae^{-\sin t}.$$

To compute A we substitute:

$$\frac{1}{2} = Ae^{-\sin 0} = A,$$

so the solutions is

$$y = \frac{1}{2}e^{-\sin t}.$$

For the second problem,

$$\begin{aligned} \frac{1}{2} &= Ae^{-\sin 2} \\ A &= \frac{1}{2}e^{\sin 2} \end{aligned}$$

so the solution is

$$y = \frac{1}{2}e^{\sin 2}e^{-\sin t}.$$

♣

Example 10.15:

Solve the initial value problem $ty' + 3y = 0$, $y(1) = 2$, assuming $t > 0$.

Solution. We write the equation in standard form: $y' + 3y/t = 0$. Then

$$P(t) = \int -\frac{3}{t}\, dt = -3\ln t$$

and

$$y = Ae^{-3\ln t} = At^{-3}.$$

Substituting to find A: $2 = A(1)^{-3} = A$, so the solution is $y = 2t^{-3}$. ♣

Exercises for 10.2

Find the general solution of each equation in the following exercises.

10.2.1 $y' + 5y = 0$

10.2.2 $y' - 2y = 0$

10.2.3 $y' + \dfrac{y}{1+t^2} = 0$

10.2.4 $y' + t^2 y = 0$

In the following exercises, solve the initial value problem.

10.2.5 $y' + y = 0,\ y(0) = 4$

10.2.6 $y' - 3y = 0,\ y(1) = -2$

10.2.7 $y' + y\sin t = 0,\ y(\pi) = 1$

10.2.8 $y' + ye^t = 0,\ y(0) = e$

10.2.9 $y' + y\sqrt{1+t^4} = 0,\ y(0) = 0$

10.2.10 $y' + y\cos(e^t) = 0,\ y(0) = 0$

10.2.11 $ty' - 2y = 0,\ y(1) = 4$

10.2.12 $t^2 y' + y = 0,\ y(1) = -2,\ t > 0$

10.2.13 $t^3 y' = 2y,\ y(1) = 1,\ t > 0$

10.2.14 $t^3 y' = 2y,\ y(1) = 0,\ t > 0$

10.2.15 *A function $y(t)$ is a solution of $y' + ky = 0$. Suppose that $y(0) = 100$ and $y(2) = 4$. Find k and find $y(t)$.*

10.2.16 *A function $y(t)$ is a solution of $y' + t^k y = 0$. Suppose that $y(0) = 1$ and $y(1) = e^{-13}$. Find k and find $y(t)$.*

10.2.17 *A bacterial culture grows at a rate proportional to its population. If the population is one million at $t = 0$ and 1.5 million at $t = 1$ hour, find the population as a function of time.*

10.2.18 *A radioactive element decays with a half-life of 6 years. If a mass of the element weighs ten pounds at $t = 0$, find the amount of the element at time t.*

10.3 First Order Linear Equations

As you might guess, a first order linear differential equation has the form $y' + p(t)y = f(t)$. Not only is this closely related in form to the first order homogeneous linear equation, we can use what we know about solving homogeneous equations to solve the general linear equation.

Suppose that $y_1(t)$ and $y_2(t)$ are solutions to $y' + p(t)y = f(t)$. Let $g(t) = y_1 - y_2$. Then

$$\begin{aligned} g'(t) + p(t)g(t) &= y_1' - y_2' + p(t)(y_1 - y_2) \\ &= (y_1' + p(t)y_1) - (y_2' + p(t)y_2) \\ &= f(t) - f(t) = 0. \end{aligned}$$

In other words, $g(t) = y_1 - y_2$ is a solution to the homogeneous equation $y' + p(t)y = 0$. Turning this around, any solution to the linear equation $y' + p(t)y = f(t)$, call it y_1, can be written as $y_2 + g(t)$, for some particular y_2 and some solution $g(t)$ of the homogeneous equation $y' + p(t)y = 0$. Since we already know how to find all solutions of the homogeneous equation, finding just one solution to the equation $y' + p(t)y = f(t)$ will give us all of them.

How might we find that one particular solution to $y' + p(t)y = f(t)$? Again, it turns out that what we already know helps. We know that the general solution to the homogeneous equation $y' + p(t)y = 0$ looks like $Ae^{P(t)}$. We now make an inspired guess: Consider the function $v(t)e^{P(t)}$, in which we have replaced the constant parameter A with the function $v(t)$. This technique is called **variation of parameters**. For convenience write this as $s(t) = v(t)h(t)$, where $h(t) = e^{P(t)}$ is a solution to the homogeneous equation. Now let's compute a bit with $s(t)$:

$$\begin{aligned} s'(t) + p(t)s(t) &= v(t)h'(t) + v'(t)h(t) + p(t)v(t)h(t) \\ &= v(t)(h'(t) + p(t)h(t)) + v'(t)h(t) \\ &= v'(t)h(t). \end{aligned}$$

The last equality is true because $h'(t) + p(t)h(t) = 0$. Since $h(t)$ is a solution to the homogeneous equation. We are hoping to find a function $s(t)$ so that $s'(t) + p(t)s(t) = f(t)$; we will have such a function if we can arrange to have $v'(t)h(t) = f(t)$, that is, $v'(t) = f(t)/h(t)$. But this is as easy (or hard) as finding an anti-derivative of $f(t)/h(t)$. Putting this all together, the general solution to $y' + p(t)y = f(t)$ is

$$v(t)h(t) + Ae^{P(t)} = v(t)e^{P(t)} + Ae^{P(t)}.$$

Example 10.16: Solving an IVP

Find the solution of the initial value problem $y' + 3y/t = t^2$, $y(1) = 1/2$.

Solution. First we find the general solution; since we are interested in a solution with a given condition at $t = 1$, we may assume $t > 0$. We start by solving the homogeneous equation as usual; call the solution g:

$$g = Ae^{-\int (3/t)\,dt} = Ae^{-3\ln t} = At^{-3}.$$

Then as in the discussion, $h(t) = t^{-3}$ and $v'(t) = t^2/t^{-3} = t^5$, so $v(t) = t^6/6$. We know that every solution to the equation looks like

$$v(t)t^{-3} + At^{-3} = \frac{t^6}{6}t^{-3} + At^{-3} = \frac{t^3}{6} + At^{-3}.$$

Finally we substitute to find A:

$$\begin{aligned} \frac{1}{2} &= \frac{(1)^3}{6} + A(1)^{-3} = \frac{1}{6} + A \\ A &= \frac{1}{2} - \frac{1}{6} = \frac{1}{3}. \end{aligned}$$

The solution is then

$$y = \frac{t^3}{6} + \frac{1}{3}t^{-3}.$$

♣

Another common method for solving such a differential equation is by means of an **integrating factor**.

Using an Integrating Factor

Linear equations of the form $y' + p(t)y = q(t)$ can always be solved by multiplying both sides of the equation with a specially chosen function called the *integrating factor,* $I(t)$. It is defined by

$$I(t) = e^{\int p(t)\, dt}. \tag{10.6}$$

It looks like we just pulled this definition of $I(t)$ out of a hat.

Multiply the equation by the integrating factor $I(t)$ to get

$$I(t)\frac{dy}{dt} + p(t)I(t)y = I(t)q(t).$$

By the chain rule the integrating factor satisfies

$$\frac{d}{dt}I(t) = \frac{d}{dt}e^{\int p(t)\, dt} = \underbrace{\frac{d}{dt}\left(\int p(t)\, dt\right)}_{=p(t)} \underbrace{e^{\int p(t)\, dt}}_{=I(t)} = p(t)I(t).$$

Therefore one has

$$\begin{aligned}\frac{d}{dt}I(t)y &= I(t)\frac{d}{dt}y + p(t)I(t)y \\ &= I(t)\left\{\frac{d}{dt}y + p(t)y\right\} \\ &= I(t)q(t).\end{aligned}$$

Integrating and then dividing by the integrating factor gives the solution

$$y = \frac{1}{I(t)}\left(\int I(t)q(t)\, dt + C\right).$$

In this derivation we have to divide by $I(t)$, but since $I(t) = e^{\int p(t)\, dt}$ and since exponentials never vanish we know that $I(t) \neq 0$, so we can always divide by $I(t)$.

If you look carefully, you will see that this is exactly the same solution we found by variation of parameters, because $e^{-P(t)}q(t) = q(t)/h(t)$.

Some people find it easier to remember how to use the integrating factor method, rather than variation of parameters. Since ultimately they require the same calculation, you should use whichever of the two methods appeals to you more strongly.

Example 10.17:

Find the general solution to the differential equation

$$\frac{dy}{dx} = y + x.$$

Then find the solution that satisfies

$$y(2) = 0. \tag{10.7}$$

Solution. We first write the equation in the standard linear form

$$\frac{dy}{dx} - y = x, \tag{10.8}$$

and then multiply by the integrating factor $I(x)$. We could of course memorize the formula

$$I(x) = e^{\int p(x)\,dx}$$

but the following procedure will always give us the integrating factor.
Assuming that $I(x)$ is as yet unknown we multiply the differential equation (10.8) by I,

$$I(x)\frac{dy}{dx} - I(x)y = I(x)x. \tag{10.9}$$

If $I(x)$ is such that

$$-I(x) = \frac{dI(x)}{dx}, \tag{10.10}$$

then equation (10.9) implies

$$I(x)\frac{dy}{dx} + \frac{dI(x)}{dx}y = I(x)x.$$

The expression on the left is exactly what comes out of the product rule – this is the point of multiplying with $I(x)$ and then insisting on (10.10). So, if $I(x)$ satisfies (10.10), then the differential equation for y is equivalent with

$$\frac{dI(x)y}{dx} = I(x)x.$$

We can integrate this equation,

$$I(x)y = \int I(x)x\ \ dx,$$

and thus find the solution

$$y(x) = \frac{1}{I(x)}\int I(x)x\ \ dx. \tag{10.11}$$

All we have to do is find the integrating factor I. This factor can be any function that satisfies (10.10). Equation (10.10) is a differential equation for I, but it is separable, and we can easily solve it:

$$\frac{dI}{dx} = -I \iff \frac{1}{I}\,dI = -\,dx \iff \ln|I| = -x + C.$$

Since *we only need one integrating factor* I we are not interested in finding all solutions of (10.10), and therefore we can choose the constant C. The simplest choice is $C = 0$, which leads to

$$\ln|I| = -x \iff |I| = e^{-x} \iff I = \pm e^{-x}.$$

Again, we only need one integrating factor, so we may choose the $\pm$ sign: the simplest choice for I here is

$$m(x) = e^{-x}.$$

With this choice of integrating factor we can now complete the calculation that led to (10.11). The solution to the differential equation is

$$\begin{aligned} y(x) &= \frac{1}{I(x)}\int I(x)x \;\, dx \\ &= \frac{1}{e^{-x}}\int e^{-x}x \;\, dx && \text{(integrate by parts)} \\ &= e^{x}\Big\{-e^{-x}x - e^{-x} + C\Big\} \\ &= -x - 1 + Ce^{x}. \end{aligned}$$

This is the general solution.

To find the solution that satisfies not just the differential equation, but also the "initial condition" (10.7), i.e. $y(2) = 0$, we compute $y(2)$ for the general solution,

$$y(2) = -2 - 1 + Ce^{2} = -3 + Ce^{2}.$$

The requirement $y(2) = 0$ then tells us that $C = 3e^{-2}$. The solution of the differential equation that satisfies the prescribed initial condition is therefore

$$y(x) = -x - 1 + 3e^{x-2}.$$

♣

Example 10.18: U

e the Integrating Factor method to solve the IVP in Example 10.16

Solution. Given $y' + 3y/t = t^2$, we have $p(t) = \frac{3}{t}$, $q(t) = t^2$, the integrating factor is $I(t) = e^{\int (p(t)\,dt}$, and the solution to the differential equation is

$$y = \frac{1}{I(t)}\int I(t)q(t)\,dt$$

where

$$I(t) = e^{\int 3/t} = e^{3\ln|t|} = |t^3| = \pm t^3$$

Again, we only need one integrating factor, so we may choose $I(t) = t^3$. So the general solution is

$$y = \frac{1}{t^3}\int t^3 \cdot t^2\,dt = \frac{1}{t^3}\int t^5\,dt = \frac{1}{t^3}\left(\frac{t^6}{6} + C\right) = \frac{t^3}{6} + \frac{C}{t^3}$$

The initial value $y(1) = \frac{1}{2}$ then gives $C = \frac{1}{3}$, giving the same answer as before.

♣

Exercises for 10.3

In the following exercises, find the general solution of the equation.

10.3.1 $y' + 4y = 8$

10.3.2 $y' - 2y = 6$

10.3.3 $y' + ty = 5t$

10.3.4 $y' + e^t y = -2e^t$

10.3.5 $y' - y = t^2$

10.3.6 $2y' + y = t$

10.3.7 $ty' - 2y = 1/t,\ t > 0$

10.3.8 $ty' + y = \sqrt{t},\ t > 0$

10.3.9 $y' \cos t + y \sin t = 1,\ -\pi/2 < t < \pi/2$

10.3.10 $y' + y \sec t = \tan t,\ -\pi/2 < t < \pi/2$

10.4 Approximation

We have seen how to solve a restricted collection of differential equations, or more accurately, how to attempt to solve them—we still may not be able to find the required anti-derivatives. Not surprisingly, non-linear equations can be even more difficult to solve. Yet much is known about solutions to some more general equations.

Suppose $\phi(t, y)$ is a function of two variables. A more general class of first order differential equations has the form $y' = \phi(t, y)$. This is not necessarily a linear first order equation, since ϕ may depend on y in some complicated way; note however that y' appears in a very simple form. Under suitable conditions on the function ϕ, it can be shown that every such differential equation has a solution, and moreover that for each initial condition the associated initial value problem has exactly one solution. In practical applications this is obviously a very desirable property.

Example 10.19: First Order Non-linear

The equation $y' = t - y^2$ is a first order non-linear equation, because y appears to the second power. We will not be able to solve this equation.

Example 10.20: Non-linear and Separable

The equation $y' = y^2$ is also non-linear, but it is separable and can be solved by separation of variables.

Not all differential equations that are important in practice can be solved exactly, so techniques have been developed to approximate solutions. We describe one such technique, **Euler's Method**, which is simple though not particularly useful compared to some more sophisticated techniques.

Suppose we wish to approximate a solution to the initial value problem $y' = \phi(t, y)$, $y(t_0) = y_0$, for $t \geq t_0$. Under reasonable conditions on ϕ, we know the solution exists, represented by a curve in the t-y plane; call this solution $f(t)$. The point (t_0, y_0) is of course on this curve. We also know the slope of the curve at this point, namely $\phi(t_0, y_0)$. If we follow the tangent line for a brief distance, we arrive at a point that should be almost on the graph of $f(t)$, namely $(t_0 + \Delta t, y_0 + \phi(t_0, y_0)\Delta t)$; call this point (t_1, y_1). Now we pretend, in effect, that this point really is on the graph of $f(t)$, in which case we again know the slope of the curve through (t_1, y_1), namely $\phi(t_1, y_1)$. So we can compute a new point, $(t_2, y_2) = (t_1 + \Delta t, y_1 + \phi(t_1, y_1)\Delta t)$ that is a little farther along, still close to the graph of $f(t)$ but probably not quite so close as (t_1, y_1). We can continue in this way, doing a sequence of straightforward calculations, until we have an approximation (t_n, y_n) for whatever time t_n we need. At each step we do essentially the same calculation, namely:

$$(t_{i+1}, y_{i+1}) = (t_i + \Delta t, y_i + \phi(t_i, y_i)\Delta t).$$

We expect that smaller time steps Δt will give better approximations, but of course it will require more work to compute to a specified time. It is possible to compute a guaranteed upper bound on how far off the approximation might be, that is, how far y_n is from $f(t_n)$. Suffice it to say that the bound is not particularly good and that there are other more complicated approximation techniques that do better.

Example 10.21: Approximating a Solution

Compute an approximation to the solution for $y' = t - y^2$, $y(0) = 0$, when $t = 1$.

Solution. We will use $\Delta t = 0.2$, which is easy to do even by hand, though we should not expect the resulting approximation to be very good. We get

$$\begin{aligned}
(t_1, y_1) &= (0 + 0.2, 0 + (0 - 0^2)0.2) = (0.2, 0)\\
(t_2, y_2) &= (0.2 + 0.2, 0 + (0.2 - 0^2)0.2) = (0.4, 0.04)\\
(t_3, y_3) &= (0.6, 0.04 + (0.4 - 0.04^2)0.2) = (0.6, 0.11968)\\
(t_4, y_4) &= (0.8, 0.11968 + (0.6 - 0.11968^2)0.2) = (0.8, 0.23681533952)\\
(t_5, y_5) &= (1.0, 0.23681533952 + (0.6 - 0.23681533952^2)0.2) = (1.0, 0.385599038513605)
\end{aligned}$$

So $y(1) \approx 0.3856$. As it turns out, this is not accurate to even one decimal place. Figure 10.1 shows these points connected by line segments (the lower curve) compared to a solution obtained by a much better approximation technique. Note that the shape is approximately correct even though the end points are quite far apart.

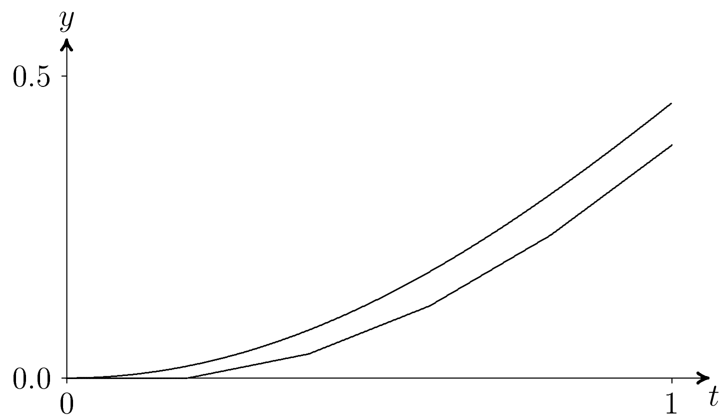


Figure 10.1: Approximating a solution to $y' = t - y^2$, $y(0) = 0$.

If you need to do Euler's method by hand, it is useful to construct a table to keep track of the work, as shown in Figure 10.2. Each row holds the computation for a single step: The starting point (t_i, y_i); the stepsize Δt; the computed slope $\phi(t_i, y_i)$; the change in y, $\Delta y = \phi(t_i, y_i)\Delta t$; and the new point, $(t_{i+1}, y_{i+1}) = (t_i + \Delta t, y_i + \Delta y)$. The starting point in each row is the newly computed point from the end of the previous row.

(t, y)	Δt	$\phi(t, y)$	$\Delta y = \phi(t, y)\Delta t$	$(t + \Delta t, y + \Delta y)$
$(0, 0)$	0.2	0	0	$(0.2, 0)$
$(0.2, 0)$	0.2	0.2	0.04	$(0.4, 0.04)$
$(0.4, 0.04)$	0.2	0.3984	0.07968	$(0.6, 0.11968)$
$(0.6, 0.11968)$	0.2	$0.58\ldots$	$0.117\ldots$	$(0.8, 0.236\ldots)$
$(0.8, 0.236\ldots)$	0.2	$0.743\ldots$	$0.148\ldots$	$(1.0, 0.385\ldots)$

Figure 10.2: Computing with Euler's Method.

♣

Euler's method is related to another technique that can help in understanding a differential equation in a qualitative way. Euler's method is based on the ability to compute the slope of a solution curve at any point in the plane, simply by computing $\phi(t, y)$. If we compute $\phi(t, y)$ at many points, say in a grid, and plot a small line segment with that slope at the point, we can get an idea of how solution curves must look. Such a plot is called a **slope field**. A slope field for $\phi = t - y^2$ is shown in Figure 10.3; compare this to figure 10.1. With a little practice, one can sketch reasonably accurate solution curves based on the slope field, in essence doing Euler's method visually.

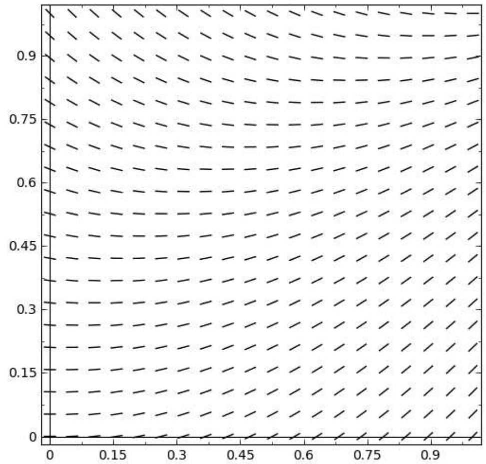


Figure 10.3: A slope field for $y' = t - y^2$.

Even when a differential equation can be solved explicitly, the slope field can help in understanding what the solutions look like with various initial conditions. Recall the logistic equation $y' = ky(M - y)$: y is a population at time t, M is a measure of how large a population the environment can support, and k measures the reproduction rate of the population. Figure 10.4 shows a slope field for this equation that is quite informative. It is apparent that if the initial population is smaller than M it rises to M over the long term, while if the initial population is greater than M it decreases to M.

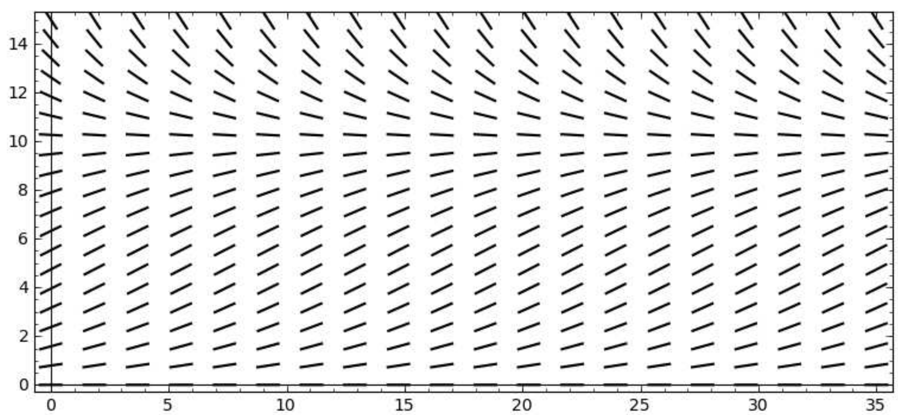


Figure 10.4: A slope field for $y' = 0.2y(10 - y)$.

Exercises for 10.4

In the following exercises, compute the Euler approximations for the initial value problem for $0 \le t \le 1$ and $\Delta t = 0.2$. If you have access to Sage, generate the slope field first and attempt to sketch the solution curve. Then use Sage to compute better approximations with smaller values of Δt.

10.4.1 $y' = t/y,\ y(0) = 1$

10.4.2 $y' = t + y^3,\ y(0) = 1$

10.4.3 $y' = \cos(t + y),\ y(0) = 1$

10.4.4 $y' = t \ln y,\ y(0) = 2$

10.5 Second Order Homogeneous Equations

A second order differential equation is one containing the second derivative y''. These are in general quite complicated, but one fairly simple type is useful: The second order linear equation with constant coefficients.

Example 10.22: Second Order Homogeneous Equation

Analyze the intial value problem $y'' - y' - 2y = 0$, $y(0) = 5$, $y'(0) = 0$.

Solution. We make an inspired guess: might there be a solution of the form e^{rt}? This seems at least plausible, since in this case y'', y', and y all involve e^{rt}.

If such a function is a solution then

$$\begin{aligned} r^2e^{rt} - re^{rt} - 2e^{rt} &= 0 \\ e^{rt}(r^2 - r - 2) &= 0 \\ (r^2 - r - 2) &= 0 \\ (r-2)(r+1) &= 0, \end{aligned}$$

so r is 2 or -1. Not only are $f = e^{2t}$ and $g = e^{-t}$ solutions, but notice that $y = Af + Bg$ is also, for any constants A and B:

$$\begin{aligned} (Af + Bg)'' - (Af + Bg)' - 2(Af + Bg) &= Af'' + Bg'' - Af' - Bg' - 2Af - 2Bg \\ &= A(f'' - f' - 2f) + B(g'' - g' - 2g) \\ &= A(0) + B(0) = 0. \end{aligned}$$

Can we find A and B so that this is a solution to the initial value problem? Let's substitute:

$$5 = y(0) = Af(0) + Bg(0) = Ae^0 + Be^0 = A + B$$

and

$$0 = y'(0) = Af'(0) + Bg'(0) = A2e^0 + B(-1)e^0 = 2A - B.$$

So we need to solve this system of <u>two</u> equations with <u>two</u> unknowns:

$$\begin{cases} A + B &= 5 \\ 2A - B &= 0 \end{cases}$$

Let $B = 2A$, substitute into the first equation to get $5 = A + 2A = 3A$. Then $A = 5/3$ and $B = 10/3$, and the desired solution is $(5/3)e^{2t} + (10/3)e^{-t}$. You now see why the initial condition in this case included both $y(0)$ and $y'(0)$: We needed two equations in the two unknowns A and B ♣

You should of course wonder whether there might be other solutions, but as it turns out, the answer is no. We will not prove this, but here is the theorem that tells us what we need to know:

Theorem 10.1: Solutions to Second Order Homogeneous

Given the differential equation $ay'' + by' + cy = 0$, $a \neq 0$, consider the quadratic polynomial $ar^2 + br + c = 0$, called the **characteristic polynomial**. *Using the quadratic formula, this polynomial always has one or two roots, call them r_1 and r_2. The general solution of the differential equation is:*

(a) $y = Ae^{r_1 t} + Be^{r_2 t}$, if the roots r_1 and r_2 are real numbers, and $r_1 \neq r_2$.

(b) $y = Ae^{r_1 t} + Bte^{r_2 t}$, if $r_1 = r_2$ is a real, repeated root.

(c) $y = A\cos(\beta t)e^{\alpha t} + B\sin(\beta t)e^{\alpha t}$, if the roots are complex numbers, $r = \alpha \pm \beta i$.

Example 10.23:

Solve the differential equation $y'' + A^2 y = 0$.

Solution. First we write the characteristic equation, $r^2 + A^2 = 0$. Then we find the roots of the characteristic equation:

$$r^2 = -A^2 \implies r = \pm Ai$$

These are imaginary roots, so the solution of the differential equation is in the form:

$$y = c_1 \cos(At) + c_2 \sin(At)$$

♣

Example 10.24:

Solve the differential equation $y'' - A^2 y = 0$.

Solution. First we write the characteristic equation, $r^2 - A^2 = 0$. Then we find the roots of the characteristic equation:

$$r^2 = A^2 \implies r = \pm A$$

These are imaginary roots, so the solution of the differential equation is in the form:

$$y = c_1 e^{At} + c_2 e^{-At} = c_1 e^{At} + \frac{c_2}{e^{At}}$$

♣

Example 10.25: Damped Spring Oscillation

Use a differential equation to describe the position of a mass hung on a spring.

Solution. Suppose a mass m is hung on a spring with spring constant k. If the spring is compressed or stretched and then released, the mass will oscillate up and down. Due to friction, the oscillation will be damped: Eventually the motion will cease. The damping will depend on the amount of friction; for example, if the system is suspended in oil the motion will cease sooner than if the system is in air. Using some simple physics, it is not hard to see that the position of the mass is described by the differential equation: $my'' + by' + ky = 0$. Using $m = 1$, $b = 4$, and $k = 5$ we find the motion of the mass. The characteristic polynomial is $r^2 + 4r + 5 = 0$, with roots $r = (-4 \pm \sqrt{16-20})/2 = -2 \pm i$. Thus the general solution is $y = A\cos(t)e^{-2t} + B\sin(t)e^{-2t}$. Suppose we know that $y(0) = 1$ and $y'(0) = 2$. Then as before we form two simultaneous equations: From $y(0) = 1$ we get $1 = A\cos(0)e^0 + B\sin(0)e^0 = A$. For the second we compute

$$y'' = -2Ae^{-2t}\cos(t) + Ae^{-2t}(-\sin(t)) - 2Be^{-2t}\sin(t) + Be^{-2t}\cos(t),$$

and then

$$2 = -2Ae^0\cos(0) - Ae^0\sin(0) - 2Be^0\sin(0) + Be^0\cos(0) = -2A + B.$$

So we get $A = 1$, $B = 4$, and $y = \cos(t)e^{-2t} + 4\sin(t)e^{-2t}$.

Here is a useful trick that makes this easier to understand: We have $y = (\cos t + 4\sin t)e^{-2t}$. The expression $\cos t + 4\sin t$ is a bit reminiscent of the trigonometric formula $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ with $\alpha = t$. Let's rewrite it a bit as

$$\sqrt{17}\left(\frac{1}{\sqrt{17}}\cos t + \frac{4}{\sqrt{17}}\sin t\right).$$

Note that $(1/\sqrt{17})^2 + (4/\sqrt{17})^2 = 1$, which means that there is an angle β with $\cos\beta = 1/\sqrt{17}$ and $\sin\beta = 4/\sqrt{17}$ (of course, β may not be a "nice" angle). Then

$$\cos t + 4\sin t = \sqrt{17}(\cos t\cos\beta + \sin\beta\sin t) = \sqrt{17}\cos(t-\beta).$$

Thus, the solution may also be written $y = \sqrt{17}e^{-2t}\cos(t-\beta)$. This is a cosine curve that has been shifted β to the right; the $\sqrt{17}e^{-2t}$ has the effect of diminishing the amplitude of the cosine as t increases. ♣

Other physical systems that oscillate can also be described by such differential equations. Some electric circuits, for example, generate oscillating current.

Example 10.26:

Find the solution to the intial value problem $y'' - 4y' + 4y = 0$, $y(0) = -3$, $y'(0) = 1$.

Solution. The characteristic polynomial is $r^2 - 4r + 4 = (r-2)^2$, so there is one root, $r = 2$, and the general solution is $Ae^{2t} + Bte^{2t}$. Substituting $t = 0$ we get $-3 = A + 0 = A$. The first derivative is $2Ae^{2t} + 2Bte^{2t} + Be^{2t}$; substituting $t = 0$ gives $1 = 2A + 0 + B = 2A + B = 2(-3) + B = -6 + B$, so $B = 7$. The solution is $-3e^{2t} + 7te^{2t}$. ♣

Exercises for 10.5

10.5.1 *Solve the initial value problem* $y'' - \omega^2 y = 0$, $y(0) = 1$, $y'(0) = 1$, *assuming* $\omega \neq 0$.

10.5.2 *Solve the initial value problem* $2y'' + 18y = 0$, $y(0) = 2$, $y'(0) = 15$.

10.5.3 *Solve the initial value problem* $y'' + 6y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$.

10.5.4 *Solve the initial value problem* $y'' - y' - 12y = 0$, $y(0) = 0$, $y'(0) = 14$.

10.5.5 *Solve the initial value problem* $y'' + 12y' + 36y = 0$, $y(0) = 5$, $y'(0) = -10$.

10.5.6 *Solve the initial value problem* $y'' - 8y' + 16y = 0$, $y(0) = -3$, $y'(0) = 4$.

10.5.7 *Solve the initial value problem* $y'' + 5y = 0$, $y(0) = -2$, $y'(0) = 5$.

10.5.8 *Solve the initial value problem* $y'' + y = 0$, $y(\pi/4) = 0$, $y'(\pi/4) = 2$.

10.5.9 *Solve the initial value problem* $y'' + 12y' + 37y = 0$, $y(0) = 4$, $y'(0) = 0$.

10.5.10 *Solve the initial value problem* $y'' + 6y' + 18y = 0$, $y(0) = 0$, $y'(0) = 6$.

10.5.11 *Solve the initial value problem* $y'' + 4y = 0$, $y(0) = \sqrt{3}$, $y'(0) = 2$.

10.5.12 *Solve the initial value problem* $y'' + 100y = 0$, $y(0) = 5$, $y'(0) = 50$.

10.5.13 *Solve the initial value problem* $y'' + 4y' + 13y = 0$, $y(0) = 1$, $y'(0) = 1$.

10.5.14 *Solve the initial value problem* $y'' - 8y' + 25y = 0$, $y(0) = 3$, $y'(0) = 0$.

10.5.15 *A mass-spring system* $my'' + by' + kx$ *has* $k = 29$, $b = 4$, *and* $m = 1$. *At time* $t = 0$ *the position is* $y(0) = 2$ *and the velocity is* $y'(0) = 1$. *Find* $y(t)$.

10.5.16 *A mass-spring system* $my'' + by' + kx$ *has* $k = 24$, $b = 12$, *and* $m = 3$. *At time* $t = 0$ *the position is* $y(0) = 0$ *and the velocity is* $y'(0) = -1$. *Find* $y(t)$.

10.5.17 *Consider the differential equation* $ay'' + by' = 0$, *with* a *and* b *both non-zero. Find the general solution by the method of this section. Now let* $g = y'$; *the equation may be written as* $ag' + bg = 0$, *a first order linear homogeneous equation. Solve this for* g, *then use the relationship* $g = y'$ *to find* y.

10.5.18 *Suppose that* $y(t)$ *is a solution to* $ay'' + by' + cy = 0$, $y(t_0) = 0$, $y'(t_0) = 0$. *Show that* $y(t) = 0$.

10.6 Second Order Linear Equations - Method of Undetermined Coefficients

Now we consider second order equations of the form $ay'' + by' + cy = f(t)$, with a, b, and c constant. Of course, if $a = 0$ this is really a first order equation, so we assume $a \neq 0$. Also, if $c = 0$ we can solve the related first order equation $ah' + bh = f(t)$, and then solve $h = y'$ for y. So we will only examine examples in which $c \neq 0$.

Suppose that $y_1(t)$ and $y_2(t)$ are solutions to $ay'' + by' + cy = f(t)$, and consider the function $h = y_1 - y_2$. We substitute this function into the left hand side of the differential equation and simplify:

$$a(y_1 - y_2)'' + b(y_1 - y_2)' + c(y_1 - y_2) = ay_1'' + by_1' + cy_1 - (ay_2'' + by_2' + cy_2) = f(t) - f(t) = 0.$$

So h is a solution to the homogeneous equation $ay'' + by' + cy = 0$. Since we know how to find all such h, then with just one particular solution y_2 we can express all possible solutions y_1, namely, $y_1 = h + y_2$, where now h is the general solution to the homogeneous equation. Of course, this is exactly how we approached the first order linear equation.

To make use of this observation we need a method to find a single solution y_2. This turns out to be somewhat more difficult than the first order case, but if $f(t)$ is of a certain simple form, we can find a solution using the **method of undetermined coefficients**, sometimes more whimsically called the **method of judicious guessing**.

Example 10.27: Second Order Linear Equation

Solve the differential equation $y'' - y' - 6y = 18t^2 + 5$.

Solution. The general solution of the homogeneous equation is $Ae^{3t} + Be^{-2t}$. We guess that a solution to the non-homogeneous equation might look like $f(t)$ itself, namely, a quadratic $y = at^2 + bt + c$. Substituting this guess into the differential equation we get

$$y'' - y' - 6y = 2a - (2at + b) - 6(at^2 + bt + c) = -6at^2 + (-2a - 6b)t + (2a - b - 6c).$$

We want this to equal $18t^2 + 5$, so we need

$$\begin{aligned} -6a &= 18 \\ -2a - 6b &= 0 \\ 2a - b - 6c &= 5 \end{aligned}$$

This is a system of three equations in three unknowns and is not hard to solve: $a = -3$, $b = 1$, $c = -2$. Thus the general solution to the differential equation is $Ae^{3t} + Be^{-2t} - 3t^2 + t - 2$. ♣

So the "judicious guess" is a function with the same form as $f(t)$ but with undetermined (or better, yet to be determined) coefficients. This works whenever $f(t)$ is a polynomial.

Example 10.28: Mass-Spring System with No Damping

Analyze the initial value problem $my'' + ky = -mg$, $y(0) = 2$, $y'(0) = 50$.

Solution. The left hand side represents a mass-spring system with no damping, i.e., $b = 0$. Unlike the homogeneous case, we now consider the force due to gravity, $-mg$, assuming the spring is vertical at the surface of the earth, so that $g = 980$. To be specific, let us take $m = 1$ and $k = 100$. The general solution to the homogeneous equation is $A\cos(10t) + B\sin(10t)$. For the solution to the non-homogeneous equation we guess simply a constant $y = a$, since $-mg = -980$ is a constant. Then $y'' + 100y = 100a$ so $a = -980/100 = -9.8$. The desired general solution is then $A\cos(10t) + B\sin(10t) - 9.8$. Substituting the initial conditions we get

$$\begin{aligned} 2 &= A - 9.8 \\ 50 &= 10B \end{aligned}$$

so $A = 11.8$ and $B = 5$ and the solution is $11.8\cos(10t) + 5\sin(10t) - 9.8$. ♣

More generally, this method can be used when a function similar to $f(t)$ has derivatives that are also similar to $f(t)$; in the examples so far, since $f(t)$ was a polynomial, so were its derivatives. The method will work if $f(t)$ has the form $p(t)e^{\alpha t}\cos(\beta t) + q(t)e^{\alpha t}\sin(\beta t)$, where $p(t)$ and $q(t)$ are polynomials; when $\alpha = \beta = 0$ this is simply $p(t)$, a polynomial. In the most general form it is not simple to describe the appropriate judicious guess; we content ourselves with some examples to illustrate the process.

Example 10.29: Solving a Second Order Linear Equation

Find the general solution to $y'' + 7y' + 10y = e^{3t}$.

Solution. The characteristic equation is $r^2 + 7r + 10 = (r + 5)(r + 2)$, so the solution to the homogeneous equation is $Ae^{-5t} + Be^{-2t}$. For a particular solution to the inhomogeneous equation we guess Ce^{3t}. Substituting we get

$$9Ce^{3t} + 21Ce^{3t} + 10Ce^{3t} = e^{3t}40C.$$

When $C = 1/40$ this is equal to $f(t) = e^{3t}$, so the solution is $Ae^{-5t} + Be^{-2t} + (1/40)e^{3t}$. ♣

Example 10.30: Solving a Second Order Linear Equation

Find the general solution to $y'' + 7y' + 10y = e^{-2t}$.

Solution. Following the last example we might guess Ce^{-2t}, but since this is a solution to the homogeneous equation it cannot work. Instead we guess Cte^{-2t}. Then

$$(-2Ce^{-2t} - 2Ce^{-2t} + 4Cte^{-2t}) + 7(Ce^{-2t} - 2Cte^{-2t}) + 10Cte^{-2t} = e^{-2t}(-3C).$$

Then $C = -1/3$ and the solution is $Ae^{-5t} + Be^{-2t} - (1/3)te^{-2t}$. ♣

In general, if $f(t) = e^{kt}$ and k is one of the roots of the characteristic equation, then we guess Cte^{kt} instead of Ce^{kt}. If k is the only root of the characteristic equation, then Cte^{kt} will also not work, so we must guess Ct^2e^{kt}.

Example 10.31: Solving a Second Order Linear Equation

Find the general solution to $y'' - 6y' + 9y = e^{3t}$.

Solution. The characteristic equation is $r^2 - 6r + 9 = (r-3)^2$, so the general solution to the homogeneous equation is $Ae^{3t} + Bte^{3t}$. Guessing Ct^2e^{3t} for the particular solution, we get

$$(9Ct^2e^{3t} + 6Cte^{3t} + 6Cte^{3t} + 2Ce^{3t}) - 6(3Ct^2e^{3t} + 2Cte^{3t}) + 9Ct^2e^{3t} = e^{3t}2C.$$

Thus, the solution is $Ae^{3t} + Bte^{3t} + (1/2)t^2e^{3t}$. ♣

It is common in various physical systems to encounter an $f(t)$ of the form $a\cos(\omega t) + b\sin(\omega t)$.

Example 10.32: Solving a Second Order Linear Equation

Find the general solution to $y'' + 6y' + 25y = \cos(4t)$.

Solution. The roots of the characteristic equation are $-3 \pm 4i$, so the solution to the homogeneous equation is $e^{-3t}(A\cos(4t) + B\sin(4t))$. For a particular solution, we guess $C\cos(4t) + D\sin(4t)$. Substituting as usual:

$$(-16C\cos(4t) + -16D\sin(4t)) + 6(-4C\sin(4t) + 4D\cos(4t)) + 25(C\cos(4t) + D\sin(4t))$$

$$= (24D + 9C)\cos(4t) + (-24C + 9D)\sin(4t).$$

To make this equal to $\cos(4t)$ we need

$$\begin{aligned} 24D + 9C &= 1 \\ 9D - 24C &= 0 \end{aligned}$$

which gives $C = 1/73$ and $D = 8/219$. The full solution is then $e^{-3t}(A\cos(4t) + B\sin(4t)) + (1/73)\cos(4t) + (8/219)\sin(4t)$.

The function $e^{-3t}(A\cos(4t)+B\sin(4t))$ is a damped oscillation as in example 10.28, while $(1/73)\cos(4t)+(8/219)\sin(4t)$ is a simple undamped oscillation. As t increases, the sum $e^{-3t}(A\cos(4t)+B\sin(4t))$ approaches zero, so the solution

$$e^{-3t}(A\cos(4t) + B\sin(4t)) + (1/73)\cos(4t) + (8/219)\sin(4t)$$

becomes more and more like the simple oscillation $(1/73)\cos(4t) + (8/219)\sin(4t)$—notice that the initial conditions don't matter to this long term behavior. The damped portion is called the **transient** [part of the] **solution**, and the simple oscillation is called the **steady state** [part of the] **solution**. A physical example is a mass-spring system. If the only force on the mass is due to the spring, then the behavior of the system is a damped oscillation. If in addition an external force is applied to the mass, and if the force varies according to a function of the form $a\cos(\omega t) + b\sin(\omega t)$, then the long term behavior will be a simple oscillation determined by the steady state portion of the general solution; the initial position of the mass will not matter. ♣

As with the exponential form, such a simple guess may not work.

Example 10.33: Solving a Second Order Linear Equation

Find the general solution to $y'' + 16y = -\sin(4t)$.

Solution. The roots of the characteristic equation are $\pm 4i$, so the solution to the homogeneous equation is $A\cos(4t) + B\sin(4t)$. Since both $\cos(4t)$ and $\sin(4t)$ are solutions to the homogeneous equation, $C\cos(4t) + D\sin(4t)$ is also, so it cannot be a solution to the non-homogeneous equation. Instead, we guess $Ct\cos(4t) + Dt\sin(4t)$. Then substituting:

$$(-16Ct\cos(4t) - 16D\sin(4t) + 8D\cos(4t) - 8C\sin(4t))) + 16(Ct\cos(4t) + Dt\sin(4t))$$

$$= 8D\cos(4t) - 8C\sin(4t).$$

Thus $C = 1/8$, $D = 0$, and the solution is $C\cos(4t) + D\sin(4t) + (1/8)t\cos(4t)$. ♣

In general, if $f(t) = a\cos(\omega t) + b\sin(\omega t)$, and $\pm\omega i$ are the roots of the characteristic equation, then instead of $C\cos(\omega t) + D\sin(\omega t)$ we guess $Ct\cos(\omega t) + Dt\sin(\omega t)$.

Exercises for 10.6

Find the general solution to the differential equation.

10.6.1 $y'' - 10y' + 25y = \cos t$

10.6.2 $y'' + 2\sqrt{2}y' + 2y = 10$

10.6.3 $y'' + 16y = 8t^2 + 3t - 4$

10.6.4 $y'' + 2y = \cos(5t) + \sin(5t)$

10.6.5 $y'' - 2y' + 2y = e^{2t}$

10.6.6 $y'' - 6y + 13 = 1 + 2t + e^{-t}$

10.6.7 $y'' + y' - 6y = e^{-3t}$

10.6.8 $y'' - 4y' + 3y = e^{3t}$

10.6.9 $y'' + 16y = \cos(4t)$

10.6.10 $y'' + 9y = 3\sin(3t)$

10.6.11 $y'' + 12y' + 36y = 6e^{-6t}$

10.6.12 $y'' - 8y' + 16y = -2e^{4t}$

10.6.13 $y'' + 6y' + 5y = 4$

10.6.14 $y'' - y' - 12y = t$

10.6.15 $y'' + 5y = 8\sin(2t)$

10.6.16 $y'' - 4y = 4e^{2t}$

Solve the initial value problem.

10.6.17 $y'' - y = 3t + 5,\ y(0) = 0,\ y'(0) = 0$

10.6.18 $y'' + 9y = 4t,\ y(0) = 0,\ y'(0) = 0$

10.6.19 $y'' + 12y' + 37y = 10e^{-4t},\ y(0) = 4,\ y'(0) = 0$

10.6.20 $y'' + 6y' + 18y = \cos t - \sin t,\ y(0) = 0,\ y'(0) = 2$

10.6.21 *Find the solution for the mass-spring equation* $y'' + 4y' + 29y = 689\cos(2t)$.

10.6.22 *Find the solution for the mass-spring equation* $3y'' + 12y' + 24y = 2\sin t$.

10.6.23 *Consider the differential equation* $my'' + by' + ky = \cos(\omega t)$, *with* m, b, *and* k *all positive and* $b^2 < 2mk$; *this equation is a model for a damped mass-spring system with external driving force* $\cos(\omega t)$. *Show that the steady state part of the solution has amplitude*

$$\frac{1}{\sqrt{(k - m\omega^2)^2 + \omega^2 b^2}}.$$

Show that this amplitude is largest when $\omega = \dfrac{\sqrt{4mk - 2b^2}}{2m}$. *This is the* **resonant frequency** *of the system.*

10.7 Second Order Linear Equations - Variation of Parameters

The method of the last section works only when the function $f(t)$ in $ay'' + by' + cy = f(t)$ has a particularly nice form, namely, when the derivatives of f look much like f itself. In other cases we can try variation of parameters as we did in the first order case.

Since as before $a \neq 0$, we can always divide by a to make the coefficient of y'' equal to 1. Thus, to simplify the discussion, we assume $a = 1$. We know that the differential equation $y'' + by' + cy = 0$ has a general solution $y = Ay_1 + By_2$. As before, we guess a particular solution to $y'' + by' + cy = f(t)$; this time we use the guess $y = u(t)y_1 + v(t)y_2$. Compute the derivatives:

$$y' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

$$y'' = u''y_1 + u'y_1' + u'y_1' + uy_1'' + v''y_2 + v'y_2' + v'y_2' + vy_2''.$$

Now substituting:

$$\begin{aligned} y'' + by' + cy &= u''y_1 + u'y_1' + u'y_1' + uy_1'' + v''y_2 + v'y_2' + v'y_2' + vy_2'' \\ &\quad + bu'y_1 + buy_1' + bv'y_2 + bvy_2' + cuy_1 + cvy_2 \\ &= (uy_1'' + buy_1' + cuy_1) + (vy_2'' + bvy_2' + cvy_2) \\ &\quad + b(u'y_1 + v'y_2) + (u''y_1 + u'y_1' + v''y_2 + v'y_2') + (u'y_1' + v'y_2') \\ &= 0 + 0 + b(u'y_1 + v'y_2) + (u''y_1 + u'y_1' + v''y_2 + v'y_2') + (u'y_1' + v'y_2'). \end{aligned}$$

The first two terms in parentheses are zero because y_1 and y_2 are solutions to the associated homogeneous equation. Now we engage in some wishful thinking. If $u'y_1 + v'y_2 = 0$, then we also have $u''y_1 + u'y_1' + v''y_2 + v'y_2' = 0$ by taking derivatives of both sides. This reduces the entire expression to $u'y_1' + v'y_2' = 0$. We want this to be $f(t)$, that is, we need $u'y_1' + v'y_2' = f(t)$. So we would very much like these equations to be true:

$$\begin{aligned} u'y_1 + v'y_2 &= 0 \\ u'y_1' + v'y_2' &= f(t). \end{aligned}$$

This is a system of two equations in the two unknowns u' and v', so we can solve as usual to get $u' = g(t)$ and $v' = h(t)$. Then we can find u and v by computing antiderivatives. This is of course the sticking point in the whole plan, since the antiderivatives may be impossible to find. Nevertheless, this sometimes works out and is worth a try.

Example 10.34: Variation of Parameters

Consider the equation $y'' - 5y' + 6y = \sin t$. Solve using variation of parameters.

Solution. The solution to the homogeneous equation is $Ae^{2t} + Be^{3t}$, so the simultaneous equations to be solved are

$$\begin{aligned} u'e^{2t} + v'e^{3t} &= 0 \\ 2u'e^{2t} + 3v'e^{3t} &= \sin t. \end{aligned}$$

If we multiply the first equation by 2 and subtract it from the second equation we get

$$\begin{aligned} v'e^{3t} &= \sin t \\ v' &= e^{-3t}\sin t \\ v &= -\frac{1}{10}(3\sin t + \cos t)e^{-3t}, \end{aligned}$$

using integration by parts. Then from the first equation:

$$\begin{aligned} u' &= -e^{-2t}v'e^{3t} = -e^{-2t}e^{-3t}\sin(t)e^{3t} = -e^{-2t}\sin t \\ u &= \frac{1}{5}(2\sin t + \cos t)e^{-2t}. \end{aligned}$$

Now the particular solution we seek is

$$ue^{2t} + ve^{3t} = \frac{1}{5}(2\sin t + \cos t)e^{-2t}e^{2t} - \frac{1}{10}(3\sin t + \cos t)e^{-3t}e^{3t}$$

$$\begin{aligned} &= \frac{1}{5}(2\sin t + \cos t) - \frac{1}{10}(3\sin t + \cos t) \\ &= \frac{1}{10}(\sin t + \cos t), \end{aligned}$$

and the solution to the differential equation is $Ae^{2t} + Be^{3t} + (\sin t + \cos t)/10$. For comparison (and practice) you might want to solve this using the method of undetermined coefficients—both techniques should yield the same result. ♣

Example 10.35: Variation of Parameters

The differential equation $y'' - 5y' + 6y = e^t \sin t$ can be solved using the method of undetermined coefficients, though we have not seen any examples of such a solution. Again, we will solve it by variation of parameters.

Solution. The equations to be solved are

$$\begin{aligned} u'e^{2t} + v'e^{3t} &= 0 \\ 2u'e^{2t} + 3v'e^{3t} &= e^t \sin t. \end{aligned}$$

If we multiply the first equation by 2 and subtract it from the second equation we get

$$\begin{aligned} v'e^{3t} &= e^t \sin t \\ v' &= e^{-3t}e^t \sin t = e^{-2t}\sin t \\ v &= -\frac{1}{5}(2\sin t + \cos t)e^{-2t}. \end{aligned}$$

Then substituting we get

$$\begin{aligned} u' &= -e^{-2t}v'e^{3t} = -e^{-2t}e^{-2t}\sin(t)e^{3t} = -e^{-t}\sin t \\ u &= \frac{1}{2}(\sin t + \cos t)e^{-t}. \end{aligned}$$

The particular solution is

$$\begin{aligned} ue^{2t} + ve^{3t} &= \frac{1}{2}(\sin t + \cos t)e^{-t}e^{2t} - \frac{1}{5}(2\sin t + \cos t)e^{-2t}e^{3t} \\ &= \frac{1}{2}(\sin t + \cos t)e^{t} - \frac{1}{5}(2\sin t + \cos t)e^{t} \\ &= \frac{1}{10}(\sin t + 3\cos t)e^{t}, \end{aligned}$$

and the solution to the differential equation is $Ae^{2t} + Be^{3t} + e^t(\sin t + 3\cos t)/10$. ♣

Example 10.36: Solving a DE

The differential equation $y'' - 2y' + y = e^t/t^2$ is not of the form amenable to the method of undetermined coefficients. Solve it using variation of parameters.

Solution. The solution to the homogeneous equation is $Ae^t + Bte^t$ and so the simultaneous equations are

$$\begin{aligned} u'e^t + v'te^t &= 0 \\ u'e^t + v'te^t + v'e^t &= \frac{e^t}{t^2}. \end{aligned}$$

Subtracting the equations gives

$$\begin{aligned} v'e^t &= \frac{e^t}{t^2} \\ v' &= \frac{1}{t^2} \\ v &= -\frac{1}{t}. \end{aligned}$$

Then substituting we get

$$\begin{aligned} u'e^t &= -v'te^t = -\frac{1}{t^2}te^t \\ u' &= -\frac{1}{t} \\ u &= -\ln t. \end{aligned}$$

The solution is $Ae^t + Bte^t - e^t \ln t - e^t$. ♣

Exercises for 10.7

Find the general solution to the differential equation using variation of parameters.

10.7.1 $y'' + y = \tan x$

10.7.2 $y'' + y = e^{2t}$

10.7.3 $y'' + 4y = \sec x$

10.7.4 $y'' + 4y = \tan x$

10.7.5 $y'' + y' - 6y = t^2 e^{2t}$

10.7.6 $y'' - 2y' + 2y = e^t \tan(t)$

10.7.7 $y'' - 2y' + 2y = \sin(t)\cos(t)$ *(This is rather messy when done by variation of parameters; compare to undetermined coefficients.)*

Selected Exercise Answers

1.1.1 (a) $\dfrac{x}{y}$

(b) $\sqrt{x}y$

(c) $\dfrac{2}{\sqrt[3]{x}}$

1.1.2 $a = 2$, $b = -\frac{5}{3}$, $c = \frac{3}{2}$.

1.1.3 $x = -4$ and $x = 6$.

1.1.5 (a) $(5/3, \infty)$

(b) $[1/7, 2/7]$

(c) $(-\infty, -3) \cup (-2, 1]$

(d) $(-\infty, \infty)$

(e) No solution

(f) $(-\infty, 1) \cup (1, \infty)$

(g) $(-2, 0) \cup (2, \infty)$

(h) $[4, \infty) \cup \{0\}$

(i) $(0, \frac{1}{2})$

(j) $(-2, -1] \cup (1, 4]$

1.1.6 $x = -\frac{1}{2}$ and $x = -\frac{1}{6}$.

1.1.7 (a) $(-\infty, -2] \cup [2, \infty)$

(b) $[2, 4]$

(c) $(-\infty, -9/2] \cup [-1/2, \infty)$

(d) $(4, \infty)$

(e) $(-\infty, \infty)$

(f) $(-9, -6) \cup (4, 7)$

1.2.1 (a) $(2/3)x + (1/3)$

(b) $y = -2x$

(c) $y = (-2/3)x + (1/3)$

(d) $y = -x/3 + 17/3$

(e) $y = -1/2x + 5/2$

1.2.2 (a) $y = 2x + 2$, 2, -1

(b) $y = -x + 6$, 6, 6

(c) $y = x/2 + 1/2$, $1/2$, -1

(d) $y = 3/2$, y-intercept: $3/2$, no x-intercept

(e) $y = (-2/3)x - 2$, -2, -3

1.2.3 Yes, the lines are parallel as they have the same slope of $-1/2$

1.2.4 $y = 0$, $y = -2x + 2$, $y = 2x + 2$

1.2.5 $y = (9/5)x + 32$, $(-40, -40)$

1.2.6 $y = 0.15x + 10$

1.2.7 $0.03x + 1.2$

1.2.8 (a) $P = -0.0001x + 2$
(b) $x = -10000P + 20000$

1.2.9 $(2/25)x - (16/5)$

1.2.10 (a) 2

(b) $\sqrt{2}$

(c) $\sqrt{2}$

1.2.12 (a) $x^2 + y^2 = 9$

(b) $(x - 5)^2 + (y - 6)^2 = 9$

(c) $(x + 5)^2 + (y + 6)^2 = 9$

1.2.14 (a) circle

(b) ellipse

(c) horizontal parabola

1.2.15 $(x + 2/7)^2 + (y - 41/7)^2 = 1300/49$

1.3.1 $2n\pi - \pi/2$, any integer n

1.3.2 $n\pi \pm \pi/6$, any integer n

1.3.4 $-\frac{5}{4}$

1.3.5 $\sin\theta = -x/\sqrt{x^2+1}$, $\cos\theta = -1/\sqrt{x^2+1}$.

1.3.6 $-\frac{2\pi}{7}$ is the unique answer.

1.3.7 $(\sqrt{2}+\sqrt{6})/4$

1.3.8 $-(1+\sqrt{3})/(1-\sqrt{3}) = 2+\sqrt{3}$

1.3.11 $t = \pi/2$

1.4.1 (a) $\dfrac{\sqrt{2}}{2}$

(b) $3(\sqrt{x+h+1}+\sqrt{x+1})$

1.4.2 (a) $-13/5$

(b) $-1/2, 3$

(c) $(1 \pm \sqrt{13})/2$

(d) No real solutions

(e) $\sqrt{3}$

1.4.3 Counter-examples may vary.

(a) $x = 3$

(b) $x = h = 1$

(c) $x = y = 1$

1.4.4 $x + 3y - 13 = 0$, or equivalents such as $y = -\frac{1}{3}x + \frac{13}{3}$

1.4.5 $(-\infty, 0] \cup (\frac{1}{3}, 3]$

1.4.6 It is impossible for both $x-2$ and $1-x$ to be non-negative for the same real number x.

1.4.7 $6x + 3h$

1.4.8 $-1/\left[(2x+2h-1)(2x-1)\right]$

1.4.9 -3

1.4.10 $\pi/6$, $5\pi/6$

1.4.11 $2\pi/5$

1.4.12 It is equal to 2 for all x larger than 4.

1.4.13 $(x+2)^2 + (y-3)^2 = 25$.

1.4.14 Centre is $(-3, 2)$ and radius is 2.

1.4.15 y could be any real number greater than or equal to 6.

1.4.16 $x^6y^4/(36z^8)$

1.4.17 $x = (y-2)/(3+4y)$

1.4.18 $Q(x) = x + 1$, $R = -7$

2.1.1 (a) $\{x \mid x \in \mathbb{R}\}$, i.e., all x

(b) $\{x \mid x \geq 3/2\}$

(c) $\{x \mid x \neq -1\}$

(d) $\{x \mid x \neq 1 \text{ and } x \neq -1\}$

(e) $\{x \mid x < 0\}$

(f) $\{x \mid x \in \mathbb{R}\}$, i.e., all x

(g) $\{x \mid h - r \leq x \leq h + r\}$

(h) $\{x \mid x \geq 0\}$

(i) $\{x \mid -1 \leq x \leq 1\}$

(j) $\{x \mid x \geq 1\}$

(k) $\{x \mid -1/3 < x < 1/3\}$

(l) $\{x \mid x \geq 0 \text{ and } x \neq 1\}$

(m) $\{x \mid x \geq 0 \text{ and } x \neq 1\}$

2.1.2 $A = x(500 - 2x)$, $\{x \mid 0 \leq x \leq 250\}$

2.1.3 $V = r(50 - \pi r^2)$, $\{r \mid 0 < r \leq \sqrt{50/\pi}\}$

2.1.4 $A = 2\pi r^2 + 2000/r$, $\{r \mid 0 < r < \infty\}$

2.2.3 $\{x \mid x \geq 3\}$, $\{x \mid x \geq 0\}$

2.3.1 $y = 2^x$

2.3.2 $y = 7$

2.3.3 $y = 2$

2.3.4 $x \neq 0$

2.6.1 (a) $\pi/3$

(b) $3\pi/4$

2.6.2 (a) $\pi/4$

(b) $-\pi/3$

(c) $1/3$

(d) $-3/4$

2.6.3 $\sqrt{1-x^2}/x$ with domain $[-1, 0) \cup (0, 1]$.

2.7.1 (a)

$$\begin{aligned} \coth^2 x - \operatorname{csch}^2 x &= \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - \left(\frac{2}{e^x - e^{-x}}\right)^2 \\ &= \frac{(e^{2x} + 2 + e^{-2x}) - (4)}{e^{2x} - 2 + e^{-2x}} \\ &= \frac{e^{2x} - 2 + e^{-2x}}{e^{2x} - 2 + e^{-2x}} \\ &= 1 \end{aligned}$$

(b)

$$\begin{aligned} \cosh^2 x + \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} + \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{2e^{2x} + 2e^{-2x}}{4} \\ &= \frac{e^{2x} + e^{-2x}}{2} \\ &= \cosh 2x. \end{aligned}$$

(c)

$$\begin{aligned} \cosh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} \\ &= \frac{1}{2}\frac{(e^{2x} + e^{-2x}) + 2}{2} \\ &= \frac{1}{2}\left(\frac{e^{2x} + e^{-2x}}{2} + 1\right) \\ &= \frac{\cosh 2x + 1}{2}. \end{aligned}$$

(d)
$$\begin{aligned}\sinh^2 x &= \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{1}{2}\frac{\left(e^{2x} + e^{-2x}\right) - 2}{2} \\ &= \frac{1}{2}\left(\frac{e^{2x} + e^{-2x}}{2} - 1\right) \\ &= \frac{\cosh 2x - 1}{2}.\end{aligned}$$

2.8.1 (d)

2.8.2 $3/\left[5(x+3)\right]$

2.8.3 $6 + h$

2.8.4 (a) $[2,3) \cup (3,\infty)$

(b) $(-\infty,-3) \cup (-3,3) \cup (3,\infty)$

2.8.5 $\{x : x \neq 0\}$

2.8.6 $g(x) = (5x + 26)/3$

2.8.7 (c)

2.8.8 $f^{-1}(x) = f(x) = \ln\left(\dfrac{e^x}{e^x - 1}\right)$ and its domain is $(0,\infty)$.

2.8.9 (a) $2 - \ln 3$

(b) 1,3

(c) $(e^2 - 1)^2$

(d) 3

2.8.10 $-\pi$

2.8.11 $2\pi/5$

2.8.12 1

3.2.1 (a) Answers will vary.

(b) An indeterminate form.

(c) F

(d) The function may approach different values from the left and right, the function may grow without bound, or the function might oscillate.

(e) Answers will vary.

(f) F

(g) F

(h) T

3.2.2 (a) 8

(b) 6

(c) dne

(d) -2

(e) -1

(f) 8

(g) 7

(h) 6

(i) 3

(j) $-3/2$

(k) 6

(l) 2

3.2.3 (a) $\lim\limits_{x\to 1} x^2 + 3x - 5$

(b) -5

(c) Limit does not exist

(d) 2

(e) 1.5

(f) Limit does not exist.

(g) Limit does not exist.

(h) 7

(i) 1

(j) Limit does not exist.

3.2.4 (a)

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-7
-0.01	-7
0.01	-7
0.1	-7

The limit seems to be exactly 7.

(b)

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	9
-0.01	9
0.01	9
0.1	9

The limit seems to be exactly 9.

(c)

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	4.9
-0.01	4.99
0.01	5.01
0.1	5.1

The limit is approx. 5.

(d)

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.114943
-0.01	-0.111483
0.01	-0.110742
0.1	-0.107527

The limit is approx. -0.11.

(e)

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	29.4
-0.01	29.04
0.01	28.96
0.1	28.6

The limit is approx. 29.

(f)

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	0.202027
-0.01	0.2002
0.01	0.1998
0.1	0.198026

The limit is approx. 0.2.

(g)

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.998334
-0.01	-0.999983
0.01	-0.999983
0.1	-0.998334

The limit is approx. -1.

(h)

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.0499583
-0.01	-0.00499996
0.01	0.00499996
0.1	0.0499583

The limit is approx. 0.005.

3.2.5 (a) (a) 2

(b) 2

(c) 2

(d) 1

(e) As f is not defined for $x < 0$, this limit is not defined.

(f) 1

(b) (a) 1

(b) 2

- (c) Does not exist.
- (d) 2
- (e) 0
- (f) As f is not defined for $x < 0$, this limit is not defined.

(c)
- (a) Does not exist.
- (b) Does not exist.
- (c) Does not exist.
- (d) Not defined.
- (e) 0
- (f) 0

(d)
- (a) 2
- (b) 0
- (c) Does not exist.
- (d) 1

(e)
- (a) 2
- (b) 2
- (c) 2
- (d) 2

(f)
- (a) 4
- (b) -4
- (c) Does not exist.
- (d) 0

(g)
- (a) 2
- (b) 2
- (c) 2
- (d) 0
- (e) 2
- (f) 2
- (g) 2
- (h) Not defined

(h)
- (a) $a - 1$
- (b) a
- (c) Does not exist.

(d) a

3.2.6 (a)

(b)

(c)

(d)

3.3.1 (a) ϵ should be given first, and the restriction $|x - a| < \delta$ implies $|f(x) - K| < \epsilon$, not the other way around.

(b) The y–tolerance.

(c) T

(d) T

3.3.2 (a) Let $\epsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x - 2| < \delta$, $|f(x) - 5| < \epsilon$. However, since $f(x) = 5$, a constant function, the latter inequality is simply $|5 - 5| < \epsilon$, which is always true. Thus we can choose any δ we like; we arbitrarily choose $\delta = \epsilon$.

(b) Let $\epsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x - 5| < \delta$, $|f(x) - (-2)| < \epsilon$. Consider $|f(x) - (-2)| < \epsilon$:

$$
\begin{gathered}
|f(x) + 2| < \epsilon \\
|(3 - x) + 2| < \epsilon \\
|5 - x| < \epsilon \\
-\epsilon < 5 - x < \epsilon \\
-\epsilon < x - 5 < \epsilon.
\end{gathered}
$$

This implies we can let $\delta = \epsilon$. Then:

$$
\begin{gathered}
|x - 5| < \delta \\
-\delta < x - 5 < \delta \\
-\epsilon < x - 5 < \epsilon \\
-\epsilon < (x - 3) - 2 < \epsilon \\
-\epsilon < (-x + 3) - (-2) < \epsilon \\
|3 - x - (-2)| < \epsilon,
\end{gathered}
$$

which is what we wanted to prove.

(c) Let $\epsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-3| < \delta$, $|f(x)-6| < \epsilon$. Consider $|f(x)-6| < \epsilon$, keeping in mind we want to make a statement about $|x-3|$:

$$\begin{gathered}
|f(x)-6| < \epsilon \\
|x^2-3-6| < \epsilon \\
|x^2-9| < \epsilon \\
|x-3|\cdot|x+3| < \epsilon \\
|x-3| < \epsilon/|x+3|
\end{gathered}$$

Since x is near 3, we can safely assume that, for instance, $2 < x < 4$. Thus

$$\begin{gathered}
2+3 < x+3 < 4+3 \\
5 < x+3 < 7 \\
\frac{1}{7} < \frac{1}{x+3} < \frac{1}{5} \\
\frac{\epsilon}{7} < \frac{\epsilon}{x+3} < \frac{\epsilon}{5}
\end{gathered}$$

Let $\delta = \frac{\epsilon}{7}$. Then:

$$\begin{gathered}
|x-3| < \delta \\
|x-3| < \frac{\epsilon}{7} \\
|x-3| < \frac{\epsilon}{x+3} \\
|x-3|\cdot|x+3| < \frac{\epsilon}{x+3}\cdot|x+3|
\end{gathered}$$

Assuming x is near 3, $x+3$ is positive and we can drop the absolute value signs on the right.

$$\begin{gathered}
|x-3|\cdot|x+3| < \frac{\epsilon}{x+3}\cdot(x+3) \\
|x^2-9| < \epsilon \\
|(x^2-3)-6| < \epsilon,
\end{gathered}$$

which is what we wanted to prove.

(d) Let $\epsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-2| < \delta$, $|f(x)-7| < \epsilon$. Consider $|f(x)-7| < \epsilon$, keeping in mind we want to make a statement about $|x-2|$:

$$\begin{gathered}
|f(x)-7| < \epsilon \\
|x^3-1-7| < \epsilon \\
|x^3-8| < \epsilon
\end{gathered}$$

$$
\begin{aligned}
|x-2| \cdot |x^2+2x+4| &< \epsilon \\
|x-3| &< \epsilon/|x^2+2x+4|
\end{aligned}
$$

Since x is near 2, we can safely assume that, for instance, $1 < x < 3$. Thus

$$
\begin{aligned}
1^2+2\cdot 1+4 < x^2 &+ 2x+4 < 3^2+2\cdot 3+4 \\
7 < x^2 &+ 2x+4 < 19 \\
\frac{1}{19} < &\frac{1}{x^2+2x+4} < \frac{1}{7} \\
\frac{\epsilon}{19} < &\frac{\epsilon}{x^2+2x+4} < \frac{\epsilon}{7}
\end{aligned}
$$

Let $\delta = \frac{\epsilon}{19}$. Then:

$$
\begin{aligned}
|x-2| &< \delta \\
|x-2| &< \frac{\epsilon}{19} \\
|x-2| &< \frac{\epsilon}{x^2+2x+4} \\
|x-2| \cdot |x^2+2x+4| &< \frac{\epsilon}{x^2+2x+4} \cdot |x^2+2x+4|
\end{aligned}
$$

Assuming x is near 2, x^2+2x+4 is positive and we can drop the absolute value signs on the right.

$$
\begin{aligned}
|x-2| \cdot |x^2+2x+4| &< \frac{\epsilon}{x^2+2x+4} \cdot (x^2+2x+4) \\
|x^3-8| &< \epsilon \\
|(x^3-1)-7| &< \epsilon,
\end{aligned}
$$

which is what we wanted to prove.

(e) Let $\epsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-0| < \delta$, $|f(x)-0| < \epsilon$.

Consider $|f(x)-0| < \epsilon$, keeping in mind we want to make a statement about $|x-0|$ (i.e., $|x|$):

$$
\begin{aligned}
|f(x)-0| &< \epsilon \\
|e^{2x}-1| &< \epsilon \\
-\epsilon < e^{2x} &- 1 < \epsilon \\
1-\epsilon < e^{2x} &< 1+\epsilon \\
\ln(1-\epsilon) < 2x &< \ln(1+\epsilon) \\
\frac{\ln(1-\epsilon)}{2} < x &< \frac{\ln(1+\epsilon)}{2}
\end{aligned}
$$

Let $\delta = \min\left\{\left|\frac{\ln(1-\epsilon)}{2}\right|, \frac{\ln(1+\epsilon)}{2}\right\} = \frac{\ln(1+\epsilon)}{2}$.

Thus:

$$\begin{gathered}|x| < \delta \\ |x| < \frac{\ln(1+\epsilon)}{2} < \left|\frac{\ln(1-\epsilon)}{2}\right| \\ \frac{\ln(1-\epsilon)}{2} < x < \frac{\ln(1+\epsilon)}{2} \\ \ln(1-\epsilon) < 2x < \ln(1+\epsilon) \\ 1-\epsilon < e^{2x} < 1+\epsilon \\ -\epsilon < e^{2x} - 1 < \epsilon \\ |e^{2x} - 1 - (0)| < \epsilon,\end{gathered}$$

which is what we wanted to prove.

(f) Let $\epsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x - 0| < \delta$, $|f(x) - 0| < \epsilon$. In simpler terms, we want to show that when $|x| < \delta$, $|\sin x| < \epsilon$.

Set $\delta = \epsilon$. We start with assuming that $|x| < \delta$. Using the hint, we have that $|\sin x| < |x| < \delta = \epsilon$. Hence if $|x| < \delta$, we know immediately that $|\sin x| < \epsilon$.

(g) Let $\epsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x - 4| < \delta$, $|f(x) - 15| < \epsilon$.

Consider $|f(x) - 15| < \epsilon$, keeping in mind we want to make a statement about $|x - 4|$:

$$\begin{gathered}|f(x) - 15| < \epsilon \\ |x^2 + x - 5 - 15| < \epsilon \\ |x^2 + x - 20| < \epsilon \\ |x - 4| \cdot |x + 5| < \epsilon \\ |x - 4| < \epsilon/|x + 5|\end{gathered}$$

Since x is near 4, we can safely assume that, for instance, $3 < x < 5$. Thus

$$\begin{gathered}3 + 5 < x + 5 < 5 + 5 \\ 8 < x + 5 < 10 \\ \frac{1}{10} < \frac{1}{x+5} < \frac{1}{8} \\ \frac{\epsilon}{10} < \frac{\epsilon}{x+5} < \frac{\epsilon}{8}\end{gathered}$$

Let $\delta = \frac{\epsilon}{10}$. Then:

$$\begin{gathered}|x - 4| < \delta \\ |x - 4| < \frac{\epsilon}{10}\end{gathered}$$

$$\begin{aligned}|x-4| &< \frac{\epsilon}{x+5}\\ |x-4|\cdot|x+5| &< \frac{\epsilon}{x+5}\cdot|x+5|\end{aligned}$$

Assuming x is near 4, $x+5$ is positive and we can drop the absolute value signs on the right.

$$\begin{aligned}|x-4|\cdot|x+5| &< \frac{\epsilon}{x+5}\cdot(x+5)\\ |x^2+x-20| &< \epsilon\\ |(x^2+x-5)-15| &< \epsilon,\end{aligned}$$

which is what we wanted to prove.

3.4.1 Answers will vary.

3.4.2 Answers will vary.

3.4.3 Answers will vary.

3.4.4 (a) 9

(b) 6

(c) 0

(d) Limit does not exist.

(e) 3

(f) Not possible to know.

(g) 3

(h) -45

3.4.5 (a) 1

(b) -1

(c) 0

(d) π

3.4.6 (a) 7

(b) 5

(c) 0

(d) $\frac{3\pi+1}{1-\pi}$

(e) $\frac{\pi^2+3\pi+5}{5\pi^2-2\pi-3} \approx 0.6064$

(f) $-0.000000015 \approx 0$

(g) 1/2

(h) Limit does not exist

(i) 64

(j) undefined

(k) 1/6

(l) 0

(m) 3

(n) 172

(o) 0

(p) 2

(q) does not exist

(r) $\sqrt{2}$

(s) $3a^2$

(t) 512

3.4.7 $L = 0$ and $M = 1$. No.

3.5.1 (a) 5

(b) 7/2

(c) 3/4

(d) 1

(e) $-\sqrt{2}/2$

3.5.2 7

3.5.3 2

3.5.4 (a) 0

(b) 0

(c) 0

(d) 9

3.5.6 3

3.5.7 (a) 3

(b) 5/8

(c) 1

(d) $\pi/180$

3.6.1 (a) Consider the function $h(x) = g(x) - f(x)$, and use the Bisection Method to find a root of h.

(b) A root of a function f is a value c such that $f(c) = 0$.

(c) Answers will vary.

(d) Answers will vary.

(e) T

(f) F

(g) F

(h) T

(i) T

(j) F

3.6.2 (a) No; $\lim\limits_{x\to 1} f(x) = 2$, while $f(1) = 1$.

(b) No; $\lim\limits_{x\to 1} f(x)$ does not exist.

(c) No; $f(1)$ does not exist.

(d) Yes

(e) Yes

(f) Yes

(g) (a) No; $\lim\limits_{x\to -2} f(x) \neq f(-2)$

(b) Yes

(c) No; $f(2)$ is not defined.

(h) No.

3.6.3 (a) (a) Yes

(b) Yes

(b) (a) Yes

(b) No; the left and right hand limits at 1 are not equal.

(c) (a) Yes

(b) Yes

(d) (a) Yes

(b) No. $\lim_{x\to 8} f(x) = 16/5 \neq f(8) = 5$.

3.6.4 (a) $(-\infty, \infty)$

(b) $(-\infty, -2] \cup [2, \infty)$

(c) $[-1, 1]$

(d) $(-\infty, -\sqrt{6}] \cup [\sqrt{6}, \infty)$

(e) $(-1, 1)$

(f) $(-\infty, \infty)$

(g) $(-\infty, \infty)$

(h) $(0, \infty)$

(i) $(-\infty, \infty)$

(j) $(-\infty, 0]$

(k) $(-\infty, \infty)$

3.6.8 Yes, by the Intermediate Value Theorem.

3.6.9 Yes, by the Intermediate Value Theorem. In fact, we can be more specific and state such a value c exists in $(0, 2)$, not just in $(-3, 7)$.

3.6.10 We cannot say; the Intermediate Value Theorem only applies to function values between -10 and 10; as 11 is outside this range, we do not know.

3.6.11 We cannot say; the Intermediate Value Theorem only applies to continuous functions. As we do know know if h is continuous, we cannot say.

3.6.12 (a) Approximate root is $x = 1.23$. The intervals used are: $[1, 1.5]$ $[1, 1.25]$ $[1.125, 1.25]$ $[1.1875, 1.25]$ $[1.21875, 1.25]$ $[1.234375, 1.25]$ $[1.234375, 1.2421875]$ $[1.234375, 1.2382813]$

(b) Approximate root is $x = 0.52$. The intervals used are: $[0.5, 0.55]$ $[0.5, 0.525]$ $[0.5125, 0.525]$ $[0.51875, 0.525]$ $[0.521875, 0.525]$

(c) Approximate root is $x = 0.69$. The intervals used are: $[0.65, 0.7]$ $[0.675, 0.7]$ $[0.6875, 0.7]$ $[0.6875, 0.69375]$ $[0.690625, 0.69375]$

(d) Approximate root is $x = 0.78$. The intervals used are: $[0.7, 0.8]$ $[0.75, 0.8]$ $[0.775, 0.8]$ $[0.775, 0.7875]$ $[0.78125, 0.7875]$

(A few more steps show that 0.79 is better as the root is $\pi/4 \approx 0.78539$.)

3.6.13 (a) 20

(b) 25

(c) Limit does not exist

(d) 25

3.6.14

x	$f(x)$
-0.81	-2.34129
-0.801	-2.33413
-0.79	-2.32542
-0.799	-2.33254

The top two lines give an approximation of the limit from the left: -2.33. The bottom two lines give an approximation from the right: -2.33 as well.

3.7.1 (a) 1

(b) 1

(c) $-\infty$

(d) $1/3$

(e) 0

(f) ∞

(g) ∞

(h) $2/7$

(i) 2

(j) $-\infty$

(k) ∞

(l) 0

(m) $1/2$

(n) 5

(o) $2\sqrt{2}$

(p) $3/2$

(q) ∞

(r) does not exist

3.7.2 $y = 1$ and $y = -1$

3.7.3 $x = 0$ and $x = 2$.

3.7.5 $y = x + 4$

3.7.6 (a) $-\infty$

(b) $\pi/2$

(c) 0

(d) ∞

(e) -5

(f) $1/3$

(g) 0

4.1.1 -5, -2.47106145, -2.4067927, -2.400676, -2.4

4.1.2 $-4/3$, $-24/7$, $7/24$, $3/4$

4.1.3 -0.107526881, -0.11074197, -0.1110741, $\dfrac{-1}{3(3+\Delta x)} \to \dfrac{-1}{9}$

4.1.4 $\dfrac{3+3\Delta x+\Delta x^2}{1+\Delta x} \to 3$

4.1.5 3.31, 3.003001, 3.0000,
$3+3\Delta x+\Delta x^2 \to 3$

4.1.6 m

4.1.9 10, 25/2, 20, 15, 25, 35.

4.1.10 5, 4.1, 4.01, 4.001, $4+\Delta t \to 4$

4.1.11 -10.29, -9.849, -9.8049,
$-9.8-4.9\Delta t \to -9.8$

4.2.1 (a) $-x/\sqrt{169-x^2}$

(b) $-9.8t$

(c) $2x+1/x^2$

(d) $2ax+b$

(e) $3x^2$

(f) $-2/(2x+1)^{3/2}$

(g) $5/(t+2)^2$

4.2.4 $y=-13x+17$

4.2.5 -8

4.3.1 (a) $100x^{99}$

(b) $-100x^{-101}$

(c) $-5x^{-6}$

(d) $\pi x^{\pi-1}$

(e) $(3/4)x^{-1/4}$

(f) $-(9/7)x^{-16/7}$

(g) $15x^2+24x$

(h) $-20x^4+6x+10/x^3$

(i) $-30x+25$

(j) $3x^2+6x-1$

(k) $-\dfrac{x^2+2x+5}{(x^2+2x-3)^2}$

(l) $3x^2(x^3-5x+10)+x^3(3x^2-5)$

(m) $x^4(7x^2+30x-15)$

(n) $\dfrac{-3x^2-20x+15}{x^6}$

(o) $-\dfrac{3x(5x+8)}{(5x^3+12x^2-15)^2}$

4.3.2 $y=13x/4+5$

4.3.3 $y=24x-48-\pi^3$

4.3.4 $-49t/5+5$, $-49/5$

4.3.6 $\displaystyle\sum_{k=1}^{n} k a_k x^{k-1}$

4.3.7 $x^3/16 - 3x/4 + 4$

4.3.10 $f' = 4(2x-3)$, $y = 4x - 7$

4.3.12 $\dfrac{3x^2}{x^3 - 5x + 10} - \dfrac{x^3(3x^2-5)}{(x^3-5x+10)^2}$

4.3.13 $\dfrac{2x+5}{x^5 - 6x^3 + 3x^2 - 7x + 1} - \dfrac{(x^2+5x-3)(5x^4 - 18x^2 + 6x - 7)}{(x^5 - 6x^3 + 3x^2 - 7x + 1)^2}$

4.3.14 $\dfrac{x - 1250}{2(x-625)^{3/2}}$

4.3.15 $\dfrac{200 - 39x}{2x^{21}\sqrt{x-5}}$

4.3.16 $y = 17x/4 - 41/4$

4.3.17 $y = 11x/16 - 15/16$

4.3.18 $13/18$

4.4.2 $\pi/6 + 2n\pi$, $5\pi/6 + 2n\pi$, any integer n

4.5.1 $4x^3 - 9x^2 + x + 7$

4.5.2 $3x^2 - 4x + 2/\sqrt{x}$

4.5.3 $6(x^2+1)^2 x$

4.5.4 $\sqrt{169 - x^2} - x^2/\sqrt{169-x^2}$

4.5.5 $(2x-4)\sqrt{25-x^2} -$
$(x^2 - 4x + 5)x/\sqrt{25-x^2}$

4.5.6 $-x/\sqrt{r^2 - x^2}$

4.5.7 $2x^3/\sqrt{1+x^4}$

4.5.8 $\dfrac{1}{4\sqrt{x}(5-\sqrt{x})^{3/2}}$

4.5.9 $6 + 18x$

4.5.10 $\dfrac{2x+1}{1-x}+\dfrac{x^2+x+1}{(1-x)^2}$

4.5.11 $-1/\sqrt{25-x^2}-\sqrt{25-x^2}/x^2$

4.5.12 $\dfrac{1}{2}\left(\dfrac{-169}{x^2}-1\right)\Big/\sqrt{\dfrac{169}{x}-x}$

4.5.13 $\dfrac{3x^2-2x+1/x^2}{2\sqrt{x^3-x^2-(1/x)}}$

4.5.14 $\dfrac{300x}{(100-x^2)^{5/2}}$

4.5.15 $\dfrac{1+3x^2}{3(x+x^3)^{2/3}}$

4.5.16 $\left(4x(x^2+1)+\dfrac{4x^3+4x}{2\sqrt{1+(x^2+1)^2}}\right)\Big/$
$2\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$

4.5.17 $5(x+8)^4$

4.5.18 $-3(4-x)^2$

4.5.19 $6x(x^2+5)^2$

4.5.20 $-12x(6-2x^2)^2$

4.5.21 $24x^2(1-4x^3)^{-3}$

4.5.22 $5+5/x^2$

4.5.23 $-8(4x-1)(2x^2-x+3)^{-3}$

4.5.24 $1/(x+1)^2$

4.5.25 $3(8x-2)/(4x^2-2x+1)^2$

4.5.26 $-3x^2+5x-1$

4.5.27 $6x(2x-4)^3+6(3x^2+1)(2x-4)^2$

4.5.28 $-2/(x-1)^2$

4.5.29 $4x/(x^2+1)^2$

4.5.30 $(x^2-6x+7)/(x-3)^2$

4.5.31 $-5/(3x-4)^2$

4.5.32 $60x^4+72x^3+18x^2+18x-6$

4.5.33 $(5-4x)/((2x+1)^2(x-3)^2)$

4.5.34 $1/(2(2+3x)^2)$

4.5.35 $56x^6+72x^5+110x^4+100x^3+60x^2+28x+6$

4.5.36 $y=23x/96-29/96$

4.5.37 $y=3-2x/3$

4.5.38 $y=13x/2-23/2$

4.5.39 $y=2x-11$

4.5.40 $y=\dfrac{20+2\sqrt{5}}{5\sqrt{4+\sqrt{5}}}\,x+\dfrac{3\sqrt{5}}{5\sqrt{4+\sqrt{5}}}$

4.5.41 $(f(g(1)))'=20$

4.5.42 $g'(x)=2x(f(x^2)+x^2f'(x^2))$

4.6.1 $2\ln(3)x3^{x^2}$

4.6.2 $\dfrac{\cos x-\sin x}{e^x}$

4.6.3 $2e^{2x}$

4.6.4 $e^x\cos(e^x)$

4.6.5 $\cos(x)e^{\sin x}$

4.6.6 $x^{\sin x}\left(\cos x\ln x+\dfrac{\sin x}{x}\right)$

4.6.7 $3x^2e^x+x^3e^x$

4.6.8 $1+2^x\ln(2)$

4.6.9 $-2x\ln(3)(1/3)^{x^2}$

4.6.10 $e^{4x}(4x-1)/x^2$

4.6.11 $(3x^2+3)/(x^3+3x)$

4.6.12 $-\tan(x)$

4.6.13 $(1-\ln(x^2))/(x^2\sqrt{\ln(x^2)})$

4.6.14 $\sec(x)$

4.6.15 $x^{\cos(x)}(\cos(x)/x-\cos(x)\ln(x))$

4.6.19 e

4.7.1 (a) x/y

(b) $-(2x+y)/(x+2y)$

(c) $(2xy-3x^2-y^2)/(2xy-3y^2-x^2)$

(d) $\sin(x)\sin(y)/(\cos(x)\cos(y))$

(e) $-\sqrt{y}/\sqrt{x}$

(f) $(y\sec^2(x/y)-y^2)/(x\sec^2(x/y)+y^2)$

(g) $(y-\cos(x+y))/(\cos(x+y)-x)$

(h) $-y^2/x^2$

4.7.2 1

4.7.3 $y=2x\pm 6$

4.7.4 $y=x/2\pm 3$

4.7.6 $(\sqrt{3},2\sqrt{3})$, $(-\sqrt{3},-2\sqrt{3})$, $(2\sqrt{3},\sqrt{3})$, $(-2\sqrt{3},-\sqrt{3})$

4.7.7 $y=7x/\sqrt{3}-8/\sqrt{3}$

4.7.8 $y=(-y_1^{1/3}x+y_1^{1/3}x_1+x_1^{1/3}y_1)/x_1^{1/3}$

4.7.9 $(y-y_1)/(x-x_1)=(2x_1^3+2x_1y_1^2-x_1)/(2y_1^3+2y_1x_1^2+y_1)$

4.8.1 1

4.9.1 (a) $4(2x+3)$

(b) $\frac{3}{2}x^{1/2}$

4.9.2 3

4.9.3 (a) $28x^3 - \dfrac{1}{3\pi x^{4/3}}$

(b) $-\dfrac{1}{\sqrt{x}(1+\sqrt{x})^2}$

(c) $f'(x) = \begin{cases} -2 & \text{if } x < -2 \\ 0 & \text{if } -2 < x < 1 \\ 2 & \text{if } x > 1 \end{cases}$

(d) $2x\sin x\cos x + x^2\cos^2 x - x^2\sin^2 x$

(e) $\dfrac{(\sin x + x\cos x)(1+\sin x) - x\sin x\cos x}{(1+\sin x)^2}$

(f) $-\dfrac{3}{4x^{3/2}}\left(2+\dfrac{3}{\sqrt{x}}\right)^{-1/2}$

(g) $\frac{1}{3}(x^4+x^2+1)^{-2/3}(4x^3+2x) - \dfrac{5(3x^2-1)}{(x^3-x+4)^6}$

(h) $3\sin^2 x\cos x - 3x^2\cos(x^3)$

(i) $4\sec^4 x\tan x + 4\tan^3 x\sec^2 x$

(j) $\dfrac{4}{(1+x)^2}\cos\left(\dfrac{1-x}{1+x}\right)\sin\left(\dfrac{1-x}{1+x}\right)$

(k) $(2x+2\sec^2 x\tan x)\sec^2(\sin(x^2+\sec^2 x))\cos(x^2+\sec^2 x)$

(l) $-\dfrac{\pi\cos\frac{\pi}{x}}{x^2(2+\sin\frac{\pi}{x})^2}$

4.9.4 (a) $3e^{3x} - e^{-x}$

(b) $2e^{2x}\cos 3x - 3e^{2x}\sin 3x$

(c) $(1+e^x)\sec^2(x+e^x)$

(d) $2e^x/(e^x+2)^2$

(e) $\dfrac{\cos x}{2+\sin x} - \dfrac{\cos(2+\ln x)}{x}$

(f) $e^{x^\pi}\cdot\pi x^{\pi-1} + \pi^e x^{\pi^e-1} + \pi^{e^x}\ln\pi\cdot e^x$

(g) $\log_a b + (\log_a b)x^{(\log_a b)-1}$

(h) $(x^2+1)^{x^3+1}\left(3x^2\ln(x^2+1)+\dfrac{2x(x^3+1)}{x^2+1}\right)$

(i) $\dfrac{(x^2+e^x)^{1/\ln x}}{(\ln x)^2}\left(\dfrac{2x+e^x}{x^2+e^x}\ln x - \dfrac{x^2+e^x}{x}\right)$

(j) $\dfrac{x\sqrt{x^2+x+1}}{(2+\sin x)^4(3x+5)^7}\left(\dfrac{1}{x}+\dfrac{2x+1}{2(x^2+x+1)}-\dfrac{4\cos x}{2+\sin x}-\dfrac{21}{3x+5}\right)$

4.9.5 (a) $-(2x+y)/(x+2y)$

(b) $\dfrac{x-(2x^2+2y^2-x)(4x-1)}{4y(2x^2+2y^2-x)-y}$

(c) $-\dfrac{\sin x + 2x\sin y}{x^2\cos y + 3y^2}$

(d) $\dfrac{2x+e^y-e^x}{2-xe^y}$

4.9.6 (a) $\sin^{-1}x + x/\sqrt{1-x^2}$

(b) $\dfrac{\cos^{-1}x+\sin^{-1}x}{(\cos^{-1}x)^2\sqrt{1-x^2}}$

(c) $a/(x^2+a^2)$

(d) $\tan^{-1}x$

5.1.2 $1/(16\pi)$ cm/s

5.1.3 $3/(1000\pi)$ meters/second

5.1.4 $1/4$ m/s

5.1.5 $-6/25$ m/s

5.1.6 80π mi/min

5.1.7 $3\sqrt{5}$ ft/s

5.1.8 $20/(3\pi)$ cm/s

5.1.9 $13/20$ ft/s

5.1.10 $5\sqrt{10}/2$ m/s

5.1.11 75/64 m/min

5.1.12 tip: 6 ft/s, length: 5/2 ft/s

5.1.13 tip: 20/11 m/s, length: 9/11 m/s

5.1.14 $380/\sqrt{3} - 150 \approx 69.4$ mph

5.1.15 $500/\sqrt{3} - 200 \approx 88.7$ km/hr

5.1.16 4000/49 m/s

5.2.1 min at $x = 1/2$

5.2.2 min at $x = -1$, max at $x = 1$

5.2.3 max at $x = 2$, min at $x = 4$

5.2.4 min at $x = \pm 1$, max at $x = 0$.

5.2.5 min at $x = 1$

5.2.6 none

5.2.7 none

5.2.8 min at $x = 7\pi/12 + k\pi$, max at $x = -\pi/12 + k\pi$, for integer k.

5.2.9 local min at $x = 49$

5.2.12 one

5.2.16 Absolute maximum $(3, 7)$; Absolute minimum $(0, 1)$.

5.2.17 Absolute maximum $(3, 7)$; Absolute minimum $(0, 1)$.

5.2.18 Absolute minimum $(\pi/2, 1)$; No absolute maximum.

5.2.19 Absolute minimum $(1, 0)$; Absolute maximum $(e^{1/2}, \frac{1}{2e})$.

5.2.20 Absolute minimum $(1, 0)$; Absolute maximum $(e^{1/2}, \frac{1}{2e})$.

5.2.21 Absolute minimum $(0, 0)$; Absolute maximum $(2, 2e^{1/8})$.

5.2.22 Absolute minimum $(1/2, \frac{2-\pi}{4})$; Absolute maximum $(2, 2 - \tan^{-1}(4))$.

5.2.23 Absolute maximum $(1, 1/2)$; Absolute minimum $(-1, -1/2)$.

5.3.1 $c = 1/2$

5.3.2 $c = \sqrt{18} - 2$

5.3.6 $x^3/3 + 47x^2/2 - 5x + k$

5.3.7 $\arctan x + k$

5.3.8 $x^4/4 - \ln x + k$

5.3.9 $-\cos(2x)/2 + k$

5.4.1 $L(x) = x$, $f(0.1) \approx L(0.1) = 0.1$

5.4.2 Choose $f(x) = x^3$ and $a = 2$, the closest integer to 1.9. The linearization of f at a is $L(x) = 12(x-2) + 8$, and $(1.9)^3 = f(1.9) \approx L(1.9) = 12(1.9 - 2) + 8 = 6.8$.

5.4.4 Choose $a = 7$ since $f(7) = \sqrt[3]{7+1} = \sqrt[3]{8} = 2$ is an integer close to $\sqrt[3]{9}$. The linearization of f at $a = 7$ is $L(x) = {}^1\!/_{12}(x-7) + 2$. Then $f(8) = \sqrt[3]{8+1} = \sqrt[3]{9} \approx L(8) = {}^1\!/_{12}(8-7) + 2 = 2.08\overline{3}$. We are over-estimating $\sqrt[3]{9}$ since $L(x) > f(x)$ for all x around $a = 7$.

5.4.5 $\Delta y = 65/16$, $dy = 2$

5.4.6 $\Delta y = \sqrt{11/10} - 1$, $dy = 0.05$

5.4.7 $\Delta y = \sin(\pi/50)$, $dy = \pi/50$

5.4.8 $dV = 8\pi/25$

5.4.9 $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

(a) $\sin(0.1) \approx T_5(0.1) \approx 0.10016675$

(b) $\sin(0.1) = 0.0998334\ldots$ using a calculator. Our approximation is accurate to $0.10016675 - 0.0998334\ldots = 0.000\overline{3}$.

5.4.11 $T_3(x) = x + x^2 + x^3$. The point $x = 5$ is not close to $x = 0$, and f is not continuous at $x = 1$.

5.4.12 (a) $f^{(n)}(x) = \frac{(-1)^{(n-1)}(n-1)!}{x^n}$

(b) $T_n(x) = \ln(1) + \displaystyle\sum_{i=1}^{n} \frac{\left(\frac{(-1)^{(i-1)}(i-1)!}{1^n}\right)}{i!}(x-1)^i = \sum_{i=1}^{n} \left(\frac{(-1)^{(i-1)}(i-1)!}{i!}\right)(x-1)^i$ since $\ln(1) = 0$ and $1^n = 1$.

5.4.13 Notice that $f(-2) = 19$, $f(0) = -11$, and $f(5) = 19$ and f is a continuous function. By the Intermediate Value Theorem there exists a root in $[-2, 0]$ and $[0, 5]$. Choose $x_0 = 0$, then $x_4 \approx -0.93242$. Choose $x_0 = 5$, then $x_4 \approx 3.93242$.

5.4.14 (a) $x_4 \approx 1.00022\ldots$

(b) $x = 1$ is the root of f. Our approximation in part (a) was correct to 3 decimal places.

(c) $x_1 = 1$. The root is found in one iteration of Newton's Method.

5.4.15 $\cos(\pi/2) = 0$, so x_1 is undefined.

5.5.1 0

5.5.2 ∞

5.5.3 0

5.5.4 0

5.5.5 1/6

5.5.6 1/16

5.5.7 3/2

5.5.8 $-1/4$

5.5.9 -3

5.5.10 1/2

5.5.11 0

5.5.12 -1

5.5.13 $-1/2$

5.5.14 5

5.5.15 1

5.5.16 1

5.5.17 2

5.5.18 1

5.5.19 0

5.5.20 $1/2$

5.5.21 2

5.5.22 0

5.5.23 $1/2$

5.5.24 $-1/2$

5.5.25 2

5.5.26 0

5.5.27 ∞

5.5.28 0

5.5.29 5

5.5.30 $-1/2$

5.5.31 -1

5.5.32 $-\sqrt{2}/2$

5.5.33 5

5.5.34 ∞

5.5.35 $1/2$

5.5.36 0

5.5.37 0

5.5.38 ∞

5.5.39 ∞

5.5.40 0

5.5.41 1

5.5.42 1

5.5.43 1

5.5.44 1

5.5.45 1

5.5.46 1

5.5.47 1

5.5.48 2

5.5.49 1/2

5.5.50 $-\infty$

5.5.51 1

5.5.52 0

5.5.53 3

5.5.54 ∞

5.6.1 min at $x = 1/2$

5.6.2 min at $x = -1$, max at $x = 1$

5.6.3 max at $x = 2$, min at $x = 4$

5.6.4 min at $x = \pm 1$, max at $x = 0$.

5.6.5 min at $x = 1$

5.6.6 none

5.6.7 none

5.6.8 min at $x = 7\pi/12 + k\pi$, max at $x = -\pi/12 + k\pi$, for integer k.

5.6.9 none

5.6.10 max at $x = 0$, min at $x = \pm 11$

5.6.11 min at $x = -3/2$, neither at $x = 0$

5.6.12 min at $n\pi$, max at $\pi/2 + n\pi$

5.6.13 min at $2n\pi$, max at $(2n+1)\pi$

5.6.14 min at $\pi/2 + 2n\pi$, max at $3\pi/2 + 2n\pi$

5.6.17 min at $x = 1/2$

5.6.18 min at $x = -1$, max at $x = 1$

5.6.19 max at $x = 2$, min at $x = 4$

5.6.20 min at $x = \pm 1$, max at $x = 0$.

5.6.21 min at $x = 1$

5.6.22 none

5.6.23 none

5.6.24 min at $x = 7\pi/12 + n\pi$, max at $x = -\pi/12 + n\pi$, for integer n.

5.6.25 max at $x = 63/64$

5.6.26 max at $x = 7$

5.6.27 max at $-5^{-1/4}$, min at $5^{-1/4}$

5.6.28 none

5.6.29 max at -1, min at 1

5.6.30 min at $2^{-1/3}$

5.6.31 none

5.6.32 min at $n\pi$

5.6.33 max at $n\pi$, min at $\pi/2 + n\pi$

5.6.34 max at $\pi/2 + 2n\pi$, min at $3\pi/2 + 2n\pi$

5.6.35 concave up everywhere

5.6.36 concave up when $x < 0$, concave down when $x > 0$

5.6.37 concave down when $x < 3$, concave up when $x > 3$

5.6.38 concave up when $x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$, concave down when $-1/\sqrt{3} < x < 1/\sqrt{3}$

5.6.39 concave up when $x < 0$ or $x > 2/3$, concave down when $0 < x < 2/3$

5.6.40 concave up when $x < 0$, concave down when $x > 0$

5.6.41 concave up when $x < -1$ or $x > 1$, concave down when $-1 < x < 0$ or $0 < x < 1$

5.6.42 concave down on $((8n-1)\pi/4, (8n+3)\pi/4)$, concave up on $((8n+3)\pi/4, (8n+7)\pi/4)$, for integer n

5.6.43 concave down everywhere

5.6.44 concave up on $(-\infty, (21-\sqrt{497})/4)$ and $(21+\sqrt{497})/4, \infty)$

5.6.45 concave up on $(0, \infty)$

5.6.46 concave down on $(2n\pi/3, (2n+1)\pi/3)$

5.6.47 concave up on $(0, \infty)$

5.6.48 concave up on $(-\infty, -1)$ and $(0, \infty)$

5.6.49 concave down everywhere

5.6.50 concave up everywhere

5.6.51 concave up on $(\pi/4 + n\pi, 3\pi/4 + n\pi)$

5.6.52 inflection points at $n\pi$, $\pm\arcsin(\sqrt{2/3}) + n\pi$

5.6.53 up/incr: $(3, \infty)$, up/decr: $(-\infty, 0)$, $(2, 3)$, down/decr: $(0, 2)$

5.7.1 25×25

5.7.2 $P/4 \times P/4$

5.7.3 $w = l = 2 \cdot 5^{2/3}$, $h = 5^{2/3}$, $h/w = 1/2$

5.7.4 $\sqrt[3]{100} \times \sqrt[3]{100} \times 2\sqrt[3]{100}$, $h/s = 2$

5.7.5 $w = l = 2^{1/3}V^{1/3}$, $h = V^{1/3}/2^{2/3}$, $h/w = 1/2$

5.7.6 1250 square feet

5.7.7 $l^2/8$ square feet

5.7.8 \$5000

5.7.9 100

5.7.10 r^2

5.7.11 $h/r = 2$

5.7.12 $h/r = 2$

5.7.13 $r = 5$, $h = 40/\pi$, $h/r = 8/\pi$

5.7.14 $8/\pi$

5.7.15 $4/27$

5.7.16 (a) 2, (b) $7/2$

5.7.17 $\dfrac{\sqrt{3}}{6} \times \dfrac{\sqrt{3}}{6} + \dfrac{1}{2} \times \dfrac{1}{4} - \dfrac{\sqrt{3}}{12}$

5.7.18 (a) $a/6$, (b) $(a + b - \sqrt{a^2 - ab + b^2})/6$

5.7.19 1.5 meters wide by 1.25 meters tall

5.7.20 If $k \leq 2/\pi$ the ratio is $(2 - k\pi)/4$; if $k \geq 2/\pi$, the ratio is zero: the window should be semicircular with no rectangular part.

5.7.21 a/b

5.7.22 $1/\sqrt{3} \approx 58\%$

5.7.23 $18 \times 18 \times 36$

5.7.24 $r = 5/(2\pi)^{1/3} \approx 2.7$ cm,
$h = 5 \cdot 2^{5/3}/\pi^{1/3} = 4r \approx 10.8$ cm

5.7.25 $h = \dfrac{750}{\pi}\left(\dfrac{2\pi^2}{750^2}\right)^{1/3}$, $r = \left(\dfrac{750^2}{2\pi^2}\right)^{1/6}$

5.7.26 $h/r = \sqrt{2}$

5.7.27 $1/2$

5.7.28 \$7000

6.2.1 10

6.2.2 $35/3$

6.2.3 x^2

6.2.4 $2x^2$

6.2.5 $2x^2 - 8$

6.2.6 $2b^2 - 2a^2$

6.2.7 4 rectangles: $41/4 = 10.25$, 8 rectangles: $183/16 = 11.4375$

6.2.8 $23/4$

6.3.1 $87/2$

6.3.2 2

6.3.3 $\ln(10)$

6.3.4 $e^5 - 1$

6.3.5 $3^4/4$

6.3.6 $2^6/6 - 1/6$

6.3.7 $x^2 - 3x$

6.3.8 $2x(x^4 - 3x^2)$

6.3.9 e^{x^2}

6.3.10 $2xe^{x^4}$

6.3.11 $\tan(x^2)$

6.3.12 $2x\tan(x^4) - 10\tan(100x^2)$

6.3.13 31, 14

6.3.14 5

6.3.15 (a) $2/3$

(b) $24/5$

7.1.1 $-(1-t)^{10}/10 + C$

7.1.2 $x^5/5 + 2x^3/3 + x + C$

7.1.3 $(x^2+1)^{101}/202 + C$

7.1.4 $-3(1-5t)^{2/3}/10 + C$

7.1.5 $(\sin^4 x)/4 + C$

7.1.6 $-(100-x^2)^{3/2}/3 + C$

7.1.7 $-2\sqrt{1-x^3}/3 + C$

7.1.8 $\sin(\sin \pi t)/\pi + C$

7.1.9 $1/(2\cos^2 x) = (1/2)\sec^2 x + C$

7.1.10 $-\ln|\cos x| + C$

7.1.11 0

7.1.12 $\tan^2(x)/2 + C$

7.1.13 $1/4$

7.1.14 $-\cos(\tan x) + C$

7.1.15 $1/10$

7.1.16 $\sqrt{3}/4$

7.1.17 $(27/8)(x^2-7)^{8/9}$

7.1.18 $-(3^7+1)/14$

7.1.19 0

7.1.20 $f(x)^2/2$

7.2.1 $x/2-\sin(2x)/4+C$

7.2.2 $-\cos x+(\cos^3 x)/3+C$

7.2.3 $3x/8-(\sin 2x)/4+(\sin 4x)/32+C$

7.2.4 $(\cos^5 x)/5-(\cos^3 x)/3+C$

7.2.5 $\sin x-(\sin^3 x)/3+C$

7.2.6 $(\sin^3 x)/3-(\sin^5 x)/5+C$

7.2.7 $-2(\cos x)^{5/2}/5+C$

7.2.8 $\tan x-\cot x+C$

7.2.9 $(\sec^3 x)/3-\sec x+C$

7.2.10 $-\cos x+\sin x+C$

7.2.11 $\frac{3}{2}\ln|\sec x+\tan x|+\tan x+\frac{1}{2}\sec x\tan x+C$

7.2.12 $\frac{\tan^5(x^2)}{10}+C$

7.3.1 $x\sqrt{x^2-1}/2-\ln|x+\sqrt{x^2-1}|/2+C$

7.3.2 $x\sqrt{9+4x^2}/2+(9/4)\ln|2x+\sqrt{9+4x^2}|+C$

7.3.3 $-(1-x^2)^{3/2}/3+C$

7.3.4 $\arcsin(x)/8-\sin(4\arcsin x)/32+C$

7.3.5 $\ln|x+\sqrt{1+x^2}|+C$

7.3.6 $(x+1)\sqrt{x^2+2x}/2-\ln|x+1+\sqrt{x^2+2x}|/2+C$

7.3.7 $-\arctan x-1/x+C$

7.3.8 $2\arcsin(x/2) - x\sqrt{4-x^2}/2 + C$

7.3.9 $\arcsin(\sqrt{x}) - \sqrt{x}\sqrt{1-x} + C$

7.3.10 $(2x^2+1)\sqrt{4x^2-1}/24 + C$

7.4.1 $\cos x + x\sin x + C$

7.4.2 $x^2\sin x - 2\sin x + 2x\cos x + C$

7.4.3 $(x-1)e^x + C$

7.4.4 $(1/2)e^{x^2} + C$

7.4.5 $(x/2) - \sin(2x)/4 + C =$
$(x/2) - (\sin x\cos x)/2 + C$

7.4.6 $x\ln x - x + C$

7.4.7 $(x^2\arctan x + \arctan x - x)/2 + C$

7.4.8 $-x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$

7.4.9 $x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x + C$

7.4.10 $x^2/4 - (\cos^2 x)/4 - (x\sin x\cos x)/2 + C$

7.4.11 $x/4 - (x\cos^2 x)/2 + (\cos x\sin x)/4 + C$

7.4.12 $x\arctan(\sqrt{x}) + \arctan(\sqrt{x}) - \sqrt{x} + C$

7.4.13 $2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x}) + C$

7.4.14 $\sec x\csc x - 2\cot x + C$

7.5.1 $-\ln|x-2|/4 + \ln|x+2|/4 + C$

7.5.2 $-x^3/3 - 4x - 4\ln|x-2| +$
$4\ln|x+2| + C$

7.5.3 $-1/(x+5) + C$

7.5.4 $-x - \ln|x-2| + \ln|x+2| + C$

7.5.5 $-4x + x^3/3 + 8\arctan(x/2) + C$

7.5.6 $(1/2)\arctan(x/2+5/2)+C$

7.5.7 $x^2/2-2\ln(4+x^2)+C$

7.5.8 $(1/4)\ln|x+3|-(1/4)\ln|x+7|+C$

7.5.9 $(1/5)\ln|2x-3|-(1/5)\ln|1+x|+C$

7.5.10 $(1/3)\ln|x|-(1/3)\ln|x+3|+C$

7.6.1 (a) (a) Area is 30.8667 cm^2.
(b) Area is $308,667$ m^2.

(b) (a) Area is 25.0667 cm^2
(b) Area is 250,667 yd^2

7.6.2 (a) (a) $3/4$
(b) $2/3$
(c) $2/3$

(b) (a) 250
(b) 250
(c) 250

(c) (a) $\frac{1}{4}(1+\sqrt{2})\pi \approx 1.896$
(b) $\frac{1}{6}(1+2\sqrt{2})\pi \approx 2.005$
(c) 2

(d) (a) $2+\sqrt{2}+\sqrt{3} \approx 5.15$
(b) $2/3(3+\sqrt{2}+2\sqrt{3}) \approx 5.25$
(c) $16/3 \approx 5.33$

(e) (a) 38.5781
(b) $147/4 \approx 36.75$
(c) $147/4 \approx 36.75$

(f) (a) 0.2207
(b) 0.2005
(c) $1/5$

(g) (a) 0
(b) 0
(c) 0

(h) (a) $9/2(1+\sqrt{3}) \approx 12.294$

(b) $3+6\sqrt{3} \approx 13.392$

(c) $9\pi/2 \approx 14.137$

7.6.3 (a) Trapezoidal Rule: 0.9006

Simpson's Rule: 0.90452

(b) Trapezoidal Rule: 3.0241

Simpson's Rule: 2.9315

(c) Trapezoidal Rule: 13.9604

Simpson's Rule: 13.9066

(d) Trapezoidal Rule: 3.0695

Simpson's Rule: 3.14295

(e) Trapezoidal Rule: 1.1703

Simpson's Rule: 1.1873

(f) Trapezoidal Rule: 2.52971

Simpson's Rule: 2.5447

(g) Trapezoidal Rule: 1.0803

Simpson's Rule: 1.077

(h) Trapezoidal Rule: 3.5472

Simpson's Rule: 3.6133

7.6.4 (a) (a) $n = 161$ (using $\max\big(f''(x)\big) = 1$)

(b) $n = 12$ (using $\max\big(f^{(4)}(x)\big) = 1$)

(b) (a) $n = 150$ (using $\max\big(f''(x)\big) = 1$)

(b) $n = 18$ (using $\max\big(f^{(4)}(x)\big) = 7$)

(c) (a) $n = 1004$ (using $\max\big(f''(x)\big) = 39$)

(b) $n = 62$ (using $\max\big(f^{(4)}(x)\big) = 800$)

(d) (a) $n = 5591$ (using $\max\big(f''(x)\big) = 300$)

(b) $n = 46$ (using $\max\big(f^{(4)}(x)\big) = 24$)

7.6.5 T,S: 4 ± 0

7.6.6 T: 9.28125 ± 0.281125; S: 9 ± 0

7.6.7 T: 60.75 ± 1; S: 60 ± 0

7.6.8 T: 1.1167 ± 0.0833; S: 1.1000 ± 0.0167

7.6.9 T: 0.3235 ± 0.0026; S: 0.3217 ± 0.000065

7.6.10 T: 0.6478 ± 0.0052; S: 0.6438 ± 0.000033

7.6.11 T: 2.8833 ± 0.0834; S: 2.9000 ± 0.0167

7.6.12 T: 1.1170 ± 0.0077; S: 1.1114 ± 0.0002

7.6.13 T: 1.097 ± 0.0147; S: 1.089 ± 0.0003

7.6.14 T: 3.63 ± 0.087; S: 3.62 ± 0.032

7.7.1 Converges to 1.

7.7.2 Diverges.

7.7.3 $1/3$

7.7.4 Divergent.

7.7.7 (a) $\pi/2$

(b) divergent (to ∞)

(c) 1

(d) divergent (to ∞)

(e) $\frac{5}{3}(4^{3/5})$

7.7.9 $0 < p < 1$

7.8.1 (a)

$$\begin{aligned}
\frac{d}{dx}[\text{sech} x] &= \frac{d}{dx}\left[\frac{2}{e^x + e^{-x}}\right] \\
&= \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
&= -\frac{2(e^x - e^{-x})}{(e^x + e^{-x})(e^x + e^{-x})} \\
&= -\frac{2}{e^x + e^{-x}} \cdot \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
&= -\text{sech} x \tanh x
\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dx}[\coth x] &= \frac{d}{dx}\left[\frac{e^x+e^{-x}}{e^x-e^{-x}}\right]\\ &= \frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2}\\ &= \frac{e^{2x}+e^{-2x}-2-(e^{2x}+e^{-2x}+2)}{(e^x-e^{-x})^2}\\ &= -\frac{4}{(e^x-e^{-x})^2}\\ &= -\text{csch}^2 x\end{aligned}$$

(c) $\displaystyle\int \tanh x\ dx = \int \frac{\sinh x}{\cosh x}\ dx$

Let $u = \cosh x$; $du = (\sinh x)dx$

$\displaystyle= \int \frac{1}{u}\ du$

$= \ln|u| + C$

$= \ln(\cosh x) + C.$

(d)
$$\int \coth x\ dx = \int \frac{\cosh x}{\sinh x}\ dx$$

Let $u = \sinh x$; $du = (\cosh x)dx$

$$\begin{aligned}&= \int \frac{1}{u}\ du\\ &= \ln|u| + C\\ &= \ln|\sinh x| + C.\end{aligned}$$

7.8.2 (a) $2\sinh 2x$

(b) $2x\sec^2(x^2)$

(c) $\coth x$

(d) $\sinh^2 x + \cosh^2 x$

(e) $x\cosh x$

(f) $\frac{-2x}{(x^2)\sqrt{1-x^4}}$

(g) $\frac{3}{\sqrt{9x^2+1}}$

(h) $\frac{4x}{\sqrt{4x^4-1}}$

(i) $\frac{1}{1-(x+5)^2}$

(j) $-\csc x$

(k) $\sec x$

7.8.3 (a) $y = x$

(b) $y = 3/4(x - \ln 2) + 5/4$

(c) $y = -72/125(x - \ln 3) + 9/25$

(d) $y = x$

(e) $y = (x - \sqrt{2}) + \cosh^{-1}(\sqrt{2}) \approx (x - 1.414) + 0.881$

7.8.4 (a) $1/2 \ln(\cosh(2x)) + C$

(b) $1/3 \sinh(3x - 7) + C$

(c) $1/2 \sinh^2 x + C$ or $1/2 \cosh^2 x + C$

(d) $x \sinh(x) - \cosh(x) + C$

(e) $x \cosh(x) - \sinh(x) + C$

(f) $\begin{cases} \frac{1}{3} \tanh^{-1}\left(\frac{x}{3}\right) + C & x^2 < 9 \\ \frac{1}{3} \coth^{-1}\left(\frac{x}{3}\right) + C & 9 < x^2 \end{cases} = \frac{1}{2} \ln|x+1| - \frac{1}{2} \ln|x-1| + C$

(g) $\cosh^{-1}(x^2/2) + C = \ln(x^2 + \sqrt{x^4 - 4}) + C$

(h) $2/3 \sinh^{-1} x^{3/2} + C = 2/3 \ln(x^{3/2} + \sqrt{x^3 + 1}) + C$

(i) $\frac{1}{16} \tan^{-1}(x/2) + \frac{1}{32} \ln|x-2| + \frac{1}{32} \ln|x+2| + C$

(j) $\ln x - \ln|x+1| + C$

(k) $\tan^{-1}(e^x) + C$

(l) $x \sinh^{-1} x - \sqrt{x^2 + 1} + C$

(m) $x \tanh^{-1} x + 1/2 \ln|x^2 - 1| + C$

(n) $\tan^{-1}(\sinh x) + C$

7.8.5 (a) 0

(b) $3/2$

(c) 2

7.9.1 $\dfrac{(t+4)^4}{4} + C$

7.9.2 $\dfrac{(t^2-9)^{5/2}}{5} + C$

7.9.3 $\dfrac{(e^{t^2}+16)^2}{4}+C$

7.9.4 $\cos t-\dfrac{2}{3}\cos^3 t+C$

7.9.5 $\dfrac{\tan^2 t}{2}+C$

7.9.6 $\ln|t^2+t+3|+C$

7.9.7 $\dfrac{1}{8}\ln|1-4/t^2|+C$

7.9.8 $\dfrac{1}{25}\tan(\arcsin(t/5))+C=\dfrac{t}{25\sqrt{25-t^2}}+C$

7.9.9 $\dfrac{2}{3}\sqrt{\sin 3t}+C$

7.9.10 $t\tan t+\ln|\cos t|+C$

7.9.11 $2\sqrt{e^t+1}+C$

7.9.12 $\dfrac{3t}{8}+\dfrac{\sin 2t}{4}+\dfrac{\sin 4t}{32}+C$

7.9.13 $\dfrac{\ln|t|}{3}-\dfrac{\ln|t+3|}{3}+C$

7.9.14 $\dfrac{-1}{\sin\arctan t}+C=-\sqrt{1+t^2}/t+C$

7.9.15 $\dfrac{-1}{2(1+\tan t)^2}+C$

7.9.16 $\dfrac{(t^2+1)^{5/2}}{5}-\dfrac{(t^2+1)^{3/2}}{3}+C$

7.9.17 $\dfrac{e^t\sin t-e^t\cos t}{2}+C$

7.9.18 $\dfrac{(t^{3/2}+47)^4}{6}+C$

7.9.19 $\dfrac{2}{3(2-t^2)^{3/2}}-\dfrac{1}{(2-t^2)^{1/2}}+C$

7.9.20 $\dfrac{\ln|\sin(\arctan(2t/3))|}{9} + C = (\ln(4t^2) - \ln(9 + 4t^2))/18 + C$

7.9.21 $\dfrac{(\arctan(2t))^2}{4} + C$

7.9.22 $\dfrac{3\ln|t+3|}{4} + \dfrac{\ln|t-1|}{4} + C$

7.9.23 $\dfrac{\cos^7 t}{7} - \dfrac{\cos^5 t}{5} + C$

7.9.24 $\dfrac{-1}{t-3} + C$

7.9.25 $\dfrac{-1}{\ln t} + C$

7.9.26 $\dfrac{t^2(\ln t)^2}{2} - \dfrac{t^2 \ln t}{2} + \dfrac{t^2}{4} + C$

7.9.27 $(t^3 - 3t^2 + 6t - 6)e^t + C$

7.9.28 $\dfrac{5+\sqrt{5}}{10}\ln(2t+1-\sqrt{5}) + \dfrac{5-\sqrt{5}}{10}\ln(2t+1+\sqrt{5}) + C$

8.1.1 It rises until $t = 100/49$, then falls. The position of the object at time t is $s(t) = -4.9t^2 + 20t + k$. The net distance traveled is $-45/2$, that is, it ends up $45/2$ meters below where it started. The total distance traveled is $6205/98$ meters.

8.1.2 $\displaystyle\int_0^{2\pi} \sin t\, dt = 0$

8.1.3 net: 2π, total: $2\pi/3 + 4\sqrt{3}$

8.1.4 8

8.1.5 $17/3$

8.1.6 $A = 18$, $B = 44/3$, $C = 10/3$

8.2.1 (a) π

(b) $4\pi + \pi^2 \approx 22.436$

(c) $16/3$

(d) π

(e) $1/2$

(f) $2\sqrt{2}$

(g) $1/\ln 4$

8.2.2 (a) 1

(b) $5/3$

(c) $9/2$

(d) $9/4$

(e) $1/12(9-2\sqrt{2}) \approx 0.514$

8.2.3 (a) 1

(b) 5

(c) 4

(d) $\frac{133}{20}$

8.2.4 $8\sqrt{2}/15$

8.2.5 $1/12$

8.2.6 $9/2$

8.2.7 $4/3$

8.2.8 $2/3-2/\pi$

8.2.9 $3/\pi-3\sqrt{3}/(2\pi)-1/8$

8.2.10 $1/3$

8.2.11 $10\sqrt{5}/3-6$

8.2.12 $500/3$

8.2.13 2

8.2.14 $1/5$

8.2.15 $1/6$

8.2.16 219,000 m^2

8.2.17 623,333 m^2

8.3.5 $8\pi/3$

8.3.6 $\pi/30$

8.3.7 $\pi(\pi/2-1)$

8.3.8

(a) $114\pi/5$

(b) $74\pi/5$

(c) 20π

(d) 4π

8.3.9 16π, 24π

8.3.11 $\pi h^2(3r-h)/3$

8.3.13 2π

8.4.1 $2/\pi$; $2/\pi$; 0

8.4.2 $4/3$

8.4.3 $1/A$

8.4.4 $\pi/4$

8.4.5 $-1/3$, 1

8.4.6 $-4\sqrt{1224}$ ft/s; $-8\sqrt{1224}$ ft/s

8.5.1 $\approx 5,305,028,516$ N-m

8.5.2 $\approx 4,457,854,041$ N-m

8.5.3 $367,500\pi$ N-m

8.5.4 $49000\pi + 196000/3$ N-m

8.5.5 2450π N-m

8.5.6 0.05 N-m

8.5.7 6/5 N-m

8.5.8 3920 N-m

8.5.9 23520 N-m

8.5.10 12740 N-m

8.6.1 $15/2$

8.6.2 5

8.6.3 16/5

8.6.5 $\bar{x} = 45/28$, $\bar{y} = 93/70$

8.6.6 $\bar{x} = 0$, $\bar{y} = 4/(3\pi)$

8.6.7 $\bar{x} = 1/2$, $\bar{y} = 2/5$

8.6.8 $\bar{x} = 0$, $\bar{y} = 8/5$

8.6.9 $\bar{x} = 4/7$, $\bar{y} = 2/5$

8.6.10 $\bar{x} = \bar{y} = 1/5$

8.6.11 $\bar{x} = 0$, $\bar{y} = 28/(9\pi)$

8.6.12 $\bar{x} = \bar{y} = 28/(9\pi)$

8.6.13 $\bar{x} = 0$, $\bar{y} = 244/(27\pi) \approx 2.88$

8.7.1 (a) $\sqrt{2}$

(b) 6

(c) 4/3

(d) 6

(e) 109/2

(f) 3/2

(g) 12/5

(h) $79953333/400000 \approx 199.883$

(i) $-\ln(2-\sqrt{3}) \approx 1.31696$

(j) $\sinh^{-1} 1$

8.7.2 (a) $\int_0^1 \sqrt{1+4x^2}\,dx$

(b) $\int_0^1 \sqrt{1+100x^{18}}\,dx$

(c) $\int_1^e \sqrt{1+\frac{1}{x^2}}\,dx$

(d) $\int_0^1 \sqrt{1+\frac{1}{4x}}\,dx$

(e) $\int_{-1}^{1} \sqrt{1+\frac{x^2}{1-x^2}}\ dx$

(f) $\int_{-3}^{3} \sqrt{1+\frac{x^2}{81-9x^2}}\ dx$

(g) $\int_{1}^{2} \sqrt{1+\frac{1}{x^4}}\ dx$

(h) $\int_{-\pi/4}^{\pi/4} \sqrt{1+\sec^2 x \tan^2 x}\ dx$

8.7.3 (a) 1.4790

(b) 1.8377

(c) 2.1300

(d)

(e)

(f)

(g) 1.4058

(h) 1.7625

8.7.4 $(22\sqrt{22}-8)/27$

8.7.5 $\ln(2)+3/8$

8.7.6 $a+a^3/3$

8.7.7 $\ln((\sqrt{2}+1)/\sqrt{3})$

8.7.9 $3/4$

8.7.10 ≈ 3.82

8.7.11 ≈ 1.01

8.7.12 $\sqrt{1+e^2}-\sqrt{2}+\frac{1}{2}\ln\left(\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right)+\frac{1}{2}\ln(3+2\sqrt{2})$

8.8.1 $8\pi\sqrt{3}-\frac{16\pi\sqrt{2}}{3}$

8.8.3 $\frac{730\pi\sqrt{730}}{27}-\frac{10\pi\sqrt{10}}{27}$

8.8.4 $\pi + 2\pi e + \frac{1}{4}\pi e^2 - \frac{\pi}{4e^2} - \frac{2\pi}{e}$

8.8.6 $8\pi^2$

8.8.7 $2\pi + \frac{8\pi^2}{3\sqrt{3}}$

8.8.8 $a > b$: $2\pi b^2 +$
$\frac{2\pi a^2 b}{\sqrt{a^2-b^2}} \arcsin(\sqrt{a^2-b^2}/a)$,
$a < b$: $2\pi b^2 +$
$\frac{2\pi a^2 b}{\sqrt{b^2-a^2}} \ln\left(\frac{b}{a} + \frac{\sqrt{b^2-a^2}}{a}\right)$

9.1.1 1

9.1.3 0

9.1.4 1

9.1.5 1

9.1.6 0

9.2.1 $\lim_{n\to\infty} n^2/(2n^2+1) = 1/2$

9.2.2 $\lim_{n\to\infty} 5/(2^{1/n}+14) = 1/3$

9.2.3 $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\displaystyle\sum_{n=1}^{\infty} 3\frac{1}{n}$ diverges

9.2.4 $-3/2$

9.2.5 11

9.2.6 20

9.2.7 3/4

9.2.8 3/2

9.2.9 3/10

9.3.1 diverges

9.3.2 diverges

9.3.3 converges

9.3.4 converges

9.3.5 converges

9.3.6 converges

9.3.7 diverges

9.3.8 converges

9.3.9 $N = 5$

9.3.10 $N = 10$

9.3.11 $N = 1687$

9.3.12 any integer greater than e^{200}

9.4.1 converges

9.4.2 converges

9.4.3 diverges

9.4.4 converges

9.4.5 0.90

9.4.6 0.95

9.5.1 converges

9.5.2 converges

9.5.3 converges

9.5.4 diverges

9.5.5 diverges

9.5.6 diverges

9.5.7 converges

9.5.8 diverges

9.5.9 converges

9.5.10 diverges

9.6.1 converges absolutely

9.6.2 diverges

9.6.3 converges conditionally

9.6.4 converges absolutely

9.6.5 converges conditionally

9.6.6 converges absolutely

9.6.7 diverges

9.6.8 converges conditionally

9.7.5 (a) converges

(b) converges

(c) converges

(d) diverges

9.8.1 (a) $R = 1$, $I = (-1, 1)$

(b) $R = \infty$, $I = (-\infty, \infty)$

(c) $R = e$, $I = (2 - e, 2 + e)$

(d) $R = 0$, converges only when $x = 2$

(e) $R = 1$, $I = [-6, -4]$

9.8.2 $R = e$

9.9.1 the alternating harmonic series

9.9.2 $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$

9.9.3 $\displaystyle\sum_{n=0}^{\infty}(n+1)(n+2)x^n$

9.9.4 $\displaystyle\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n,\ R=1$

9.9.5 $\displaystyle C+\sum_{n=0}^{\infty}\frac{-1}{(n+1)(n+2)}x^{n+2}$

9.10.1 (a) $p_3(x)=1-x+\frac{1}{2}x^3-\frac{1}{6}x^3$

(b) $p_8(x)=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7$

(c) $p_8(x)=x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4+\frac{1}{24}x^5$

(d) $p_6(x)=\frac{2x^5}{15}+\frac{x^3}{3}+x$

(e) $p_4(x)=\frac{2x^4}{3}+\frac{4x^3}{3}+2x^2+2x+1$

(f) $p_4(x)=x^4+x^3+x^2+x+1$

(g) $p_4(x)=x^4-x^3+x^2-x+1$

(h) $p_7(x)=-\frac{x^7}{7}+\frac{x^5}{5}-\frac{x^3}{3}+x$

9.10.2 (a) $p_4(x)=1+\frac{1}{2}(-1+x)-\frac{1}{8}(-1+x)^2+\frac{1}{16}(-1+x)^3-\frac{5}{128}(-1+x)^4$

(b) $p_4(x)=\ln(2)+\frac{1}{2}(-1+x)-\frac{1}{8}(-1+x)^2+\frac{1}{24}(-1+x)^3-\frac{1}{64}(-1+x)^4$

(c) $p_6(x)=\frac{1}{\sqrt{2}}-\frac{-\frac{\pi}{4}+x}{\sqrt{2}}-\frac{\left(-\frac{\pi}{4}+x\right)^2}{2\sqrt{2}}+\frac{\left(-\frac{\pi}{4}+x\right)^3}{6\sqrt{2}}+\frac{\left(-\frac{\pi}{4}+x\right)^4}{24\sqrt{2}}-\frac{\left(-\frac{\pi}{4}+x\right)^5}{120\sqrt{2}}-\frac{\left(-\frac{\pi}{4}+x\right)^6}{720\sqrt{2}}$

(d) $p_5(x)=\frac{1}{2}+\frac{1}{2}\sqrt{3}\left(-\frac{\pi}{6}+x\right)-\frac{1}{4}\left(-\frac{\pi}{6}+x\right)^2-\frac{\left(-\frac{\pi}{6}+x\right)^3}{4\sqrt{3}}+\frac{1}{48}\left(-\frac{\pi}{6}+x\right)^4+\frac{\left(-\frac{\pi}{6}+x\right)^5}{80\sqrt{3}}$

(e) $p_5(x)=\frac{1}{2}-\frac{x-2}{4}+\frac{1}{8}(x-2)^2-\frac{1}{16}(x-2)^3+\frac{1}{32}(x-2)^4-\frac{1}{64}(x-2)^5$

(f) $p_8(x)=1-2(-1+x)+3(-1+x)^2-4(-1+x)^3+5(-1+x)^4-6(-1+x)^5+7(-1+x)^6-8(-1+x)^7+9(-1+x)^8$

(g) $p_3(x)=\frac{1}{2}+\frac{1+x}{2}+\frac{1}{4}(1+x)^2$

(h) $p_2(x)=-\pi^2-2\pi(x-\pi)+\frac{1}{2}\left(\pi^2-2\right)(x-\pi)^2$

9.10.3 (a) $p_3(x)=x-\frac{x^3}{6}$; $p_3(0.1)=0.09983$. Error is bounded by $\pm\frac{1}{4!}\cdot 0.1^4\approx\pm 0.000004167$.

(b) $p_4(x)=1-\frac{x^2}{2}+\frac{x^4}{24}$; $p_4(1)=13/24\approx 0.54167$. Error is bounded by $\pm\frac{1}{5!}\cdot 1^5\approx\pm 0.00833$

(c) $p_2(x)=3+\frac{1}{6}(-9+x)-\frac{1}{216}(-9+x)^2$; $p_2(10)=3.16204$. The third derivative of $f(x)=\sqrt{x}$ is bounded on $(8,11)$ by 0.003. Error is bounded by $\pm\frac{0.003}{3!}\cdot 1^3=\pm 0.0005$.

(d) $p_3(x) = -1 + x - \frac{1}{2}(-1+x)^2 + \frac{1}{3}(-1+x)^3$; $p_3(1.5) = 0.41667$. The fourth derivative of $f(x) = \ln x$ is bounded on $(.9, 2)$ by 10. Error is bounded by $\pm\frac{10}{4!} \cdot .5^4 = \pm 0.026$.

9.10.4 (a) The n^{th} derivative of $f(x) = e^x$ is bounded by 3 on intervals containing 0 and 1. Thus $|R_n(1)| \le \frac{3}{(n+1)!}1^{(n+1)}$. When $n = 7$, this is less than 0.0001.

(b) The n^{th} derivative of $f(x) = \sqrt{x}$ is bounded by 0.1 on intervals containing 3 and 4. Thus $|R_n(\pi)| \le \frac{0.1}{(n+1)!}(1)^{(n+1)}$. When $n = 4$, this is less than 0.0001.

(c) The n^{th} derivative of $f(x) = \cos x$ is bounded by 1 on intervals containing 0 and $\pi/3$. Thus $|R_n(\pi/3)| \le \frac{1}{(n+1)!}(\pi/3)^{(n+1)}$. When $n = 7$, this is less than 0.0001. Since the Maclaurin polynomial of $\cos x$ only uses even powers, we can actually just use $n = 6$.

(d) The n^{th} derivative of $f(x) = \sin x$ is bounded by 1 on intervals containing 0 and π. Thus $|R_n(\pi)| \le \frac{1}{(n+1)!}(\pi)^{(n+1)}$. When $n = 12$, this is less than 0.0001. Since the Maclaurin polynomial of $\sin x$ only uses odd powers, we can actually just use $n = 11$.

9.10.5 (a) The n^{th} term is $\frac{1}{n!}x^n$.

(b) The n^{th} term is: when n is even, $\frac{(-1)^{n/2}}{n!}x^n$; when n is odd, 0.

(c) The n^{th} term is x^n.

(d) The n^{th} term is $(-1)^n x^n$.

(e) The n^{th} term is $(-1)^n \frac{(x-1)^n}{n}$.

9.10.6 (a) $1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \dfrac{1}{24}x^4$

(b) $3 + 15x + \dfrac{75}{2}x^2 + \dfrac{375}{6}x^3 + \dfrac{1875}{24}x^4$

(c) $1 + 2x - 2x^2 + 4x^3 - 10x^4$

9.11.1 (a) All derivatives of e^x are e^x which evaluate to 1 at $x = 0$.

The Taylor series starts $1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$;

the Taylor series is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(b) All derivatives of $\sin x$ are either $\pm\cos x$ or $\pm\sin x$, which evaluate to ± 1 or 0 at $x = 0$. The Taylor series starts $0 + x + 0x^2 - \frac{1}{6}x^3 + 0x^4 + \frac{1}{120}x^5$;

the Taylor series is $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

(c) The n^{th} derivative of $1/(1-x)$ is $f^{(n)}(x) = (n)!/(1-x)^{n+1}$, which evaluates to $n!$ at $x=0$.
The Taylor series starts $1 + x + x^2 + x^3 + \cdots$;

the Taylor series is $\displaystyle\sum_{n=0}^{\infty} x^n$

(d) The derivative of $\tan^{-1} x$ is $1/(1+x^2)$. Taking successive derivatives using the Quotient Rule, the derivatives of $\tan^{-1} x$ fall into two categories in terms of their evaluation at $x=0$.

When n is even, $f^{(n)}(x) = (-1)^{(n-1)/2}\dfrac{p(x)}{(1+x^2)^n}$, where $p(x)$ is a polynomial such that $p(0) = 0$. Hence $f^{(n)}(0) = 0$ when n is even.

When n is odd, $f^{(n)}(x) = (-1)^{(n-1)/2}\dfrac{p(x)}{(1+x^2)^n}$, where $p(x)$ is a polynomial such that $p(0) = (n-1)!$. Hence $f^{(n)}(0) = (-1)^{(n-1)/2}(n-1)!$ when n is odd. (The unusual power of (-1) is such that every other odd term is negative.)

The Taylor series starts $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$; by reindexing to only obtain odd powers of x, we get

the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

9.11.2 (a) The Taylor series starts $0 - (x-\pi/2) + 0x^2 + \frac{1}{6}(x-\pi/2)^3 + 0x^4 - \frac{1}{120}(x-\pi/2)^5$;

the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi/2)^{2n+1}}{(2n+1)!}$

(b) The Taylor series starts $1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5$;

the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^n(x-1)^n$

(c) $f^{(n)}(x) = (-1)^n e^{-x}$; at $x=0$, $f^{(n)}(0) = -1$ when n is odd and $f^{(n)}(0) = 1$ when n is even.
The Taylor series starts $1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \cdots$;

the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$.

(d) $f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(1+x)^n}$; at $x=0$, $f^{(n)}(0) = (-1)^{n+1}(n-1)!$

The Taylor series starts $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$;

the Taylor series is $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$.

(e) $f^{(n)}(x) = (-1)^{n+1}\frac{n!}{(x+1)^{n+1}}$; at $x=1$, $f^{(n)}(1) = (-1)^{n+1}\frac{n!}{2^{n+1}}$

The Taylor series starts $\frac{1}{2} + \frac{1}{4}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 \cdots$;

the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{2^{n+1}}$.

(f) The derivatives of $\sin x$ are $\pm\cos x$ and $\pm\sin x$; at $x = \pi/4$, these derivatives evaluate to $\pm\sqrt{2}/2$.

The Taylor series starts $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\pi/4)-\frac{\sqrt{2}}{2}\frac{(x-\pi/4)^2}{2}-\frac{\sqrt{2}}{2}\frac{(x-\pi/4)^3}{3!}+\frac{\sqrt{2}}{2}\frac{(x-\pi/4)^4}{4!}+\frac{\sqrt{2}}{2}\frac{(x-\pi/4)^5}{5!}\cdots$.
Note how the signs are "even, even, odd, odd, even, even, odd, odd,... We saw signs like these in Example ?? of Section 9.1; one way of producing such signs is to raise (-1) to a special quadratic power. While many possibilities exist, one such quadratic is $(n+3)(n+4)/2$.

Thus the Taylor series is $\displaystyle\sum_{n=0}^{\infty}(-1)^{\frac{(n+3)(n+4)}{2}}\frac{\sqrt{2}}{2}\frac{(x-\pi/4)^n}{n!}$.

9.11.3 (a) The following argument is essentially the same as that given for $f(x) = \cos x$ in Example ??.

Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$\big|R_n(x)\big| \le \frac{\max\big|f^{(n+1)}(z)\big|}{(n+1)!}\big|x^{(n+1)}\big|.$$

Since all derivatives of $\sin x$ are $\pm\cos x$ or $\pm\sin x$, whose magnitudes are bounded by 1, we can state

$$\big|R_n(x)\big| \le \frac{1}{(n+1)!}\big|x^{(n+1)}\big|.$$

For any x, $\displaystyle\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)!} = 0$. Thus by the Squeeze Theorem, we conclude that $\displaystyle\lim_{n\to\infty}R_n(x) = 0$ for all x, and hence

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x.$$

(b) Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$\big|R_n(x)\big| \le \frac{\max\big|f^{(n+1)}(z)\big|}{(n+1)!}\big|x^{(n+1)}\big|,$$

where z is between 0 and x.

If $x > 0$, then $z < x$ and $f^{(n+1)}(z) = e^z < e^x$. If $x < 0$, then $x < z < 0$ and $f^{(n+1)}(z) = e^z < 1$. So given a fixed x value, let $M = \max\{e^x, 1\}$; $f^{(n)}(z) < M$. This allows us to state

$$\big|R_n(x)\big| \le \frac{M}{(n+1)!}\big|x^{(n+1)}\big|.$$

For any x, $\displaystyle\lim_{n\to\infty}\frac{M}{(n+1)!}\big|x^{(n+1)}\big| = 0$. Thus by the Squeeze Theorem, we conclude that $\displaystyle\lim_{n\to\infty}R_n(x) = 0$ for all x, and hence

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \text{for all } x.$$

(c) Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$\left|R_n(x)\right| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|(x-1)^{(n+1)}\right|,$$

where z is between 1 and x.

Note that $\left|f^{(n+1)}(x)\right| = \frac{n!}{x^{n+1}}$.

We consider the cases when $x > 1$ and when $x < 1$ separately.

If $x > 1$, then $1 < z < x$ and $f^{(n+1)}(z) = \frac{n!}{z^{n+1}} < n!$. Thus

$$\left|R_n(x)\right| \leq \frac{n!}{(n+1)!}\left|(x-1)^{(n+1)}\right| = \frac{(x-1)^{n+1}}{n+1}.$$

For a fixed x,

$$\lim_{n\to\infty} \frac{(x-1)^{n+1}}{n+1} = 0.$$

If $0 < x < 1$, then $x < z < 1$ and $f^{(n+1)}(z) = \frac{n!}{z^{n+1}} < \frac{n!}{x^{n+1}}$. Thus

$$\left|R_n(x)\right| \leq \frac{n!/x^{n+1}}{(n+1)!}\left|(x-1)^{(n+1)}\right| = \frac{x^{n+1}}{n+1}(1-x)^{n+1}.$$

Since $0 < x < 1$, $x^{n+1} < 1$ and $(1-x)^{n+1} < 1$. We can then extend the inequality from above to state

$$\left|R_n(x)\right| \leq \frac{x^{n+1}}{n+1}(1-x)^{n+1} < \frac{1}{n+1}.$$

As $n \to \infty$, $1/(n+1) \to 0$. Thus by the Squeeze Theorem, we conclude that $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x, and hence

$$\ln x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} \quad \text{for all } 0 < x \leq 2.$$

(d) Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$\left|R_n(x)\right| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{(n+1)}\right|,$$

where z is between 0 and x.

Note that $\left|f^{(n+1)}(x)\right| = \frac{(n+1)!}{(1-x)^{n+2}}$.

If $0 < x < 1$, then $0 < z < x$ and $f^{(n+1)}(z) = \frac{(n+1)!}{(1-z)^{n+2}} < \frac{(n+1)!}{(1-x)^{n+2}}$. Thus

$$\left|R_n(x)\right| \leq \frac{(n+1)!}{(1-x)^{n+2}}\frac{1}{(n+1)!}\left|x^{n+1}\right| = \frac{(x-1)^{n+1}}{n+1}.$$

For a fixed x,

$$\lim_{n\to\infty} \frac{(x-1)^{n+1}}{n+1} = 0,$$

hence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ on } (-1,0).$$

9.11.4 (a) Given $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$,

$\cos(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{(-x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$, as all powers in the series are even.

(b) Given $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$,

$\sin(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{(-x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{-x^{2n+1}}{(2n+1)!} = -\sin x$, as all powers in the series are odd.

(c) Given $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$,

$\frac{d}{dx}\big(\sin x\big) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$. (The summation still starts at $n = 0$ as there was no constant term in the expansion of $\sin x$).

(d) Given $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$,

$\frac{d}{dx}\big(\cos x\big) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right) = \sum_{n=1}^{\infty} (-1)^n \frac{(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!}$. We can re-index this summation to start at $n = 0$ by replacing n with $n+1$ in the summation:

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}.$$

Note that this series has the opposite sign of the Taylor series for $\sin x$; thus $\frac{d}{dx}(\cos x) = -\sin x$.

9.11.5 (a) $\sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$.

(b) $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$.

(c) $\sum_{n=0}^{\infty} (-1)^n \frac{(2x+3)^{2n+1}}{(2n+1)!}$.

(d) $\sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+1}}{(2n+1)}$.

(e) $x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}$

(f) $1+\dfrac{x}{2}-\dfrac{5x^2}{8}-\dfrac{3x^3}{16}$

9.11.6 (a) $1+\dfrac{x}{2}-\dfrac{x^2}{8}+\dfrac{x^3}{16}-\dfrac{5x^4}{128}$

(b) $1-\dfrac{x}{2}+\dfrac{3x^2}{8}-\dfrac{5x^3}{16}+\dfrac{35x^4}{128}$

(c) $1+\dfrac{x}{3}-\dfrac{x^2}{9}+\dfrac{5x^3}{81}-\dfrac{10x^4}{243}$

(d) $1+4x+6x^2+4x^3+x^4$ (note the series is finite, and the formula still applies)

9.11.7 (a) $\displaystyle\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\,dx \approx \int_0^{\sqrt{\pi}} \left(x^2-\frac{x^6}{6}+\frac{x^{10}}{120}-\frac{x^{14}}{5040}\right)dx = 0.8877$

(b) $\displaystyle\int_0^{\pi^2/4} \cos\left(\sqrt{x}\right)\,dx \approx \int_0^{\pi^2/4} \left(1-\frac{x}{2}+\frac{x^2}{24}-\frac{x^3}{720}\right)\,dx = 1.1412.$ (Actual answer: $\pi-2$)

9.11.8 (a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!,\ R=\infty$

(b) $\displaystyle\sum_{n=0}^{\infty} x^n/n!,\ R=\infty$

(c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x-5)^n}{5^{n+1}},\ R=5$

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n},\ R=1$

(e) $\displaystyle\ln(2)+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-2)^n}{n2^n},\ R=2$

(f) $\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n,\ R=1$

(g) $\displaystyle 1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!2^n}x^n = 1+\sum_{n=1}^{\infty}\frac{(2n-1)!}{2^{2n-1}(n-1)!\,n!}x^n,\ R=1$

(h) $x+x^3/3$

(i) $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{4n+1}/(2n)!$

(j) $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{n+1}/n!$

10.1.2 $y = \arctan t + C$

10.1.3 $y = \dfrac{t^{n+1}}{n+1} + 1$

10.1.4 $y = t\ln t - t + C$

10.1.5 $y = n\pi$, for any integer n.

10.1.6 none

10.1.7 $y = \pm\sqrt{t^2 + C}$

10.1.8 $y = \pm 1$, $y = (1 + Ae^{2t})/(1 - Ae^{2t})$

10.1.9 $y^4/4 - 5y = t^2/2 + C$

10.1.10 $y = (2t/3)^{3/2}$

10.1.11 $y = M + Ae^{-kt}$

10.1.12 $\dfrac{10\ln(15/2)}{\ln 5} \approx 2.52$ minutes

10.1.13 $y = \dfrac{M}{1 + Ae^{-Mkt}}$

10.1.14 $y = 2e^{3t/2}$

10.1.15 $t = -\dfrac{\ln 2}{k}$

10.1.16 $600e^{-6\ln 2/5} \approx 261$ mg; $\dfrac{5\ln 300}{\ln 2} \approx 41$ days

10.1.17 $100e^{-200\ln 2/191} \approx 48$ mg; $\dfrac{5730\ln 50}{\ln 2} \approx 32339$ years

10.1.18 $y = y_0 e^{t\ln 2}$

10.1.19 $500e^{-5\ln 2/4} \approx 210$ g

10.2.1 $y = Ae^{-5t}$

10.2.2 $y = Ae^{2t}$

10.2.3 $y = Ae^{-\arctan t}$

10.2.4 $y = Ae^{-t^3/3}$

10.2.5 $y = 4e^{-t}$

10.2.6 $y = -2e^{3t-3}$

10.2.7 $y = e^{1+\cos t}$

10.2.8 $y = e^2 e^{-e^t}$

10.2.9 $y = 0$

10.2.10 $y = 0$

10.2.11 $y = 4t^2$

10.2.12 $y = -2e^{(1/t)-1}$

10.2.13 $y = e^{1-t^{-2}}$

10.2.14 $y = 0$

10.2.15 $k = \ln 5$, $y = 100e^{-t\ln 5}$

10.2.16 $k = -12/13$, $y = \exp(-13t^{1/13})$

10.2.17 $y = 10^6 e^{t\ln(3/2)}$

10.2.18 $y = 10e^{-t\ln(2)/6}$

10.3.1 $y = Ae^{-4t} + 2$

10.3.2 $y = Ae^{2t} - 3$

10.3.3 $y = Ae^{-(1/2)t^2} + 5$

10.3.4 $y = Ae^{-e^t} - 2$

10.3.5 $y = Ae^t - t^2 - 2t - 2$

10.3.6 $y = Ae^{-t/2} + t - 2$

10.3.7 $y = At^2 - \dfrac{1}{3t}$

10.3.8 $y = \dfrac{c}{t} + \dfrac{2}{3}\sqrt{t}$

10.3.9 $y = A\cos t + \sin t$

10.3.10 $y = \dfrac{A}{\sec t + \tan t} + 1 - \dfrac{t}{\sec t + \tan t}$

10.4.1 $y(1) \approx 1.355$

10.4.2 $y(1) \approx 40.31$

10.4.3 $y(1) \approx 1.05$

10.4.4 $y(1) \approx 2.30$

10.5.1 $\dfrac{\omega + 1}{2\omega}e^{\omega t} + \dfrac{\omega - 1}{2\omega}e^{-\omega t}$

10.5.2 $2\cos(3t) + 5\sin(3t)$

10.5.3 $-(1/4)e^{-5t} + (5/4)e^{-t}$

10.5.4 $-2e^{-3t} + 2e^{4t}$

10.5.5 $5e^{-6t} + 20te^{-6t}$

10.5.6 $(16t - 3)e^{4t}$

10.5.7 $-2\cos(\sqrt{5}t) + \sqrt{5}\sin(\sqrt{5}t)$

10.5.8 $-\sqrt{2}\cos t + \sqrt{2}\sin t$

10.5.9 $e^{-6t}\,(4\cos t + 24\sin t)$

10.5.10 $2e^{-3t}\sin(3t)$

10.5.11 $2\cos(2t - \pi/6)$

10.5.12 $5\sqrt{2}\cos(10t - \pi/4)$

10.5.13 $\sqrt{2}e^{-2t}\cos(3t - \pi/4)$

10.5.14 $5e^{4t}\cos(3t + \arcsin(4/5))$

10.5.15 $(2\cos(5t) + \sin(5t))e^{-2t}$

10.5.16 $-(1/2)e^{-2t}\sin(2t)$

10.6.1 $Ae^{5t} + Bte^{5t} + (6/169)\cos t - (5/338)\sin t$

10.6.2 $Ae^{-\sqrt{2}t} + Bte^{-\sqrt{2}t} + 5$

10.6.3 $A\cos(4t) + B\sin(4t) + (1/2)t^2 + (3/16)t - 5/16$

10.6.4 $A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t) - (\cos(5t) + \sin(5t))/23$

10.6.5 $e^t(A\cos t + B\sin t) + e^{2t}/2$

10.6.6 $Ae^{\sqrt{6}t} + Be^{-\sqrt{6}t} + 2 - t/3 - e^{-t}/5$

10.6.7 $Ae^{-3t} + Be^{2t} - (1/5)te^{-3t}$

10.6.8 $Ae^t + Be^{3t} + (1/2)te^{3t}$

10.6.9 $A\cos(4t) + B\sin(4t) + (1/8)t\sin(4t)$

10.6.10 $A\cos(3t) + B\sin(3t) - (1/2)t\cos(3t)$

10.6.11 $Ae^{-6t} + Bte^{-6t} + 3t^2e^{-6t}$

10.6.12 $Ae^{4t} + Bte^{4t} - t^2e^{4t}$

10.6.13 $Ae^{-t} + Be^{-5t} + (4/5)$

10.6.14 $Ae^{4t} + Be^{-3t} + (1/144) - (t/12)$

10.6.15 $A\cos(\sqrt{5}t) + B\sin(\sqrt{5}t) + 8\sin(2t)$

10.6.16 $Ae^{2t} + Be^{-2t} + te^{2t}$

10.6.17 $4e^t + e^{-t} - 3t - 5$

10.6.18 $-(4/27)\sin(3t) + (4/9)t$

10.6.19 $e^{-6t}(2\cos t + 20\sin t) + 2e^{-4t}$

10.6.20 $\left(-\frac{23}{325}\cos(3t) + \frac{592}{975}\sin(3t)\right) + \frac{23}{325}\cos t - \frac{11}{325}\sin t$

10.6.21 $e^{-2t}(A\sin(5t) + B\cos(5t)) + 8\sin(2t) + 25\cos(2t)$

10.6.22 $e^{-2t}(A\sin(2t) + B\cos(2t)) + (14/195)\sin t - (8/195)\cos t$

10.7.1 $A\sin(t) + B\cos(t) -$
$\cos t \ln|\sec t + \tan t|$

10.7.2 $A\sin(t) + B\cos(t) + \dfrac{1}{5}e^{2t}$

10.7.3 $A\sin(2t) + B\cos(2t) + \cos t - \sin t\cos t\ln|\sec t + \tan t|$

10.7.4 $A\sin(2t) + B\cos(2t) + \dfrac{1}{2}\sin(2t)\sin^2(t) + \dfrac{1}{2}\sin(2t)\ln|\cos t| - \dfrac{t}{2}\cos(2t) + \dfrac{1}{4}\sin(2t)\cos(2t)$

10.7.5 $Ae^{2t} + Be^{-3t} + \dfrac{t^3}{15}e^{2t} - \left(\dfrac{t^2}{5} - \dfrac{2t}{25} + \dfrac{2}{125}\right)\dfrac{e^{2t}}{5}$

10.7.6 $Ae^t\sin t + Be^t\cos t - e^t\cos t\ln|\sec t + \tan t|$

10.7.7 $Ae^t\sin t + Be^t\cos t - \dfrac{1}{10}\cos t(\cos^3 t + 3\sin^3 t - 2\cos t - \sin t) + \dfrac{1}{10}\sin t(\sin^3 t - 3\cos^3 t -$
$2\sin t + \cos t) = \dfrac{1}{10}\cos(2t) - \dfrac{1}{20}\sin(2t)$

Inside Cover Material

Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$
28. $\frac{d}{dx}(\text{sech} x) = -\text{sech} x \tanh x$
29. $\frac{d}{dx}(\text{csch} x) = -\text{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\text{csch}^2 x$
31. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}\left(\text{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}\left(\text{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Integration Rules

1. $\int c \cdot f(x)\ dx = c \int f(x)\ dx$

2. $\int f(x) \pm g(x)\ dx = \int f(x)\ dx \pm \int g(x)\ dx$

3. $\int 0\ dx = C$

4. $\int 1\ dx = x + C$

5. $\int x^n\ dx = \frac{1}{n+1} x^{n+1} + C,\ n \neq -1$
$n \neq -1$

6. $\int e^x\ dx = e^x + C$

7. $\int a^x\ dx = \frac{1}{\ln a} \cdot a^x + C$

8. $\int \frac{1}{x}\ dx = \ln |x| + C$

9. $\int \cos x\ dx = \sin x + C$

10. $\int \sin x\ dx = -\cos x + C$

11. $\int \tan x\ dx = -\ln|\cos x| + C$

12. $\int \sec x\ dx = \ln|\sec x + \tan x| + C$

13. $\int \csc x\ dx = -\ln|\csc x + \cot x| + C$

14. $\int \cot x\ dx = \ln|\sin x| + C$

15. $\int \sec^2 x\ dx = \tan x + C$

16. $\int \csc^2 x\ dx = -\cot x + C$

17. $\int \sec x \tan x\ dx = \sec x + C$

18. $\int \csc x \cot x\ dx = -\csc x + C$

19. $\int \cos^2 x\ dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

20. $\int \sin^2 x\ dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

21. $\int \frac{1}{x^2+a^2}\ dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

22. $\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

23. $\int \frac{1}{x\sqrt{x^2-a^2}}\ dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

24. $\int \cosh x\ dx = \sinh x + C$

25. $\int \sinh x\ dx = \cosh x + C$

26. $\int \tanh x\ dx = \ln(\cosh x) + C$

27. $\int \coth x\ dx = \ln|\sinh x| + C$

28. $\int \frac{1}{\sqrt{x^2-a^2}}\ dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$

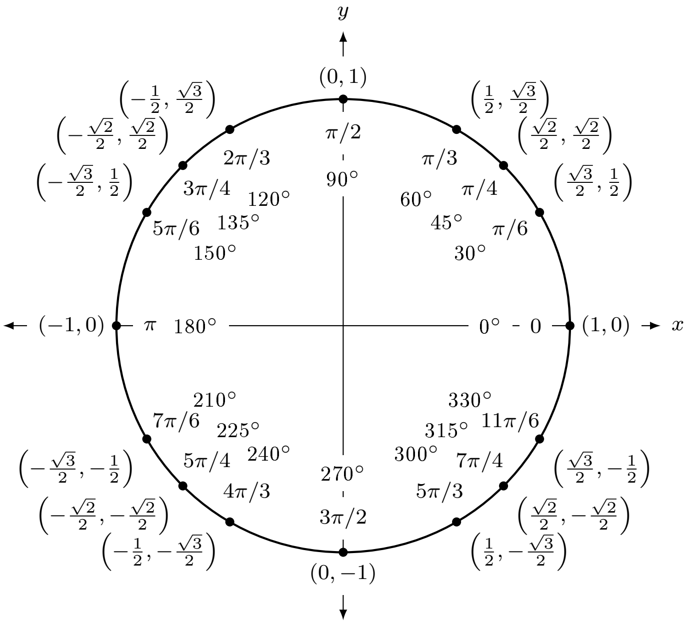
29. $\int \frac{1}{\sqrt{x^2+a^2}}\ dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$

30. $\int \frac{1}{a^2-x^2}\ dx = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right| + C$

31. $\int \frac{1}{x\sqrt{a^2-x^2}}\ dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$

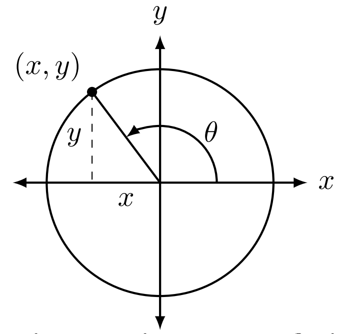
32. $\int \frac{1}{x\sqrt{x^2+a^2}}\ dx = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

The Unit Circle



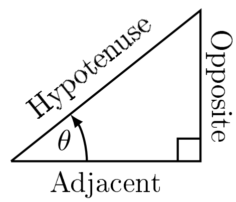
Definitions of the Trigonometric Functions

Unit Circle Definition



$$\sin\theta = y \qquad \cos\theta = x$$
$$\csc\theta = \frac{1}{y} \qquad \sec\theta = \frac{1}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Right Triangle Definition



$$\sin\theta = \frac{\mathrm{O}}{\mathrm{H}} \qquad \csc\theta = \frac{\mathrm{H}}{\mathrm{O}}$$
$$\cos\theta = \frac{\mathrm{A}}{\mathrm{H}} \qquad \sec\theta = \frac{\mathrm{H}}{\mathrm{A}}$$
$$\tan\theta = \frac{\mathrm{O}}{\mathrm{A}} \qquad \cot\theta = \frac{\mathrm{A}}{\mathrm{O}}$$

Common Trigonometric Identities

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Power–Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Even/Odd Identities

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x$$
$$\sec(-x) = \sec x$$
$$\cot(-x) = -\cot x$$

Product to Sum Formulas

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

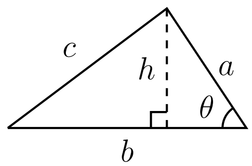
Areas and Volumes

Triangles

$h = a \sin\theta$

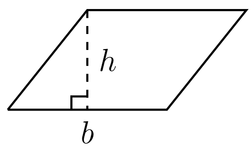
Area $= \frac{1}{2}bh$

Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$



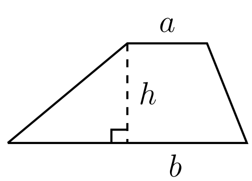
Parallelograms

Area $= bh$



Trapezoids

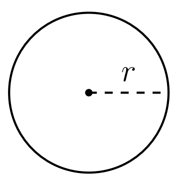
Area $= \frac{1}{2}(a+b)h$



Circles

Area $= \pi r^2$

Circumference $= 2\pi r$

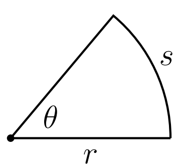


Sectors of Circles

θ in radians

Area $= \frac{1}{2}\theta r^2$

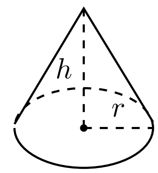
$s = r\theta$



Right Circular Cone

Volume $= \frac{1}{3}\pi r^2 h$

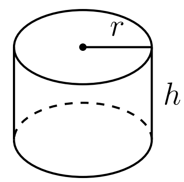
Surface Area $= \pi r\sqrt{r^2+h^2} + \pi r^2$



Right Circular Cylinder

Volume $= \pi r^2 h$

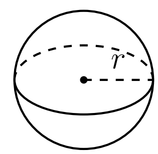
Surface Area $= 2\pi rh + 2\pi r^2$



Sphere

Volume $= \frac{4}{3}\pi r^3$

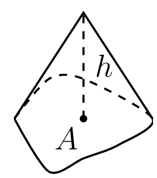
Surface Area $= 4\pi r^2$



General Cone

Area of Base $= A$

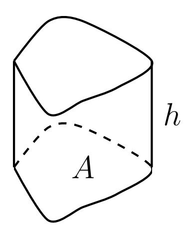
Volume $= \frac{1}{3}Ah$



General Right Cylinder

Area of Base $= A$

Volume $= Ah$



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \leq b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x - a)(x + a)$ $\qquad$ $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$ $\qquad$ $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x + y)^2 = x^2 + 2xy + y^2$ $\qquad$ $(x - y)^2 = x^2 - 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ $\qquad$ $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ $\qquad$ $(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$$ab + ac = a(b+c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a-b}{c-d} = \frac{b-a}{d-c} \qquad \frac{ab+ac}{a} = b+c$$

Exponents and Radicals

$$a^0 = 1, \quad a \neq 0 \qquad (ab)^x = a^x b^x \qquad a^x a^y = a^{x+y} \qquad \sqrt{a} = a^{1/2} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n} \qquad a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{2}\big[f(x_1) + 2f(x_2) + 2f(x_3) + ... + 2f(x_n) + f(x_{n+1})\big]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\big[\max|f''(x)|\big]$

Simpson's Rule:

$$\int_a^b f(x)\ dx \approx \frac{\Delta x}{3}\big[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + ... + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\big]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\big[\max|f^{(4)}(x)|\big]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\ dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\ dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\ dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\, d(y)\, \ell(y)\ dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + ... + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum $= \dfrac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		Sum $= \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\ dn$ is convergent	$\int_1^{\infty} a(n)\ dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$

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