Math 1003

Section 3.9

Dr. Tim Alderson

University of New Brunswick Saint John

Fall 2015

Outline

Related Rates

- Suppose we are pumping a balloon with air.
- The balloon's volume is increasing.
- The balloon's radius is increasing.
- The rates of increase of these quantities are related to one another.
- It is easier to measure the rate of increase of volume.
- In a related rates problem, we compute the rate of change of one quantity in terms of the rate of change of another (which may be more easily measured).

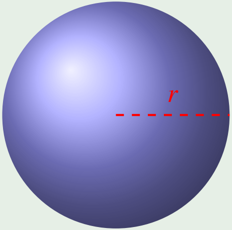
Related Rates: Guideline

- **Draw a picture, introduce notation:** If possible, draw a schematic picture with all the relevant information, introduce variables/labels.
- **Identify:** Identify quantities whose rates of change are either given or are required.
- **Find an equation:** Find an equation that relates only the quantities with rates that are given or required.
- **Differentiate the equation with respect to t (time):** This will often involve implicit differentiation.
- **Evaluate the appropriate equation at the desired values:** The known/given values should allow you solve for the required rate.

Example

Air is being pumped into a spherical balloon such that its volume changes at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

- **Draw and introduce notation:**
- At time t seconds, let $V = V(t)$ be the balloon's volume (cm^3).
- Let $r = r(t)$ denote its radius (cm).
- **Identify given/req'd rates:** Given: $V'(t) = 100$ cm^3/s.
- Required: $r'(t)$ when $r = 25$ cm.



- **Find an equation** relating the quantities with given/required rates.

$$V = \frac{4}{3}\pi r^3 \ \text{cm}^3$$

- Now: **Differentiate** (implicitly) this equation with respect to time.

Example

Air is being pumped into a spherical balloon such that its volume changes at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

$$V = \frac{4}{3}\pi r^3 \quad \longrightarrow \quad \begin{aligned} V'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi r^3\right) \\ &= \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \\ &= 4\pi r^2 \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \end{aligned}$$

Now **evaluate** the above expression when $r = 25$ and $\mathrm{d}V/\mathrm{d}t = 100$ in order to solve for $\frac{\mathrm{d}r}{\mathrm{d}t}$:

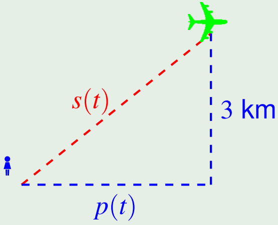
$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{4\pi r^2} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{4\pi(25^2)} \cdot 100 = \frac{1}{25\pi} \text{ cm/s}$$

Therefore the radius is increasing at a rate of $\frac{1}{25\pi}$ cm/s when $r = 25$ cm.

Example

A plane flies directly overhead of you at 500 km/h maintaining an altitude of 3 km. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 km from you?

- We'll use our general guideline to solve this problem. To start, **draw a picture** and introduce variables/labels.



At time t (hours), let

$p = p(t)$ be the distance (km) between you and the point on the ground directly below the plane,

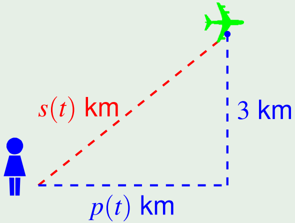
$s = s(t)$ be the distance (km) between yourself and the plane.

Identify and given and required rates:

Given: $\frac{dp}{dt} = p'(t) = 500$ km/h

Required: Find $\frac{ds}{dt} = s'(t)$ when $p = 4$.

Example



Given: $p'(t) = 500$ km/h
Required: $s'(t)$ when $p = 4$.
Find an equation relating quantities with given/required rates: (ty Pythagoras)

$$p^2 + 3^2 = s^2$$

Differentiate (implicitly with respect to t):

$$2p \cdot p'(t) + 0 = 2s \cdot s'(t)$$

Now **evaluate** when $p(t) = 4$ and $p'(t) = 500$.
Additionally, we know that when $p(t) = 4$ we have $s^2 = p^2 + 3^2 = 16 + 9 = 25$, so $s = 5$. Putting this all into our last equation we get

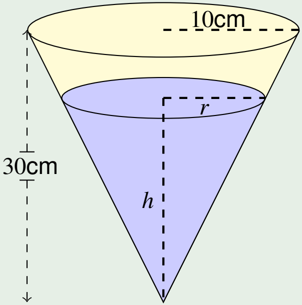
$$2(4)(500) = 2(5)s'(t),$$

thus $s'(t) = 400$ km/h.

Example

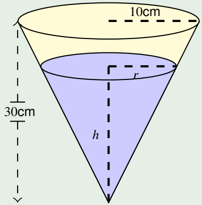
Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm. How fast is the water level rising when the water is 4 cm deep?

- To start, **draw a picture** and introduce variables/labels.



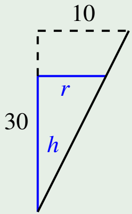
At time $= t$ seconds let
$r = r(t)$ be the surface radius (cm) of the water,
$h = h(t)$ be the height (cm) of the water,
$V = V(t)$ be the volume of water in the tank.
Identify and given and required rates:
Given: $\frac{dV}{dt} = 10$ cm^3/s
Required: Find $\frac{dh}{dt}$ when $h = 4$.

Example



Given: $\frac{\mathrm{d}V}{\mathrm{d}t} = 10$ cm/s, Find: $\frac{\mathrm{d}h}{\mathrm{d}t}$ when $h = 4$.
Find an equation relating quantities whose rates are given or required:

$$V = \frac{1}{3}\pi r^2 h$$



We do not have the rate of r either given or required, so we eliminate it from this equation. Similar triangles (or equivalently, by taking $\tan$ of the bottom angle) gives $\frac{r}{h} = \frac{10}{30}$, so $r = \frac{1}{3}h$.

This gives us

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 \cdot h = \frac{1}{27}\pi h^3$$

Example

$$V = \frac{\pi}{27} h^3$$

Differentiate (implicitly with respect to t):

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{9} h^2 \cdot h'(t)$$

We are interested in **evaluating** $h'(t)$ when $h = 4$cm and $V'(t) = 10$ cm^3/s. Putting this information into our last equation gives:

$$h'(t) = \frac{9}{\pi h^2} \cdot V'(t) = \frac{9}{\pi \cdot 4^2} \cdot 10 = \frac{90}{16\pi} \text{ cm/s}$$

thus $s'(t) = 400$ km/h.