# Math 1003

## Implicit Differentiation

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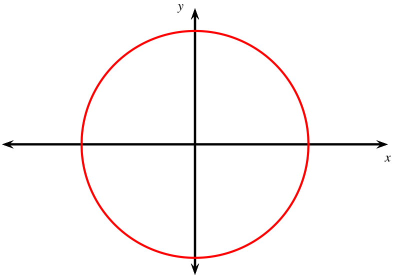
# Outline

1 Implicit Differentiation

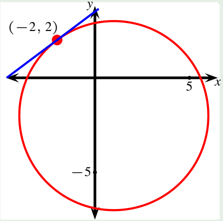
# Implicit Differentiation

- So far, we have seen functions with formulas that express one varable explicitly in terms of the other.
- $y = \sqrt{x^3 + 1}$, $y = x \sin x$, etc.
- Some functions are given implicitly by a relation between $x$ and $y$.
- $x^2 + y^2 = 1$ isn't the equation of any one function.
- Implicitly it gives two functions: $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$.
- How do we differentiate these functions?
- Differentiate both sides with respect to $x$, and then solve for $y'$.



## Example

Find an equation of the tangent line to $(x-1)^2+(y+2)^2=25$ at $(-2,2)$.




Find $\frac{dy}{dx}$, given $(x-1)^2 + (y+2)^2 = 25$ :

$$\frac{d}{dx}\left((x-1)^2\right) + \frac{d}{dx}\left((y+2)^2\right) = \frac{d}{dx}(25)$$

$$2(x-1)\frac{d}{dx}(x-1) + 2(y+2)\frac{d}{dx}(y+2) = 0$$

$$2(x-1)(1) + 2(y+2)\left(\frac{dy}{dx}\right) = 0$$

$$2(y+2)\left(\frac{dy}{dx}\right) = 2(1-x)$$

$$\frac{dy}{dx} = \frac{1-x}{y+2}$$

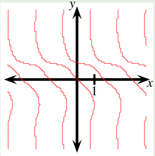
Plug in $(-2,2)$ :

$$\frac{dy}{dx} = \frac{1-(-2)}{2+2} = \frac{3}{4}$$

Point-slope form:

$$y-2 = \frac{3}{4}(x+2)$$

## Example




Find $y'$ as an expression of $x$ and $y$.

$$
\begin{aligned}
\sin(2(x+y)) &= y^2\cos(2x).\\
\frac{d}{dx}\left(\sin(2(x+y))\right) &= \frac{d}{dx}\left(y^2\cos(2x)\right)\\
\cos(2(x+y))\frac{d}{dx}(2(x+y)) &= \frac{d}{dx}(y^2)\cos(2x) + (y^2)\frac{d}{dx}(\cos(2x))\\
\cos(2(x+y))\left(2+2y'\right) &= 2yy'\cos(2x) + y^2\left(-\sin(2x)\right)\frac{d}{dx}(2x)\\
2\cos(2(x+y))\left(1+y'\right) &= 2yy'\cos(2x) - y^2\sin(2x)\,2\\
\cos(2(x+y)) + y'\cos(2(x+y)) &= yy'\cos(2x) - y^2\sin(2x)\\
y'\cos(2(x+y)) - yy'\cos(2x) &= -\cos(2(x+y)) - y^2\sin(2x)\\
y'\left(\cos(2(x+y)) - y\cos(2x)\right) &= -\cos(2(x+y)) - y^2\sin(2x)\\
y' &= \frac{-\cos(2(x+y)) - y^2\sin(2x)}{\cos(2(x+y)) - y\cos(2x)}.
\end{aligned}
$$

## Example

Let $x^4 + y^4 = 16$. Find $y''$.

$$4x^3 + 4y^3y' = 0$$

$$y' = -\frac{x^3}{y^3}.$$

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{x^3}{y^3}\right) = -\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)y^3 - x^3\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)}{\left(y^3\right)^2}$$

$$= -\frac{(3x^2)y^3 - x^3(3y^2y')}{y^6} = -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3x^2(y^3 + \frac{x^4}{y})}{y^6} = -\frac{3x^2\left(\frac{y^4+x^4}{y}\right)}{y^6}$$

$$= -\frac{3x^2(y^4 + x^4)}{y^7} = -\frac{3x^2(16)}{y^7} = -48\frac{x^2}{y^7}.$$

## Example (Implicit Differentiation)

Find $y'$ if $\tan xy = x^2 - y^2$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\tan xy) &= \frac{\mathrm{d}}{\mathrm{d}x}(x^2 - y^2) \\
(\sec^2 xy)\frac{\mathrm{d}}{\mathrm{d}x}(xy) &= 2x - 2yy' \\
\left(y\frac{\mathrm{d}}{\mathrm{d}x}(x) + x\frac{\mathrm{d}}{\mathrm{d}x}(y)\right)\sec^2 xy &= 2x - 2yy' \\
(y(1) + x(y'))\sec^2 xy &= 2x - 2yy' \\
y\sec^2 xy + xy'\sec^2 xy &= 2x - 2yy' \\
xy'\sec^2 xy + 2yy' &= 2x - y\sec^2 xy \\
y'(x\sec^2 xy + 2y) &= 2x - y\sec^2 xy \\
y' &= \frac{2x - y\sec^2 xy}{x\sec^2 xy + 2y}.
\end{aligned}$$