Math 1003

Trigonometry

Dr. Tim Alderson

University of New Brunswick Saint John

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Outline

Angles

Angles can be measured in degrees or radians (abbreviated as rad). The angle of a complete rotation contains 360°, which is the same as 2π rad. Therefore

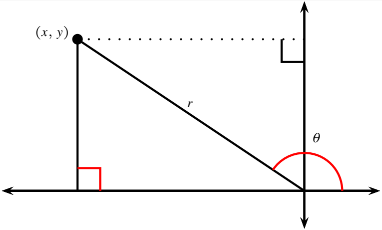
$$\pi \text{ rad} = 180^\circ.$$

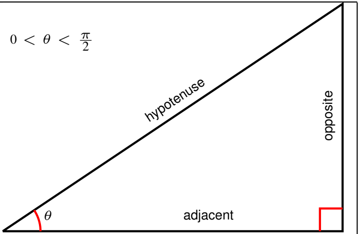
$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ, \qquad 1^\circ = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad}.$$

The following table shows the correspondence between degrees and radians for some common angles.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Trigonometric Functions and Right Angle Triangles

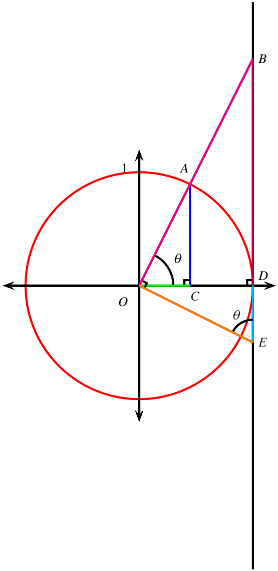




All angles		Acute angles	
$\cos\theta = \frac{x}{r}$	$\sec\theta = \frac{r}{x}$	$\cos\theta = \frac{\text{adj}}{\text{hyp}}$	$\sec\theta = \frac{\text{hyp}}{\text{adj}}$
$\sin\theta = \frac{y}{r}$	$\csc\theta = \frac{r}{y}$	$\sin\theta = \frac{\text{opp}}{\text{hyp}}$	$\csc\theta = \frac{\text{hyp}}{\text{opp}}$
$\tan\theta = \frac{y}{x}$	$\cot\theta = \frac{x}{y}$	$\tan\theta = \frac{\text{opp}}{\text{adj}}$	$\cot\theta = \frac{\text{adj}}{\text{opp}}$

- The trigonometric functions can be defined without requesting that the pt. (x, y) on the terminal arm of the angle lie on the unit circle.
- To do so we rescale by the distance r from the origin.
- The trig functions of acute θ (between 0 and $\frac{\pi}{2}$) can be interpreted as ratios of sides of right angle triangle with angle θ.

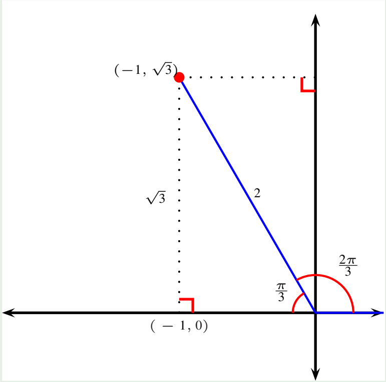
Geometric interpretation of all trigonometric functions



On picture: circle of radius 1 centered at point O with coordinates $(0, 0)$. Let $\angle DOB = \theta$. Let OB intersect the circle at point A. Then the coordinates of A are $(\cos\theta, \sin\theta)$.

$$\sin\theta = \frac{|AC|}{|OA|} = \frac{|AC|}{1} = |AC|.$$
$$\cos\theta = \frac{|OC|}{|OA|} = \frac{|OC|}{1} = |OC|.$$
$$\tan\theta = \frac{|BD|}{|OD|} = \frac{|BD|}{1} = |BD|.$$
$$\cot\theta = \frac{|DE|}{|OD|} = \frac{|DE|}{1} = |DE|.$$
$$\sec\theta = \frac{|OB|}{|OD|} = \frac{|OB|}{1} = |OB|.$$
$$\csc\theta = \frac{|OE|}{|DO|} = \frac{|OE|}{1} = |OE|.$$

Example



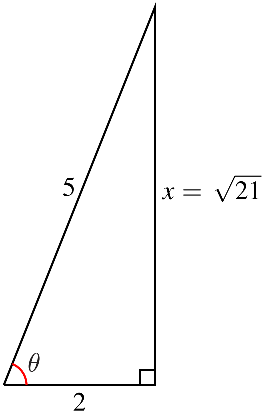
Find the exact values of the trigonometric functions of $\theta = \dfrac{2\pi}{3} = 120^\circ$.

$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \quad \tan\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\csc\left(\frac{2\pi}{3}\right) = \frac{2}{\sqrt{3}} \quad \sec\left(\frac{2\pi}{3}\right) = -\frac{2}{1} = -2 \quad \cot\left(\frac{2\pi}{3}\right) = -\frac{1}{\sqrt{3}}$$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \frac{\pi}{2}$, find the other five trigonometric functions of θ.

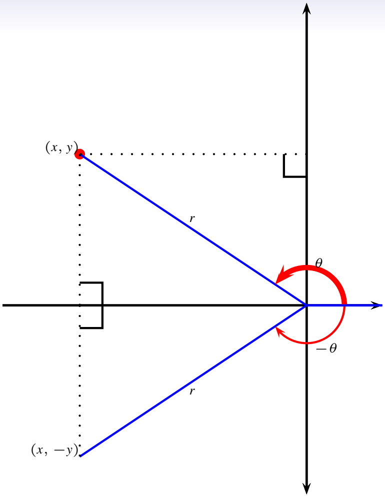


- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

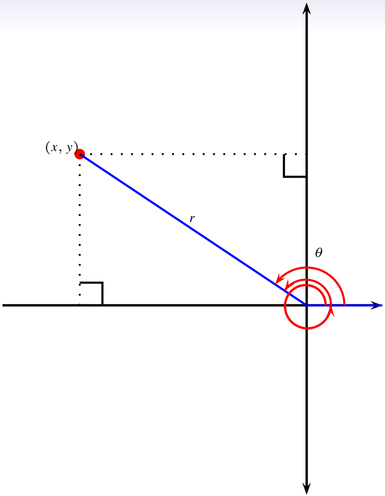
$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2}$$

$$\cot\theta = \frac{2}{\sqrt{21}}$$



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If (x, y) is on the terminal arm of the angle θ, then $(x, -y)$ is on the terminal arm of $-\theta$.
- $\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$.
- $\cos(-\theta) = \frac{x}{r} = \cos\theta$.
- $\sin$ is an odd function.
- $\cos$ is an even function.



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

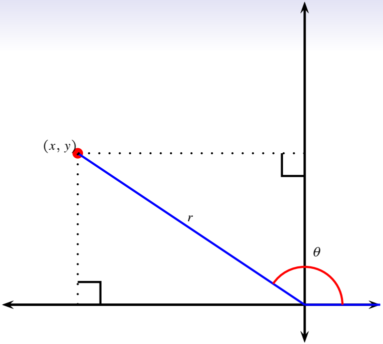
- 2π represents a full rotation.
- $\theta + 2\pi$ has the same terminal arm as θ.
- $\theta + 2\pi$ uses the same point (x, y) and the same length r.
- $\sin(\theta + 2\pi) = \sin\theta$.
- $\cos(\theta + 2\pi) = \cos\theta$.
- We say $\sin$ and $\cos$ are 2π-periodic.

Trigonometric Identities

Definition (Trigonometric Identity)

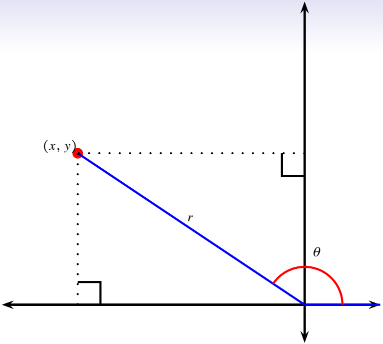
A trigonometric identity is an equality between the trigonometric functions in one or more variables that holds for all values of the involved variables in the domains of all of the expressions.

- By convention, when dealing with trigonometric identities we do not account for the domains of the involved expressions.
- For example, $\frac{\sin\theta}{\sin\theta} = 1$ is considered a valid trigonometric identity, although, when considered as a function, the left hand side is not defined for $\theta \neq 0$.



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

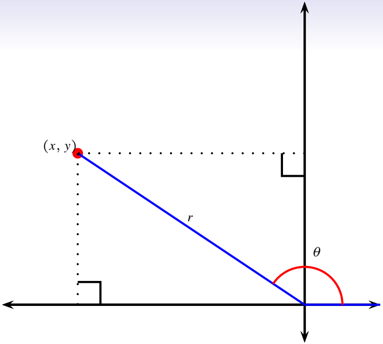
- $\csc\theta = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta}$
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$



$$\sin\theta = \tfrac{y}{r} \quad \csc\theta = \tfrac{r}{y}$$
$$\cos\theta = \tfrac{x}{r} \quad \sec\theta = \tfrac{r}{x}$$
$$\tan\theta = \tfrac{y}{x} \quad \cot\theta = \tfrac{x}{y}$$

$$\begin{aligned} & \sin^2\theta + \cos^2\theta \\ = \quad & \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ = \quad & \frac{y^2 + x^2}{r^2} \\ = \quad & \frac{r^2}{r^2} \\ = \quad & 1 \end{aligned}$$

Therefore $\sin^2\theta + \cos^2\theta = 1$.



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

Example ($\tan^2\theta + 1 = \sec^2\theta$)

Prove the identity $\tan^2\theta + 1 = \sec^2\theta$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\ \tan^2\theta + 1 &= \sec^2\theta \end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \end{aligned}$$

Substitute $-y$ for y, and use the fact that $\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$:

$$\begin{aligned} \sin(x-y) &= \sin x \cos y - \cos x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y \end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{array}{rcrcl} \sin(x+y) & = & \sin x \cos y & + & \cos x \sin y \\ \cos(x+y) & = & \cos x \cos y & - & \sin x \sin y \end{array}$$

To get the double angle formulas, substitute x for y:

$$\begin{array}{rcl} \sin(2x) & = & 2 \sin x \cos x \\ \cos(2x) & = & \cos^2 x - \sin^2 x \end{array}$$

Rewrite the second double angle formula in two ways, using $\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$:

$$\begin{array}{rcl} \cos(2x) & = & 2\cos^2 x - 1 \\ \cos(2x) & = & 1 - 2\sin^2 x \end{array}$$

To get the half-angle formulas, solve these equations for $\cos^2 x$ and $\sin^2 x$ respectively.

$$\cos^2 x = \frac{1+\cos(2x)}{2}, \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \end{aligned}$$

Divide the first equation by the second, and then cancel $\cos x \cos y$ from the top and bottom:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Do the same for the subtraction formulas:

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Here we explicitly permit the use of the Pythagorean identities and the double angle f-las:

$$\begin{aligned}\cos^2\theta + \sin^2\theta &= 1\\ \sin(2\theta) &= 2\sin\theta\cos\theta\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Example

Prove the trigonometric identity.

$$(\sin\theta + \cos\theta)^2 = 1 + \sin(2\theta)$$

We need to transform both sides to the same expression. In this case, we choose to transform the left hand side to the right:

$$\begin{aligned}(\sin\theta + \cos\theta)^2 &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\\ &= 1 + \sin(2\theta)\end{aligned} \quad \left| \quad \begin{aligned}&(A+B)^2 =\\ &A^2 + 2AB + B^2\end{aligned}\right.$$

Complex numbers definition

Definition

The set of complex numbers $\mathbb{C}$ is defined as the set

$$\{a + bi | a, b - \text{real numbers}\},$$

where the number i is a number for which

$$i^2 = -1 \quad .$$

The number i is called the imaginary unit.

- Complex numbers are added/subtracted according to the rule

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i \quad .$$

- Complex numbers are multiplied according to the rule

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (bc + ad)i \quad .$$

You will not be tested on the material in the following slide.

Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ *is Euler's/Napier's constant .*

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots$$

$\square$

Euler's Formula

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where $e \approx 2.71828$ *is Euler's/Napier's constant .*

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\begin{array}{rclllllll}
i \sin x & = & & ix & & -i\frac{x^3}{3!} & & +i\frac{x^5}{5!} & - \dots \\
\cos x & = & 1 & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & + \dots \\
\hline
e^{ix} & = & 1 & +ix & -\frac{x^2}{2!} & -i\frac{x^3}{3!} & +\frac{x^4}{4!} & +i\frac{x^5}{5!} & - \dots
\end{array}$$

Rearrange. Plug-in $z = ix$. Use $i^2 = -1$. Multiply $\sin x$ by i. Add to get $e^{ix} = \cos x + i \sin x$. $\square$

You will not be tested on the material in the following slide.

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix}e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
- $e^0 = 1$ (exponentiation rule).
- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ (easy to remember).

Example

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \sin y\cos x\\ \cos(x+y) &= \cos x\cos y - \sin x\sin y\end{aligned} .$$

Proof.

$$\begin{aligned}e^{i(x+y)} &= \cos(x+y) + i\sin(x+y)\\ e^{ix}e^{iy} &= \cos(x+y) + i\sin(x+y)\\ (\cos x + i\sin x)(\cos y + i\sin y) &= \cos(x+y) + i\sin(x+y)\\ \cos x\cos y - \sin x\sin y + i(\sin x\cos y + \sin y\cos x) &= \cos(x+y) + i\sin(x+y)\end{aligned}$$

Compare coefficient in front of i and remaining terms to get the desired equalities. □

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
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- $e^0 = 1$ (exponentiation rule).
- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ (easy to remember).

Example

$$\sin^2 x + \cos^2 x = 1$$

Proof.

$$\begin{aligned} 1 &= e^0 \\ &= e^{ix-ix} = e^{ix}e^{-ix} = (\cos x + i \sin x)(\cos(-x) + i \sin(-x)) \\ &= (\cos x + i \sin x)(\cos x - i \sin x) = \cos^2 x - i^2 \sin^2 x \\ &= \cos^2 x + \sin^2 x \quad . \end{aligned}$$

□

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
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Example

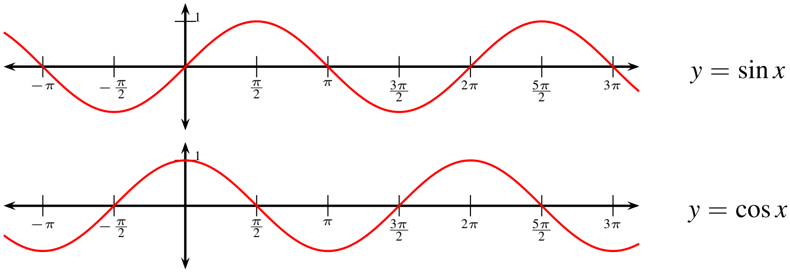
$$\begin{array}{rcl} \sin(2x) & = & 2\sin x \cos x \\ \cos(2x) & = & \cos^2 x - \sin^2 x \end{array} .$$

Proof.

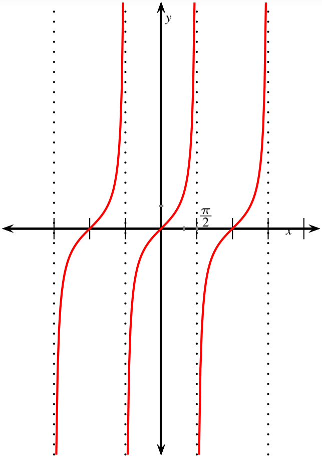
$$\begin{array}{rcl} e^{i(2x)} & = & \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} & = & \cos(2x) + i\sin(2x) \\ (\cos x + i\sin x)^2 = (\cos x + i\sin x)(\cos x + i\sin x) & = & \cos(2x) + i\sin(2x) \\ \cos^2 x - \sin^2 x + i(2\sin x \cos x) & = & \cos(2x) + i\sin(2x) \end{array}$$

Compare coefficient in front of i and remaining terms to get the desired equalities. $\square$

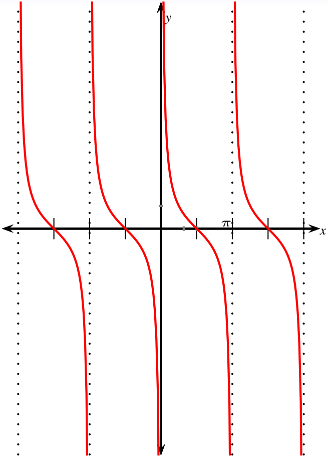
Graphs of the Trigonometric Functions



- $\sin x$ has zeroes at $n\pi$ for all integers n.
- $\cos x$ has zeroes at $\frac{\pi}{2} + n\pi$ for all integers n.
- $-1 \leq \sin x \leq 1$.
- $-1 \leq \cos x \leq 1$.
- If we translate the graph of $\cos x$ by $\frac{\pi}{2}$ units to the right we get the graph of $\sin x$. This is a consequence of $\cos\left(x - \frac{\pi}{2}\right) = \sin x$.

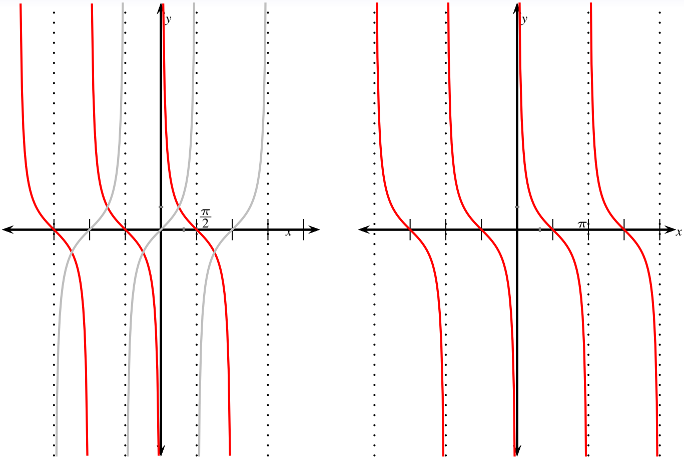


$y = \tan x$



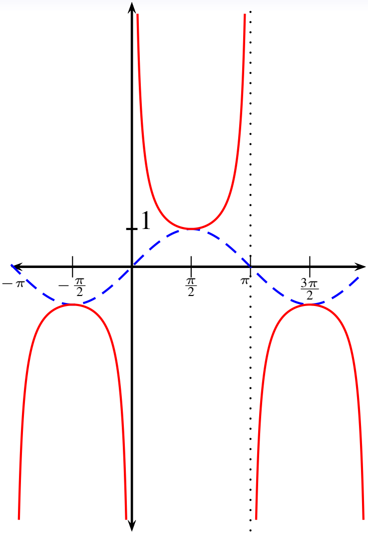
$y = \cot x$

If we move the graph of $\tan x$ by $\frac{\pi}{2}$ units to the left (or right) and reflect across the x axis, we get the graph of $\cot x$. This follows from $\tan\left(x \pm \frac{\pi}{2}\right) = -\cot x$.

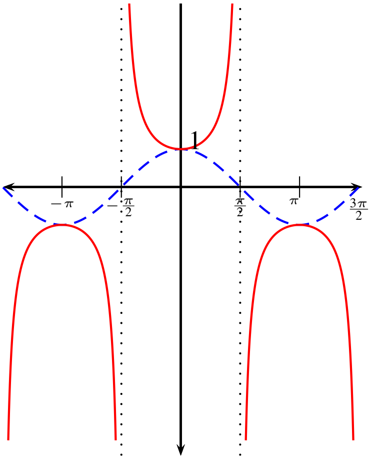


$y = \tan x$ $y = \cot x$

If we move the graph of $\tan x$ by $\frac{\pi}{2}$ units to the left (or right) and reflect across the x axis, we get the graph of $\cot x$. This follows from $\tan\left(x \pm \frac{\pi}{2}\right) = -\cot x$.


1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

$y = \csc x$


1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

$y = \sec x$