Math 1003

Sections 3.1 and 3.2

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Outline

Differentiation Formulas

Let c be a constant and consider the constant function $f(x) = c$. Let us calculate the derivative of f:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$

Theorem (Derivative of a Constant Function)

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0$$

Power Functions

Now consider functions of the form $f(x) = x^n$, where n is a positive integer. For $f(x) = x$, the graph is the line $y = x$, which has slope 1. So

$$\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1.$$

What about $n = 2$ and $n = 3$?

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(x^2) \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \\
&= \lim_{h\to 0}(2x+h) = 2x.
\end{aligned}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(x^3) \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} \\
&= \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2.
\end{aligned}$$

Theorem (The Power Rule)

If n is a positive integer, then $$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

Proof.

Use this formula (which you can verify):

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

Let $f(x) = x^n$. Then

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^n - a^n}{x-a} \\ &= \lim_{x\to a} \frac{\cancel{(x-a)}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{\cancel{x-a}} \\ &= \lim_{x\to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\ &= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} = na^{n-1}. \end{aligned}$$

□

Example (Power Rule)

If $f(x) = x^5$,
Then $f'(x) = 5x^4$.

If $y = x^{1000}$,
Then $y' = 1000x^{999}$.

If $u = t^{22}$,
Then $\frac{\mathrm{d}u}{\mathrm{d}t} = 22t^{21}$.

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^3) = 3r^2.$$

Theorem (The Power Rule (General Version))

If n is any real number, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

Example (Power Rule, negative exponent)

$$\text{Differentiate} \quad y = \frac{1}{x}.$$

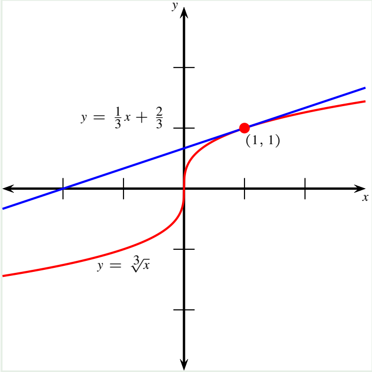
$$y = x^{-1}.$$

$$\text{Power Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = (-1)x^{-2}$$

$$= -\frac{1}{x^2}.$$

Example (Calculating the tangent line using the Power Rule)

Find an equation for the tangent line to the cubic $y = \sqrt[3]{x}$ at the point $P = (1, 1)$.



Here $a = 1$ and $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$.

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{\frac{1}{3}-1} \\ &= \frac{1}{3}x^{\frac{-2}{3}} \\ &= \frac{1}{3\sqrt[3]{x^2}}. \\ f'(1) &= \frac{1}{3}. \end{aligned}$$

Point-slope form: $y - 1 = \frac{1}{3}(x - 1)$, or $y = \frac{1}{3}x + \frac{2}{3}$.

Example (Power Rule, fractional exponent)

Differentiate $y = \sqrt[6]{x^5}$.

$$y = x^{\frac{5}{6}}.$$

Power Rule: $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5x^{-\frac{1}{6}}}{6} = \frac{5}{6\sqrt[6]{x}}.$$

Theorem (The Constant Multiple Rule)

If c is a constant and f is a differentiable function, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = c\frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

Proof.

$$\begin{aligned}
\text{Let} \quad g(x) &= cf(x). \\
\text{Then} \quad g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\
&= \lim_{h\to 0}\frac{c(f(x+h)-f(x))}{h} \\
\text{Limit Law 3:} \quad &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= cf'(x).
\end{aligned}$$

□

Example (Constant Multiple Rule, Power Rule)

Find the derivative of $y = \dfrac{2x^5}{7}$.

$$y = \left(\frac{2}{7}\right)\left(x^5\right).$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{2}{7}\right)\left(x^5\right)\right]$$

Constant Multiple Rule: $= \left(\dfrac{2}{7}\right)\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^5\right)$

$$= \left(\frac{2}{7}\right)\left(5x^4\right)$$

$$= \frac{10x^4}{7}.$$

Example (Constant Multiple Rule, Power Rule, Negative Exponent)

Find the derivative of $t = \dfrac{2\pi}{x^4}$.

$$t = (2\pi)\left(x^{-4}\right).$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[(2\pi)\left(x^{-4}\right)\right]$$

Constant Multiple Rule: $$= (2\pi)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-4}\right)$$

$$= (2\pi)\left(-4x^{-5}\right)$$

$$= -\frac{8\pi}{x^5}.$$

Theorem (The Sum Rule)

If f and g are both differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

Proof.

$$\begin{aligned}
\text{Let} \quad F(x) &= f(x)+g(x). \\
\text{Then} \quad F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] \\
\text{Limit Law 1:} \quad &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x).
\end{aligned}$$

□

The Sum Rule can be extended to any number of summands. For instance, using the theorem twice, we get

$$(f+g+h)' = [(f+g)+h]' = (f+g)' + h' = f' + g' + h'.$$

By writing $f-g$ as $f+(-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get

Theorem (The Difference Rule)

If f and g are both differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)-g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}f(x) - \frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

The Constant Multiple Rule, the Sum Rule, the Difference Rule, and the Power Rule can be combined to differentiate any polynomial.

Example (Derivative of a Polynomial)

$$\text{If}\quad y = x^{16} + 2\sqrt{3}x^7 - 4x^3 + \frac{x}{8} - 5,$$

$$\begin{aligned}
\text{Then}\quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{16} + 2\sqrt{3}x^7 - 4x^3 + \frac{x}{8} - 5\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{16}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(2\sqrt{3}x^7\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(4x^3\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{8}\right) - \frac{\mathrm{d}}{\mathrm{d}x}(5) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{16}\right) + 2\sqrt{3}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^7\right) - 4\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) + \frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}x}(x) - \frac{\mathrm{d}}{\mathrm{d}x}(5) \\
&= \left(16x^{15}\right) + 2\sqrt{3}\left(7x^6\right) - 4\left(3x^2\right) + \frac{1}{8}(1) - (0) \\
&= 16x^{15} + 14\sqrt{3}x^6 - 12x^2 + \frac{1}{8}.
\end{aligned}$$

Example (Difference Rule, Negative Fractional Exponents)

Differentiate $v = \dfrac{3\sqrt{x} - \sqrt[3]{x}}{x}$.

$$v = 3\frac{\sqrt{x}}{x} - \frac{\sqrt[3]{x}}{x}$$

$$v = 3x^{-\frac{1}{2}} - x^{-\frac{2}{3}}.$$

Difference Rule: $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(3x^{-\frac{1}{2}}\right) - \dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\frac{2}{3}}\right)$

Constant Multiple Rule: $= 3\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\frac{1}{2}}\right) - \dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\frac{2}{3}}\right)$

Power Rule: $= 3\left(-\dfrac{1}{2}x^{-\frac{3}{2}}\right) - \left(-\dfrac{2}{3}x^{-\frac{5}{3}}\right)$

$$= \frac{2}{3}x^{-\frac{5}{3}} - \frac{3}{2}x^{-\frac{3}{2}}.$$

Derivatives of Exponential Functions

Compute the derivative of $f(x) = a^x$ using the definition:

$$\begin{aligned} f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x \left(a^h - 1\right)}{h} \\ &= a^x \lim_{h \to 0} \frac{a^h - 1}{h} \\ &= a^x \lim_{h \to 0} \frac{a^h - a^0}{h} \\ &= a^x f'(0). \end{aligned}$$

We have shown that, if $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere, and

$$f'(x) = f'(0)a^x.$$

We leave the following theorem without proof.

Theorem

Let a be a positive number and let $f(x) = a^x$. Then the limit

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}$$

exists.

We will later show that

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h} = \ln(a).$$

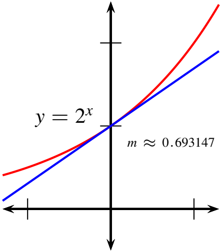
Here, $\ln$ is the natural logarithm function.

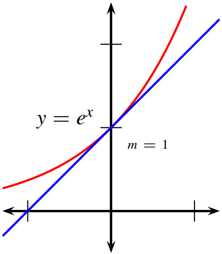
$$\text{If} \quad f(x) = a^x, \quad \text{then} \quad f'(x) = f'(0)a^x.$$

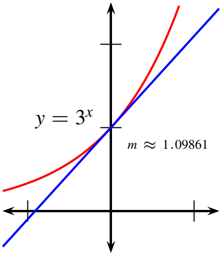
The formula above is simplest when $f'(0) = 1$. Since $\lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.69$ and $\lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.10$, we expect there is a number a between 2 and 3 such that $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$.

Definition (e)

e is the number such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.







Definition (Natural Exponential Function)

e^x is called the natural exponential function. Its derivative is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x.$$

Example (Derivative of a Polynomial and the Natural Exponential Function)

Differentiate $y = e^x + x^7.$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(e^x) + \frac{\mathrm{d}}{\mathrm{d}x}(x^7)\\ &= e^x + 7x^6.\end{aligned}$$

We need a formula for the derivative of the product of two functions. One might guess that the derivative of a product is the product of the derivatives; however, this is wrong.

Example (Not the Product Rule)

Let $f(x) = x$ and $g(x) = x^2$.

$$\begin{aligned} f'(x) &= 1. & (fg)(x) = f(x)g(x) &= x^3. \\ g'(x) &= 2x. & (fg)'(x) &= 3x^2. \\ f'(x)g'(x) &= 2x. \end{aligned}$$

Therefore

$$f'(x)g'(x) \neq (fg)'(x) \quad .$$

The correct formula is called the Product Rule.

Theorem (The Product Rule)

If f and g are both differentiable, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Proof.

Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left(f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right) \\
&= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&\quad + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f(x)g'(x) + g(x)f'(x). \quad \square
\end{aligned}$$

Example (Product Rule, polynomial times the Natural Exponential Function)

$$\begin{aligned}
\text{Differentiate} \quad f(x) &= x^3 e^x. \\
\text{Product Rule:} \quad f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)\left(e^x\right) + \left(x^3\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) \\
&= \left(3x^2\right)\left(e^x\right) + \left(x^3\right)\left(e^x\right) \\
&= e^x\left(x^3 + 3x^2\right).
\end{aligned}$$

Theorem (The Quotient Rule)

If f and g are differentiable and $g(x) \neq 0$, then

$$
\begin{array}{rcl|l}
\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{f(x)}{g(x)}\right) & = & \dfrac{\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right)g(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right)}{\left(g(x)\right)^2} & \textit{(Leibniz notation)} \\
\left(\dfrac{f(x)}{g(x)}\right)' & = & \dfrac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2} & {}'\ \textit{notation} \\
\left(\dfrac{f}{g}\right)' & = & \dfrac{f'g - fg'}{g^2} & \textit{abbreviated}
\end{array}
$$

- The proof of the Quotient Rule is similar to the proof of the Product Rule.
- There is an alternative algebraic proof via the Product Rule, the Power Rule and the (not yet studied) Chain Rule.

Example (Quotient Rule, rational function)

Differentiate $y = \frac{x^5 + 2x}{-x^6 + 2}$.

Quotient Rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^5 + 2x\right)\left(-x^6 + 2\right) - \left(x^5 + 2x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(-x^6 + 2\right)}{\left(-x^6 + 2\right)^2}$$

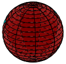
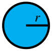
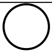
$$= \frac{\left(5x^4 + 2\right)\left(-x^6 + 2\right) - \left(x^5 + 2x\right)\left(-6x^5\right)}{\left(-x^6 + 2\right)^2}$$

$$= \frac{(-5x^{10} - 2x^6 + 10x^4 + 4) - (-6x^{10} - 12x^6)}{(-x^6 + 2)^2}$$

$$= \frac{x^{10} + 10x^6 + 10x^4 + 4}{(-x^6 + 2)^2}.$$

The Relation between Ball Volume and Surface Area

There is a relationship between the surface area and the volume of a ball (in any dimension).

Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\pi r^2\right) = 2\pi r$
1	 interval	length	$2r$	 endpts.	2	$\frac{\mathrm{d}}{\mathrm{d}r}(2r) = 2$