Math 1013

Improper Integrals and Arc Length

Dr. Tim Alderson

University of New Brunswick Saint John

Week 9

Outline

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Improper Integrals and Arc Length

Differentiation..WT*?

The derivative of x^2, with respect to x, is $2x$. However, suppose we write x^2 as the sum of x x's, and then take the derivative:

$$\text{Let } f(x) = x + x + \cdots + x \ (x \text{ times})$$

Then

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x + x + \cdots + x] \ (x \text{ times}) \\ &= \frac{d}{dx}(x) + \frac{d}{dx}(x) + \cdots + \frac{d}{dx}(x) \ (x \text{ times}) \\ &= 1 + 1 + \cdots + 1 \ (x \text{ times}) \\ &= x \end{aligned}$$

This argument appears to show that the derivative of x^2, with respect to x, is actually x. Where is the fallacy?

Improper Integrals

- The definition of $\int_a^b f(x)\mathrm{d}x$, where f is defined on $[a,b]$, has two requirements:
  1. $[a,b]$ is a finite interval.
  2. f has no infinite discontinuities in $[a,b]$.

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Definition (Improper Integral)

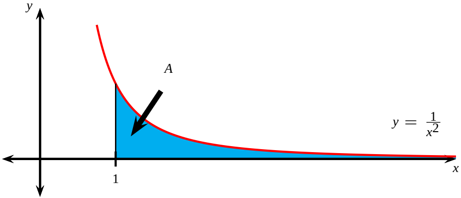
The integral

$$\int_a^b f(x)\mathrm{d}x$$

is called improper if one or more of the endpoints a and b is infinite, or if f has an infinite discontinuity on $[a,b]$.

Type I: Infinite Intervals

- Consider the region A that lies under $y = 1/x^2$, above the x-axis, and to the right of $x = 1$.



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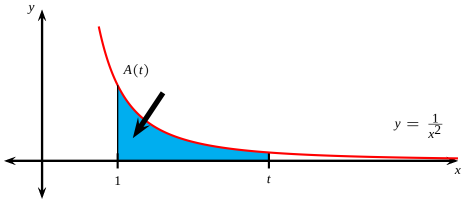

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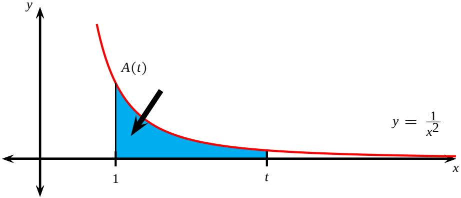

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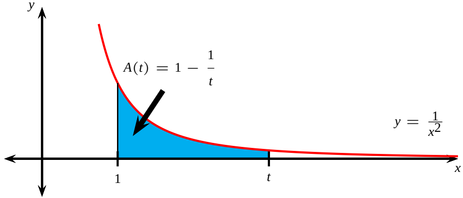

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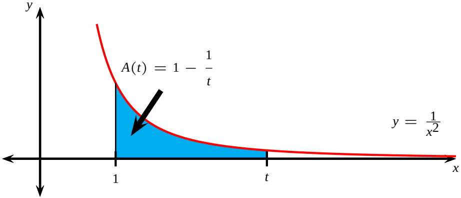

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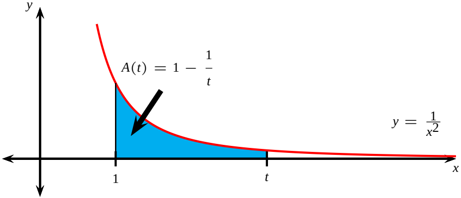

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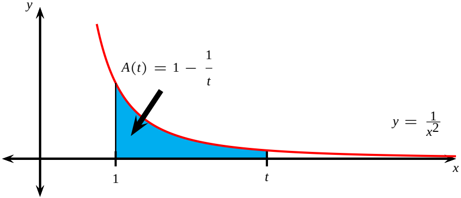

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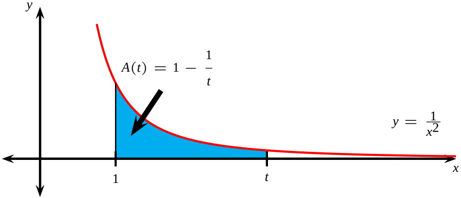

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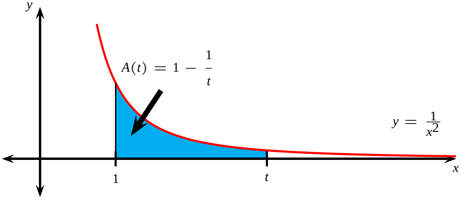

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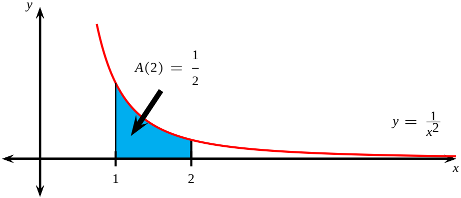

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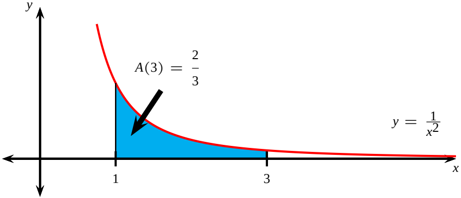

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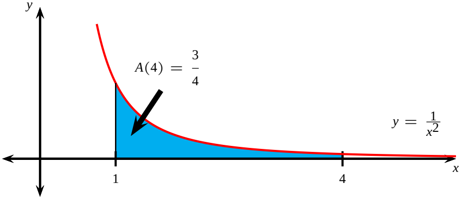

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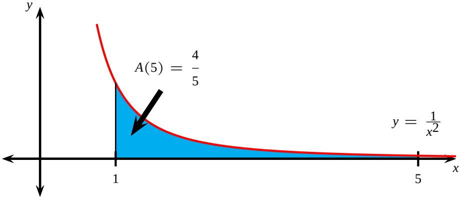

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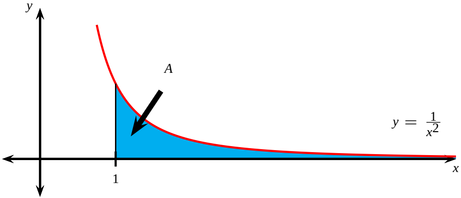

- Notice $A(t) < 1$ no matter how big t is.
- Also notice $\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$.
- We say that the area A is equal to 1 and write

$$\int_1^\infty \frac{1}{x^2}\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\mathrm{d}x = 1.$$

Definition (Improper Integral of Type I)

1. If $\int_a^t f(x)\mathrm{d}x$ exists for every $t \geq a$, then
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$\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^b f(x)\mathrm{d}x$ are called **convergent** if the corresponding limit exists and **divergent** if it doesn't exist.

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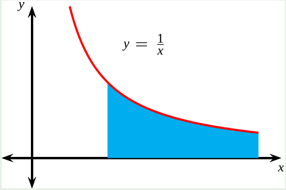
3. If both $\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^a f(x)\mathrm{d}x$ are convergent, then we define
$$\int_{-\infty}^\infty f(x)\mathrm{d}x = \int_{-\infty}^a f(x)\mathrm{d}x + \int_a^\infty f(x)\mathrm{d}x.$$

Example

Determine whether $\int_1^\infty \frac{1}{x}\mathrm{d}x$ is convergent or divergent.



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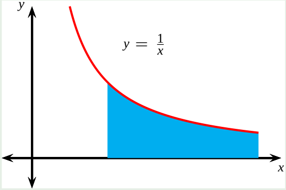

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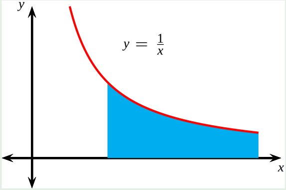

$$\begin{aligned}\int_1^\infty \frac{1}{x}\mathrm{d}x &= \lim_{t\to\infty}\int_1^t \frac{1}{x}\mathrm{d}x \\ &= \lim_{t\to\infty}\left[\ln|x|\right]_1^t\end{aligned}$$

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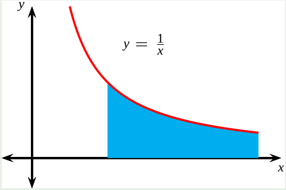

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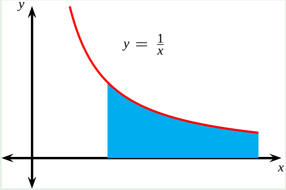

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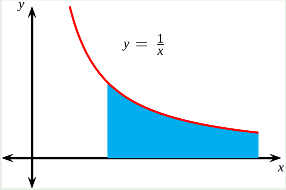

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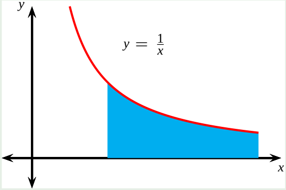

Infinite area

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Therefore the improper integral is divergent.

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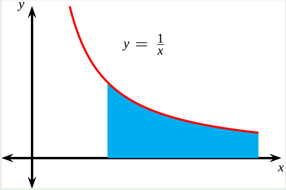

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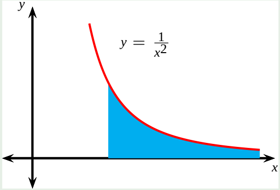

Finite area

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\int_1^\infty \frac{1}{x} \mathrm{d}x &= \lim_{t\to\infty} \int_1^t \frac{1}{x} \mathrm{d}x \\
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$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \int_{-\infty}^{0} \frac{1}{1+x^2}\mathrm{d}x + \int_{0}^{\infty} \frac{1}{1+x^2}\mathrm{d}x$$

Evaluate the two integrals separately:

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\mathrm{d}x = \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{1+x^2}\mathrm{d}x$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \lim_{t\to\infty}\int_{0}^{t} \frac{1}{1+x^2}\mathrm{d}x$$

Example

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \int_{-\infty}^{0} \frac{1}{1+x^2}\mathrm{d}x + \int_{0}^{\infty} \frac{1}{1+x^2}\mathrm{d}x$$

Evaluate the two integrals separately:

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\mathrm{d}x = \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{1+x^2}\mathrm{d}x = \lim_{t\to-\infty}\left[\tan^{-1} x\right]_t^0$$

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&= \lim_{t\to\infty} (\tan^{-1} t - \tan^{-1} 0) = \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}
\end{aligned}$$

Therefore $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \mathrm{d}x = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

Example

For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\mathrm{d}x$ convergent?

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- Therefore $\int_1^\infty \frac{1}{x^p}\mathrm{d}x = \frac{1}{p-1}$ if $p > 1$, so the integral is convergent.

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- Therefore $\displaystyle\int_1^{\infty} \frac{1}{x^p}\mathrm{d}x = \frac{1}{p-1}$ if $p > 1$, so the integral is convergent.
- If $p < 1$, then $p - 1 < 0$, so $\dfrac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t \to \infty$.

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- If $p < 1$, then $p - 1 < 0$, so $\dfrac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t \to \infty$.
- Therefore $\displaystyle\int_1^\infty \frac{1}{x^p}\mathrm{d}x$ is divergent if $p < 1$.

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- Therefore $\int_1^\infty \frac{1}{x^p}\mathrm{d}x = \frac{1}{p-1}$ if $p>1$, so the integral is convergent.
- If $p<1$, then $p-1<0$, so $\frac{1}{t^{p-1}} = t^{1-p}\to\infty$ as $t\to\infty$.
- Therefore $\int_1^\infty \frac{1}{x^p}\mathrm{d}x$ is divergent if $p<1$.

Theorem

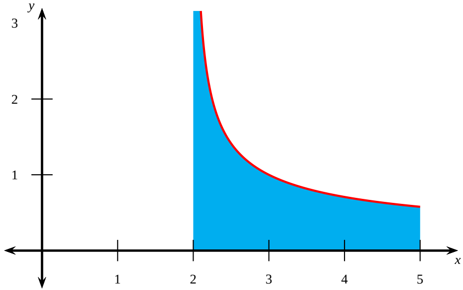
$\int_1^\infty \frac{1}{x^p}\mathrm{d}x$ *converges if* $p>1$ *and diverges if* $p\leq 1$.

Type II: Discontinuous Integrands

We can use the same approach if the function f is discontinuous at one of the endpoints a and b in the integral $\int_a^b f(x)\mathrm{d}x$.
For example, $\frac{1}{\sqrt{x-2}}$ is discontinuous at 2, so we might wonder if the integral

$$\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x$$

exists.



Definition (Improper Integral of Type II)

1. If f is continuous on $[a, b)$ and discontinuous at b, then
$$\int_a^b f(x)\mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x)\mathrm{d}x$$
if the limit exists.

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2. If f is continuous on $(a, b]$ and discontinuous at a, then
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$\int_a^b f(x)\mathrm{d}x$ is called **convergent** if the corresponding limit exists and **divergent** if it doesn't exist.

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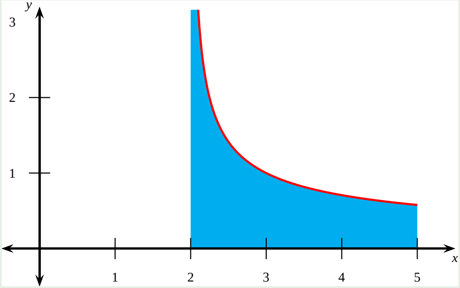
$\int_a^b f(x)\mathrm{d}x$ is called **convergent** if the corresponding limit exists and **divergent** if it doesn't exist.

3. If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\mathrm{d}x$ and $\int_c^b f(x)\mathrm{d}x$ are convergent, then we define
$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$

Example

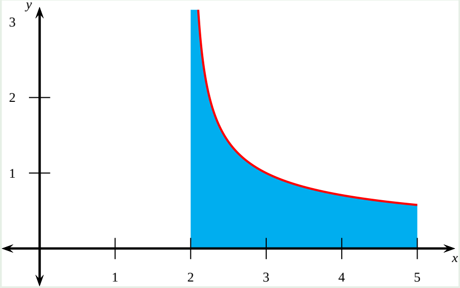
Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,\mathrm{d}x$.



Example

Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,\mathrm{d}x$.

Observe that $x = 2$ is a vertical asymptote for the integrand.



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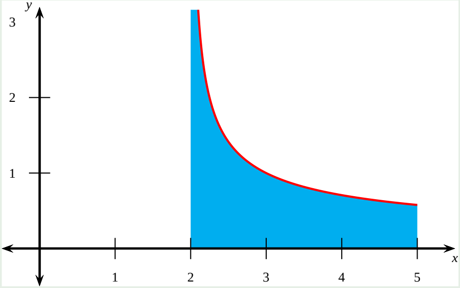

$$\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x$$

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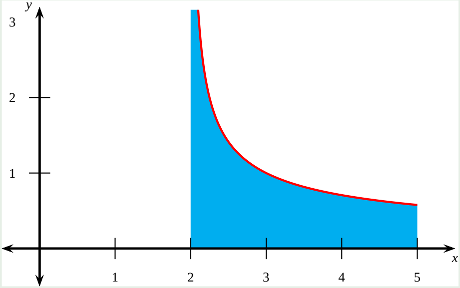

$$
\begin{aligned}
&\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x \\
= \quad & \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x \\
= \quad & \lim_{t\to 2^+} \left[2\sqrt{x-2}\right]_t^5
\end{aligned}
$$

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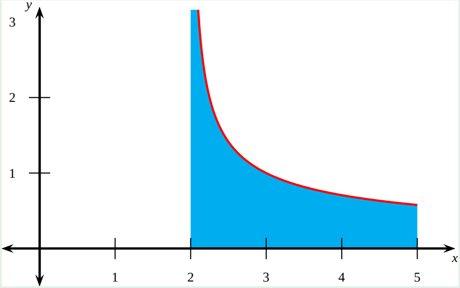

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= \quad & \lim_{t\to 2^+} \left[2\sqrt{x-2}\right]_t^5 \\
= \quad & \lim_{t\to 2^+} 2(\sqrt{5-2} - \sqrt{t-2})
\end{aligned}
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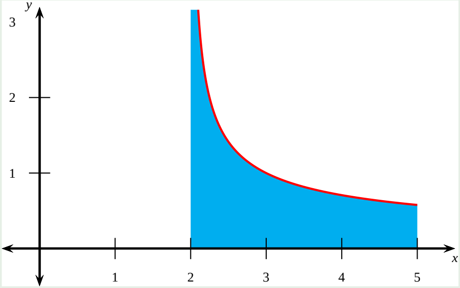

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= \quad & \lim_{t\to 2^+} 2(\sqrt{5-2} - \sqrt{t-2}) \\
= \quad & 2\sqrt{3}
\end{aligned}
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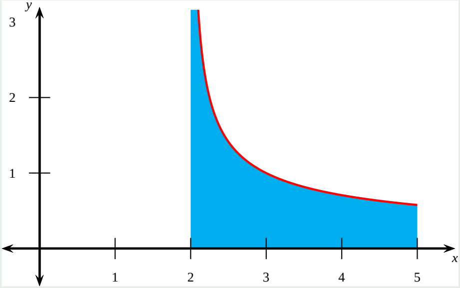

Area = $2\sqrt{3}$

$$
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= \quad & \lim_{t\to 2^+} \Big[2\sqrt{x-2}\Big]_t^5 \\
= \quad & \lim_{t\to 2^+} 2(\sqrt{5-2} - \sqrt{t-2}) \\
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\end{aligned}
$$

Example

Evaluate $\displaystyle\int_0^3 \frac{1}{x-1}\mathrm{d}x$.

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Evaluate $\displaystyle\int_0^3 \frac{1}{x-1}\mathrm{d}x$.
Observe that $x = 1$ is a vertical asymptote for the integrand.

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- Therefore the integral diverges.
- If we had not noticed the vertical asymptote, we might have made the following mistake:

$$\int_0^3 \frac{\mathrm{d}x}{x-1} = \left[\ln|x-1|\right]_0^3 = \ln 2 - \ln 1 = \ln 2.$$

A Comparison Test for Improper Integrals

Sometimes it's impossible to find the exact value of an integral, but we still want to know if it's convergent or divergent. For such cases, we can sometimes use the following theorem.

Theorem (Comparison Theorem)

Suppose f and g are continuous and $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. *If $\int_a^\infty f(x)\mathrm{d}x$ is convergent, then $\int_a^\infty g(x)\mathrm{d}x$ is convergent.*
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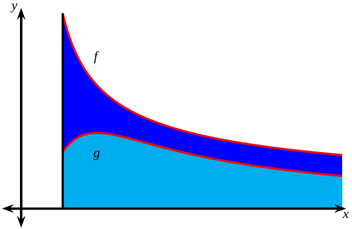

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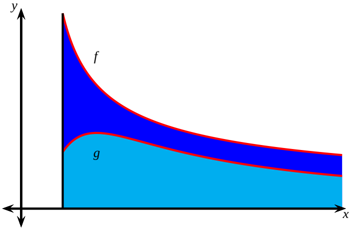

A similar theorem holds for Type II improper integrals.

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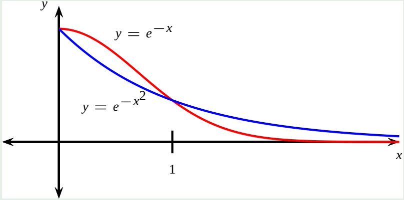
- Notice that $e^{-x^2} \leq e^{-x}$ for $x \geq 1$.



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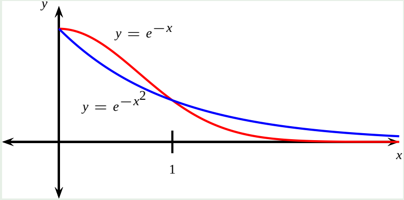

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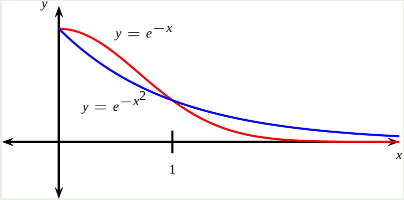

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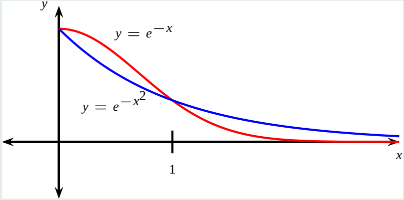

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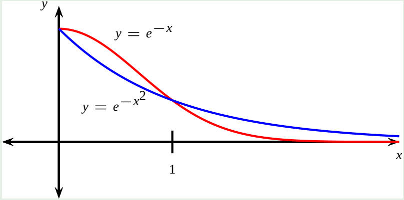

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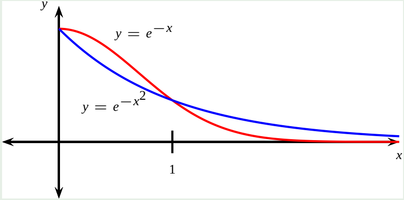

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Therefore by the Comparison Theorem, $\int_0^\infty e^{-x^2}\,\mathrm{d}x$ converges.

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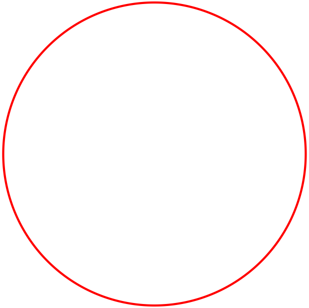
- By a previously studied example, $\int_1^{\infty} \frac{\mathrm{d}x}{x}$ is divergent.
- Therefore $\displaystyle\int_1^{\infty} \frac{1 + e^{-x}}{x} \mathrm{d}x$ is divergent by the Comparison Theorem.

Arc Length



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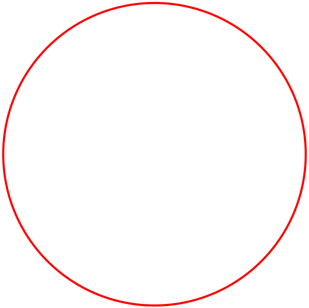

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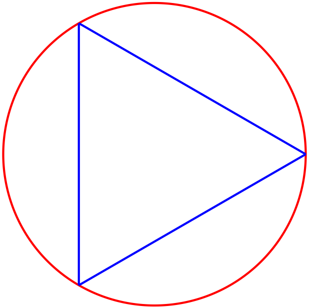

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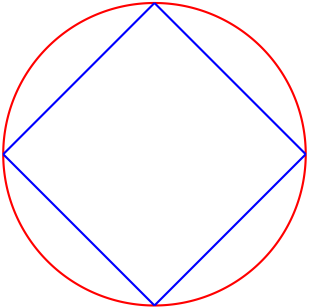

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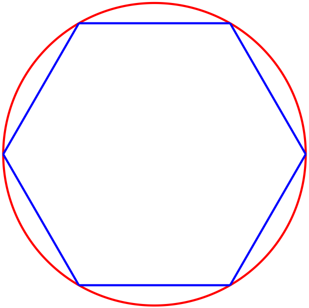

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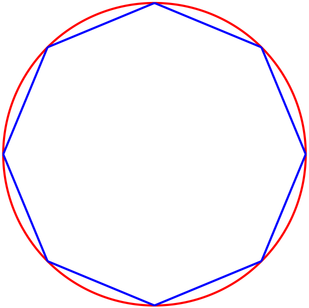

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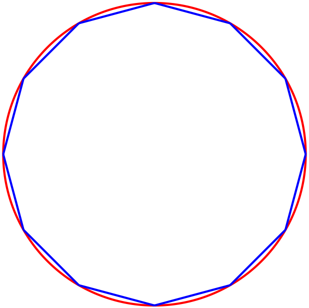

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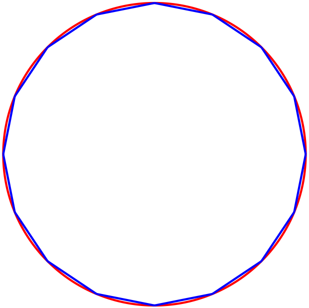

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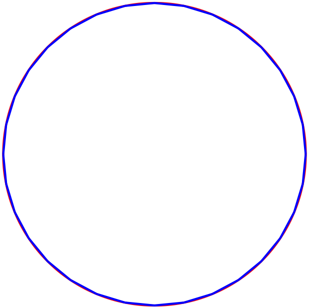

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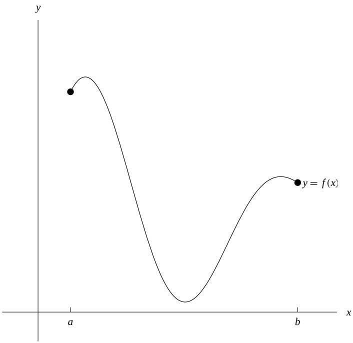

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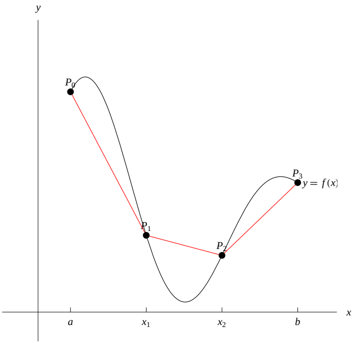

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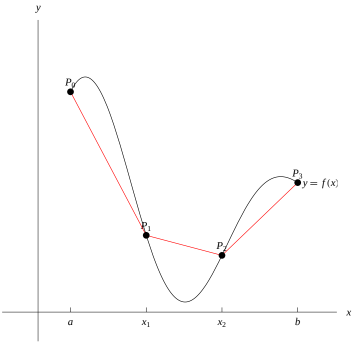

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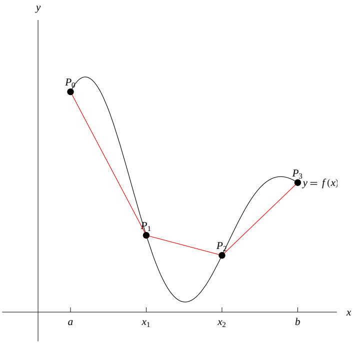

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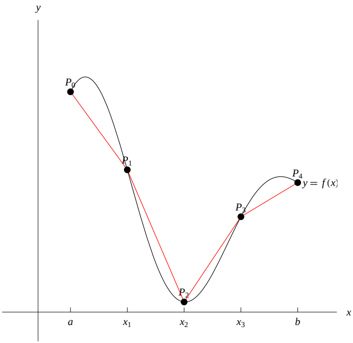

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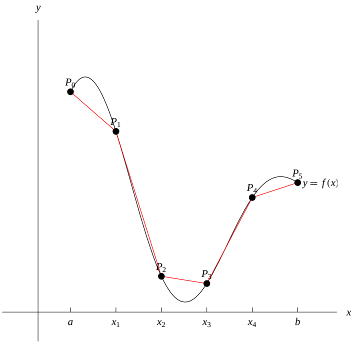

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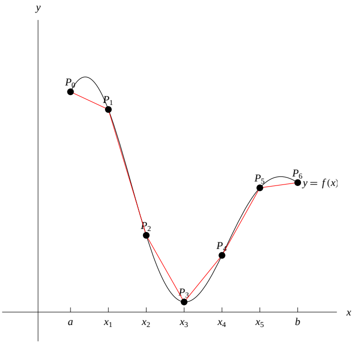

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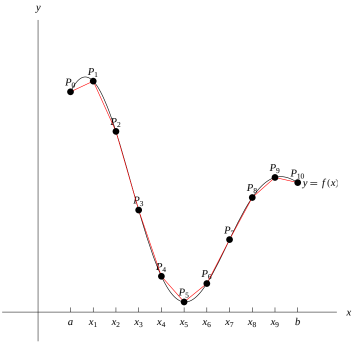

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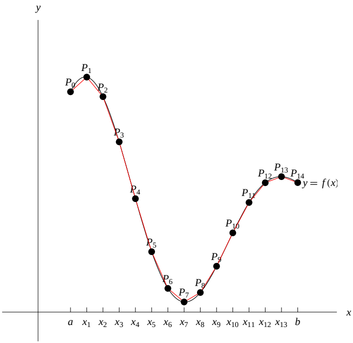

- What about $y = f(x)$ for a continuous function f on $[a, b]$?
- Divide $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx.
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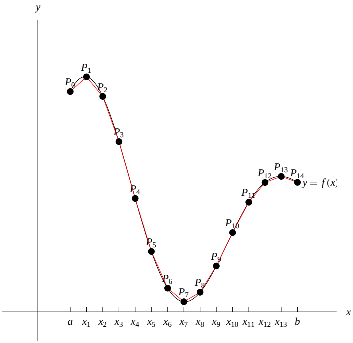

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The Arc Length Formula

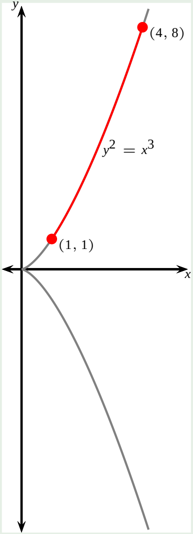
Definition

Suppose f' exists and is continuous on $[a, b]$. Then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x \\ &= \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x \qquad \text{(in Leibniz notation)} \quad . \end{aligned}$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.



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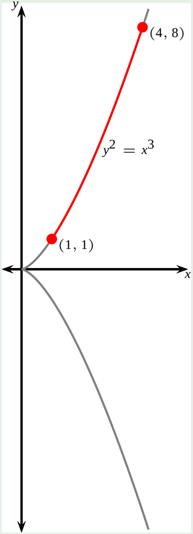

- For the top half of the curve we have:
- $y =$ and $y' =$.

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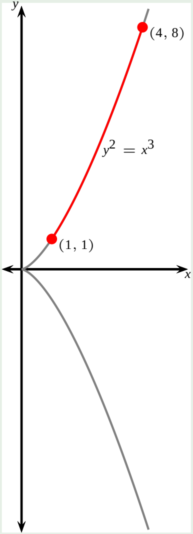

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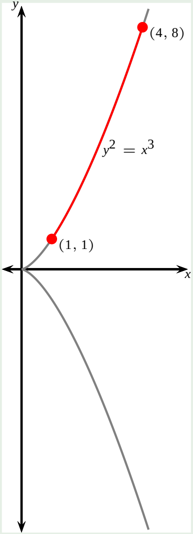

- For the top half of the curve we have:
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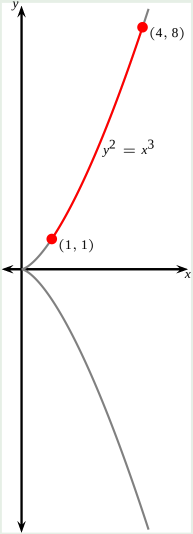

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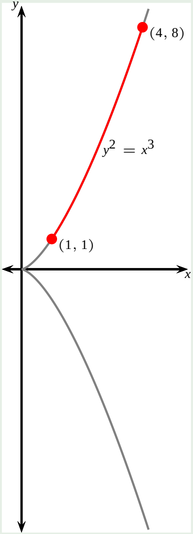

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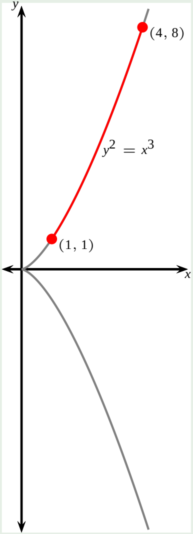

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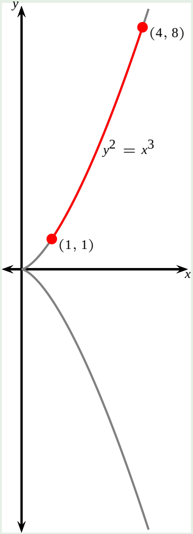

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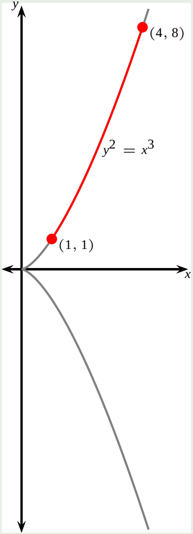

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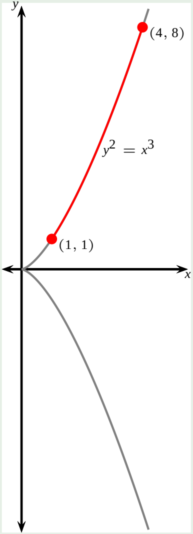

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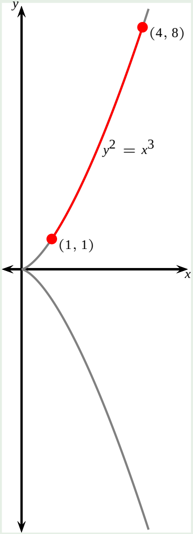

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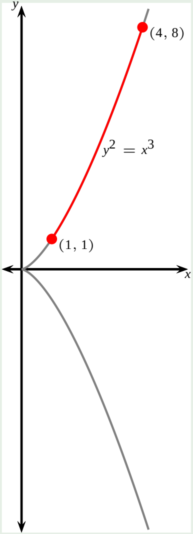

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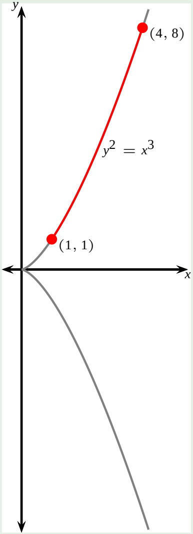

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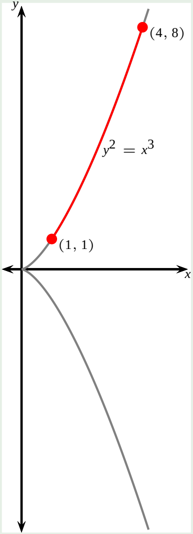

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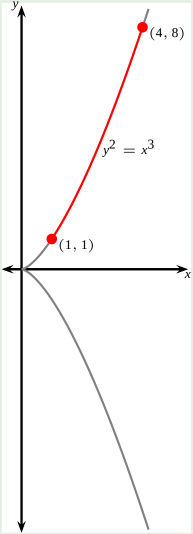

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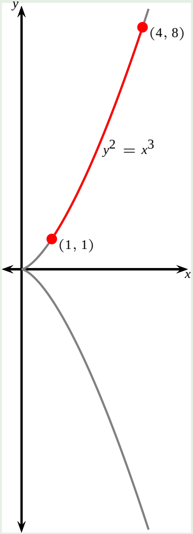
- $u = 1 + \frac{9}{4}x$ and $\mathrm{d}x = \frac{4}{9}\mathrm{d}u$.
- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u =$.

$$
\begin{aligned}
L &= \int_1^4 \sqrt{1 + (y')^2}\,\mathrm{d}x \\
&= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4} \frac{4}{9}\sqrt{u}\,\mathrm{d}u
\end{aligned}
$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.

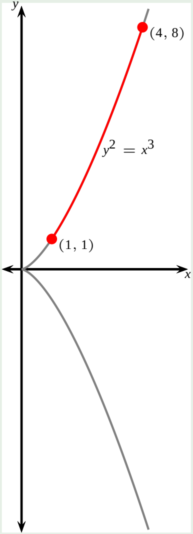


- For the top half of the curve we have:
- $y = x^{3/2}$ and $y' = \frac{3}{2}x^{1/2}$.
- $u = 1 + \frac{9}{4}x$ and $\mathrm{d}x = \frac{4}{9}\mathrm{d}u$.
- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u =$.

$$
\begin{aligned}
L &= \int_1^4 \sqrt{1 + (y')^2}\,\mathrm{d}x \\
&= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4} \frac{4}{9}\sqrt{u}\,\mathrm{d}u
\end{aligned}
$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1,1)$ and $(4,8)$.

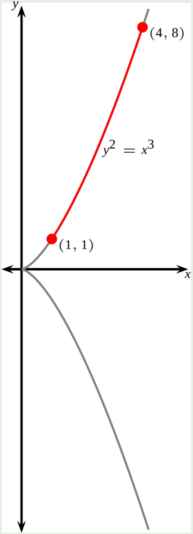


- For the top half of the curve we have:
- $y = x^{3/2}$ and $y' = \frac{3}{2}x^{1/2}$.
- $u = 1 + \frac{9}{4}x$ and $\mathrm{d}x = \frac{4}{9}\mathrm{d}u$.
- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u = 10$.

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + (y')^2}\,\mathrm{d}x \\ &= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,\mathrm{d}u \end{aligned}$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.

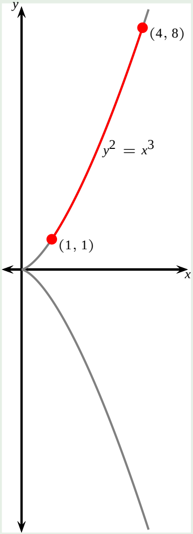


- For the top half of the curve we have:
- $y = x^{3/2}$ and $y' = \frac{3}{2}x^{1/2}$.
- $u = 1 + \frac{9}{4}x$ and $\mathrm{d}x = \frac{4}{9}\mathrm{d}u$.
- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u = 10$.

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + (y')^2}\,\mathrm{d}x \\ &= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,\mathrm{d}u \\ &= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{10} \end{aligned}$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.



- For the top half of the curve we have:
- $y = x^{3/2}$ and $y' = \frac{3}{2}x^{1/2}$.
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- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u = 10$.

$$
\begin{aligned}
L &= \int_1^4 \sqrt{1 + (y')^2}\,\mathrm{d}x \\
&= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,\mathrm{d}u \\
&= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{10} = \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right)
\end{aligned}
$$

If a curve has equation $x = g(y)$, $c \leq y \leq d$, and $g'(y)$ is continuous, then we can get the length of the curve by interchanging the roles of x and y in the arc length formula:

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\,\mathrm{d}y = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

$$L = \int_0^1 \sqrt{1 + (\mathsf{d}x/\mathsf{d}y)^2}\,\mathsf{d}y$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y =$.

$$L = \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1 + \quad}\,\mathrm{d}y$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = \quad$.

$$L \quad = \quad \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1 + \quad}\,\mathrm{d}y$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.

$$L = \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1 + 4y^2}\,\mathrm{d}y$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y =$, so $\mathrm{d}y =$, and $\sqrt{1+4y^2} =$.

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
&= \int
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y =$, so $\mathrm{d}y =$, and $\sqrt{1+4y^2} =$.

$$\begin{aligned} L &= \int_0^1 \sqrt{1+\left(\mathrm{d}x/\mathrm{d}y\right)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\ &= \int \end{aligned}$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y =$, and $\sqrt{1+4y^2} =$.

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
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Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
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$$\begin{aligned} L &= \int_0^1 \sqrt{1+(\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\ &= \int \qquad \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta \end{aligned}$$

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Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} =$.

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
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Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
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$$\begin{aligned} L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\ &= \int \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta \end{aligned}$$

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- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
- When $y = 0$, $\tan\theta =$, so $\theta =$.
- When $y = 1$, $\tan\theta =$, so $\theta =$.

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\ &= \int \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta \end{aligned}$$

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- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta =$, so $\theta =$.

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
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- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta =$, so $\theta =$.

$$
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- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta = 2$, so $\theta =$.

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\ &= \int_0 \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta \end{aligned}$$

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- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta = 2$, so $\theta =$.

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\ &= \int_0 \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta \end{aligned}$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta = 2$, so $\theta = \tan^{-1}(2)$ (call this α).

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\ &= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta \end{aligned}$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta = 2$, so $\theta = \tan^{-1}(2)$ (call this α).

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
&= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\,\mathrm{d}\theta
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta\mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta = 2$, so $\theta = \tan^{-1}(2)$ (call this α).

$$\begin{aligned}
L &= \int_0^1 \sqrt{1+(\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
&= \int_0^\alpha \sec\theta\cdot\frac{1}{2}\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right]_0^\alpha
\end{aligned}$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
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$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
&= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^\alpha \\
&= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta\mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
- When $y = 0$, $\tan\theta = 0$, so $\theta = 0$.
- When $y = 1$, $\tan\theta = 2$, so $\theta = \tan^{-1}(2)$ (call this α).

$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
&= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right]_0^\alpha \\
&= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right) \\
&= \frac{1}{4}\Big(\qquad + \ln|\qquad + \quad|\Big)
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta \mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
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$$
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L &= \int_0^1 \sqrt{1+(\mathrm{d}x/\mathrm{d}y)^2}\,\mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\,\mathrm{d}y \\
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&= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right]_0^\alpha \\
&= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right) \\
&= \frac{1}{4}\left(2 \qquad + \ln| \qquad + 2|\right)
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0, 0)$ to $(1, 1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
- Substitute $y = \frac{1}{2}\tan\theta$, so $\mathrm{d}y = \frac{1}{2}\sec^2\theta\mathrm{d}\theta$, and $\sqrt{1+4y^2} = \sec\theta$.
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&= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^\alpha \\
&= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right) \\
&= \frac{1}{4}\Big(2 \qquad + \ln| \qquad + 2|\Big)
\end{aligned}
$$

Example

Find the length of the arc of $x = y^2$ from $(0,0)$ to $(1,1)$.

- $x = y^2$, so $\mathrm{d}x/\mathrm{d}y = 2y$.
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$$
\begin{aligned}
L &= \int_0^1 \sqrt{1 + (\mathrm{d}x/\mathrm{d}y)^2}\, \mathrm{d}y = \int_0^1 \sqrt{1+4y^2}\, \mathrm{d}y \\
&= \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta\, \mathrm{d}\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\, \mathrm{d}\theta \\
&= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^\alpha \\
&= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right) \\
&= \frac{1}{4}\left(2\sqrt{5} + \ln|\sqrt{5}+2|\right)
\end{aligned}
$$

Sometimes an arc length problem can be simplified significantly if the function $y = f(x)$ has a certain form.
In particular, if $y' = a - b$, where $2ab = \frac{1}{2}$, then $1 + (y')^2$ is a perfect square, simplifying the arc length formula.

$$\begin{aligned} 1 + (y')^2 &= 1 + (a-b)^2 \\ &= 1 + \left(a^2 - \frac{1}{2} + b^2\right) \\ &= a^2 + \frac{1}{2} + b^2 \\ &= a^2 + 2ab + b^2 \\ &= (a+b)^2. \end{aligned}$$

There are several exercises like this in the textbook (such as numbers 9 and 10) and on WebWork (question 2 from this section) of this type.
Consider the following examples:

Example: Find the arc length of $y = \frac{x^7}{14} + \frac{1}{10x^5}$ between $x = 1$ and $x = 2$.

Solution: Arc length formula: $L = \int_1^2 \sqrt{1 + (y')^2}\mathrm{d}x$, so first find y'.

$$\begin{aligned} y' &= 7\frac{x^6}{14} + (-5)\frac{1}{10x^6} \\ &= \frac{1}{2}x^6 - \frac{1}{2}x^{-6} \end{aligned}$$

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$$\begin{aligned} y' &= 7\frac{x^6}{14} + (-5)\frac{1}{10x^6} \\ &= \frac{1}{2}x^6 - \frac{1}{2}x^{-6} \end{aligned}$$

Observe that if $a = \frac{1}{2}x^6$ and $b = \frac{1}{2}x^{-6}$, then $y' = a - b$, and $2ab = 2\left(\frac{1}{2}x^6\right)\left(\frac{1}{2}x^{-6}\right) = \frac{1}{2}$, so

$$1 + (y')^2 = \left(\frac{1}{2}x^6 + \frac{1}{2}x^{-6}\right)^2$$

Now apply this to the arc length formula:

$$L = \int_1^2 \sqrt{1 + (y')^2}\,\mathsf{d}x = \int_1^2 \sqrt{\left(\frac{1}{2}x^6 + \frac{1}{2}x^{-6}\right)^2}\,\mathsf{d}x$$

$$L = \int_1^2 \sqrt{1 + (y')^2}\,\mathrm{d}x = \int_1^2 \sqrt{\left(\frac{1}{2}x^6 + \frac{1}{2}x^{-6}\right)^2}\,\mathrm{d}x$$
$$= \int_1^2 \left(\frac{1}{2}x^6 + \frac{1}{2}x^{-6}\right)\mathrm{d}x$$

$$\begin{aligned} L = \int_1^2 \sqrt{1+(y')^2}\,\mathsf{d}x &= \int_1^2 \sqrt{\left(\frac{1}{2}x^6 + \frac{1}{2}x^{-6}\right)^2}\,\mathsf{d}x \\ &= \int_1^2 \left(\frac{1}{2}x^6 + \frac{1}{2}x^{-6}\right)\mathsf{d}x \\ &= \left[\frac{1}{2}\cdot\frac{1}{7}x^7 + \frac{1}{2}\cdot\frac{1}{-5}x^{-5}\right]_1^2 \end{aligned}$$

$$
\begin{aligned}
L = \int_1^2 \sqrt{1+(y')^2}\,\mathrm{d}x &= \int_1^2 \sqrt{\left(\frac{1}{2}x^6 + \frac{1}{2}x^{-6}\right)^2}\,\mathrm{d}x \\
&= \int_1^2 \left(\frac{1}{2}x^6 + \frac{1}{2}x^{-6}\right)\mathrm{d}x \\
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\end{aligned}
$$

$$
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&= \left[\frac{1}{2} \cdot \frac{1}{7}x^7 + \frac{1}{2} \cdot \frac{1}{-5}x^{-5}\right]_1^2 \\
&= \left[\frac{1}{14}x^7 - \frac{1}{10}x^{-5}\right]_1^2 \\
&= \left(\frac{128}{14} - \frac{1}{10} \cdot \frac{1}{32}\right) - \left(\frac{1}{14} - \frac{1}{10}\right)
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$$
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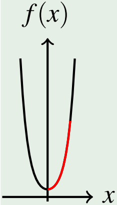
&= \left(\frac{128}{14} - \frac{1}{10}\cdot\frac{1}{32}\right) - \left(\frac{1}{14} - \frac{1}{10}\right) = \frac{20537}{2240}.
\end{aligned}
$$

Example (Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.)



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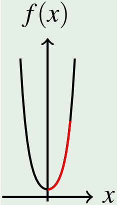

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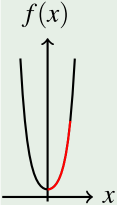

$$y' = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.$$

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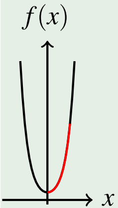

$$y' = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.$$

$$= a - b \text{ where } a = \frac{1}{2}e^{3x}, b = \frac{1}{2}e^{-3x}, \text{ and } 2ab = \frac{1}{2}$$

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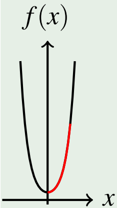

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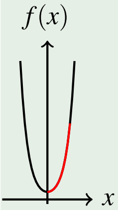

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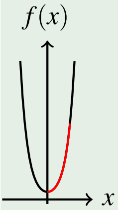

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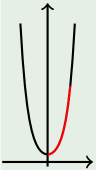

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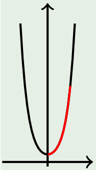

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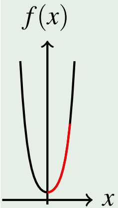

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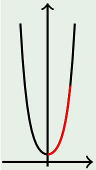

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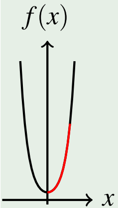

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\end{aligned}
$$