Math 1003

Section 1.5

Dr. Tim Alderson

University of New Brunswick Saint John

Fall 2015

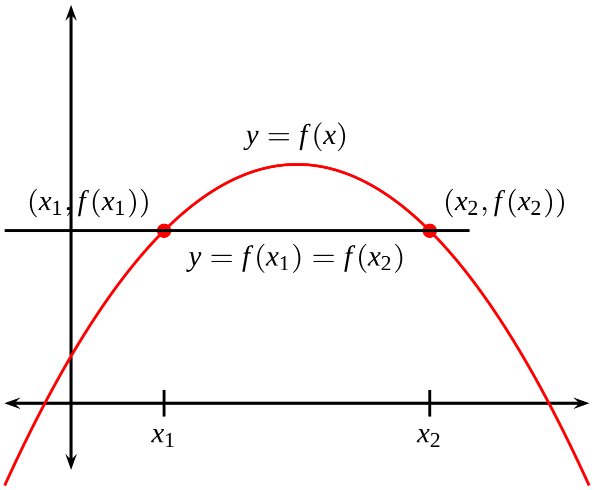
Outline

One-to-one Functions

Definition (One-to-one Function)

A function f is a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

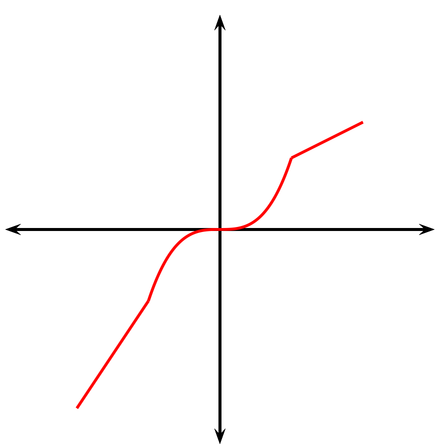


$\leftarrow$ This function is not one-to-one.

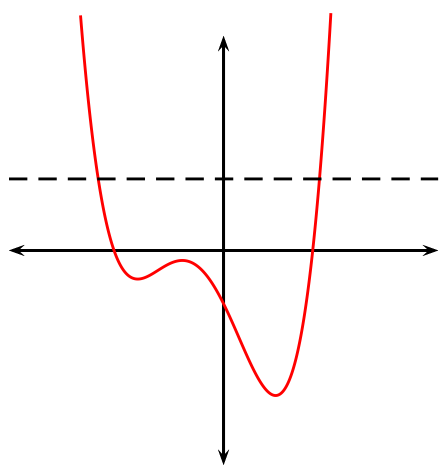
Question: How can we tell from the graph of a function whether it is one-to-one or not?
Answer: Use the horizontal line test.

The Horizontal Line Test.

A function is one-to-one if and only if no horizontal line intersects it more than once. □



One-to-one



Not one-to-one

The Definition of the Inverse of f

Definition (f^{-1})

Let f be a one-to-one function with domain A and range B. Then the inverse of f is the function f^{-1} that has domain B and range A and is defined by

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad f(x) = y$$

for all y in B.

Note:

- Only one-to-one functions have inverses.
- f^{-1} reverses the effect of f.
- domain of f^{-1} = range of f.
- range of f^{-1} = domain of f.

Example ($f(x) = x^3$)

The inverse of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$. This is because if $y = x^3$, then

$$f^{-1}(y) = \sqrt[3]{y} = \sqrt[3]{x^3} = x.$$

The inverse of f is denoted as f^{-1}. This notation is one of the most frequent causes of student confusion. **WARNING:**

$$f^{-1}(x) \text{ does not mean } (f(x))^{-1} = \frac{1}{f(x)} \quad .$$

The notations are different: the superscript -1 has different positions.

- f^{-1} is the compositional inverse of f.
- $\frac{1}{f(x)}$ is the multiplicative inverse of $f(x)$.
- $f^2(x)$ is an abbreviation for $(f(x))^2$, $f^3(x)$ is an abbreviation of $(f(x))^3$, and so on.
- However, $f^{-1}(x)$ is not the abbreviation of $(f(x))^{-1}$ and does not follow this pattern.

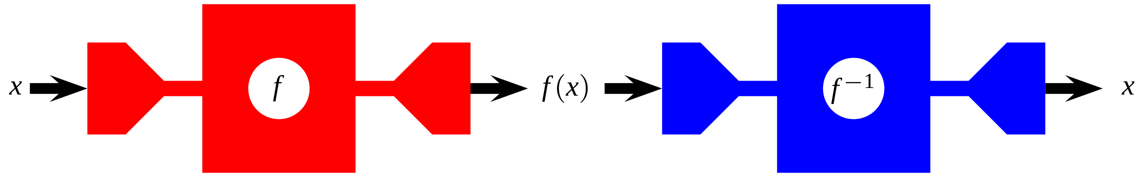
$$f^n(x) = \begin{cases} \text{stands for } (f(x))^n & \text{when } n = 1, 2, 3, \ldots \\ \text{stands for inverse of } f \text{ applied to } x & \text{when } n = -1 \\ \text{should be avoided} & \text{when } n \neq -1, 1, 2, 3, \ldots. \end{cases}$$

To reduce confusion, if possible, use $\frac{1}{f(x)}$ instead of $(f(x))^{-1}$.

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad f(x) = y.$$

Therefore

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x.$$

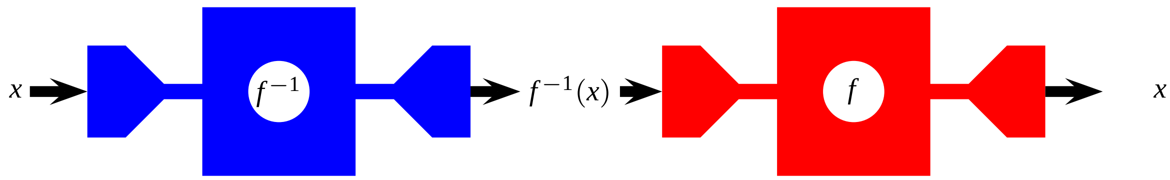


Switch the roles of x and y:

$$f^{-1}(x) = y \qquad \Leftrightarrow \qquad f(y) = x.$$

Therefore

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(y) = x.$$



How to Find the Inverse of a One-to-one Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).

Example

If $f(x) = x^3 + 2$, find a formula for $f^{-1}(y)$.

$$\begin{aligned} y &= x^3 + 2 \\ x^3 &= y - 2 \\ x &= \sqrt[3]{y-2} \end{aligned}$$

Therefore $x = f^{-1}(y) = \sqrt[3]{y-2}$. Usually we relabel x and y and write $f^{-1}(x) = \sqrt[3]{x-2}$. Unless asked for $f^{-1}(x)$, do not relabel anything.

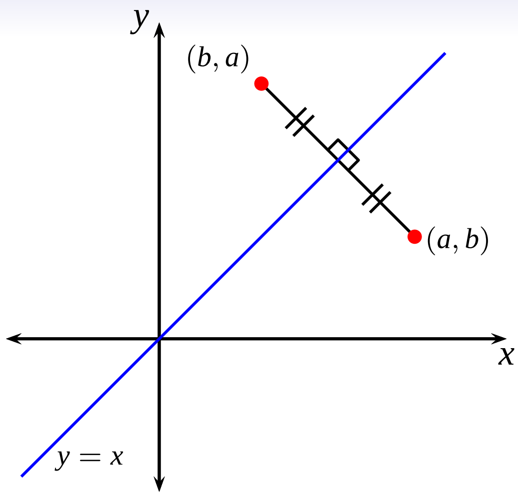
Shoes-socks

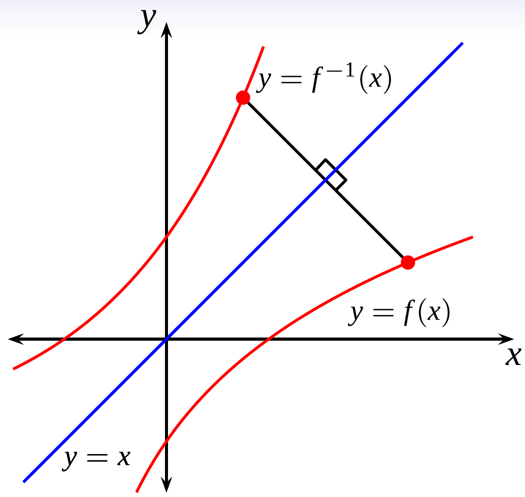
Example (Guess and Check)

If $f(x) = 2x + \sin 2x + e^{x/2}$, find $f^{-1}(1)$.

$$f(\quad) = 2(\quad) + \sin 2(\quad) + e^{(\quad)/2}$$
$$=$$
$$= 1.$$

Therefore $f^{-1}(1) =$

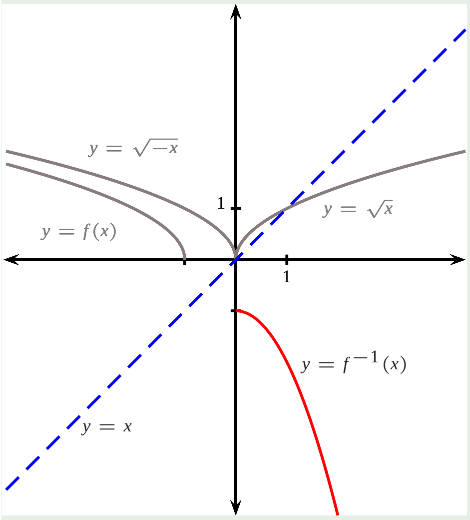




Interchanging x and y suggests a relation between the graphs of f^{-1} and f:

- Suppose (a, b) is on the graph of f.
- Then $f(a) = b$.
- Then $f^{-1}(b) = a$.
- Then (b, a) is on the graph of f^{-1}.
- (b, a) is the reflection of (a, b) in the line $y = x$.
- Thus the graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

Example

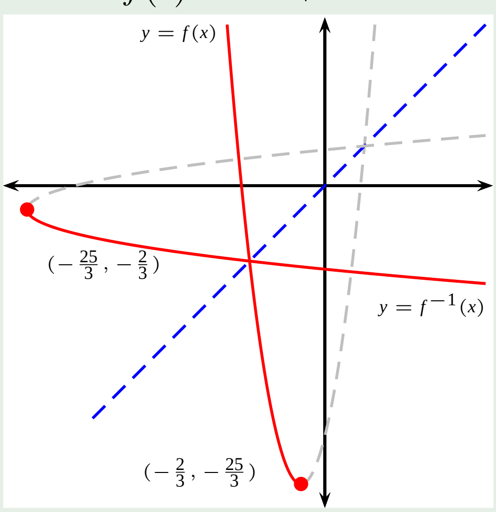


Sketch the graph of $f(x) = \sqrt{-x-1}$ and its inverse function.

- First draw the graph of $y = \sqrt{x}$.
- $y = \sqrt{-x}$ is the reflection of $y = \sqrt{x}$ in the y-axis.
- $y = f(x) = \sqrt{-x-1}$ is the shift of $y = \sqrt{-x}$ one unit to the left.
- $y = f^{-1}(x)$ is the reflection of $y = f(x)$ across the line $y = x$.

Example (What if we change the problem to $x \leq -\frac{2}{3}$?)

Given: $f(x) = 3x^2 + 4x - 7$ with domain $x \leq -\frac{2}{3}$. Find $f^{-1}(x)$.



$$\begin{aligned} 3x^2 + 4x - 7 &= y \\ 3x^2 + 4x + (-7 - y) &= 0 \end{aligned}$$

That's a quadratic equation in x. Solve:

$$\frac{-4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot (-y - 7)}}{2 \cdot 3}$$

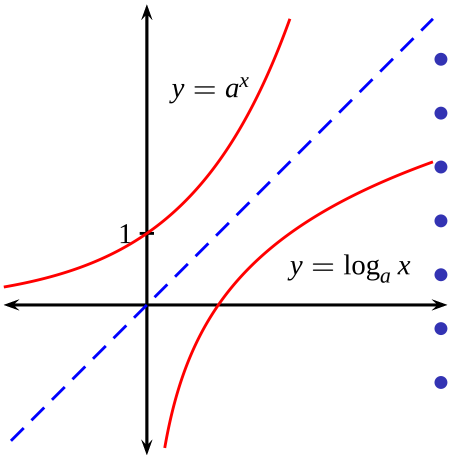
$$= -\frac{2 \pm \sqrt{25 + 3y}}{3} = -\frac{2}{3} \pm \frac{\sqrt{25 + 3y}}{3} .$$

We are given $x \leq -\frac{2}{3}$, therefore $x = -\frac{2}{3} - \frac{\sqrt{25+3y}}{3} = f^{-1}(y)$.

Final answer, relabelled:

$$f^{-1}(x) = -\frac{2}{3} - \frac{\sqrt{25 + 3x}}{3} .$$

Logarithmic Functions



- Suppose $a > 0$, $a \neq 1$.
- Let $f(x) = a^x$.
- Then f is either increasing or decreasing.
- Therefore f is one-to-one.
- Therefore f has an inverse function, f^{-1}.
- The graph shows $y = a^x$ for $a > 1$.
- The graph of $y = \log_a x$ is the reflection of this in the line $y = x$.

Definition ($\log_a x$)

The inverse function of $f(x) = a^x$ is called the logarithmic function with base a, and is written $\log_a x$. It is defined by the formula

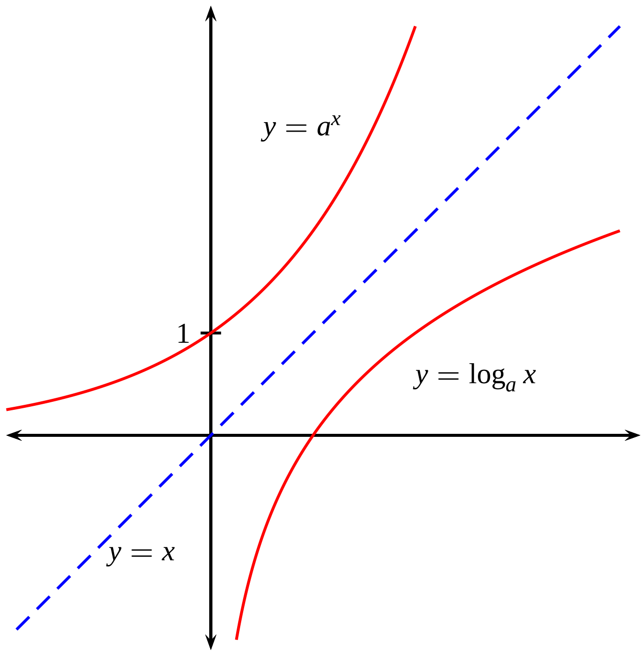
$$\log_a x = y \qquad \Leftrightarrow \qquad a^y = x.$$

If $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

Example

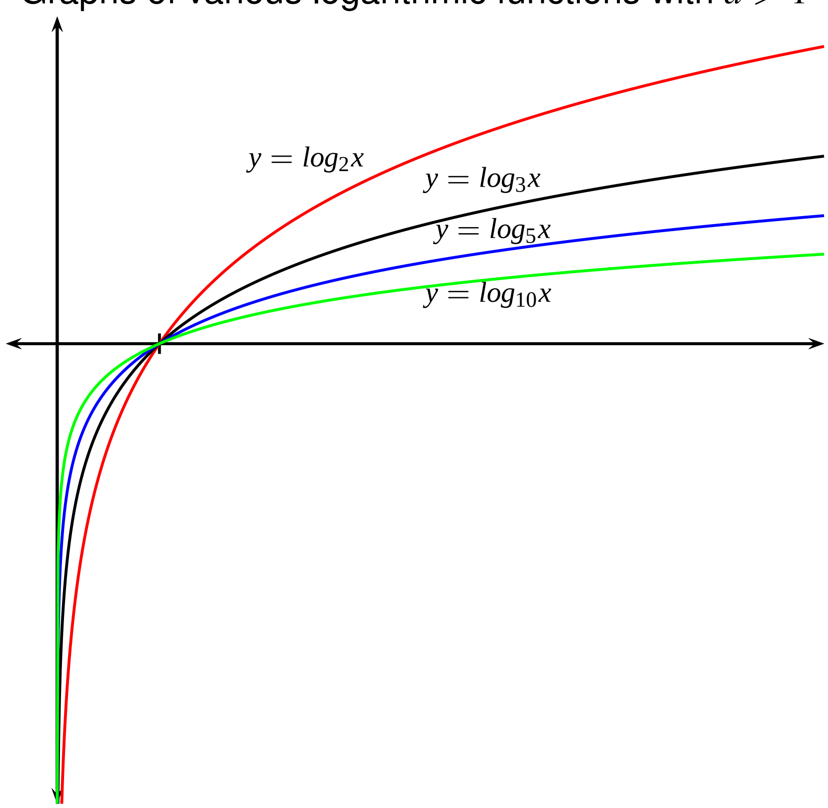
Evaluate:

1. $\log_3 81 = 4$ because $3^4 = 81$.
2. $\log_{25} 5 = \frac{1}{2}$ because $25^{1/2} = 5$.
3. $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.



- Suppose $a > 1$.
- Domain of a^x: $\mathbb{R}$.
- Range of a^x: $(0, \infty)$.
- Domain of $\log_a x$: $(0, \infty)$.
- Range of $\log_a x$: $\mathbb{R}$.
- $\log_a(a^x) = x$ for $x \in \mathbb{R}$.
- $a^{\log_a x} = x$ for $x > 0$.

Graphs of various logarithmic functions with $a > 1$



Theorem (Properties of Logarithmic Functions)

If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y, a, b > 0$ and r is any real number, then

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a(x^r) = r\log_a x$.
4. $\log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$

Example

Use the properties of logarithms to evaluate the following:

$$\begin{aligned} &\log_4 2 + \log_4 32 \\ = & \\ = & \\ = & \end{aligned}$$

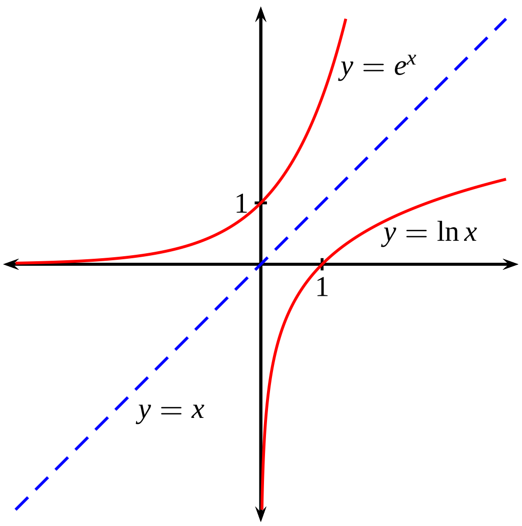
$$\begin{aligned} &\log_2 80 - \log_2 5 \\ = & \\ = & \\ = & \end{aligned}$$

Natural Logarithms

Definition ($\ln x$)

The logarithm with base e is called the natural logarithm, and has a special notation:

$$\log_e x = \ln x.$$



- $\ln x = y \quad \Leftrightarrow \quad e^y = x$.
- $\ln(e^x) = x$ for $x \in \mathbb{R}$.
- $e^{\ln x} = x$ for $x > 0$.

Example

Solve the equation $e^{5-3x} = 10$

$=$

$=$

$=$

$x =$

Calculator: $x \approx 0.8991.$

Example

Solve the equation

$$e^{2x} - 3e^x - 4 = 0$$

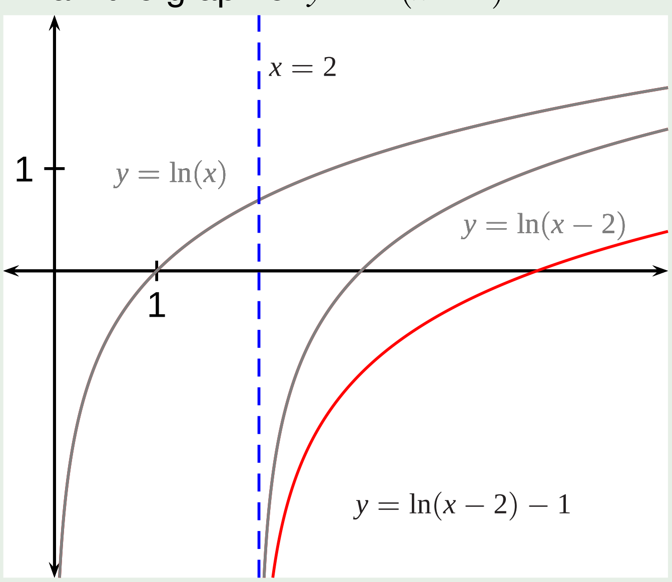
Set $e^x = u$. Then $e^{2x} = u^2$.

$$u^2 - 3u - 4 = 0$$
$$(u-4)\,(u+1) = 0$$

$$\begin{aligned} u &= 4 && \text{or} && u = -1 \\ e^x &= 4 && \text{or} && e^x = -1 \\ x &= \ln 4 && \text{or} && \text{no real solution} \\ x &\approx 1.386294361 \end{aligned}$$

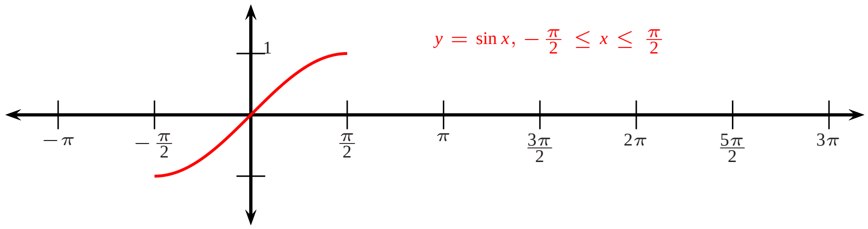
Example

Draw the graph of $y = \ln(x-2) - 1$.

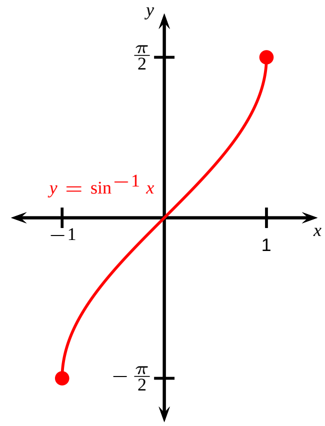


- Graph $y = \ln(x)$ assumed given.
- $f(x-2)$ shifts graph 2 units to the right.
- $g(x) - 1$ shifts graph 1 unit down.

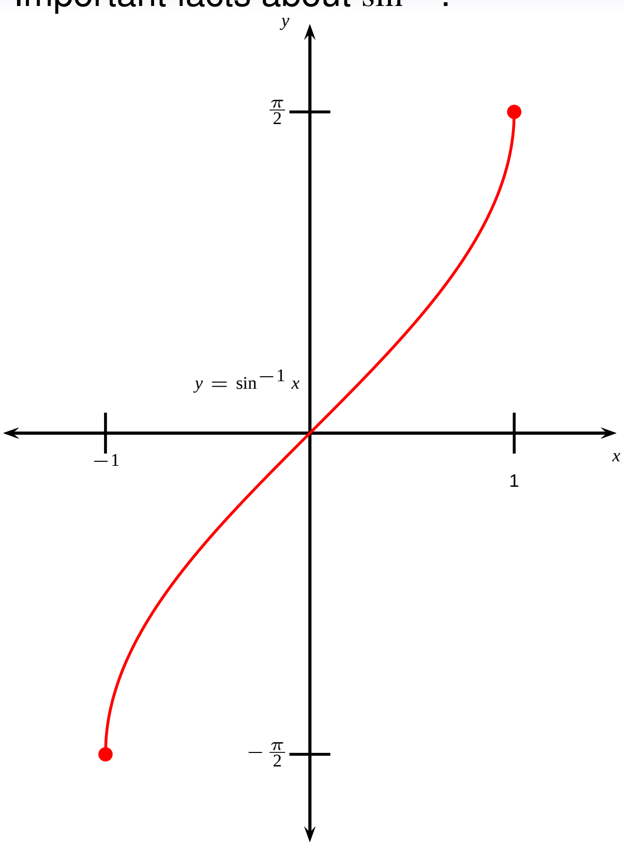
Inverse Trigonometric Functions



- $\sin x$ isn't one-to-one.
- It is if we restrict the domain to $[-\pi/2, \pi/2]$.
- Then it has an inverse function.
- We call it arcsin or $\sin^{-1}$.
- $\sin^{-1} x = y \Leftrightarrow \sin y = x$ and $-\pi/2 \leq y \leq \pi/2$.



Important facts about $\sin^{-1}$:



1. Domain:
2. Range:
3. $\sin^{-1} x = y \Leftrightarrow \sin y = x$ and $-\pi/2 \leq y \leq \pi/2$.
4. $\sin^{-1}(\sin x) = x$ for $-\pi/2 \leq x \leq \pi/2$.
5. $\sin(\sin^{-1} x) = x$ for $-1 \leq x \leq 1$.
6. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

Example

Find $\sin^{-1}(\sin 1.5)$.

- $\pi/2 \approx 1.57.$
- Therefore $-\pi/2 \leq 1.5 \leq \pi/2$.
- Therefore $\sin^{-1}(\sin 1.5) = 1.5$.

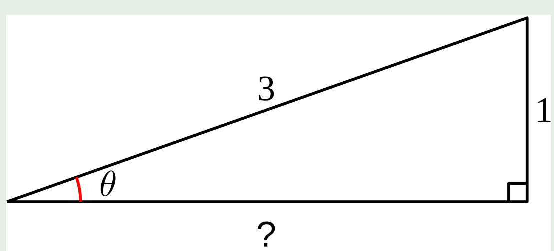
Example

Find $\sin^{-1}\left(\dfrac{1}{2}\right)$

- $\sin(\quad) = 1/2$.
- $-\pi/2 \leq \quad \leq \pi/2$.
- Therefore $\sin^{-1}\left(\frac{1}{2}\right) = \quad$.

Find $\tan\left(\sin^{-1}\left(\dfrac{1}{3}\right)\right)$

- Let $\theta = \sin^{-1}(1/3)$, so $\sin\theta = 1/3$.
- Draw a right triangle with opposite side 1 and hypotenuse 3.
- Length of adjacent side
  $=$
- Then $\tan(\sin^{-1}\frac{1}{3}) =$



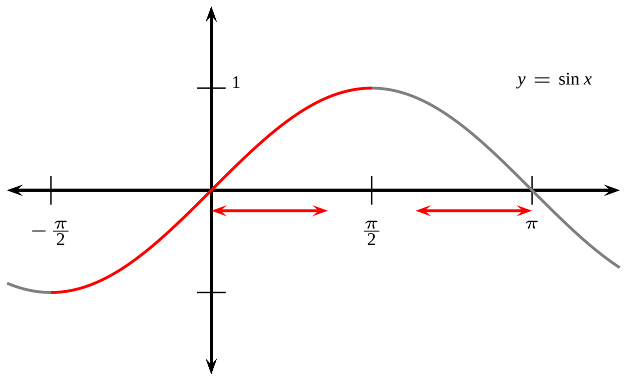
Example

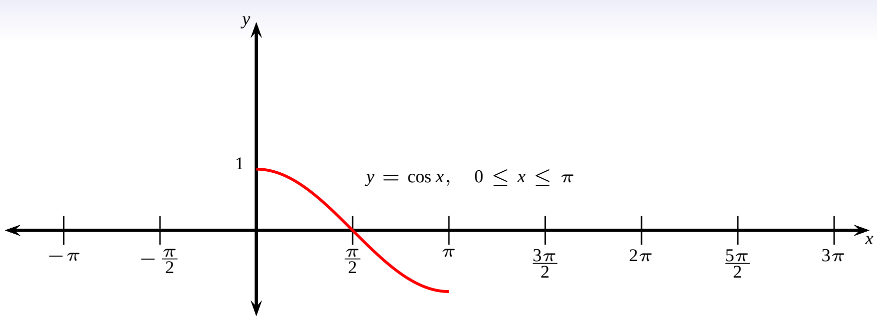
Find $\sin^{-1}(\sin 2)$.

- 2 is not between $-\pi/2$ and $\pi/2$.
- $\sin 2 = \sin a$ for some a between $-\pi/2$ and $\pi/2$.

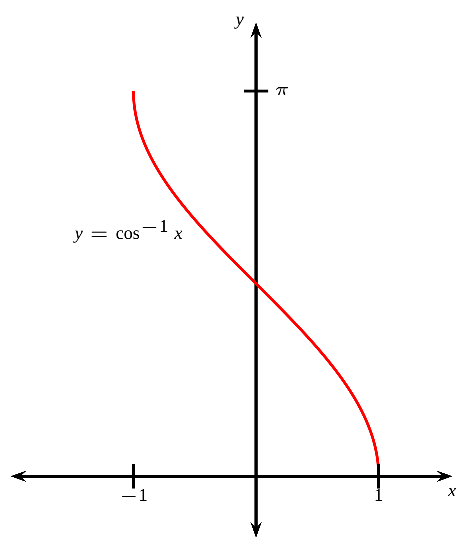
$$a - 0 = \pi - 2.$$

Therefore $\sin^{-1}(\sin 2) = \sin^{-1}(\sin a)$
$= a = \pi - 2.$

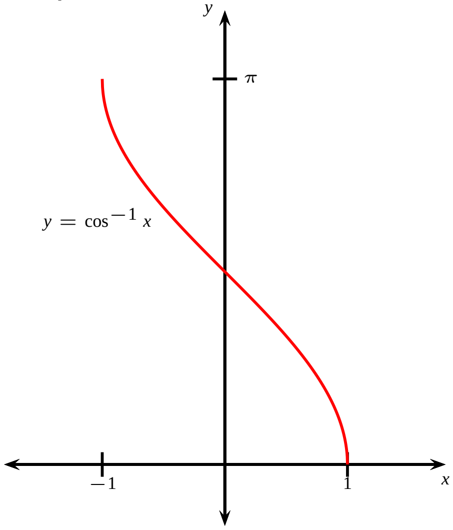




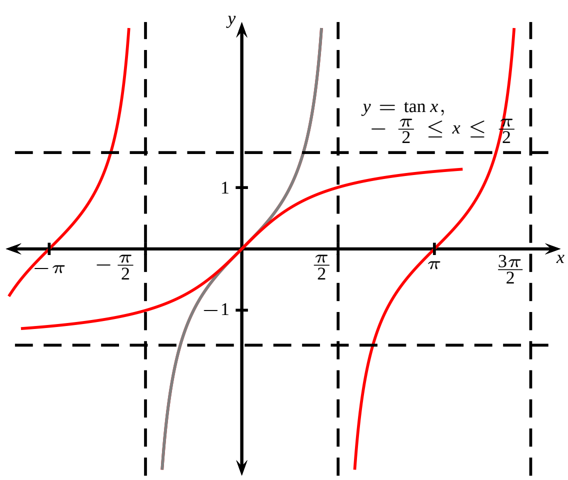
- Same for $\cos x$.
- Restrict the domain to $[0, \pi]$.
- The inverse is called arccos or $\cos^{-1}$.
- $\cos^{-1}(x) = y \Leftrightarrow \cos y = x$ and $0 \leq y \leq \pi$.



Important facts about $\cos^{-1}$:



1. Domain:
2. Range:
3. $\cos^{-1} x = y \Leftrightarrow \cos y = x$ and $0 \leq y \leq \pi$.
4. $\cos^{-1}(\cos x) = x$ for $0 \leq x \leq \pi$.
5. $\cos(\cos^{-1} x) = x$ for $-1 \leq x \leq 1$.
6. $\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$. (The proof is similar to the proof of the formula for the derivative of $\sin^{-1} x$.)

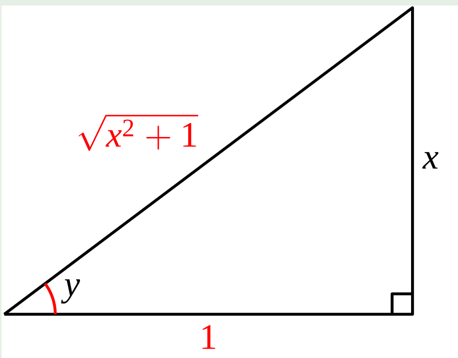


- $\tan x$ isn't one-to-one.
- Restrict the domain to $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- The inverse is called $\tan^{-1}$ or $\arctan$.
- $\tan^{-1} x = y \Leftrightarrow \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
- Domain of $\tan^{-1}$:
- Range of $\tan^{-1}$:
- $\lim_{x \to \infty} \tan^{-1} x =$
- $\lim_{x \to -\infty} \tan^{-1} x =$

Example

Simplify the expression $\cos(\tan^{-1} x)$.

- Let $y = \tan^{-1} x$, so $\tan y = x$.
- Draw a right triangle with opposite x and adjacent 1.
- Length of hypotenuse $= \sqrt{1^2 + x^2}$.
- Then $\cos(\tan^{-1} x) =$



Example

Evaluate

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right).$$

$$\frac{1}{x-2} \to \infty \qquad \text{as} \qquad x \to 2^+.$$

Therefore

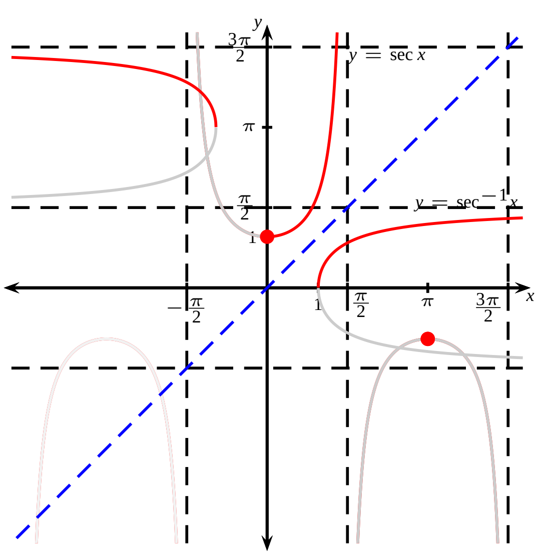
$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) =$$

The remaining inverse trigonometric functions aren't used as often:

$$\begin{array}{llllll} y=\csc^{-1}x & (|x|\geq 1) & \Leftrightarrow & \csc y = x & \text{and} & y\in\left(0,\frac{\pi}{2}\right]\cup\left(\pi,\frac{3\pi}{2}\right] \\ y=\sec^{-1}x & (|x|\geq 1) & \Leftrightarrow & \sec y = x & \text{and} & y\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right) \\ y=\cot^{-1}x & (|x|\in\mathbb{R}) & \Leftrightarrow & \cot y = x & \text{and} & y\in(0,\pi) \end{array}$$

We will however make use of $\sec^{-1}x$: we discuss in detail its domain.

$$y=\sec^{-1}x \quad (|x|\geq 1) \quad \Leftrightarrow \quad \sec y = x \quad \text{and} \quad y\in \textbf{?}\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$$



- Plot $\sec x$.
- Restrict domain to make one-to-one: Two common choices:
  $x\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ and
  $x\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$.
- $x\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ is good because the domain is easiest to remember: an interval without a point. **NOT our choice.**
- $x\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$ is good because $\tan x$ is positive on both intervals, resulting in easier differentiation and integration formulas. **Our choice.**

Table of derivatives of inverse trigonometric functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cot^{-1} x) = -\frac{1}{1+x^2}$$