The SdS Scalar Spectrum

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In this paper, we calculate a semiclassical approximation to the eigenvalues of the eigenvalue equation $-\Box u = \lambda_S u$ for Euclidean Schwarzschild-de Sitter. Here, $\Box = \nabla^\sigma \nabla_\sigma$ is the Laplace-Beltrami operator, u the scalar eigenfunction and λ_S its eigenvalues. It is λ_S we will obtain.
Note that in this paper I shall use $h = c = k = 1$.

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1.1 The 1-loop Determinant

In order to calculate 1-loop quantum corrections to the Schwarzschild-de Sitter partition function, we must calculate its 1-loop determinant. For Euclidean Schwarzschild-de Sitter, which has a constraint functional added to the action, the 1-loop determinant consists of the eigenvalues of the following three equations:

1. The tensor modes:
$$-\Box\phi_{\mu\nu} - 2R_{\mu\alpha\nu\beta}\phi^{\alpha\beta} = \lambda_T\phi_{\mu\nu}. \tag{1}$$
2. The vector modes:
$$\left(-\Box - \tilde{\Lambda}\right)\eta_\mu = \lambda_V\eta_\mu. \tag{2}$$
3. The scalar modes:
$$-\Box u = \lambda_S u. \tag{3}$$

Here, $\Box = \nabla^\sigma\nabla_\sigma$ and λ_T, λ_V and λ_S are the eigenvalues of the tensor, vector and scalar differential equations, respectively. The 1-loop determinant also consists of non-differential terms that we integrate over (terms that are, for example, functions of the black hole mass $u(r_b)$, where $u(r_b)$ the scalar eigenfunction evaluated on the black hole horizon r_b), but since these follow from the above equations and are easy to evaluate, the challenge comes the most from determining the eigenvalues.
In this paper, we will obtain a semiclassical approximation to λ_S.

1.2 The SdS Laplace-Beltrami Equation

The differential operator for the Schwarzschild-de Sitter scalar modes is the Laplace-Beltrami operator $\Box$, which can be conveniently expressed as

$$\begin{aligned}\Box u &= \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu u\right) &(4)\\ &= f(r)^{-1}\partial_\tau^2 u + r^{-2}\partial_r\left(r^2 f(r)\partial_r u\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta u\right) + \frac{1}{r^2\sin\theta^2}\partial_\phi^2 u, &(5)\end{aligned}$$

where we plugged in the Euclidean Schwarzschild-de Sitter metric

$$ds^2 = f(r)d\tau^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin\theta^2 d\phi^2\right), \tag{6}$$

where

$$f(r) = \left(1 - \frac{2GM}{r} - \frac{r^2}{l_{dS}^2}\right), \tag{7}$$

with $l_{dS}^2 = \frac{3}{\Lambda}$.

This equation is separable in variables by taking an Ansatz

$$u = T(\tau)R(r)Y(\theta,\phi) = e^{-i\omega\tau}R(r)Y_l^m(\theta,\phi), \tag{8}$$

where $Y_l^m(\theta,\phi)$ the spherical harmonics on the S^2 and $R(r)$ the radial function. In Lorentzian signature, ω would now be a parameter that is to be determined from the equation satisfied by u, however, in Euclidean signature, ω's form follows immediately from the periodicity of Euclidean time τ. Namely, since time τ in Euclidean signature is periodic with $\tau \sim \tau + \beta$, u naturally has to respect and remain invariant under this periodicity as well. This constrains ω to be

$$\boxed{\omega = \frac{2\pi n}{\beta}, \qquad n \in \mathbb{Z}}, \tag{9}$$

since then

$$T(\tau) = e^{-i\omega\tau} \to e^{-i\frac{2\pi n}{\beta}(\tau+\beta)} = e^{-i\frac{2\pi n}{\beta}\tau}e^{-i2\pi n} = e^{-i\omega\tau} = T(\tau). \tag{10}$$

Inserting the Ansatz into the Laplace-Beltrami operator (4), it becomes

$$\Box u = -f(r)^{-1}\omega^2 u + \frac{e^{-i\omega\tau}Y_l^m(\theta,\phi)}{r^2}\partial_r\left(r^2 f(r)\partial R\right) - \frac{l(l+1)}{r^2}u, \tag{11}$$

where $l(l+1)$ are the eigenvalues of the spherical harmonic $Y_l^m(\theta,\phi)$ from

$$\frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta Y\right) + \frac{1}{r^2\sin\theta^2}\partial_\phi^2 Y = -\frac{l(l+1)}{r^2}Y. \tag{12}$$

Now, the eigenvalue equation $-\Box u = \lambda u$, separated in variables, can be written as

$$-f(r)^{-1}\omega^2 u + \frac{e^{-i\omega\tau}Y_l^m(\theta,\phi)}{r^2}\partial_r\left(r^2 f(r)\partial_r R\right) - \frac{l(l+1)}{r^2}u = -\lambda u, \tag{13}$$

and if we divide by $e^{-i\omega\tau}Y_l^m(\theta,\phi)$ it can be expressed purely in what I will call **the Euclidean SdS radial equation**

$$\boxed{\left(\frac{1}{r^2}\partial_r\left(r^2 f(r)\partial_r\right) - f(r)^{-1}\omega^2 - \frac{l(l+1)}{r^2} + \lambda\right)R = 0}. \tag{14}$$

Finding the eigenfunction u now boils down to finding the solution $R(r)$ to this equation. The eigenvalues λ_S follows, normally, from placing boundary conditions on this solution, however, we will have to take a different approach.

1.3 The Full Radial Equation

It can be found that the aforementioned radial equation can be rewritten to what, in the mathematical literature, is called a 'generalized Lamé equation' [1]:

$$\frac{d^2w}{dz^2} + \left(\sum_{j=1}^{N} \frac{\gamma_j}{z-a_j}\right)\frac{dw}{dz} + \frac{\Phi(z)}{\prod_{j=1}^{N}(z-a_j)}w = 0, \tag{15}$$

where N the number of regular poles in the differential equation, γ_i are constant factors and $\Phi(z)$ is a polynomial of at most degree $N-2$. For Schwarzschild-de Sitter, $N = 4$ and the four a_j correspond to the roots $r = 0$, $r = r_-$, $r = r_b$ and $r = r_c$, where r_- the negative root of the emblackening factor $f(r)$.

Unfortunately, the solution to this equation does not have an analytic expression. The best what the literature has to offer is conditions on γ_j and $\Phi(z)$ for when polynomial solutions exist; none of which, *guarantee* a polynomial solution. Since we don't know the solution, we can't calculate the eigenvalues. From this, we can therefore not obtain the eigenvalue spectrum. We will have to turn to approximations.

2 Matched Asymptotic Expansion

2.1 Why not WKB?

One of the first methods physicists turn to to simplify a differential equation is the WKB approximation. While the radial equation (14) can indeed be recast to a Regge-Wheeler-Zerilli-type equation[1], for WKB to be mathematically sensible, we would need a small parameter multiplying *only* the second derivative in the radial equation [3]. In our theory, we have a natural candidate for such a small parameter. To see this, realize that in order to perform calculations in Schwarzschild-de Sitter in semiclassical gravity consistently, we need to require that the size of the black hole is well above the Planck scale, that is

$$r_b \gg l_{Planck}. \tag{16}$$

This condition becomes important for small mass black holes. Expanding for small mass M, this condition becomes[2]

$$2GM + \frac{8}{3}G^3M^3\Lambda + \mathcal{O}\left(M^7\right) \gg \sqrt{G}. \tag{17}$$

From here, it seems easiest to take M very large and use $1/M$ as a small parameter, but remember that M is bounded above by the Nariai mass $M_N = \frac{1}{3G\sqrt{\Lambda}}$. This means that for M to be able to be very large, $G\sqrt{\Lambda}$ needs to be very small. It is therefore easier to take

$$G\sqrt{\Lambda} \ll 1, \tag{18}$$

and have (from condition (17))

$$M \gg \frac{1}{2}M_{Planck}. \tag{19}$$

It is furthermore more convenient to work with a single small parameter, so I shall put $G = 1$ for the rest of this paper, for convenience. We can always later restore it by dimensional analysis. We therefore have, for consistent semiclassical calculations, the small parameter

$$\boxed{\sqrt{\Lambda} \ll 1}. \tag{20}$$

The problem that now arises is that this small parameter pops up everywhere in the radial equation and the equation cannot be massaged such that the parameter only multiplies the second derivative. WKB is therefore not a good candidate.

[1] Or a Schrödinger-type equation, if you prefer to call it that.

[2] Remember that we're working in natural units where $c = k = \hbar = 1$.

2.2 Unnecessary structure

One of the things that severely complicates the Laplace-Beltrami differential equation is that it takes into account the *global* structure of Schwarzschild-de Sitter, meaning it will take into account *all* the poles the metric possesses, even those for negative radius (like r_-), outside of the valid range of the radial coordinate r. For Euclidean Schwarzschild-de Sitter, which rounds off at the black hole horizon at r_b and the cosmological horizon at r_c, and for which nothing exists outside of the range of $r_b \leq r \leq r_c$, this is a lot of unnecessary structure. For us, it suffices to have a solution for the differential equation in the coordinate range $r_b \leq r \leq r_c$. To this end, we can use **matched asymptotic expansion**.

My use of this technique is heavily inspired by a paper by Baumann et al., which uses matched asymptotic expansion in order to calculate the quasi-bound state spectra of scalar and vector fields around Kerr black holes [2]. The idea is the following: we obtain an approximate solution of the radial equation near r_b and then, separately, an approximate solution of the radial equation near r_c. We then want these two approximate solutions to match for any r in the range $r_b \leq r \leq r_c$. We do this matching by expanding the solutions in the small parameter $\sqrt{\Lambda}$ and matching the first terms in the expansion up to order $\sqrt{\Lambda}$. An expression for λ_S then follows from the demand that these terms match. Let us obtain the approximate solutions and see how this works in practice.

2.3 Approximate solution near r_b

For convenience, I will define a new variable x, defined as

$$x \equiv \frac{r - r_b}{r_c - r_b}. \tag{21}$$

$x = 0$ will then correspond to $r = r_b$ and $x = 1$ to $r = r_c$. Writing out the whole radial equation (14) in terms of this coordinate gives us an expression that is not very illuminating, so this will be relegated to the Appendix. It can be shown however that the full equation, near $x = 0$, becomes

$$\left[\partial_x^2 + \left(\frac{1}{x} + S\right)\partial_x + \frac{A}{x} + \frac{B}{x^2} + E\right] R_b(x) = 0, \tag{22}$$

where $R_b(x)$ is the approximate solution near the black hole horizon and

$$A = -\frac{l_{dS}^2 r_b}{r_c + 2r_b}\left(\frac{2\omega^2\left(r_b^2 + r_b r_c + r_c^2\right) l_{dS}^2}{\left(r_c^2 + r_b\left(r_c - 2r_b\right)\right)^2} + \frac{l(l+1)}{r_b^2} - \lambda_S\right), \tag{23}$$

$$B = -\left(\frac{\omega r_b l_{dS}^2}{\left(r_c^2 + r_b\left(r_c - 2r_b\right)\right)}\right)^2, \tag{24}$$

$$S = \frac{r_c^2 + 3r_b r_c - r_b^2}{r_b(r_c + 2r_b)} - 2, \tag{25}$$

$$E = -\omega^2 l_{dS}^4 \frac{3r_b^4 + 10r_b^3 r_c + 11r_b^2 r_c^2 + 2r_b r_c^3 + r_c^4}{(r_c - r_b)^2(2r_b + r_c)^4} + l(l+1)l_{dS}^2 \frac{-5r_b^2 + r_b r_c + r_c^2}{r_b^2(2r_b + r_c)^2} \tag{26}$$

$$+ \lambda_S l_{dS}^2 \frac{r_b^2 + r_b r_c + r_c^2}{(2r_b + r_c)^2}. \tag{27}$$

I have retained the x-dependence for terms that have at least a $1/x$ pole, which will blow up near $x = 0$ and contribute the greatest. Additionally, for terms that do not exhibit a $1/x$ pole, I have added them to the differential equation, but evaluated them at $x = 0$. So, for example, a term that goes as $\frac{r_c - r_b}{(r_c - r_b)x + r_b}$, I have added at $x = 0$, meaning it will contribute to equation (22) as $\frac{r_c - r_b}{r_b}$. The reason for this is that some terms, while not exhibiting a pole at $x = 0$ for general mass M, will blow up in the limit $M \to M_N$, where M_N the Nariai mass, and $x \to 0$. In this limit we have that $r_b \to r_c$, and so it is easy that terms like $\frac{1}{(r_c - r_b)(1-x)}$ will blow up. Terms like this will affect how accurate the solution is near the Nariai limit, so they need to be included. The inclusion of these non-pole terms changes the mathematical structure and produces a different solution to the differential equation than without them. Since these non-pole terms need to be included, we might as well include all other terms evaluated at $x = 0$, since, as it turns out, including one or all of them results in the same solution: the only difference is that the parameters in the approximate solution contain more information about the structure of the differential equation near $x = 0$.

The solution to equation (22) is

$$R_b(x) = e^{\frac{1}{2}\left(-x\left(S + \sqrt{S^2 - 4E}\right) + 2i\sqrt{B}\log x\right)}\left[c_1 U\left(-m, 1 + 2i\sqrt{B}, x\sqrt{S^2 - 4E}\right)\right. \tag{28}$$

$$\left. + c_2 L_m^{2i\sqrt{B}}\left(x\sqrt{S^2 - 4E}\right)\right], \tag{29}$$

where

$$m \equiv \frac{2A - S - \sqrt{S^2 - 4E} - 2i\sqrt{B}\sqrt{S^2 - 4E}}{2\sqrt{S^2 - 4E}}, \tag{30}$$

$U(a,b,z)$ the Tricomi confluent hypergeometric function and $L_a^b(z)$ the generalized Laguerre polynomial. We want this function to behave regular near $x=0$; the reasoning is that, since we have added constraints to the path integral to regularize the conical singularities appearing in the Schwarzschild-de Sitter solution that shows up in the saddle point approximation, the geometry should be regular near both horizons. In addition, we want the scalar eigenfunction to be regular at the horizons, such that it's regular and normalizable with respect to the inner product

$$\left\langle h_{\mu\nu(n)}, h^{\mu\nu}_{(m)} \right\rangle = \frac{1}{32\pi G} \int d\Omega_2 \int_0^\beta d\tau \int_{r_b}^{r_c} dr\, h_{\mu\nu(n)} h^{\mu\nu}_{(m)}\, d^4x\sqrt{g}, \tag{31}$$

as this will lead to a well-defined 1-loop determinant.

To investigate the regularity of the function $R_b(x)$ near $x=0$, we perform a Taylor expansion around this point. For extra clarity, we will define

$$\sqrt{B} = iP_{\omega,b}, \tag{32}$$

such that

$$P_{\omega,b} = \frac{\omega r_b l_{dS}^2}{\left(r_c^2 + r_b\left(r_c - 2r_b\right)\right)}. \tag{33}$$

The Taylor expansion around $x=0$ of $R_b(x)$ is

$$R_b(x) \approx c_1 \left[x^{-P_{\omega,b}} \left(\text{Constant} + \mathcal{O}(x)\right) + \left(\sqrt{S^2-4E}x\right)^{2P_{\omega,b}} \left(\text{Constant} + \mathcal{O}(x)\right) \right] \tag{34}$$

$$+ c_2 x^{-P_{\omega,b}} \left(\text{Constant} + \mathcal{O}(x)\right). \tag{35}$$

Notice that the expression multiplied by c_1 will blow up as $x \to 0$ regardless of whether $P_{\omega,b}$ is positive or negative (unless, trivially, $\omega=0$). c_1 therefore needs to be put to zero. However, the expression multiplied by c_2 can be made to behave regular near $x=0$ if $P_{\omega,b} \leq 0$. Since $\frac{r_b}{(r_c^2 + r_b(r_c - 2r_b))} > 0$ for $0 \leq M \leq M_N$, we would therefore need $\omega \leq 0$. We therefore restrict $\omega = 2\pi n/\beta$ to be negative by restricting $n \in \mathbb{Z}^{\leq}$, where $\mathbb{Z}^{\leq}$ all non-positive integers. We can do this without losing any generality: all that was required of the eigenfunction u was for it to be invariant under $\tau \sim \tau + \beta$. For $n \leq 0$ this is just as well realized. In addition, ω appears only as ω^2 in the radial equation, so whether ω is positive or negative should not matter for the spectrum of λ_S. We therefore put $c_1 = 0$ and $n \leq 0$ and obtain

$$\boxed{R_b(x) = c_2 e^{\frac{1}{2}\left(-x\left(S+\sqrt{S^2-4E}\right)+2i\sqrt{B}\log x\right)} L_m^{2i\sqrt{B}}\left(x\sqrt{S^2-4E}\right)}. \tag{36}$$

2.4 Approximate solution near r_c

The radial equation near r_c, at $x = 1$, is significantly simpler:

$$\left[\partial_x^2 - \frac{1}{1-x}\partial_x + \frac{C}{1-x} + \frac{D}{(1-x)^2}\right] R_c(x) = 0, \tag{37}$$

where

$$C = -\frac{l_{dS}^2 r_c}{r_b + 2r_c}\left(\frac{2\omega^2\left(r_b^2 + r_b r_c + r_c^2\right) l_{dS}^2}{\left(r_b^2 + r_c\left(r_b - 2r_c\right)\right)^2} + \frac{l(l+1)}{r_c^2} - \lambda_S\right), \tag{38}$$

$$D = -\left(\frac{\omega r_c l_{dS}^2}{\left(r_b^2 + r_c\left(r_b - 2r_c\right)\right)}\right)^2 \equiv -P_{\omega,c}^2, \tag{39}$$

where we have immediately defined $P_{\omega,c}$ for later convenience. The reason that the solution near the cosmological horizon is simpler, is because we do not have to include any non-pole terms: there is only one non-pole term which becomes relevant in the de Sitter ($M \to 0$) limit, but since we're doing a semiclassical approximation and therefore do not expect our result to hold below the Planck mass, we do not care for this term. We therefore only have to pay attention to the pole terms. The solution to this equation is

$$R_c = c_3 J_{2P_{\omega,c}}\left(2\sqrt{C}\left(1 - 2x + x^2\right)^{1/4}\right)\Gamma\left(1 + 2P_{\omega,c}\right) \tag{40}$$

$$+ c_4 J_{-2P_{\omega,c}}\left(2\sqrt{C}\left(1 - 2x + x^2\right)^{1/4}\right)\Gamma\left(1 - 2P_{\omega,c}\right), \tag{41}$$

where $J_n(x)$ the Bessel function of the first kind and $\Gamma(x)$ the gamma function. We expand near the cosmological horizon, near $x = 1$, to see which function will diverge there. Near $x = 1$, $R_c(x)$ goes as

$$R_c(x) \approx c_3\left(C\sqrt{1 - 2x + x^2}\right)^{P_{\omega,c}}\left(1 + \mathcal{O}\left(1 - 2x + x^2\right)\right) \tag{42}$$

$$+ c_4\left(C\sqrt{1 - 2x + x^2}\right)^{-P_{\omega,c}}\left(1 + \mathcal{O}\left(1 - 2x + x^2\right)\right). \tag{43}$$

Since we had chosen $n \leq 0$ and $\frac{r_c}{\left(r_b^2 + r_c(r_b - 2r_c)\right)} < 0$ for $0 \leq M \leq M_N$, we have that $P_{\omega,c} \geq 0$. Therefore, for the solution to be regular as $x \to 1$, we have to put $c_4 = 0$. We thus obtain

$$\boxed{R_c(x) = c_3 \Gamma\left(1 + 2P_{\omega,c}\right) J_{2P_{\omega,c}}\left(2\sqrt{C}\left(1 - 2x + x^2\right)^{1/4}\right)}. \tag{44}$$

2.5 Matching R_b and R_c

We are now in a position to start matching the two solutions to obtain a condition on λ_S. As is explained in Baumann et al.'s paper [2], we can't naively do a Taylor expansion in

$\sqrt{\Lambda}$ on $R_b(x)$ and $R_c(x)$: for the pole terms, such as $P_{\omega,b}$ and $P_{\omega,c}$, *all* terms in such an expansion would become important as $x \to 0$ or $x \to 1$, respectively. In addition, we had seen that the form of $P_{\omega,b}$ and $P_{\omega,c}$ are important to determine the boundary conditions put on R_b and R_c (which constants of c_1, c_2, c_3 and c_4 to put to zero) for regularity. Expanding these terms in $\sqrt{\Lambda}$ would disturb the form of these terms, to the point that we would have to impose different boundary conditions.

Instead, we introduce a *matching parameter*

$$\xi = \frac{x}{\sqrt{\Lambda}} \tag{45}$$

through which we will carry out the expansion in $\sqrt{\Lambda}$, while keeping ξ fixed. Rewriting the solutions in terms of this parameter, by inserting $x = \sqrt{\Lambda}\xi$, and expanding in $\sqrt{\Lambda}$, we obtain the matching equation

$$c_2 \left(\sqrt{\Lambda}\xi\right)^{-P_{\omega,b}} \left[L_m^{-2P_{\omega,b}}(0)\right. \tag{46}$$

$$\left. - \xi\sqrt{\Lambda}\left[\frac{1}{2}\left(S + \sqrt{S^2 - 4E}\right) L_m^{-2P_{\omega,b}}(0) + \sqrt{S^2 - 4E}L_{m-1}^{1-2P_{\omega,b}}(0)\right] + \mathcal{O}\left(\xi^2\Lambda\right)\right] \tag{47}$$

$$= c_3\left[J_{2P_{\omega,c}}\left(2\sqrt{C}\right)\Gamma(1 + 2P_{\omega,c}) + \xi\sqrt{\Lambda}\Gamma(1 + 2P_{\omega,c})\left[\sqrt{C}J_{1+2P_{\omega,c}}\left(2\sqrt{C}\right)\right.\right. \tag{48}$$

$$\left.\left. - P_{\omega,c}J_{2P_{\omega,c}}\left(2\sqrt{C}\right)\right] + \mathcal{O}\left(\xi^2\Lambda\right)\right]. \tag{49}$$

The left-hand side of the equation comes from the expansion of R_b and the right-hand side from the expansion of R_c. We can write the $\left(\sqrt{\Lambda}\xi\right)^{-P_{\omega,b}}$ term on the left-hand side as

$$\left(\sqrt{\Lambda}\xi\right)^{-P_{\omega,b}} = e^{-P_{\omega,c}\log\left(\sqrt{\Lambda}\xi\right)} = 1 - P_{\omega,c}\log\left(\sqrt{\Lambda}\xi\right) + \mathcal{O}\left(\log^2\sqrt{\Lambda}\xi\right). \tag{50}$$

This term will therefore lead to logarithmic terms, but since the right-hand side of the matching equation has no logarithmic terms to match to, we can ignore these terms to order of $\sqrt{\Lambda}$. If we would take into account higher orders of the expansion, we would be able to match these logarithmic terms, something we will not do in this paper.

We now put $c_3 = c_2$ and obtain to order 1 the matching condition

$$L_m^{-2P_{\omega,b}}(0) = J_{2P_{\omega,c}}\left(2\sqrt{C}\right)\Gamma\left(1 + 2P_{\omega,c}\right) \tag{51}$$

and to order $\sqrt{\Lambda}$ the matching condition

$$\frac{1}{2}\left(S + \sqrt{S^2 - 4E}\right) L_m^{-2P_{\omega,b}}(0) + \sqrt{S^2 - 4E}L_{m-1}^{1-2P_{\omega,b}}(0) \tag{52}$$

$$= \Gamma(1 + 2P_{\omega,c})\left[P_{\omega,c}J_{2P_{\omega,c}}\left(2\sqrt{C}\right) - \sqrt{C}J_{1+2P_{\omega,c}}\left(2\sqrt{C}\right)\right]. \tag{53}$$

We divide the order $\sqrt{\Lambda}$ matching condition by the order 1 matching condition to obtain a single matching condition for both conditions. We get

$$\frac{1}{2}\left(S+\sqrt{S^2-4E}\right)+\sqrt{S^2-4E}\frac{L_{m-1}^{1-2P_{\omega,b}}(0)}{L_m^{-2P_{\omega,b}}(0)} \tag{54}$$

$$=P_{\omega,c}-\sqrt{C}\frac{J_{1+2P_{\omega,c}}\left(2\sqrt{C}\right)}{J_{2P_{\omega,c}}\left(2\sqrt{C}\right)}. \tag{55}$$

Using the identity $L_n^{(\alpha)}(0)=\binom{n+\alpha}{n}$ this can be further simplified to

$$\boxed{\frac{A-SP_{\omega,b}}{1-2P_{\omega,b}}-P_{\omega,c}=\sqrt{C}\frac{J_{1+2P_{\omega,c}}\left(2\sqrt{C}\right)}{J_{2P_{\omega,c}}\left(2\sqrt{C}\right)}}. \tag{56}$$

This is the **matching equation**. Unfortunately, this equation cannot be analytically solved for λ_S. However, we will show that we can solve this equation approximately and can obtain a formula that describes the *entire* spectrum within the semiclassical approximation.

3 A formula for λ_S and its properties

3.1 Obtaining λ_S

The reason the matching equation (56) is not analytically solvable for λ_S is because of the ratio of the Bessel functions. We will therefore approximate this ratio by using the recurrence relation of the Bessel functions to write it as a continued fraction:

$$\frac{J_{1+2P_{\omega,c}}\left(2\sqrt{C}\right)}{J_{2P_{\omega,c}}\left(2\sqrt{C}\right)}=\cfrac{\frac{\sqrt{C}}{1+2P_{\omega,c}}}{1-\cfrac{\frac{C}{(1+2P_{\omega,c})(2+2P_{\omega,c})}}{1-\cfrac{\frac{C}{(2+2P_{\omega,c})(3+2P_{\omega,c})}}{1-\cdots}}}. \tag{57}$$

Truncating this continued fraction at the first term, so at $\frac{J_{1+2P_{\omega,c}}\left(2\sqrt{C}\right)}{J_{2P_{\omega,c}}\left(2\sqrt{C}\right)}=\frac{\sqrt{C}}{1+2P_{\omega,c}}$, we obtain an approximation that works well for all values of ω and l as long as $\omega>0$. However, if $\omega=0$, it becomes wildly inaccurate as compared with numerical calculations and actually becomes infinitely inaccurate as $l\to\infty$ for all masses, except when very close to the Nariai mass. To approximate the full spectrum, therefore, we need to include one term more. Truncating the continued fraction to the next term, we get the matching equation

$$\frac{A-SP_{\omega,b}}{1-2P_{\omega,b}}-P_{\omega,c}=\frac{\frac{\sqrt{C}}{1+2P_{\omega,c}}}{1-\frac{C}{(1+2P_{\omega,c})(2+2P_{\omega,c})}}, \tag{58}$$

for which one solution is consistent with numerical calculations, namely

$$\boxed{\lambda_S=\frac{1}{2l_{dS}^2r_br_c}\left[\gamma-\sqrt{\gamma^2-4\beta}\right]}, \tag{59}$$

where

$$\gamma=r_c^2\left(2+A_b+P_{\omega,b}\left(S-4-6P_{\omega,c}\right)+3P_{\omega,c}\right)+r_b^2\left(2+A_c+6P_{\omega,c}^2+4P_{\omega,c}^2\right) \tag{60}$$

$$+2r_br_c\left(4+A_c+A_b+P_{\omega,b}\left(S-4-6P_{\omega,c}\right)\right)+P_{\omega,c}\left(9+4P_{\omega,c}\right) \tag{61}$$

and

$$\beta=r_br_c\left(2r_b+r_c\right)\left(r_b+2r_c\right)\left[2A_b+A_c\left(2+A_b\right)+P_{\omega,c}\left(2+3A_c+6A_b+2P_{\omega,c}\left(3+2A_b+2P_{\omega,c}\right)\right)\right. \tag{62}$$

$$\left.+P_{\omega,b}\left(A_c\left(S-4\right)+2S-2P_{\omega,c}\left(2+3A_c-3S+2P_{\omega,c}\left(3-S+2P_{\omega,c}\right)\right)\right)\right], \tag{63}$$

where

$$\omega=\frac{2\pi n}{\beta},\ n\in\mathbb{Z}^{\leq} \tag{64}$$

$$P_{\omega,c}=\frac{\omega r_c l_{dS}^2}{\left(r_b^2+r_c\left(r_b-2r_c\right)\right)} \tag{65}$$

$$P_{\omega,b}=\frac{\omega r_b l_{dS}^2}{\left(r_c^2+r_b\left(r_c-2r_b\right)\right)} \tag{66}$$

$$S=\frac{r_c^2+3r_br_c-r_b^2}{r_b(r_c+2r_b)}-2 \tag{67}$$

$$A_b=\frac{l_{dS}^2r_b}{r_c+2r_b}\left(\frac{2\omega^2\left(r_b^2+r_br_c+r_c^2\right)l_{dS}^2}{\left(r_c^2+r_b\left(r_c-2r_b\right)\right)^2}+\frac{l(l+1)}{r_b^2}\right) \tag{68}$$

$$A_c=\frac{l_{dS}^2r_c}{r_b+2r_c}\left(\frac{2\omega^2\left(r_b^2+r_br_c+r_c^2\right)l_{dS}^2}{\left(r_b^2+r_c\left(r_b-2r_c\right)\right)^2}+\frac{l(l+1)}{r_c^2}\right). \tag{69}$$

This is our approximation to the SdS scalar spectrum. This approximation of the Bessel function ratio is valid as long as the next term in the continued fraction, that we did not include, is small. The next term in the continued fraction is $\frac{C}{(3+2P_{\omega,c})(2+2P_{\omega,c})}$. The approximation therefore holds as long as[3] $|C| \ll (2+2P_{\omega,c})(3+2P_{\omega,c})$, or, written out,

$$\left|-\frac{l_{dS}^2 r_c}{r_b+2r_c}\left(\frac{2\omega^2\left(r_b^2+r_b r_c+r_c^2\right)l_{dS}^2}{\left(r_b^2+r_c\left(r_b-2r_c\right)\right)^2}+\frac{l(l+1)}{r_c^2}-\lambda_S\right)\right| \ll$$
$$\left(3+2\frac{\omega r_c l_{dS}^2}{\left(r_b^2+r_c\left(r_b-2r_c\right)\right)}\right)\left(2+2\frac{\omega r_c l_{dS}^2}{\left(r_b^2+r_c\left(r_b-2r_c\right)\right)}\right) \tag{70}$$

This is a rather odd range of validity, as it depends on the solution λ_S that we've derived. λ_S is itself a function of ω and l, so we will just have to insert λ_S into the range of validity to see for what range of n and l it holds. We will show in the next section that our approximation holds for the entire range of l and n and is therefore a good approximation to the entire spectrum. Furthermore, in section (3.4) we will plot comparisons with numerical calculations to show exactly how accurate this semiclassical approximation is. We will see there that our approximation is about 80% accurate across the spectrum for which we've done numerical calculations.

Interestingly, for masses $5M_N/4 \leq M < M_N$ and $\omega > 0$, this approximation is less accurate than the one derived for the first-order truncation of the Bessel function ratio, which I shall dub $\lambda_{S,0}$ just for this little side note. For most masses this difference is of the order of a few percent, but around $M = M_N/2 + M_{Planck}$ and $\Lambda = 0.001$ (for smaller Λ the difference is smaller) the approximation $\lambda_{S,0}$ is 11% more accurate than the λ_S of equation (59), agreeing up to 99.3% with numerical calculations. However, the need for $\omega > 0$ for $\lambda_{S,0}$ to work would require us to split up the scalar spectrum in two ranges of (n, l) and approximate each range with a different approximation. This seems technically possible, but will likely proof very cumbersome in practice; something that does not seem worth it for a few percentages of extra accuracy. We will therefore approximate the whole spectrum with the formula (59), which we will now show is possible.

[3]You might worry what happens if λ_S is such that $\sqrt{C}$ turns imaginary. Bessel functions of the first kind are defined for real arguments, after all. This is where the absolute sign in the equation for the range of validity comes from: it turns out it doesn't matter. If there is some λ_S for which C turns negative, the Bessel functions turn into modified Bessel functions, for which the exact same continued fraction holds (but for which $\sqrt{C}^2 = -C$). It therefore doesn't matter whether C is positive or negative, the range of validity remains the same.

3.2 The range of validity of λ_S

To make sure our approximation stays valid over the entire range of $n \in [0, -\infty)$ and $l \in [0, \infty)$, we will have to check if our obtained λ_S satisfies equation (70) over this entire range for all masses $M_{Planck} \leq M \leq M_N$. To this end, we insert our approximation into equation (70) and see if it is satisfied for the case $(n, l) \to (0, \infty)$, the case $(n, l) \to (-\infty, 0)$ and finally the case $(n, l) \to (-\infty, \infty)$. If λ_S satisfies the equation in all these cases, then it is valid over the entire range, as either n, or l, or both will be smaller in between these edge cases, meaning the equation is then automatically satisfied.

3.2.1 The case $(n, l) \to (0, \infty)$

For $n = 0$ and $l \to \infty$ we drop all but the highest order of l in equation (70) and find it becomes trivial, namely

$$|0| \ll 6. \tag{71}$$

While zero is not much smaller than six, it at least indicates that the equation is reasonably satisfied for all masses. We conclude λ_S is a valid approximation in this limit.

3.2.2 The case $(n, l) \to (-\infty, 0)$

This case is slightly less trivial. We put $l = 0$ and drop all but the highest orders of ω and find that equation (59) becomes

$$\begin{aligned} &\Bigg| \frac{1}{(r_b + r_c)^3 (r_b - r_c)^2 (2r_b + r_c)^2} \Bigg(\left(-4r_b^4 + 17r_b^3 r_c + 60r_b^2 r_c^2 + 48r_b r_c^3 + 13r_c^4\right) \\ &- \sqrt{9r_b^8 - 6r_b^7 r_c + 185r_b^6 r_c^2 + 1488r_b^5 r_c^3 + 3594r_b^4 r_c^4 + 4010r_b^3 r_c^5 + 2328r_b^2 r_c^6 + 696r_b r_c^7 + 89r_c^8} \Bigg) \Bigg| \\ &\ll \frac{4l_{dS}^2 r_c}{(r_b^2 + (r_b - 2r_c) r_c)^2}. \end{aligned} \tag{72}$$

Note that since both sides go as ω^2, since we dropped all lower orders, I have divide this term out. The only variable in this equation is therefore the mass. Plotting the LHS of this inequality, C, minus the RHS of this inequality, which we will call Lim, we can see whether the resulting plot is negative or positive. If it is negative, $C <$ Lim and the approximation holds. The plot is figure (1), where we have plotted the difference for relatively large Λ, namely $\Lambda = 0.01$, since we expect the approximation to only get better the smaller Λ becomes. From the figure we can see the inequality is satisfied and λ_S is a good approximation in this case.

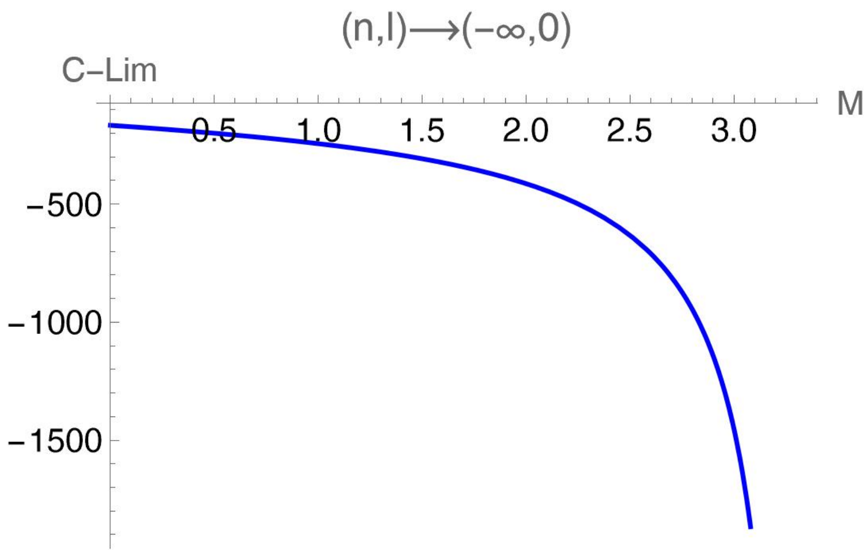

Figure 1: C minus Lim plotted for $(n, l) \to (-\infty, 0)$, $\Lambda = 0.01$ and $G = 1$ for a mass range of $0 \leq M \leq M_N$. At $M = 0$ a value of -166 is achieved, meaning that even at this mass λ_S should prove a good approximation to a solution to the matching equation. Naturally, it will not be a good approximation to the scalar SdS spectrum at this mass, as we are far beyond the semiclassical regime at this point. Clearly, λ_S stays within the range of validity in the limit $(n, l) \to (-\infty, 0)$.

3.2.3 The case $(n, l) \to (-\infty, \infty)$

We put $\omega = l$ and drop all but the highest orders of ω. This gives a formula for C that is very lengthy and not all that illuminating to display fully, however, we can nevertheless plot the difference between C and Lim (where the latter is the same as in the $(n, l) \to (-\infty, 0)$ case), which gives us figure (2). There it is clear to see that our λ_S stays within the range of validity and so, λ_S holds for all n and l within the desired mass range.

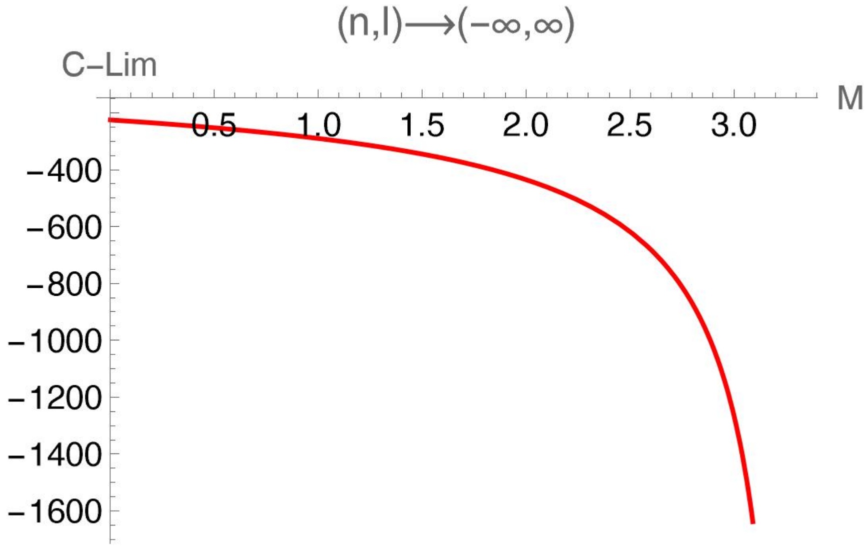

Figure 2: C minus Lim plotted for $(n,l) \to (-\infty, \infty)$, $\Lambda = 0.01$ and $G = 1$ for a mass range of $0 \leq M \leq M_N$. At $M = 0$ a value of -255 is achieved. Clearly, λ_S stays within the range of validity in this limit.

3.3 A connection with the Nariai scalar spectrum

If we take the de Sitter limit of $M \to 0$ limit, which is the same as taking $r_b \to 0$, we find our λ_S diverges. This is to be expected: our approximation only holds up to order $\sqrt{\Lambda}$, so we expect that below the Planck mass it is not well-defined anyway.

We would like to check, however, if our semiclassical approximation to the SdS scalar spectrum interpolates to the Nariai scalar spectrum. After all, Nariai black holes, of all black hole masses, should be the best described by semiclassical gravity. The Nariai scalar spectrum was obtained by Volkov & Wipf and is

$$\lambda_{S,N} = \left(j_1 \left(j_1 + 1\right) + j_2 \left(j_2 + 1\right)\right) \Lambda, \tag{73}$$

where j_1 and j_2 the spherical harmonics eigenvalues of $Y_{j_1}^{m_1}(\theta_1, \phi_1)$ and $Y_{j_2}^{m_2}(\theta_2, \phi_2)$ of each respective 2-sphere (recall that Nariai is topologically $S^2 \times S^2$) [4].

We would like to compare this result to our formula obtained for the SdS scalar spectrum. If we naively take the $M \to M_N$ limit, which is the same as $r_b \to r_c$, we'd find that

λ_S once again diverges. However, interestingly, if we put $\omega = 0$, we find

$$\lambda_S\Big|_{\omega=0} = \frac{1}{2l_{dS}^2 r_b^2 r_c^2}$$
$$\Bigg(l(1+l)l_{dS}^2(r_b^2 + r_c^2) + 2r_b r_c(r_b^2 + 4r_b r_c + r_c^2)$$
$$- r_b r_c \sqrt{\frac{(l^2(1+l)^2 l_{dS}^4 (r_b^2 - r_c^2)^2)}{r_b^2 r_c^2} + \frac{4l(1+l)l_{dS}^2(r_b^2 - r_c^2)^2}{r_b r_c} + 4(r_b^2 + 4r_b r_c + r_c^2)^2}\Bigg). \qquad (74)$$

If we now take the limit $r_b \to r_c$ we obtain

$$\boxed{\lambda_S\Big|_{\omega=0} \to l(l+1)r_N^{-2} = l(l+1)\Lambda}. \qquad (75)$$

We have recovered *exactly half* of the Nariai scalar spectrum! How can this be? The reason is something similar we saw for the zero mode volume. SdS has an $U(1) \times SO(3)$ isometry group while Nariai has a $SO(3) \times SO(3)$ isometry group. We find that the $U(1)$ part of the spectrum, associated with ω, does not interpolate to the $SO(3)$ part of the Nariai group, as $U(1)$ is not a subgroup of $SO(3)$. However, the $SO(3)$ part of the SdS isometry group interpolates perfectly well to the $SO(3)$ of the Nariai isometry group. Because SdS only has a single $SO(3)$ in its isometry group, it interpolates to exactly half of the Nariai spectrum.

3.4 Comparison with numerical data

Using Mathematica 14, we can determine how accurate our result is compared to numerical calculations. In the below figures, I plot the relative difference $\frac{|\lambda_{S,A} - \lambda_{S,N}|}{\lambda_{S,N}} \cdot 100\%$, where $\lambda_{S,A}$ our formula obtained for the eigenvalues and $\lambda_{S,N}$ the eigenvalues obtained numerically. I have imposed Dirichlet boundary conditions $u(r_b) = 0$ and $u(r_c) = 0$ on the numerical eigenfunction to mimic the boundary conditions that our approximate solution satisfies. In addition, I have put $\Lambda = 0.001$ and $G = \beta = 1$. Because of this, $\omega = 2\pi n$, $M_{Planck} = 1$ and $M_N = \frac{1}{3\sqrt{\Lambda}} = \frac{\sqrt{1000}}{3} = 10.54$. I do not plot the comparison for $n = l = 0$, as then trivially $\lambda_{S,A} = \lambda_{S,N} = 0$ for all masses. I also do not plot for l larger than 15000, as then the numerical simulation starts to break down.

In the below figures it can be seen that for $n \neq 0$ the approximation is fairly accurate up to the Planck mass, below which it starts to derail pretty quickly. It performs the worst when $\omega \approx l/2$, though even then performs good when $M \gg M_{Pl}$. It seems the approximation, for non-zero n, is the most accurate when $l \gg |n|$, but is still fairly accurate when $n = l$. Near the Nariai mass the approximation is still consistently about 70% accurate, though it blows up at exactly the Nariai mass.

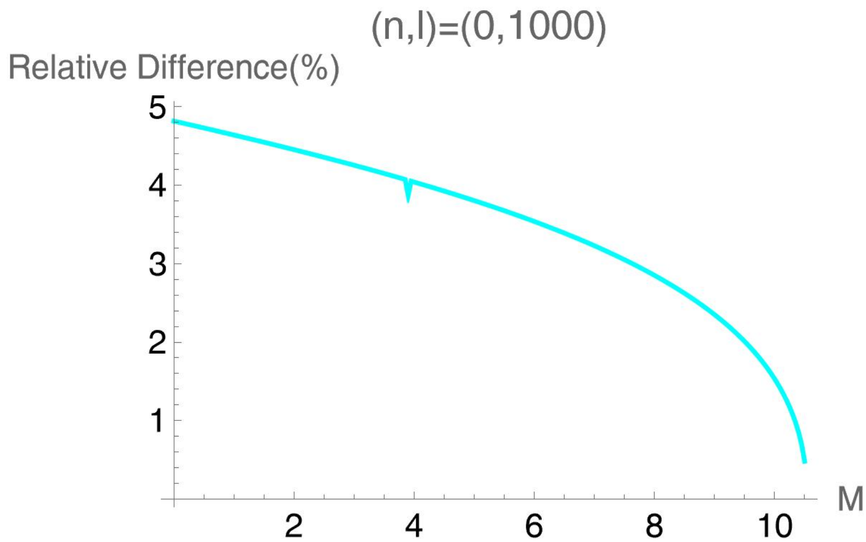

Figure 3: The relative difference between our analytical formula and numerical calculations. We see that for large l and $n = 0$ the formula is very accurate. At M_{Pl}, the relative difference is 4.64%.

18.3%.

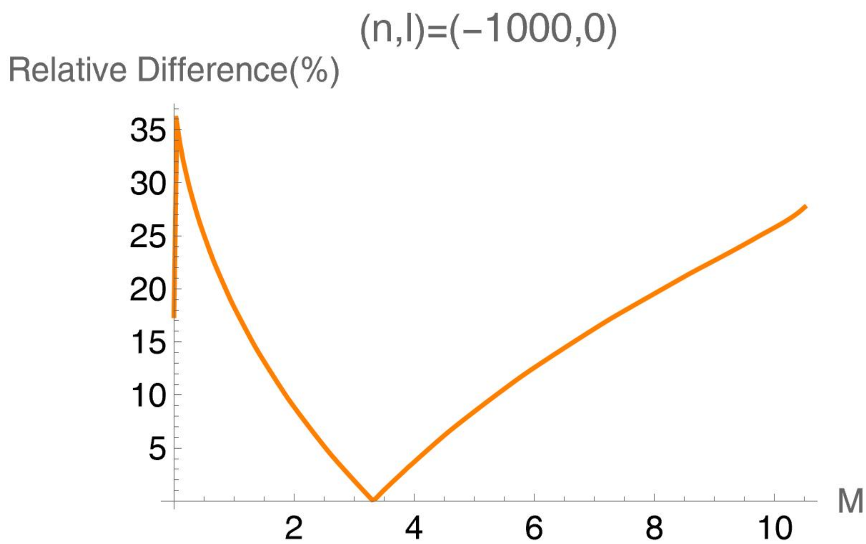

Figure 4: We see that for large $|n|$ and $l = 0$ the formula becomes less accurate, but captures, on average, the spectrum fairly well, never dipping below 70% accuracy for $M_{Pl} \leq M \leq M_N$. At M_{Pl}, the relative difference is 18.3%, which is still pretty good for a semiclassical approximation.

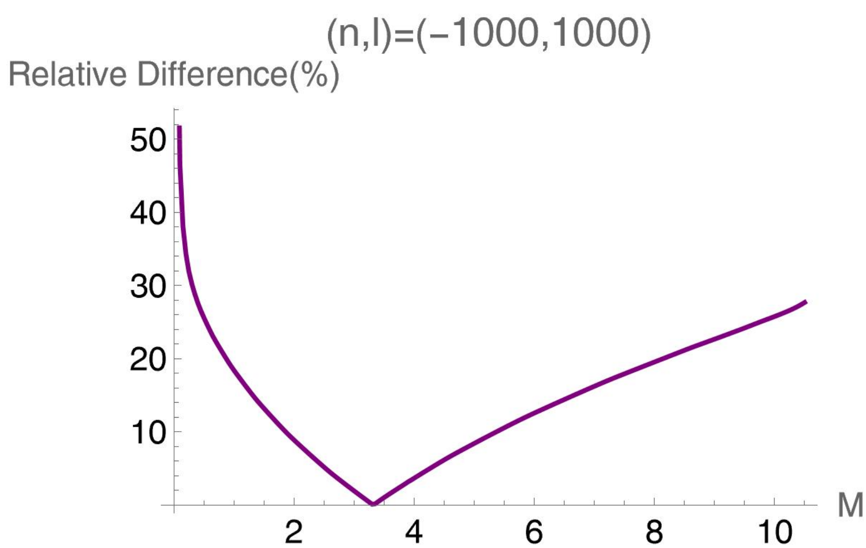

Figure 5: For large $|n| = l$ the formula is similarly accurate, though slightly less so, to the case for $l = 0$ and $|n|$ large, but becomes less accurate. Still, it captures on average, the spectrum fairly well, never dipping below 70% accuracy for $M_{Pl} \leq M \leq M_N$. At M_{Pl}, the relative difference is 18.4%.

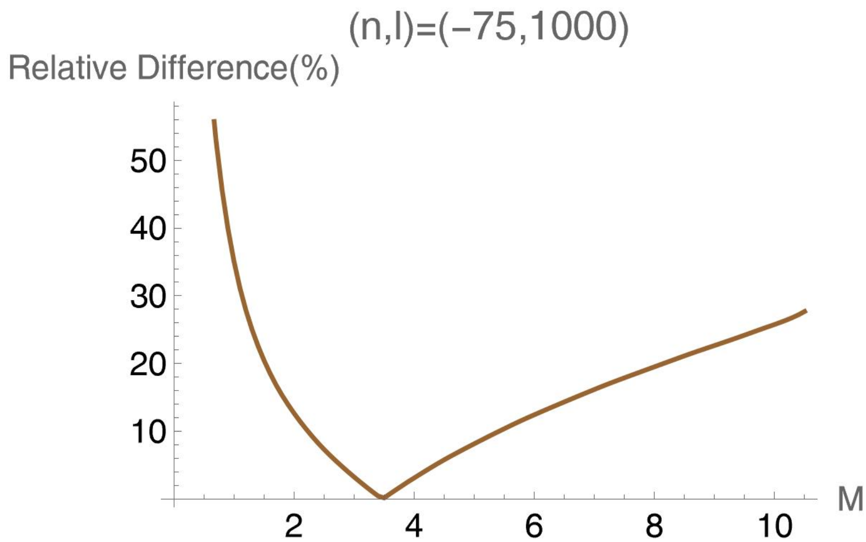

Figure 6: For small, but non-zero n, and large l, the formula performs the worst: for mass $M > M_{Planck}$ it is similarly accurate to the case for $l = 0$ and $|n|$ large, but it starts to really break down near the Planck limit, where the relative difference is 35.1%. Nevertheless, it quickly dips down and still is over 70% accurate just below the Nariai mass. In the next figure we shall see that the higher l goes, the approximation regains its accuracy.

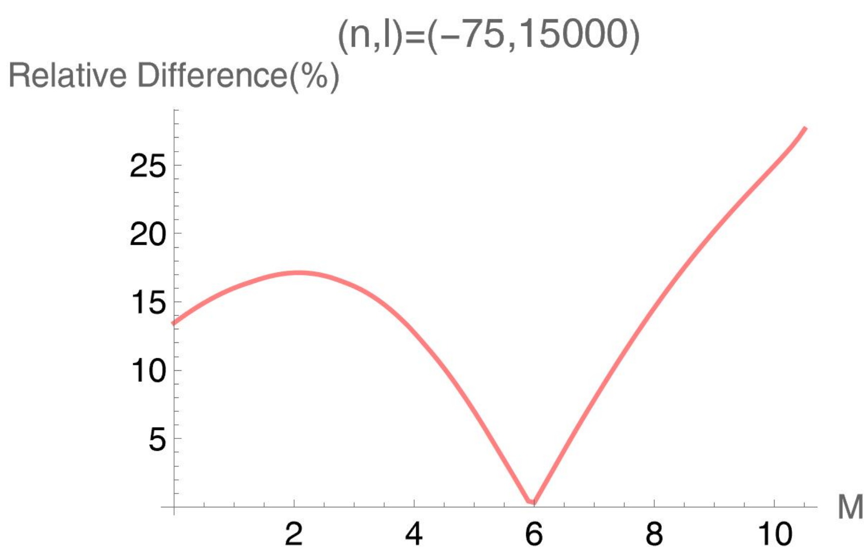

Figure 7: For small, but non-zero n, and really large l, the formula performs better as it did compared to large l: now at $M = M_{Pl}$ the relative difference is 16.0%. As we see for all comparisons with $n \neq 0$, the approximation is over 70% accurate just below the Nariai mass.

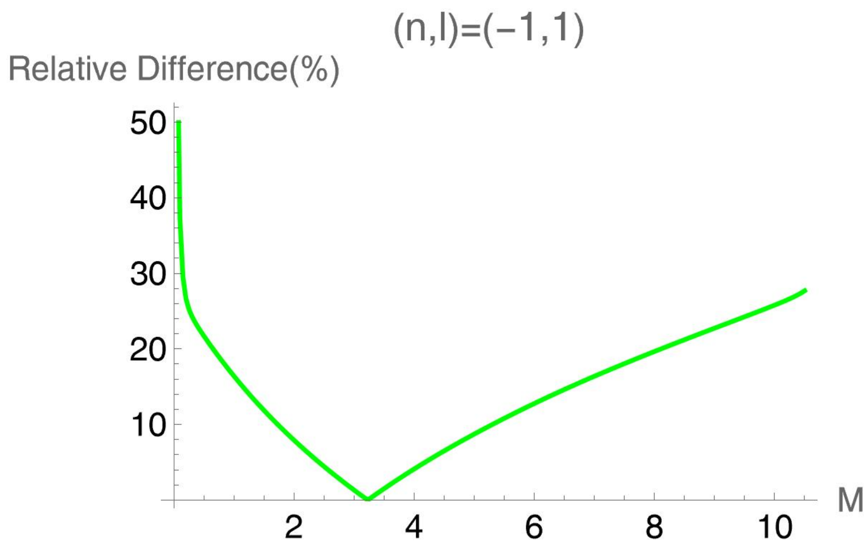

Figure 8: For small $n = l$ the formula performs almost exactly the same as for large $n = l$, with a relative accuracy of 16.4% at $M = M_{Pl}$.

4 Conclusion

We have obtained a semiclassical approximation to the scalar spectrum of Euclidean Schwarzschild-de Sitter through the use of matched asymptotic expansion. The semi-classical approximation is valid between the black hole horizon and cosmological horizon, where Euclidean Schwarzschild-de Sitter rounds off. From comparison with numerical simulations and analyses of the range of validity of our approximation, we determined that our expression for λ_S is sufficiently accurate to describe the entire SdS scalar spectrum. Furthermore, we found an interesting connection with the Nariai scalar spectrum, where, if we put the part of the scalar SdS spectrum associated with $U(1)$ to zero, we recovered half of the Nariai spectrum in the Nariai limit, but found this same limit singular if we kept the $U(1)$ section. We interpreted this as a result of $U(1)$ not being a subgroup of $SO(3)$.

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5 Appendix: The Radial Equation in x-coordinates

When writing out the radial equation (14) in terms of the coordinate

$$x \equiv \frac{r - r_b}{r_c - r_b}, \tag{76}$$

we obtain

$$\left[\partial_x^2 + \left(\frac{1-2x}{(1-x)x} + \bar{r}\frac{2\bar{r}x + r_c + 3r_b}{(\bar{r}x + r_c + 2r_b)(\bar{r}x + r_b)}\right)\partial_x - \frac{l_{dS}^4(\bar{r}x + r_b)^2\omega^2}{\bar{r}^2(1-x)^2x^2(\bar{r}x + r_c + 2r_b)^2}\right. \tag{77}$$

$$\left. - \frac{l(l+1)l_{dS}^2}{(1-x)x(\bar{r}x + r_c + 2r_b)(\bar{r}x + r_b)} + \frac{\lambda l_{dS}^2(\bar{r}x + r_b)}{(1-x)x(\bar{r}x + r_c + 2r_b)}\right]R(x) = 0, \tag{78}$$

where $\bar{r} = r_c - r_b$. From here, when pulling apart the fractions to isolate the poles, it is easy to see which terms blow up near r_b and r_c.