The 1-loop Schwarzschild-de Sitter proposal

Tim Blankenstein

July 23, 2025

A brief recap

In this section we will investigate a proposal regarding Schwarzschild-de Sitter at one-loop order made in [5]. Recall that in the constrained path integral,

$$Z = \int \mathcal{D}g \frac{d\lambda d\zeta}{2\pi} e^{-I_E[g]+i\lambda(C[g]-\zeta)}, \tag{1}$$

where I_E the Euclidean Einstein-Hilbert action, we treat $I_{tot} = I_E[g] - i\lambda(C[g]-\zeta)$ as the total action. Saddle points of this action are constrained instanton solutions that consist of a solution for λ and a solution for the metric tensor $g_{\mu\nu}$. One-loop corrections arise from a functional Taylor series of this total action about the background metric $\overline{g}$ and Lagrange multiplier $\overline{\lambda}$

$$I_{tot} = I_{tot}[\overline{g}] + \delta I_{tot}[\overline{g}, \overline{\lambda}] + \delta^2 I_{tot}[\overline{g}, \overline{\lambda}] + \cdots, \tag{2}$$

and are contained within the third term $\delta^2 I_{tot}[\overline{g}, \overline{\lambda}]$. Specified to the Schwarzschild-de Sitter constrained instanton, for a metric perturbation $g_{\mu\nu} = g_{\mu\nu,SdS} + h_{\mu\nu}$ and multiplier perturbation $\lambda = \lambda_{SdS} + \delta\lambda$, this term is

$$\delta^2 I_{SdS} = \frac{1}{2}\int d^4x\, h^{\mu\nu} \frac{\delta^2 I_{tot}[g_{SdS}, \lambda_{SdS}]}{\delta g^{\mu\nu}\delta g^{\alpha\gamma}} h^{\alpha\gamma} - i\delta\lambda \int d^4x \frac{\delta C[g_{SdS}]}{\delta g^{\mu\nu}} h^{\mu\nu}, \tag{3}$$

where

$$\frac{\delta^2 I_{tot}[g_{SdS}, \lambda_{SdS}]}{\delta g^{\mu\nu}\delta g^{\alpha\gamma}} = \frac{\delta^2 I_E[g_{SdS}]}{\delta g^{\mu\nu}\delta g^{\alpha\gamma}} - i\lambda_{SdS}\frac{\delta^2 C[g_{SdS}]}{\delta g^{\mu\nu}\delta g^{\alpha\gamma}}. \tag{4}$$

and

$$\lambda_{SdS} = -\frac{i\beta GM}{2\pi r_b^2}. \tag{5}$$

Here, we are using the area constraint

$$C[g] = \frac{1}{16\pi G}\int d^4x\sqrt{g}\left(4\pi\delta_{\Sigma_b} + 4\pi\delta_{\Sigma_c}\right), \tag{6}$$

to include Schwarzschild-de Sitter as constrained instanton into the saddle point sum of the path integral. The delta functions $\delta_{\Sigma_b}, \delta_{\Sigma_c}$ within this constraint have the defining property $\int_{\mathcal{M}} f\delta_{\Sigma_h} = \int_{\Sigma_h} f$.

The equations of motion we obtain from the first-order variation of the total action I_{tot} is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + i2\pi\lambda g_{\mu\nu}\left(\delta_{\Sigma_b} + \delta_{\Sigma_c}\right) = 0. \tag{7}$$

The proposal

It was proposed that, in the second-order variation of the total action, we should leave out the contribution of the second-order variation of the area constraint functional, but keep the first-order variation. That is, effectively,

$$\boxed{\frac{\delta^2 C[g_{SdS}]}{\delta g^{\mu\nu}\delta g^{\alpha\gamma}} = 0}, \tag{8}$$

from here on denoted as $\delta^2 C$. The reasoning for this is that this term would constitute a second-order metric perturbation, whilst we are only considering first-order perturbations. The proposal in addition claims that this term should be discarded because it represents a second-order fluctuation of the horizons, which we have fixed through the area constraint. This short paper aims to show that the second order constraint functional fluctuations cannot be discarded based on this proposal. Specifically, I will show that the first- and second-order area constraint variations do not constitute metric perturbations and therefore should not be discarded based on this argument.

Testing the proposal

The set-up

We will now show that variations of the area constraint functional do not relate to metric perturbations. First, we will express the metric tensor as a perturbation series about the Schwarzschild-de Sitter metric tensor, which from here we will simply denote as $\overline{g}$:

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \varepsilon h_{\mu\nu} + \varepsilon^2 k_{\mu\nu} + \mathcal{O}(\varepsilon^3), \tag{9}$$

where ε a dimensionless small parameter and $h_{\mu\nu}$ and $k_{\mu\nu}$ represent first-order and second-order metric perturbations, respectively. This seems different from the quantum field theory approach to metric perturbations, within which we would expand the metric tensor as

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}, \tag{10}$$

upon which all higher-order pieces are contained within $h_{\mu\nu}$. They are, however, ultimately the same. In the first method we expand in terms of the parameter ε, in the second method this expansion is done implicitly in terms of the amplitude $\mathcal{A}$ of the metric perturbations (see section 35.13 in [6]).

Solutions for the perturbations $h_{\mu\nu}$ and $k_{\mu\nu}$ can be found by inserting the perturbation series into the Einstein field equations and solving these equations order-to-order. Denote the constrained Einstein field equations (7) by

$$\mathscr{E}_{\mu\nu}[g,\lambda] = G_{\mu\nu} + \Lambda g_{\mu\nu} + i2\pi\lambda g_{\mu\nu}\left(\delta_{\Sigma_b} + \delta_{\Sigma_c}\right) = 0. \tag{11}$$

Inserting the perturbation series (9) into these equations, we obtain the expansion

$$\mathscr{E}_{\mu\nu}[g,\lambda] = \overline{\mathscr{E}}_{\mu\nu}[\overline{g},\overline{\lambda}] + \varepsilon\mathscr{E}^{(1)}_{\mu\nu}[h,\overline{\lambda}] + \varepsilon^2\left(\mathscr{E}^{(2)}_{\mu\nu}[h,\overline{\lambda}] + \mathscr{E}^{(1)}_{\mu\nu}[k,\overline{\lambda}]\right) + \mathcal{O}(\varepsilon^3), \tag{12}$$

where an overline denotes that quantities are calculated with respect to the Schwarzschild-de Sitter metric tensor $\overline{g}_{\mu\nu}$. $\overline{\lambda}$ denotes the Schwarzschild-de Sitter Lagrange multiplier for the area constraint (6). We could include the perturbation series of the Lagrange multiplier around the Schwarzschild-de Sitter Lagrange multiplier

$$\lambda = \overline{\lambda} + \epsilon\varsigma + \epsilon^2\chi + \mathcal{O}(\epsilon^3), \tag{13}$$

where ϵ another small, dimensionless parameter, but this would not alter the equations of $h_{\mu\nu}$ relevant to the subject of this paper. We therefore choose to not include it.

First-order metric perturbation

The first-order linearized constrained Einstein field equations can be found to be

$$\mathscr{E}^{(1)}_{\mu\nu}[h,\overline{\lambda}] = (R_{\mu\nu})^{(1)} - \frac{1}{2}\overline{g}_{\mu\nu}\,(R)^{(1)} - \frac{1}{2}h_{\mu\nu}\overline{R} + \Lambda h_{\mu\nu} + i2\pi\overline{\lambda}h_{\mu\nu}\left(\delta_{\Sigma_b} + \delta_{\Sigma_c}\right) = 0, \tag{14}$$

where

$$(R_{\mu\nu})^{(1)} = \frac{1}{2}\left(\nabla_\nu\nabla^\alpha h_{\alpha\mu} + \nabla^\alpha\nabla_\mu h_{\alpha\nu} - \Delta h_{\mu\nu} - \nabla_\mu\nabla_\nu h\right) \tag{15}$$

and

$$(R)^{(1)} = \nabla_\alpha\nabla^\beta h^\alpha_\beta - \Delta h - h^{\alpha\beta}\overline{R}_{\alpha\beta}, \tag{16}$$

where Δ the Laplacian and h the trace of $h_{\mu\nu}$ [4, 1]. Any first-order metric perturbation around the Schwarzschild-de Sitter constrained instanton must satisfy equation (14). We can now explicitly check whether the first-order variation of the area constraint functional, as it appears in the total action, satisfies this equation. The first-order variation of the area constraint is

$$\frac{\delta C[\overline{g}]}{\delta g_{\mu\nu}} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}2\pi\overline{g}_{\mu\nu}\left(\delta_{\Sigma_b} + \delta_{\Sigma_c}\right). \tag{17}$$

Therefore, as this variation appears within the integral $\int d^4x/(16\pi G)$ of the total action, the claim is that

$$h_{\mu\nu} = -2\pi\overline{g}_{\mu\nu}\left(\delta_{\Sigma_b} + \delta_{\Sigma_c}\right) \tag{18}$$

is a first-order metric perturbation. Inserting this tensor into the linearized equations (14), we get

$$\boxed{\mathscr{E}^{(1)}_{\mu\nu}[h,\overline{\lambda}] = (\text{derivatives of } \delta_{\Sigma_b} \text{ and } \delta_{\Sigma_c}) - 2\pi\overline{g}_{\mu\nu}\Lambda\left(\delta_{\Sigma_b} + \delta_{\Sigma_c}\right) - \frac{2\pi\beta GM}{r_b^2}\overline{g}_{\mu\nu}\left(\delta_{\Sigma_b} + \delta_{\Sigma_c}\right)^2} \tag{19}$$

The terms proportional to the derivatives of the delta functions stem from the covariant derivatives in $(R_{\mu\nu})^{(1)}$ and $(R)^{(1)}$ working on the delta functions. Clearly, the metric perturbation (18) does not satisfy the linearized equations. $\delta C[\overline{g}]$ is therefore not a first-order metric perturbation.

Second-order metric perturbation

From here, we could calculate the second-order linearized constrained Einstein field equations and see if $\delta^2 C[\overline{g}]$ satisfies it. There is an easier way, however. Note that the second-order linearized constrained field equations are

$$\boxed{\mathscr{E}^{(2)}_{\mu\nu}[h,\overline{\lambda}] + \mathscr{E}^{(1)}_{\mu\nu}[k,\overline{\lambda}] = 0} \tag{20}$$

This equation tells us that second-order perturbations are sourced by the first-order perturbations. Since the first-order variation of the area constraint was not a metric perturbation, this means that a term stemming from the area constraint will not enter into higher-order metric perturbations. This means that a second-order metric perturbation stemming from the constraint functional is not possible.

Other doubts about the proposal

After showing variations of the constraint functional do not constitute metric perturbations, we aim to furthermore make the case that it doesn't seem logical to discard $\delta^2 C$ based on the assumption that it would be a variation of the fixed geometry. While it is true that the constraint functional must relate to the geometry to be fixed in order to actually fix a part of the geometry, this does not automatically mean that the variation of this constraint functional is a direct variation of the fixed geometry. If that were true, it seems there wouldn't be a reason to include the first-order variation of the constraint functional either: after all, both constitute a variation of something that is not allowed to vary. If we wouldn't include either first-order or second-order variations of $C[g]$, then the equations of motion of the constrained path integral wouldn't be altered at all and all constrained instanton would simply be instantons. This can therefore not be the case.

Finally, we remark that assuming $\delta^2 C = 0$ seems at odds with the literature, for example, with the stability analyses by Colter and Jensen in [2, 3], which do include $\delta^2 C$. These would be rendered incorrect.

Conclusion

Based on the above analysis, we conclude we cannot discard the second variation of the constraint functional based on the assertion that this is a second-order metric perturbation. I therefore see no further reason to leave out this quantity from the one-loop Schwarzschild-de Sitter differential equations.

References

[1] E. Altas and B. Tekin. Second order perturbation theory in general relativity: Taub charges as integral constraints. *Physical Review D*, 99(10), May 2019.

[2] J. Cotler and K. Jensen. Gravitational constrained instantons. *Physical Review D*, 104(8), oct 2021.

[3] J. Cotler and K. Jensen. Wormholes and black hole microstates in ads/cft, April 2021.

[4] E. Lifshitz. Republication of: On the gravitational stability of the expanding universe. *General Relativity and Gravitation*, 49(2):18, Jan. 2017.

[5] C. V. Meinersen. *de Sitter Quantum Gravity: Non-perturbative Effects, Conical Singularities & Quantum Chaos*. Master's thesis, University of Amsterdam, 2023.

[6] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. 1973.