No Gauge Dependency from Covariant Constraint Functionals

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I show that manifestly covariant constraint functionals lead to gauge *independent* partition functions.

1 Constraint choice and setup

Consider the following constraint functional

$$\boxed{C[g] = \frac{1}{16\pi G}\int d^4x\sqrt{g}\,\bar{\Lambda}}, \tag{1}$$

where $\bar{\Lambda}$ a constant. Clearly, this constraint functional is manifestly covariant. Recall that in the constrained path integral, we have

$$Z = \int \mathcal{D}g \frac{d\lambda d\zeta}{2\pi} e^{-I_E[g]+i\lambda(C[g]-\zeta)}, \tag{2}$$

where I_E the Euclidean Einstein-Hilbert action, we treat $I_{tot} = I_E[g] - i\lambda(C[g]-\zeta)$ as the total action. With the addition of the constraint functional of equation (1), we have

$$I_{tot} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(R - 2\Lambda - i\lambda\bar{\Lambda}\right) + i\lambda\zeta. \tag{3}$$

Clearly, all that $C[g]$ does is rescale Λ. In fact, if we define $2\tilde{\Lambda} \equiv 2\Lambda + i\lambda\bar{\Lambda}$, then equation (3) is simply the Einstein-Hilbert action shifted by an unimportant constant:

$$I_{tot} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\left(R - 2\tilde{\Lambda}\right) + i\lambda\zeta = I_E + i\lambda\zeta. \tag{4}$$

From this observation it is clear that the first order variation will simply give us the vacuum Einstein field equations,

$$R_{\mu\nu} = \tilde{\Lambda} g_{\mu\nu}. \tag{5}$$

We now furthermore assume that $\lambda < 2i\frac{\Lambda}{\tilde{\Lambda}}$; this guarantees that $\tilde{\Lambda} > 0$ and so that solutions of this equations have the same properties as those of the vacuum Einstein equations with $\Lambda > 0$. In particular, the manifolds of the metric tensor solutions of this equation are compact, simply connected[1], satisfy equation (5), and we will add the same gauge-fixing term (see next paragraph), such that our 1-loop analysis carries out the same as that of the authors Volkov & Wipf's 1-loop calculations for Nariai black holes in [1], aside from one crucial difference, which we will shortly mention.

Just as the first order variation with respect to $g_{\mu\nu}$ is the same as for vacuum (unconstrained) Einstein gravity, so will the second order variation of I_{tot} (equation (4)) be the same as that of the $\Lambda > 0$ Einstein-Hilbert action. There is one crucial difference: since we vary λ too, to second order in variations we will obtain a linear term in the metric perturbations. To go to 1-loop order, consider a metric perturbation $g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}$ and multiplier perturbation $\lambda = \overline{\lambda} + \delta\lambda$, around background metric $\overline{g}$, which is a solution of the EFE of (5), and background Lagrange multiplier $\overline{\lambda}$, respectively. Then to second order in fluctuations we have:

$$I_{tot} = I_E[\overline{g}] + \frac{1}{2}\int d^4x\, h^{\mu\nu}\frac{\delta^2 I_E[\overline{g},\overline{\lambda}]}{\delta g^{\mu\nu}\delta g^{\alpha\gamma}}h^{\alpha\gamma} - i\delta\lambda\int d^4x \frac{\delta C[\overline{g}]}{\delta g^{\mu\nu}}h^{\mu\nu}. \tag{6}$$

Which we write as

$$I_{tot} = I + \delta^2 I - i\delta\lambda\delta I_c, \tag{7}$$

such that the 1-loop partition function Z_{SdS} is

$$Z_{SdS} = e^{-I}\int D\left[h_{\mu\nu}\right]\, \exp\left(-\delta^2 I + i\delta\lambda\delta I_c\right). \tag{8}$$

We need to gauge fix lest our path integral will diverge or become unmanageable. We opt to add the de Donder gauge-fixing term, which is standard in gravity:

$$\delta^2 I_g = \frac{\gamma}{32\pi G}\int_{\mathcal{M}}\left(\nabla_\sigma h^\sigma_\rho - \frac{\gamma+1}{4\gamma}\nabla_\rho h\right)\left(\nabla^\alpha h^\rho_\alpha - \frac{\gamma+1}{4\gamma}\nabla^\rho h\right). \tag{9}$$

[1]This is stated in Volkov & Wipf and follows from an application of Myers's theorem, the Hopf-Rinow theorem and Cheng's diameter rigidity theorem to Einstein manifolds.

Here γ is a real parameter which is usually taken to be $\gamma = 1$, but which is kept general to check for gauge invariance. This will contribute the Faddeev-Popov determinant, defined by

$$(\mathcal{D}_{FP})^{-1} = \int D\left[\xi_\mu\right] \exp\left(-\delta^2 I_g\right), \tag{10}$$

to Z_{SdS}, resulting in the final 1-loop expression being

$$\boxed{Z_{SdS} = e^{-I} \int D\left[h_{\mu\nu}\right]\ \mathcal{D}_{FP} \exp\left(-\delta^2 I_{gf} + i\delta\lambda\delta I_c\right)}. \tag{11}$$

The linear $\delta\lambda$ term will seemingly introduce a gauge dependency.

2 The naive gauge dependency

From here, we will seemingly obtain a gauge dependency. We have

$$\delta I_c = \int d^4x \frac{\delta C[\overline{g}]}{\delta g^{\mu\nu}} h^{\mu\nu} = \frac{\bar{\Lambda}}{16\pi G} \int d^4x \frac{\delta\sqrt{g}}{\delta g^{\mu\nu}} h^{\mu\nu} = -\frac{1}{2}\frac{\bar{\Lambda}}{16\pi G} \int d^4x \sqrt{g}\, h. \tag{12}$$

Since, aside from this one term, our setup is the same as Volkov & Wipf's, we can skip right to just before performing the mode integration. We can decompose the metric perturbation as

$$h_{\mu\nu} = \phi_{\mu\nu} + \frac{1}{4} h g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \xi^\sigma \tag{13}$$

and expand the various components in the decomposition in terms of eigenfunctions of the differential operator stemming from the second order variation of the action. The only mode expansion we care about is

$$h = \sum_p C_p^h \alpha^{(p)}, \tag{14}$$

which we then take orthonormal with respect to the inner product

$$\left\langle h_{\mu\nu(n)}, h^{\mu\nu}_{(m)} \right\rangle = \frac{1}{32\pi G} \int_{\mathcal{M}} h_{\mu\nu(n)} h^{\mu\nu}_{(m)}\, d^4x \sqrt{g} = \delta_{nm}, \tag{15}$$

and where $\alpha^{(p)}$ is an eigenfunction of

$$\tilde{\Delta}_0^\gamma \alpha^{(p)} = \left(\gamma(-3\Box - 4\tilde{\Lambda}) + \Box\right) \alpha^{(p)} = \tilde{\lambda}_p^\gamma \alpha^{(p)} = \left(\gamma\tilde{\lambda}_p - \lambda_p\right) \alpha^{(p)}, \tag{16}$$

with $\Box = \nabla_\sigma \nabla^\sigma$ the Laplace-Beltrami operator and λ_p the eigenvalue of $-\Box u = \lambda_p u$.

The integration over the remaining modes carries out the exact same as Volkov & Wipf. The measure over the modes is the same, too. We focus just on h and ignore the Faddeev-Popov determinant and all other modes not related to h; they are unimportant as they do not relate to the gauge dependency. The 1-loop factor F in $Z = e^{-I}F$ we then get is proportional to

$$F \propto \int d(\delta\lambda) \prod_p dC_p^h \exp\left\{ \left(\frac{1}{16\gamma} \left\langle h, \tilde{\Delta}_0^\gamma h \right\rangle - i\delta\lambda\bar{\Lambda} \left\langle h \right\rangle \right) \right\} \tag{17}$$

$$= \int d(\delta\lambda) \prod_p dC_p^h \exp\left\{ \left(\sum_n \frac{1}{16\gamma} \tilde{\lambda}_p^\gamma (C_n^h)^2 - i\delta\lambda\bar{\Lambda}\kappa_n C_n^h \right) \right\}, \tag{18}$$

where $d(\delta\lambda)$ comes from the saddle point approximation in λ. It is manifest that the h modes have the wrong sign for Gaussian integration, which Volkov & Wipf remedy by rotating the offending modes. This will be implicitly done throughout the remainder of this paper. κ_n is simply the integral over $\alpha^{(n)}$:

$$\langle h \rangle = \frac{1}{32\pi G} \sum_n \int d^4x \sqrt{g}\, C_n^h \alpha^{(n)} = \sum_n \kappa_n C_n^h. \tag{19}$$

When the integration is performed over the C^h modes, this seemingly gives a gauge dependency

$$F \propto \int d\left(\delta\lambda\right) \prod_m \sqrt{\frac{2\gamma}{\tilde{\lambda}_m^\gamma}} \exp\left(-4\gamma \left(\delta\lambda\right)^2 \sum_n \frac{\kappa_n^2}{\tilde{\lambda}_n^\gamma} \right) \tag{20}$$

$$= \prod_m \sqrt{\frac{2\gamma}{\tilde{\lambda}_m^\gamma}} \sqrt{\frac{\pi}{4\gamma} \sum_n \frac{\tilde{\lambda}_n^\gamma}{\kappa_n^2}}. \tag{21}$$

While the $\sqrt{\frac{2\gamma}{\tilde{\lambda}_m^\gamma}}$ term will cancel through the Faddeev-Popov determinant, the linear term from the λ perturbation has seemingly given a dependence on γ and the gauge-dependent eigenvalue $\tilde{\lambda}_n^\gamma$.

However, this does not take a crucial aspect into account. As mentioned, $\alpha^{(n)}$ is an eigenfunction of the second order variation of the action and so, in particular, an eigenfunction of equation (16). Depending on the metric tensor solution around which we're perturbing, these differential equations take on different forms, and therefore specify different eigenfunctions in the eigenmode sum (for h modes, different metric tensors lead to different Laplace-Beltrami operators $\Box$ in equation (16)). We shall see that using the particular form $\alpha^{(p)}$ takes on for de Sitter and Nariai shows that, actually, *the gauge dependency precisely drops out.*

3 Nariai with a covariant constraint functional

When perturbing around the Nariai metric tensor,

$$ds^2 = \frac{1}{\tilde{\Lambda}}\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right), \tag{22}$$

Volkov & Wipf obtain for the eigenvalue equation of the h modes

$$-\Box h = \lambda_n h \tag{23}$$

the eigenfunction

$$h = Y_{j_1 m_1}(\theta_1, \phi_1) Y_{j_2 m_2}(\theta_2, \phi_2), \tag{24}$$

with eigenvalues

$$\lambda_{j_1 j_2} = j_1(j_1+1) + j_2(j_2+1). \tag{25}$$

$Y_{jm}(\theta, \phi)$ is a spherical harmonic function. From this it is therefore to be understood that, for perturbations around Nariai (with or without constraint functional), the scalar modes are expanded as

$$h = \sum_{j_1, j_2} Y_{j_1 m_1}(\theta_1, \phi_1) Y_{j_2 m_2}(\theta_2, \phi_2) C^h_{j_1 j_2}, \tag{26}$$

where the sum is in ascending order of j_1 and j_2. If we now calculate κ_n, we see that

$$\int d^4x \sqrt{g}\, h = \frac{1}{\tilde{\Lambda}^2} \int_0^{\pi} d\theta_1 \int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_2 \int_0^{2\pi} d\phi_2 \sin\theta_1 \sin\theta_2 h \tag{27}$$

$$= \frac{1}{\tilde{\Lambda}^2} \sum_{j_1, j_2} C^h_{j_1 j_2} \int_0^{\pi} \int_0^{2\pi} d\theta_1 d\phi_1 \sin(\theta_1) Y_{j_1 m_1}(\theta_1, \phi_1) \int_0^{\pi} \int_0^{2\pi} d\theta_2 d\phi_2 \sin(\theta_2) Y_{j_2 m_2}(\theta_2, \phi_2) \tag{28}$$

$$= \frac{1}{\tilde{\Lambda}^2} \sum_{j_1, j_2} C^h_{j_1 j_2} \delta_{j_1,0} \delta_{m_1,0} \delta_{j_2,0} \delta_{m_2,0} \tag{29}$$

$$= \frac{1}{\tilde{\Lambda}^2} C^h_{00} \tag{30}$$

$$= \frac{1}{\tilde{\Lambda}^2} C^h_0. \tag{31}$$

So the only scalar mode that contributes to κ_n is C_0^h, with $\kappa_0 = \frac{1}{32\pi G\tilde{\Lambda}^2}$ and so, actually, for Nariai, the offending gauge introducing term of (17) goes as

$$F_{Nariai} \propto \int d(\delta\lambda) \prod_p dC_p^h \exp\left\{ \left(\sum_n \frac{1}{16\gamma} \tilde{\lambda}_p^\gamma (C_n^h)^2 - i\delta\lambda\bar{\Lambda}\kappa_n C_n^h \right) \right\} \tag{32}$$

$$= \int d(\delta\lambda) dC_0^h {\prod_p}' \sqrt{\frac{2\gamma}{\tilde{\lambda}_p^\gamma}} \exp\left\{ \left(\frac{1}{16\gamma} \tilde{\lambda}_0^\gamma (C_0^h)^2 - i\delta\lambda\bar{\Lambda}\kappa_0 C_0^h \right) \right\}, \tag{33}$$

where the prime near the product indicates the zero'th mode C_0^h is not included in the product. Now, crucially

$$\boxed{\tilde{\lambda}_0^\gamma = \gamma(3\lambda_0 - 4\tilde{\Lambda}) - \lambda_0 = -4\gamma\tilde{\Lambda}}, \tag{34}$$

as $\lambda_0 = 0$, as can be read off from equation (25). This one eigenvalue, picked out by the constraint functional, means that we are left with

$$F_{Nariai} \propto \int d(\delta\lambda) dC_0^h {\prod_p}' \sqrt{\frac{2\gamma}{\tilde{\lambda}_p^\gamma}} \exp\left\{ \left(-\frac{1}{4} \tilde{\Lambda} (C_0^h)^2 - i\delta\lambda\bar{\Lambda}\kappa_0 C_0^h \right) \right\}. \tag{35}$$

Almost by magic, the constraint functional picks out precisely the only eigenvalue for which the path integral both *works and has no gauge dependency.* Upon performing the last integration, we get

$$F_{Nariai} \propto \prod_p \sqrt{\frac{2\gamma}{\tilde{\lambda}_p^\gamma}} \left(32G\sqrt{\frac{\pi^3\tilde{\Lambda}^5}{3\bar{\Lambda}^2}} \right), \tag{36}$$

the contribution from the constraint functional is the term in brackets. The product term will cancel through the Faddeev-Popov determinant and so the final partition function, with a covariant constraint, will be *gauge independent.*

4 de Sitter with a covariant constraint functional

The situation for the de Sitter metric tensor

$$ds^2 = \frac{3}{\tilde{\Lambda}} \left(d\psi^2 + \sin^2\psi \, d\Omega_3^2 \right) \tag{37}$$

is slightly more tricky, as the resulting eigenfunctions are less straightforward to work with. Volkov & Wipf obtain for the h modes eigenvalue equation

$$-\Box h = \lambda_j h \tag{38}$$

for de Sitter the eigenfunction

$$h = h_{(a_1 \cdots a_j)} x^{a_1} \cdots x^{a_j}, \tag{39}$$

which is a representation in terms of homogeneous polynomials on the S^4, where x^a the Cartesian coordinates of the five-dimensional flat space embedding of de Sitter. The eigenvalues are

$$\lambda_j = \frac{\tilde{\Lambda}}{3} j(j+1). \tag{40}$$

Since homogeneous polynomials are less straightforward to work with, we use that, since h is a square-integrable function on the 4-sphere, it can be decomposed uniquely into a series of 4-dimensional spherical harmonics as

$$h = \sum_{L,l,p,m} Y_{Llpm} C^h_{Llpm}. \tag{41}$$

In four dimensions, the spherical harmonic functions are

$$Y_{l_1 l_2 l_3 l_4}(\theta_1, \theta_2, \theta_3, \theta_4) = \frac{1}{\sqrt{2\pi}} e^{i l_1 \psi} \prod_{j=2}^{4} \sqrt{\frac{2l_j + j - 1}{2} \frac{(l_j + l_{j-1} + j - 2)!}{(l_j - l_{j-1})!}} \sin^{\frac{2-j}{2}}(\theta_j) P^{-\left(l_{j-1} + \frac{j-2}{2}\right)}_{l_j + \frac{j-2}{2}}(\cos\theta_j), \tag{42}$$

where P^l_L are Legendre functions. From this formula, we can calculate that

$$Y_{0000}(\psi, \chi, \theta, \phi) = \frac{1}{2\sqrt{2}\pi}. \tag{43}$$

The four-dimensional spherical harmonics are orthonormal with respect to one another on the 4-sphere,

$$\int d^4 g \sqrt{g}\, Y_{Llpm} Y^*_{L'l'p'm'} = \delta_{L,L'} \delta_{l,l'} \delta_{p,p'} \delta_{m,m'}, \tag{44}$$

which we can use to calculate that

$$\int d^4 g \sqrt{g}\, Y_{Llpm} Y^*_{0000} = \frac{1}{2\sqrt{2}\pi} \int d^4 g \sqrt{g}\, Y_{Llpm} = \delta_{L,0} \delta_{l,0} \delta_{p,0} \delta_{m,0} \tag{45}$$

and therefore

$$\int d^4 x \sqrt{g}\, Y_{Llpm} = 2\sqrt{2}\pi \delta_{L,0} \delta_{l,0} \delta_{p,0} \delta_{m,0}. \tag{46}$$

Since the 4-sphere of de Sitter has a radius $l = \sqrt{\frac{3}{\tilde{\Lambda}}}$, we have, for de Sitter,

$$\int_{dS} d^4 x \sqrt{g}\, Y_{Llpm} = \frac{2\sqrt{2}\tilde{\Lambda}^2 \pi}{9} \delta_{L,0} \delta_{l,0} \delta_{p,0} \delta_{m,0}. \tag{47}$$

Therefore, calculating κ_n,

$$\int d^4x\sqrt{g}\,h = \sum_{L,l,p,m} C^h_{Llpm}\int d^4x\sqrt{g}Y_{Llpm} = \frac{2\sqrt{2}\tilde{\Lambda}^2\pi}{9}C^h_0. \tag{48}$$

Once again, as if by magic, the only scalar mode that contributes to κ_n is C^h_0, with $\kappa_0 = \frac{\sqrt{2}\tilde{\Lambda}^2}{16G}$. The 1-loop corrections to the de Sitter partition function will therefore once again go as

$$F_{dS} \propto \int d(\delta\lambda)dC^h_0 \prod_p{}' \sqrt{\frac{2\gamma}{\tilde{\lambda}^\gamma_p}}\exp\left\{\left(-\frac{1}{4}\tilde{\Lambda}(C^h_0)^2 - i\delta\lambda\bar{\Lambda}\kappa_0 C^h_0\right)\right\}, \tag{49}$$

though with a different value of κ_0. Once evaluated, we will obtain

$$F_{dS} \propto \prod_p \sqrt{\frac{2\gamma}{\tilde{\lambda}^\gamma_p}}\left(G\sqrt{\frac{128\pi}{3\tilde{\Lambda}^3\bar{\Lambda}^2}}\right). \tag{50}$$

The term between brackets is the contribution from the constrained instanton, while the other product term will cancel by the Faddeev-Popov determinant. Once again, we can conclude that the final partition function Z_{dS}, with a covariant constraint, is *gauge independent*.

It is interesting to note that the factors of G stemming from the constraint functional have no effect on the perturbativity of the Nariai nucleation rate, since both partition functions Z_{dS} and Z_N have an additional factor G which will drop out in the division Z_N/Z_{dS}.

5 Conclusion

Though the introduction of a covariant constraint functional naively gives a gauge dependency, we have found that this is intimitately tied to which metric tensor one perturbs around and that the dependency is likely only an illusion. For the Nariai and de Sitter solutions it is clear that a covariant constraint functional does not introduce a gauge dependency and it seems likely that a covariant constraint functional will not, in general, introduce a gauge dependency. These findings strengthen the consistency of the constrained instanton formalism to 1-loop order and give a good reason to investigate covariant constraint functionals if one runs into a gauge dependency at 1-loop order.

References

[1] M. S. Volkov and A. Wipf. Black hole pair creation in de sitter space: a complete one-loop analysis. *Nuclear Physics B*, 582(1-3):313–362, aug 2000.