The Schwarzschild-de Sitter Zero Mode Volume

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Under the assumption that, after gauge-fixing, only the Killing vectors contribute to the zero mode volume, I calculate the zero mode volume of Schwarzschild-de Sitter. I then inspect its interpolation behavior between the de Sitter and Nariai zero mode volume. It is found that, using the techniques presented, the SdS zero mode volume interpolates to a fraction of de Sitter, whereas it is undefined for Nariai in both SdS and Nariai coordinates. It is argued that the zero mode volume, as a function of mass, cannot account for the *change* in background isometries that occurs in the de Sitter and Nariai mass limit.

For this calculation, I will follow the calculation of the de Sitter and Nariai zero mode volume by Volkov & Wipf [2], where the zero mode volume is referred to as the 'isometry factor'. As the technique used to handle the zero modes is explained in detail in this paper, I will not re-explain it here; instead, I will, for the most part, simply use the technique and make comments about it when needed. In addition, I shall assume the reader is familiar with terms and expressions I have explained in previous papers.

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1 Assumptions

Whether or not zero modes contribute to our path integral depends on whether the measure, for the integration over the modes, actually includes the zero modes. For the measure $D[h_{\mu\nu}]$ over the metric perturbation in

$$Z_{SdS} = e^{-I} \int D\left[h_{\mu\nu}\right] \mathcal{D}_{FP} \exp\left(-\delta^2 I_{gf}\right), \tag{1}$$

we obtain, for Schwarzschild-de Sitter,

$$D[h_{\mu\nu}] = \left(\prod_k \frac{dC_k^\phi}{\sqrt{2\pi}}\right)\left(\prod_s{}' \sqrt{\frac{\sigma_s}{\pi}} dC_s^\eta\right)\left(\frac{1}{2}\frac{dC_0^h}{\sqrt{2\pi}}\right)\left(\prod_n \frac{1}{2}\frac{dC_n^h}{\sqrt{2\pi}}\right)\left(\prod_p{}' \frac{dC_p^\chi}{2\pi}\sqrt{\tilde{\lambda}_p^\Lambda - 32\varrho_p}\right), \tag{2}$$

with $n > 0$. The integration with respect to this measure therefore goes only over the zero modes of the TT tensor $\phi_{\mu\nu}$, if they exist. For the measure $D[\xi_\mu]$ in the Faddeev-Popov factor

$$\mathcal{D}_{FP} = \left(\int D\left[\xi_\mu\right] \exp\left(-\delta^2 I_g\right)\right)^{-1}, \tag{3}$$

we obtain

$$D[\xi_\mu] = \left(\prod_s \frac{dC_s^\eta}{\sqrt{\pi}}\right)\left(\prod_p{}' \sqrt{\frac{\lambda_p}{\pi}} dC_p^\chi\right), \tag{4}$$

which goes over the zero modes of the vector modes η_μ. The measures therefore go over the zero modes in the same way as Volkov & Wipf. However, Volkov & Wipf show by explicitly calculating the spectra of the modes, that only the vector modes contribute. Moreover, they show that Nariai has six zero vector modes and de Sitter has ten zero vector modes, corresponding to the six Killing vectors of Nariai and the ten Killing vectors of de Sitter.

Having not yet calculated the spectrum of the modes of Schwarzschild-de Sitter, we do not know if it will only be the Killing vectors that contribute to our zero modes. However, we shall assume this is the case. We shall assume that the zero modes are the Killing vectors only. Take note, though, that this is definitely an assumption.

2 The zero mode volume

The isometries of the background manifold form the isometry group, which Volkov & Wipf denote $\mathcal{H}$. Since the isometries do not change $h_{\mu\nu}$, the generators of the isometries, the Killing vectors, are zero modes at 1-loop order. As can be seen from equation (4), they will contribute a factor of $1/\Omega_1$ to the 1-loop partition function, where Ω_1 is

$$\Omega_1 = \int \prod_j \frac{dC^\eta_{0j}}{\sqrt{\pi}}, \tag{5}$$

where j counts the degeneracy of the zero modes and our normalization does not include the factor μ_0^2 like Volkov & Wipf. Note that, with how zero mode volume is defined,

$$V_{zm} = (\Omega_1)^{-1}. \tag{6}$$

We will call Ω_1 the isometry factor. Volkov & Wipf explain that, if the zero modes originate purely from the Killing vectors, that zero mode integration as in (5) should really be done as

$$\Omega_1 = \left(\prod_j \frac{\Lambda}{\sqrt{\pi}} ||K_j|| \right) Vol\left(\mathcal{H}\right). \tag{7}$$

Here, $||K_j||$ is the norm of the Killing vectors of the background manifold. The factor Λ is introduced to make the Killing vectors dimensionless, as all eigenmodes in their analysis (and our analysis) have unit norm. Note, however, that there is some ambiguity in making the Killing vectors dimensionless, as shall be elaborated later on. $Vol\left(\mathcal{H}\right)$ is the volume of the isometry group, which comes from the integration of the Haar measure of $\mathcal{H}$ to render the zero mode integration convergent.

2.1 Schwarzschild-de Sitter zero mode volume

From here, we can calculate the zero mode volume for Schwarzschild-de Sitter. Schwarzschild-de Sitter has isometry group $\mathcal{H} = U(1) \times SO(3)$. As is explained by Volkov & Wipf on page 39, the norm of each of the three $SO(3)$ generators is the square root of

$$\langle \partial_\phi, \partial_\phi \rangle = \frac{1}{32\pi G} \int_{SdS} g_{\phi\phi} \sqrt{g} d^4x \tag{8}$$

$$= \frac{1}{32\pi G} \int_0^\beta d\tau \int_{r_b}^{r_c} r^4 dr \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} d\phi \tag{9}$$

$$= \frac{\beta}{60G} \left[r_c^5 - r_b^5 \right]. \tag{10}$$

The $U(1)$ component of the isometry group is generated by the Killing vector $K^\mu = (1,0,0,0)$, which in covariant form is therefore $K_\mu = \delta_\mu^\tau g_{\tau\tau} = \delta_\mu^\tau f(r)$, where $f(r)$ the SdS emblackening factor. The norm of the $U(1)$ generator is therefore the square root of

$$\langle \partial_\tau, \partial_\tau \rangle = \frac{1}{32\pi G} \int_{SdS} f(r) \sqrt{g} d^4x \tag{11}$$

$$= 4\pi\beta \left(\frac{1}{3} \left[r_c^3 - r_b^3 \right] - \frac{1}{4l^2} \left[r_c^5 - r_b^5 \right] - GM \left[r_c^2 - r_b^2 \right] \right) \tag{12}$$

$$\equiv 4\pi\beta F(M), \tag{13}$$

where l the de Sitter radius. As for the isometry group volume, it is known that $Vol(SO(3)) = 8\pi^2$. Furthermore, the elements of $U(1)$ with radius β can be parameterized as $\beta e^{i\varphi}$, $\varphi \in [0, 2\pi)$, which leads to a rescaling of the Haar measure by β, in turn leading to $Vol(U(1)) = 2\pi\beta$. We can therefore, using equation (7), calculate that

$$\Omega_{1,SdS} = \frac{\sqrt{30}}{450} \frac{\beta^3 \Lambda^4 \pi}{G^2} \sqrt{F(M)} \left(\sqrt{r_c^5 - r_b^5} \right)^3. \tag{14}$$

We now check if this reduces to the de Sitter and Nariai isometry factors in their respective mass limits.

2.2 The de Sitter limit $(M \to 0)$

De Sitter does not have a $U(1) \times SO(3)$ isometry; it enjoys an $SO(5)$ isometry, so it is interesting to see whether the Schwarzschild-de Sitter isometry factor smoothly reduces to the de Sitter isometry factor in the de Sitter mass limit $(M \to 0)$. This would be remarkable, since this would imply that, purely through the mass parameter, the zero mode volume 'knows' the isometries of the background. We can calculate that

$$\lim_{M\to 0}\lim_{\beta\to 2\pi l}\Omega_{1,SdS} = 32\pi^4\mu_0^8\sqrt{\left(\frac{9}{10}\right)^3}\sqrt{\frac{9}{16}}\left(\Lambda G\right)^{-2}, \tag{15}$$

where we have restored the μ_0 factor for comparison with Volkov & Wipf. They obtain

$$\Omega_{1,dS} = \left(\frac{9}{10}\right)^5\frac{128\pi^6}{3}\frac{(\mu_0)^{20}}{(\Lambda G)^5}. \tag{16}$$

Clearly, these expressions do not agree. Here, it should be mentioned that there is some arbitrariness with respect to making the Killing vector norm in the isometry factor dimensionless. Volkov & Wipf multiply by Λ to accomplish this, but this is not the only length scale in our theory. We could, for example, also divide the Killing vector norm by the Planck length squared, as:

$$\Omega_1 = \left(\prod_j \frac{1}{G\sqrt{\pi}}||K_j||\right)Vol\left(\mathcal{H}\right). \tag{17}$$

This will result in a factor $(\Lambda G)^{-6}$ in $\Omega_{1,SdS}$ and a different ΛG factor overall for all isometry factors and nucleation rates calculated. This is to say that there is some wiggle room in the factors present in the isometry factors. However, we must face that we do not recover the de Sitter isometry factor from our SdS one, even though we would like to. We shall further comment on this reason for this after inspecting the Nariai limit.

2.3 The Nariai limit ($M \to M_N$)

The Nariai limit, $M \to M_N$, is even worse: in this limit the entire SdS isometry factor vanishes to zero. This happens as the $U(1)$ norm vanishes in the Nariai limit. This limit is therefore not well-defined. This would make sense, since the static Schwarzschild-de Sitter coordinates are not well-defined in this limit to begin with, were it not that the situation doesn't change when switching to Nariai coordinates. We can perform a coordinate transformation to the Nariai coordinates as they were proposed by Bousso and Hawking [1],

$$\tau = \frac{1}{\epsilon\sqrt{\Lambda}}\left(1-\frac{1}{2}\epsilon^2\right)\psi; \quad r = \frac{1}{\sqrt{\Lambda}}\left[1+\epsilon\cos\chi-\frac{1}{6}\epsilon^2+\frac{4}{9}\epsilon^3\cos\chi\right] \tag{18}$$

where $\epsilon \to 0$ is the $M \to M_N$ limit. However, doing so after the integration in the SdS isometry factor has already been performed, does not elucidate either, since $r_{c,b}$ will simply tend to $r_{c,b} \to (\sqrt{\Lambda})^{-1}$ as $M \to M_N$ and the factors $r_c^n - r_b^n$ in (11) will tend to zero, sending $\Omega_{1,SdS}$ to zero, as well.

We could try performing the coordinate transformation before taking the integral of the $U(1)$ generators, but then we get the same problem, which we will show now. Expressed in Nariai coordinates, the SdS emblackening factor becomes

$$f(\chi) = \sin^2\chi\epsilon^2\left(1 - \frac{2}{3}\epsilon\cos\chi + \frac{2}{3}\epsilon^2\cos^2\chi + \frac{8}{9}\epsilon^2\right). \tag{19}$$

We have the differential elements

$$d\tau = \frac{1}{\epsilon\sqrt{\Lambda}}\left(1 - \frac{1}{2}\epsilon^2\right)d\psi; \quad dr = -\frac{\epsilon\sin\chi}{\Lambda}\left(1 + \frac{4}{9}\epsilon^2\right)d\chi. \tag{20}$$

From here, it is not hard to see that the $U(1)$ norm integral in Nariai coordinates becomes, to leading order in ϵ,

$$\langle\partial_\tau, \partial_\tau\rangle = \frac{-\epsilon^2}{32\pi G\Lambda\sqrt{\Lambda}}\int_{SdS}\sin^3\chi\sin\theta d\tau d\chi d\theta d\phi, \tag{21}$$

which vanishes in the limit $\epsilon \to 0$. It seems that, no matter what, the $U(1)$ isometry factor is ill-defined in the Nariai limit. When calculated, the $SO(3)$ norm is perfectly well-defined in the Nariai limit, after coordinate transformations, however. Why is this?

2.4 How does this happen?

The exact reason why the zero mode volume cannot interpolate between de Sitter, Schwarzschild-de Sitter and Nariai is, as of yet, unknown to me. Why the Nariai limit is ill-defined, while the de Sitter limit is well-defined, can be explained, however.

Inspecting the norms, it can be readily seen that in the limit $M \to 0$ the norm squared of the $SO(3)$ generator of equation (8) reduces to the norm squared of the de Sitter $SO(5)$ generator, as calculated by Volkov & Wipf on page 43, *if* β is taken to be the inverse temperature of de Sitter, so $\beta \to 2\pi l$. The $U(1)$ generator norm seemingly reduces to no part of the de Sitter isometry factor. However, it does reduce to a part of the $SO(5)$ generators norm.

Why? The simple reason for the difference in how well-defined both limits of the SdS zero mode volume are comes down to this: $U(1)$ and $SO(3)$ are subgroups of $SO(5)$, the de Sitter isometry group, while $U(1)$ is not a subgroup of $SO(3)$, the isometry group of one of the 2-spheres in Nariai. Therefore, when we calculate the norm of the $U(1)$ generators, the de Sitter mass limit is well-defined, as $U(1)$ is a subgroup of the de Sitter isometry group. When we then calculate the $U(1)$ norm in the Nariai limit, *the vanishing of this norm is telling us that Nariai does not have the $U(1)$ isometry.* This works the same way for the $SO(3)$ generator norm of SdS: $SO(3)$ is a subgroup of both Nariai and de Sitter, so its limit is in both cases well-defined. In this sense, the zero mode volume knows *how* the isometries of SdS scale to other isometries, but *cannot correct* for the *changes* in the isometries.

3 Conclusion

Having calculated the zero mode volume for SdS, we disappointingly find that it does not interpolate between that of de Sitter and Nariai. Since the reason for this depends on whether the SdS isometry group is a subgroup of the isometry group of the spacetimes in the two limits, we conclude that the SdS zero mode volume knows about the scaling behavior of its own isometry group, but somehow either does not have the information of all the isometries of its limiting cases, or fails to correct for the change in the isometries at those limits. Perhaps performing the complete 1-loop analysis will bring clarity to this issue.

References

[1] R. Bousso and S. W. Hawking. Pair creation of black holes during inflation. *Physical Review D*, 54(10):6312–6322, nov 1996.

[2] M. S. Volkov and A. Wipf. Black hole pair creation in de sitter space: a complete one-loop analysis. *Nuclear Physics B*, 582(1-3):313–362, aug 2000.