The Gauge Dependency of Z_{SdS}

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In this paper, we delve into gauge issues that arise when calculating the 1-loop corrections F_{SdS} to Schwarzschild-de Sitter's partition function $Z_{SdS} = F_{SdS}e^{-I_{SdS}}$. We will see that a possible gauge dependency arises, for which we offer several solutions.

Contents

1 A brief recap

Before calculating Schwarzschild-de Sitter's 1-loop differential equations, we will briefly recap some relevant facts. Recall that in the constrained path integral, we have

$$Z = \int \mathcal{D}g \frac{d\lambda d\zeta}{2\pi} e^{-I_E[g]+i\lambda(C[g]-\zeta)}, \tag{1}$$

where I_E the Euclidean Einstein-Hilbert action, we treat $I_{tot} = I_E[g] - i\lambda(C[g]-\zeta)$ as the total action. Saddle points of this action are constrained instanton solutions that consist of a solution for λ and a solution for the metric tensor $g_{\mu\nu}$. One-loop corrections arise from a functional Taylor series of this total action about the background metric $\overline{g}$ and Lagrange multiplier $\overline{\lambda}$

$$I_{tot} = I_{tot}[\overline{g}] + \delta^2 I[\overline{g}, \overline{\lambda}] + \cdots, \tag{2}$$

and are contained within the term $\delta^2 I_{tot}[\overline{g}, \overline{\lambda}]$. Specified to the Schwarzschild-de Sitter constrained instanton, for a metric perturbation $g_{\mu\nu} = g_{\mu\nu,SdS} + h_{\mu\nu}$ and multiplier perturbation $\lambda = \lambda_{SdS} + \delta\lambda$, $\delta^2 I[\overline{g}, \overline{\lambda}]$ is equal to

$$\delta^2 I = \delta^2 I_{tot} - i\delta\lambda\delta I_c. \tag{3}$$

We defined

$$\delta^2 I_{tot} = \frac{1}{2}\int d^4x\, h^{\mu\nu} \frac{\delta^2 I_{tot}[g_{SdS}, \lambda_{SdS}]}{\delta g^{\mu\nu}\delta g^{\alpha\gamma}} h^{\alpha\gamma}, \tag{4}$$

with

$$\frac{\delta^2 I_{tot}[g_{SdS}, \lambda_{SdS}]}{\delta g^{\mu\nu}\delta g^{\alpha\gamma}} = \frac{\delta^2 I_E[g_{SdS}]}{\delta g^{\mu\nu}\delta g^{\alpha\gamma}} - i\lambda_{SdS}\frac{\delta^2 C[g_{SdS}]}{\delta g^{\mu\nu}\delta g^{\alpha\gamma}}, \tag{5}$$

and $\lambda_{SdS} = -\frac{iGM}{2\pi r_b^2}$. We also defined

$$\delta I_c = \int d^4x \frac{\delta C[g_{SdS}]}{\delta g^{\mu\nu}} h^{\mu\nu}. \tag{6}$$

Here, we are using the area constraint

$$C[g] = \frac{1}{16\pi G}\int d^4x\sqrt{g}\left(4\pi\delta_{\Sigma_b} + 4\pi\delta_{\Sigma_c}\right), \tag{7}$$

to include Schwarzschild-de Sitter as constrained instanton into the saddle point sum of the path integral. The delta functions $\delta_{\Sigma_b}, \delta_{\Sigma_c}$ within this constraint have the defining property $\int_{\mathcal{M}} f\delta_{\Sigma_h} = \int_{\Sigma_h} f$. We will refer to these delta functions as *Solodukhin delta functions*, based on their appearance in [1, 4].

The trace-reversed equations of motion we obtain from the first-order variation of the total action I_{tot} is

$$R_{\mu\nu} = g_{\mu\nu} \left[\Lambda - i2\pi\lambda \left(\delta_{\Sigma_b} + \delta_{\Sigma_c} \right) \right]. \tag{8}$$

We will refer to these equations as the *constrained Einstein field equations* (conEFE). For convenience, we will write these equations as

$$\boxed{R_{\mu\nu} = \tilde{\Lambda} g_{\mu\nu}}, \tag{9}$$

where $\tilde{\Lambda} \equiv \Lambda - i2\pi\lambda \left(\delta_{\Sigma_b} + \delta_{\Sigma_c} \right)$. It is these equations that Schwarzschild-de Sitter truly satisfies, as it accounts for its conical singularities.

It is important to note that $\tilde{\Lambda}$ is a function of spacetime coordinates due to the functions $\delta_{\Sigma_b}, \delta_{\Sigma_c}$ present within it. This will have important consequences once we calculate the differential equations that govern the 1-loop corrections of Schwarzschild-de Sitter.

2 One-loop constrained instantons

In this section we will largely follow Volkov and Wipf [5] in setup for the 1-loop calculations. Along the way, we will explain any results that differ from their results.

2.1 Path integral setup

We are interested in calculating

$$F_{SdS} = \int D[h_{\mu\nu}] \exp\left(-\delta^2 I\right). \tag{10}$$

While we could express $\delta^2 I$ directly in terms of the metric perturbation $h_{\mu\nu}$, it is more convenient to decompose $h_{\mu\nu}$ first. We perform a decomposition:

$$h_{\mu\nu} = \phi_{\mu\nu} + \frac{1}{4} h g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \xi^\sigma. \tag{11}$$

Here $\phi_{\mu\nu}$ is the transverse, traceless tensor part ($\nabla_\mu \phi^\mu_\nu = \phi^\mu_\mu = 0$), h is the trace, and ξ_μ is the longitudinal, traceless vector part that generates general diffeomorphisms (general relativity's gauge transformations). We can Hodge decompose ξ_μ further as

$$\xi_\mu = \eta_\mu + \nabla_\mu \chi, \tag{12}$$

where $\nabla_\mu \eta^\mu = 0$. Technically, the harmonic vector piece ξ^H_μ is also included in this decomposition, though it does not contribute for Schwarzschild-de Sitter. The number of square-integrable harmonic vectors is equal to the first Betti number b_1 of the manifold. It is easy to see that Schwarzschild-de Sitter has $b_1 = 0$. Schwarzschild-de Sitter is a warped product $S^2 \times_f S^2$, with $f : S^2 \to \mathbb{R}$ a warping function, which is homeomorphic to $S^2 \times S^2$. We therefore have

$$b_1(S^2 \times_f S^2) = b_1(S^2 \times S^2) = b_1(S^2) + b_1(S^2) = 0. \tag{13}$$

Thus, in our analysis, ξ^H_μ does not contribute to equation (12).

Before attempting to evaluate the path integral (16), we must add a gauge-fixing term to account for gauge freedom; if we would not, our path integral would diverge. This stems from the ξ_μ that correspond to diffeomorphisms, for which the metric perturbation corresponds to zero, which would result in zero modes and with that, in divergent Gaussian integrals. We add the standard gauge-fixing term $\delta^2 I_g$ to $\delta^2 I$:

$$\delta^2 I_g = \frac{\gamma}{32\pi G} \int_{\mathcal{M}} \left(\nabla_\sigma h^\sigma_\rho - \frac{\gamma+1}{4\gamma} \nabla_\rho h\right) \left(\nabla^\alpha h^\rho_\alpha - \frac{\gamma+1}{4\gamma} \nabla^\rho h\right). \tag{14}$$

Here γ is a real parameter which is usually taken to be $\gamma = 1$. We shall, however, not fix the value of γ to make sure our final answer is gauge-independent. If γ is not present at the end of the calculation, the answer is gauge-invariant. We will denote the gauge-fixed action as

$$\delta^2 I_{gf} = \delta^2 I + \delta^2 I_g. \tag{15}$$

The gauge-fixing term will, besides modifying the second-order variation $\delta^2 I$, also add a compensating factor to the path integral (16) to make sure it doesn't depend on the gauge-fixing condition. This factor is referred to as the *Faddeev-Popov determinant*, denoted $\mathcal{D}_{FP}$. After compensating for gauge freedom, the path integral (16) will therefore become

$$\boxed{F_{SdS} = \int D\left[h_{\mu\nu}\right] \mathcal{D}_{FP} \exp\left(-\delta^2 I_{gf}\right)}, \tag{16}$$

where the Faddeev-Popov determinant is defined by

$$\left(\mathcal{D}_{FP}\right)^{-1} = \int D\left[\xi_\mu\right] \exp\left(-\delta^2 I_g\right). \tag{17}$$

We are therefore interested in evaluating $\delta^2 I_{gf}$, $\delta^2 I_g$ and $\mathcal{D}_{FP}$.

Both $\delta^2 I_{gf}$ and $\delta^2 I_g$ will take the form of a second-order partial differential operator P working on $h_{\mu\nu}$. Due to the Hodge decomposition, we will find P decomposes into a linear combination of differential operators working on the various components of $h_{\mu\nu}$. To evaluate it, we expand the terms in $h_{\mu\nu}$ as eigenfunctions, or *eigenmodes*, of said differential operators, namely

$$\phi_{\mu\nu} = \sum_k C_k^\phi \phi_{\mu\nu}^{(k)}, \quad \eta_\mu = \sum_s C_s^\eta \eta_\mu^{(s)} \tag{18}$$

and

$$\chi = \sum_p C_p^\chi \alpha^{(p)}, \quad h = \sum_p C_p^h \alpha^{(p)}, \tag{19}$$

where we followed Volkov and Wipf [5] for the eigenmode expansion. We then take the eigenfunctions orthonormal with respect to the inner product

$$\left\langle h_{\mu\nu(n)}, h^{\mu\nu}_{(m)} \right\rangle = \frac{1}{32\pi G} \int_{\mathcal{M}} h_{\mu\nu(n)} h^{\mu\nu}_{(m)}\, d^4x \sqrt{g} = \delta_{nm}. \tag{20}$$

When we do as so, P becomes a sum of eigenvalues. Simplifying the measures $D[h_{\mu\nu}]$ and $D[\xi_\mu]$ as a measure over the Fourier coefficients $C_n^\phi, C_n^\eta, C_n^\chi, C_n^\phi$, we can then evaluate F_{SdS} as an infinite product of Gaussian functions. We shall later specify some differential operators that belong to some of the eigenfunctions when we will write out $\delta^2 I_{gf}$. We will also specify the measures $D[h_{\mu\nu}]$ and $D[\xi_\mu]$ later on.

2.2 The second variation of the action

We now proceed with detailing $\delta^2 I_{gf}$ and $\delta^2 I_g$. To do this, we first give the full expression of $\delta^2 I$. We have

$$\delta^2 I = \frac{1}{32\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}\Bigg(-\frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} + \frac{1}{4}h\Box h - \left[\nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h\right]^2 - h^{\mu\lambda}h^{\nu\sigma}R_{\mu\nu\lambda\sigma} \tag{21}$$

$$+ h^{\nu\sigma}R_{\nu\sigma}h - h^{\mu\lambda}h^\nu_\lambda R_{\mu\nu} + \frac{1}{2}\left[h^{\mu\nu}h_{\mu\nu} - \frac{1}{4}h^2\right]\left[R - 2\tilde{\Lambda}\right]\Bigg) \tag{22}$$

$$- i\delta\lambda\frac{1}{32\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}\,4\pi\left(\delta_{\Sigma_b} + \delta_{\Sigma_c}\right)h, \tag{23}$$

where the Riemann tensor, Ricci tensor and Ricci scalar are evaluated with respect to g_{SdS}, and $\Box = \nabla_\alpha\nabla^\alpha$ the Laplace-Beltrami operator. The last term and the $\tilde{\Lambda}$ stem from the first and second variation of the constraint functional, respectively. We can also see why, conventionally, $\gamma \to 1$ in $\delta^2 I_g$ (14): this will cancel one of the terms within $\delta^2 I$ once added.

Before we give the decomposed expression for $\delta^2 I$, we will preface that we will encounter many 'crossterms', also called off-diagonal terms, of the form $\tilde{\Lambda}\eta^\alpha\nabla_\alpha\tilde{\Lambda}\chi$, unlike Volkov and Wipf. We would like to explain where these come from. To make the expression as diagonal as possible, we use:

- The generalized Stokes' theorem, which allows us to write

$$\int_{\mathcal{M}} d^4x\sqrt{g}\,V_\mu\nabla_\nu\left(T^{\mu\nu}\right) = -\int_{\mathcal{M}} d^4x\sqrt{g}\,T^{\mu\nu}\nabla_\nu\left(V_\mu\right), \tag{24}$$

  for some general tensor $T_{\mu\nu}$ and vector V_μ. The boundary integral vanishes due to Schwarzschild-de Sitter being compact. Note that this also means that the $\Box$ operator is self-adjoint with respect to the inner product (20).

- The definition of the Riemann tensor

$$\left[\nabla_\mu, \nabla_\nu\right]V^\rho = R^\rho_{\ \lambda\mu\nu}V^\lambda \tag{25}$$

  in addition to the second Bianchi identity. Note that this implies that

$$\nabla^\alpha\left[\nabla_\mu, \nabla_\nu\right]V^\mu = \nabla^\alpha\left(R^\mu_{\ \lambda\mu\nu}V^\lambda\right) = \nabla^\alpha\left(\tilde{\Lambda}V_\nu\right), \tag{26}$$

  where, because $\tilde{\Lambda}$ is a function, ∇_α acts on it. It is from here the aforementioned crossterms appear.

Though these crossterms seem troublesome, we will see later on that they might save gauge-invariance.

2.2.1 $\delta^2 I$ decomposed

We now insert the decomposition of equation (11) to turn it into a more manageable form. We obtain

$$\begin{aligned}\delta^2 I =& \frac{1}{2}\left\langle \phi^{\mu\nu}, \Delta_2 \phi_{\mu\nu}\right\rangle - \frac{1}{16}\left\langle \tilde{h}, \tilde{\Delta}_0 \tilde{h}\right\rangle - 2\left\langle \eta^\sigma, \nabla_\sigma \tilde{\Lambda}\left(\Box + 3\tilde{\Lambda}\right)\chi\right\rangle \\ &- \left\langle \tilde{h}, \nabla^\sigma \tilde{\Lambda}\xi_\sigma\right\rangle - i4\pi\delta\lambda\left\langle\left(\delta_{\Sigma_b} + \delta_{\Sigma_c}\right), h\right\rangle,\end{aligned} \tag{27}$$

where we have defined, in line with Volkov and Wipf, the differential operators

$$\Delta_2 \phi_{\mu\nu} = -\Box\phi_{\mu\nu} - 2R_{\mu\alpha\nu\beta}\phi^{\alpha\beta} \tag{28}$$

$$\tilde{\Delta}_0 = -3\Box - 4\tilde{\Lambda}. \tag{29}$$

Looking at equation (27), it can be readily seen that upon sending $\tilde{\Lambda} \to \Lambda$, corresponding to perturbing around a solution of the conEFE to a solution of the vacuum EFE, the crossterms involving $\nabla_\sigma\tilde{\Lambda}$ disappear and we recover Volkov and Wipf's expression for $\delta^2 I$ (minus the linear $\delta\lambda$ term stemming from the λ perturbation). Ultimately, it will turn out more convenient to express $\delta^2 I$ in terms of h. Upon doing so, we recover the much lengthier expression

$$\begin{aligned}\delta^2 I =& \frac{1}{2}\left\langle \phi^{\mu\nu}, \Delta_2 \phi_{\mu\nu}\right\rangle - \frac{1}{16}\left\langle h, \tilde{\Delta}_0 h\right\rangle + \frac{1}{4}\left\langle \chi, \left(3\Box^3 + 4\tilde{\Lambda}\Box^2 + 4\Box\tilde{\Lambda}\Box\right)\chi\right\rangle \\ &- \frac{1}{4}\left\langle h, \left(3\Box^2 + 4\tilde{\Lambda}\Box\right)\chi\right\rangle - 6\left\langle \eta^\sigma, \tilde{\Lambda}\nabla_\sigma\tilde{\Lambda}\chi\right\rangle - \left\langle h, \nabla^\sigma\tilde{\Lambda}\xi_\sigma\right\rangle \\ &- i4\pi\delta\lambda\left\langle\left(\delta_{\Sigma_b} + \delta_{\Sigma_c}\right), h\right\rangle.\end{aligned} \tag{30}$$

From this, we will calculate the gauge-fixed action $\delta^2 I_{gf}$. To do this, however, we must first decompose the gauge term $\delta^2 I_g$.

2.2.2 $\delta^2 I_g$ decomposed

We expand the gauge term $\delta^2 I_g$ (equation (14)) and, after inserting the decomposition of equation (11) and a lengthy calculation, obtain

$$\begin{aligned}\delta^2 I_g =& -\frac{1}{16\gamma}\left\langle h, \Box h\right\rangle + \frac{1}{4}\left\langle h, \left(3\Box^2 + 4\tilde{\Lambda}\Box\right)\chi\right\rangle - \frac{\gamma}{4}\left\langle \chi, \left(9\Box^3 + 24\tilde{\Lambda}\Box^2 + 16\tilde{\Lambda}^2\Box\right)\chi\right\rangle \\ &+ \gamma\left\langle \eta_\mu, \Delta_1^2\eta^\mu\right\rangle + \left\langle h, \nabla^\alpha\tilde{\Lambda}\xi_\alpha\right\rangle - 2\gamma\left\langle \chi, \nabla_\rho\tilde{\Lambda}\left(3\Box + \tilde{\Lambda}\right)\nabla^\rho\chi\right\rangle \\ &- 4\gamma\left\langle \Box\chi, \nabla^\rho\tilde{\Lambda}\eta_\rho\right\rangle - 4\gamma\left\langle \chi, \left(3\tilde{\Lambda}\nabla^\rho\tilde{\Lambda} + \nabla^\rho\tilde{\Lambda}\Box\right)\eta_\rho\right\rangle,\end{aligned} \tag{31}$$

where we have defined, following Volkov and Wipf, the differential operator

$$\Delta_1 = -\Box - \tilde{\Lambda}. \tag{32}$$

Our expression is much longer than Volkov and Wipf's, as we have used h instead of $\tilde{h}$ and have expressed it not in a diagonal manner. Once again, however, if expressed in terms of $\tilde{h}$ and sending $\tilde{\Lambda} \to \Lambda$, their result is recovered. Our way of expressing $\delta^2 I_g$ has a reason, as we shall see later on in this paper. From this expression, we calculate the gauge-fixed action.

2.2.3 $\delta^2 I_{gf}$ decomposed

We obtain for the decomposed gauge-fixed action:

$$\begin{aligned}
\delta^2 I_{gf} =& \frac{1}{2} \langle \phi^{\mu\nu}, \Delta_2 \phi_{\mu\nu} \rangle + \gamma \langle \eta_\mu, \Delta_1^2 \eta^\mu \rangle - \frac{1}{16\gamma} \left\langle h, \tilde{\Delta}_0^\gamma h \right\rangle \\
& - \frac{1}{4} \left\langle \chi, \left(-3\Box^3 - 4\tilde{\Lambda}\Box^2 + 24\gamma\tilde{\Lambda}\Box^2 + 9\gamma\Box^3 + 16\gamma\tilde{\Lambda}^2\Box \right) \chi \right\rangle \\
& - \left\langle \chi, \left(\Box\tilde{\Lambda}\Box + 2\gamma\nabla_\mu\tilde{\Lambda} \left[3\Box\nabla^\mu + \tilde{\Lambda}\nabla^\mu \right] \right) \chi \right\rangle \\
& - \left\langle \eta_\mu, \nabla^\mu\tilde{\Lambda} \left(6\tilde{\Lambda} + 5\gamma\Box + 13\gamma\tilde{\Lambda} \right) \chi \right\rangle - i4\pi\delta\lambda \left\langle \left(\delta_{\Sigma_b} + \delta_{\Sigma_c} \right), h \right\rangle ,
\end{aligned} \tag{33}$$

where we have defined, like Volkov and Wipf, the differential operator

$$\tilde{\Delta}_0^\gamma = \gamma\tilde{\Delta}_0 + \Box. \tag{34}$$

Note that in Volkov and Wipf's expression of $\delta^2 I_{gf}$, the differential operator $\tilde{\Delta}_0$ is working on h. This is an error, and it can be seen that further in the article, when they consider the conformal modes on page 18, they consider the eigenvalues of the operator $\tilde{\Delta}_0^\gamma$.

Though less manifest from our expression, it can once again be confirmed that upon sending $\tilde{\Lambda} \to \Lambda$ we recover Volkov and Wipf's expression. In our case we have not written the diagonal χ term as $\left\langle \chi, \Delta_0\tilde{\Delta}_0\tilde{\Delta}_0^\gamma \chi \right\rangle$, where $\Delta_0 = -\Box$, as this would give terms involving $\nabla_\mu\tilde{\Lambda}$ which do not match ours. However, when written out for $\tilde{\Lambda} \to \Lambda$, the second line of our expression of $\delta^2 I_{gf}$ would match with $\left\langle \chi, \Delta_0\tilde{\Delta}_0\tilde{\Delta}_0^\gamma \chi \right\rangle$.

In order to perform the integration in F_{SdS} (equation (16)) and $\mathcal{D}_{FP}$ (equation (17)), we must first define the measures $D[h_{\mu\nu}]$ and $D[\xi_\mu]$.

2.3 Path integral measures

Volkov and Wipf define the perturbative measures as

$$D[h_{\mu\nu}] \sim \sqrt{\mathrm{Det}\left[\langle dh_{\mu\nu}, dh^{\mu\nu}\rangle\right]}, \quad D[\xi_\mu] \sim \sqrt{\mathrm{Det}\left[\langle d\xi_\mu, d\xi^\mu\rangle\right]}. \tag{35}$$

Here, $dh_{\mu\nu}$ and $d\xi_\mu$ means we take the differential of the Fourier coefficients, dC_n, as they appear in the mode expansions of equations (18) and (19). The proportionality sign is because we will endow terms within the measure with normalization factors. Volkov and Wipf also include a parameter μ_0 such that their measures have the correct dimension and are normalized, which we choose to not include, as it does not qualitatively affect physical quantities one can calculate. We shall ignore this normalization throughout this paper.

We have

$$\langle d\xi_\mu, d\xi^\mu\rangle = \langle d\eta_\mu, d\eta^\mu\rangle - \langle d\chi, \Box d\chi\rangle = \sum_s \left(dC_s^\eta\right)^2 + \sum_p{}' \lambda_p \left(dC_p^\chi\right)^2, \tag{36}$$

$$\begin{aligned}\langle dh_{\mu\nu}, dh^{\mu\nu}\rangle &= \langle d\phi^{\mu\nu}, d\phi_{\mu\nu}\rangle + 2\langle d\eta_\mu, \Delta_1 d\eta^\mu\rangle + \frac{1}{4}\langle dh, dh\rangle \\ &\quad + \left\langle d\chi, \tilde{\Delta}^\Lambda d\chi\right\rangle + 8\left\langle d\chi, \nabla_\mu \tilde{\Lambda} d\eta^\mu\right\rangle \\ &= \sum_k \left(dC_k^\phi\right)^2 + 2\sum_s{}' \sigma_s \left(dC_s^\eta\right)^2 + \frac{1}{4}\sum_n \left(dC_n^h\right)^2 + \sum_p{}' \tilde{\lambda}_p^\Lambda \left(dC_p^\chi\right)^2 \\ &\quad + 8\sum_p{}' \varrho_p dC_p^\chi dC_p^\eta,\end{aligned} \tag{37}$$

where we have defined

$$\tilde{\Delta}^\Lambda = 3\Box^2 + 4\nabla^\mu \tilde{\Lambda}\nabla_\mu + 4\tilde{\Lambda}\Box, \tag{38}$$

which can be seen to reduce to $\Delta_0\tilde{\Delta}_0$ when $\tilde{\Lambda} \to \Lambda$. The prime next to the sums indicate we do not include the first term, which is the zero mode, as they get multiplied by zero. We inserted the mode expansions of equations (18) and (19), chose the eigenfunction $\alpha^{(p)}, \eta_\mu^{(p)}$ such that

$$-\Box\alpha^{(p)} = \lambda_p \alpha^{(p)}, \tag{39}$$

$$\tilde{\Delta}^\Lambda \alpha^{(p)} = \tilde{\lambda}_p^\Lambda \alpha^{(p)}, \tag{40}$$

$$\Delta_1 \eta_\mu^{(s)} = \sigma_s \eta_\mu^{(s)}, \tag{41}$$

$$8\left\langle \nabla^\mu \tilde{\Lambda}\alpha^{(p)}, \eta_\mu^{(p)}\right\rangle = 8\varrho_p, \tag{42}$$

and used the orthonormality of the eigenfunctions. We note that ϱ_p is just Λ minus the eigenfunctions $\alpha^{(p)}, \eta_\mu^{(p)}$ evaluated on the value specified by the Solodukhin delta functions. It is also useful to define the eigenvalues

$$\tilde{\Delta}_0^\gamma \alpha^{(p)} = \tilde{\lambda}_p^\gamma \alpha^{(p)} = \left(\gamma \tilde{\lambda}_p - \lambda_p\right) \alpha^{(p)}, \tag{43}$$

$$4\pi \left\langle \left(\delta_{\Sigma_b} + \delta_{\Sigma_c}\right), \alpha^{(p)} \right\rangle = \kappa_p. \tag{44}$$

Many of the above eigenvalue equations involve Solodukhin delta functions. These equations can be solved by using the identity that the Solodukhin delta function for spherically symmetric, static metrics reduces to

$$\delta_{\Sigma_h} = \delta(\tau - c)\delta(r - r_h). \tag{45}$$

We can then use a Laplace transform, for which we have

$$\mathcal{L}\{\delta(r - r_h)\}(s) = e^{-r_h s} \tag{46}$$

and the condition that, since SdS is compact, $h_{\mu\nu}(r = r_b) = h_{\mu\nu}(r = r_b) = 0$, to turn the differential equations into solvable forms.

We should now comment on how the meaning of the determinant is not clear, as $\langle dh_{\mu\nu}, dh^{\mu\nu}\rangle$ and $\langle d\xi_\mu, d\xi^\mu\rangle$ are scalars. If we interpret the sum $\sum_n (dC_n)^2$ as a diagonal matrix, with a matrix element for every n, we arrive at the measure

$$D[\xi_\mu] = \left(\prod_s \frac{dC_s^\eta}{\sqrt{\pi}}\right) \left({\prod_p}' \sqrt{\frac{\lambda_p}{\pi}} dC_p^\chi\right), \tag{47}$$

where we have endowed with a weight factor $1/\sqrt{\pi}$ such that

$$\int D[\xi_\mu] \exp\left(- \langle \xi_\mu, \xi^\mu \rangle\right) = 1. \tag{48}$$

Note that Volkov and Wipf split the λ_p eigenvalues into a product over the smallest positive eigenvalues and the remaining eigenvalues through the Lichnerowicz-Obata theorem, which does not hold for the Schwarzschild-de Sitter constrained instanton. The Lichnerowicz-Obata theorem states that, for a metric tensor in n-dimensions satisfying $R_{\mu\nu} \geq \kappa g_{\mu\nu}$, with $\kappa > 0$ a positive constant, the first positive eigenvalue of $-\Box u = \lambda u$ is bounded as $\lambda_1 \geq \frac{n}{n-1}\kappa$. For our constrained instanton we have that κ is not a constant, so the theorem does not apply.

Our interpretation of the determinant $\mathrm{Det}\left[\langle dh_{\mu\nu}, dh^{\mu\nu}\rangle\right]$ produces an ill-defined measure for $D[h_{\mu\nu}]$, as it would include a measure that goes as $\left(dC_p^{\chi} dC_p^{\eta}\right)^{3/2}$ due to off-diagonal terms. Therefore, we use a trick that is employed to find the functional measure in string theory [2, p. 559-561]: we define the measure implicitly by interpreting the inner product $\langle \cdot, \cdot \rangle$ as a metric[1] and demanding

$$\int D[h_{\mu\nu}] \exp\left(-\frac{1}{2}\langle h_{\mu\nu}, h^{\mu\nu}\rangle\right) = 1. \tag{49}$$

From here, we arrive at the measure

$$D[h_{\mu\nu}] = \left(\prod_k \frac{dC_k^{\phi}}{\sqrt{2\pi}}\right)\left(\prod_s{}' \sqrt{\frac{\sigma_s}{\pi}} dC_s^{\eta}\right)\left(\frac{1}{2}\frac{dC_0^h}{\sqrt{2\pi}}\right)\left(\prod_n \frac{1}{2}\frac{dC_n^h}{\sqrt{2\pi}}\right)\left(\prod_p{}' \frac{dC_p^{\chi}}{2\pi}\sqrt{\tilde{\lambda}_p^{\Lambda} - 32\varrho_p}\right), \tag{50}$$

where $n > 0$.

[1]Note that in equation (14.54a) on page 561, our inner product agrees in four dimensions with the metric for $u = 1$ up to a factor of 2.

3 Gauge issues

Now all elements are in place, we should be able to calculate the modes and thus Z_{SdS}. However, this section shall sketch that we run into a problem of gauge dependency: unless we perform the calculation in a very particular fashion, the final answer will depend on the gauge parameter γ. Recall that we had defined the 1-loop contribution to Z_{SdS} as

$$F_{SdS} = \int D\left[h_{\mu\nu}\right] \mathcal{D}_{FP} \exp\left(-\delta^2 I_{gf}\right), \tag{51}$$

for which we have obtained the decomposed action $\delta^2 I_{gf}$. The issue arises from the h modes. We shall suppress all other modes besides these particular modes. We have

$$F_{SdS} \propto \int d\left(\delta\lambda\right) \prod_m \frac{1}{2}\frac{dC_m^h}{\sqrt{2\pi}} \exp\left(\sum_n -\frac{1}{16\gamma}\tilde{\lambda}_n^\gamma \left(dC_n^h\right)^2 + i\delta\lambda\,\kappa_n dC_n^h\right), \tag{52}$$

where we have already rotated the $\tilde{\lambda}_n^\gamma \left(dC_n^h\right)^2$ to a minus sign, like Volkov and Wipf. This gives

$$F_{SdS} \propto \int d\left(\delta\lambda\right) \prod_m \sqrt{\frac{2\gamma}{\tilde{\lambda}_m^\gamma}} \exp\left(-4\gamma\left(\delta\lambda\right)^2 \sum_n \frac{\kappa_n^2}{\tilde{\lambda}_n^\gamma}\right) \tag{53}$$

$$= \prod_m \sqrt{\frac{2\gamma}{\tilde{\lambda}_m^\gamma}} \sqrt{\frac{\pi}{4\gamma}\sum_n \frac{\tilde{\lambda}_n^\gamma}{\kappa_n^2}}. \tag{54}$$

While the $\sqrt{\frac{2\gamma}{\tilde{\lambda}_m^\gamma}}$ term will cancel through the Faddeev-Popov determinant, the linear term from the λ perturbation has given a dependence on γ and the gauge-dependent eigenvalue $\tilde{\lambda}_n^\gamma$.

3.1 A solution

Luckily, there seems to be a solution for this problem. In the Faddeev-Popov determinant, we have the term

$$\left(\mathcal{D}_{FP}\right)^{-1} \propto \int D\left[\xi_\mu\right] \exp\left(-\frac{1}{16\gamma}\left\langle h, \Box h\right\rangle\right). \tag{55}$$

Since the integral is not taken over h and $\mathcal{D}_{FP}$ appears as one over the above expression in F_{SdS}, we can use it to cancel the term $\langle h, \Box h\rangle$ in $-\delta^2 I_{gf}$ such that we are left with

$$\delta^2 I_{gf} \propto -\frac{1}{16\gamma}\left\langle h, \tilde{\Delta}_0^\gamma h\right\rangle + \frac{1}{16\gamma}\left\langle h, \Box h\right\rangle = -\frac{1}{16}\left\langle h, \tilde{\Delta}_0 h\right\rangle. \tag{56}$$

This will result in F_{SdS} being gauge-independent. This is unlike Volkov and Wipf, who integrate out the h (or equivalently $\tilde{h}$) dependency. The Faddeev-Popov determinant being a functional of the gauge field and altering the gauge-fixed action is not uncommon: the same happens in non-Abelian gauge theories [3, p. 512-515]. The standard procedure, however, is to write the determinant as ghost fields, upon which these fields add terms to the action. In our case, it is terms from the metric perturbation decomposition adding to the action. This particular procedure therefore is not common in standard literature. Because of its unusual nature, this procedure requires careful thought. Additionally, we must be wary, because in the Faddeev-Popov determinant there are many more crossterms involving the h modes, which could add gauge-dependent terms back into the gauge-fixed action.

It must be made clear though: because there are no further h terms in the gauge-fixed action $\delta^2 I_{gf}$, and because the linear λ term adds a sum $\sum_n \tilde{\lambda}_n^\gamma$ to F_{SdS}, which cannot be cancelled by the product $\prod_n \tilde{\lambda}_n^\gamma$ of the Faddeev-Popov determinant, *the Faddeev-Popov determinant has to add terms to the gauge-fixed action to remedy this problem, unless we modify the problem.*

3.2 Other solutions

Other possible solutions exist, but, as mentioned, they require modification of the problem. They are

1. We ignore gauge-invariance of the problem and simply send $\gamma \to 1$. This approach to the problem is not entirely senseless, as the constraint functional terms actively break covariance, which could lead to a gauge dependency no matter what. The question then, however, is if we allow the final answer to be gauge dependent, what the meaning is of that answer. We would be able to add any gauge to cancel any term, seemingly making the final answer meaningless. Preferable would be to take the following approach:

2. We choose a different constraint functional, one that is covariant. The possibility exists that, since our constraint is not covariant, it results in a gauge dependency.

3. Another solution is that we decide that, since we have added a constraint to the path integral such that the conical singularities in Schwarzschild-de Sitter are held fixed,

Schwarzschild-de Sitter in this 1-loop constrained instanton problem is a solution of the vacuum EFE. This will remove close to all crossterms we encountered and could also solve the problem by leaving a small select few of crossterms that remove the gauge dependency.

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