

# Curvature based remeshing for phase field based topology optimization

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A thesis presented for the degree of  
Master of Science

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November 2019

*I, Tim Burg confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.*

# Abstract

This work deals with a novel approach to remesh triangular meshes that facilitates a surface interpolation by Radial Basis Functions (RBFs). While serving a representation of the surface for remeshing this interpolation also allows to obtain a normalvector to the surface which is used in a higher-dimensional embedding scheme to yield a curvature adapted mesh ie. smaller triangle-sizes where the curvature is larger.

A remeshing-program that incorporates the resulting algorithm was implemented and a parameter study for the interpolation conducted.

Consequently the program was used to remesh the isosurfaces obtained from several phase-field topology optimizations.

# Acknowledgements

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# Chapter 1

## Introduction & Overview

Triangular meshes are the most prominent representation of 3D surfaces in computer graphics and the go-to format for computer-aided-design. Aside from the topological aspects of the described surfaces or non-manifold errors that can cause issues with remeshing and additive manufacturing, These meshes, being piecewise linear, can only approximate curved surfaces and often such approximations may be severely ill-conditioned for computations and inefficient in their surface-description. This for example the case when the mesh is generated from scattered data. If this is the case then remeshing techniques can be used to yield a better mesh.

One such case which is the one treated here arises after the extraction of an isosurface-mesh of a scalar-valued function defined on a finite-element mesh. The isovalue-intersections may cut the Finite-element-simplices close to corners and edges resulting in very small and possibly distorted ie. non-equilateral triangles.

The Finite-element application considered in this work is a topology optimization with the phase-field method. The phase-field distinguishes material from void but has a continuous range from 0 to 1 where 0 is void and 1 is material. Usually then the isosurface to the value 0.5 is used to determine the boundary of the solid-body.

## 1.1 common approaches for remeshing

There are different approaches for remeshing a given triangular mesh that each involve some form of interpolation or approximation to find new vertices for the mesh or generate a whole new mesh altogether. Interpolation schemes do, as the name implies respect the original data(here: the mesh) in the sense that the vertices are still points on the surface.

## 1.2 Radial basis function surface interpolation

For remeshing a surface one has to have a representation that can be queried for new on-surface points. Here implicit surface descriptions, ie.  $F(x) = 0$  for an interpolating space-function  $F : D \rightarrow \mathbb{R}$  where  $D$  is the data space have proven to be practical. Radial-basis-functions interpolation falls into that category and it is, besides polynomial-interpolation, among the most widely used interpolation-scheme nowadays. Radial-basis-functions are especially suited for multivariate data (read: arbitrary dimensions) as they only depend on the space-dimensions through the associated vector norm. The interpolant is given as a simple sum over basis functions centered at the data points  $i$ :

$$F(x) = \sum_i \alpha_i \varphi(\|x - x_i\|)$$

### 1.2.1 HIGHER DIMENSIONAL EMBEDDING FOR CURVATURE ADAPTION

The RBF-interpolant can be derived analytically to obtain the gradient. This gradient, standing perpendicular on the 0-level-set describes the curvature of the mesh and used for curvature adaptive remeshing. This is accomplished with something called a higher dimensional embedding. The idea is that



# Chapter 2

## Theoretical backgrounds

### 2.1 Topology optimization via the phase-field method

#### 2.1.1 THE EQUATIONS OF LINEAR ELASTICITY

#### 2.1.2 COMPLIANCE MINIMIZATION

#### 2.1.3 PHASE-FIELD FORMULATION AND ADVANCING-FRONT ALGORITHMS

#### 2.1.4 ISOSURFACE EXTRACTION

Since the Finite-Element-Mesh is providing a 3D-tessellation of the domain, which in this case consists of tetrahedra, the generation of an isosurface is handled as in the marching-tetrahedra algorithm. Tetrahedra, as opposed to cubes, can only have 3 distinct cases of edge intersections that differ in terms of their makeup of triangular faces. No intersections, intersection at 3 edges(1 triangle) and intersection at 4 edges (2 triangles). See figure 2.1 for an illustration.

For the edge intersections, a linear interpolation of the values between two vertices is used. The intersections are then found via simple line intersections

comparable to the section of the x-axis for a line. If 3 intersections are found one triangle is generated with the vertices of the intersections and if 4 intersections are found 2 triangles are generated. Subsequently the ordering of the vertices is checked so that looking from the outside, the vertices are ordered counterclockwise in accordance with the stl-specification. For this, the function values at the tetrahedra-nodes are considered to find a point that is inside (has a value greater than 0) This is especially important since the orientation is used in the remeshing procedure and can only be correctly determined at this step.

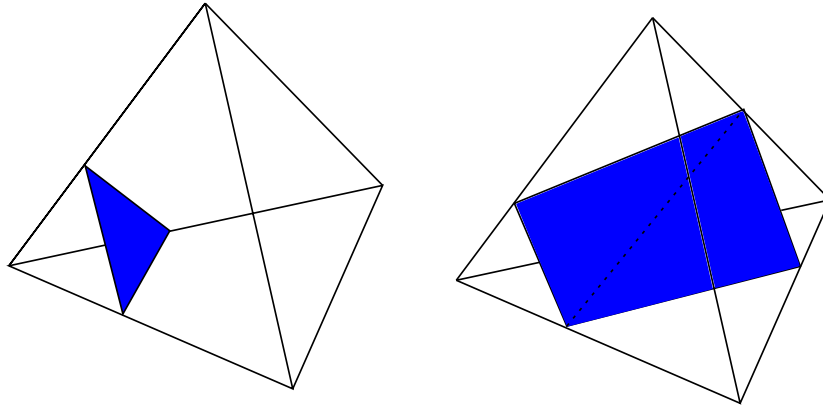


Figure 2.1: In the case of a 3D-Tetrahedra tessellation only 3 distinct cases can appear a) intersection at three edges(left) or b) intersection at 4 edges(right) or c) no intersections(not displayed)

## 2.2 Radial-basis-function theory

### 2.2.1 RBF INTERPOLATION

Interpolation can be viewed as a special kind of approximation in which, for an approximant  $S$  to some function  $F$ , it is demanded that the interpolant reproduces the original functions values at special points  $x_i$  ie.:

$$S(x_i) = F(x_i) \quad \forall i \in \Xi \quad (2.1)$$

Where  $\Xi$  is some finite (possibly scattered) dataset in  $\mathbb{R}^N$  (multivariate) ie. a set  $(\xi, f_\xi) \in \mathbb{R}^N \times \mathbb{R}$ . The functions considered here are scalar valued. A

construction of vector-valued interpolants from scalar-valued components is straightforward.

Interpolants are usually constructed from some function space which in this case is made up of Radial-basis-functions. Radial-basis-functions are special in that they allow easy interpolation of scattered multivariate data with guaranteed existence and uniqueness results. In general, different interpolants do behave differently for the space in between the data sites and are distinguished by their approximation and or convergence properties for special classes or cases of  $F$ . However, when no original function (just the values  $F(x_i)$ ) is given the accuracy of an interpolation can not generally be assessed. Because this is the case here, determining qualities for the RBF-interpolant are discussed in section ...

Radial-basis-function interpolation constructs the interpolant  $S$  as a linear combination of scaled Radial-basis-functions centered at the data sites:

$$S(x) = \sum_i \alpha_i \varphi(\|x - x_i\|)$$

The norm denotes the standard euclidian norm which is essential for the convergence results (see Buhmann n.d. p)

The Radial-basis-functions themselves are functions of the form  $\varphi : \mathbb{R} \mapsto \mathbb{R}$

By introducing the interpolation matrix  $A$  as:

$$A = \varphi(\|x_i - x_j\|)|_{i,j}$$

we can write the interpolation condition 2.1 as:

$$A\alpha = F$$

## 2.2.2 EXISTENCE AND UNIQUENESS RESULTS

The invertibility of the interpolation matrix has been investigated thoroughly in the 1970's and 1980's. Key results rely on complete monotonicity of the

Radial-Basis-function, ie. the property:

$$(-1)^l g^l(t) \geq 0 \quad \forall l \in \mathbb{N} \quad \forall t > 0$$

Then it can be shown that the interpolation matrix is always positive definite. The proof relies on the Bernstein representation theorem for monotone functions. (see Buhmann n.d., pp.11–14 for the case of multiquadratics) A weaker requirement is that only one of the derivatives must be completely monotone. This then leads to the concept of conditionally positive definiteness of a function in which polynomial is added to the interpolant.

- complete monotonicity yields a positive definite A (result due to Michelli 1986)
- weaker concept: complete monotonicity of some derivative:  
 $(-1)^k \frac{d^k}{dt^k} \varphi(\sqrt{t})$ 
  - introduces conditionally positive definite functions that use an added polynomial that vanishes on the data sites for interpolation and thus the interpolant is again unique and exists
- called “unisolvence”

### 2.2.3 COMMONLY USED RADIAL BASIS FUNCTIONS

It remains for us to recite some of the more often used RBFs and state if they are positive definite or conditionally so. During the course of writing this thesis it became clear that only local basis functions would be good candidates for a feasible surface interpolation due to the number of vertices common in triangular meshes.

Now commonly used are the Wendland functions (see Wendland 1995) which are piecewise polynomial, of minimal degree and positive definite. For the surface interpolation I use the C2 continuous function and its derivative.

Table 2.1: RBF functions with global support

function	name	definiteness
$e^{-r^2}$	gaussian	pd
$\sqrt{1+r^2}$	multiquadratics	pd
$1/\sqrt{1+r^2}$	inverse multiquadratics	pd
$r^3$	polyharmonic spline	cpd

Table 2.2: Local RBF functions introduced by Wendland (Wendland 1995)

function	name	definiteness
$(1-r)_+^4(4r+1)$	$\varphi_{3,1}(r)$	pd
$(1-r)_+^6(35r^2+18r+3)$	$\varphi_{3,2}(r)$	pd
$(1-r)_+^8(32r^3+25r^2+8r+1)$	$\varphi_{3,3}(r)$	pd

## 2.2.4 SURFACE INTERPOLATION

Surface descriptions are either explicit or implicit. Explicit usually means that the surface is the graph of a function  $F : \Omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$  which can be very complicated to construct. Implicit surfaces on the other hand are defined via a functions level set (usually the zero level) ie.  $F(x) = 0$  which might be easier to construct but is harder to visualise. Usually then for visualization either marching-cubes or raytracing methods are used.

For the surface interpolation this translates to the interpolant being zero at the datasites:  $S(x_i) = 0$ . Since the zero function would be a trivial solution to this as well off-surface constraints must be given. This is usually done with off-surface points generated from normalvectors to the surface given the value of the signed distance function ie. the value of the distance to the surface:

$$S(\mathbf{x}_i + \epsilon \mathbf{n}_i) = F(\mathbf{x}_i + \epsilon \mathbf{n}_i) = \epsilon \quad (2.2)$$

If not given, those normalvectors can be generated from a cotangent plane that is constructed via a principal component analysis of nearest neighbors. This however is a nontrivial problem. In my case the vectors could be obtained from an average of the normals of the adjacent triangles scaled with the inverse of the corresponding edgelengths. These offset-points were generated for every vertex of the original mesh and in both directions (on the inside and on the outside) such as to give the interpolant a constant slope of one around the surface. This is done to have an area of convergence for a simple gradient-descent projection algorithm.

## 2.3 Remeshing operations

Different approaches exist to remesh a surface. Most fall into one of the following categories: - triangulate a completely new mesh, usually with delauney triangulation and go from there - incremental triangulation, with new nodes inserted or removed one at a time. - local mesh modifications / pliant remeshing

Additionally most methods utilize some form of vertex-smoothing as this is an straightforward iterative procedure that improves themesh globally and is guaranteed to converge.

The approach used here falls into the latter category and uses consecutive loops of local mesh modifications of the following kinds: - Edge collapse - Edge split - Edge flip - Vertex smoothing

Which of the modification is applied depends on an edges length in comparison to a target-edge-length.

### 2.3.1 EDGE COLLAPSE

Edge collapse, as the name suggests removes an edge from the mesh thereby deleting two adjacent triangles and removing one point. Special conditions have to be checked as there are certain configurations that would result in an illegal triangulation. See figures 2.3 and 2.4 To avoid having to project a

new midpoint to the surface, the two vertices of the edge are joined at either one of them.

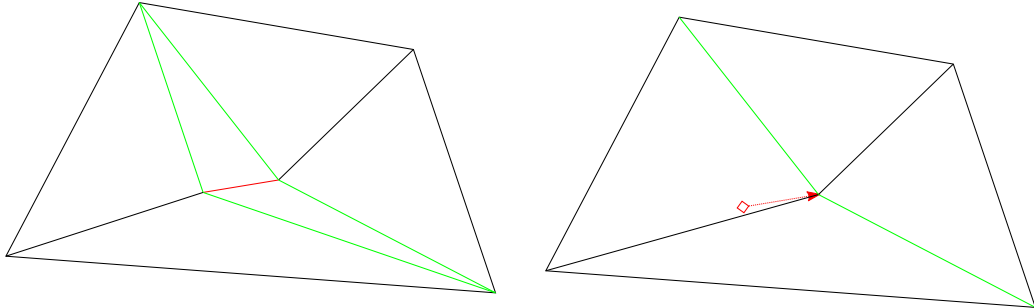


Figure 2.2: Edge collapse with the new point at one of the endpoints

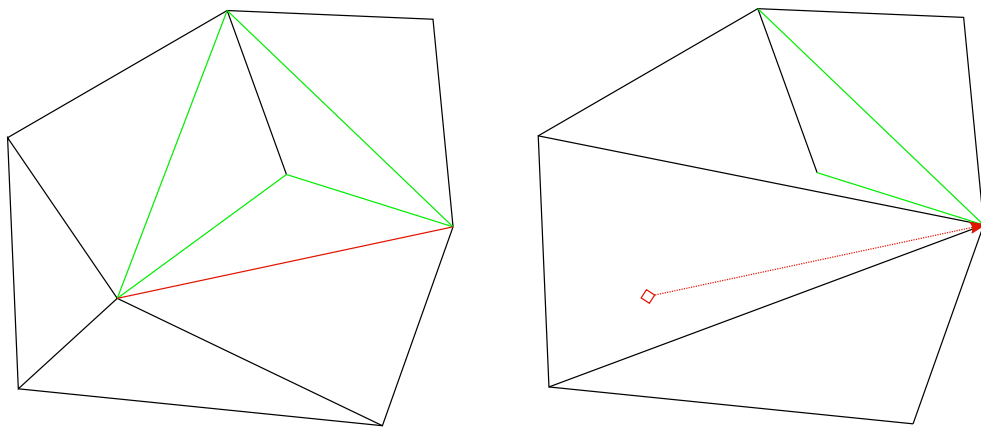


Figure 2.3: Illegal edge collapse with more than two common neighbors for the edges endpoints

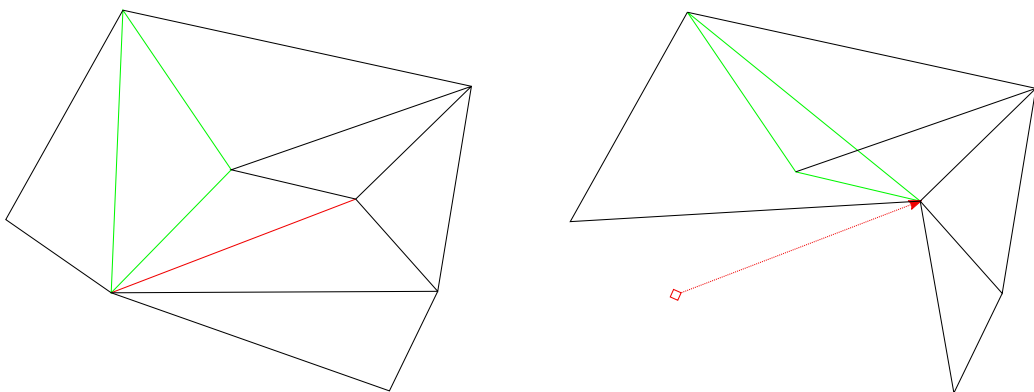


Figure 2.4: Illegal edge collapse with a triangle flip

### 2.3.2 EDGE SPLIT

The edge split is a straightforward operation as no special cases have to be taken care of. A new vertex is put at the surface projected midpoint of the existing edge and 4 new edges as well as 4 new triangles replace the split edge and its adjacent triangles.

### 2.3.3 EDGE FLIP

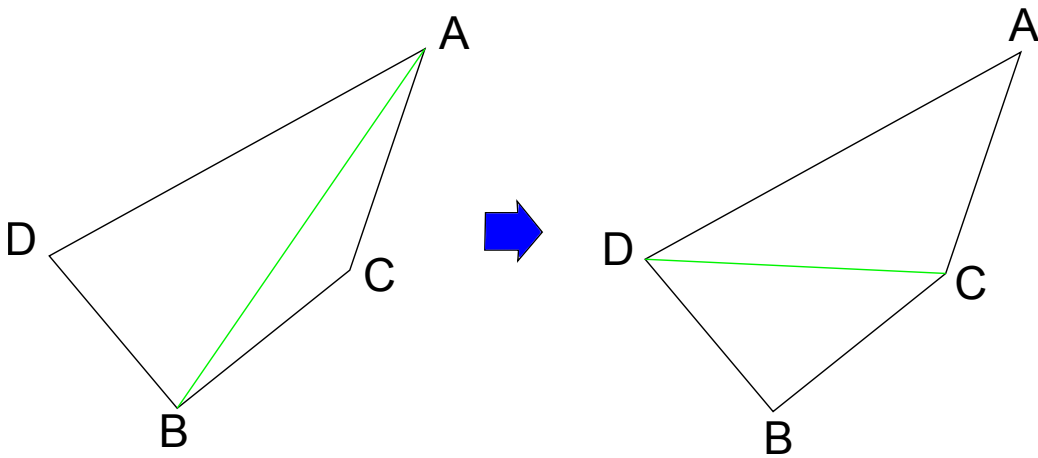


Figure 2.5: Edge flip

An edge flip can dramatically increase the aspect ratio of a triangle if the right conditions are met. Consider the edge in figure ?? Such an edge is flippable if:

- The edge does not belong to the boundary of the mesh
- The edge CD does not already belong to the mesh
- $\phi_{ABC} + \phi_{ABD} < \pi$  and  $\phi_{BAC} + \phi_{BAD} < \pi$
- The angle between the normals of the triangles is not too big to not cast “ridges”

I do a flip based on the following criteria:



### 2.3.4 VERTEX SMOOTHING

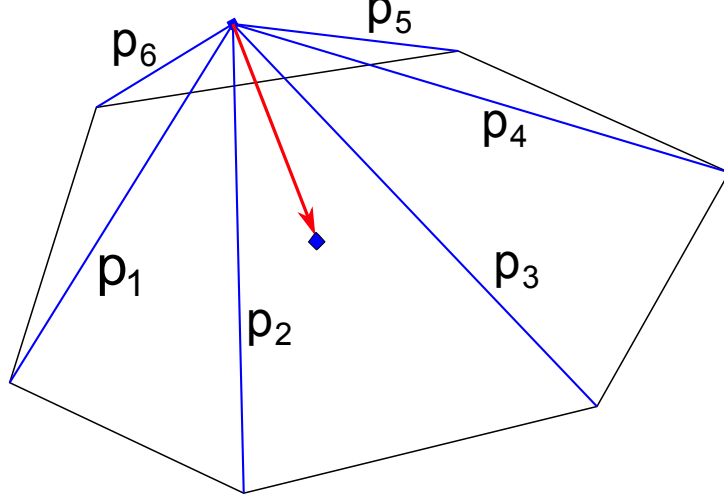


Figure 2.6: Vertex smoothing

Vertex smoothing finds a new position for a given vertex based on the distance to its neighbors according to the following formula:

$$\mathbf{p}' = \mathbf{p} + \alpha \sum_{j \in \mathcal{N}} f(\|\mathbf{p} - \mathbf{p}_j\|)(\mathbf{p} - \mathbf{p}_j)$$

Wherein  $\mathcal{N}$  stands for the neighbors,  $\alpha$  is a normalization constant and  $f$  is a weight function. Different weights have been investigated in (Bossen & Heckbert n.d.) where they constructed a well performing weight function. Given a target edge length  $t$  and an actual edge length  $l$  a normalized edge length is defined as  $d = l/t$  and the weight function reads:

$$f(d) = (1 - d^4) \cdot e^{-d^4}$$

This function pushes if  $l < t$  and slightly pulls if  $t > l$ . The function is plotted in figure 2.7 versus the frequently used laplace weights. Additionally, I clipped the movedistance to 80% of the minimum of the adjacent triangles heights. This is done because moves that exceed this distance are likely to cause unacceptable triangles. What unacceptable means is defined in the algorithm section but is basically implemented as triangles with excess tilt

versus the surface normal.

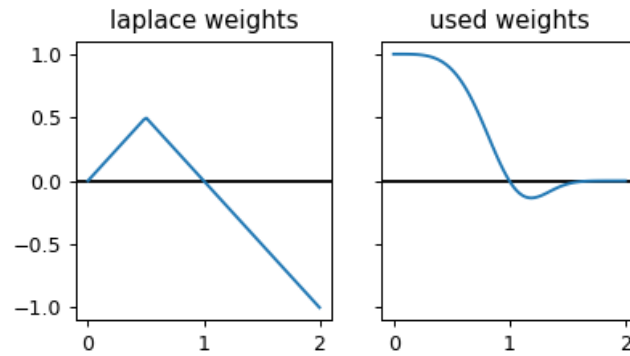


Figure 2.7: The weight function used compared to the laplace weights

### 2.3.5 PROJECTION OF VERTICES ONTO THE SURFACE

Both in an edge split as well as in vertex smoothing a constructed new vertex must be projected onto the surface. To this end I use a simple gradient descent with a fixed steplength of one. This is a rather heuristical result that has proven much better convergence than the exact steplength. This is most probably due to the fact that around the surface the slope of the function is one by construction.

## 2.4 Higher dimensional embedding

## Chapter 3

### Algorithm description and implemenation details

# Chapter 4

## Results

As mentioned before there is a fundamental challenge in the surface interpolation with radial basis functions that is due to the fact that the surface is given implicitly by the zero-level of the 3-dimensional interpolant. Since the interpolant is only guaranteed to have a zero crossing at the interpolation points and not in between them, the surface can be non-contiguous. More precisely there is no topological guarantee for a manifold surface over a cluster of islands.

Several parameters influence that situation. Those being:

- the spacing and values of the offset points and if they are uniform or not
- the scale-factor(s) of the radial-basis-functions and if they are uniform or not

To assess the acceptance of different parameter-combinations in that regard I conducted a parameter-survey. The aim was to have a general heuristic for an always working or at least ‘as good as it gets’ parameter set for the following remeshings.

As an essential feature the values of the RBF-interpolant along an outward line through the triangle-centroids were probed for: a) the existence of a zero-crossing (mandatory) b) the width between minima and maxima adjacent

to the zero crossing (convergence area of the projection)

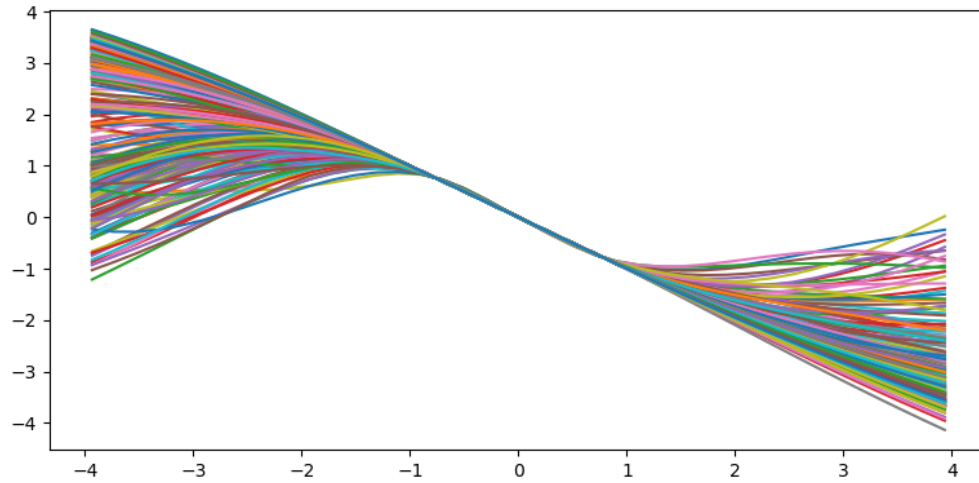


Figure 4.1: At the vertices of the mesh in between the offset interpolation points the RBF-interpolant is well behaved. The offset interpolation points are located at  $\pm 0.78$  in units of the actual mesh

#### 4.1 Parameter survey

#### 4.2 convergence behaviour

## Chapter 5

# Problems, outlook and future work

Combine interpolation and smoothing into one step with a lower order RBF function that is fitted rather than interpolates. See Buhmann Chapter 8 or >Reconstruction and Representation of 3D Objects with Radial Basis Functions paper<

# Chapter 6

## References

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