

1. beta-binomial conjugation

prior: $P(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$

likelihood: $P(k, n | \theta) = \binom{n}{k} \theta^k (1-\theta)^{n-k}$

$$\begin{aligned} \Rightarrow \text{posterior: } P(\theta | k, n) &= \frac{P(k, n | \theta) P(\theta)}{P(k, n)} \\ &= \frac{\binom{n}{k} \frac{1}{B(\alpha, \beta)} \theta^{\alpha+k-1} (1-\theta)^{\beta+n-k-1}}{\binom{n}{k} \frac{1}{B(\alpha, \beta)} \int_0^1 \theta^{\alpha+k-1} (1-\theta)^{\beta+n-k-1} d\theta} \\ &= \frac{\theta^{\alpha+k-1} (1-\theta)^{\beta+n-k-1}}{B(\alpha+k-1, \beta+n-k-1)} \\ &= \beta(\alpha+k-1, \beta+n-k-1) \end{aligned}$$

note: $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

2. Gamma - Poisson conjugation

-prior: $p(\lambda | \alpha, \beta) = \text{Gamma}(\lambda; \alpha, \beta)$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

likelihood: $P(X=x_i | \lambda) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$

$$\Rightarrow P(x | \lambda) = \prod_{i=1}^n P(x_i | \lambda) = \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$
$$= \frac{\lambda^{\sum x_i} e^{-n\lambda}}{\prod x_i!}$$

posterior:

$$P(\lambda | x) \propto P(x | \lambda) P(\lambda)$$

$$\propto (\lambda^{\sum x_i} e^{-n\lambda}) (\lambda^{\alpha-1} e^{-\beta\lambda})$$

$$\propto \lambda^{(\sum x_i + \alpha) - 1} e^{-(\beta + n)\lambda}$$

$$\Rightarrow P(\lambda | x) = \text{Gamma}(\lambda; \alpha' = \alpha + \sum x_i, \beta' = \beta + n)$$