2020 North American Qualifier Solution Outlines

The Judges

Feb 11, 2021

Note

These notes discuss only the original problems developed for this contest. 4 of the problems originally appeared in earlier NAQ contests.

Using Digits – First solved at 0:06

Problem

- Find a least-cost path from $(1, 1)$ to (X, Y) on a square grid, moving only right and up, usually one square at a time.
- Each square has a cost.
- A sequence of longer hops, specified by a string, is allowed, which may be used or ignored.

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Solution

- Use Dynamic Programming. (Un-memoized recursion times out.)
- Let $f(a, b, s)$ denote the least cost to hop from (a, b) to (X, Y) starting with string s on square (a, b).

Using Digits – First solved at 0:06

Solution

Let $f(a, b, s)$ denote the least cost to hop from (a, b) to (X, Y) starting with string s on square (a, b).
$(1, 1)$ is the lower-left corner, and (X, Y) is the upper-right.

The Recursion

$f(a, b, s)$ is the minimum of up to four possible values:

- $c(a, b) + f(a + 1, b, s)$ if it is possible to step right
- $c(a, b) + f(a, b + 1, s)$ if it is possible to step up
- $c(a, b) + f(a + h, b, s')$ if it is possible to hop right
- $c(a, b) + f(a, b + h, s')$ if it is possible to hop up

Here h is the first character/digit of s,
s' is s with h removed,
$c(a, b) =$ the cost on square (a, b).

Perfect Path Patrol – First solved at 0:08

Problem

- Given a tree with a requirement $p_e \geq 0$ for each edge e, compute the fewest paths possible such that each edge e lies on *exactly* p_e paths.
- View each edge e as a bundle of p_e parallel edges.
- Key observation: for each vertex v we should pair the maximum number of copies of edges incident to v such that no edge is paired with a parallel copy of itself.
- Any unpaired edge incident to v will be the endpoint of a path. So if $b(v)$ denotes the minimum number of edges we can leave unpaired then the final answer is $\frac{1}{2}\sum_{v \in V} b(v)$.
- The fact the neighbourhood is a tree ensures such a pairing will not create cycles.

Perfect Path Patrol – First solved at 0:08

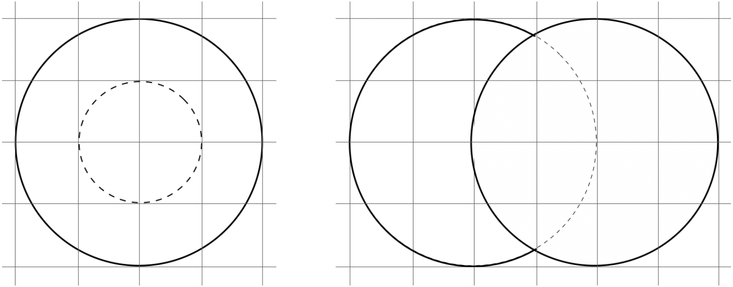
Problem continued on 2nd slide

- How to compute $b(v)$? Let D_v be all (original) edges in the tree having v as an endpoint and let $P_v := \sum_{e \in D_v} p_e$.
- If $2 \cdot p_e > P_v$ for some $e \in D_v$, the best we can do is pair all copies of each $e' \in D_v - \{e\}$ with copies of p_e so $b(v) = 2 \cdot p_e - P_v$.
- Otherwise, we can leave at most 1 unpaired edge (depending on the parity of P_v). To prove this, pair a single copy of the two largest bundles. Show that $2 \cdot p_e \leq P_v$ continues to hold for all $e \in D_v$ if we reduce the two bundles by 1. As long $P_v \geq 2$, this invariant ensures there are two nonempty bundles to choose from.
- Can compute $b(v)$ in $O(|D_v|)$ time which leads to a total running time of $O(n)$.

Drawing Circles – First solved at 0:41

Problem

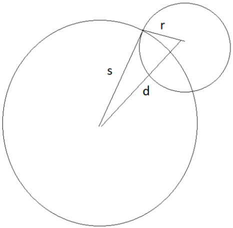
- Given a ordered sequence of closed disks in the plane, find the total length of all boundary circle segments that are not occluded by a disk drawn later.

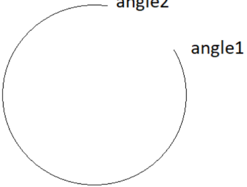


Drawing Circles – First solved at 0:41

Preliminaries

- Must compute circle-circle intersection to find the segment erased by a later-drawn circle.
- Can use law of cosines to find angle $\theta = \measuredangle(s, d) = \arccos(\frac{r^2 - s^2 - d^2}{-2sd})$.
- Represent erasure as a range $\in [-\pi, \pi)$ based on atan2 angle from center.





Drawing Circles – First solved at 0:41

Finding the amount of a circle that is visible

- For each circle $j = i + 1, \ldots, n$ drawn after circle i, calculate the angle range of circle i that is erased (if any).
- Sort ranges and run a sweep through the whole range, i.e., increment/decrement a counter to find overlapping ranges. The areas where the counter is 0 are visible.
- Take care to handle the endpoints.

Drawing Circles – First solved at 0:41

Complexity Analysis

- For each circle, calculate the ranges of the erasures caused by circles drawn after it: $O(n)$.
- Sort and sweep at most $2n$ values, take $O(n \log n)$ time.
- Repeat for all n circles for a total complexity of $O(n^2 \log n)$.
- For $n \leq 2000$ fits into time budget.

Sky's The Limit – First solved at 1:05

Problem

- Given: line of houses with initial height h_i
- (Left of leftmost house and right of rightmost house have height zero)
- Repeatedly compute:

$$h_i \leftarrow \max\left(h_i, \frac{h_{i-1} + h_{i+1}}{2} + k\right).$$

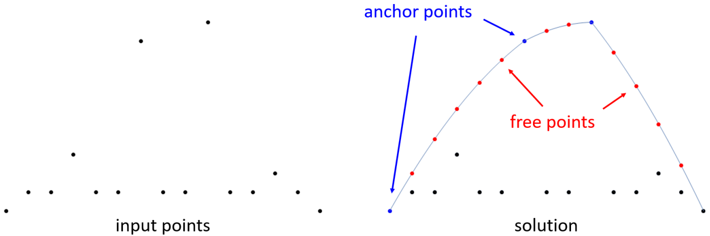
Naive solution

- Many teams tried direct simulation of the above equation
- Doomed approach: takes $O(n^2)$ time for information to propagate from one end of the line of houses to the other end and back
- Problem bounds restrict solutions to $O(n^{1+\epsilon})$

Sky's The Limit – First solved at 1:05

Some observations

- House heights go up, never down
- So at steady state, every house belongs to one of two groups:
  1. Those whose height has *never* changed (*anchor points*)
  2. Those with exactly
  $$h_i = \frac{h_{i-1} + h_{i+1}}{2} + k$$
  (*free points*).



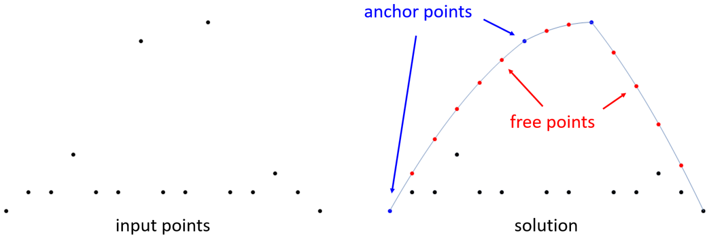
Sky's The Limit – First solved at 1:05

Some observations

- Free points :

$$h_i = \frac{h_{i-1} + h_{i+1}}{2} + k$$

- Constant second differences: heights h_i of consecutive free points lie on parabola determined by anchor point positions



Sky's The Limit – First solved at 1:05

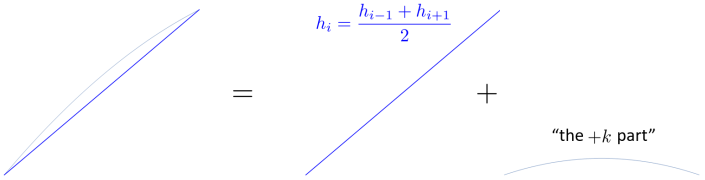
Some observations

- Free points between (x_1, y_1) and (x_2, y_2):

$$h_{x_1,y_1,x_2,y_2}(x) = -kx^2 + \left[\frac{y_2 - y_1}{x_2 - x_1} + k(x_1 + x_2)\right] x + \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} - kx_1 x_2$$

- Key fact: parabola decomposes into line plus homogeneous solution

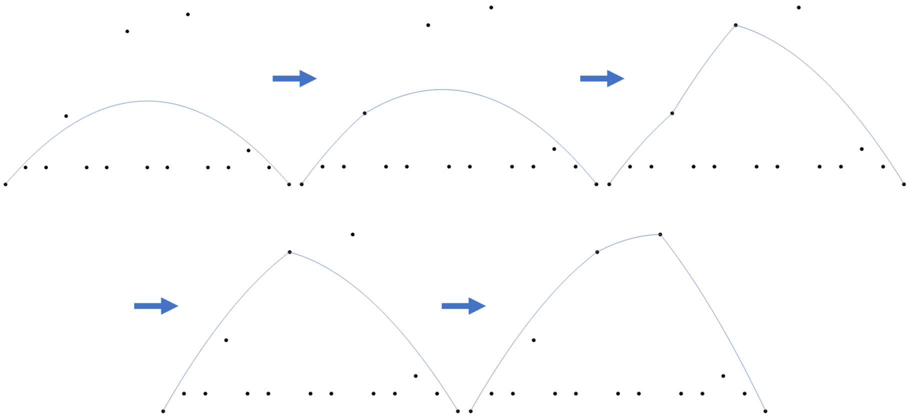
$$h_{x_1,y_1,x_2,y_2}(x) = h_{x_1,0,x_2,0}(x) + \frac{x_2 - x}{x_2 - x_1} y_1 + \frac{x - x_1}{x_2 - x_1} y_2$$



Sky's The Limit – First solved at 1:05

Solution 1

Perform Graham scan with piecewise parabolas



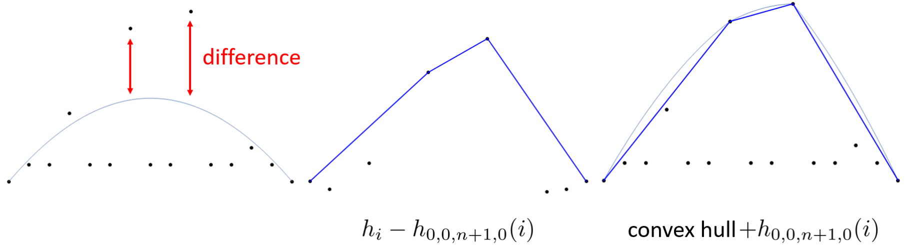
Sky's The Limit – First solved at 1:05

Solution 1

Perform Graham scan with piecewise parabolas

Solution 2

- Subtract $h_{0,0,n+1,0}(i)$ from the h_i
- Take the (ordinary, linear) convex hull
- Add back in $h_{0,0,n+1,0}(i)$



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Other Note

- Recursive divide-and-conquer approaches can also pass. In theory worst-case $O(n^2)$, but only with infinite precision
- Sequences of free points satisfy the well-known *1D Poisson equation*:

$$\Delta h = -\frac{k}{2}.$$