CG

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Closest Points

```
struct pt {
  int x, y, id;
};

struct cmp_x {
  bool operator()(const pt& a, const pt& b) const {
    return a.x < b.x || (a.x == b.x && a.y < b.y);
  }
};

struct cmp_y {
  bool operator()(const pt& a, const pt& b) const { return a.y < b.y; }
};

int n;
vector<pt> a;

double mindist;
int ansa, ansb;

// compute distance between points & update answer
inline void upd_ans(const pt& a, const pt& b) {
  double dist =
      sqrt((a.x - b.x) * (a.x - b.x) + (a.y - b.y) * (a.y - b.y) + .0);
  if (dist < mindist) mindist = dist, ansa = a.id, ansb = b.id;
}

void rec(int l, int r) {
  if (r - l <= 3) {
    for (int i = l; i <= r; ++i)
      for (int j = i + 1; j <= r; ++j) upd_ans(a[i], a[j]);
    sort(a + l, a + r + 1, &cmp_y);
    return;
  }

  int m = (l + r) >> 1;
  int midx = a[m].x;
  rec(l, m), rec(m + 1, r);
  inplace_merge(a + l, a + m + 1, a + r + 1, &cmp_y);

  static pt t[MAXN];
  int tsz = 0;
  for (int i = l; i <= r; ++i)
    if (abs(a[i].x - midx) < mindist) {
      for (int j = tsz - 1; j >= 0 && a[i].y - t[j].y < mindist; --j)
        upd_ans(a[i], t[j]);
      t[tsz++] = a[i];
    }
}

int main() {
    // a: list of points

    sort(a, a + n, &cmp_x);
    mindist = 1E20;
    rec(0, n - 1);
}
```

Convex Hull (Graham Scan)

C++:

```
struct pt {
    double x, y;
};

int orientation(pt a, pt b, pt c) {
    double v = a.x*(b.y-c.y)+b.x*(c.y-a.y)+c.x*(a.y-b.y);
    if (v < 0) return -1; // clockwise
    if (v > 0) return +1; // counter-clockwise
    return 0;
}

bool cw(pt a, pt b, pt c, bool include_collinear) {
    int o = orientation(a, b, c);
    return o < 0 || (include_collinear && o == 0);
}
bool collinear(pt a, pt b, pt c) { return orientation(a, b, c) == 0; }

// use the new values in vector a as the final result
void convex_hull(vector<pt>& a, bool include_collinear = false) {
    pt p0 = *min_element(a.begin(), a.end(), [](pt a, pt b) {
        return make_pair(a.y, a.x) < make_pair(b.y, b.x);
    });
    sort(a.begin(), a.end(), [&p0](const pt& a, const pt& b) {
        int o = orientation(p0, a, b);
        if (o == 0)
            return (p0.x-a.x)*(p0.x-a.x) + (p0.y-a.y)*(p0.y-a.y)
                < (p0.x-b.x)*(p0.x-b.x) + (p0.y-b.y)*(p0.y-b.y);
        return o < 0;
    });
    if (include_collinear) {
        int i = (int)a.size()-1;
        while (i >= 0 && collinear(p0, a[i], a.back())) i--;
        reverse(a.begin()+i+1, a.end());
    }

    vector<pt> st;
    for (int i = 0; i < (int)a.size(); i++) {
        while (st.size() > 1 && !cw(st[st.size()-2], st.back(), a[i], include_collinear))
            st.pop_back();
        st.push_back(a[i]);
    }

    a = st;
}
```

Python:

```
p = [] # points (input)
stk = [] # stack of indices
tp = 0 # initialize stack
p.sort()
# adding 1st point without updating array used
stk[tp] = 1
tp = tp + 1
for i in range(2, n + 1):
    # "*" operation is cross product
    while tp >= 2 and (p[stk[tp]] - p[stk[tp - 1]]) * (p[i] - p[stk[tp]]) <= 0:
```

```
        used[stk[tp]] = 0
        tp = tp - 1
        used[i] = 1 # used = 1 => point on convex hull
        stk[tp] = i
        tp = tp + 1
tmp = tp # tmp = size of lower hull
for i in range(n - 1, 0, -1):
    if used[i] == False:
        # finding upper hull without affecting lower hull
        while tp > tmp and (p[stk[tp]] - p[stk[tp - 1]]) * (p[i] - p[stk[tp]]) <= 0:
            used[stk[tp]] = 0
            tp = tp - 1
            used[i] = 1
            stk[tp] = i
            tp = tp + 1
# array h finally has ans + 1 points (duplicated first point) in c.c.w.
for i in range(1, tp + 1):
    h[i] = p[stk[i]]
ans = tp - 1 # number of points in the convex hull
```

Finding Intersections (Sweeping Line)

```
const double EPS = 1E-9;

struct pt {
    double x, y;
};

struct seg {
    pt p, q;
    int id;

    double get_y(double x) const {
        if (abs(p.x - q.x) < EPS)
            return p.y;
        return p.y + (q.y - p.y) * (x - p.x) / (q.x - p.x);
    }
};

bool intersect1d(double l1, double r1, double l2, double r2) {
    if (l1 > r1)
        swap(l1, r1);
    if (l2 > r2)
        swap(l2, r2);
    return max(l1, l2) <= min(r1, r2) + EPS;
}

int vec(const pt& a, const pt& b, const pt& c) {
    double s = (b.x - a.x) * (c.y - a.y) - (b.y - a.y) * (c.x - a.x);
    return abs(s) < EPS ? 0 : s > 0 ? +1 : -1;
}

bool intersect(const seg& a, const seg& b)
{
    return intersect1d(a.p.x, a.q.x, b.p.x, b.q.x) &&
           intersect1d(a.p.y, a.q.y, b.p.y, b.q.y) &&
           vec(a.p, a.q, b.p) * vec(a.p, a.q, b.q) <= 0 &&
           vec(b.p, b.q, a.p) * vec(b.p, b.q, a.q) <= 0;
}

bool operator<(const seg& a, const seg& b)
{
    double x = max(min(a.p.x, a.q.x), min(b.p.x, b.q.x));
    return a.get_y(x) < b.get_y(x) - EPS;
}

struct event {
    double x;
    int tp, id;

    event() {}
    event(double x, int tp, int id) : x(x), tp(tp), id(id) {}

    bool operator<(const event& e) const {
        if (abs(x - e.x) > EPS)
            return x < e.x;
        return tp > e.tp;
    }
};
```

```
set<seg> s;
vector<set<seg>::iterator> where;

set<seg>::iterator prev(set<seg>::iterator it) {
    return it == s.begin() ? s.end() : --it;
}

set<seg>::iterator next(set<seg>::iterator it) {
    return ++it;
}

pair<int, int> solve(const vector<seg>& a) {
    int n = (int)a.size();
    vector<event> e;
    for (int i = 0; i < n; ++i) {
        e.push_back(event(min(a[i].p.x, a[i].q.x), +1, i));
        e.push_back(event(max(a[i].p.x, a[i].q.x), -1, i));
    }
    sort(e.begin(), e.end());

    s.clear();
    where.resize(a.size());
    for (size_t i = 0; i < e.size(); ++i) {
        int id = e[i].id;
        if (e[i].tp == +1) {
            set<seg>::iterator nxt = s.lower_bound(a[id]), prv = prev(nxt);
            if (nxt != s.end() && intersect(*nxt, a[id]))
                return make_pair(nxt->id, id);
            if (prv != s.end() && intersect(*prv, a[id]))
                return make_pair(prv->id, id);
            where[id] = s.insert(nxt, a[id]);
        } else {
            set<seg>::iterator nxt = next(where[id]), prv = prev(where[id]);
            if (nxt != s.end() && prv != s.end() && intersect(*nxt, *prv))
                return make_pair(prv->id, nxt->id);
            s.erase(where[id]);
        }
    }

    return make_pair(-1, -1);
}
```

Half-plane Intersection

Intersection is trivially convex, adding an outer frame avoids infinite intersection shape

```
// Redefine epsilon and infinity as necessary. Be mindful of precision errors.
const long double eps = 1e-9, inf = 1e9;

// Basic point/vector struct.
struct Point {
    long double x, y;
    explicit Point(long double x = 0, long double y = 0) : x(x), y(y) {}

    friend Point operator + (const Point& p, const Point& q) {
        return Point(p.x + q.x, p.y + q.y);
    }
    friend Point operator - (const Point& p, const Point& q) {
        return Point(p.x - q.x, p.y - q.y);
    }
    friend Point operator * (const Point& p, const long double& k) {
        return Point(p.x * k, p.y * k);
    }
    friend long double dot(const Point& p, const Point& q) {
        return p.x * q.x + p.y * q.y;
    }
    friend long double cross(const Point& p, const Point& q) {
        return p.x * q.y - p.y * q.x;
    }
};

// Basic half-plane struct.
struct Halfplane {

    // 'p' is a passing point of the line and 'pq' is the direction vector of the line.
    Point p, pq;
    long double angle;

    Halfplane() {}
    Halfplane(const Point& a, const Point& b) : p(a), pq(b - a) {
        angle = atan2l(pq.y, pq.x);
    }

    // Check if point 'r' is outside this half-plane.
    // Every half-plane allows the region to the LEFT of its line.
    bool out(const Point& r) {
        return cross(pq, r - p) < -eps;
    }

    // Comparator for sorting.
    bool operator < (const Halfplane& e) const {
        return angle < e.angle;
    }

    // Intersection point of the lines of two half-planes. It is assumed they're never parallel.
    friend Point inter(const Halfplane& s, const Halfplane& t) {
        long double alpha = cross((t.p - s.p), t.pq) / cross(s.pq, t.pq);
        return s.p + (s.pq * alpha);
    }
};

// Actual algorithm
vector<Point> hp_intersect(vector<Halfplane>& H) {
```

```
Point box[4] = {  // Bounding box in CCW order
    Point(inf, inf),
    Point(-inf, inf),
    Point(-inf, -inf),
    Point(inf, -inf)
};

for(int i = 0; i<4; i++) { // Add bounding box half-planes.
    Halfplane aux(box[i], box[(i+1) % 4]);
    H.push_back(aux);
}

// Sort by angle and start algorithm
sort(H.begin(), H.end());
deque<Halfplane> dq;
int len = 0;
for(int i = 0; i < int(H.size()); i++) {

    // Remove from the back of the deque while last half-plane is redundant
    while (len > 1 && H[i].out(inter(dq[len-1], dq[len-2]))) {
        dq.pop_back();
        --len;
    }

    // Remove from the front of the deque while first half-plane is redundant
    while (len > 1 && H[i].out(inter(dq[0], dq[1]))) {
        dq.pop_front();
        --len;
    }

    // Special case check: Parallel half-planes
    if (len > 0 && fabsl(cross(H[i].pq, dq[len-1].pq)) < eps) {
        // Opposite parallel half-planes that ended up checked against each other.
        if (dot(H[i].pq, dq[len-1].pq) < 0.0)
            return vector<Point>();

        // Same direction half-plane: keep only the leftmost half-plane.
        if (H[i].out(dq[len-1].p)) {
            dq.pop_back();
            --len;
        }
        else continue;
    }

    // Add new half-plane
    dq.push_back(H[i]);
    ++len;
}

// Final cleanup: Check half-planes at the front against the back and vice-versa
while (len > 2 && dq[0].out(inter(dq[len-1], dq[len-2]))) {
    dq.pop_back();
    --len;
}

while (len > 2 && dq[len-1].out(inter(dq[0], dq[1]))) {
    dq.pop_front();
    --len;
}
```

```
    // Report empty intersection if necessary
    if (len < 3) return vector<Point>();

    // Reconstruct the convex polygon from the remaining half-planes.
    vector<Point> ret(len);
    for(int i = 0; i+1 < len; i++) {
        ret[i] = inter(dq[i], dq[i+1]);
    }
    ret.back() = inter(dq[len-1], dq[0]);
    return ret;
}
```

Smallest Covering Circle

Smallest circles (inclusive boundary) covering all points.

```
#include <cmath>
#include <cstdio>
#include <cstdlib>
#include <cstring>
#include <iostream>

using namespace std;

int n;
double r;

struct point {
  double x, y;
} p[100005], o;

inline double sqr(double x) { return x * x; }

inline double dis(point a, point b) {
  return sqrt(sqr(a.x - b.x) + sqr(a.y - b.y));
}

inline bool cmp(double a, double b) { return fabs(a - b) < 1e-8; }

point geto(point a, point b, point c) {
  double a1, a2, b1, b2, c1, c2;
  point ans;
  a1 = 2 * (b.x - a.x), b1 = 2 * (b.y - a.y),
  c1 = sqr(b.x) - sqr(a.x) + sqr(b.y) - sqr(a.y);
  a2 = 2 * (c.x - a.x), b2 = 2 * (c.y - a.y),
  c2 = sqr(c.x) - sqr(a.x) + sqr(c.y) - sqr(a.y);
  if (cmp(a1, 0)) {
    ans.y = c1 / b1;
    ans.x = (c2 - ans.y * b2) / a2;
  } else if (cmp(b1, 0)) {
    ans.x = c1 / a1;
    ans.y = (c2 - ans.x * a2) / b2;
  } else {
    ans.x = (c2 * b1 - c1 * b2) / (a2 * b1 - a1 * b2);
    ans.y = (c2 * a1 - c1 * a2) / (b2 * a1 - b1 * a2);
  }
  return ans;
}

int main() {
  scanf("%d", &n);
  for (int i = 1; i <= n; i++) scanf("%lf%lf", &p[i].x, &p[i].y);
  // swap randomly
  for (int i = 1; i <= n; i++) swap(p[rand() % n + 1], p[rand() % n + 1]);

  o = p[1];
  for (int i = 1; i <= n; i++) {
    if (dis(o, p[i]) < r || cmp(dis(o, p[i]), r)) continue;
    o.x = (p[i].x + p[1].x) / 2;
    o.y = (p[i].y + p[1].y) / 2;
    r = dis(p[i], p[1]) / 2;
    for (int j = 2; j < i; j++) {
      if (dis(o, p[j]) < r || cmp(dis(o, p[j]), r)) continue;
```

```
      o.x = (p[i].x + p[j].x) / 2;
      o.y = (p[i].y + p[j].y) / 2;
      r = dis(p[i], p[j]) / 2;
      for (int k = 1; k < j; k++) {
        if (dis(o, p[k]) < r || cmp(dis(o, p[k]), r)) continue;
        o = geto(p[i], p[j], p[k]);
        r = dis(o, p[i]);
      }
    }
  }
  printf("%.10lf\n%.10lf %.10lf", r, o.x, o.y);
  return 0;
}
```

Triangulation

```
typedef long long ll;

bool ge(const ll& a, const ll& b) { return a >= b; }
bool le(const ll& a, const ll& b) { return a <= b; }
bool eq(const ll& a, const ll& b) { return a == b; }
bool gt(const ll& a, const ll& b) { return a > b; }
bool lt(const ll& a, const ll& b) { return a < b; }
int sgn(const ll& a) { return a >= 0 ? a ? 1 : 0 : -1; }

struct pt {
    ll x, y;
    pt() { }
    pt(ll _x, ll _y) : x(_x), y(_y) { }
    pt operator-(const pt& p) const {
        return pt(x - p.x, y - p.y);
    }
    ll cross(const pt& p) const {
        return x * p.y - y * p.x;
    }
    ll cross(const pt& a, const pt& b) const {
        return (a - *this).cross(b - *this);
    }
    ll dot(const pt& p) const {
        return x * p.x + y * p.y;
    }
    ll dot(const pt& a, const pt& b) const {
        return (a - *this).dot(b - *this);
    }
    ll sqrLength() const {
        return this->dot(*this);
    }
    bool operator==(const pt& p) const {
        return eq(x, p.x) && eq(y, p.y);
    }
};

const pt inf_pt = pt(1e18, 1e18);

struct QuadEdge {
    pt origin;
    QuadEdge* rot = nullptr;
    QuadEdge* onext = nullptr;
    bool used = false;
    QuadEdge* rev() const {
        return rot->rot;
    }
    QuadEdge* lnext() const {
        return rot->rev()->onext->rot;
    }
    QuadEdge* oprev() const {
        return rot->onext->rot;
    }
    pt dest() const {
        return rev()->origin;
    }
};

QuadEdge* make_edge(pt from, pt to) {
```

```
    QuadEdge* e1 = new QuadEdge;
    QuadEdge* e2 = new QuadEdge;
    QuadEdge* e3 = new QuadEdge;
    QuadEdge* e4 = new QuadEdge;
    e1->origin = from;
    e2->origin = to;
    e3->origin = e4->origin = inf_pt;
    e1->rot = e3;
    e2->rot = e4;
    e3->rot = e2;
    e4->rot = e1;
    e1->onext = e1;
    e2->onext = e2;
    e3->onext = e4;
    e4->onext = e3;
    return e1;
}

void splice(QuadEdge* a, QuadEdge* b) {
    swap(a->onext->rot->onext, b->onext->rot->onext);
    swap(a->onext, b->onext);
}

void delete_edge(QuadEdge* e) {
    splice(e, e->oprev());
    splice(e->rev(), e->rev()->oprev());
    delete e->rev()->rot;
    delete e->rev();
    delete e->rot;
    delete e;
}

QuadEdge* connect(QuadEdge* a, QuadEdge* b) {
    QuadEdge* e = make_edge(a->dest(), b->origin);
    splice(e, a->lnext());
    splice(e->rev(), b);
    return e;
}

bool left_of(pt p, QuadEdge* e) {
    return gt(p.cross(e->origin, e->dest()), 0);
}

bool right_of(pt p, QuadEdge* e) {
    return lt(p.cross(e->origin, e->dest()), 0);
}

template <class T>
T det3(T a1, T a2, T a3, T b1, T b2, T b3, T c1, T c2, T c3) {
    return a1 * (b2 * c3 - c2 * b3) - a2 * (b1 * c3 - c1 * b3) +
           a3 * (b1 * c2 - c1 * b2);
}

bool in_circle(pt a, pt b, pt c, pt d) {
// If there is __int128, calculate directly. Otherwise, calculate angles.
#if defined(__LP64__) || defined(_WIN64)
    __int128 det = -det3<__int128>(b.x, b.y, b.sqrLength(), c.x, c.y,
                                   c.sqrLength(), d.x, d.y, d.sqrLength());
    det += det3<__int128>(a.x, a.y, a.sqrLength(), c.x, c.y, c.sqrLength(), d.x,
                          d.y, d.sqrLength());
```

```
    det -= det3<__int128>(a.x, a.y, a.sqrLength(), b.x, b.y, b.sqrLength(), d.x,
                          d.y, d.sqrLength());
    det += det3<__int128>(a.x, a.y, a.sqrLength(), b.x, b.y, b.sqrLength(), c.x,
                          c.y, c.sqrLength());
    return det > 0;
#else
    auto ang = [](pt l, pt mid, pt r) {
        ll x = mid.dot(l, r);
        ll y = mid.cross(l, r);
        long double res = atan2((long double)x, (long double)y);
        return res;
    };
    long double kek = ang(a, b, c) + ang(c, d, a) - ang(b, c, d) - ang(d, a, b);
    if (kek > 1e-8)
        return true;
    else
        return false;
#endif
}

pair<QuadEdge*, QuadEdge*> build_tr(int l, int r, vector<pt>& p) {
    if (r - l + 1 == 2) {
        QuadEdge* res = make_edge(p[l], p[r]);
        return make_pair(res, res->rev());
    }
    if (r - l + 1 == 3) {
        QuadEdge *a = make_edge(p[l], p[l + 1]), *b = make_edge(p[l + 1], p[r]);
        splice(a->rev(), b);
        int sg = sgn(p[l].cross(p[l + 1], p[r]));
        if (sg == 0)
            return make_pair(a, b->rev());
        QuadEdge* c = connect(b, a);
        if (sg == 1)
            return make_pair(a, b->rev());
        else
            return make_pair(c->rev(), c);
    }
    int mid = (l + r) / 2;
    QuadEdge *ldo, *ldi, *rdo, *rdi;
    tie(ldo, ldi) = build_tr(l, mid, p);
    tie(rdi, rdo) = build_tr(mid + 1, r, p);
    while (true) {
        if (left_of(rdi->origin, ldi)) {
            ldi = ldi->lnext();
            continue;
        }
        if (right_of(ldi->origin, rdi)) {
            rdi = rdi->rev()->onext;
            continue;
        }
        break;
    }
    QuadEdge* basel = connect(rdi->rev(), ldi);
    auto valid = [&basel](QuadEdge* e) { return right_of(e->dest(), basel); };
    if (ldi->origin == ldo->origin)
        ldo = basel->rev();
    if (rdi->origin == rdo->origin)
        rdo = basel;
    while (true) {
        QuadEdge* lcand = basel->rev()->onext;
```

```
        if (valid(lcand)) {
            while (in_circle(basel->dest(), basel->origin, lcand->dest(),
                             lcand->onext->dest())) {
                QuadEdge* t = lcand->onext;
                delete_edge(lcand);
                lcand = t;
            }
        }
        QuadEdge* rcand = basel->oprev();
        if (valid(rcand)) {
            while (in_circle(basel->dest(), basel->origin, rcand->dest(),
                             rcand->oprev()->dest())) {
                QuadEdge* t = rcand->oprev();
                delete_edge(rcand);
                rcand = t;
            }
        }
        if (!valid(lcand) && !valid(rcand))
            break;
        if (!valid(lcand) ||
            (valid(rcand) && in_circle(lcand->dest(), lcand->origin,
                                       rcand->origin, rcand->dest())))
            basel = connect(rcand, basel->rev());
        else
            basel = connect(basel->rev(), lcand->rev());
    }
    return make_pair(ldo, rdo);
}

vector<tuple<pt, pt, pt>> delaunay(vector<pt> p) {
    sort(p.begin(), p.end(), [](const pt& a, const pt& b) {
        return lt(a.x, b.x) || (eq(a.x, b.x) && lt(a.y, b.y));
    });
    auto res = build_tr(0, (int)p.size() - 1, p);
    QuadEdge* e = res.first;
    vector<QuadEdge*> edges = {e};
    while (lt(e->onext->dest().cross(e->dest(), e->origin), 0))
        e = e->onext;
    auto add = [&p, &e, &edges]() {
        QuadEdge* curr = e;
        do {
            curr->used = true;
            p.push_back(curr->origin);
            edges.push_back(curr->rev());
            curr = curr->lnext();
        } while (curr != e);
    };
    add();
    p.clear();
    int kek = 0;
    while (kek < (int)edges.size()) {
        if (!(e = edges[kek++])->used)
            add();
    }
    vector<tuple<pt, pt, pt>> ans;
    for (int i = 0; i < (int)p.size(); i += 3) {
        ans.push_back(make_tuple(p[i], p[i + 1], p[i + 2]));
    }
    return ans;
}
```

Theorems

Manhattan Dist (L_1) & Chebyshev Dist(L_∞)

$$d_{Manhattan}(x, y) = d_{Chebyshev}(x + y, x - y)$$
$$d_{Cheby}(x, y) = d_{Man}\left(\frac{x + y}{2}, \frac{x - y}{2}\right)$$

Euler Formula

$$V - E + F = 2$$

Pick Theorem

For a **simple** polygon on a (square/parallelogram) grid, its area A relates to the inner grid points i and boundary grid points b with

$$A = i + \frac{b}{2} - 1$$

For a triangular grid,

$$A = 2i + b - 2$$

For a **non-simple** polygon on a square grid, area also relates to faces, edges and vertices numbers:

$$A = i + \frac{b}{2} - \chi, \, \chi = F - E + V$$