Exam 1

Thursday, October 7, 2021 1:22 AM

1.

Show that

$$f(\mathbf{x}) = (x_2 - x_1^2)^2 + x_1^5$$

has only one stationary point which is neither a minimizer or a maximizer.

Find gradient of f(x)

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial f(x)}{\partial x_1} = 2(x_2 - x_1^2)\cdot(-2x_1) + 5x_1^4$$
$$= -4x_1(x_2 - x_1^2) + 5x_1^4$$
$$= -4x_1x_2 + 4x_1^3 + 5x_1^4$$

$$\frac{\partial f(x)}{\partial x_2} = 2(x_2 - x_1^2)$$
$$= 2x_2 - 2x_1^2$$

$$\Rightarrow \nabla f(x) = \begin{bmatrix} 5x_1^4 + 4x_1^3 - 4x_1x_2 \\ -2x_1^2 + 2x_2 \end{bmatrix}$$

Set equal to 0 & solve

① $5x_1^4 + 4x_1^3 - 4x_1x_2 = 0$

② $-2x_1^2 + 2x_2 = 0$

$2x_2 = 2x_1^2 \Rightarrow \underline{x_2 = x_1^2}$

$4x_1x_2 = 5x_1^4 + 4x_1^3$

$x_1x_2 = \frac{5}{4}x_1^4 + x_1^3 \Rightarrow \underline{x_2 = \frac{5}{4}x_1^3 + x_1^2}$

$x_1^2 = \frac{5}{4}x_1^3 + x_1^2$

$\frac{5}{4}x_1^3 = 0$

$x_1^3 = 0 \Rightarrow \underline{x_1 = 0} \Rightarrow \underline{x_2 = 0}$

$f(0,0) = (0-0^2)^2 + 0^5 = 0$

$\Rightarrow$ There is 1 stationary point $\underline{x}^*$ at $x_1 = 0$ + $x_2 = 0$

At these points $\nabla(x) = 0$.

Calculate the Hessian $H(x)$

$$H(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x^2} & \frac{\partial^2 f(x)}{\partial x \partial y} \\ \frac{\partial^2 f(x)}{\partial y \partial x} & \frac{\partial^2 f(x)}{\partial y^2} \end{bmatrix}$$

$\frac{\partial^2 f(x)}{\partial x^2} = 20x_1^3 + 12x_1^2 - 4x_2$

$\frac{\partial^2 f(x)}{\partial x \partial y} = -4x_1$

$\frac{\partial^2 f(x)}{\partial y \partial x} = -4x_1$

$$\frac{\partial^2 f(x)}{\partial y^2} = 2$$

$$\Rightarrow H(x) = \begin{bmatrix} 20x_1^3 + 12x_1^2 - 4x_2 & -4x_1 \\ -4x_1 & 0 \end{bmatrix}$$

$$H_1 = h_{11} = 20x_1^3 + 12x_1^2 - 4x_2$$
$$= 20(0)^3 + 12(0)^2 - 4(0) = 0$$
$$\Rightarrow \underline{H_1 = 0}$$

$$H_2 = \begin{vmatrix} 20x_1^3 + 12x_1^2 - 4x_2 & -4x_1 \\ -4x_1 & 0 \end{vmatrix}$$
$$= (20x_1^3 + 12x_1^2 - 4x_2)(0) - (-4x_1)(-4x_1)$$
$$= 0 - 16x_1^2$$
$$= -16x_1^2$$
$$= -16(0)^2 = 0$$
$$\Rightarrow \underline{H_2 = 0}$$

Since $H_1 = H_2 = 0$, $H(\underline{x}^*)$ where $\underline{x}^* = [0\ 0]$ is indefinite $\Rightarrow$ $\underline{x}^*$ is a saddle point + neither a min nor a max.

2. Check if the following functions are convex. Show your computations.

$$
\begin{aligned}
&(a)\ \ f(\mathbf{x}) = e^{x_1} + x_2^2 + 5 \\
&(b)\ \ f(\mathbf{x}) = 3x_1^2 - 5x_1x_2 + x_2^2 \\
&(c)\ \ f(\mathbf{x}) = \tfrac{1}{4}x_1^4 - x_1^2 + x_2^2 \\
&(d)\ \ f(\mathbf{x}) = 50 + 10x_1 + x_2 - 6x_1^2 - 3x_2^2
\end{aligned}
$$

(a) $f(x) = e^{x_1} + x_2^2 + 5$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial f(x)}{\partial x_1} = e^{x_1} \qquad \frac{\partial f(x)}{\partial x_2} = 2x_2 \quad \Rightarrow \nabla f(x) = \begin{bmatrix} e^{x_1} \\ 2x_1 \end{bmatrix}$$

$$\text{Hessian } H(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 x_2} \\ \frac{\partial^2 f(x)}{\partial x_2 x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} \end{bmatrix}$$

$$\Rightarrow H(x) = \begin{bmatrix} e^{x_1} & 0 \\ 0 & 2 \end{bmatrix}$$

$H_1 = h_{11} = e^{x_1}$ Is this > 0 ?

e^{x_1} has to b $> 0 \Rightarrow H_1 > 0$

$$H_2 = \begin{vmatrix} e^{x_1} & 0 \\ 0 & 2 \end{vmatrix} = (e^{x_1})(2) - (0)(0) = 2e^{x_1}$$

$2e^{x_1}$ has to be $> 0 \Rightarrow H_2 > 0$

$\Rightarrow$ Since $H_1 > 0$ + $H_2 > 0$,
$H(x)$ is positive definite

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$$\Rightarrow \boxed{f(x) \text{ is strictly convex}}$$

(b) $f(x) = 3x_1^2 - 5x_1x_2 + x_2^2$

$$\frac{\partial f(x)}{\partial x_1} = 6x_1 - 5x_2$$
$$\frac{\partial f(x)}{\partial x_2} = -5x_1 + 2x_2$$
$$\Rightarrow \nabla f(x) = \begin{bmatrix} 6x_1 - 5x_2 \\ -5x_1 + 2x_2 \end{bmatrix}$$

$$\frac{\partial^2 f(x)}{\partial x_1^2} = 6$$
$$\frac{\partial^2 f(x)}{\partial x_2^2} = 2 \quad \Rightarrow H(x) = \begin{bmatrix} 6 & -5 \\ -5 & 2 \end{bmatrix}$$
$$\frac{\partial^2 f(x)}{\partial x_1 x_2} = -5$$

$$H_1 = h_{11} = 6 > 0$$
$$H_2 = \begin{vmatrix} 6 & -5 \\ -5 & 2 \end{vmatrix} = (6)(2) - (-5)(-5) = 12 - 25 = -13 < 0$$

Since $H_1 > 0$ & $H_2 < 0$,
$H(x)$ is indefinite

$$\Rightarrow \boxed{f(x) \text{ is neither convex nor concave}}$$

(c) $f(x) = \frac{1}{4}x_1^4 - x_1^2 + x_2^2$

$$\frac{\partial f(x)}{\partial x_1} = x_1^3 - 2x_1 \quad \nabla f(x) = \begin{bmatrix} x_1^3 - 2x_1 \end{bmatrix}$$

$\frac{\partial f(x)}{\partial x_1} = x_1^3 - 2x_1$
$\frac{\partial f(x)}{\partial x_2} = 2x_2$ $\Rightarrow \nabla f(x) = \begin{bmatrix} x_1^3 - 2x_1 \\ 2x_2 \end{bmatrix}$

$\frac{\partial^2 f(x)}{\partial x_1^2} = 3x_1^2 - 2$
$\frac{\partial^2 f(x)}{\partial x_2^2} = 2$ $\Rightarrow H(x) = \begin{bmatrix} 3x_1^2 - 2 & 0 \\ 0 & 2 \end{bmatrix}$
$\frac{\partial^2 f(x)}{\partial x_1 x_2} = 0$

$H_1 = h_{11} = 3x_1^2 - 2 \geq 0$
$3x_1^2 \geq 2 \rightarrow x_1^2 \geq 2/3 \Rightarrow \underline{x_1 \geq \sqrt{2/3} \text{ or } x_1 \leq -\sqrt{2/3}}$
$H_2 = \begin{vmatrix} 3x_1^2 - 2 & 0 \\ 0 & 2 \end{vmatrix} = (3x_1^2 - 2)(2) - (0)(0)$
$H_2 = 6x_1^2 - 4 \geq 0 \rightarrow 6x_1^2 \geq 4$
$x_1^2 \geq 2/3 \Rightarrow \underline{x_1 \geq \sqrt{2/3} \text{ or } x_1 \leq -\sqrt{2/3}}$

Case 1: $H(x)$ is positive semi-definite

$\Rightarrow \boxed{f(x) \text{ is convex}}$

$H_1 = h_{11} = 3x_1^2 - 2 > 0$ *Strictly > 0
$3x_1^2 > 2 \rightarrow x_1^2 > 2/3 \Rightarrow \underline{x_1 > \sqrt{2/3} \text{ or } x_1 < -\sqrt{2/3}}$

Case 2: $H(x)$ is positive definite

$\Rightarrow \boxed{f(x) \text{ is strictly convex}}$

$-\sqrt{2/3} < x_1 < \sqrt{2/3}$

Case 3: $H(x)$ is indefinite

$\Rightarrow$ $\boxed{f(x) \text{ is neither convex nor concave}}$

(d) $f(x) = 50 + 10x_1 + x_2 - 6x_1^2 - 3x_2^2$

$$\frac{\partial f(x)}{\partial x_1} = 10 - 12x_1$$
$$\frac{\partial f(x)}{\partial x_2} = 1 - 6x_2 \quad \Rightarrow \nabla f(x) = \begin{bmatrix} 10-12x_1 \\ 1-6x_2 \end{bmatrix}$$

$$\frac{\partial^2 f(x)}{\partial x_1^2} = -12$$
$$\frac{\partial^2 f(x)}{\partial x_2^2} = -6 \quad \Rightarrow H(x) = \begin{bmatrix} -12 & 0 \\ 0 & -6 \end{bmatrix}$$
$$\frac{\partial^2 f(x)}{\partial x_1 \partial x_2} = 0$$

$H_1 = h_{11} = -12 < 0$

$H_2 = \begin{vmatrix} -12 & 0 \\ 0 & -6 \end{vmatrix} = (-12)(-6) - (0)(0) = 72 > 0$

Since $H_1 < 0$ + $H_2 > 0$,
$H(x)$ is negative definite

$\Rightarrow$ $\boxed{f(x) \text{ is strictly concave}}$

3. Dr. C wants to invest in two projects, A and B. The total investment budget is $100. He does not want to invest more than $40 in project A. The investment goal is the maximization of satisfaction measured as the product of the amount invested in projects A and B. (a) Does this problem have a maximum? Give an argument to support your answer and (b) compute the optimal solution, if it exists.

Objective function: $A \cdot B$
Constraint: $A \leq 40$, $A + B = 100$, $A \geq 0$, $B \geq 0$

(a) Does this have a maximum?

From Constraint: $B = 100 - A$
Plug into Objective Function: $A \cdot (100 - A)$
$\rightarrow 100A - A^2$

Take derivative of this wrt A:
$100 - 2A$

Since $0 \leq A \leq 40$: $100 - 2A > 0$
=> Since derivative is > 0, the objective function will always be increasing within these constraints.

Based on the Extreme Value Theorem, since $100A - A^2$ is continuous & the domain is a compact set (closed & bounded), the function will have a min/max.

$\Rightarrow \boxed{\text{This problem has a maximum}}$

(b) Compute the optimal solution.

$\frac{\partial f}{\partial A} = 100 - 2A = 0$

$\rightarrow 100 = 2A$

$\Rightarrow A = 50$, but $0 \leq A \leq 40$

Try $A = 0: 100(0) - (0)^2 = 0$

Try $A = 40: 100(40) - (40)^2 = 2{,}400$

If $A = 40 \rightarrow B = 100 - A$

$\Rightarrow B = 60$

$A \cdot B = 40 \cdot 60 = 2{,}400$

$\Rightarrow$ $\boxed{A = \$40,\ B = \$60 \text{ give the max of } \$2{,}400}$