HW2

Thursday, October 7, 2021 3:18 PM

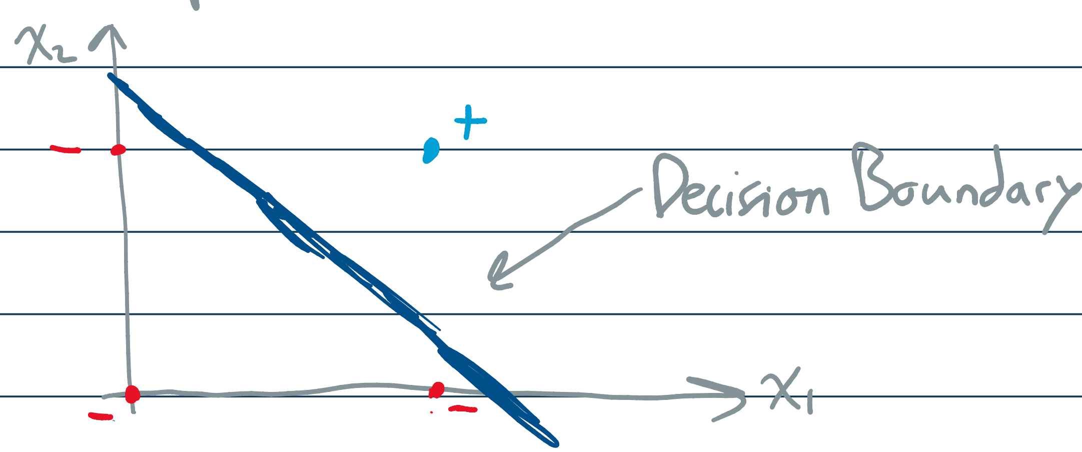
Problem 1 (30pt): [Perceptron Algorithm]

In this problem, we learn the linear discriminant function for boolean NAND function. Suppose we have two dimensional $x = (x_1, x_2)$, x_1 and x_2 can be either 0 (false) or 1 (true). The boolean NAND function is defined as: $f(x_1, x_2) = x_1$ **NAND** x_2. Specifically, $f(0,0) =$ **true**, $f(1,0) =$ **true**, $f(0,1) =$ **true**, and $f(1,1) =$ **false** where **false** can be treated as *positive class* and **true** can be treated as *negative class*. You can think of this function as having 4 points on the 2D plane (x_1 being the horizontal axis and x_2 being the vertical axis): $P_1 = (0,1), P_2 = (1,1), P_3 = (1,0), P_4 = (0,0)$, P_2 in *positive class* and P_1, P_3, P_4 in *negative class*.

(1) [5pt] For boolean NAND function, is the negative class and positive class linearly separable?

(2) [25pt] Is it possible to apply the **perceptron algorithm** to obtain the linear decision boundary that correctly classify both the positive and negative classes? If so, write down the updation steps and the obtained linear decision boundary. (You may assume the initial decision boundary is $x_1 + x_2 - \frac{1}{2} = 0$, and sweep the 4 points in clockwise order, i.e., $(P_1, P_2, P_3, P_4, P_1, P_2, \ldots)$, please **do not** directly write down the arbitrary linear boundary **without** update steps.)

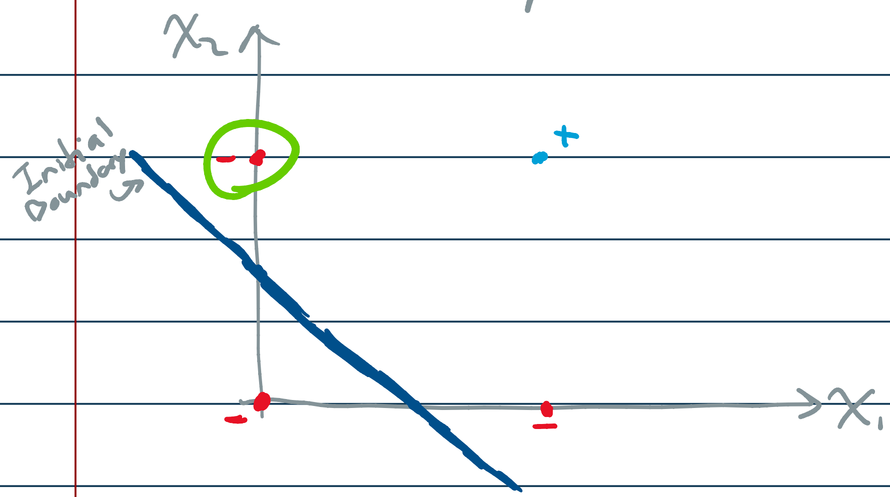
(1) Graphed, the NAND function looks like this:



Clearly a linear decision boundary can be drawn to fully separate these two classes.

=> $\boxed{\text{The classes of NAND are linearly separable.}}$

(2) Initial Boundary: $x_1 + x_2 - \frac{1}{2} = 0 \rightarrow x_2 > -x_1 + 1/2$



Since this is linearly seperable, we can use the perceptron algorithm to obtain the linear decision boundary. Initially we can see that the initial boundary incorrectly classifies 2 negative points as positive. Let's start with $P_1 = (0,1)$.

Update Step $\rightarrow w^{(\tau+1)} = \begin{cases} w^{(\tau)} + x_k & \text{if } x_k \text{ is } + \\ w^{(\tau)} - x_k & \text{if } x_k \text{ is } - \end{cases}$

$w^0 = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}, \; x = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}$ $x_1 = 0$, $x_2 = 1$

Update Step 1: $w_0^1 = -1/2 - 1 = -1.5$

$w_1^1 = 1 - 0 = 1$

$w_2^1 = 1 - 1 = 0$

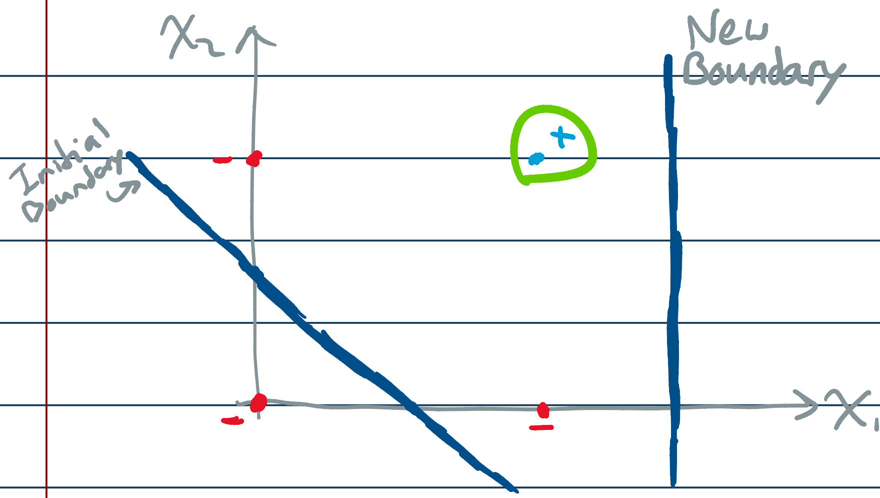
$\ast\; w_0 + w_1 x_1 + w_2 x_2 > 0$

$\Rightarrow (-1.5) + (1)x_1 + (0)x_2 > 0$

$$\Rightarrow (-1.5) + (1)x_1 + (0)x_2 > 0$$

$$-1.5 + x_1 > 0$$

$$x_1 > 1.5$$



Update Step 2: $w_0^2 = -1.5 + 1 = -0.5$

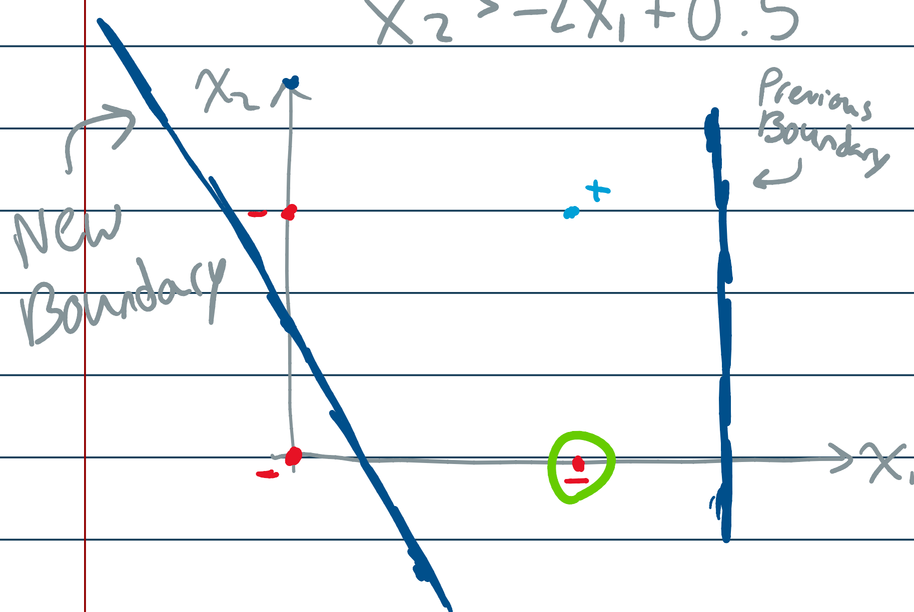
$$w_1^2 = 1 + 1 = 2$$

$$w_2^2 = 0 + 1 = 1$$

$$\Rightarrow -0.5 + (2)x_1 + (1)x_2 > 0$$

$$-0.5 + 2x_1 + x_2 > 0$$

$$x_2 > -2x_1 + 0.5$$



Update Step 3: $w_0^3 = -0.5 - 1 = -1.5$

Update Step 3: $w_0^3 = -0.5 - 1 = -1.5$

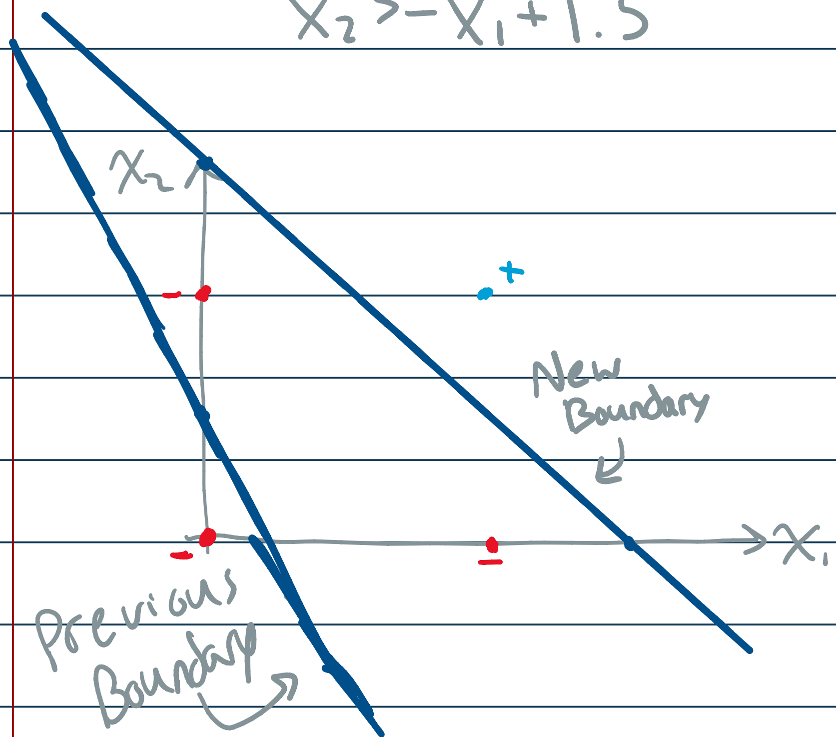
$w_1^3 = 2 - 1 = 1$

$w_2^3 = 1 - 0 = 1$

$\Rightarrow -1.5 + (1)x_1 + (1)x_2 > 0$

$-1.5 + x_1 + x_2 > 0$

$x_2 > -x_1 + 1.5$



Here, after 3 update steps, you can see that we have find a linear decision boundary that completely separates the 2 classes.

$$\boxed{\text{Linear Decision Boundary: } x_1 + x_2 - 1.5 = 0}$$