HW2

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Problem 1: For a one dimensional Rayleigh distribution

$$p(x \mid \theta) = \begin{cases} 2\theta x e^{-\theta x^2} & x \geq 0 \\ 0 & \textbf{otherwise} \end{cases}, \quad \theta > 0$$

Given n training samples $\{x_1, x_2, \ldots, x_n\}$,

1. Calculate the maximum likelihood estimation of parameter θ (follow the example in CPE646-4 pp. 15-16).
2. Assume a prior density for θ as a uniform distribution

$$p(\theta) \sim U(0, \lambda) = \begin{cases} \frac{1}{\lambda} & 0 \leq \theta \leq \lambda \\ 0 & \textbf{otherwise} \end{cases}, \quad \lambda > 0 \textbf{ and fixed}$$

Calculate the Bayesian estimation of parameter θ (follow the example in CPE646-4 pp. 29-32).

1) Maximum likelihood estimation of θ.

Likelihood of θ wrt set samples $\rightarrow p(D \mid \theta)$

We want to find the value of θ that maximizes this.

Easier to work with the log so we will find log likelihood.

Log likelihood $\rightarrow \ln p(D \mid \theta)$

$\rightarrow \ln [\prod_{k=1}^{n} p(x_k \mid \theta)]$

$\rightarrow \sum_{k=1}^{n} \ln p(x_k \mid \theta)$

To maximize this we can calculate the gradient wrt θ and set it equal to 0.

Gradient of log likelihood $\rightarrow \nabla_\theta = \sum_{k=1}^{n} \nabla_\theta \ln p(x_k|\theta)$

$$\rightarrow \sum_{k=1}^{n} \nabla_\theta \ln(2\theta x_k e^{-\theta x_k^2})$$

$$\rightarrow \sum_{k=1}^{n} \nabla_\theta \ln(2\theta x_k) + \nabla_\theta \ln(e^{-\theta x_k^2})$$ *product rule

$$\rightarrow \sum_{k=1}^{n} \frac{1}{2\theta x_k} \cdot 2x_k + \frac{1}{e^{-\theta x_k^2}} \cdot (-x_k^2 e^{-\theta x_k^2})$$

$$\rightarrow \sum_{k=1}^{n} \frac{1}{\theta} - x_k^2$$

Now we can set this equal to 0 & solve for θ.

$$\sum_{k=1}^{n} \frac{1}{\theta} - x_k^2 = 0$$

$$\rightarrow \sum_{k=1}^{n} \frac{1}{\theta} - \sum_{k=1}^{n} x_k^2 = 0$$

$$\rightarrow \sum_{k=1}^{n} \frac{1}{\theta} = \sum_{k=1}^{n} x_k^2$$

$$\rightarrow \frac{n}{\theta} = \sum_{k=1}^{n} x_k^2$$

$$\Rightarrow \boxed{\theta = n \,/\, \sum_{k=1}^{n} x_k^2}$$

2) Bayesian estimation of θ.

Posterior density of $\theta \rightarrow p(\theta|D)$

We want to find the value of θ that maximizes this.

$$p(\theta|D) = \frac{p(D|\theta)p(\theta)}{\int p(D|\theta)p(\theta)d\theta} = \frac{\prod_{k=1}^{n} p(x_k|\theta)p(\theta)}{\int p(D|\theta)p(\theta)d\theta \rightarrow = \alpha}$$

$$= \alpha \prod_{k=1}^{n} p(x_k|\theta)p(\theta)$$

$$= \alpha \prod_{k=1}^{n} [2\theta x_k e^{-\theta x_k^2}][1/\lambda]$$

We can take the natural log of both sides to simplify.

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$$\ln p(\theta|D) = \ln\alpha + \sum_{k=1}^{n} [\ln\theta + \ln 2x_k - \theta x_k^2 - \ln\lambda]$$

The gradient of this is now calculated & set equal to 0.

$$\nabla_\theta p(\theta|D) = \sum_{k=1}^{n} \nabla_\theta [\ln\theta + \ln 2x_k - \theta x_k^2 - \ln\lambda]$$

$$= \sum_{k=1}^{n} \frac{1}{\theta} - x_k^2$$

$$\rightarrow \sum_{k=1}^{n} \frac{1}{\theta} - x_k^2 = 0$$

$$\rightarrow \sum_{k=1}^{n} \frac{1}{\theta} - \sum_{k=1}^{n} x_k^2 = 0$$

$$\rightarrow \sum_{k=1}^{n} \frac{1}{\theta} = \sum_{k=1}^{n} x_k^2$$

$$\rightarrow \frac{n}{\theta} = \sum_{k=1}^{n} x_k^2$$

$$\Rightarrow \boxed{\theta = n / \sum_{k=1}^{n} x_k^2 ,\ 0 \le \theta \le \lambda}$$

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