HW1

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Problem 1:
For a one dimensional 2-class problem with 2 actions a_1, a_2, (a_1: choose ω_1; a_2 choose ω_2). We assume $P(\omega_1) = 3/5$, $P(\omega_2) = 2/5$, conditional densities
$p(x|\omega_1) \sim N(0, 1)$ and $p(x|\omega_2) \sim N(2, 4)$,
where $N(\mu, \sigma^2)$ is Normal distribution, μ is mean and σ^2 is variance.

5. Calculate the likelihood ratio threshold for zero-one loss function, i.e. maximum posterior classification

Zero-One Loss : $\lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Likelihood Ratio: $\frac{p(x|\omega_1)}{p(x|\omega_2)} > \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \cdot \frac{P(\omega_2)}{P(\omega_1)} = \theta_\lambda$

If this is true → take action a_1 (ω_1)

Otherwise → take action a_2 (ω_2)

Gaussian : $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$p(x|\omega_1) = \frac{1}{\sqrt{2\pi}} \cdot e^{-1/2 (x)^2}$, $p(x|\omega_2) = \frac{1}{2\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-2}{2}\right)^2}$

$\frac{p(x|\omega_1)}{p(x|\omega_2)} = \frac{1/\sqrt{2\pi} \cdot e^{-1/2(x)^2}}{1/2\sqrt{2\pi} \cdot e^{-1/2\left(\frac{x-2}{2}\right)^2}} = 2e^{-\frac{1}{2}(x)^2 + \frac{1}{2}\left(\frac{x-2}{2}\right)^2}$
$= 2e^{\frac{(3x-2)(-x-2)}{8}}$

$\frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \cdot \frac{P(\omega_2)}{P(\omega_1)} = \frac{1}{1} \cdot \frac{2/5}{3/5} = 2/3$

$\Rightarrow 2e^{\frac{(3x-2)(-x-2)}{8}} > 2/3 \rightarrow \ln\left(2e^{\frac{(3x-2)(-x-2)}{8}}\right) > \ln(2/3)$

$\Rightarrow -\frac{(3x-2)(x+2)}{8} + \ln(2) > \ln(2/3) \rightarrow (3x-2)(x+2) > 8\ln(2/3) - 8\ln(2)$

$\Rightarrow \boxed{-2.8363 < x < 1.5030}$ ← When x is between these choose ω_1

6. Calculate the likelihood ratio threshold for the following risk matrix

	ω_1	ω_2
a_1	0	4
a_2	2	0

$\frac{p(x|\omega_1)}{p(x|\omega_2)} = 2e^{\frac{(3x-2)(-x-2)}{8}}$ ← same as question 5

$\frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \cdot \frac{P(\omega_2)}{P(\omega_1)} = \frac{4-0}{2-0} \cdot \frac{2/5}{3/5} = 2 \cdot 2/3 = 4/3$

$\Rightarrow 2e^{\frac{(3x-2)(-x-2)}{8}} > 4/3$

$\Rightarrow \boxed{-2.5 < x < 1}$ ← When x is between these choose ω_1

Problem 2:
For a two dimensional 2-class problem, assume both class conditional densities are Gaussian, with the following estimated distribution parameters

mean, covariance

$$\mu_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } \mu_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \Sigma_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

assume $P(\omega_1) = 0.25$, **and** $P(\omega_2) = 0.75$

Prior

1. Calculate the decision boundary.
2. Calculate the Bhattacharyya error bound.

cancels out

1) Discriminant: $g_i(x) = -\frac{1}{2}(x-\mu_i)^t \Sigma_i^{-1}(x-\mu_i) - \cancel{\frac{d}{2}\ln(2\pi)} - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$

$g_1(x) = -\frac{1}{2}(x-\mu_1)^t \Sigma_1^{-1}(x-\mu_1) - \frac{1}{2}\ln|\Sigma_1| + \ln P(\omega_1)$
$g_2(x) = -\frac{1}{2}(x-\mu_2)^t \Sigma_2^{-1}(x-\mu_2) - \frac{1}{2}\ln|\Sigma_2| + \ln P(\omega_2)$

$g_1(x) = g_2(x)$
$-\frac{1}{2}(x_1-\mu_1)^t \Sigma_1^{-1}(x_1-\mu_1) - \frac{1}{2}\ln|\Sigma_1| + \ln P(\omega_1) = -\frac{1}{2}(x_2-\mu_2)^t \Sigma_2^{-1}(x_2-\mu_2) - \frac{1}{2}\ln|\Sigma_2| + \ln P(\omega_2)$

Inverse of Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\Sigma_1^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

$\Sigma_2^{-1} = \frac{1}{2-1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Determinant of Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow |A| = ad - bc$

$|\Sigma_1| = (1)(2) - (0)(0) = 2$

$|\Sigma_2| = (1)(2) - (1)(1) = 1$

Since Σ_i is arbitrary $\rightarrow$ discriminant is quadratic

$g_i(x) = x^t W_i x + w_i^t x + w_{i0}$

$\rightarrow W_i = -\frac{1}{2}\Sigma_i^{-1} \quad \rightarrow w_i = \Sigma_i^{-1}\mu_i$

$\rightarrow w_{i0} = -\frac{1}{2}\mu_i^t \Sigma_i^{-1}\mu_i - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$

$W_1 = -\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ 0 & -1/4 \end{bmatrix}$

$w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$w_{10} = -\frac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix}^t \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \frac{1}{2}\ln(2) + \ln(0.25) = -\frac{1}{2}\ln(2) + \ln(0.25) = -1.7329$

$W_2 = -\frac{1}{2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$w_2 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$w_{20} = -\frac{1}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix}^t \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{1}{2}\ln(1) + \ln(0.75) = -2 - 0 + \ln(0.75) = -2.2877$

$g_1(x) = x^t \begin{bmatrix} -1/2 & 0 \\ 0 & -1/4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix}^t x - 1.7329$ $\qquad$ * $g_1(x) = g_2(x)$

$g_2(x) = x^t \begin{bmatrix} -1 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix}^t x - 2.2877$ $\qquad$ * $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^t \begin{bmatrix} -1/2 & 0 \\ 0 & -1/4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1.7329 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^t \begin{bmatrix} -1 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}^t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2.2877$

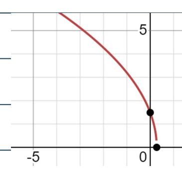
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^t \begin{bmatrix} -1/2\, x_1 \\ -1/4\, x_2 \end{bmatrix} - 1.7329 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^t \begin{bmatrix} -1/2\, x_1 \\ 0 \end{bmatrix} + 2x_1 - 2.2877$

$-\frac{1}{2}x_1^2 - \frac{1}{4}x_2^2 - 1.7329 = -\frac{1}{2}x_1^2 + 2x_1 - 2.2877$

$-\frac{1}{4}x_2^2 = 2x_1 - 0.5548$

$x_2^2 = -8x_1 + 2.2192$

$\Rightarrow \boxed{x_2 = \sqrt{-8x_1 + 2.2192}}$ $\leftarrow$ Decision Boundary



2) Error Bound: $P(error) \le P^{\beta}(\omega_1) P^{1-\beta}(\omega_2) \int p^{\beta}(x|\omega_1) p^{1-\beta}(x|\omega_2)\, dx$

Since $p(x|\omega_i)$ is normal: $\int p^{\beta}(x|\omega_1) p^{1-\beta}(x|\omega_2)\, dx = e^{-k(\beta)}$

$\rightarrow k(\beta) = \frac{\beta(1-\beta)}{2}(\mu_2 - \mu_1)^t [\beta\Sigma_1 + (1-\beta)\Sigma_2]^{-1}(\mu_2 - \mu_1) + \frac{1}{2}\ln\left(\frac{|\beta\Sigma_1 + (1-\beta)\Sigma_2|}{|\Sigma_1|^{\beta}|\Sigma_2|^{1-\beta}}\right)$

Since Bhattacharyya bound $\rightarrow \beta = 0.5$

$\Rightarrow k(0.5) = \frac{1}{8}(\mu_2 - \mu_1)^t [(\Sigma_1 + \Sigma_2)/2]^{-1}(\mu_2 - \mu_1) + \frac{1}{2}\ln\left(\frac{|(\Sigma_1 + \Sigma_2)/2|}{\sqrt{|\Sigma_1||\Sigma_2|}}\right)$

$k(0.5) = \frac{1}{8}\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)^t \left[\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right)/2\right]^{-1}\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) + \frac{1}{2}\ln\left(\frac{|(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix})/2|}{\sqrt{|\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}||\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}|}}\right)$

$= \frac{1}{8}\begin{bmatrix} 2 \\ 2 \end{bmatrix}^t \begin{bmatrix} 1 & 1/2 \\ 1/2 & \ldots \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \frac{1}{2}\ln\left(\frac{|\begin{bmatrix} 1 & 1/2 \\ 1/2 & 2 \end{bmatrix}|}{\ldots}\right)$

$$K(0.5) = \frac{1}{8}\left(\begin{bmatrix}2\\2\end{bmatrix}-\begin{bmatrix}0\\0\end{bmatrix}\right)^t\left[\left(\begin{bmatrix}1&0\\0&2\end{bmatrix}+\begin{bmatrix}1&1\\1&2\end{bmatrix}\right)/2\right]^{-1}\left(\begin{bmatrix}2\\2\end{bmatrix}-\begin{bmatrix}0\\0\end{bmatrix}\right)+\frac{1}{2}\ln\left(\frac{\left|\left(\begin{bmatrix}1&0\\0&2\end{bmatrix}+\begin{bmatrix}1&1\\1&2\end{bmatrix}\right)/2\right|}{\sqrt{\left|\begin{bmatrix}1&0\\0&2\end{bmatrix}\right|\left|\begin{bmatrix}1&1\\1&2\end{bmatrix}\right|}}\right)$$

$$= \frac{1}{8}\begin{bmatrix}2\\2\end{bmatrix}^t\begin{bmatrix}1&1/2\\1/2&2\end{bmatrix}^{-1}\begin{bmatrix}2\\2\end{bmatrix}+\frac{1}{2}\ln\left(\frac{\left|\begin{bmatrix}1&1/2\\1/2&2\end{bmatrix}\right|}{\sqrt{\left|\begin{bmatrix}1&0\\0&2\end{bmatrix}\right|\left|\begin{bmatrix}1&1\\1&2\end{bmatrix}\right|}}\right)$$

$$= \frac{1}{8}\begin{bmatrix}2\\2\end{bmatrix}^t\begin{bmatrix}8/7&-2/7\\-2/7&4/7\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}+\frac{1}{2}\ln\left(\frac{7/4}{\sqrt{2}}\right)$$

$$= \frac{1}{8}\begin{bmatrix}2\\2\end{bmatrix}^t\begin{bmatrix}12/7\\4/7\end{bmatrix}+0.1065$$

$$= \frac{1}{8}\left(\frac{32}{7}\right)+0.1065$$

$$= 4/7 + 0.1065$$

$$\Rightarrow \underline{K(0.5) = 0.6779}$$

$$P(error) \le \sqrt{P(\omega_1)P(\omega_2)}\cdot e^{-K(0.5)}$$

$$\le \sqrt{0.25\cdot 0.75}\cdot e^{-0.6779}$$

$$\le \sqrt{3/16}\cdot e^{-0.6779}$$

$$\le 0.4330\cdot 0.5077$$

$$\Rightarrow \boxed{P(error) \le 0.2198}$$

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