

# Lecture 3: The OLG Model

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Workshop on Life-cycle Models and Pensions

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# The Model

- ▶ Time is discrete and indexed by  $t \in \{0, 1, 2, \dots\}$
- ▶ There is no growth in technology and population size

## Households

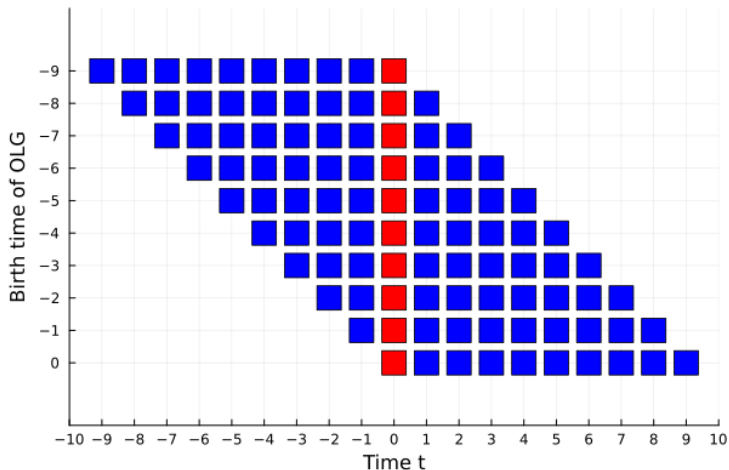
- ▶ Agents live for  $T$  periods and retire in period  $T_R$
- ▶ Household age is discrete and indexed by  $j \in \{1, 2, \dots, T\}$
- ▶ Own/save in capital and supply labor exogenously

## OLG

In every period a new generation  $t$  is born with associated individual variables

1. Consumption:  $c_{j,t}$  (endogenous)
2. Saving (in bonds):  $a_{j,t}$  (endogenous)
3. Labour supply:  $l_{j,t}$  (exogenous)

# Graphical Illustration of $T = 10$ OLG



$\Rightarrow$  to calculate aggregate variables at time  $t = 0$  we have to sum over individual variables market in red.

# The Model

## Aggregate variables

1. Capital:  $K_t$
2. Labor supply:  $L_t$
3. Output:  $Y_t$
4. Consumption:  $C_t$

**Prices** are determined in equilibrium but taken as given by households and firms

1. Wage rate:  $w_t$
2. Interest rate:  $r_t$

The rental rate on capital  $z_t = r_t + \delta$  where  $\delta \in [0, 1]$  is the depreciation rate

# The Household Problem

Agent  $i$  born at time  $t$ , who starts to consume in period  $t + 1$ , faces the following problem:

$$\begin{aligned} \max_{\{c_t, a_t\}} \quad & U = \sum_{j=1}^T \beta^j u(c_{j,t}) \\ \text{s.t.} \quad & \\ & c_{j,t} + a_{j,t} = w_{t+j} l_{j,t} + (1 + r_{t+j}) a_{j-1,t} \end{aligned} \tag{1}$$

$$\text{with } l_{j,t} = \begin{cases} 1 & \text{if } j \leq T_r \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } a_{0,t} = 0$$

# Constant Relative Risk Aversion (CRRA) Utility

$$u(c) = \begin{cases} \frac{c^{1-\rho} - 1}{1-\rho} & \rho \geq 0, \rho \neq 1 \\ \log(c) & \rho = 1, \end{cases}$$

where intertemporal elasticity of substitution (IES) of consumption is define by

$$IES \equiv \frac{d\left(\frac{c_{j+1,t+1}}{c_{j,t}}\right)}{d(1+r)} \cdot \frac{1+r}{\frac{c_{j+1,t+1}}{c_{j,t}}} = \frac{1}{\rho},$$

which is non-separable from the relative risk aversion (RRA)

$$RRA \equiv -c \cdot \frac{u''(c)}{u'(c)} = \rho.$$

Use [Epstein-Zin Preferences](#) if you want to separate the two!

# First Order Condition (FOC)

Derive the period-by-period Euler-Equation:

$$c_{j+1,t} = c_{j,t} [\beta (1 + r_{t+j+1})]^{\frac{1}{\rho}}$$

Derive the long-term Euler-Equation by iterating the period-by-period Euler-Equation forward:

$$c_{j,t} = c_{1,t} \left[ \beta^{j-1} \prod_{x=2}^j (1 + r_{t+x}) \right]^{\frac{1}{\rho}} \quad \text{for } j > 1 \quad (2)$$

## Closed-form Solution

Derive the consolidated budget constraint at birth:

$$\underbrace{\sum_{j=1}^T \frac{c_{j,t}}{\prod_{x=1}^j (1 + r_{t+x})}}_{\text{NPV of consumption}} = \underbrace{\sum_{j=1}^T \frac{w_{t+j} l_{j,t}}{\prod_{x=1}^j (1 + r_{t+x})}}_{\text{NPV of labor income}}$$

Solve for  $c_{1,t}$  by eliminating  $c_{j,t}$  using the long-term Euler-Equation:

$$c_{1,t}^* = \frac{\sum_{j=1}^T \frac{w_{t+j} l_{j,t}}{\prod_{x=1}^j (1 + r_{t+x})}}{\sum_{j=1}^T \frac{\left[ \frac{\beta^{j-1}}{(1 + r_{t+1})} \prod_{x=1}^j (1 + r_{t+x}) \right]^{\frac{1}{\rho}}}{\prod_{x=1}^j (1 + r_{t+x})}}. \quad (3)$$

Insert the solution in (6) into the long-term Euler-Equation (2) to solve for  $c_{j,t}^*$  for  $j > 1$ .



# Firms

- ▶ There is a single consumption good, which is produced using a neoclassical aggregate production function  $Y_t = F(K_t, L_t)$  from  $K_t$  units of capital and  $L_t$  labor units supplied at time  $t$ .
- ▶ We assume that  $F$  is CRS and Cobb-Douglas,  $Y_t = K_t^\alpha L_t^{1-\alpha}$ .

Firms maximise their profits:

$$\max_{\{K_t, L_t\}} \Pi_t = Y_t - (r_t + \delta) K_t - w_t L_t,$$

where  $\delta$  is the depreciation rate of capital and  $r_t + \delta$  is the capital rental rate. FOCs imply

$$\begin{aligned} F_K(K_t, L_t) - (r_t + \delta) &= 0 \quad \Rightarrow \quad r_t = \alpha \left( \frac{K_t}{L_t} \right)^{\alpha-1} - \delta \\ F_L(K_t, L_t) - w_t &= 0 \quad \Rightarrow \quad w_t = (1 - \alpha) \left( \frac{K_t}{L_t} \right)^{\alpha} \end{aligned}$$

# Labour and Capital Income Share

The capital income share yields

$$\frac{(r_t + \delta) K_t}{Y_t} = \frac{\alpha \left( \frac{K_t}{L_t} \right)^{\alpha-1} K_t}{K_t^\alpha L_t^{1-\alpha}} = \alpha,$$

while the labour income share is

$$\frac{w_t L_t}{Y_t} = \frac{(1 - \alpha) \left( \frac{K_t}{L_t} \right)^\alpha L_t}{K^\alpha L^{1-\alpha}} = 1 - \alpha$$

# Equilibrium

- ▶ Households maximize utility, while firms maximize profits
- ▶ The the capital market, the labor market as well as the goods market clear in equilibrium:

$$K_{t+1} = \sum_{j=1}^T a_{j,t-j}$$

$$L_t = \sum_{j=1}^T l_{j,t-j}$$

$$Y_t = C_t + I_t = C_t + K_{t+1} - (1 - \delta)K_t$$

- ▶ Equilibrium factor prices equal

$$r_t = \alpha \left( \frac{K_t}{L_t} \right)^{\alpha-1} - \delta$$

$$w_t = (1 - \alpha) \left( \frac{K_t}{L_t} \right)^{\alpha}$$

- ▶ Note that  $C_t = \sum_{j=1}^T c_{j,t-j}$ .

# Steady State

In a steady-state equilibrium, all variables remain constant over time, allowing us to omit the time subscript  $t$ :

$$K = \sum_{j=1}^T a_j$$

$$L = \sum_{j=1}^T l_j$$

$$Y = C + I = C + K' - (1 - \delta)K$$

$$C = \sum_{j=1}^T c_j$$

$$r = \alpha \left( \frac{K}{L} \right)^{\alpha-1} - \delta \quad (4)$$

$$w = (1 - \alpha) \left( \frac{K}{L} \right)^{\alpha} \quad (5)$$

## Steady-State Solution

Since  $r_t = r$  and  $w_t = w$  holds in steady state, the solution for consumption simplifies to

$$c_1^* = \frac{w \sum_{j=1}^T \frac{l_j}{(1+r)^j} + \frac{a_{0,t}}{1+r}}{\sum_{j=1}^T \frac{[\beta(1+r)]^{\frac{j-1}{\rho}}}{(1+r)^j}} \quad (6)$$

$$c_j^* = c_1^* [\beta(1+r)]^{\frac{j-1}{\rho}} \quad \text{for } j > 1 \quad (7)$$

Insert the solution for consumption  $c_j^*$  into the budget constraint (1) to back out the solution for savings:

$$a_j^* = w l_j + (1+r) a_{j-1} - c_j^* \quad (8)$$

$\Rightarrow$  final steps are calibration and simulation

## Algorithm to Find Steady-State Capital Level $K^{ss}$

1. Guess a steady-state capital stock  $K'$
2. Calculate factor prices  $r'$  and  $w'$  given  $K'$  as in equations (4) and (5)
3. Solve for  $c_1$  in equation (6) given  $w'$  and  $r'$
4. Solve  $c_j \forall j > 1$  using the Long-run Euler equation in (7):

$$c_j^* = c_1^* [\beta (1 + r')]^{\frac{j-1}{\rho}} \quad \text{for } j > 1$$

5. Solve  $a_j \forall j > 1$  using the budget constraint:

$$a'_j = w' l_j + (1 + r') a_{j-1} - c_j$$

6. Compute implied aggregate capital:  $K'' = \sum_{j=1}^T a'_j$
7. Check whether aggregate capital equals the guess  $K'$
8. Update guess of capital until convergence,  $K' \approx K''$

## Discipline Guess

Since

$$r_t = \alpha \left( \frac{K_t}{L_t} \right)^{\alpha-1} - \delta \quad (9)$$

it must hold that

$$K_t = L_t \left( \frac{\alpha}{r_t + \delta} \right)^{\frac{1}{1-\alpha}} \quad (10)$$

which we can use to discipline the upper bound of the guess of capital since if  $r_t + \delta < 0$  capital become imaginary.

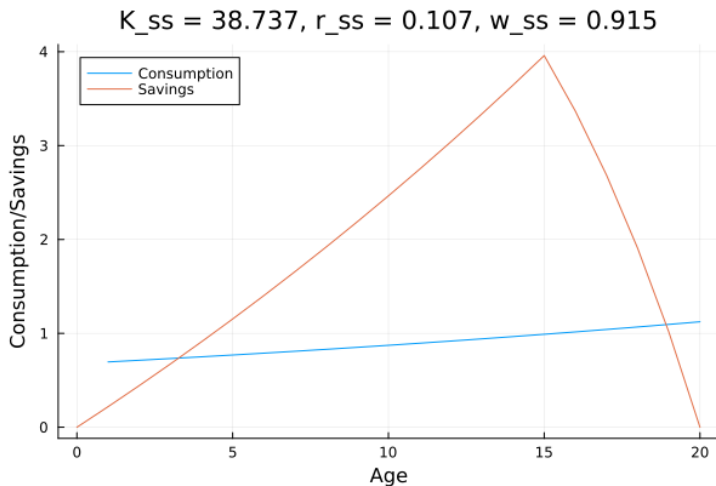
# Calibration

Parameter	Description	Value	Origin
<i>Timing</i>			
$T$	Maximum age of life	20	Model period $\approx$ 4 years
$T_R$	Retirement Age	15	
<i>Preferences</i>			
$\beta$	Subj. discount factor	0.96	To match annual factor of 0.99 Standard value
$\rho$	RRA / Inverse IES	2.0	
<i>Technology</i>			
$\alpha$	Capital income share	1/3	
$\delta$	Yearly depreciation rate	0.07	



# Result

## Steady State



# Potential Overaccumulation of Capital

In OLG models, overaccumulation of capital may occur because

- ▶ When each individual saves, they only think about their own future consumption and retirement needs
- ▶ agents do not consider how their savings contribute to the aggregate capital stock and drive down the future return on capital
- ▶ This externality may lead to an inefficient outcome for the economy as a whole.

Hence, overaccumulation of capital is directly linked to the dynamic efficiency of the economy

- ▶ **Dynamic Efficiency:**  $r > g$ , where  $r$  is the return on capital (interest rate), and  $g$  is the growth rate of the economy.
- ▶ **Dynamic Inefficiency:**  $r < g$ , indicating overaccumulation of capital.
  - Here reducing capital accumulation (saving less and consuming more) can increase welfare across generations.

# Introducing Pay-as-you-go (PAYG) Pensions

A PAYG pension scheme is a retirement system in which current workers' contributions are used to pay for the pension benefits of current retirees

Let us assume that

1. Workers contribute a constant share of their wage income  $\tau$
2. Retirees receive a constant benefit  $b_t$

Then, the balanced-budget condition within PAYG pensions implies

$$\tau w_t \sum_{j=1}^{T_r} l_{j,t-j} = \sum_{j=T_r+1}^T b_t \quad (11)$$

# The household problem

with a pension system

With a pension system, the household problem reads:

$$\begin{aligned} \max_{\{c_t^i, s_t^i\}} \quad & U = \sum_{j=1}^T \beta^j u(c_{j,t}) \\ \text{s.t.} \quad & c_{j,t} + a_{j,t} = (1 - \tau) w_{t+j} l_{j,t} + (1 + r_{t+j}) a_{j-1,t} + (1 - l_{j,t}) b_t \\ & \text{with } l_{j,t} = \begin{cases} 1 & \text{if } j \leq T_r \\ 0 & \text{otherwise} \end{cases} \\ & \text{and } a_{0,t} = 0 \end{aligned} \tag{12}$$

## Solution with Pensions

Deriving the consolidated budget constraint at birth yields:

$$\sum_{j=1}^T \frac{c_{j,t}}{\prod_{x=1}^j (1 + r_{t+x})} = \sum_{j=1}^T \frac{(1 - \tau) w_{t+j} l_{j,t} + (1 - l_{j,t}) b_{t+j}}{\prod_{x=1}^j (1 + r_{t+x})} \quad (13)$$

Solve for  $c_1$  Solving for  $c_{1,t}$  by eliminating  $c_{j,t}$  using the long-term Euler-Equation:

$$c_{1,t} = \frac{\sum_{j=1}^T \frac{(1 - \tau) w_{t+j} l_{j,t} + (1 - l_{j,t}) b_{t+j}}{\prod_{x=1}^j (1 + r_{t+x})}}{\sum_{j=1}^T \frac{\left[ \frac{\beta^{j-1}}{(1 + r_{t+j})} \prod_{x=1}^j (1 + r_{t+x}) \right]^{\frac{1}{\rho}}}{\prod_{x=1}^j (1 + r_{t+x})}} \quad (14)$$

## Steady-state Solution

The closed-form solution for first-period consumption for given steady-state factor prices reads:

$$c_1^* = \frac{\sum_{j=1}^T \frac{(1-\tau) w l_j + (1-l_j) b}{(1+r)^j}}{\sum_{j=1}^T \frac{[\beta (1+r)]^{\frac{j-1}{\rho}}}{(1+r)^j}} \quad (15)$$

$$c_j^* = c_1^* [\beta (1+r)]^{\frac{j-1}{\rho}} \quad \text{for } j > 1 \quad (16)$$

where:

$$b = \tau w \frac{\sum_{j=1}^T l_j}{T - T_r} \quad (17)$$

Insert the solution for consumption  $c_j^*$  into the budget constraint (12) to back out the solution for savings:

$$a_j^* = (1-\tau) w l_j + (1-l_j) b + (1+r) a_{j-1} - c_j^* \quad (18)$$

## Algorithm to Find Steady-State Capital Level $K^{ss}$

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6. Compute implied aggregate capital:  $K'' = \sum_{j=1}^T a'_j$
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# Result

## Steady State Comparison

