

# Lecture 1: The Life-cycle Model

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Workshop on Life-cycle Models and Pensions

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# The Model

- ▶ Time is discrete and indexed by  $t \in \{0, 1, 2, \dots\}$
- ▶ There is no growth in technology and population size

## Households

- ▶ Agents live for  $T$  periods and retire in period  $T_R$
- ▶ Household age is discrete and indexed by  $j \in \{1, 2, \dots, T\}$
- ▶ Save in bonds and supply labor exogenously

## Variables

1. Consumption:  $c_{j,t}$  (endogenous)
2. Saving (in bonds):  $a_{j,t}$  (endogenous)
3. Labour supply:  $l_{j,t}$  (exogenous)

## Exogenous Prices (partial equilibrium)

1. Wage:  $w_t = w$
2. Interest rate (on bonds):  $r_t = r$

⇒ amounts to assuming a small open economy taking prices as given

# The Household Problem

Agent  $i$  born at time  $t$ , who starts to consume in period  $t + 1$ , faces the following problem:

$$\begin{aligned} \max_{\{c_{j,t}^i, a_{j,t}^i\}} \quad & U = \sum_{j=1}^T \beta^j u(c_{j,t}^i) \\ \text{s.t.} \quad & \\ & c_{j,t}^i + a_{j,t}^i = w_{t+j} l_{j,t} + (1 + r_{t+j}) a_{j-1,t}^i \end{aligned} \tag{1}$$

$$\text{with } l_{j,t} = \begin{cases} 1 & \text{if } j \leq T_r \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\text{and } a_{0,t} \sim \text{Lognormal}(\mu, \sigma^2) \tag{3}$$

See [Log-normal Distribution!](#)

## Constant Relative Risk Aversion (CRRA) Utility

$$u(c) = \begin{cases} \frac{c^{1-\rho} - 1}{1-\rho} & \rho \geq 0, \rho \neq 1 \\ \log(c) & \rho = 1, \end{cases} \quad (4)$$

where intertemporal elasticity of substitution (IES) of consumption is define by

$$IES \equiv \frac{d\left(\frac{c_{j+1,t+1}}{c_{j,t}}\right)}{d(1+r)} \cdot \frac{1+r}{\frac{c_{j+1,t+1}}{c_{j,t}}} = \frac{1}{\rho},$$

which is non-separable from the relative risk aversion (RRA)

$$RRA \equiv -c \cdot \frac{u''(c)}{u'(c)} = \rho.$$

Use [Epstein-Zin Preferences](#) if you want to separate the two!

# First Order Condition (FOC) and Consolidated Budget

Derive the period-by-period Euler-Equation:

$$c_{j+1,t}^i = c_{j,t}^i [\beta (1 + r_{t+j+1})]^{\frac{1}{\rho}} \quad (5)$$

Derive the long-term Euler-Equation by iterating the period-by-period Euler-Equation forward:

$$c_{j,t}^i = c_{1,t}^i \left[ \beta^{j-1} \prod_{x=2}^j (1 + r_{t+x}) \right]^{\frac{1}{\rho}} \quad \text{for } j > 1 \quad (6)$$

Derive the consolidated budget constraint at birth:

$$\underbrace{\sum_{j=1}^T \frac{c_{j,t}^i}{\prod_{x=1}^j (1 + r_{t+x})}}_{\text{NPV of consumption}} = \underbrace{\sum_{j=1}^T \frac{w_{t+j} l_{j,t}}{\prod_{x=1}^j (1 + r_{t+x})}}_{\text{NPV of labor income}} + \underbrace{\frac{a_{0,t}^i}{1 + r_{t+1}}}_{\text{Initial wealth}}$$

## Closed-form Solution (1)

Use the consolidated budget constraint to solve for  $c_{1,t}^i$  by eliminating  $c_{j,t}^i$  using the long-term Euler-Equation:

$$c_{1,t}^{i,*} = \frac{\sum_{j=1}^T \frac{w_{t+j} l_{j,t}}{\prod_{x=1}^j (1 + r_{t+x})} + \frac{a_{0,t}^i}{1 + r_{t+1}}}{\sum_{j=1}^T \frac{\left[ \frac{\beta^{j-1}}{(1 + r_{t+1})} \prod_{x=1}^j (1 + r_{t+x}) \right]^{\frac{1}{\rho}}}{\prod_{x=1}^j (1 + r_{t+x})}}. \quad (7)$$

Insert the solution in (8) into the long-term Euler-Equation (6) to solve for  $c_{j,t}^{i,*}$  for  $j > 1$ :

$$c_{j,t}^{i,*} = c_{1,t}^{i,*} \left[ \beta^{j-1} \prod_{x=2}^j (1 + r_{t+x}) \right]^{\frac{1}{\rho}} \quad \text{for } j > 1$$

## Closed-form Solution (2)

Since  $r_t = r$  and  $w_t = w$  holds, the solution simplifies to

$$c_{1,t}^{i,*} = \frac{w \sum_{j=1}^T \frac{l_{j,t}}{(1+r)^j} + \frac{a_{0,t}^i}{1+r}}{\sum_{j=1}^T \frac{[\beta(1+r)]^{\frac{j-1}{\rho}}}{(1+r)^j}} = \frac{X + Y^i}{Z} \quad (8)$$

with

$$X \equiv w \sum_{j=1}^T \frac{l_{j,t}}{(1+r)^j} \quad Y^i \equiv \frac{a_{0,t}^i}{1+r} \quad Z = \sum_{j=1}^T \frac{[\beta(1+r)]^{\frac{j-1}{\rho}}}{(1+r)^j} \quad (9)$$

Using the Long-run Euler, we can solve for

$$c_{j,t}^{i,*} = c_{1,t}^{i,*} [\beta(1+r)]^{\frac{j-1}{\rho}} \quad \text{for } j > 1 \quad (10)$$

## Closed-form Solution (3)

Insert the solution for consumption  $c_{j,t}^*$  into the budget constraint (1) to back out the solution for savings:

$$a_{j,t}^* = w_{t+j} l_{j,t} + (1 + r_{t+j}) a_{j-1,t} - c_{j,t}^* \quad (11)$$

⇒ final steps are calibration and simulation



# Calibration

Parameter	Description	Value	Origin
<i>Timing</i>			
$T$	Maximum age of life	20	Model period $\approx$ 4 years
$T_R$	Retirement Age	15	
<i>Prices</i>			
$r$	Interest rate	0.13	To match an annual int. rate of 3% Normalization
$w$	Wage	1	
<i>Preferences</i>			
$\beta$	Subj. discount factor	0.96	To match annual factor of 0.99 Standard value
$\rho$	RRA / Inverse IES	2.0	
<i>Distribution</i>			
$\mu$	Location	0	
$\sigma$	Scale	1	

# Result

## Simulated Consumption and Savings

