Lecture 3: The OLG Model

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Workshop on Life-cycle Models and Pensions

September 27, 2024

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The Model

- Time is discrete and indexed by $t \in \{0, 1, 2, ..\}$
- There is no growth in technology and population size

Households

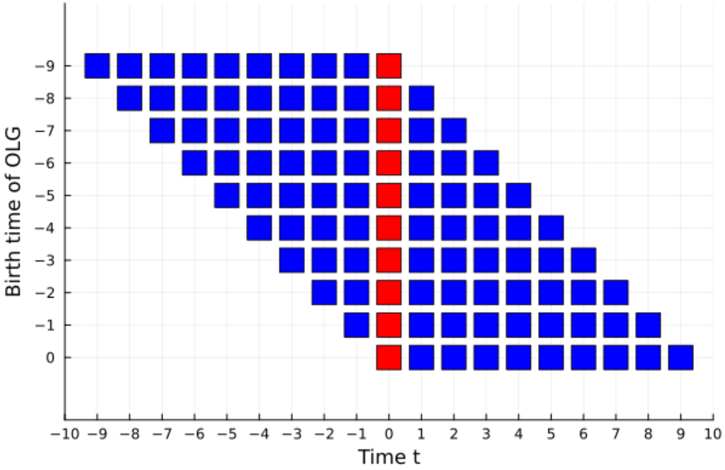
- Agents live for T periods and retire in period T_R
- Household age is discrete and indexed by $j \in \{1, 2, .., T\}$
- Own/save in capital and supply labor exogenously

OLG

In every period a new generation t is born with associated individual variables

1. Consumption: $c_{j,t}$ (endogenous)
2. Saving (in bonds): $a_{j,t}$ (endogenous)
3. Labour supply: $l_{j,t}$ (exogenous)

Graphical Illustration of $T = 10$ OLG



$\Rightarrow$ to calculate aggregate variables at time $t = 0$ we have to sum over individiual variables market in red.

The Model

Aggregate variables

1. Capital: K_t
2. Labor supply: L_t
3. Output: Y_t
4. Consumption: C_t

Prices are determined in equilibrium but taken as given by households and firms

1. Wage rate: w_t
2. Interest rate: r_t

The rental rate on capital $z_t = r_t + \delta$ where $\delta \in [0, 1]$ is the depreciation rate

The Household Problem

Agent i born at time t, who start to consume in period $t+1$, face the following problem:

$$\max_{\{c_t, a_t\}} \; U = \sum_{j=1}^{T} \beta^t u(c_{j,t})$$

$$s.t.$$

$$c_{j,t} + a_{j,t} = w_{t+j} l_{j,t} + (1 + r_{t+j})\, a_{j-1,t} \tag{1}$$

$$\text{with } l_{j,t} = \begin{cases} 1 & \text{if } j \leq T_r \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } a_{0,t} = 0$$

Constant Relative Risk Aversion (CRRA) Utility

$$u(c) = \begin{cases} \dfrac{c^{1-\rho} - 1}{1-\rho} & \rho \geq 0, \rho \neq 1 \\ \log(c) & \rho = 1, \end{cases}$$

where intertemporal elasticity of substitution (IES) of consumption is define by

$$IES \equiv \frac{d\left(\frac{c_{j+1,t+1}}{c_{j,t}}\right)}{d(1+r)} \cdot \frac{1+r}{\frac{c_{j+1,t+1}}{c_{j,t}}} = \frac{1}{\rho},$$

which is non-separable from the relative risk aversion (RRA)

$$RRA \equiv -c \cdot \frac{u''(c)}{u'(c)} = \rho.$$

Use Epstein-Zin Preferences if you want to separate the two!

First Order Condition (FOC)

Derive the period-by-period Euler-Equation:

$$c_{j+1,t} = c_{j,t} \left[\beta \left(1 + r_{t+j+1}\right)\right]^{\frac{1}{\rho}}$$

Derive the long-term Euler-Equation by iterating the period-by-period Euler-Equation forward:

$$c_{j,t} = c_{1,t} \left[\beta^{j-1} \prod_{x=2}^{j} \left(1 + r_{t+x}\right)\right]^{\frac{1}{\rho}} \quad \text{for } j > 1 \tag{2}$$

Closed-form Solution

Derive the consolidated budget constraint at birth:

$$\underbrace{\sum_{j=1}^{T} \frac{c_{j,t}}{\prod_{x=1}^{j}(1+r_{t+x})}}_{\text{NPV of consumption}} = \underbrace{\sum_{j=1}^{T} \frac{w_{t+j} l_{j,t}}{\prod_{x=1}^{j}(1+r_{t+x})}}_{\text{NPV of labor income}} + \underbrace{\frac{a_{0,t}}{1+r_{t+1}}}_{\text{Initial wealth}}$$

Solve for $c_{1,t}$ by eliminating $c_{j,t}$ using the long-term Euler-Equation:

$$c^*_{1,t} = \frac{\sum_{j=1}^{T} \frac{w_{t+j} l_{j,t}}{\prod_{x=1}^{j}(1+r_{t+x})} + \frac{a_{0,t}}{1+r_{t+1}}}{\sum_{j=1}^{T} \frac{\left[\frac{\beta^{j-1}}{(1+r_{t+1})} \prod_{x=1}^{j}(1+r_{t+x})\right]^{\frac{1}{\rho}}}{\prod_{x=1}^{j}(1+r_{t+x})}}. \tag{3}$$

Insert the solution in (6) into the long-term Euler-Equation (2) to solve for $c^*_{j,t}$ for $j > 1$.

Firms

- There is a single consumption good, which is produced using a neoclassical aggregate production function $Y_t = F(K_t, L_t)$ from K_t units of capital and L_t labor units supplied at time t.
- We assume that F is CRS and Cobb-Douglas, $Y_t = K_t^{\alpha} L_t^{1-\alpha}$.

Firms maximise their profits:

$$\max_{\{K_t, L_t\}} \Pi_t = Y_t - (r_t + \delta) K_t - w_t L_t,$$

where δ is the depreciation rate of captial and $r_t + \delta$ is the capital rental rate. FOCs imply

$$F_K(K_t, L_t) - (r_t + \delta) = 0 \quad \Rightarrow \quad r_t = \alpha \left(\frac{K_t}{L_t} \right)^{\alpha - 1} - \delta$$

$$F_L(K_t, L_t) - w_t = 0 \quad \Rightarrow \quad w_t = (1 - \alpha) \left(\frac{K_t}{L_t} \right)^{\alpha}$$

Labour an Capital Income Share

The capital income share yields

$$\frac{(r_t + \delta)\, K_t}{Y_t} = \frac{\alpha \left(\frac{K_t}{L_t}\right)^{\alpha - 1} K_t}{K_t^{\alpha} L_t^{1-\alpha}} = \alpha,$$

while the labour income share is

$$\frac{w_t L_t}{Y_t} = \frac{(1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha} L_t}{K^{\alpha} L^{1-\alpha}} = 1 - \alpha$$

Equilibrium

- Households maximize utility, while firms maximize profits
- The the capital market, the labor market as well as the goods market clear in equilibrium:

$$K_{t+1} = \sum_{j=1}^{T} a_{j,t-j}$$

$$L_t = \sum_{j=1}^{T} l_{j,t-j}$$

$$Y_t = C_t + I_t = C_t + K_{t+1} - (1-\delta)K_t$$

- Equilibrium factor prices equal

$$r_t = \alpha \left(\frac{K_t}{L_t}\right)^{\alpha-1} - \delta$$

$$w_t = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha}$$

- Note that $C_t = \sum_{j=1}^{T} c_{j,t-j}$.

Steady State

In a steady-state equilibrium, all variables remain constant over time, allowing us to omit the time subscript t:

$$
\begin{aligned}
K &= \sum_{j=1}^{T} a_j \\
L &= \sum_{j=1}^{T} l_j \\
Y &= C + I = C + K' - (1-\delta)K \\
C &= \sum_{j=1}^{T} c_j \\
r &= \alpha \left(\frac{K}{L}\right)^{\alpha-1} - \delta \qquad (4) \\
w &= (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} \qquad (5)
\end{aligned}
$$

Steady-State Solution

Since $r_t = r$ and $w_t = w$ holds in steady state, the solution in equation 6 simplifies to

$$c_1^* = \frac{w \sum_{j=1}^{T} \frac{l_j}{(1+r)^j} + \frac{a_{0,t}}{1+r}}{\sum_{j=1}^{T} \frac{[\beta(1+r)]^{\frac{j-1}{\rho}}}{(1+r)^j}} \tag{6}$$

Insert the solution for consumption c_j^* into the budget constraint (1) to back out the solution for savings:

$$a_j^* = wl_j + (1+r)\, a_{j-1} - c_j^* \tag{7}$$

$\Rightarrow$ final steps are calibration and simulation

Algorithm to Find Steady-State Capital Level K^{ss}

1. Guess a steady-state capital stock K'
2. Calculate factor prices r' and w' given K' as in equations (4) and (5)
3. Solve for c_1 in equation (6) given w' and r'
4. Solve $c_j \,\forall\, j > 1$ using the Long-run Euler equation in (2):

$$c_{j,t} = c_{1,t} \left[\beta^{j-1} \prod_{x=2}^{j} \left(1 + r_{t+x}\right) \right]^{\frac{1}{\rho}} \quad \text{for } j > 1$$

5. Solve $a_j \,\forall\, j > 1$ using the budget constraint:

$$a_j' = w' l_j + \left(1 + r'\right) a_{j-1} - c_j$$

6. Compute implied aggregate capital: $K'' = \sum_{j=1}^{T} a_j'$
7. Check whether aggregate capital equals the guess K'
8. Update guess of capital until convergence, $K' \approx K''$

Discipline Guess

Since

$$r_t = \alpha \left(\frac{K_t}{L_t} \right)^{\alpha - 1} - \delta \tag{8}$$

it must hold that

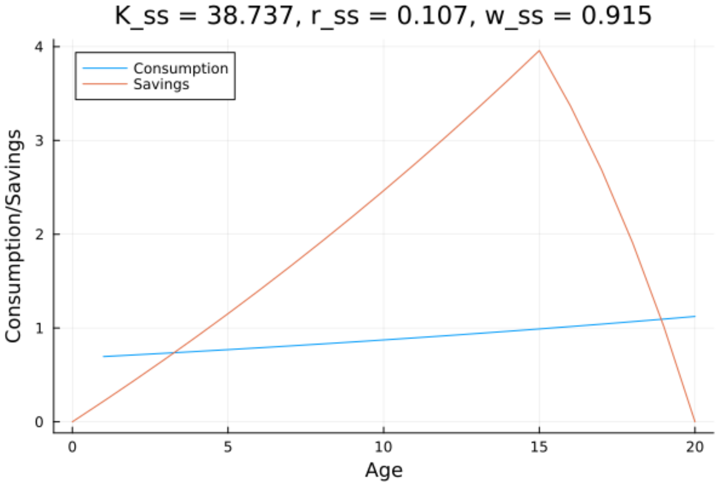
$$K_t = L_t \left(\frac{\alpha}{r_t + \delta} \right)^{\frac{1}{1-\alpha}} \tag{9}$$

which we can use to discipline the upper bound of the guess of capital since if $r_t + \delta < 0$ capital become imaginary.

Calibration

Parameter	Description	Value	Origin
Timing			
T	Maximum age of life	20	Model period $\approx$ 4 years
T_R	Retirement Age	15	
Preferences			
β	Subj. discount factor	0.96	To match annual factor of 0.99
ρ	RRA / Inverse IES	2.0	Standard value
Technology			
α	Capital income share	$1/3$	
δ	Yearly depreciation rate	0.07	

Result
Steady State



Introducing Pay-as-you-go (PAYG) Pensions

A PAYG pension scheme is a retirement system in which current workers' contributions are used to pay for the pension benefits of current retirees

Let us assume that

1. Workers contribute a constant share of their wage income τ
2. Retirees receive a constant benefit b_t

Then, the balanced-budget condition within PAYG pensions implies

$$\tau w_t \sum_{j=1}^{R} l_{j,t-j} = \sum_{j=R+1}^{T} b_t \tag{10}$$

The household problem

with a pension system

With a pension system, the household problem reads:

$$\max_{\{c_t^i, s_t^i\}} \quad U = \sum_{j=1}^{T} \beta^t u(c_{j,t})$$

$$s.t.$$

$$c_{j,t} + a_{j,t} = (1-\tau)\, w_{t+j} l_{j,t} + (1 + r_{t+j})\, a_{j-1,t} + (1 - l_{j,t})\, b_t \tag{11}$$

$$\text{with } l_{j,t} = \begin{cases} 1 & \text{if } j \leq T_r \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } a_{0,t} = 0$$

Solution with Pensions

Deriving the consolidated budget constraint at birth yields:

$$\sum_{j=1}^{T} \frac{c_{j,t}}{\prod_{x=1}^{j} (1 + r_{t+x})} = \sum_{j=1}^{T} \frac{(1-\tau)\, w_{t+j} l_{j,t} + (1 - l_{j,t})\, b_{t+j}}{\prod_{x=1}^{j} (1 + r_{t+x})} \tag{12}$$

Solve for c_1 Solving for $c_{1,t}$ by eliminating $c_{j,t}$ using the long-term Euler-Equation:

$$c_{1,t} = \frac{\sum_{j=1}^{T} \frac{(1-\tau)\, w_{t+j} l_{j,t} + (1 - l_{j,t})\, b_{t+j}}{\prod_{x=1}^{j} (1 + r_{t+x})}}{\sum_{j=1}^{T} \frac{\left[\frac{\beta^{j-1}}{(1 + r_{t+j})} \prod_{x=1}^{j} (1 + r_{t+x}) \right]^{\frac{1}{\rho}}}{\prod_{x=1}^{j} (1 + r_{t+x})}} \tag{13}$$

Steady-state Solution

The closed-form solution for first-period consumption for given steady-state factor prices reads:

$$c_1^* = \frac{\sum_{j=1}^{T} \frac{(1-\tau)\, w l_j + (1-l_j)\, b}{(1+r)^j}}{\sum_{j=1}^{T} \frac{[\beta\,(1+r)]^{\frac{j-1}{\rho}}}{(1+r)^j}} \tag{14}$$

where:

$$b = \tau w \frac{\sum_{j=1}^{T} l_j}{T-R} \tag{15}$$

Insert the solution in (14) into the long-term Euler-Equation (2) (which hasn't changed) to solve for $c_{j,t}^*$ for $j > 1$. Insert the solution for consumption c_j^* into the budget constraint (11) to back out the solution for savings.

Algorithm to Find Steady-State Capital Level K^{ss}

1. Guess a steady-state capital stock K'
2. Calculate factor prices r' and w' given K' as in equations (4) and (5)
3. Solve for c_1 in equation (14) given w' and r'
4. Solve $c_j \, \forall \, j > 1$ using the Long-run Euler equation in (2):

$$c_{j,t} = c_{1,t} \left[\beta^{j-1} \prod_{x=2}^{j} (1 + r_{t+x}) \right]^{\frac{1}{\rho}} \quad \text{for } j > 1$$

5. Solve $a_j \, \forall \, j > 1$ using the budget constraint:

$$a_j' = (1 - \tau) w' l_j + (1 - l_j)\, b + (1 + r')\, a_{j-1} - c_j$$

6. Compute implied aggregate capital: $K'' = \sum_{j=1}^{T} a_j'$
7. Check whether aggregate capital equals the guess K'
8. Update guess of capital until convergence, $K' \approx K''$

Result

Steady State Comparison