# Lecture 3: The OLG Model

Tim D. Maurer[1]

Workshop on Life-cycle Models and Pensions

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[1] Department of Economics, Norwegian School of Economics. **E-mail**: `tim.maurer@nhh.no`.

# The Model

- Time is discrete and indexed by $t \in \{0, 1, 2, ..\}$
- There is no growth in technology and population size

**Households**

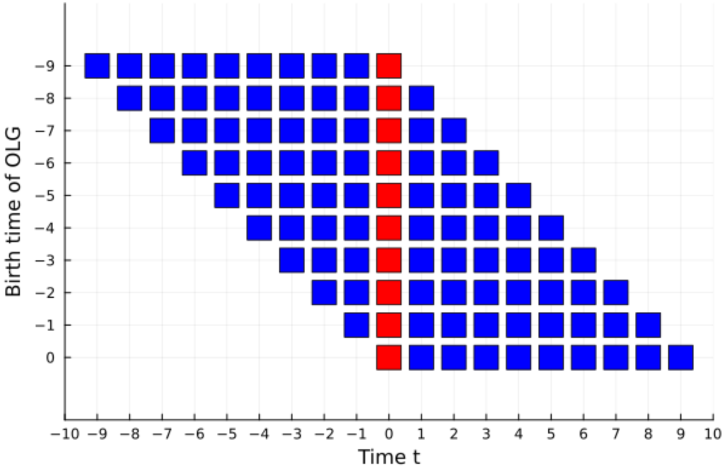
- Agents live for $T$ periods and retire in period $T_R$
- Household age is discrete and indexed by $j \in \{1, 2, .., T\}$
- Own/save in capital and supply labor exogenously

**OLG**

In every period a new generation $t$ is born with associated individual variables

1. Consumption: $c_{j,t}$ (endogenous)
2. Saving (in bonds): $a_{j,t}$ (endogenous)
3. Labour supply: $l_{j,t}$ (exogenous)

# Graphical Illustration of $T = 10$ OLG




$\Rightarrow$ to calculate aggregate variables at time $t = 0$ we have to sum over individual variables market in red.

# The Model

**Aggregate variables**

1. Capital: $K_t$
2. Labor supply: $L_t$
3. Output: $Y_t$
4. Consumption: $C_t$

**Prices** are determined in equilibrium but taken as given by households and firms

1. Wage rate: $w_t$
2. Interest rate: $r_t$

The rental rate on capital is $z_t = r_t + \delta$, where $\delta \in [0, 1]$ is the depreciation rate

# The Household Problem

Agent $i$ born at time $t$, who starts to consume in period $t+1$, faces the following problem:

$$\max_{\{c_t, a_t\}} \; U = \sum_{j=1}^{T} \beta^t u(c_{j,t})$$

$$s.t.$$

$$c_{j,t} + a_{j,t} = w_{t+j} l_{j,t} + (1 + r_{t+j})\, a_{j-1,t} \tag{1}$$

$$\text{with } l_{j,t} = \begin{cases} 1 & \text{if } j \leq T_r \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } a_{0,t} = 0$$

# Constant Relative Risk Aversion (CRRA) Utility

$$u(c) = \begin{cases} \dfrac{c^{1-\rho} - 1}{1 - \rho} & \rho \geq 0, \rho \neq 1 \\ \log(c) & \rho = 1, \end{cases}$$

where intertemporal elasticity of substitution (IES) of consumption is define by

$$IES \equiv \frac{d\left(\frac{c_{j+1,t+1}}{c_{j,t}}\right)}{d(1+r)} \cdot \frac{1+r}{\frac{c_{j+1,t+1}}{c_{j,t}}} = \frac{1}{\rho},$$

which is non-separable from the relative risk aversion (RRA)

$$RRA \equiv -c \cdot \frac{u''(c)}{u'(c)} = \rho.$$

Use Epstein-Zin Preferences if you want to separate the two!

# First Order Condition (FOC)

Derive the period-by-period Euler-Equation:

$$c_{j+1,t} = c_{j,t}\left[\beta\left(1 + r_{t+j+1}\right)\right]^{\frac{1}{\rho}}$$

Derive the long-term Euler-Equation by iterating the period-by-period Euler-Equation forward:

$$c_{j,t} = c_{1,t}\left[\beta^{j-1}\prod_{x=2}^{j}\left(1 + r_{t+x}\right)\right]^{\frac{1}{\rho}} \quad \text{for } j > 1 \tag{2}$$

# Closed-form Solution

Derive the consolidated budget constraint at birth:

$$\underbrace{\sum_{j=1}^{T} \frac{c_{j,t}}{\prod_{x=1}^{j}(1+r_{t+x})}}_{\text{NPV of consumption}} = \underbrace{\sum_{j=1}^{T} \frac{w_{t+j} l_{j,t}}{\prod_{x=1}^{j}(1+r_{t+x})}}_{\text{NPV of labor income}}$$

Solve for $c_{1,t}$ by eliminating $c_{j,t}$ using the long-term Euler-Equation:

$$c_{1,t}^{*} = \frac{\sum_{j=1}^{T} \frac{w_{t+j} l_{j,t}}{\prod_{x=1}^{j}(1+r_{t+x})}}{\sum_{j=1}^{T} \frac{\left[\frac{\beta^{j-1}}{(1+r_{t+1})} \prod_{x=1}^{j}(1+r_{t+x})\right]^{\frac{1}{\rho}}}{\prod_{x=1}^{j}(1+r_{t+x})}}. \tag{3}$$

Insert the solution in (6) into the long-term Euler-Equation (2) to solve for $c_{j,t}^{*}$ for $j > 1$.

# Firms

- There is a single consumption good, which is produced using a neoclassical aggregate production function $Y_t = F(K_t, L_t)$ from $K_t$ units of capital and $L_t$ labor units supplied at time $t$.
- We assume that $F$ is CRS and Cobb-Douglas, $Y_t = K_t^{\alpha} L_t^{1-\alpha}$.

Firms maximise their profits:

$$\max_{\{K_t, L_t\}} \Pi_t = Y_t - (r_t + \delta) K_t - w_t L_t,$$

where $\delta$ is the depreciation rate of captial and $r_t + \delta$ is the capital rental rate. FOCs imply

$$F_K(K_t, L_t) - (r_t + \delta) = 0 \quad \Rightarrow \quad r_t = \alpha \left(\frac{K_t}{L_t}\right)^{\alpha - 1} - \delta$$

$$F_L(K_t, L_t) - w_t = 0 \quad \Rightarrow \quad w_t = (1 - \alpha) \left(\frac{K_t}{L_t}\right)^{\alpha}$$

# Elasticity and Income Shares in Cobb-Douglas Function

The output elasticity and income shares for capital and labor are given by the following relationships:

- Capital Elasticity and Income Share:

$$\varepsilon_K = \frac{\partial Y}{\partial K} \cdot \frac{K}{Y} = \alpha \quad \text{and} \quad \frac{(r_t + \delta)\, K_t}{Y_t} = \alpha$$

- Labor Elasticity and Income Share:

$$\varepsilon_L = \frac{\partial Y}{\partial L} \cdot \frac{L}{Y} = 1 - \alpha \quad \text{and} \quad \frac{w_t L_t}{Y_t} = 1 - \alpha$$

# Equilibrium

- Households maximize utility, while firms maximize profits
- The the capital market, the labor market as well as the goods market clear in equilibrium:

$$K_{t+1} = \sum_{j=1}^{T} a_{j,t-j}$$

$$L_t = \sum_{j=1}^{T} l_{j,t-j}$$

$$Y_t = C_t + I_t = C_t + K_{t+1} - (1-\delta)K_t$$

- Equilibrium factor prices equal

$$r_t = \alpha \left(\frac{K_t}{L_t}\right)^{\alpha-1} - \delta$$

$$w_t = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha}$$

- Note that $C_t = \sum_{j=1}^{T} c_{j,t-j}$.

# Steady State

In a steady-state equilibrium, all variables remain constant over time, allowing us to omit the time subscript $t$:

$$K = \sum_{j=1}^{T} a_j$$

$$L = \sum_{j=1}^{T} l_j$$

$$Y = C + I = C + K' - (1-\delta)K$$

$$C = \sum_{j=1}^{T} c_j$$

$$r = \alpha \left(\frac{K}{L}\right)^{\alpha-1} - \delta \tag{4}$$

$$w = (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} \tag{5}$$

# Steady-State Solution

Since $r_t = r$ and $w_t = w$ holds in steady state, the solution for consumption simplifies to

$$c_1^* = \frac{w \sum_{j=1}^{T} \frac{l_j}{(1+r)^j} + \frac{a_{0,t}}{1+r}}{\sum_{j=1}^{T} \frac{\left[\beta\left(1+r\right)\right]^{\frac{j-1}{\rho}}}{(1+r)^j}} \tag{6}$$

$$c_j^* = c_1^* \left[\beta\left(1+r\right)\right]^{\frac{j-1}{\rho}} \quad \text{for } j > 1 \tag{7}$$

Insert the solution for consumption $c_j^*$ into the budget constraint (1) to back out the solution for savings:

$$a_j^* = wl_j + \left(1+r\right) a_{j-1} - c_j^* \tag{8}$$

# Algorithm to Find Steady-State Capital Level $K^{ss}$

1. Guess a steady-state capital stock $K'$
2. Calculate factor prices $r'$ and $w'$ given $K'$ as in equations (4) and (5)
3. Solve for $c_1$ in equation (6) given $w'$ and $r'$
4. Solve $c_j \, \forall \, j > 1$ using the Long-run Euler equation in (7):

$$c_j^* = c_1^* \left[\beta \left(1 + r'\right)\right]^{\frac{j-1}{\rho}} \quad \text{for } j > 1$$

5. Solve $a_j \, \forall \, j > 1$ using the budget constraint:

$$a_j' = w' l_j + \left(1 + r'\right) a_{j-1} - c_j$$

6. Compute implied aggregate capital: $K'' = \sum_{j=1}^{T} a_j'$
7. Check whether aggregate capital equals the guess $K'$
8. Update guess of capital until convergence, $K' \approx K''$

# Discipline Guess

Since

$$r_t = \alpha \left( \frac{K_t}{L_t} \right)^{\alpha - 1} - \delta \tag{9}$$

it must hold that

$$K_t = L_t \left( \frac{\alpha}{r_t + \delta} \right)^{\frac{1}{1-\alpha}} \tag{10}$$

which we can use to discipline the upper bound of the guess of capital since if $r_t + \delta < 0$ capital become imaginary.

# Calibration

| Parameter | Description | Value | Origin |
|---|---|---|---|
| *Timing* | | | |
| $T$ | Maximum age of life | 20 | Model period $\approx$ 4 years |
| $T_R$ | Retirement Age | 15 | |
| *Preferences* | | | |
| $\beta$ | Subj. discount factor | 0.96 | To match annual factor of 0.99 |
| $\rho$ | RRA / Inverse IES | 2.0 | Standard value |
| *Technology* | | | |
| $\alpha$ | Capital income share | 1/3 | |
| $\delta$ | Yearly depreciation rate | 0.07 | |

# Potential Overaccumulation of Capital

In OLG models, overaccumulation of capital may occur because

- When each individual saves, they only think about their own future consumption and retirement needs
- Agents do not consider how their savings contribute to the aggregate capital stock and drive down the future return on capital
- This externality may lead to an inefficient outcome for the economy as a whole.

Hence, overaccumulation of capital is directly linked to the dynamic efficiency of the economy

# Dynamic efficiency

- **Dynamic efficiency** requires

$$r \geq g,$$

where $g$ is the growth rate of the economy.

- **Dynamic Inefficiency** indicating overaccumulation of capital occurs under

$$r < g,$$

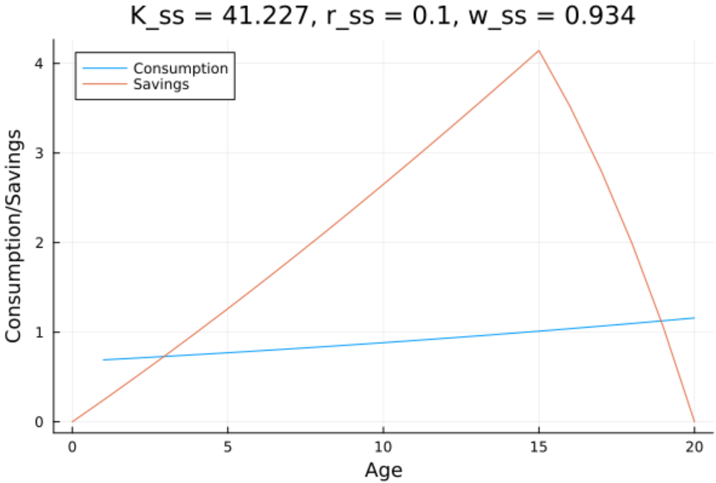
  - Here, reducing capital accumulation (saving less and consuming more) can increase welfare across generations.

- The **Golden Rule level of capital** $K^G$ maximising welfare implies

$$r(K^G) = g,$$

# Result

## Steady State



# Introducing Pay-as-you-go (PAYG) Pensions

A PAYG pension scheme is a retirement system in which current workers' contributions are used to pay for the pension benefits of current retirees

Let us assume that

1. Workers contribute a constant share of their wage income $\tau$
2. Retirees receive a constant benefit $b_t$

Then, the balanced-budget condition within PAYG pensions implies

$$\tau w_t \sum_{j=1}^{T_r} l_{j,t-j} = \sum_{j=T_r+1}^{T} b_t \tag{11}$$

# The household problem
## with a pension system

With a pension system, the household problem reads:

$$
\begin{aligned}
\max_{\{c_t^i, s_t^i\}} \quad & U = \sum_{j=1}^{T} \beta^t u(c_{j,t}) \\
& s.t. \\
c_{j,t} + a_{j,t} &= (1-\tau)\, w_{t+j} l_{j,t} + (1 + r_{t+j})\, a_{j-1,t} + (1 - l_{j,t})\, b_t \qquad (12) \\
& \text{with } l_{j,t} = \begin{cases} 1 & \text{if } j \leq T_r \\ 0 & \text{otherwise} \end{cases} \\
& \text{and } a_{0,t} = 0
\end{aligned}
$$

# Solution with Pensions

Deriving the consolidated budget constraint at birth yields:

$$\sum_{j=1}^{T} \frac{c_{j,t}}{\prod_{x=1}^{j}(1+r_{t+x})} = \sum_{j=1}^{T} \frac{(1-\tau)\, w_{t+j} l_{j,t} + (1-l_{j,t})\, b_{t+j}}{\prod_{x=1}^{j}(1+r_{t+x})} \tag{13}$$

Solve for $c_1$ Solving for $c_{1,t}$ by eliminating $c_{j,t}$ using the long-term Euler-Equation:

$$c_{1,t} = \frac{\sum_{j=1}^{T} \frac{(1-\tau)\, w_{t+j} l_{j,t} + (1-l_{j,t})\, b_{t+j}}{\prod_{x=1}^{j}(1+r_{t+x})}}{\sum_{j=1}^{T} \frac{\left[\frac{\beta^{j-1}}{(1+r_{t+j})} \prod_{x=1}^{j}(1+r_{t+x})\right]^{\frac{1}{\rho}}}{\prod_{x=1}^{j}(1+r_{t+x})}} \tag{14}$$

# Steady-state Solution

The closed-form solution for first-period consumption for given steady-state factor prices reads:

$$c_1^* = \frac{\sum_{j=1}^{T} \frac{(1-\tau)\, w l_j + (1-l_j)\, b}{(1+r)^j}}{\sum_{j=1}^{T} \frac{[\beta\,(1+r)]^{\frac{j-1}{\rho}}}{(1+r)^j}} \tag{15}$$

$$c_j^* = c_1^* \left[\beta\,(1+r)\right]^{\frac{j-1}{\rho}} \quad \text{for } j > 1 \tag{16}$$

where:

$$b = \tau w \frac{\sum_{j=1}^{T} l_j}{T - T_r} \tag{17}$$

Insert the solution for consumption $c_j^*$ into the budget constraint (12) to back out the solution for savings:

$$a_j^* = (1-\tau) w l_j + (1-l_j) b + (1+r)\, a_{j-1} - c_j^* \tag{18}$$

# Algorithm to Find Steady-State Capital Level $K^{ss}$

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$$c_j^* = c_1^* \left[ \beta \left( 1 + r' \right) \right]^{\frac{j-1}{\rho}} \quad \text{for } j > 1$$

5. Solve $a_j \, \forall \, j > 1$ using the budget constraint:

$$a_j' = (1 - \tau) w' l_j + (1 - l_j) \, b + \left( 1 + r' \right) a_{j-1} - c_j$$

6. Compute implied aggregate capital: $K'' = \sum_{j=1}^{T} a_j'$
7. Check whether aggregate capital equals the guess $K'$
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# Result

## Steady State Comparison