Lecture 1: The Life-cycle Model

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Workshop on Life-cycle Models and Pensions

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The Model

- Time is discrete and indexed by $t \in \{0, 1, 2, ..\}$
- There is no growth in technology and population size

Households

- Agents live for T periods and retire in period T_R
- Household age is discrete and indexed by $j \in \{1, 2, .., T\}$
- Save in bonds and supply labor exogenously

Variables

1. Consumption: $c_{j,t}$ (endogenous)
2. Saving (in bonds): $a_{j,t}$ (endogenous)
3. Labour supply: $l_{j,t}$ (exogenous)

Exogenous Prices (partial equilibirum)

1. Wage: $w_t = w$
2. Interest rate (on bonds): $r_t = r$

$\Rightarrow$ amounts to assuming a small open economy taking prices as given

The Household Problem

Agent i born at time t, who starts to consume in period $t+1$, faces the following problem:

$$\max_{\left\{c_t^i, a_t^i\right\}} \; U = \sum_{j=1}^{T} \beta^t u(c_{j,t}^i)$$

$$s.t.$$

$$c_{j,t}^i + a_{j,t}^i = w_{t+j} l_{j,t} + (1 + r_{t+j})\, a_{j-1,t}^i \tag{1}$$

$$\text{with } l_{j,t} = \begin{cases} 1 & \text{if } j \leq T_r \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\text{and } a_{0,t} \sim \text{Lognormal}\left(\mu, \sigma^2\right) \tag{3}$$

See Log-normal Distribution!

Constant Relative Risk Aversion (CRRA) Utility

$$u(c) = \begin{cases} \dfrac{c^{1-\rho} - 1}{1 - \rho} & \rho \geq 0, \rho \neq 1 \\ \log(c) & \rho = 1, \end{cases} \tag{4}$$

where intertemporal elasticity of substitution (IES) of consumption is define by

$$IES \equiv \frac{d\left(\frac{c_{j+1,t+1}}{c_{j,t}}\right)}{d(1+r)} \cdot \frac{1+r}{\frac{c_{j+1,t+1}}{c_{j,t}}} = \frac{1}{\rho},$$

which is non-separable from the relative risk aversion (RRA)

$$RRA \equiv -c \cdot \frac{u''(c)}{u'(c)} = \rho.$$

Use Epstein-Zin Preferences if you want to separate the two!

First Order Condition (FOC) and Consolidated Budget

Derive the period-by-period Euler-Equation:

$$c^i_{j+1,t} = c^i_{j,t} \left[\beta \left(1 + r_{t+j+1}\right)\right]^{\frac{1}{\rho}} \tag{5}$$

Derive the long-term Euler-Equation by iterating the period-by-period Euler-Equation forward:

$$c^i_{j,t} = c^i_{1,t} \left[\beta^{j-1} \prod_{x=2}^{j} \left(1 + r_{t+x}\right)\right]^{\frac{1}{\rho}} \quad \text{for } j > 1 \tag{6}$$

Derive the consolidated budget constraint at birth:

$$\underbrace{\sum_{j=1}^{T} \frac{c^i_{j,t}}{\prod_{x=1}^{j} \left(1 + r_{t+x}\right)}}_{\text{NPV of consumption}} = \underbrace{\sum_{j=1}^{T} \frac{w_{t+j} l_{j,t}}{\prod_{x=1}^{j} \left(1 + r_{t+x}\right)}}_{\text{NPV of labor income}} + \underbrace{\frac{a^i_{0,t}}{1 + r_{t+1}}}_{\text{Initial wealth}}$$

Closed-form Solution (1)

Use the consolidated budget constraint to solve for $c^i_{1,t}$ by eliminating $c^i_{j,t}$ using the long-term Euler-Equation:

$$c^{i,*}_{1,t} = \frac{\sum_{j=1}^{T} \frac{w_{t+j} l_{j,t}}{\prod_{x=1}^{j} (1+r_{t+x})} + \frac{a^i_{0,t}}{1+r_{t+1}}}{\sum_{j=1}^{T} \frac{\left[\frac{\beta^{j-1}}{(1+r_{t+1})} \prod_{x=1}^{j} (1+r_{t+x}) \right]^{\frac{1}{\rho}}}{\prod_{x=1}^{j} (1+r_{t+x})}}. \tag{7}$$

Insert the solution in (8) into the long-term Euler-Equation (6) to solve for $c^{i,*}_{j,t}$ for $j > 1$:

$$c^{i,*}_{j,t} = c^{i,*}_{1,t} \left[\beta^{j-1} \prod_{x=2}^{j} (1+r_{t+x}) \right]^{\frac{1}{\rho}} \quad \text{for } j > 1$$

Closed-form Solution (2)

Since $r_t = r$ and $w_t = w$ holds, the solution simplifies to

$$c_{1,t}^{i,*} = \frac{w \sum_{j=1}^{T} \frac{l_{j,t}}{(1+r)^j} + \frac{a_{0,t}^i}{1+r}}{\sum_{j=1}^{T} \frac{\left[\beta\,(1+r)\right]^{\frac{j-1}{\rho}}}{(1+r)^j}} = \frac{X + Y^i}{Z} \tag{8}$$

with

$$X \equiv w \sum_{j=1}^{T} \frac{l_{j,t}}{(1+r)^j} \quad Y^i \equiv \frac{a_{0,t}^i}{1+r} \quad Z = \sum_{j=1}^{T} \frac{\left[\beta\,(1+r)\right]^{\frac{j-1}{\rho}}}{(1+r)^j} \tag{9}$$

Using the Long-run Euler, we can solve for

$$c_{j,t}^{i,*} = c_{1,t}^{i,*} \left[\beta\,(1+r)\right]^{\frac{j-1}{\rho}} \quad \text{for } j > 1 \tag{10}$$

Closed-form Solution (3)

Insert the solution for consumption $c^*_{j,t}$ into the budget constraint (1) to back out the solution for savings:

$$a^*_{j,t} = w_{t+j} l_{j,t} + (1 + r_{t+j})\, a_{j-1,t} - c^*_{j,t} \tag{11}$$

$\Rightarrow$ final steps are calibration and simulation

Calibration

Parameter	Description	Value	Origin
Timing			
T	Maximum age of life	20	Model period $\approx$ 4 years
T_R	Retirement Age	15	
Prices			
R	Interest rate	0.13	To match an annual int. rate of 3%
w_t	Wage	1	Normalization
Preferences			
β	Subj. discount factor	0.96	To match annual factor of 0.99
ρ	RRA / Inverse IES	2.0	Standard value
Distribution			
μ	Location	0	
σ	Scale	1	

Result

Simulated Consumption and Savings