Lecture 4: The Transition Path

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Workshop on Life-cycle Models and Pensions

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Overview

- In Lecture 3, we analysed a steady-state equilibrium with and without a pension scheme
- We now consider the transition between two steady states before and after the introduction of a pension scheme
- We model this as an MIT shock (a shock in a world without shocks) to the pension contribution rate $\tau >$ at time t
  - Agents believe that the economy remains in the initial steady-state and will be surprised at t that τ has increased
- We will assume the exact same model as in Lecture 3

The Household Problem

Agent i born at time t, who start to consume in period $t+1$, face the following problem:

$$\max_{\{c_t^i, s_t^i\}} \; U = \sum_{j=1}^{T} \beta^t u(c_{j,t}) \quad s.t.$$

$$c_{j,t} + a_{j,t} = (1-\tau)\, w_{t+j} l_{j,t} + (1 + r_{t+j})\, a_{j-1,t} + (1 - l_{j,t})\, b_t \tag{1}$$

$$\text{with } l_{j,t} = \begin{cases} 1 & \text{if } j \leq T_r \\ 0 & \text{otherwise} \end{cases} \quad \text{and } a_{0,t} = 0 \text{ and } b_t = \tau_t w_t \frac{\sum_{j=1}^{T_r} l_{j,t-j}}{T - T_r}$$

- The decision problem of the individual is forward looking, as agents take into account expected future factor prices and benefits T periods ahead $\{w_t, r_t, b_t\}_{t=1}^{T}$
- However, as generations overlap, cohort-specific decisions are interdependent, and thus, the dynamic transition path becomes infinitely forward looking.

Truncation

The infinitely forward-looking nature of the problem poses a challenge when solving such models for a dynamic transition numerically!

- We can only work finite sequences of $\{w_t, r_{t+1}\}^t$

The Trick

We utilize an Auerbach-Kotlikoff-type algorithm that uses information on the long-run steady-state equilibrium to truncate the problem at a finite time horizon.

- In any period after this finite horizon T_{con}, the economy is assumed to have converged to the new steady state
  - Hence, due to perfect foresight all future prices are known
- To initialize the algorithm, we assume that the economy is initially in a steady state with $\tau = 0$ and has been for many periods
- Thus, we know the level of inherited capital when the shock occurs.

Algorithm for Transition Path

1. Guess a transition path vector of $\{K_t\}^t$ from one steady state to the other. For all periods on the finite transition path, $t_p \in T_{conv}$.
   – The first T entries of the vector are equal to the pre-shock, steady-state capital level, while the last T entries are equal to the post-shock, steady-state capital level.
2. Calculate associated vectors of factor prices and benefits $\{w_t, r_t, b_t\}^t$
3. For cohorts born before the shock, $t_p < t$, update savings decisions given past savings and future prices.
4. Loop over the periods after the shock and:
   4.1 Use inherited savings to produce a new guess for capital, factor prices, and pension benefits in the current period of the loop.
   4.2 For cohorts born after the shock, $t_p \geq t$, compute new savings decisions in the current period of the loop given future prices.
5. Check whether the updated transition path vector of capital is equal to the initial vector.
6. Given the updated transition path, **repeat step 3 to 5** until the entire path converges.

Adjustment

In step 3 of the solution algorithm, we allow people alive at the time of the shock to update their planned behavior given the old steady state.

Imagine that new information arrives at the start of period t. Then an individual who is aged j in that period was born in period $t-j$ and now solves:

$$\max_{\{c_{\tau,t-j},a_{\tau,t-j}\}} \quad U_{t-j} = \sum_{x=j}^{d} \beta^x u\left(c_{x,t-j}\right) \tag{2}$$

$$s.t. \tag{3}$$

$$c_{x,t-j} + a_{x,t-j} = \left(1-\tau_{t-j+x}\right) w_{t-j+x} l_{x,t-j} + \left(1+r_{t-j+x}\right) a_{x-1,t-j} + b_{t-j+x}\left(1-l_{x,t-j}\right) \tag{4}$$

for $x \geq a$ given savings, $a_{j-1,t-j}$, carried over from the previous period. The consolidated budget reads:

$$\sum_{x=j}^{d} \frac{c_{x,t-j}}{\prod_{s=j}^{x}\left(1+r_{t-j+s}\right)} = a_{j-1,t-j} + \sum_{x=j}^{d} \frac{\left(1-\tau_{t-j+x}\right) w_{t-j+x} l_{x,t-j} + b_{t-j+x}\left(1-l_{x,t-j}\right)}{\prod_{s=j}^{x}\left(1+r_{t-j+s}\right)} \tag{5}$$

Adjustment

The long-run Euler equation at age x can be written as:

$$c_{x,t-j} = \left(\beta^{x-j} \frac{\prod_{s=j}^{x} (1 + r_{t-j+s})}{1 + r_t} \right)^{\frac{1}{\rho}} c_{j,t-j} \text{ for } x > j \tag{6}$$

Thus, we can solve closed-form for consumption in the period of the shock:

$$c_{a,t-j} = \frac{a_{j-1,t-j} + \sum_{x=j}^{d} \frac{(1 - \tau_{t-j+\tau})\, w_{t-j+\tau} l_{\tau,t-j} + b_{t-j+\tau} (1 - l_{\tau,t-j})}{\prod_{s=j}^{x} (1 + r_{t-j+s})}}{\sum_{\tau=a}^{d} \frac{\left(\beta^{\tau-j} \frac{\prod_{s=j}^{\tau} (1 + r_{t-j+s})}{1 + r_t} \right)^{\frac{1}{\rho}}}{\prod_{s=j}^{\tau} (1 + r_{t-j+s})}} \tag{7}$$

and for savings

$$a_{x,t-j} = (1 - \tau_{t-j+x})\, w_{t-j+x} l_{x,t-j} + (1 + r_{t-j+x})\, a_{x-1,t-j} + b_{t-j+x} (1 - l_{x,t-j}) - c_{x,t-j} \tag{8}$$

Closed-form Solution for Agents born after shock

This is the same solution as in Lecture 3 (before assuming a steady-state):

$$c^*_{1,t} = \frac{\sum_{j=1}^{T} \frac{(1-\tau) w_{t+j} l_{j,t} + (1 - l_{j,t}) b_{t+j}}{\prod_{x=1}^{j} (1 + r_{t+x})}}{\sum_{j=1}^{T} \frac{\left[\frac{\beta^{j-1}}{(1+r_{t+j})} \prod_{x=1}^{j} (1 + r_{t+x})\right]^{\frac{1}{\rho}}}{\prod_{x=1}^{j} (1 + r_{t+x})}} \tag{9}$$

$$c^*_{j,t} = c^*_{1,t} \left[\beta^{j-1} \prod_{x=2}^{j} (1 + r_{t+x})\right]^{\frac{1}{\rho}} \quad \text{for } j > 1 \tag{10}$$

$$a^*_{j,t} = (1-\tau) w_{t+j} l_{j,t} + (1 + r_{t+j}) a^*_{j-1,t} + (1 - l_{j,t}) b_t - c^*_{j,t} \quad \text{for } j > 1 \tag{11}$$