Shaping the Energy of Mechanical Systems Without Solving Partial Differential Equations

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***Abstract*—Control of underactuated mechanical systems via energy shaping is a well-established, robust design technique. Unfortunately, its application is often stymied by the need to solve partial differential equations (PDEs). In this technical note a new, fully constructive, procedure to shape the energy for a class of mechanical systems that obviates the solution of PDEs is proposed. The control law consists of a first stage of partial feedback linearization followed by a simple proportional plus integral controller acting on two new passive outputs. The class of systems for which the procedure is applicable is identified imposing some (directly verifiable) conditions on the systems inertia matrix and its potential energy function. It is shown that these conditions are satisfied by three benchmark examples.**

***Index Terms*—Mechanical systems, nonlinear systems, passivity-based control, stabilization.**

I. Introduction

Stabilization (of a desired equilibrium) of underactuated mechanical systems by shaping their energy function and *preserving the systems structure* has attracted the attention of control researchers for several years now. Potential energy shaping for fully actuated mechanical systems was first introduced in [1] more than 30 years ago. To shape the potential energy for underactuated systems it is sometimes necessary to modify also its kinetic energy, a procedure that is called total energy shaping. The idea of total energy shaping was first introduced in [2] with the two main approaches being now: the method of controlled Lagrangians [3] and interconnection and damping assignment passivity-based control [4], see also the closely related work [5]. In both cases stabilization is achieved identifying the class of systems—Lagrangian for the first method and Hamiltonian for the latter—that can possibly be obtained via feedback.

The conditions under which such a feedback law exists are called *matching equations*, and consist of a set of nonlinear partial differential equations (PDEs). A lot of research effort has been devoted to the solution of the matching equations—see, e.g., [6]–[12]. Also, there is a large list of applications where it has been possible to solve these equations, including (almost) all basic pendular systems considered in the literature, motors, generators, power systems, power converters, level control systems, etc.—see [13] for a partial list. In spite of that, this difficult key step remains the main stumbling block for the wider application of these methods.

For mechanical systems the matching equations PDEs, which identify the kinetic and potential energies that are achievable via feedback, arise because of our desire to preserve in closed-loop the mechanical structure of the system. In this technical note we *relax* this constraint, and concentrate our attention on the energy shaping objective only. That is, we look for a static state-feedback that stabilizes the desired equilibrium assigning to the closed loop a Lyapunov function of the same form as the energy function of the open-loop system but with new, desired inertia matrix and potential energy function. However, we *do not require* that the closed-loop system is a mechanical system with this Lyapunov function qualifying as its energy function. In this way, the need to solve the matching equations is avoided.

The controller design proceeds in three steps, first the application of a (collocated) partial feedback linearization stage, *à la* [14]. Then, as done in [15], the identification of conditions on the inertia matrix and the potential energy function that ensure the Lagrangian structure is preserved. As a corollary of the Lagrangian structure preservation two new passive outputs are immediately identified [16], [17]. Finally, a proportional plus integral (PI) controller around this passive output is applied. Our main contribution is the identification of a class of mechanical systems for which this procedure is applicable. The class is identified imposing some (directly verifiable) conditions on the systems inertia matrix and its potential energy function. It is shown that these conditions are satisfied by three benchmark examples.

The remainder of the technical note is organized as follows. Section II presents the model of the system and formulates the new energy shaping problem. Some preliminaries on partial linearization and the corresponding passive outputs are briefly recalled in Section III. Section IV contains the main stabilization result. Three representative examples are presented in Section V. The technical note is wrapped-up with future research in Section VI.

Notation: I_n is the $n \times n$ identity matrix and $0_{n\times s}$ is an $n \times s$ matrix of zeros. For $x \in \mathbb{R}^n$, $S \in \mathbb{R}^{n\times n}$, $S = S^\top > 0$, we denote the Euclidean norm $|x|^2 := x^\top x$, and the weighted—norm $\|x\|_S^2 := x^\top Sx$. For any matrix $A \in \mathbb{R}^{n\times n}$, $(A)_i \in \mathbb{R}^n$ denotes the i-th column, $(A)^i$ the i-th row and $(A)_{ij}$ the ij-th element. $e_i \in \mathbb{R}^n$, $i \in \bar{n} := \{1, 2, \ldots, n\}$ is the Euclidean basis vectors. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, we define the differential operators $\nabla f := (\partial f/\partial x)^\top$, $\nabla_{x_i} f := (\partial f/\partial x_i)^\top$, where $x_i \in \mathbb{R}^p$ is an element of the vector x.

II. Formulation of the Energy Shaping Problem

In this technical note, we consider underactuated mechanical systems whose dynamics is described by the well-known Euler–Lagrange

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equations of motion

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla V(q) = G\tau \tag{1}$$

where $q \in \mathbb{R}^n$ are the configuration variables,[1] $\tau \in \mathbb{R}^m$ are the control signals, $M(q) > 0$ is the generalized inertia matrix, $C(q.\dot{q})\dot{q}$ represent the Coriolis and centrifugal forces, $V(q)$ is the systems potential energy and we assume the input matrix is of the form $G = \begin{bmatrix} I_m \\ 0_{(n-m)\times m} \end{bmatrix}$.

In 1788, Lagrange stated the following proposition, which was later proved by Dirichlet, see [18].

Proposition 1: Assume $\bar{q} = \arg\min V(q)$ and the minimum is isolated. The equilibrium $(q,\dot{q}) = (\bar{q}, 0)$ of (1) with $\tau = 0$ is Lyapunov stable with Lyapunov function the total energy of the system, that is, $H(q,\dot{q}) = (1/2)\dot{q}^\top M(q)\dot{q} + V(q)$.

Motivated by Proposition 1, the energy-shaping control problem that we consider in this technical note is formulated as follows.

Problem Formulation: Find a mapping $\hat{\tau} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ such that the system (1) in closed loop with the static state-feedback control law $\tau = \hat{\tau}(q,\dot{q})$ has a stable equilibrium at the desired point $(q,\dot{q}) = (q_*, 0)$ with Lyapunov function

$$H_d(q,\dot{q}) = \frac{1}{2}\dot{q}^\top M_d(q)\dot{q} + V_d(q) \tag{2}$$

where

C1. $M_d(q) > 0$.

C2. $q_\star = \arg\min V_d(q)$, and the minimum is isolated.

It is important to underscore that the energy shaping problem formulated above is different from the one addressed in the controlled Lagrangian [3] and the interconnection and damping assignment passivity-based control [4] literature. In these two techniques, it is additionally imposed that the closed-loop system remains a mechanical system with total energy (2). This additional, structure-preserving, requirement is obviated here.

III. Preliminaries

As indicated in the Introduction, the first two steps in our design are the application of a partial feedback linearization stage, *à la* [14]. Then, as done in [15], the identification of conditions on $M(q)$ and $V(q)$ that ensure the Lagrangian structure is preserved, which has as a corollary the definition of the two new passive outputs suggested in [16], [17]. For the sake of completeness, these two steps are recalled below.

A. Partial Linearization

To simplify the notation, and following [14], we find convenient to partition the generalized coordinates into its actuated and unactuated part, that is, $q = \mathrm{col}(q_a, q_u)$, with $q_a \in \mathbb{R}^m$ and $q_u \in \mathbb{R}^{n-m}$. Conformally, we also partition the inertia matrix as

$$M(q) = \begin{bmatrix} m_{aa}(q) & m_{au}(q) \\ m_{au}^\top(q) & m_{uu}(q) \end{bmatrix} \tag{3}$$

where $m_{aa}(q) \in \mathbb{R}^{m\times m}$, $m_{au}(q) \in \mathbb{R}^{m\times(n-m)}$ and $m_{uu}(q) \in \mathbb{R}^{(n-m)\times(n-m)}$.

Proposition 2 [14]: The underactuated system (1) is feedback equivalent to

$$\begin{aligned} &\ddot{q}_a = u \\ &m_{uu}(q)\ddot{q}_u + \begin{bmatrix} 0_{(n-m)\times m} & I_{n-m} \end{bmatrix} C(q,\dot{q})\dot{q} \\ &\quad + \nabla_{q_u} V(q) = -m_{au}^\top(q)u. \end{aligned} \tag{4}$$

[1]To simplify the notation, we assumed that the system lives in the Euclidean space. This can be modified without affecting the main results.

That is, there exists a mapping $\hat{\tau}_{pl} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ such that the system (1) in closed loop with the static state-feedback control law $\tau = \hat{\tau}_{pl}(q,\dot{q}) + u$ takes the form (4).

B. Preservation of Lagrangian Structure

As done in [15] consider the following assumptions.

A1. The inertia matrix depends only on the unactuated variables q_u, i.e., $M(q) = M(q_u)$.

A2. The sub-block matrix m_{aa} of the inertia matrix is constant.

A3. The potential energy can be written as

$$V(q) = V_a(q_a) + V_u(q_u).$$

A4. The columns of the matrix $m_{au}(q_u)$ satisfy

$$\nabla_{q_{u_j}}(m_{au})_k = \nabla_{q_{u_k}}(m_{au})_j,\ \forall j \neq k,\ j,k \in \{1,2,\ldots,n-m\}$$

or, equivalently, the rows of $m_{au}(q_u)$ satisfy

$$\nabla(m_{au})^i = \left[\nabla(m_{au})^i\right]^\top,\ \forall i \in \{1,2,\ldots,m\}.$$

That is the rows of $m_{au}(q_u)$ are gradient vector fields, which is equivalent to the existence of a function $V_N : \mathbb{R}^{n-m} \to \mathbb{R}^m$ such that

$$\dot{V}_N = -m_{au}(q_u)\dot{q}_u. \tag{5}$$

Note that under Assumptions **A1**–**A3** the left-hand side of the second equation in (4) depends only on q_u. Therefore, it is clear that if $m_{au}(q_u) = 0$ then the system is uncontrollable. Hence, in the sequel we will assume that $m_{au}(q) \neq 0$.

Proposition 3 [15]: Consider the system (4) verifying Assumptions **A1**–**A4**.

P1. The system satisfies the Euler–Lagrange equations with Lagrangian function

$$\tilde{\mathcal{L}}(q,\dot{q}) = \frac{1}{2}\begin{bmatrix} \dot{q}_a^\top & \dot{q}_u^\top \end{bmatrix} \begin{bmatrix} I_m & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & m_{uu}(q_u) \end{bmatrix} \times \begin{bmatrix} \dot{q}_a \\ \dot{q}_u \end{bmatrix} - V_u(q_u) \tag{6}$$

and input matrix $\tilde{G}(q_u) = \begin{bmatrix} I_m \\ -m_{au}^\top(q_u) \end{bmatrix}$.

P2. It may be written as

$$\begin{aligned} &\ddot{q}_a = u \\ &m_{uu}(q_u)\ddot{q}_u + c_u(q_u,\dot{q}_u)\dot{q}_u + \nabla V_u(q_u) = -\, m_{au}^\top(q_u)u \end{aligned} \tag{7}$$

where the key skew-symmetry property

$$\dot{m}_{uu}(q_u) = c_u(q_u,\dot{q}_u) + c_u^\top(q_u,\dot{q}_u)$$

is satisfied.

P3. The mapping $u \mapsto \dot{q}_a - m_{au}^\top(q_u)\dot{q}_u$ is cyclo-passive with storage function

$$\tilde{H}(q,\dot{q}) = \frac{1}{2}\begin{bmatrix} \dot{q}_a^\top & \dot{q}_u^\top \end{bmatrix} \begin{bmatrix} I_m & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & m_{uu}(q_u) \end{bmatrix} \times \begin{bmatrix} \dot{q}_a \\ \dot{q}_u \end{bmatrix} + V_u(q_u). \tag{8}$$

Moreover, the mapping is passive if $V_u(q_u)$ is bounded from below.

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C. New (Cyclo) Passive Outputs

As observed in [16] and [17], we have the following corollary.

Corollary 1: Consider the functions

$$H_a(\dot{q}_a) = \frac{1}{2}|\dot{q}_a|^2$$
$$H_u(q_u, \dot{q}_u) = \frac{1}{2}\dot{q}_u^\top m_{uu}(q_u)\dot{q}_u + V_u(q_u). \tag{9}$$

The time derivatives of these functions along the solutions of (7) verify

$$\dot{H}_a = u^\top y_a, \quad \dot{H}_u = u^\top y_u \tag{10}$$

with $y_a := \dot{q}_a$, and $y_u := -m_{au}(q_u)\dot{q}_u$.

The corollary above, which exploits the decoupled structure of the system (7), plays a central role in the controller design. Indeed, as will become clear below, to shape the energy of the open-loop system we will add a *weighted sum* of the storage functions $H_a(\dot{q}_a)$ and $H_u(q_u, \dot{q}_u)$ and add a PI controller around the weighted sum of the (cyclo)passive outputs y_a and y_u, this will give more flexibility to achieve the Lyapunov function assignment objective.

IV. Main Result

In this section, we present the proposed controller design methodology, state the assumptions required to identify the class of systems for which it is applicable and discuss their implication.

A. Proposed Controller Design

To present the main result the following additional assumption is needed.

A5. There exist

$$k_e, k_a, k_u \in \mathbb{R}, \quad K_k, K_I \in \mathbb{R}^{m\times m},\ K_k, K_I \geq 0$$

such that the following holds.

(**a**) $\det[K(q_u)] \neq 0,\ \forall q_u \in \mathbb{R}^{n-m}$, where $K : \mathbb{R}^{n-m} \to \mathbb{R}^{m\times m}$ is defined as

$$K(q_u) := k_e I_m + k_a K_k + k_u K_k m_{au}(q_u) m_{uu}^{-1}(q_u) m_{au}^\top(q_u). \tag{11}$$

(**b**) The matrix

$$M_d(q_u) := \begin{bmatrix} k_e k_a I_m + k_a^2 K_k & -k_a k_u K_k m_{au}(q_u) \\ -k_a k_u m_{au}^\top(q_u) K_k^\top & M_d^{22}(q_u) \end{bmatrix} \tag{12}$$

with

$$M_d^{22}(q_u) := k_e k_u m_{uu}(q_u) + k_u^2 m_{au}^\top(q_u) K_k m_{au}(q_u)$$

and the function

$$V_d(q) := k_e k_u V_u(q_u) + \frac{1}{2}\left\|k_a q_a + k_u V_N(q_u)\right\|^2_{K_I} \tag{13}$$

satisfy conditions **C1** and **C2** of the Problem Formulation, respectively.

Proposition 4: Given the underactuated mechanical system (1) with $M(q)$ and $V(q)$ satisfying Assumptions **A1**–**A5**. there exists a mapping $\hat{\tau} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ such that the system (1) in closed loop with the static state-feedback control law $\tau = \hat{\tau}(q, \dot{q})$ has a *globally stable* equilibrium at the desired point $(q, \dot{q}) = (q_\star, 0)$ with Lyapunov function (2), (12), (13) Moreover, $(q_\star, 0)$ is globally *asymptotically* stable if

$$y := k_a \dot{q}_a - k_u m_{au}(q_u)\dot{q}_u$$

is a detectable output of the closed-loop system. That is, if the implication below holds

$$y(t) \equiv 0 \quad \Rightarrow \quad \lim_{t\to\infty}(q(t), \dot{q}(t)) = (q_\star, 0).$$

Proof: First, we apply the partial feedback linearization of Proposition 2, that is, we set[2]

$$\tau = \hat{\tau}_{pl}(q, \dot{q}) + u.$$

This transforms the system into Spong's Normal Form (4) that, under Assumptions **A1**–**A4**, takes the decoupled form (7).

Second, some simple calculations show that the function $H_d(q, \dot{q})$ can be written as follows:

$$\begin{aligned} H_d(q, \dot{q}) &= k_e\left[k_a H_a(\dot{q}_a) + k_u H_u(q_u, \dot{q}_u)\right] \\ &\quad + \frac{1}{2}\|k_a y_a + k_u y_u\|^2_{K_k} + \frac{1}{2}\left\|k_a q_a + k_u V_N(q_u)\right\|^2_{K_I}. \end{aligned} \tag{14}$$

Then, its time derivative is

$$\begin{aligned} \dot{H}_d &= (k_a y_a + k_u y_u)^\top \\ &\quad \times \left[k_e u + K_k(k_a u + k_u \dot{y}_u)\ + K_I \int (k_a y_a + k_u y_u) dt\right] \\ &= (k_a y_a + k_u y_u)^\top \\ &\quad \times \left[(k_e I_m + k_a K_k + k_u K_k m_{au} m_{uu}^{-1} m_{au}^\top) u\right. \\ &\qquad + k_u K_k\left[-\dot{m}_{au}\dot{q}_u + m_{au} m_{uu}^{-1}(c_u \dot{q}_u + \nabla V_u)\right] \\ &\qquad \left. + K_I(k_a q_a + k_u V_N)\right] \\ &= (k_a y_a + k_u y_u)^\top\left[K(q_u) u + S(q, \dot{q})\right] \end{aligned} \tag{15}$$

where to get the first identity we invoked Corollary 1, the facts that $\dot{y}_a = u$, $y_a = \dot{q}_a$ and (5), while to get the second equation we use the dynamics of the partially linearized system (7) and defined

$$\begin{aligned} S(q, \dot{q}) := k_u K_k\left[-\dot{m}_{au}\dot{q}_u + m_{au} m_{uu}^{-1}(c_u \dot{q}_u + \nabla V_u)\right] \\ + K_I(k_a q_a + k_u V_N). \end{aligned}$$

Invoking Assumption **A5 (a)**, we can select

$$u = -K^{-1}(q_u)\left[S(q, \dot{q}) + K_p(k_a y_a + k_u y_u)\right] \tag{16}$$

with $K_p \in \mathbb{R}^{m\times m},\ K_p > 0$, which replaced in (15) yields

$$\dot{H}_d = -\|k_a y_a + k_u y_u\|^2_{K_p}. \tag{17}$$

The proof is completed noting that, under Assumption **A5 (b)**, the function $H_d(q, \dot{q})$ is a proper Lyapunov function and invoking standard Lyapunov stability arguments [19]. ■

B. Discussion

R1. The role of the tuning gains k_e, k_a, k_u and K_k, K_I in the energy shaping stage is clear from the expressions of $M_d(q_u)$ and $V_d(q)$ given in (12) and (13), respectively. It is important to highlight that there is no sign constraint on the scalar quantities, which gives a large flexibility to shape the energy functions—see (19) in the pendulum on a car example of Section V-B where the open-loop potential energy is simply flipped around to assign to $V_d(q)$ the desired minimum.

[2]See [14] for the explicit expression of $\hat{\tau}_{pl}(q, \dot{q})$.

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R2. Assumption **A4** is imposed to be able to add the new term $(1/2)\|k_a q_a + k_u V_N(q_u)\|^2_{K_I}$ to the desired potential energy function, see (13). This motivates the name $V_N(q_u)$. Noting that

$$k_a q_a(t) + k_u V_N(q_u(t)) = \int_0^t \left[k_a y_a(s) + k_u y_u(s)\right] ds$$

the energy shaping is then achieved adding an integral term, weighted by K_I, to the controller. If this additional degree of freedom is not required, K_I is set to zero, and the assumption is obviated.

R3. If $K_k = 0$ the control law u becomes a simple PI of the form

$$u = -\frac{1}{k_e}\left(K_p + \frac{1}{p}K_I\right)(k_a y_a + k_u y_u)$$

where $p := d/dt$. Moreover, as seen from (12), the inertia matrix $M_d(q_u)$ takes a simple diagonal form. This is the case of the overhead crane example of Section V-C.

V. Examples

A. Inertia-Wheel Pendulum

As shown in [4], this system has $n = 2$ and can, after partial-feedback linearization and a change of coordinates, be represented in the form

$$\begin{aligned} \ddot{q}_a &= u \\ \ddot{q}_u - m_3 \sin(q_u) &= -u \end{aligned}$$

where we have taken the moments of inertia of the disk and pendulum equal to one, and $m_3 := mg\ell$, with m the pendulum mass and ℓ its length and g the gravity constant, see [4, equation (5.1)]. The desired equilibrium position is $q_\star = (0, 0)$.

From (5), we obtain $V_N(q_u) = -q_u$. From (11)–(13) and setting $k_a = K_I = 1$, it results that

$$\begin{aligned} K &= k_e + k_u K_k + K_k \\ M_d &= \begin{bmatrix} k_e + K_k & -k_u K_k \\ -k_u K_k & k_e k_u + k_u^2 K_k \end{bmatrix} \\ V_d(q) &= k_e k_u m_3 \cos(q_u) + \frac{1}{2}(q_a - k_u q_u)^2. \end{aligned}$$

To verify Assumption **A5**, we see that

$$(k_e + K_k) > 0 \wedge k_e k_u (k_e + k_u K_k + K_k) > 0 \Rightarrow M_d > 0.$$

Moreover

$$k_e k_u < 0 \Rightarrow q_\star = \arg\min V_d(q).$$

To satisfy $K \neq 0$ and the two inequalities above, we chose $k_e > 0$, $K_k > 0$, and

$$k_u < -K_k^{-1}(k_e + K_k).$$

The control law is obtained from (16) as follows:

$$u = -K^{-1}\left[k_u K_k \nabla V(q_u) + (q_a - k_u q_u) - K_p(\dot{q}_a - k_u \dot{q}_u)\right] \tag{18}$$

with $K_p > 0$. As an example, the choice $k_e = 1/3$, $k_u = -3$, $K_k = 2/3$ yields $K = -1$

$$u = -2\nabla V + q_a + 3q_u - K_p(\dot{q}_a + 3\dot{q}_u)$$

and that satisfies Assumption **A5**.

As stated in Proposition 4, this controller, added to the partial linearizing feedback, ensures global stability of the desired equilibrium. To establish asymptotic stability we apply La Salle's invariance principle, looking at the dynamics restricted to the set where

$$y = -K_p(\dot{q}_a - k_u \dot{q}_u)^2 \equiv 0.$$

which implies that $q_a - k_u q_u = \kappa$, where κ is a constant. Then, it follows that $\ddot{q}_a - k_u \ddot{q}_u = 0$ that, in its turn, implies $\nabla V(q)$ is constant, which implies that q_u and q_a are constant, completing the proof.

A globally asymptotically stabilizing interconnection and damping assignment controller for this system was reported in [4]. As always with this design technique, it was necessary to solve a PDE. In this case, with the desired constant inertia matrix

$$\tilde{M}_d = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} > 0,$$

the PDE takes the form

$$G^\perp \left[\nabla V(q) - \tilde{M}_d M^{-1} \nabla \tilde{V}_d(q)\right] = 0$$

where $G^\perp$ is a full rank, left annihilator of G and $\tilde{V}_d(q)$ is the desired potential energy. As shown in [4], this PDE becomes

$$-m_3 \sin(q_u) = (a_1 + a_2)\nabla_{q_a} \tilde{V}_d(q) + (a_2 + a_3)\nabla_{q_u} \tilde{V}_d(q)$$

that can easily be solved. The resulting controller is also of the form (18) but with different ranges for the tuning parameters. The important point here is that with the proposed design method there is no need to solve PDEs.

B. Inverted Pendulum on a Car

For this classical example, we have $n = 2$

$$\begin{aligned} M(q_u) &= \begin{bmatrix} M_c + m & m\ell\cos(q_u) \\ m\ell\cos(q_u) & m\ell^2 \end{bmatrix} \\ V(q_u) &= mg\ell\cos(q_u),\ G = e_1 \end{aligned}$$

where q_a is the position of the car and q_u denotes the angle of the pendulum with respect to the up-right vertical position, M_c is the mass of the car, m is the mass of the pendulum and ℓ its length. We want to stabilise the up-right vertical position of the pendulum and place the cart at the origin, consequently, $q_\star = (0, 0)$.

From (12) and (13) and setting $k_a = 1$ and $K_I = 0$, it results

$$\begin{aligned} K(q_u) &= k_e + K_k + k_u K_k m \cos^2(q_u) \\ M_d(q_u) &= \begin{bmatrix} k_e + K_k & -k_u K_k m\ell\cos(q_u) \\ -k_u K_k m\ell\cos(q_u) & M_d^{22}(q_u) \end{bmatrix} \\ V_d(q_u) &= k_e k_u mg\ell\cos(q_u) \end{aligned} \tag{19}$$

where $M_d^{22}(q_u) = k_e k_u m\ell^2 + k_u^2 K_k [m\ell\cos(q_u)]^2$. Note that[3]

$$k_e k_u < 0 \Rightarrow 0 = \arg\min V_d(q).$$

Also

$$\left.\begin{array}{c} k_e + K_k > 0 \\ k_e K_k k_u^2 m \cos^2(q_u) > -k_e k_u (k_e + K_k) \end{array}\right\} \Rightarrow M_d(q_u) > 0.$$

[3]From (5), we obtain $V_N(q_u) = -ml\sin(q_u)$. Hence, taking $K_I \neq 0$ generates a new family of desired potential energy functions. This additional degree of freedom is not exploited here.

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The control law is obtained from (16) as follows:

$$u = \frac{1}{k_e + K_k + k_u K_k m \cos^2(q_u)} \left[-k_u K_k m \sin(q_u)\right. \times \left[\ell \dot{q}_u^2 - g\cos(q_u)\right] - K_p(\dot{q}_a - k_u m\ell \cos q_u \dot{q}_u)\Big]. \quad (20)$$

It is easy to show that there is no choice of the gains k_e, k_u and K_k that will satisfy the three inequalities above and ensure $|K(q_u)| > 0$ for all $q_u \in [-\pi, \pi]$—being this possible only for $q_u \in (-\pi/2, \pi/2)$. Therefore, stability of the desired equilibrium is only local.

Stabilizing energy shaping controllers for this system have already been reported in the literature. In [6], the interconnection and damping assignment technique was used, while in [3] the design was done with the controlled Lagrangian method. Both designs involve the solution of PDEs.[4] It is interesting to note that the controller reported in [3] *exactly coincides* with (20)—see equation (1.18) in that paper. This similarity suggests a deeper connection between the controlled Lagrangian method and the one proposed here, which we are currently investigating. On the other hand, the controller reported in [6], given in equation (47) of that paper, is completely different from (20).

C. 4-DOF Overhead Crane

This is a 4-dof underactuated mechanical system analysed in [16]. The generalized coordinates are $q = \mathrm{col}(x, y, \theta_x, \theta_y)$, with x, y the horizontal and vertical position of the troley, and θ_x and θ_y the payload angles. The control signals are the forces acting on the trolley, that is, $u = \mathrm{col}(F_x, F_y)$. The inertia and input matrices are

$$M = \begin{bmatrix} m+m_x & 0 & mlC_xC_y & -mlS_xS_y \\ 0 & m+m_y & 0 & mlC_y \\ mlC_xC_y & 0 & ml^2C_y^2 & 0 \\ -mlS_xS_y & mlC_y & 0 & ml^2 \end{bmatrix}$$
$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the potential energy is

$$V(\theta_x, \theta_y) = mgl(1 - C_xCy) \quad (21)$$

where S_x, C_x, S_y and C_y stand for $\sin(\theta_x)$, $\cos(\theta_x)$, $\sin(\theta_y)$, and $\cos(\theta_y)$, respectively. A full description of the system can be found in [20].

The control objective is to stabilize the system at the desired position $q_\star = (x_\star, y_\star, 0, 0)$.

From (5), we obtain

$$V_N(\theta_x, \theta_y) = \mathrm{col}(mlS_xC_y + c_1, mlS_y + c_2)$$

where c_1 and c_2 are integration constants to be chosen. From (12) and (13), setting $K_k = 0$ and $k_a = 1$, it results

$$M_d = \begin{bmatrix} k_e & 0 & 0 & 0 \\ 0 & k_e & 0 & 0 \\ 0 & 0 & k_e k_u ml^2 C_y^2 & 0 \\ 0 & 0 & 0 & k_e k_u ml^2 \end{bmatrix}$$
$$V_d = \frac{1}{2}\left\|\mathrm{col}(x - x_\star - k_u mlS_xC_y, y - y_\star - k_u mlS_y)\right\|_{K_I}^2 + k_e k_u mgl(1 - C_xCy).$$

Note that for physical reasons C_y cannot be zero, then $M_d > 0$ if and only if $k_e > 0$ and $k_u > 0$. We choose $k_u = 1$, and we obtain the control law from (16) as follows:

$$u = -\frac{1}{k_e}\begin{bmatrix} K_{I1}(x - x_\star - mlS_xC_y) \\ K_{I2}(y - y_\star - mlS_y) \end{bmatrix} - \frac{K_p}{k_e}\begin{bmatrix} \dot{x} - mlC_xC_y\dot{\theta}_x + mlS_xS_y\dot{\theta}_y \\ \dot{y} - mlC_y\dot{\theta}_y \end{bmatrix}.$$

As stated in Proposition 4, this controller, added to the partial linearizing feedback, ensures global stability of the desired equilibrium.

VI. Future Work

Two lines of research that are currently being pursued are: first, to explore the preservation of *port-Hamiltonian*, instead of Lagrangian, structure after partial feedback linearization. More precisely, from Proposition 3 it is clear that the systems satisfying Assumptions **A1**–**A4** can also be written in port-Hamiltonian form as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ -I_n & 0_{n\times n} \end{bmatrix} \nabla W(q,p) + \begin{bmatrix} 0_{n\times m} \\ \tilde{G}(q_u) \end{bmatrix} u$$

with energy function

$$W(q,p) := \frac{1}{2} p^\top \begin{bmatrix} I_m & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & m_{uu}^{-1}(q_u) \end{bmatrix} p + V_u(q_u).$$

The question is whether we can relax Assumptions **A1**–**A4** if we consider the more general class of systems

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ -I_n & J(q,p) \end{bmatrix} \nabla W(q,p) + \begin{bmatrix} 0_{n\times m} \\ \tilde{G}(q_u) \end{bmatrix} u$$

where $J(q,p)$ is skew-symmetric.

Second, we are investigating the use of non-collocated, instead of collocated, partial feedback linearization, whose construction has been reported in [14]. The first step in this direction is the derivation of the equivalent of Proposition 3, that is, the identification of the systems whose Lagrangian (or Hamiltonian) structure is invariant to this kind of feedback.

A geometric interpretation of the proposed method has been reported in [16], [17]. As discussed in Section V-B, there seems to be a deep connection between our construction and the controlled Lagrangian method of [3]. Further investigations are needed to clarify this issue.

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References

[1] M. Takegaki and S. Arimoto, "A new feedback method for dynamic control of manipulators," *J. Dynam. Syst., Measur., Control*, vol. 103, no. 2, pp. 119–125, 1981.

[2] A. Ailon and R. Ortega, "An observer-based controller for robot manipulators with flexible joints," *Syst. & Control Lett.*, vol. 21, no. 4, pp. 329–335, 1993.

[3] A. Bloch, N. Leonard, and J. Marsden, "Controlled Lagrangians and the stabilization of mechanical systems," *IEEE Trans. Autom. Control*, vol. 45, no. 12, pp. 2253–2270, Dec. 2000.

[4] R. Ortega, M. Spong, F. Gomez-Estern, and G. Blankenstein, "Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment," *IEEE Trans. Autom. Control*, vol. 47, no. 8, pp. 1218–1233, Aug. 2002.

[4] It should be mentioned that in [6] *explicit solutions* of the PDEs for a class of mechanical systems, which includes the pendulum on a car, are given.

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[5] K. Fujimoto and T. Sugie, "Canonical transformations and stabilization of generalized Hamiltonian systems," *Syst. & Control Lett.*, vol. 42, no. 3, pp. 217–227, 2001.

[6] J. Acosta, R. Ortega, A. Astolfi, and A. Mahindrakar, "Interconnection and damping assignment passivity-based control of mechanical systems with underactuation degree one," *IEEE Trans. Autom. Control*, vol. 50, no. 12, pp. 1936–1955, Dec. 2005.

[7] D. Auckly and L. Kapitanski, "On the λ-equations for matching control laws," *SIAM J. Control and Optimiz.*, vol. 41, no. 5, pp. 1372–1388, 2002.

[8] G. Blankenstein, R. Ortega, and A. V. D. Schaft, "The matching conditions of controlled Lagrangians and interconnection assigment passivity based control," *Int. J. Control*, vol. 75, no. 9, pp. 645–665, 2002.

[9] A. Lewis, "Notes on energy shaping," in *Proc. 43rd IEEE Conf. Decision and Control*, Paradise Island, Bahamas, 2004.

[10] A. Mahindrakar, A. Astolfi, R. Ortega, and G. Viola, "Further constructive results on interconnection and damping assignment control of mechanical systems: The acrobot example," *Int. J. Robust and Nonlin. Control*, vol. 16, no. 14, pp. 671–685, 2006.

[11] M. S. K. Nunna and A. Astolfi, "Constructive interconnection and damping assignment for port-controlled Hamiltonian systems," *IEEE Trans. Autom. Control*, vol. 60, no. 9, pp. 2350–2361, Sep. 2015.

[12] G. Viola, R. Ortega, R. Banavar, J. Acosta, and A. Astolfi, "Total energy shaping control of mechanical systems: Simplifying the matching equations via coordinate changes," *IEEE Trans. Autom. Control*, vol. 52, no. 6, pp. 1093–1099, Jun. 2007.

[13] R. Ortega and E. Garcia-Canseco, "Interconnection and damping assignment passivity-based control: A survey," *Eur. J. Control*, vol. 10, no. 5, pp. 432–450, 2004.

[14] M. Spong, "Partial feedback linearization of underactuated mechanical systems," in *Proc. 1994 IEEE/RSJ Int. Conf. Intelligent Robots and Systems*, Munich, Germany, 1994, pp. 314–321.

[15] I. Sarras, J. Acosta, R. Ortega, and A. Mahindrakar, "Constructive immersion and invariance stabilization for a class of underactuated mechanical systems," *Automatica*, vol. 49, no. 5, pp. 1442–1448, 2013.

[16] R. Mehra, S. Satpute, F. Kazi, and N. Singh, "Geometric-PBC based control of 4-dof underactuated overhead crane system," in *Proc. 21st Int. Symp. Mathematical Theory of Networks and Systems*, Groningen, The Netherlands, 2014.

[17] R. Mehra, S. Satpute, F. Kazi, and N. Singh, "Geometric-PBC approach for control of circular ball and beam system," in *Proc. 21st Int. Symp. Mathematical Theory of Networks and Systems*, Groningen, The Netherlands, 2014.

[18] R. Ortega, A. Loria, P. Nicklasson, and H. Sira-Ramirez, *Passivity-Based Control of Euler-Lagrange Systems: Mechanical, Electrical, Electromechanical Applications*. London, U.K.: Springer-Verlag, 1998, ser. Communications and Control Engineering.

[19] H. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 2000.

[20] N. Sun, Y. Fang, and X. Zhang, "Energy coupling output feedback control of a 4-dof underactuated cranes with saturated inputs," *Automatica*, vol. 49, no. 5, pp. 1318–1325, 2013.

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