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Physical Damping in IDA-PBC Controlled Underactuated Mechanical Systems

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Energy shaping and passivity-based control designs have proven to be effective in solving control problems for underactuated mechanical systems. In recent works, interconnection and damping assignment passivity-based control (IDA-PBC) has been successfully applied to open-loop conservative models, that is, with no physical damping (e.g. friction) present. In a number of cases, in particular when IDA-PBC control only involves potential energy shaping, the actual presence of physical damping will not compromise the achieved closed-loop stability. However, when IDA-PBC control also includes the shaping of the kinetic energy, *closed-loop stability or even passivity for the model without physical damping may be lost if physical damping is present. This raises two fundamental questions. First, in which cases is the IDA-PBC controlled system designed on the basis of the undamped model still stable and passive when physical damping is present? Second, if this is not the case, when is it possible to redesign the IDA-PBC control law for the undamped systems such that stability and passivity are regained? This paper provides necessary and sufficient conditions for the existence of such a control redesign for a particular choice of the closed-loop energy function. Furthermore, if these conditions are satisfied then two methods for redesign are presented, which can be chosen depending on the problem structure and the parameter uncertainties.*

Finally, even in the cases where the addition of physical damping does not hamper the stability properties of the IDA-PBC design based on the undamped model, we show that the aforementioned redesign is still useful in order to reduce the mathematical complexity in exponential and asymptotic stability analysis.

Keywords: Energy Shaping; Passivity; Physical Damping; Port-Hamiltonian Systems; Mechanical Systems

1. Introduction

In energy shaping control techniques for underactuated systems, physical energy dissipation effects that are neglected in the control design procedure may enter the system in a way that can compromise stability. It has been proved that in control methods that modify the kinetic energy (thus leading to a non-canonical symplectic structure), such as interconnection and damping assignment passivity-based control (IDA-PBC) control (Ortega et al. 2002a, Van der Schaft 2000) and controlled Lagrangians (Bloch et al. 2000, 2001), physical damping can affect stability in closed loop. The terms related to dissipation in unactuated directions have been commonly left unmatched (Acosta et al. 2004, Ortega et al. 2002b). Hence the

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usual approach has been to solve the matching conditions for an open-loop model *without* physical damping, and just hope that actual physical damping will be only beneficial in order to reach the desired equilibrium point. However, if the kinetic energy is modified, physical damping terms do not necessarily enter as dissipation with respect to the closed-loop energy function. And even if they do, but only locally, the global asymptotic stability (GAS) results or the domain of attraction may be affected. In other cases, even the linearization at the equilibrium point may not preserve stability after the introduction of a physical damping term, as observed in Reddy et al. (2004).

However, in this paper, we will show that under certain conditions the effect of physical damping can be *partially* modified by feedback in order to achieve passivity in closed loop globally. In these cases, physical damping in non-actuated directions actually acts in favor of real closed loop dissipation. This implies that all results on local or non-local stability for the undamped model are preserved.

Moreover, in underactuated systems, exploiting physical damping is the only way to obtain *strong dissipation* (meaning that the closed-loop dissipation matrix has full rank). This strong dissipation property easily ensures global asymptotic convergence to the minima of the closed-loop potential energy function, and local exponential stability at the target equilibrium point. An additional interesting fact is that since the physical damping terms are not exactly matched, they also need not be exactly known. Hence robustness to friction parameter uncertainty is provided without paying any price regarding stability. This will be studied for IDA-PBC control of mechanical systems with arbitrary underactuation degree.

The rest of the paper is organized as follows. Section 2 describes the problem related to unmatched physical damping in the IDA-PBC method: stability issues and the need for a control redesign. Section 3 contains the main propositions of the paper, providing a necessary and sufficient condition for passivation in the presence of physical damping, which will be called the *dissipation condition*. This section is divided in two subsections where two different methods are given for stabilization in the presence of physical damping in case the *dissipation condition* is satisfied. One of these should be chosen depending on the particular problem structure and the robustness requirements. The methods are called, respectively, *passivation by interconnection assignment* (Section 3.1) and *passivation by damping injection* (Section 3.2). Section 4 analyzes an interesting property of the closed-loop dynamics that is highly related to the existence of physical damping: *strong dissipation*. In Section 5, the problem of Coulomb friction in underactuated control is studied in our framework, by approximating the discontinuous ideal Coulomb friction by a continuous function. Section 6 deals with the well-known Ball and Beam problem in the presence of friction. The recently developed IDA-PBC control law for this system is redefined to handle uncertain friction parameters, by introducing a nonlinear passive output feedback. This result is illustrated by numerical simulation. Section 7 deals with the Vertical Take-off and Landing Aircraft problem with physical damping. This problem has special interest in the sense that the control matrix is state-dependent. We provide a solution to the damped control problem that is again illustrated by simulation.

Conclusive remarks are given in the final section, while the paper ends with an appendix where a new asymptotic stability proof for the usual IDA-PBC controlled Ball and Beam is given. This proof is to be compared with the much simpler analysis that is implied by the *strong dissipation* property.

2. Problem Statement

We briefly recall the rationale of IDA-PBC of underactuated mechanical systems. Next we highlight the drawbacks of not properly handling the physical damping aspects. The open-loop Hamiltonian formulation for a mechanical system with physical damping is

$$\Sigma_1(M, V, G, R):$$

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & -R(q) \end{bmatrix} \begin{bmatrix} \dfrac{\partial H}{\partial q} \\ \dfrac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} u, \quad (1)$$

where $q \in \mathbb{R}^n$ are the generalized position coordinates, $p \in \mathbb{R}^n$ are the momenta, given as $p = M\dot{q}$ with M the open-loop inertia matrix, $R(q)$ the physical damping matrix (smooth and bounded as function of q), and the Hamiltonian is given by the total energy

$$H(q,p) = \tfrac{1}{2} p^\top M^{-1}(q) p + V(q).$$

We consider the class of *underactuated* mechanical systems where $G = G(q)$ has constant rank $m < n$. Hence a matrix $G^\perp$ of rank $n - m$ exists such that

$$G^\perp G = 0, \quad \operatorname{rank}[G^\perp | G^\top] = n.$$

The first results of this paper are obtained for a particular constant form of G, for which explicit control laws are given. Then, the analysis will be extended to more general situations.

The forthcoming discussion is based on the matching equations for IDA-PBC for mechanical systems first presented in Ortega and Spong (2000). In subsequent papers (Ortega et al. 2002b, Gómez-Estern et al. 2001) the stabilization problem has been addressed for the energy conservative open-loop models of the form $\Sigma_1(M,\ V,\ G,\ 0)$. In this paper, we will extend the study to the case where $R \neq 0$ in Σ_1.

IDA-PBC control is based on designing a control law to transform Σ_1 into a closed-loop system which is *again* a port-Hamiltonian system with dissipation of the form

$$\Sigma_2(M_\mathrm{d}, V_\mathrm{d}, J_\mathrm{d}, G, R_\mathrm{d}):\quad \begin{bmatrix}\dot q\\ \dot p\end{bmatrix} = (J_\mathrm{d}(q,p) - R_\mathrm{d}(q,p))\begin{bmatrix}\dfrac{\partial H_\mathrm{d}}{\partial q}\\[2mm] \dfrac{\partial H_\mathrm{d}}{\partial p}\end{bmatrix} + \begin{bmatrix}0\\ G\end{bmatrix} v, \tag{2}$$

where $H_\mathrm{d}(q,p) = \frac{1}{2}p^\top M_\mathrm{d}^{-1}(q)p + V_\mathrm{d}(q)$ is the closed-loop energy function, $M_\mathrm{d} > 0$ is the closed-loop inertia matrix and $V_\mathrm{d}(q)$ the closed-loop potential energy function. The skew-symmetric matrix J_d and symmetric semi-positive matrix $R_\mathrm{d} \geq 0$ are obtained by setting

$$J_\mathrm{d} = \begin{bmatrix} 0 & M^{-1}M_\mathrm{d} \\ -M_\mathrm{d}M^{-1} & J_2(q,p)\end{bmatrix},\quad R_\mathrm{d} = \begin{bmatrix} 0 & 0\\ 0 & R_2(q)\end{bmatrix}, \tag{3}$$

with $J_2 = -J_2^\top$, and then equating the open- and closed-loop state equations to solve the set of PDEs in the non-actuated projected space

$$G^\perp\{\nabla_q H + R\nabla_p H - M_\mathrm{d}M^{-1}\nabla_q H_\mathrm{d} + (J_2 - R_2)M_\mathrm{d}^{-1}p\} = 0. \tag{4}$$

The usual IDA-PBC approach assumes $R = 0$ (undamped open-loop model), allowing us to split (4) into p-dependent and p-independent terms, giving rise to the kinetic and potential energy shaping equations given as

$$G^\perp\{\nabla_q(p^\top M^{-1}p) - M_\mathrm{d}M^{-1}\nabla_q(p^\top M_\mathrm{d}^{-1}p) + 2(J_2 - R_2)M_\mathrm{d}^{-1}p\} = 0, \tag{5}$$

$$G^\perp\{\nabla_q V - M_\mathrm{d}M^{-1}\nabla_q V_\mathrm{d}\} = 0. \tag{6}$$

In this equation, R_2 has been only included for generality of the presentation. In fact, if it is required to be positive semi-definite, it should be removed from (5). The reason for this is that the elements of R_2 in that equation should be linear in p, and this is inconsistent with a positive semi-definite matrix, because when p changes sign independently, R_2 becomes $-R_2$ [see Gómez-Estern et al. (2001) and Remark 2].

In our case, new matching equations due to the appearance of open-loop damping will admit the inclusion of a sign-semidefinite R_2. Indeed, if $R \neq 0$ we have a third set of matching equations containing new terms that are linear in p, that is,

$$G^\perp\{RM^{-1}p + (J_{20} - R_2)M_\mathrm{d}^{-1}p\} = 0, \tag{7}$$

where J_{20} is a new design parameter that is introduced by splitting the free matrix J_2 in terms of its dependence on p as

$$J_2 = J_{20}(q) + J_{21}(q,p).$$

2.1. New Stability Conditions

The main problem related to the appearance of R in Σ_1 concerns stability. If physical damping is present in the system but is not taken into account in the controller design procedure, we can obtain unstable behavior as has been pointed out in Reddy et al. (2004). To illustrate this point, observe that IDA-PBC control applied to the *undamped* model leads to an energy shaping feedback[1] $u = u_\mathrm{es} + v$ with

$$u_\mathrm{es} = (G^\top G)^{-1}G^\top\{\nabla_q H - M_\mathrm{d}M^{-1}\Delta_q H_\mathrm{d} + J_2 M_\mathrm{d}^{-1}p\}, \tag{8}$$

that yields Σ_2 as a passive system *with* with respect to the triplet[2] $\{H_\mathrm{d}, v, G^\top\nabla_p H_\mathrm{d}\}$, the parameters of (8) being a solution of (5) and (6). Then the system is subsequently stabilized with a negative feedback of the passive output

$$v = -K_v y_\mathrm{d} = -K_v G^\top\nabla_p H_\mathrm{d},$$

where $K_v > 0$. If no physical damping is present, this is enough to ensure that in the closed-loop system the time derivative of H_d is at least negative semi-definite, that is,

$$\dot H_\mathrm{d} = -y_\mathrm{d}^\top K_v y_\mathrm{d} \leq 0.$$

However if physical damping is present, $K_v > 0$ does *not* guarantee that H_d is decreasing, as can be seen in the following simple second-order example. Let the parameters of Σ_1 and Σ_2 be given as

$$M = I_2,\quad G = [0 \quad 1]^\top,\quad M_\mathrm{d} = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix},$$
$$R = \mathrm{diag}\{1, 9\},\quad K_v = k_v > 0\ (\in \mathbb{R})$$

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[1]After the energy shaping step, v will denote the new input to the transformed system.

[2]In the sequel we will denote that a system is *passive with respect to* the triplet $\{H,\ u,\ y\}$ if it is *passive* with *storage function H, input u* and *output y*.

where M, M_d and R are positive definite as required. If R were equal to zero, $k_v > 0$ will ensure closed-loop stability. However for non-zero R,

$$\begin{aligned}\dot{H}_\mathrm{d} &= -y_\mathrm{d}^\top K_v y_\mathrm{d} - \left(\frac{\partial H_\mathrm{d}}{\partial p}\right)^\top R \frac{\partial H}{\partial p} \\ &= -p^\top \begin{bmatrix} 2 & 5 \\ 5 & 9 + k_v \end{bmatrix} p\end{aligned}$$

which is negative semi-definite if and only if $k_v \geq 7$. Hence, if physical damping is neglected, the system can become unstable unless *new* conditions are satisfied.

2.2. Strong Dissipation

The second issue stems from the fact that in the underactuated case the presence of physical damping is *necessary* to achieve an important property that will be called *strong dissipation*.

Definition 2.1. (*Strong dissipation*). A Hamiltonian system Σ_2 defined on an open set $\{q \in \mathcal{X} \subset \mathbb{R}^n, p \in \mathbb{R}^n\}$ as in (2) with R_d in the form (3), is said to be strongly dissipative if $R_2(q) > 0 \ \forall q \in \mathcal{X}$.

For a strongly dissipative system it is easy to check that there exists a positive function $\alpha(q) > 0$ such that the rate of energy dissipation

$$\dot{H}_\mathrm{d} = -\left(\frac{\partial H_\mathrm{d}}{\partial p}\right)^\top R_2 \frac{\partial H_\mathrm{d}}{\partial p} < -\alpha(q)\ \|p\|^2$$

provided that $M_\mathrm{d}(q) > 0$ in the domain of interest. This property is extremely useful for stability analysis. As we will see in the following sections, strong dissipation in IDA-PBC controlled underactuated mechanical systems *can only be achieved* with the aid of physical damping.

3. Main Result

In this section, we will deal with four systems; Σ_1 as defined in (1), Σ_2 from (2) and the following two:

$\Sigma_3 = \Sigma_1(M, V, G, 0)$,
Undamped open-loop system,

$\Sigma_4 = \Sigma_2(M_\mathrm{d}, V_\mathrm{d}, J_\mathrm{d}, G, 0)$,
Undamped closed-loop system.

In order to achive closed-loop stability, the IDA-PBC method builds a closed-loop system that is *passive* with storage function the desired closed-loop Hamiltonian H_d. In recent papers on underactuated control, IDA-PBC control laws were provided such that the undamped system becomes *passive* after the application of an *energy shaping* control law $u = u_\mathrm{es} + v$, with input v, storage function

$$H_\mathrm{d} = \tfrac{1}{2} p^\top M_\mathrm{d}^{-1}(q) p + V_\mathrm{d}(q)$$

and passive output $y_\mathrm{d} = G^\top M_\mathrm{d}^{-1} p$, that is,

$$\dot{H}_\mathrm{d} < v^\top y_\mathrm{d}.$$

However, if we apply the same control law $u = u_\mathrm{es} + v$ to the physically damped system Σ_1 the closed-loop system is unlikely to be passive with respect to the triplet $\{H_\mathrm{d}, v, y_\mathrm{d}\}$. Instead of searching for these rare cases (see Remark 1), we will investigate the conditions for finding a *new* control law u_es for Σ_1 such that the closed-loop system is passive with respect to the storage function H_d.

3.1. Passivation by Interconnection Assignment

Although this will be relaxed in subsequent sections, we first assume for simplicity of presentation that G has been brought (by canonical coordinate transformations) into the form

$$G = \begin{bmatrix} 0_{(n-m)\times m} \\ I_m \end{bmatrix}. \tag{9}$$

Then, clearly the left annihilator of G is given as

$$G^\perp = [I_{n-m} \ \ 0_{(n-m)\times m}]. \tag{10}$$

Assuming the existence of a control law $u = u_\mathrm{es} + v$ that transforms Σ_3 (undamped) into Σ_4, the latter being passive with respect to $\{H_\mathrm{d}, v, G^\top \nabla_p H_\mathrm{d}\}$, the following proposition establishes the condition for the existence of a state feedback u_es^d, *generally different* from u_es, that transforms the damped system Σ_1 into a passive system Σ_2 with respect to the same storage function H_d.

First, for ease of notation, we define the operator that extracts an upper-left square block of given dimension from a matrix, and then takes its symmetric part.

Definition 3.1. For $k, j \in \mathbb{N}$ with $k \leq j$, let $\psi(\cdot) : \mathbb{R}^{j\times j} \to \mathbb{R}^{k\times k}$ be the symmetric part of the k-order upper-left square submatrix of its argument, that is, for a matrix $A \in \mathbb{R}^{j\times j}$ we have

$$\psi_k(A) = \tfrac{1}{2}[A + A^\top]_{(1\cdots k, 1\cdots k)}.$$

Then, provided that the control matrix is in the form (9), the following identities for any matrix $A \in \mathbb{R}^{n\times n}$ are easy to check:

$$\psi_{n-m}(A) = \tfrac{1}{2} G^{\perp}(A + A^{\top})(G^{\perp})^{\top}, \tag{11}$$

$$A = \begin{bmatrix} G^{\perp}A(G^{\perp})^{\top} & G^{\perp}AG \\ G^{\top}A(G^{\perp})^{\top} & G^{\top}AG \end{bmatrix}. \tag{12}$$

Proposition 3.1 (*Passivation by interconnection*). Consider the system Σ_1 defined on an open set $\{q \in \mathcal{X} \subset \mathbb{R}^n, p \in \mathbb{R}^n\}$, with G in the form (9). Assume there are smooth matrices $M_{\rm d}, J_2$ and a smooth function $V_{\rm d}$ satisfying Eqs (5 and 6) with $R = R_2 = 0$, that is, such that (8) transforms Σ_3 into Σ_4.

Then there exists an energy shaping control law $u_{\rm es}^d$ for the damped system Σ_1 such that the closed-loop system is passive with storage function $H_{\rm d}$, if and only if the following dissipation condition holds:

$$\psi_{n-m}(RM^{-1}M_{\rm d}) \geq 0, \quad \forall q \in \mathcal{X}. \tag{13}$$

Proof (*Necessity*). Given $H_{\rm d}$ coming as a solution of the undamped problem (corresponding to a certain $M_{\rm d}$), we will investigate if there is any solution $u = u_{\rm es} + v$ that makes Σ_2 passive with storage function $H_{\rm d}$. Then, $u_{\rm es}$ must satisfy the extended matching conditions (5, 6, 7). In particular, the p-linearly dependent matching equation becomes (leaving out all functional dependences)

$$\begin{aligned} & G^{\perp}[R\nabla_p H + (J_{20} - R_2)\nabla_p H_{\rm d}] = 0 \\ & \quad \Rightarrow G^{\perp}[RM^{-1} + (J_{20} - R_2)M_{\rm d}^{-1}]p = 0, \quad \forall p \\ & \quad \Rightarrow G^{\perp}[RM^{-1}M_{\rm d} + (J_{20} - R_2)](G^{\perp})^{\top} = 0 \\ & \quad \Rightarrow \mathrm{symm}\{G^{\perp}[RM^{-1}M_{\rm d} - R_2](G^{\perp})^{\top}\} = 0 \\ & \quad \Rightarrow \psi_{n-m}(R_2) = \psi_{n-m}(RM^{-1}M_{\rm d}) \end{aligned}$$

because J_{20} is skew symmetric, and $G^{\perp}$ has the particular form (10). In order to check the passivity of Σ_2 we observe that along its trajectories

$$\begin{aligned} \dot{H}_{\rm d} &= -\left(\frac{\partial H_{\rm d}}{\partial p}\right)^{\top} R_2 \frac{\partial H_{\rm d}}{\partial p} + \left(\frac{\partial H_{\rm d}}{\partial x}\right)^{\top} Gv \\ &= -p^{\top}M_{\rm d}^{-1}R_2M_{\rm d}^{-1}p + M_{\rm d}^{-1}p\, Gv \\ &= -p^{\top}M_{\rm d}^{-1}R_2M_{\rm d}^{-1}p + v^{\top}y_{\rm d}. \end{aligned}$$

Now assume that there exists some $q^* \in \mathcal{X}$ not satisfying the dissipation condition, that is, there exists some vector $z \in \mathbb{R}^{n-m}$ such that

$$z^{\top}[\psi_{n-m}(R(q^*)M^{-1}(q^*)M_{\rm d}(q^*))]z < 0.$$

Defining the state $(q^*, p^*) \in (\mathcal{X} \times \mathbb{R}^n)$ with

$$p^* = M_{\rm d}[z^{\top}, 0_{1\times m}]^{\top},$$

we have

$$\begin{aligned} \dot{H}_{\rm d}(q^*, p^*) &= -z^{\top}[\psi_{n-m}(R_2)]z + v^{\top}y_{\rm d} \\ &= -z^{\top}[\psi_{n-m}(R(q^*)M^{-1}(q^*)M_{\rm d}(q^*))]z \\ &\quad + v^{\top}y_{\rm d} > v^{\top}y_{\rm d} \end{aligned}$$

which means that the system is not passive in $(\mathcal{X} \times \mathbb{R}^n)$. Since this holds for any possible IDA-PBC control law $u_{\rm es}^d$ solution of (5, 6, 7), we conclude that (13) is necessary for the passivation of Σ_1 for a fixed $M_{\rm d}$.

Sufficiency. Assume that (13) holds on $(\mathcal{X} \times \mathbb{R}^n)$. Then we will construct an input $u = u_{\rm es}^d + v$ that passivates system Σ_3 with storage function $H_{\rm d}$, input v and output $y_{\rm d} = G^{\top}\nabla_p H_{\rm d}$. The choice

$$\begin{aligned} J_{20} &= \begin{bmatrix} -\frac{1}{2}G^{\perp}(RM^{-1}M_{\rm d} - M_{\rm d}M^{-1}R)(G^{\perp})^{\top} & -G^{\perp}RM^{-1}M_{\rm d}G \\ G^{\top}M_{\rm d}M^{-1}R(G^{\perp})^{\top} & 0 \end{bmatrix} \\ &= -J_{20}^{\top}, \end{aligned}$$

cancels the non-actuated terms of $RM^{-1}M_{\rm d}$ outside the $(n-m)$-order upper-left block using the decomposition (12) and removes the skew-symmetric part of the latter.[3] Hence the p-linearly dependent equation is solved with

$$R_2 = \begin{bmatrix} \psi_{n-m}(RM^{-1}M_{\rm d}) & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, taking the matrices $M_{\rm d}, J_{21}$ and the function $V_{\rm d}$ from the solution of the undamped problem, we obtain a control law

$$\begin{aligned} u_{\rm es}^d = (G^{\top}G)^{-1}G^{\top}\{&\nabla_q H + R\nabla_p H - M_{\rm d}M^{-1}\nabla_q H_{\rm d} \\ &+ (J_2 - R_2)M_{\rm d}^{-1}p\} + v \end{aligned}$$

such that along the closed-loop trajectories

$$\begin{aligned} \dot{H}_{\rm d} = &-\left(\frac{\partial H_{\rm d}}{\partial p}\right)^{\top} \psi_{n-m}(RM^{-1}M_{\rm d}) \frac{\partial H_{\rm d}}{\partial p} \\ &+ v^{\top}y_{\rm d} \leq v^{\top}y_{\rm d}, \quad \forall (q,p) \in \mathcal{X} \times \mathbb{R}^n, \end{aligned}$$

provided that the *dissipation condition* holds on $\mathcal{X}$. This completes the proof. □

The above proposition is instrumental as it provides a criterion to check if physical damping *is* or *is not* an obstacle to achieve passivity in closed loop. Furthermore, if physical damping does *not* pose an

[3]The upper left block of J_{20} is introduced to produce a solution of (7) where R_2 is symmetric, and could be neglected without modifying the passivity result.

obstacle it also provides a simple method to construct the passivating control law. For stabilizing the passivated system it is sufficient to add a damping term of the form

$$v = -K_v y_{\rm d},$$

with $K_v > 0$. This will ensure asymptotic convergence to the set where $y_{\rm d} = 0$. The relation between this and the stability of the origin of the state space will be discussed further below.

The procedure illustrated in Proposition 3.1 for obtaining $u_{\rm es}^d$ will be called *passivation by interconnection assignment*, because it exploits the interconnection matrix J_{20} to cancel the elements of $RM^{-1}M_{\rm d}$ outside the critical block $\psi_{n-m}(RM^{-1}M_{\rm d})$. This straightforward procedure has a main drawback: since it is a cancellation of the friction terms it requires exact knowledge of some elements of the matrix R, which are friction parameters that normally are nonconstant and hard to identify.

Remark 1. Proposition 3.1 provides a condition which is much less conservative than simply hoping that the addition of damping does not affect passivity. Indeed, if we are to design an IDA-PBC law for the damped model in a single step (without redesign), then it is easy to see that $M_{\rm d}$ must be such that

$$RM^{-1}M_{\rm d} + M_{\rm d}M^{-1}R \geq 0.$$

Moreover, as has been shown in Gómez-Estern et al. (2004) there are systems (e.g. the Inertia Wheel Pendulum) for which there is no solution of the IDA-PBC problem with $M_{\rm d}$ satisfying (13), that is, there is no passivating redesign for any of the reachable closed-loop energy functions.

Remark 2. If R depends on the momenta p, two cases should be considered. In the (physically not very likely) case that the dependence is linear in p, R would enter Eq. (5). However it has been shown in Gómez-Estern et al. (2001) that the kinetic energy shaping equation cannot introduce any new term in the unactuated block of R_2, and hence its sign is still subject to the dissipation condition. For the rest of the cases, the matrix R should enter the so called p-linearly dependent equation, where J_{20} and R_2 should assume some type of nonlinear dependence on p, leaving the main result unchanged.

3.2. Passivation by Damping Injection

An alternative approach that significantly relaxes the parameter identification requirements is the *passivation by damping injection* method. This technique increases the computational effort in favor of robustness to uncertainty in damping parameters. The main proposition of this section states the conditions for passivation with physical damping via a suitable output feedback (without modifying the interconnection matrix). It assumes again the existence of a solution of the energy shaping step for the undamped problem, with a closed-loop energy function $H_{\rm d}$.

The main result will be presented here in a slightly different form as Proposition 3.1 regarding three aspects:

1. The dissipation condition (13) must hold strictly. No conclusive result is given when the critical block of (13) is positive semi-definite.
2. Necessity and sufficiency conditions are only given for *strong dissipation*, which is a stronger property than simple passivity (no further damping injection is needed). The reason for this is that the goal of the energy shaping step in IDA-PBC is obtaining a closed loop *passive* system in order to make it stabilizable by output feedback; now, as we are only interested in the damping injection (last) step, we will simply require *dissipativity* (decrease of the Lyapunov function) in closed loop.
3. Only conservative estimates of the damping parameters are needed to stabilize without compromising the domains of attraction obtained in Proposition 3.1.
4. The control matrix G is no longer required to be neither constant nor in the form (9) in order to investigate the existence of a stabilizing *damping injection* term. However, an *explicit* form of this term is only provided if G is actually in that form.

To illustrate the procedure, consider $u_{\rm es}$ as an energy shaping control law transforming Σ_3 into the passive system Σ_4 with storage function $H_{\rm d}$. Then, defining the passive output feedback

$$u_{di} = -K_v(q,p)y_{\rm d} = -K_v(q,p)G^\top M_{\rm d}^{-1}p,$$

with $K_v = K_v^\top \geq 0$ and applying $u = u_{\rm es} + u_{\rm di}$ to the damped system Σ_1 yields

$$\begin{aligned}
\begin{bmatrix}\dot q\\ \dot p\end{bmatrix} &= \begin{bmatrix}0 & I_n\\ -I_n & -R(q)\end{bmatrix}\begin{bmatrix}\dfrac{\partial H}{\partial q}\\ \dfrac{\partial H}{\partial p}\end{bmatrix} + \begin{bmatrix}0\\ G\end{bmatrix}(u_{\rm es}+u_{\rm di})\\
&= J_{\rm d}\frac{\partial H_{\rm d}}{\partial x} + \begin{bmatrix}0 & 0\\ 0 & -R\end{bmatrix}\frac{\partial H}{\partial x} + \begin{bmatrix}0\\ G\end{bmatrix}u_{\rm di}\\
&= J_{\rm d}\frac{\partial H_{\rm d}}{\partial x} + \begin{bmatrix}0 & 0\\ 0 & -RM^{-1}M_{\rm d}\end{bmatrix}\begin{bmatrix}\dfrac{\partial H_{\rm d}}{\partial q}\\ M_{\rm d}^{-1}p\end{bmatrix}\\
&\quad - \begin{bmatrix}0\\ GK_vG^\top M_{\rm d}^{-1}p\end{bmatrix}\\
&= J_{\rm d}\frac{\partial H_{\rm d}}{\partial x} + \begin{bmatrix}0 & 0\\ 0 & -RM^{-1}M_{\rm d} - GK_vG^\top\end{bmatrix}\frac{\partial H_{\rm d}}{\partial x}.
\end{aligned}$$

Now defining the matrices

$$C \triangleq \tfrac{1}{2}(RM^{-1}M_\mathrm{d} + M_\mathrm{d}M^{-1}R), \quad D \triangleq GK_vG^\top, \tag{14}$$

the closed-loop Hamiltonian is such that

$$\dot{H}_\mathrm{d} = -\left(\frac{\partial H_\mathrm{d}}{\partial p}\right)^\top (C+D)\frac{\partial H_\mathrm{d}}{\partial p}, \tag{15}$$

which will be *strongly dissipative* if and only if $(C+D) > 0$ in $\mathcal{X} \times \mathbb{R}^n$. The following proposition provides the conditions for this based on Lemma 12.31 of Nijmeijer and Van der Schaft (1990) and does neither require the control matrix to be in the form (9) nor constant. Actually, G is only assumed to be of *constant rank*, which is a requirement of the IDA-PBC method for the computation of an $(n-m) \times n$ matrix $G^\perp(q)$ whose rows span *ker G*.

Proposition 3.2 (*Passivation by damping injection*). Assume there is an IDA-PBC control law $u = u_\mathrm{es} + v$ that transforms system Σ_3 with $G = G(q)$ into a passive system Σ_4 with respect to $\{H_\mathrm{d}, v, G^\top\nabla_p H_\mathrm{d}\}$. Then, there exists an output feedback

$$u_\mathrm{di} = -K_v(q)y_\mathrm{d}$$

such that $u = u_\mathrm{es} + u_\mathrm{di}$ transforms the damped system Σ_1 into a strongly dissipative ($R_2 > 0$) system Σ_2 if and only if

$$A \triangleq \mathrm{symm}[G^\top(RM^{-1}M_\mathrm{d})^\top] > 0, \quad \forall q \in \mathcal{X}. \tag{16}$$

Furthermore, K_v can be taken to be diagonal.

Proof (*Necessity*). Assume that for some q^*, $A(q^*)$ is not positive definite, that is, there is a non-zero vector x such that $x^\top A x \le 0$. Defining the vector $z = (G^\perp)^\top x$ and using definitions (14) we have

$$\begin{aligned} z^\top R_2 z = z^\top (C+D)z &= x^\top G^\perp C (G^\perp)^\top x \\ &= x^\top A x \le 0, \end{aligned}$$

whence R_2 cannot be positive definite.

Sufficiency. Assume that $A(q) > 0\ \forall q \in \mathcal{X}$. Let V be a an $m \times n$ matrix whose columns span the orthogonal complement of *C(ker G)*. First we prove that the $n \times n$ matrix $[V|(G^\perp)^\top]$ is non-singular. Let $V\alpha + (G^\perp)^\top\beta = 0$, with $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^{n-m}$. Then,

$$0 = G^\perp C(V\alpha + (G^\perp)^\top\beta) = G^\perp C (G^\perp)^\top \beta = A\beta.$$

Since A is assumed to be positive definite, this implies that $\beta = 0$ and $\alpha = 0$. Then we observe that

$$\begin{aligned} &[V|(G^\perp)^\top]^\top (C+D)[V|(G^\perp)^\top] \\ &= \begin{bmatrix} V^\top C V + V^\top D V & 0 \\ 0 & G^\perp C (G^\perp)^\top \end{bmatrix}. \end{aligned}$$

Since

$$\begin{aligned} \mathrm{rank}\ V^\top D V &= \mathrm{rank}\ [V|(G^\perp)^\top]^\top D[V|(G^\perp)^\top] \\ &= \mathrm{rank}\ D, \end{aligned}$$

we conclude that since $D = GK_vG^\top$, $R_2 = C + D$ can be made positive definite by choosing an appropriate K_v (if necessary diagonal). This yields a *strongly dissipative* closed-loop system. □

The following corollary states that the output injection matrix K_v required for strong dissipation is not unique. Indeed, if the dissipation condition holds, a *sufficiently large* damping injection matrix will achieve strong dissipation.

Corollary 3.3 (*Robustness to uncertain damping terms*). Assume that all conditions for Proposition 3.2 hold and consider a matrix K_v computed accordingly. Then, for any $\bar{K}_v \ge K_v$, the control law

$$\begin{aligned} u &= u_\mathrm{es} + u_\mathrm{di}, \quad \text{where} \quad u_\mathrm{di} = -\bar{K}_v y_\mathrm{d}, \\ y_\mathrm{d} &= G^\top \nabla_p H_\mathrm{d}, \end{aligned}$$

applied to Σ_1 yields a strongly dissipative system in the form Σ_2.

Proof. Defining the positive semi-definite matrix $\Lambda = \bar{K}_v - K_v$, with the new damping injection u_di, Eq. (15) turns into

$$\begin{aligned} \dot{H}_\mathrm{d} &= -\left(\frac{\partial H_\mathrm{d}}{\partial p}\right)^\top (C + GK_vG^\top)\frac{\partial H_\mathrm{d}}{\partial p} \\ &\quad - \left(\frac{\partial H_\mathrm{d}}{\partial p}\right)^\top G\Lambda G^\top \frac{\partial H_\mathrm{d}}{\partial p} \\ &< -\left(\frac{\partial H_\mathrm{d}}{\partial p}\right)^\top (C + GK_vG^\top)\frac{\partial H_\mathrm{d}}{\partial p} \end{aligned}$$

hence strong dissipation is guaranteed as long as K_v yields strong dissipation. □

While the existence of the stabilizing K_v is guaranteed if the dissipation condition holds, we may only provide it in explicit form in the cases where the control matrix is in the form (9).

Corollary 3.4 (*Explicit form of* K_v). Assume that all conditions for Proposition 3.2 hold and that the control matrix G has form (9). Then, the damping injection feedback $u_\mathrm{di} = -K_v y_\mathrm{d}$ with $K_v = \mathrm{diag}\{k_1 \cdots k_m\}$ and

$$\begin{aligned} k_1 &= \frac{\lambda - \det(\psi_{n-m+1}(RM^{-1}M_d))}{\det(\psi_{n-m}(RM^{-1}M_d))}, \\ k_i &= \frac{\lambda - \det(\psi_{i+n-m}(RM^{-1}M_d))}{\lambda}, \quad i = 2, \ldots, m \end{aligned}$$

with $\lambda > 0$ as a free parameter, achieves strong dissipation in closed loop.

Proof. Some straightforward calculations show that with such a choice of K_v the principal minors of $C+D$ become

$$\det(\psi_k(C+D)) = \begin{cases} \det(\psi_k(RM^{-1}M_{\rm d})) & k=1,\dots,n-m, \\ & \text{all positive according to (16)}, \\ \lambda & k=(n-m+1),\dots,n. \end{cases} \tag{17}$$

Hence $C+D$ is positive definite and the system is *strongly dissipative*. □

Remark 3 (*Constant versus nonlinear output injection*). It is worth to note that whereas in standard IDA-PBC (and also in Proposition 3.1), a constant output injection of the form $u_{\rm di}=-Ky_{\rm d}$ with $K>0\in\mathbb{R}^{m\times m}$ constant is sufficient for stability, the matrix K_v from Proposition 3.2 must be computed at each particular state, hence the damping injection term becomes, in general, a nonlinear output injection term of the form $u_{di}=-K_v(q)y_{\rm d}$. However, when the problem is restricted to a set of states $(q,p)\in\mathcal{X}\times\mathbb{R}^n$, with $\mathcal{X}$ bounded, there exists a constant matrix $\bar{K}_v>K_v(q)$ for all q on that set. Then, based on corollary $K_v(q)$, we can state that the constant output feedback $u_{di}=-\bar{K}_v y_{\rm d}$ achieves strong dissipation in closed loop in $\mathcal{X}\times\mathbb{R}^n$.

Remark 4 (*The dissipation condition for diagonal inertia matrices*). In some cases of underactuation degree one ($m=n-1$), the strict dissipation condition (16) is trivially satisfied. To view this, assume that the physical dissipation on each axis of motion is

$$f_i=-\beta_i(q,p)\dot{q}_i,\quad i=1,\dots,n \tag{18}$$

with $\beta_i>0$ since the damping force should be passive with storage function the mechanical energy H. Equivalently, in matrix form

$$f=-\begin{bmatrix}\beta_1(q,p) & 0 & \dots & 0\\ 0 & \beta_2(q,p) & \dots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \dots & \beta_n(q,p)\end{bmatrix}\underbrace{\dot{q}_1}_{M^{-1}p}$$
$$=-R\frac{\partial H}{\partial p}.$$

Hence it is reasonable to expect that R is diagonal. If we further assume that G is in the form (9), the dissipation condition can be written independently of β as

$$\psi_{n-m}(RM^{-1}M_{\rm d})=\beta_1(q,p)\sum_{k=1}^{n}[M^{-1}]_{1k}[M_{\rm d}]_{k1}>0$$
$$\Rightarrow\sum_{k=1}^{n}[M^{-1}]_{1k}[M_{\rm d}]_{k1}>0.$$

Then, it is easy to check that the *dissipation condition* is automatically satisfied if: (i) M is diagonal, or (ii) $M_{\rm d}$ is diagonal, because in both cases the above summation is reduced to the term $[M^{-1}]_{11}[M_{\rm d}]_{11}$ which is a product of diagonal terms of two positive definite matrices.

4. Stability of Strongly Dissipative Systems

This section shows two key advantages of the property called *strong dissipation*: asymptotic stability proofs become simpler and the useful local exponential stability property is guaranteed. We will first observe that strong dissipation is directly related to the aforementioned dissipation condition. The following corollary stems directly from Propositions 3.1 and 3.2. *No matter which of previously presented methods is used*, the dissipation condition must be held *strictly* for the existence of a control law that achieves *strong dissipation* and underlines that it is only possible with the aid of physical damping.

Corollary 4.1. Given a solution to the undamped problem with closed-loop energy function $H_{\rm d}$, the system Σ_1 can be rendered strongly dissipative in closed loop on $\{(q,p)\in\mathcal{X}\times\mathbb{R}^n\}$ with respect to $H_{\rm d}$ via the above methods if and only if the dissipation condition holds strictly, that is,

$$\mathrm{symm}[G^{\perp}(RM^{-1}M_{\rm d})(G^{\perp})^{\top}]>0,\quad\forall q\in\mathcal{X}.$$

In particular, if rank $G<n$, a necessary condition for strong dissipation is $R\neq 0$, that is, physical damping must be present.

4.1. Nonlinear Asymptotic Stability

For IDA-PBC controlled mechanical systems where the control law is obtained as a solution to the matching PDEs, the time-derivative of the closed-loop Hamiltonian is

$$\dot{H}_{\rm d}=-\left(\frac{\partial H_{\rm d}}{\partial p}\right)^{\top}R_2\frac{\partial H_{\rm d}}{\partial p}=-p^{\top}M_{\rm d}^{-1}R_2M_{\rm d}^{-1}p.$$

Stability theorems guarantee that the system will converge to the largest invariant set where $\dot{H}_{\rm d}=0$. If we can ensure that $R_2>0$, that is, the system is *strongly dissipative*, this set happens to be $p\equiv 0$. Moreover,

$$\dot{p}=0\Rightarrow -M_{\rm d}M^{-1}\frac{\partial H_{\rm d}}{\partial q}=0$$
$$\Rightarrow -M_{\rm d}M^{-1}\frac{\partial V_{\rm d}}{\partial q}=0\Rightarrow\frac{\partial V_{\rm d}}{\partial q}=0,$$

where we have used that $\partial H_\mathrm{d}/\partial q = \partial V_\mathrm{d}/\partial q$ since $p = 0$. We immediately obtain the following lemma.

Lemma 4.2. The trajectories of an IDA-PBC controlled mechanical system with strong dissipation asymptotically converge to the set of states $\{(q_i^*, 0)\}$ where

$$\left.\frac{\partial V_\mathrm{d}}{\partial q}\right|_{q^*} = 0.$$

Hence if the closed-loop potential energy function has only isolated equilibrium points, the state of the system will converge to one of these equilibria. We note that if the closed-loop system is not strongly dissipative the asymptotic stability analysis becomes much more difficult. To illustrate this we have included in the appendix a (new) asymptotic stability proof for the *undamped* Ball and Beam controlled via IDA-PBC.

4.2. Local Exponential Stability

A second important property of strongly dissipative systems is the local exponential stability (LES) of the origin. As is well known, this is instrumental in the analysis of nested and cascade systems (Khalil 2002). For this we will take recourse to the fact that LES is not implied by nonlinear asymptotic stability, but instead *is* implied by asymptotic stability of the linearized system. Hence, the Jacobian linearization at the desired equilibrium point should be investigated. For this we consider the closed-loop dynamics

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = (J_\mathrm{d} - R_\mathrm{d}) \begin{bmatrix} \dfrac{\partial H_\mathrm{d}}{\partial q} \\ \dfrac{\partial H_\mathrm{d}}{\partial q} \end{bmatrix},$$

where $H_\mathrm{d} = \frac{1}{2} p^\top M_\mathrm{d}^{-1}(q) p + V_\mathrm{d}(q)$, and $M_\mathrm{d}(q) > 0$. We will assume that there is an equilibrium point at the origin, for which the IDA-PBC method requires that

$$\frac{\partial^2 V_\mathrm{d}}{\partial q^2}(0) > 0.$$

The linearization at the origin is given by

$$\begin{bmatrix} \dot{z}_q \\ \dot{z}_p \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}(0) \\ -M_\mathrm{d} M^{-1} \dfrac{\partial^2 V_\mathrm{d}}{\partial q^2}(0) & (J_{20} - R_2) M_\mathrm{d}^{-1}(0) \end{bmatrix}_{x=0} \begin{bmatrix} z_q \\ z_p \end{bmatrix}.$$

Asymptotic stability of this system will be investigated by defining the positive Lyapunov function

$$V = \frac{1}{2} z^\mathrm{T} Q z,$$
$$Q \triangleq \frac{\partial^2 H_\mathrm{d}}{\partial z_p^2}(0) = \begin{bmatrix} \dfrac{\partial^2 V_\mathrm{d}}{\partial q^2}(0) & 0 \\ 0 & M_\mathrm{d}^{-1}(0) \end{bmatrix}.$$

Clearly, $Q > 0$ for any well designed controller. The time derivative of V is

$$\begin{aligned} \dot{V} &= z^\top Q \dot{z} \\ &= z_p^\top M_\mathrm{d}^{-1}(0)[J_2(0) - R_2(0)] M_\mathrm{d}^{-1}(0) z_p \\ &= -z_p^\top M_\mathrm{d}^{-1}(0) R_2(0) M_\mathrm{d}^{-1}(0) z_p < 0, \quad \forall z_p \neq 0. \end{aligned}$$

Since R_2 is positive definite, the linearized system will converge asymptotically to the largest invariant set where $z_p \equiv 0$. This set is such that

$$\dot{z}_p = 0 \;\Rightarrow\; -M_\mathrm{d} M^{-1} \frac{\partial^2 V_\mathrm{d}}{\partial q^2}(0) z_q = 0 \;\Rightarrow\; z_q = 0.$$

Hence we deduce linear asymptotic stability, from which local exponential stability follows.

5. Coulomb Friction

In this section we study the situation where damping is replaced by Coulomb friction. In this case the elements of the matrix R in Eq. (1) are dependent on $\dot{q}$. According to Remark 2 this does not affect the previous results as long as R remains bounded as a function of $\dot{q}$. This is *not* the case for the idealized (discontinuous) Coulomb friction characteristic depicted in the *solid* plot of Fig. 1, where the friction force on each coordinate takes the form

$$f_i = -\beta_i \,\mathrm{sgn}\,(\dot{q}_i), \quad i = 1, \ldots, n, \;\; \beta_i > 0. \tag{19}$$

In this case we have R in the form

$$R = \;\mathrm{diag}\left\{ \frac{\beta_1}{|\dot{q}_1|} \cdots \frac{\beta_n}{|\dot{q}_n|} \right\} \tag{20}$$

which is unbounded on any set containing the origin. Although our results do not apply to idealized cases in the form (19), they *do* apply to arbitrarily close approximations of it. The possible approximations are depicted as smooths curves in Fig. 1, where the friction forces have been defined as

$$f_i = -\frac{\beta_i \dot{q}_i}{\sqrt{\alpha_i + \dot{q}_i^2}}, \quad \alpha_i > 0, \;\; \beta_i > 0, \;\; i = 1, \ldots, n.$$

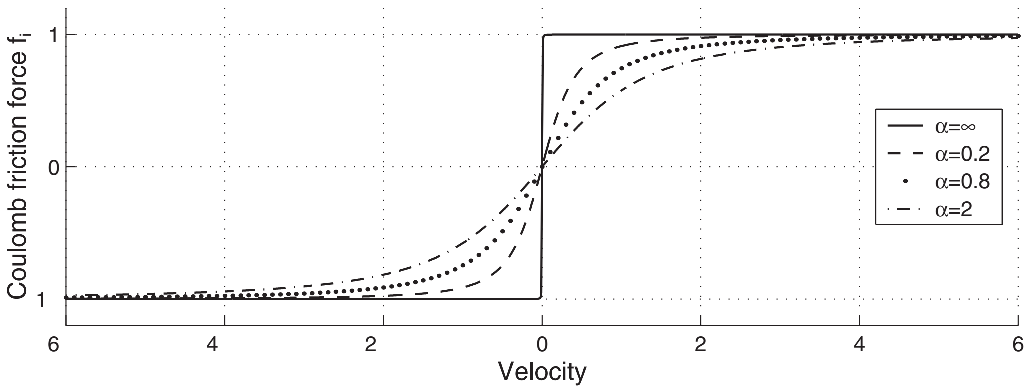

Fig. 1. "Idealized" and approximate models of Coulomb friction forces.

The parameters $\{\alpha_i\}$ are measures of the width of the transition regions in Fig. 1. In that graph, it can be observed that as $\alpha_i \to 0$, the corresponding friction model approaches the discontinuous one (solid line). The damping matrix R for Eq. (1) becomes

$$R(\dot{q}, \alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n) = \mathrm{diag}\left\{ \frac{\beta_1}{\sqrt{\alpha_1 + \dot{q}_1^2}} \cdots \frac{\beta_n}{\sqrt{\alpha_n + \dot{q}_n^2}} \right\}.$$

In this case the dissipation condition takes the form

$$\mathrm{symm}\,(G^{\perp}(R(\dot{q},\alpha_1\cdots\alpha_n,\beta_1\cdots\beta_n)M^{-1}M_{\mathrm{d}})(G^{\perp})^{\top})>0, \quad \forall(q,p)\in\mathcal{X}\times\mathbb{R}^n,$$

where α can be made arbitrarily small.

The shortcoming of this approach is that the value of f_i close to $\dot{q}=0$ is highly unpredictable, whereas the dissipation condition is very sensitive to changes in R.

This motivates an extended statement of the *dissipation condition*. Assume that there is an unknown Coulomb friction force (possibly dependent on internal states), that lies between two of the smooth continuous first-third quadrant curves of Fig. 1, that is,

$$\left|\frac{\beta_i^- \dot{q}_i}{\sqrt{\alpha_i^+ + \dot{q}_i^2}}\right| < |f_i| < \left|\frac{\beta_i^+ \dot{q}_i}{\sqrt{\alpha_i^- + \dot{q}_i^2}}\right|, \quad \alpha_i^+ > \alpha_i^- > 0,\ \beta_i^+ > \beta_i^- > 0,\ i=1,\ldots,n. \tag{21}$$

The proposed scenario is depicted in Fig. 2. In this figure, it is assumed that the friction data have been collected experimentally, and that as a function of the internal states it may vary between the two curves that represent the lower and upper bounds of (21). (Note that this allows to take into account the Stribeck effect.) Now let us define the vectors $\alpha^+ \triangleq [\alpha_1^+ \cdots \alpha_n^+]^\top$, $\alpha^- \triangleq [\alpha_1^- \cdots \alpha_n^-]^\top$, $\beta^+ \triangleq [\beta_1^+ \cdots \beta_n^+]^\top$ and $\beta^- \triangleq [\beta_1^- \cdots \beta_n^-]^\top$, and the hypercube

$$\mathcal{S} \triangleq \{(\alpha,\beta) \in \mathbb{R}^{2n} | \alpha_i \in [\alpha_i^-, \alpha_i^+], \beta_i \in [\beta_i^-, \beta_i^+], i = 1,\ldots,n\}.$$

Lemma 5.1. Assume that there is an energy shaping control law u_{es} that fulfills the conditions of Proposition 3.2. Further assume that there are four positive vectors α^+, α^-, β^+ and β^- satisfying (21). Then, if the dissipation condition holds on the hypercube $\mathcal{S}$, that is,

$$\mathrm{symm}(G^{\perp}(R(\dot{q},\alpha,\beta)M^{-1}M_{\mathrm{d}})(G^{\perp})^{\top}) > 0, \quad \forall(\alpha,\beta)\in\mathcal{S},\ \ \forall(q,p)\in\mathcal{X}\times\mathbb{R}^n \tag{22}$$

with, possibly, $\mathcal{X} \equiv \mathbb{R}^n$, there exists a matrix $K_v(q,\dot{q})$ such that a robust output feedback of the form

$$u = u_{\mathrm{es}} + u_{\mathrm{di}} + v$$

with $u_{\mathrm{di}} = -\bar{K}_v(q,\dot{q})y_d$ and $\bar{K}_v(q,\dot{q}) \geq K_v(q,\dot{q})$, passivates the damped system Σ_1 with respect to $\{H_{\mathrm{d}}, v, y\}$ in $\mathcal{X}\times\mathbb{R}^n$.

Proof. According to (21) and the Mean Value Theorem, for each (q,p) in $\mathcal{X}\times\mathbb{R}^n$ there are vectors $(\alpha^*, \beta^*) \in \mathcal{S}$ satisfying, simultaneously,

$$\mathrm{symm}(G^{\perp}(R(\dot{q},\alpha^*,\beta^*)M^{-1}M_{\mathrm{d}})(G^{\perp})^{\top}) > 0, \quad f_i = \frac{\beta_i^* \dot{q}_i}{\sqrt{\alpha_i^* + \dot{q}_i^2}}, \quad i = 1,\ldots,n. \qquad \square$$

Then, as the dissipation condition holds strictly for that specific state, Proposition 3.2 guarantees the existence of the required K_v. As the conditions of the lemma hold on whole set $\mathcal{X}\times\mathbb{R}^n$, we can assert that the required (non-constant) matrix exists on that set.

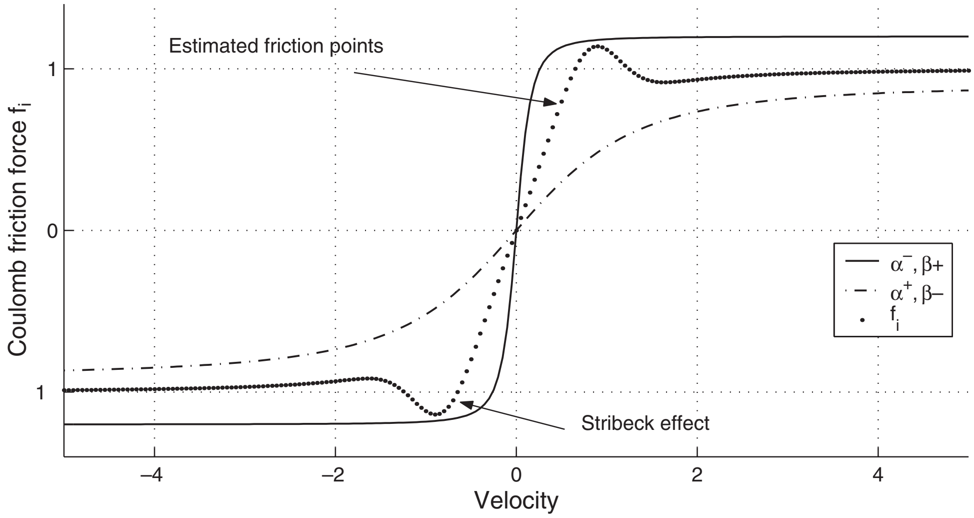

Fig. 2. Band of uncertainty in Coulomb friction.

However, it is desirable to check the dissipation condition only at specific points. The following result applies to systems with underactuation degree one, where the dissipation condition becomes a scalar function *linearly dependent* on r_i, the elements of R. This linear dependence allows us to guarantee that if the condition is fulfilled for all combinations of the maximum and minimum values of each r_i, it will be fulfilled at any point inside the hypercube $\mathcal{S}$.

Corollary 5.2. Assume that there is an energy shaping control law u_{es} for the undamped system Σ_3 fulfilling the hypotheses of Lemma 5.1. Further assume that there are four positive vectors α^+, α^-, β^+ and β^- satisfying (21). Then, if the dissipation condition holds on all vertices of the hypercube $\mathcal{S}$, that is,

$$\begin{aligned}&\mathrm{symm}(G^{\perp}(R(\dot{q},\alpha,\beta)M^{-1}M_{\mathrm{d}})(G^{\perp})^{\top})>0,\\&\forall(\alpha,\beta)\in\{\alpha^-,\alpha^+\}\times\{\beta^-,\beta^+\}\end{aligned}\tag{23}$$

within the domain of interest, the passivating output feedback defined in Lemma 5.1 u_{di} exists.

6. Example: Ball and Beam

This system has been studied in the IDA-PBC framework in Ortega et al. (2002b), where a stabilizing control law for zero initial velocities was obtained. In that paper, physical damping has been taken into account. As expected from the previous discussion, the closed-loop dissipation matrix is not of full rank; a situation leading to cumbersome stability proofs and not ensuring local exponential stability.

6.1. System Model

The commonly used physical model is the one presented in Hauser et al. (1992) where the rotational inertia of the ball is neglected. After some scaling (Gordillo et al. 2002) the Euler Lagrange equations take the form

$$\begin{aligned}&\ddot{q}_1+g\sin(q_2)-q_1\dot{q}_2^2+\beta_1(q,p)\dot{q}_1=0,\\&(L^2+q_1^2)\ddot{q}_2+2q_1\dot{q}_1\dot{q}_2+gq_1\cos(q_2)\\&\quad+\beta_2(q,p)\dot{q}_2=u,\end{aligned}\tag{24}$$

where q_1 is the position of the ball on the beam and q_2 is the angle of the bar, with the origin at the horizontal position. Here we have introduced the positive damping functions β_1 and β_2 as suggested in Reddy et al. (2004), while we also admit a possible state dependence.

6.2. Stability of the Standard IDA-PBC Controller

In Ortega et al. (2002b) an IDA-PBC control law was developed for the damping-free model (i.e. setting $\beta_i(q,p)=0$ in (24)). Hence in this case the closed-loop dissipation matrix is not full rank

$$R_2=\begin{bmatrix}0&0\\0&-k_v\end{bmatrix}$$

leading to a non-trivial asymptotic stability analysis, since vanishing of the derivative of the closed-loop Hamiltonian

$$\dot{H}_{\mathrm{d}}=-k_v\left(\frac{p_1\sqrt{L^2+{q_1}^2}-p_2\sqrt{2}}{(L^2+{q_1}^2)^{3/2}}\right)^2,$$

does not imply vanishing of p (see Section 4.1). A proof of asymptotic stability has been provided in Appendix A.

6.3. Physical Damping and Nonlinear Damping Injection

The inclusion of the $\beta_i(q,p)$ in the open-loop model (24) can induce instability in the IDA-PBC controlled ball on beam system, as pointed out in Reddy et al. (2004) by analyzing the linearization at the origin.

Since for this example the *dissipation condition* is strictly satisfied *globally*, it is possible to inject enough damping to overcome this difficulty in the whole state space.[4] As physical damping was not considered in Ortega et al. (2002b), the stability could be compromised for some values of β_i. Here we will design the damping injection terms to get a globally positive definite closed-loop dissipation matrix.

From the discussions above, there are two alternative approaches: introduce the friction compensation in the energy shaping equations (Proposition 3.1) assuming exact knowledge of β_i, or construct a $\tilde{R}$ matrix iteratively following the guidelines of Proposition 3.2. Since this is a low-dimensional problem, the last procedure will be used as it can better handle uncertainties in the dissipation terms. Instead of the constant output feedback of Ortega et al. (2002b) we will introduce a nonlinear damping control law of the form

$$u_{\mathrm{di}} = -k_v(q,p)y_d = -k_v(q,p)G^\top \nabla_p H_d.$$

For the system to be *strongly dissipative* we must set

$$k_v > \frac{1}{2\sqrt{2}} \frac{-6\beta_1(L^2+q_1^2)\beta_2 + \beta_1^2(L^2+q_1^2)^2 + \beta_2^2}{\sqrt{L^2+q_1{}^2}}.$$

According to Proposition 3.2 and Remark 3 this can be satisfied with a constant k_v^* on any compact set. However, as $\mathcal{X} \equiv \mathbb{R}^n$ there is no *constant* output feedback satisfying this equation, and thus we take recourse to a state dependent form of k_v like

$$u_{\mathrm{di}} = -\frac{\bar{\beta}_1^2(L^2+q_1^2)^2 + \bar{\beta}_2^2}{2\sqrt{L^2+q_1^2}} \left(p_1 \sqrt{\frac{L^2+q_1^2 - p_2\sqrt{2}}{(L^2+q_1^2)^{3/2}}} \right), \tag{25}$$

where

$$\bar{\beta}_1 > \max_{(q,p)}(\beta_1(q,p)), \qquad \bar{\beta}_2 > \max_{(q,p)}(\beta_2(q,p))$$

are *estimated* upper bounds on the friction parameters. This satisfies globally the conditions for strong dissipation and guarantees asymptotic and local exponential stability of the set

$$\{q \in \mathbb{R}^n | \nabla V_{\mathrm{d}}(q) = 0\} \cap \{p = 0\}$$

which is a countable set of isolated points of the form

$$\bar{q} = (L\sinh(\sqrt{2}i\pi), i\pi), \quad i \in \mathbb{N}$$

including the origin and other points outside $\{q_2 \in (\pi, \pi)\}$. This result reduces the cumbersome proof in Appendix A to Lemma A.3.

6.4. Coulomb Friction in the Ball and Beam

We will assume that we have a Coulomb friction model with uncertainties for the ball on beam system that fits inside a band of the shape depicted in Fig. 2, with eight positive constants $\alpha_i^-, \alpha_i^+, \beta_i^-, \beta_i^+, i = 1,2$ that fulfill (21). Under this assumption Corollary 5.2 leads to a dissipation condition of the form

$$\begin{gathered}\mathrm{symm}(G^\perp(R(\dot{q},\alpha,\beta)M^{-1}M_{\mathrm{d}})(G^\perp)^\top) > 0,\\ \forall(\alpha,\beta) \in \{\alpha^-,\alpha^+\} \times \{\beta^-,\beta^+\}\end{gathered}$$

for all (q,p) in the domain of interest (the whole state space in our case). This condition turns out to be, again, trivially satisfied since M is diagonal, thus guaranteeing the existence of the required output feedback. Indeed, for passivating the system with Coulomb friction it is sufficient that

$$k_v(q,\dot{q}) > \frac{1}{2\sqrt{2}} \times \frac{\left(\beta_1^+/\sqrt{\alpha_1^- + \dot{q}_1^2}\right)^2 \left(L^2+q_1^2\right)^2 + \left(\beta_2^+/\sqrt{\alpha_2^- + \dot{q}_2^2}\right)^2}{\sqrt{(L^2+q_1^2)}}$$

It is worth to note that the *no free lunch* theorem also applies here. The control effort required to passivate systems where α^- is very small (thus approaching the ideal discontinuous model) may be huge.

6.5. Ball and Beam Simulations

System (24) will be simulated with the energy shaping control law from Ortega et al. (2002b) and the two possible damping injection terms discussed in

[4]It should noted that what is achieved is *global passivity* with storage function H_d, which does not necessarily imply global asymptotic stability. Actually the geometry of H_d hampers this possibility in the Ball and Beam case.

Section 6.3. First, we will use a constant linear feedback as proposed in Ortega et al. (2002b) (setting $k_v > 0$ constant) and secondly the nonlinear output feedback (25). While for sufficiently large k_v both controllers will work fine locally, for initial conditions further away from the origin the linear output feedback will be insufficient to keep H_d always decreasing, whereas (25) ensures global dissipation.

Figure 3 depicts the simulation results of the Ball and Beam under different dissipation conditions. The three graphs in the upper row show the trajectories of q_1 and q_2 versus time. The lower row shows the time dependence of the closed-loop Hamiltonian function H_d corresponding to each trajectory in the above graph. Three different conditions have been simulated. The first case, (graphs (a1) and (a2)), illustrates the IDA-PBC controller with k_v constant applied to a damping-free model as in Ortega et al. (2002b). For any $k_v > 0$, the semidefinite dissipation matrix is sufficient to ensure stability and no further considerations have to be done. The second simulation, (b1) and (b2) shows how the performance of the constant k_v controller is downgraded when physical damping is present in the model but not considered in the design. Figure (b2) has been zoomed in to stress that the closed-loop energy is not monotonic: stability can be compromised. Graphs (c1) and (c2) show the closed-loop behavior of the physically damped system when the nonlinear damping term (25) is added to the controller. This controller recovers a monotonic Lyapunov function for every initial state, even without exact knowledge of the β parameters.

7. Variable Control Matrix Example: The VTOL Aircraft

The system model for the well-known Vertical Take-off and Landing aircraft is taken from (Acosta et al. 2004). In that paper the authors provide an IDA-PBC control law that achieves closed loop Hamiltonian dynamics for the undamped model. The open-loop model has 3 degrees of freedom and 2 actuators, and after some transformations is described as

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= \frac{g}{\epsilon}\sin\theta e_3 + G(\theta)u, \end{aligned} \tag{26}$$

where $q = [x, y, \theta]^\top$, $p = [p_1, p_2, p_3]$, $e_3 = [0, 0, 1]^\top$ and

$$G(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{\epsilon}\cos\theta & \frac{1}{\epsilon}\sin\theta \end{bmatrix}.$$

7.1. Passivation of the Damped VTOL Aircraft

This system is not yet of the form (9) since G is q-dependent. Although by canonical coordinate transformations G can be brought into the form (9), we will use, for simplicity, the form of the dissipation condition as stated in Proposition 3.2. To deal with open-loop damping we will rewrite the open-loop equations as

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= \frac{g}{\epsilon}\sin\theta e_3 - R(q)p + G(\theta)u, \end{aligned}$$

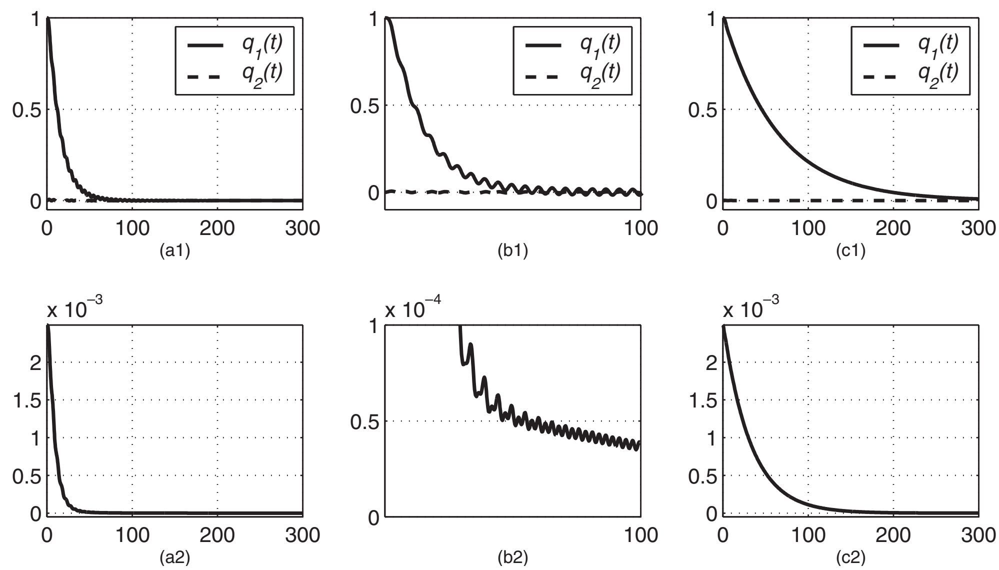

Fig. 3. Simulation results for the Ball and Beam. Up: Position versus time. Down: energy versus time.

where the open-loop damping matrix is assumed to be diagonal

$$R = \mathrm{diag}\{r_1(q), r_2(q), r_3(q)\}, \quad r_1, r_2, r_3 > 0.$$

In order to compute the dissipation condition we have $M = I$, and we also need the closed-loop inertia matrix M_{d}, which is obtained in Acosta et al. (2004), Proposition 7, as

$$M_{\mathrm{d}} = \begin{bmatrix} k_1\epsilon\cos^2\theta + k_3 & k_1\epsilon\cos\theta\sin\theta & k_1\cos\theta \\ k_1\epsilon\cos\theta\sin\theta & -k_1\epsilon\cos^2\theta + k_3 & k_1\sin\theta \\ k_1\cos\theta & k_1\sin\theta & k_2 \end{bmatrix},$$

with k_1 an arbitrary positive number and k_2, k_3 satisfying

$$k_3 > 5k_1\epsilon, \quad \frac{k_1}{\epsilon} > k_2 > \frac{k_1}{2\epsilon}.$$

$$\begin{aligned} k_3 &> \frac{1}{2}\frac{\sqrt{r_1 r_2(4r_1 r_2 k_1^2\epsilon^2\cos^4\theta + ((r_1+r_2)k_1\epsilon\cos\theta\sin\theta)^2)}}{r_1 r_2}, \\ k_3 &> \frac{1}{2}\frac{r_1\alpha_3^2 + \alpha_2^2 r_2 + \sqrt{r_1^2\alpha_3^4 + 2r_1\alpha_3^2\alpha_2^2 r_2 + \alpha_2^4 r_2^2 + 16 r_1 r_2 r_3^2 k_2^2\alpha_1^2}}{r_1 r_2 r_3 k_2} + k_1\epsilon\cos^2\theta, \end{aligned} \tag{27}$$

Proposition 3.2 provides a necessary condition for the existence of a passivating by damping injection control law in the presence of physical damping. Note that the left annihilator of G is $G^{\perp} = [\cos\theta\sin\theta - \epsilon]^{\mathrm{T}}$. Hence the dissipation condition becomes

$$\begin{aligned} &\mathrm{symm}(G^{\perp}(RM^{-1}M_{\mathrm{d}})(G^{\perp})^{\top} > 0 \\ &\quad\Rightarrow k_3[(r_1 - r_2)\cos^2\theta + r_2] + \epsilon^2 r_3 k_2 \\ &\qquad + k_1\epsilon[r_1(\cos^2\theta - 1) - r_3] > 0, \end{aligned}$$

leading to the sufficient condition

$$k_3\min(r_1, r_2) + \epsilon^2 r_3 k_2 - k_1\epsilon(r_1 + r_3) > 0.$$

Clearly, if $r_1 > 0$ and $r_2 > 0$, k_3 can be chosen large enough to fulfill this inequality. Then, Proposition 3.2 ensures the existence of a damping injection law $u_{\mathrm{di}} = -\bar{R}(q)y_{\mathrm{d}}$ for every q that makes $\dot{H}_{\mathrm{d}} \leq 0$ in closed loop. It also states that $\bar{R}(q)$ can be taken diagonal. However some difficulties could be pointed out in this regard. In order to compute exactly this matrix according to the aforementioned proposition we would need to obtain the kernel of $RM^{-1}M_{\mathrm{d}}$ for *every value of* q and this is very inefficient. The procedure to recompute the passivating damping injection term when G is q-dependent can be somewhat ad hoc. In the VTOL example we observe that with an appropriate choice of k_3, this term can be zero. Indeed, defining the matrix

$$\begin{aligned} C &\triangleq \frac{1}{2}(RM^{-1}Md + M_{\mathrm{d}}M^{-1}R) \\ &= \frac{1}{2}\begin{bmatrix} 2r_1(k_1\epsilon\cos^2\theta + k_3) & (r_1+r_2)k_1\epsilon\cos\theta\sin\theta & (r_1+r_3)k_1\cos\theta \\ (r_1+r_2)k_1\epsilon\cos\theta\sin\theta & 2r_2(-k_1\epsilon\cos^2\theta + k_3 & (r_2+r_3)k_1\sin\theta \\ (r_1+r_3)k_1\cos\theta & (r_2+r_3)k_1\sin\theta & 2r_3k_2 \end{bmatrix}, \end{aligned}$$

we have to (and we know we are able to) find a diagonal matrix $\bar{R}(q)$ such that

$$C + G\bar{R}(q)G^{\top} > 0,$$

but if we are able to tune M_{d} such that $C > 0$ the trivial choice $\bar{R}(q) = 0$ would be sufficient. In fact, any choice of k_3 satisfying the following inequalities: (27) where

$$\begin{aligned} \alpha_1 &= (r_1 + r_2)k_1\epsilon\cos\theta\sin\theta, \\ \alpha_2 &= (r_1 + r_3)k_1\cos\theta, \\ \alpha_3 &= (r_2 + r_3)k_1\sin\theta \end{aligned}$$

will guarantee that $C > 0$. With this choice, the closed-loop dissipation matrix R_2 is positive definite and hence the system becomes *strongly dissipative*. Because of this property it is straightforward to prove that the (damping neglecting) control law

$$u = (G^{\top}G)^{-1}G^{\top}\{\nabla_q H + M_{\mathrm{d}}M^{-1}\nabla_q H_{\mathrm{d}} + J_2 M_{\mathrm{d}}^{-1}p\} \tag{28}$$

with $H, V, V_{\mathrm{d}}, J_{\mathrm{d}}$ and M_{d} as defined in Acosta et al. (2004), and k_3 satisfying the above inequalities, asymptotically stabilizes the physically damped VTOL aircraft at the desired equilibrium point.

7.2. VTOL Simulations

System (26) with parameters $g = 9.8\,\mathrm{m/s^2}$ and $\epsilon = 1$ (coupling factor) has been simulated in closed loop with the control law presented in the appendix of Acosta et al. (2004). The parameters of the controller are $k_1 = 2, k_2 = 1.1$ and $k_3 = 30$ (fulfilling the required inequalities) for M_d. According to

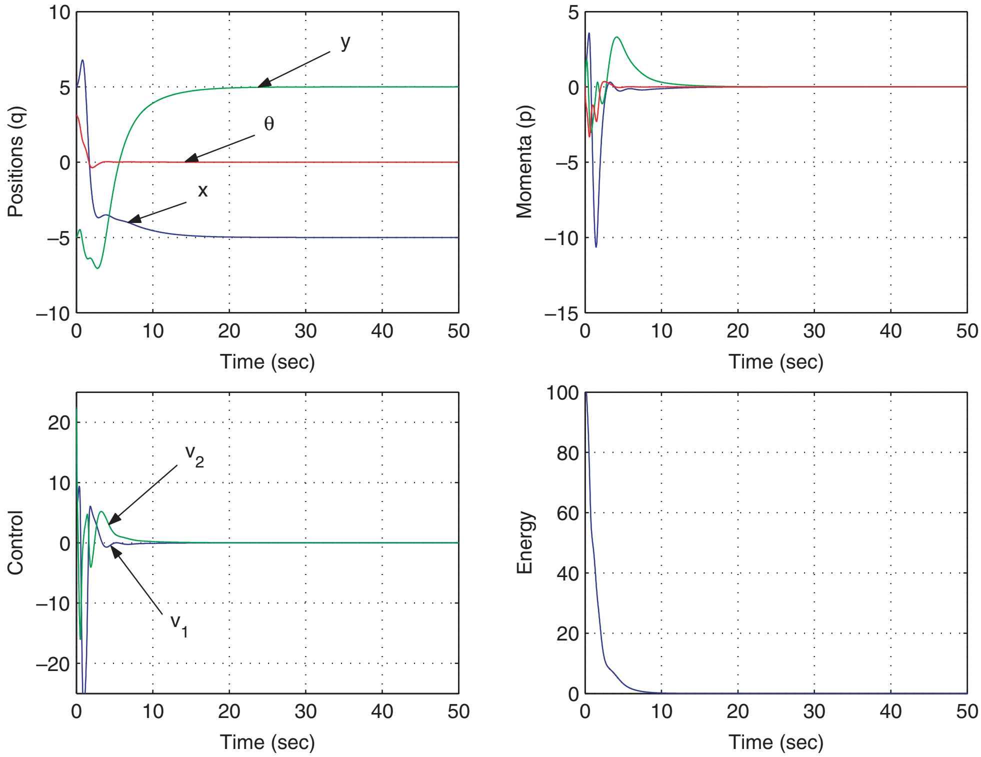

Fig. 4. Simulation results for the damped VTOL aircraft.

simulations in that paper, the potential energy parameter is set to $P = \mathrm{diag}\{0.015, 0.01\}$ and the physical damping matrix, which was inexistent thereby has been set to $R = \mathrm{diag}\{1, 1, 1\}$. The results of the simulations are depicted in Fig. 4, where the asymptotic convergence to the target point $(x, y, \theta) = (-5, 5, 0)$ is clearly viewed. With this controller, the limit values of x and y can be easily adjusted through the closed-loop potential energy function. The trajectories of the momenta, control law and closed loop Hamiltonian (which is monotonous as expected) are also illustrated.

An interesting fact that has been checked by subsequent simulations is that the enlargement of the damping coefficients (elements of R) without redesigning k_3 does not affect stability in closed loop. This is understood by observing that the right-hand sides of inequalities (27) remain bounded when the damping matrix R is multiplied by a scale factor.

8. Conclusions

In this paper, the IDA-PBC control technique for underactuated mechanical systems has been extended to incorporate open-loop damping. Given a solution of the IDA-PBC matching equations for the undamped model, this paper discusses the possible solutions that maintain passivity when damping is present. The main observation is that for a fixed closed-loop inertia matrix, there is a necessary and sufficient condition for the existence of a control redesign capable of handling the new damping terms. If this condition is fulfilled, a method to redesign the control law is given in two different versions depending on the problem structure and the robustness requirements. It is easy to check that if the damping terms are neglected, and no procedure like the one proposed hereby is applied, stability in closed loop can be compromised even locally and regions of attraction may shrink considerably. Furthermore, a remarkable fact is that damping should not only be compensated in underactuated control, but it also can be of invaluable help to prove asymptotic and local exponential stability, whose analysis often represents a bottleneck in the solution of this kind of problem. The analysis is completed with the conditions for recovering stability via damping injection in the presence of smooth but highly uncertain approximations of Coulomb friction effects. The theoretical results have been applied to two recently solved underactuated control problems, and these have been illustrated by simulation.

The work presented in this paper is mainly focused on redesigning IDA-PBC control laws where the closed-loop inertia matrix obtained as a solution of the IDA-PBC equations is left unchanged. Once we have fixed this matrix, there is the limiting factor of the *dissipation condition*. If the latter is not fulfilled, the only possible way to come to a better solution is to redo the energy shaping step and obtain a new matrix M_d, leading to a new dissipation inequality. We have found cases where a simple solution for the undamped problem exists, but no M_d can be found such that the *dissipation condition* is satisfied. This work could be enhanced by studying how the degrees of freedom remaining in the PDEs of the IDA-PBC method may be deployed to *tune* M_d in order to fulfill the dissipation condition.

Acknowledgements

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References

Acosta JA, Ortega R, Astolfi A. Interconnection and damping assignment passivity based control of mechanical systems with underactuation degree one. In: American control conference. Boston, MA, USA, 2004

Bloch A, Leonard NE, Marsden JE. Controlled Lagrangians and the stabilization of mechanical systems I: the first matching theorem. IEEE Trans Autom Control 2000; 45: 2253–2270

Bloch A, Chang DE, Leonard NE, Marsden JE. Controlled Lagrangians and the stabilization of mechanical systems II: potential shaping. IEEE Trans Autom Control 2001; 46: 1556–1571

Gómez-Estern, F, Van der Schaft AJ, Acosta JA. Passivation of underactuated systems with physical damping. In: 15th IFAC symposium on nonlinear control systems, NOLCOS'04. Stuttgart, Germany, 2004

Gómez-Estern F, Ortega R, Rubio FR, Aracil J. Stabilization of a class of underactuated mechanical systems via total energy shaping. In: Proceedings of the 40th IEEE Conference on Decision and Control. Orlando, FL, USA, 2001

Gordillo F, Gómez-Estern F, Ortega R, Aracil J. On the ball and beam problem: regulation with guaranteed transient performance and tracking periodic orbits. In: 15th international symposium on mathematical theory of networks and systems, MTNS'02. University of Notre Dame, August 12–16, 2002

Hauser J, Sastry S, Kokotovic P. Nonlinear control via approximate input-output linearization: The ball and beam example. IEEE Trans Autom Control 1992; 37(3): 392–398

Khalil HK. Nonlinear systems. Prentice Hall, 2002

Nijmeijer H, Van der Schaft AJ. Nonlinear dynamical control systems. Springer-Verlag, 1990

Ortega R, Spong M. Stabilization of underactuated mechanical systems via interconnection and damping assignment. In: IFAC Workshop on Lagrangian and Hamiltonian methods in nonlinear systems. Princeton, NJ, 2000, pp 74–79

Ortega R, Van der Schaft AJ, Maschke B, Escobar G. Interconnection and damping assignment passivity-based control of port-controlled hamiltonian systems. Automatica 2002a; 38: 585–596

Ortega R, Spong MW, Gómez-Estern F, Blankenstein G. Stabilization of underactuated mechanical systems via interconnection and damping assignment. IEEE Trans Autom Control 2002b; 47(8): 1218–1233

Reddy CK, Whitacre W, Woolsey CA. Controlled Lagrangians with gyroscopic forcing: an experimental application. In: American Control Conference. Boston, MA, USA, 2004

Van der Schaft AJ. L_2-gain and passivity techniques in nonlinear control. 2nd edn. Springer-Verlag, 2000

Appendix A. Asymptotic Stability Proof for the IDA-PBC Controlled Undamped Ball and Beam based on LaSalle's Invariance Principle

The following proof is an alternative to the proof given in Ortega et al. (2002b). In that paper, some of the hypothesis needed for Matrosov's Theorem were not satisfied thus it was not correctly applied. Here we shall only take recourse to LaSalle's Invariance Principle.

A.1. Closed-loop Dynamics

The IDA-PBC controlled Ball and Beam according to Ortega et al. (2002b) has the following state equations in closed loop:

$$\dot{q}_1 = p_1, \tag{29}$$

$$\dot{q}_2 = \frac{p_2}{L^2+{q_1}^2}, \tag{30}$$

$$\dot{p}_1 = \frac{{p_2}^2 q_1}{(L^2+{q_1}^2)^2} - g\sin q_2, \tag{31}$$

$$\begin{aligned}\dot{p}_2 = {} & \left(-\frac{1}{\sqrt{2}}q_2 + \frac{1}{2}a\sinh\left(\frac{q_1}{L}\right) - \sqrt{2}g\sin q_2\right) \\ & \times\sqrt{L^2+q_1^2} - 1/2\frac{{p_1}^2 q_1\sqrt{2}}{\sqrt{L^2+{q_1}^2}} + \frac{p_1 q_1 p_2}{L^2+{q_1}^2} \\ & + \frac{1}{2}\frac{{p_2}^2 q_1\sqrt{2}}{(L^2+{q_1}^2)^{3/2}} + \underbrace{k_v\frac{p_1\sqrt{L^2+q_1^2} - p_2\sqrt{2}}{(L^2+{q_1}^2)^{3/2}}}_{u_{\rm di}}.\end{aligned} \tag{32}$$

The term u_{di} is the *damping injection* control part needed for dissipation. The *passive* output of the system is

$$y_{\mathrm{d}} = G^{\top}\nabla_p H_{\mathrm{d}} = -\frac{p_1\sqrt{L^2+q_1^2} - p_2\sqrt{2}}{(L^2+q_1^2)^{3/2}},$$

where $G^{\top} = [0\ 1]$ is the transpose of the control matrix.

A.2. Stability Analysis

The main stability proposition in Ortega et al. (2002b) reads as follows.

Proposition A.1. Assume that system (29)–(32) has a stable equilibrium point at $(q_*, 0)$. This equilibrium is asymptotically stable if it is locally detectable from the output $G^{\top}(q)\nabla_p H_{\mathrm{d}}(q,p)$. An estimate of the domain of attraction is given by $\Omega_{\bar{c}}$ where $\Omega_c \triangleq \{(q,p)\epsilon\mathbb{R}^{2n} | H_{\mathrm{d}}(q,p) < c\}$ and

$$\bar{c} \triangleq \sup\{c > H_{\mathrm{d}}(q_*,0) \mid \Omega_c \text{ is bounded}\}. \tag{33}$$

This means that the system will converge asymptotically to the largest invariant set where $y_{\mathrm{d}} = G^{\top}(q(t))\nabla_p H_{\mathrm{d}}(q(t),p(t)) = 0$. However, in general it is not clear whether this set is formed by a unique equilibrium point, a set of them or a more complex set like a limit cycle. In Ortega et al. (2002b) the transient performance analysis guarantees that the trajectories are bounded to a set where there is actually only one equlibrium point at (0, 0), but there is no argument rejecting the existence of some orbit compatible with $y_{\mathrm{d}}(t) \equiv 0$.

A.2.1. Invariant Set Analysis

We will study the trajectories of system (29)–(32) restricted to the manifold $y_{\mathrm{d}}(t) \equiv 0\ \forall t$. This set can be characterized by three algebraic equations:

$$y_{\mathrm{d}} = 0 \ \Rightarrow\ p_2 = \frac{p_1\sqrt{L^2+q_1^2}}{\sqrt{2}}, \tag{34}$$

$$\dot{y}_{\mathrm{d}} = 0 \ \Rightarrow\ g\sin q_2 + q_2 - \frac{1}{\sqrt{2}}\ \mathrm{asinh}\left(\frac{q_1}{L}\right) + \frac{p_1^2 q_1}{L^2+q_1^2} = 0, \tag{35}$$

$$\ddot{y}_{\mathrm{d}} = 0 \ \Rightarrow\ p_1(-3p_1^2q_1^2 + p_1^2(L^2+q_1^2) + \sqrt{2}(L^2+q_1^2)^{5/2} g\cos q_2 - 2q_1 g\sin q_2(L^2+q_1^2)) = 0. \tag{36}$$

Since p_2 is uniquely determined by p_1 through (34) within the manifold $y_{\mathrm{d}}(t) \equiv 0$, we can reduce the analysis to the third-order system

$$\dot{q}_1 = p_1, \tag{37}$$

$$\dot{q}_2 = \frac{p_1}{\sqrt{2}\sqrt{L^2+q_1^2}}, \tag{38}$$

$$\dot{p}_1 = \frac{1}{2}\frac{p_1^2 q_1}{L^2+q_1^2} - g\sin q_2, \tag{39}$$

constrained to (34)–(36). We can further reduce the dimension of the problem by checking that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[q_2 - \frac{1}{\sqrt{2}}\ \mathrm{asinh}\left(\frac{q_1}{L}\right)\right] = \frac{p_1}{\sqrt{2}\sqrt{L^2+q_1^2}} - \frac{\dot{q}_1}{\sqrt{2}\sqrt{L^2+q_1^2}} = 0,$$

since $\dot{q}_1 = p_1$. As a consequence, there exists some constant α such that

$$q_2(t) - \frac{1}{\sqrt{2}}\ \mathrm{asinh}\left(\frac{q_1(t)}{L}\right) = 2\alpha, \quad \forall t,$$

which turns Eq. (35) into

$$\frac{1}{2}g\sin q_2 + \alpha + \frac{1}{2}\frac{p_1^2 q_1}{L^2+q_1^2} = 0.$$

Substituting the last term on the left-hand side into (39) yields

$$\dot{p}_1 = -\tfrac{3}{2}g\sin q_2 - \alpha.$$

The existence of an equilibrium point at the origin, which is a known fact, requires that $\alpha = 0$, and therefore $q_2 \equiv 1/\sqrt{2}\mathrm{asinh}(q_1/L)$. Now we have the following second-order state space system:

$$q_1 = L\sinh(\sqrt{2}q_2) \quad \text{(algebraic equation)}, \tag{40}$$

$$\dot{q}_2 = \frac{p_1}{\sqrt{2}\sqrt{L^2+q_1^2}}, \tag{41}$$

$$\dot{p}_1 = -\tfrac{3}{2}g\sin q_2. \tag{42}$$

Lemma A.2. The trajectories of (40)–(42) restricted to (36) are such that either $q_2(t) \geq 0\ \forall t$ or $q_2(t) \leq 0\ \forall t$.

Proof. Assume that for some t^* we have $q_2(t^*) = 0$. From (40) this implies $q_1(t^*) = 0$. Substituting $(q_1, q_2) = (0, 0)$ into (36) yields

$$p_1\left(p_1^2 + \sqrt{2}L^3 g\right) = 0 \tag{43}$$

which has no other real solution than $p_1 = 0$. Adding this to the fact that $\dot{p}_1(t^*) = 0$ from (42), we conclude that if $q_2(t^*) = 0$ the system will remain at the origin for all further time $t > t^*$. □

This lemma guarantees that either (q, p) is identically zero from some time t^* on, or $q_2(t) \neq 0 \ \forall t$. To study this possibility, we will assume that $q_2(t)\epsilon(-\pi, \pi)$ (open set). This means that $\dot{p}_1(t) \neq 0 \ \forall t$. Therefore p_1 is a strictly monotonic function of time. Using the fact that the potential energy V_d is lower bounded we can state that with finite initial energy $H_\mathrm{d}(0)$, the kinetic energy is bounded and hence p_1 is bounded.[5] As a bounded monotonic function of time it has a limit

$$\lim_{t\to\infty} p_1(t) = p \quad \Rightarrow \quad \lim_{t\to\infty} \dot{p}_1(t) = 0,$$

but this last equality implies that $q_2 \to 0$ by (42). From the previously stated fact that $q_2 = 0 \Rightarrow (q, p) = (0, 0)$ we also conclude that $p = 0$.

A.3. Conclusion of the Proof

The only assumption that has been made to prove asymptotic stability is that $q_2(t)\epsilon(-\pi, \pi)$ it is easy to verify that $p_1 \to 0$.

This condition can be easily expressed in terms of the energy of the initial state by using the following lemma, which is a trivial consequence of Lemma 1 and Proposition 5 of Ortega et al. (2002b).

Lemma A.3. For every initial condition included in the set

$$\Omega_c : \{(q, p)|q_2\epsilon(-\pi, \pi) \text{ and } p = 0\},$$

there exists a value of k_p^* such that for all $k_p > k_p^*$, system (29)–(32) will evolve in such a way that $q_2(t)\epsilon(-\pi, \pi) \ \forall t > 0$.

It is important to underscore that the condition of zero initial velocity can be further relaxed. For more details on this issue we refer the reader to Ortega et al. (2002b).

[5]Provided that M_d^{-1} is bounded away from zero, as was pointed out in Ortega et al. (2002b).