ADS – proofs goal 1 & goal 2

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1 Setting

1. $n \in \mathbb{N}$,[1]
2. $m \in \mathbb{N}$,
3. $s_i = n$ for all $i \in \{1, \ldots, m\}$,
4. $p_i \in \{1, 2, \ldots, p_{\max}\}$ with $p_{\max} \in \mathbb{N}$,
5. $h_i = \bar{h}$ with $\bar{h} \in \mathbb{Z}_{\geq 0}$ a constant.
6. Every online algorithm ALG knows n, m, $p_{\max}$ and $\bar{h}$. Also, ALG is aware of constraint 3), so that it can safely wait till the last day for sending everyone home.

Consider the following algorithm $\texttt{ALG}_{\sqrt{p_{\max}}}$:

Algorithm 1 The $\lfloor\sqrt{p_{\max}}\rfloor$-threshold algorithm $\texttt{ALG}_{\sqrt{p_{\max}}}$

```
Q ← ⌊√p_max⌋
for i ← 1 to m do
    if (p_i ≤ Q ∨ i = m) then
        Buy n tickets.
    end if
end for
```

Theorem 1. *The algorithm $\texttt{ALG}_{\sqrt{p_{\max}}}$ is an optimal deterministic algorithm.*

The proof of Theorem 1 will following straightforwardly from Lemmas 2 and 3.

Lemma 2. *The algorithm $\texttt{ALG}_{\sqrt{p_{\max}}}$ is $\sqrt{p_{\max}}$-competitive.*

Proof. Consider an arbitrary instance I satisfying the restrictions above. If $m = 1$, then the problem is trivial since both ALG and OPT buy n tickets on the first day for the same price so the competitive ratio is 1. We then make the observation that both ALG and OPT always buy all tickets on a single day be design. This means that the number of people n is irrelevant for the competitive ratio c since

$$c = \frac{\texttt{ALG}_{\sqrt{p_{\max}}}(I)}{\texttt{OPT}(I)} = \frac{p_{\texttt{ALG}_{\sqrt{p_{\max}}}} \cdot n}{p_{\texttt{OPT}} \cdot n} = \frac{p_{\texttt{ALG}_{\sqrt{p_{\max}}}}}{p_{\texttt{OPT}}},$$

where $p_{\texttt{ALG}_{\sqrt{p_{\max}}}}$ and $p_{\texttt{OPT}}$ are the prices for which $\texttt{ALG}_{\sqrt{p_{\max}}}$ and OPT buy the tickets respectively. We know distinguish two more cases:

[1] In this research $\mathbb{N}$ is the set of all strictly positive integers $1, 2, \ldots$.

i) $\forall i \in \{1, \ldots, m\}, p_i > \lfloor \sqrt{p_{\max}} \rfloor$ **:** In this case we have that $p_{\texttt{ALG}_{\sqrt{p_{\max}}}}, p_{\texttt{OPT}} \geq \lfloor \sqrt{p_{\max}} \rfloor + 1$. The competitive ratio c is then maximal if $p_{\texttt{ALG}_{\sqrt{p_{\max}}}}$ is as large as possible and $p_{\texttt{OPT}}$ as small as possible, so

$$c = \frac{p_{\texttt{ALG}_{\sqrt{p_{\max}}}}}{p_{\texttt{OPT}}} \leq \frac{p_{\max}}{\lfloor \sqrt{p_{\max}} \rfloor + 1}.$$

Now notice that $\lfloor \sqrt{p_{\max}} \rfloor + 1 \geq \sqrt{p_{\max}}$ so

$$\frac{p_{\max}}{\lfloor \sqrt{p_{\max}} \rfloor + 1} \leq \frac{p_{\max}}{\sqrt{p_{\max}}} = \sqrt{p_{\max}},$$

so we conclude that in this case the competitive ratio c is indeed bounded by $\sqrt{p_{\max}}$.
ii) $\exists i \in \{1, \ldots, m\}, p_i \leq \lfloor \sqrt{p_{\max}} \rfloor$ **:** In this case we have that $p_{\texttt{ALG}_{\sqrt{p_{\max}}}}, p_{\texttt{OPT}} \leq \lfloor \sqrt{p_{\max}} \rfloor$. Maximizing the competitive ratio yields

$$c = \frac{p_{\texttt{ALG}_{\sqrt{p_{\max}}}}}{p_{\texttt{OPT}}} \leq \frac{\lfloor \sqrt{p_{\max}} \rfloor}{1} \leq \sqrt{p_{\max}},$$

so in this case the competitive ratio is also bounded by $\sqrt{p_{\max}}$.
Since cases **i)** and **ii)** together cover all remaining cases, we conclude that the competitive ratio is indeed bounded by $\sqrt{p_{\max}}$. Note the analysis is tight if we consider an instance with for example $\sqrt{p_{\max}} = 1$. $\square$

Lemma 3. *There exists no $\epsilon > 0$ and feasible online deterministic algorithm* `ALG` *such that* `ALG` *is $(\sqrt{p_{\max}} - \epsilon)$-competitive.*

Proof. Consider the instance for which $(n, m, p_{\max}) = (1, 1, 1)$ with thus $p_1 = 1$. The cost of `ALG` and `OPT` are both equal to 1, so we have found an instance for which the competitive ratio is equal to 1, while $\sqrt{p_{\max}} - \epsilon < 1$. Therefore, we conclude that we have reached a contradiction. $\square$

Comment. In fact note that every instance with $m = 1$ results in essentially the same argument. Furthermore, if $p_{\max} = k^2$ for some number $k \in \mathbb{N}$ we can provide the instances $(p_1^1, p_2^1) = (k, 1)$ and $(p_1^2, p_2^2) = (k, k^2)$. Depending on the decision the online algorithm `ALG` makes on the first day, the adversary can always choose one of the instances so that the competitive ratio becomes $k = \sqrt{p_{\max}}$. This shows that, besides the trivial sufficient example, there are more instances for which one can not find an algorithm that is better than $\sqrt{p_{\max}}$-competitive.

Proof of Theorem 1. The fact that $\texttt{ALG}_{\sqrt{p_{\max}}}$ is an optimal algorithm follows from Lemmas 2 and 3. First of all, no online deterministic algorithm can be better than $\sqrt{p_{\max}}$-competitive, which is proven in Lemma 3. Together with the fact that $\texttt{ALG}_{\sqrt{p_{\max}}}$ is $\sqrt{p_{\max}}$-competitive, which is proven in Lemma 2, this implies that $\texttt{ALG}_{\sqrt{p_{\max}}}$ is indeed an optimal deterministic algorithm. $\square$