ADS – proofs goal 1 & goal 2

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October 2023

1 Goal 1: No Algorithm can be Constant-Competitive

Consider the Strike problem as outlined by the lecture notes:

$$n, m \in \mathbb{N} \tag{1.1a}$$
$$\forall i \in \{1, \ldots, m\} : p_i \in \mathbb{N} \wedge h_i, s_i, f_i \in \mathbb{Z}_{\geq 0} \tag{1.1b}$$
$$\sum_{i=1}^{m} s_i \geq n \tag{1.1c}$$
$$\sum_{i=1}^{m} f_i = n \tag{1.1d}$$
$$\forall i \in \{1, \ldots, m\} : f_i \leq s_i \tag{1.1e}$$

Lemma 1. *Under the above constraints no online algorithm exists that does not buy the first n available seats.*

Assume an online algorithm exists that does not buy all of the first n available seats. Now consider the instance $n = 1$, $m = 2$, $s_1 = n$, $s_2 = 0$ and $p_1, p_2, h_1, h_2 \in \mathbb{Z}_{\geq 0}$. In this instance the algorithm cannot buy the seat offered at day 1, because this would break our assumption. On the second day the algorithm cannot buy a seat due to constraint 1.1e. As a result of our assumption we thus break constraint 1.1d, proving lemma 1:

$$f_1 + f_2 = 0 + 0 \neq 1$$

2 Goal 2: Algorithms and proofs

2.1 Setting

1. $n \in \mathbb{N}$,[1]
2. $m \in \mathbb{N}$,
3. $s_i = n$ for all $i \in \{1, \ldots, m\}$,
4. $p_i \in \{1, 2, \ldots, p_{\max}\}$ with $p_{\max} \in \mathbb{N}$,
5. $h_i = 0$ with for all $i \in \{1, \ldots, m\}$.
6. Every online algorithm ALG knows n, m and $p_{\max}$.

2.2 Algorithm

Consider the following algorithm $\text{ALG}_{\sqrt{p_{\max}}}$:

Algorithm 1 The $\lfloor\sqrt{p_{\max}}\rfloor$-threshold algorithm $\text{ALG}_{\sqrt{p_{\max}}}$

```
Q ← ⌊√p_max⌋
for i ← 1 to m do
    if (p_i ≤ Q ∨ i = m) then
        Buy n tickets.
    end if
end for
```

2.3 Results

Theorem 2. *For all $p_{\max} \in \mathbb{N}$ the algorithm $\text{ALG}_{\sqrt{p_{\max}}}$ is an optimal deterministic algorithm for the setting described in 2.1.*

For the proof of Theorem 2 we will use Lemma 3 below.

Lemma 3. *Let $p_{\max} \in \mathbb{N}$. Then write $p_{\max} = k^2 + \ell$ for $k, \ell \in \mathbb{N}$ with $0 \leq \ell < 2k + 1$. Then the algorithm $\text{ALG}_{\sqrt{p_{\max}}}$ is*

- $\lfloor\sqrt{p_{\max}}\rfloor$*-competitive if* $0 \leq \ell \leq k$,
- $\frac{k^2+\ell}{k+1}$*-competitive if* $k < \ell < 2k + 1$.

Proof. Consider an arbitrary instance I satisfying the restrictions above. If $m = 1$, then the problem is trivial since both ALG and OPT buy n tickets on the first day for the same price so the competitive ratio is 1. We then make the observation that both ALG and OPT always buy all tickets on a single day be design. This means that the number of people n is irrelevant for the competitive ratio c since

$$c = \frac{\text{ALG}_{\sqrt{p_{\max}}}(I)}{\text{OPT}(I)} = \frac{p_{\text{ALG}_{\sqrt{p_{\max}}}} \cdot n}{p_{\text{OPT}} \cdot n} = \frac{p_{\text{ALG}_{\sqrt{p_{\max}}}}}{p_{\text{OPT}}},$$

where $p_{\text{ALG}_{\sqrt{p_{\max}}}}$ and p_{OPT} are the prices for which $\text{ALG}_{\sqrt{p_{\max}}}$ and OPT buy the tickets respectively.
We now distinguish two more cases in which $m > 1$:

[1] In this research $\mathbb{N}$ is the set of all strictly positive integers $1, 2, \ldots$.

- **i)** $\forall i \in \{1,\dots,m\}, p_i > \lfloor\sqrt{p_{\max}}\rfloor$ **:** In this case we have that $p_{\mathrm{ALG}_{\sqrt{p_{\max}}}}, p_{\mathrm{OPT}} \geq \lfloor\sqrt{p_{\max}}\rfloor + 1$. The competitive ratio c is then maximal if $p_{\mathrm{ALG}_{\sqrt{p_{\max}}}}$ is as large as possible and p_{OPT} as small as possible, so
$$c = \frac{p_{\mathrm{ALG}_{\sqrt{p_{\max}}}}}{p_{\mathrm{OPT}}} \leq \frac{p_{\max}}{\lfloor\sqrt{p_{\max}}\rfloor + 1}.$$

- **ii)** $\exists i \in \{1,\dots,m\}, p_i \leq \lfloor\sqrt{p_{\max}}\rfloor$ **:** In this case we have that $p_{\mathrm{ALG}_{\sqrt{p_{\max}}}}, p_{\mathrm{OPT}} \leq \lfloor\sqrt{p_{\max}}\rfloor$. Maximizing the competitive ratio yields
$$c = \frac{p_{\mathrm{ALG}_{\sqrt{p_{\max}}}}}{p_{\mathrm{OPT}}} \leq \frac{\lfloor\sqrt{p_{\max}}\rfloor}{1} = \lfloor\sqrt{p_{\max}}\rfloor.$$

Now we write $p_{\max} = k^2 + \ell$ with $k, \ell \in \mathbb{N}$ and $0 \leq \ell < 2k+1$.
If $0 \leq \ell < k$ then notice that

$$\begin{aligned} k > \ell &\implies k^2 + k > k^2 + \ell \implies k(k+1) > k^2 + \ell \\ &\implies k > \frac{k^2+\ell}{k+1} \implies \lfloor\sqrt{p_{\max}}\rfloor > \frac{p_{\max}}{\lfloor\sqrt{p_{\max}}\rfloor + 1}. \end{aligned}$$

So, in this case, the competitive ratio is determined by the value k. We have that the analysis is tight if we consider the instance $I^1 = (p_1^1, p_2^1) = (k, 1)$.
If $k \leq \ell < 2k+1$ then by analogous reasoning we find $\lfloor\sqrt{p_{\max}}\rfloor \leq \frac{p_{\max}}{\lfloor\sqrt{p_{\max}}\rfloor+1}$. So, in this case, the competitive ratio is determined by the value $\frac{k^2+\ell}{k+1}$. We have that the analysis is tight if we consider the instance $I^1 = (p_1^1, p_2^1) = (k+1, k^2+\ell)$.
Thus we conclude that we have indeed found the competitive ratios as stated in the Lemma. □

We can now prove Theorem 2.

Proof of Theorem 2. Let $p_{\max} \in \mathbb{N}$. We distinguish three cases.

1. $p_{\max} = k^2$ **for** $k \in \mathbb{N}$ **:** In this case, we have $c_{\sqrt{p_{\max}}} = \sqrt{p_{\max}} = k$ (by Lemma 3). Now assume that $c_{\mathrm{ALG}} < k$. Consider the instances $I^1 = (p_1^1, p_2^1) = (k, 1)$ and $I^2 = (p_1^1, p_2^1) = (k, k^2)$ with $n = 1$ and $m = 2$. Since $c_{\mathrm{ALG}} < k$, it is not allowed to buy the ticket on the first day because this would result in a competitive ratio of k in I^1. However, for I^2 it would then have to buy the tickets on the last day which still results in a competitive ratio of k. Thus we have reached a contradiction.

2. $p_{\max} = k^2 + \ell$ **for** $k, \ell \in \mathbb{N}$ **with** $1 \leq \ell \leq k$ **:** In this case, we have $c_{\sqrt{p_{\max}}} = \sqrt{p_{\max}} = \lfloor\sqrt{p_{\max}}\rfloor = k$ (by Lemma 3). Now assume that $c_{\mathrm{ALG}} < k$. Consider the instances $I^1 = (p_1^1, p_2^1) = (k, 1)$ and $I^2 = (p_1^1, p_2^1) = (k, k^2+\ell)$ with $n = 1$ and $m = 2$. So ALG is not allowed to buy the ticket on the first day in instance I^1 because this would result in a competitive ratio of k. But again, if ALG buys the ticket on the second day then this results in a competitive ratio of $\frac{k^2+\ell}{k} > k$ for instance I^2. Thus we again reach a contradiction.

3. $p_{\max} = k^2 + \ell$ **for** $k, \ell \in \mathbb{N}$ **with** $k < \ell < 2k+1$ **:** In this case, we have $c_{\sqrt{p_{\max}}} = \frac{p_{\max}}{\lfloor\sqrt{p_{\max}}\rfloor+1} = \frac{k^2+\ell}{k+1}$ (by Lemma 3). Assume that $c_{\mathrm{ALG}} < \frac{k^2+\ell}{k+1}$. Consider the instances $I^1 = (p_1^1, p_2^1) = (k+1, 1)$ and $I^2 = (p_1^1, p_2^1) = (k+1, k^2+\ell)$ with $n = 1$ and $m = 2$. Now if ALG waits on day 1 with buying the tickets, then in I^2 this would lead to a competitive ratio of $\frac{k^2+\ell}{k+1} = c_{\mathrm{ALG}_{\sqrt{p_{\max}}}}$. So it should buy the ticket on day 1 for a competitive ratio of $k+1$. But notice that since $\ell < 2k+1$ we have
$$(k+1)^2 = k^2 + 2k + 1 > k^2 + \ell \implies k+1 > \frac{k+\ell}{k+1},$$
thus we again reach a contradiction.

Hence, we can conclude that for all $p_{\max} \in \mathbb{N}$ every feasible online algorithm ALG has competitive ratio $c_{\text{ALG}} > c_{\text{ALG}_{\sqrt{p_{\max}}}}$. □

Proof of Theorem 2. The fact that $\text{ALG}_{\sqrt{p_{\max}}}$ is an optimal algorithm for all possible $p_{\max} \in \mathbb{N}$ now follows from Lemmas 3 and **??**. First of all, no online deterministic algorithm can be better than $\sqrt{p_{\max}}$-competitive, which is proven in Lemma **??**. Together with the fact that $\text{ALG}_{\sqrt{p_{\max}}}$ is $\sqrt{p_{\max}}$-competitive, which is proven in Lemma 3, this implies that $\text{ALG}_{\sqrt{p_{\max}}}$ is indeed an optimal deterministic algorithm. □

2.4 Setting 3

1. $n, m \in \mathbb{N}$,
2. $s_i = n$ for all $i \in \{1, \ldots, m\}$,
3. $p_i \in \{1, 2, \ldots, p_{\max}\}$ with $p_{\max} \in \mathbb{N}$,
4. $h_i \in \mathbb{N}_0$.
5. Every online algorithm ALG knows n, m, $p_{\max}$.

2.5 Algorithm

Consider the following algorithm.

Algorithm 2 The double threshold algorithm ALG_2

```
Q ← ⌊√p_max⌋
H ← ⌈√p_max⌉ − 1
for i ← 1 to m do
    if (i = m ∨ p_i + Σ_{j=1}^{i−1} h_j ≤ Q ∨ Σ_{j=1}^{i} h_j ≥ H) then
        Buy n tickets.
    end if
end for
```

2.6 Results

Theorem 4. *The algorithm* ALG_2 *is at most* $\left(\sqrt{p_{\max}} + 1 - \frac{1}{\sqrt{p_{\max}}}\right)$*-competitive.*