Eindhoven University of Technology

Magnetic confinement and MHD of fusion plasmas

3MF110

Assignment 2: Design of a fusion reactor

Tim Heiszwolf (1242343)

January 21, 2025

Contents

1 Calculations

To determine the specifications of the Tokamak reactor the approach from "Plasma Physica and Fusion Energy" by Jeffrey Freidberg [2] is used in combination with Wolfram Mathematica software to automatically do the calculations (the file can be found in the attachments).

First of all we can immediately calculate c, the coil thickness, using:

$$\xi = \frac{B_{max}}{4\mu_0 \sigma_{max}}, \quad c = \frac{2\xi}{1-\xi}(\kappa a + b) \tag{1}$$

where B_{max} is the maximum magnetic field that can be produced, σ_{max} is the maximum mechanical stress on the coil that can be allowed, κ is the elongation of the plasma ($\frac{a_z}{a}$), b is the blanket thickness (set to 1.2 meter) and a is the minor radius of the torus (which is going to be varied) [2] (equation 13.210).

Next, the (major) radius of the torus is calculated using the power output and power wall load.

$$\begin{aligned} P_E &= 0.25\eta(E_\alpha + E_n + E_{Li})n^2 < \sigma v > (2\pi^2 R_0 a^2 \kappa) \\ P_W(4\pi^2 R_0 a\sqrt{\frac{1+\kappa^2}{2}}) &= 0.25\eta E_n n^2 < \sigma v > (2\pi^2 R_0 a^2 \kappa) \end{aligned} \tag{2}$$

where P_E is the (electrical) output power of the reactor, E_α, E_n and E_{Li} respectively are the energy of the alpha particles, neutrons and Lithium, n is the density, $< \sigma v >$ is the reaction cross section and R_0 is the major radius [2] (modified equation 5.18 and 5.19). The density squared and cross-section term $n^2 < \sigma v >$ can be eliminated with substitution which then allows one to solve for R_0.

These are modified to include elongation. Specifically, the area of an ellipse is $A_{elipse} = a \cdot a_z = a^2\kappa$ [5] and the circumference can be approximated to $C_{elipse} \approx 2\pi\sqrt{\frac{a^2+a_z^2}{2}} = 2\pi a\sqrt{\frac{1+\kappa^2}{2}}$ (accurate within 5% if $\kappa < 3$) [4].

Finally the minor radius can be determined. The volume of materials used is

$$V = 2\pi^2 R_0((a+b+c)(\kappa a + b + c) - \kappa a^2) \tag{3}$$

where V is the volume of the materials [2] (equation 14.163). For costs it is ideal if the amount of materials per power is minimal so

$$\frac{\partial V/P_E}{\partial a} = 0 \tag{4}$$

of which of course the positive solution for a is going to be selected. Then the plasma dimensions can be calculated

$$A_p = 4\pi R_0\sqrt{a^2\frac{1+\kappa^2}{2}}, \quad V_p = 2\pi^2 R_0 a^2 \kappa \tag{5}$$

where A_p is the area of the plasma cross-section and V_p is its volume [2] (equation 5.33).

Now all physical dimensions are determined, the other parameters can be calculated. The density can be determined via the relation between fusion/plasma power and the plants efficiency. However the fact that the Lithium reaction doesn't take place inside the plasma does need to be taken into account

$$P_\alpha + P_n = \frac{E_\alpha + E_n}{E_\alpha + E_n + E_{Li}}\frac{P_E}{\eta}, \quad P_\alpha + P_n = 0.25(E_n + E_\alpha)n^2 < \sigma v > V_p \tag{6}$$

from which then the density can be calculated [2] (equation 5.35 and 5.36). Next the pressure can be calculated

$$p = 2nT \tag{7}$$

where p is the pressure and T is the temperature [2] (equation 5.37).

Then the confinement time can be calculated with

$$\tau_e p = 8.3 \text{ atmospheres seconds} \tag{8}$$

where τ_e is the energy confinement time [2] (chapter 5.5.6).

The magnetic field can also be calculated with a simple $\frac{1}{x}$ decay

$$B_0 = B_{max} \frac{R_0 - a - b}{R_0} \tag{9}$$

[2] (equation 5.42). Then the normalized plasma pressure is calculated

$$\beta = \frac{2\mu_0 p}{B_0^2} \tag{10}$$

where β is the normalized plasma pressure [2] (equation 5.43).

Finally to calculate the plasma current one can use a scaling law for H-mode. These scaling laws require the plasma heating power which can be calculated via the confinement time and confined energy

$$P_H = \frac{3nTV_P}{\tau_E} \tag{11}$$

[2] (equation 14.158). It should be noted that for plants which produce power most of this heating power comes from the plasma itself, particularly the alpha particles, and thus can also be calculated as

$$P_H \approx \frac{E_\alpha}{E_\alpha + E_n + E_{Li}} \frac{P_E}{\eta} \tag{12}$$

[2]. These two equations also provide an alternative method to calculate the confinement time.

However, here there are some complications because in the source material for this assignment many different scaling laws are shown as an example, each having different results

$$\begin{aligned}
\text{Assignment: } \tau_E &= 0.082 I_{p,MA} P_{H,MW}^{-0.5} R_0^{1.6} \kappa^{-0.2} B_0^{0.15} A^{0.5} \\
\text{Freidberg (IPB98(y,2)): } \tau_E &= 0.145 I_{p,MA}^{0.93} P_{H,MW}^{-0.69} R_0^{1.39} a^{0.3} \kappa^{0.78} n_{20}^{0.41} B_0^{0.15} A^{0.19} \\
\text{Lecture slides: } \tau_E &= 0.048 I_{p,MA}^{0.85} P_{H,MW}^{-0.5} R_0^{1.2} \epsilon^{0.3} \kappa^{0.5} n_{20}^{0.1} B_0^{0.2} A^{0.5} \\
\text{Wikipedia (IPB98(y,2)): } \tau_E &= 0.0562 I_{p,MA}^{0.93} P_{H,MW}^{-0.69} R_0^{1.97} \epsilon^{0.58} \kappa^{0.78} n_{20}^{0.41} B_0^{0.15} A^{0.19}
\end{aligned} \tag{13}$$

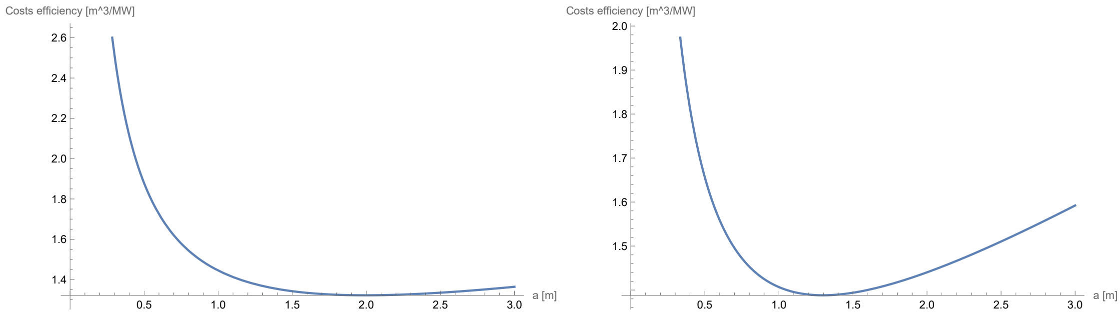
where $I_{p,ma}$ is the plasma current (in MA), $P_{H,MW}$ is the plasma heating power (in MW), A is the atomic mass of the plasma ions (in Dalton), n_{20} is the density (in $10^{20}/\text{m}^3$) and ϵ the inverse aspect ratio. [2][1][3][6]. The difference between Freidberg and Wikipedia comes from that in Freidberg the minor radius is explicit while Wikipedia accounts for it via aspect ratio (when substituting the aspect ratio of one into the other the scaling becomes identical).

However there are big differences between the other scaling laws (see table 1). Upon further investigation it was found that the scaling law from the lecture slides is ITER89-P, which is for L-mode[6]. No source was found for the scaling law from the assignment but it is assumed to be incorrect since it doesn't take into account the minor radius or density (which are very relevant). The ones based on IPB98(y,2) had the best results and had the best sources. Thus the Freidberg scaling law was used.

Table 1: Comparisons of the scaling laws for the reactor from reactor 1 (see table 2).

Scaling law	**Plasma current (MA)**
Assignment	9.96
Freidberg (IPB98(y,2))	22.13
Lecture slides	77.78
Wikipedia (IPB98(y,2))	22.22

2 Basic results



(a) No elongation $\kappa = 1$.

(b) With longation $\kappa = 1.7$.

Figure 1: Cost efficiency versus minor radius.

In table 2 one can see the results of the costs-optimized (see figure 1) reactor design. Reactor 1 is an reactor without elongation the results are very comparable to Freidberg [2] (equation 5.31 and table 5.3). It was expected that the results would be identical but this was not the case. The suspected reason for this is rounding by Freidberg done throughout his calculation. But since the differences are relatively small this is not seen as a major error. The only large difference is the plasma current (17.5 MA in the lecture slides [3]) which was discussed at the end of chapter 1.

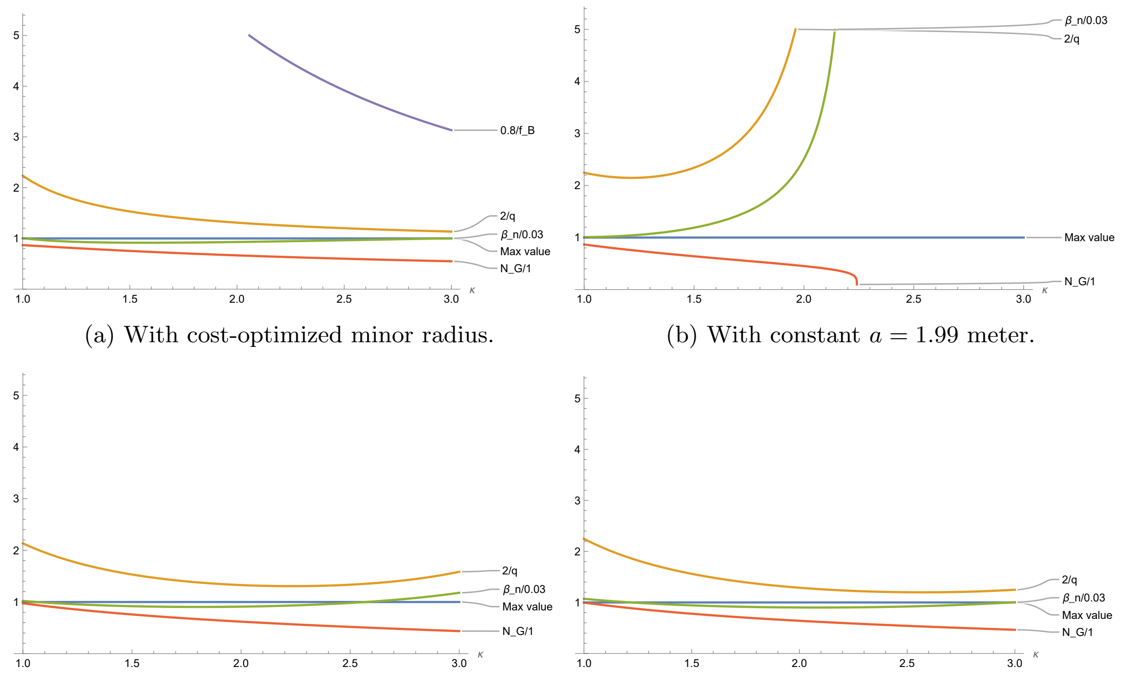
In terms of plasma limits it achieves the Greenwald limit and almost the Troyon limit but the safety limit and bootstrap fraction are not achieved. Freidberg suggests that a higher elongation might improve the results with a suggested maximum of $\kappa = 1.7$ [2]. These results can also be seen in table 2 as reactor 2. And while the costs are slightly higher the result indeed is very much improved! The Troyon limit is adhered to and while not achieved the safety factor is increased and the bootstrap fraction is greatly increased.

Table 2: Parameters of two reactors.

Property	**Reactor 1**	**Reactor 2**
Power output (P_E)	1000 MW	1000 MW
Elongation (κ)	1	1.7
Plasma temperature (T)	15 keV	15 keV
Minor radius (a)	1.99 m	1.29 m
Major radius (R_0)	5.57 m	6.14 m
Inverse aspect ratio (ϵ)	0.358	0.211
Blanket thickness (b)	1.2 m	1.2 m
Coil thickness (c)	0.806 m	0.856 m
Plasma volume (V_p)	437 m^3	346 m^3
Plasma surface area (A_p)	438 m^2	438 m^2
Magnetic field at $R = R_0$ (B_0)	5.55 T	7.71 T
Plasma current (I_p)	22.13 MA	12.76 M
Plasma pressure (p)	7.40 bar	8.31 bar
Plasma density (n)	$1.539 \cdot 10^{20}\ \text{m}^{-3}$	$1.728 \cdot 10^{20}\ \text{m}^{-3}$
Energy confinement time (τ_e)	1.12 s	0.99 s
Normalized plasma pressure (β)	6.04%	3.51%
Fusion power (P_{fusion})	2.5 GW	2.5 GW
Volume per power (V/P_E)	1.322 m^3/MW	1.388 m^3/MW
Limits		
Safety limit ($q > 2$)	1.32	1.41
Troyon limit ($\beta < 3\%$)	3.01%	2.75%
Greenwald limit ($N_G < 1$)	0.867	0.716
Bootstrap fraction ($f_B > 80\%$)	5.86%	12.41%

3 Plasma physics constraints

To find a design that (better) adheres to the plasma limits it might prove useful to see the effects of each parameter on the limits. First the elongation is investigated (see figure 2). If the minor radius is cost optimized then a higher elongation is better in terms of plasma limits. However it should be noted that a higher elongation results in the cost-optimized minor radius becoming smaller. When looking at the situation when the minor radius is kept constant with elongation there is an optimum elongation which gets 'pushed' to higher elongation for a lower minor radius (for example in figure 2b at $\kappa \approx 1.3$ and in figure 2c at $\kappa \approx 2$). Since a higher elongation thus is mostly better it was decided to keep the elongation at $\kappa = 1.7$ as recommended by Freidberg with a minor radius of 1.29 meter (reactor 2).



(a) With cost-optimized minor radius.

(b) With constant $a = 1.99$ meter.

(c) With constant $a = 1.41$ meter.

(d) With constant $a = 1.29$ meter.

Figure 2: The plasma limits versus the elongation.

3.1 Better confinement

Next, it was investigated if changing the physics constraints would affect the results. First an improvement in confinement by a factor H (see figure 3). It can be seen that:

- The safety factor improves approximately linearly ($\frac{2}{q} \propto \frac{1}{H^{1.075}}$).
- The Troyon limit gets approximately linearly worse ($\frac{\beta_n}{0.03} \propto H^{1.075}$).
- The Greenwald limit gets approximately linearly worse ($N_G \propto H^{1.075}$).
- The bootstrap fraction improves approximately quadratically ($\frac{0.8}{f_B} \propto \frac{1}{H^{2.151}}$).

So higher confinement is especially useful if the bootstrap fraction needs improvement.

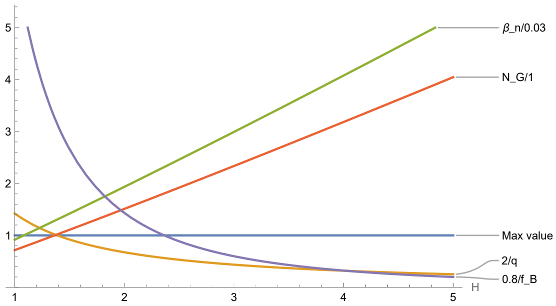


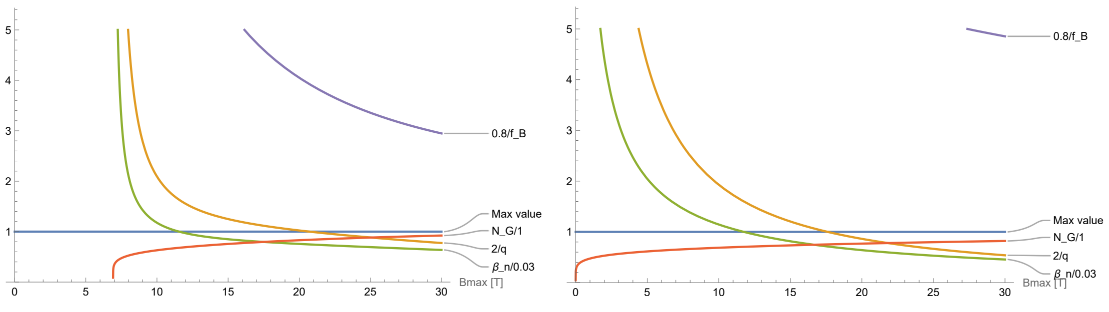
Figure 3: The plasma limits versus better confinement.

3.2 Better magnets

Better magnets could also improve the results and thus were also investigated (see figure 4):

- The safety factor improves approximately linearly ($\frac{2}{q} \propto \frac{1}{B_{max}^{1.16}}$).
- The Troyon limit improves approximately linearly ($\frac{\beta_n}{0.03} \propto \frac{1}{B_{max}^{0.84}}$).
- The Greenwald limit becomes worse very slowly ($N_G \propto B_{max}^{0.161}$).
- The bootstrap fraction improves slowly ($\frac{0.8}{f_B} \propto \frac{1}{B_{max}^{0.32}}$).

From this it can be concluded that a better magnetics are very advantageous as long as the slowly worsening Greenwald limit can be remedied.



(a) With cost-optimized minor radius.

(b) With constant $a = 1.29$ meter.

Figure 4: The plasma limits versus the maximum magnetic field strength.

3.3 Different power output

Next changing the power output (and thus the fusion power) also has an impact and was thus also investigated (see figure 5). Instead of continuously improving or worsening, the plasma constraints now show more complex behavior:

- The safety factor initially improves ($\frac{2}{q} \propto \frac{1}{P_E^{1.41}}$) and then becomes worse slowly ($\frac{2}{q} \propto P_E^{0.247}$).
- The Troyon limit initially improves ($\frac{\beta_n}{0.03} \propto \frac{1}{P_E^{1.59}}$) and then becomes worse ($\frac{\beta_n}{0.03} \propto P_E^{0.752}$).
- The Greenwald limit becomes worse ($N_G \propto P_E^{0.752}$).
- The bootstrap fraction improves linearly ($\frac{0.8}{f_B} \propto \frac{1}{P_E^{1.01}}$).

So, from this, it can be seen that (slightly) increasing the power output might be beneficial for the bootstrap current and safety factor while larger increases cause everything but the bootstrap current gets worse.

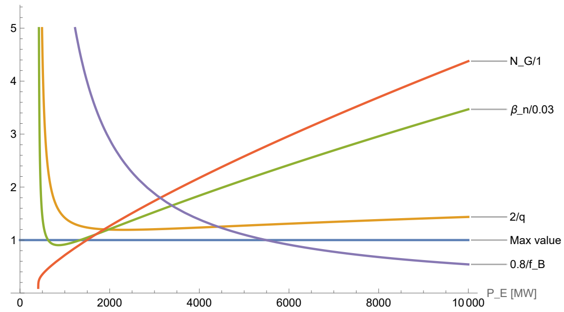


Figure 5: The plasma limits versus power output.

3.4 Varying wall load

The wall load is also relevant for the plasma constraints. In figure 6 its effects can be seen. This also has quite complex behavior:

- The safety factor initially rapidly improves ($\frac{2}{q} \propto \frac{1}{P_W^{2.59}}$) and then becomes worse ($\frac{2}{q} \propto P_W^{1.43}$).
- The Troyon limit initially improves ($\frac{\beta_n}{0.03} \propto \frac{1}{P_W^{1.91}}$) and then becomes worse ($\frac{\beta_n}{0.03} \propto P_W^{2.08}$).
- The Greenwald limit initially improves rapidly ($N_G \propto \frac{1}{P_W^{2.24}}$) and then improves slowly ($N_G \propto \frac{1}{P_W^{1.075}}$ at $P_W = 4$ MW/m^2).
- The bootstrap fraction becomes worse ($\frac{0.8}{f_B} \propto P_W^{0.15}$).

A higher wall load can help with achieving the Greenwald limit but it comes at great costs of bootstrap current and eventually also safety factor and Troyon limit. However lowering it can greatly improve the bootstrap fraction but it can't be lowered too much due to the initial steep ramp of other limits.

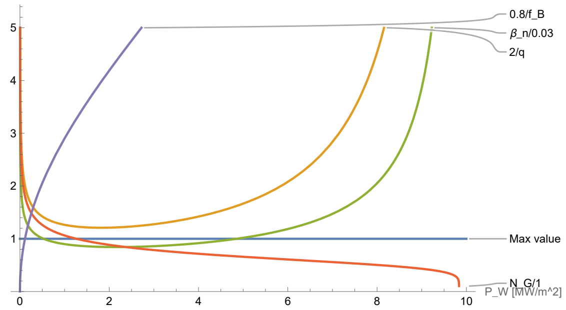


Figure 6: The plasma limits versus wall load.

3.5 Differing minor radius

Finally one could also consider varying the minor radius without regard to costs (see figure 7 and figure 1b for the effect on costs). This has a very similar effect as to varying the wall load (which is to be expected) and thus can be useful to lower the Greenwald limit but it can have large downsides if increased too much.

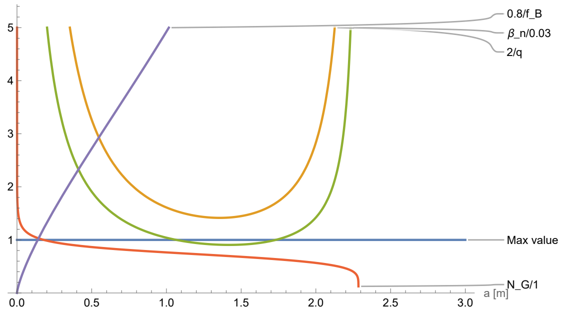
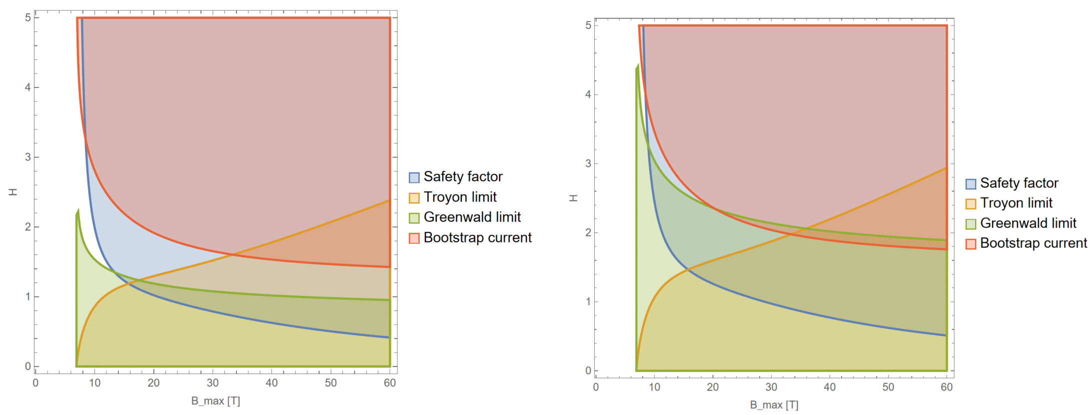


Figure 7: The plasma limits versus minor radius.

3.6 Suggestion

With this knowledge it is then interesting to attempt to find a configuration where all plasma physics constraints are satisfied. This might however prove difficult since the methods which increase the bootstrap current makes the Greenwald limit worse. However satisfying everything but the bootstrap current is relatively easy to do by for example slightly increasing the confinement ($H = 1.1$) and maximum magnetic strength ($B_{max} = 20$ T) (see figure 8a). Satisfying everything but the Greenwald limit also is not too hard, but requires a larger increase in confinement ($H = 1.6$) and an even higher magnetic strength ($B_{max} = 35$ T) (see figure 8a).

The properties of the reactor can be seen in table 3 (reactor 3) and the results are pretty decent. However the amount of material per power output is more than twice as high and the bootstrap fraction is still not reached.



(a) H versus B_{max} with $P_E = 1$ GW, $T = 15$ keV and $P_W = 4$ MW/m^2.

(b) H versus B_{max} with $P_E = 1$ GW, $T = 15$ keV and $P_W = 4$ MW/m^2.

Figure 8: Region plots of the varied parameters. Colored regions represent regions where that physics constraint is satisfied

However within this model it is possible to satisfy all 4 requirements. It requires an increase of temperature to 25 keV, a increase in confinement of $H = 2.0$ and significantly better magnets at $B_{max} = 35$ T (see figure 8b).

The parameter of can be seen in table 3 (reactor 4). While the plasma physic constraints look very good the rest of the parameter are very strange. Of note is the coil thickness which is almost twice the major radius, thus making the reactor impossible to construct. Also, the costs are 5 orders of magnitude higher which means the reactor also is unaffordable. Furthermore, it relies on increasing the temperature significantly which might mean that the Taylor approximation of the fusion cross-section might not be accurate anymore.

Table 3: Parameters of plasma physic constrained reactors.

Property	**Reactor 3**	**Reactor 4**
Elongation (κ)	1	1.7
Minor radius (a)	0.94 m	0.72 m
Major radius (R_0)	8.46 m	11.01 m
Inverse aspect ratio (ϵ)	0.111	0.065
Blanket thickness (b)	1.2 m	1.2 m
Coil thickness (c)	2.022 m	21.044 m
Plasma volume (V_p)	251 m^3	193 m^3
Plasma surface area (A_p)	438 m^2	438 m^2
Magnetic field at $R = R_0$ (B_0)	14.94 T	28.87 T
Plasma current (I_p)	6.14 MA	2.37 M
Plasma pressure (p)	9.75 bar	11.11 bar
Plasma density (n)	$2.039 \cdot 10^{20}$ m^{-3}	$1.388 \cdot 10^{20}$ m^{-3}
Energy confinement time (τ_e)	0.85 s	0.74 s
Normalized plasma pressure (β)	1.09%	0.34%
Fusion power (P_{fusion})	2.5 GW	2.5 GW
Volume per power (V/P_E)	3.101 m^3/MW	116891 m^3/MW
Adjustments		
Confinement (H)	1.1	2.0
Maximum magnetic strength (B_{max})	20 T	35
Power output (P_E)	1 GW	1 GW
Wall load (P_W)	4 MW/m^2	4 MW/m^2
Temperature (T)	15 keV	25 keV
Limits		
Safety limit ($q > 2$)	2.17	4.93
Troyon limit ($\beta < 3\%$)	2.51%	2.95%
Greenwald limit ($N_G < 1$)	0.919	0.96
Bootstrap fraction ($f_B > 80\%$)	24.23%	84.33%

4 Conclusion

In this report the dimensions/parameters and resulting plasma physics constraints were successfully modeled. A few difficulties were encountered. Of note is the use of a energy confinement scaling law different than the assignment or lecture slides. Based on this model two cost-optimized reactors were designed. These reactors however didn't satisfy the safety limit or the desired bootstrap fraction. By investigating hypothetical future developments, such as improved confinement, better magnets, etc. The most difficult to achieve is the bootstrap current but other limits could be satisfied with relatively minor changes. A solution was also found that satisfied all constraints however, this required larger magnetic coils than the major radius making it impossible to build, with a costs several orders of magnitude larger. It also relied on changing temperature to by almost a factor 2 which might have made the Taylor approximation of the cross-section invalid.

References

[1] J Freidberg, F Mangiarotti, and J Minervini. Who will save the tokamak-harry potter, arnold schwarzenegger, or shaquille o'neil? In *APS Division of Plasma Physics Meeting Abstracts*, volume 2014, pages CP8–028, 2014.

[2] Jeffrey P Freidberg. *Plasma physics and fusion energy*. Cambridge university press, 2008.

[3] Roger Jaspers. 3MF110 lecture 8: Design a Tokamak power plant.

[4] mathsisfun.com. Perimeter of Ellipse. `https://www.mathsisfun.com/geometry/ellipse-perimeter.html`. [Accessed 11-01-2025].

[5] Wikipedia. Ellipse (area). `https://en.wikipedia.org/wiki/Ellipse#Area`. [Accessed 11-01-2025].

[6] Wikipedia. High-confinement mode. `https://en.wikipedia.org/wiki/High-confinement_mode`. [Accessed 10-01-2025].