QUEEN MARY UNIVERSITY OF LONDON
SCHOOL OF PHYSICS AND ASTRONOMY

The Classical Double Copy between Electromagnetism and Gravity

Review Project

Luke Hasler
Supervisor: Dr. Christopher White

A report submitted in partial fulfilment of the
requirements for the Physics MSci (Hons) degree

April 2021

List of Contents

1 Acknowledgments

Thank you to Dr. Christopher White for taking on the challenging task of teaching me both the specific science behind the double copy and the electromagnetic principles that underpin the entirety of the subject this paper will explore.

I would also like to thank Dr. Ricardo Monteiro for teaching me his course on general relativity without which I am sure this review project could never have been completed.

Thank you also to Danai Koutsoupia Baltazani who gave me advice that led to me taking this review project for which I am grateful.

2 Introduction

As of April 2021, there is no known gauge theory of gravity. Of the four fundamental forces, gravity is the only one not to have been explained successfully by a gauge theory, so developing a link between gravity and these fundamental forces could be very important in exploring the relations between the forces and for simplifying calculations in theoretical physics. Indeed, it is the gauge theories of the other three fundamental forces that permit them to be explained by the Standard Model so a theory linking gravity to the other three fundamental forces could also contribute heavily to our understanding of particle physics (Donoghue, Golowich and Holstein (2014)).

We shall be deriving a double copy theory between the theories of electromagnetism and gravity. This is a mathematical relationship that relates the two theories when they're placed in specific conditions where we use the gauge theory for electromagnetism and Einstein's general relativity (as gravity doesn't have a known gauge theory). We shall do this by taking a gauge transform of a static point charge (Kim et al. (2020)) and by substituting a graviton field into the Einstein field equation to show that it is the Schwarzschild black hole solution. We will then depict these solutions in the Kerr-Schild form and show that they are related by a double copy relationship (Monteiro, O'Connell and White (2014)).

We will start by investigating the gauge theory of electromagnetism created by relating the electromagnetic tensor with the 4-potential and then will use this relation to derive the Maxwell equations - the fundamental equations that govern the behaviour of electric and magnetic fields. We shall then re-derive the Maxwell equations in the form of the magnetic vector potential itself.

Next, we shall use the 4-potential to demonstrate that electric and magnetic fields are gauge-invariant which results in the Maxwell equations being gauge invariant too. Then we will search for the 4-potential of a static point charge before representing this solution in the Kerr-Schild form which we shall then compare with the graviton field solution to demonstrate the double copy (Monteiro, O'Connell and White (2014)).

By showing this remarkable relationship between a static point charge and the Schwarzschild black hole we will be able to link the famous Maxwell equations of electromagnetism with the Ricci curvature tensor of gravity bringing the theories of electromagnetism and gravity into a fascinating new correspondence that is known as the classical double copy theory (Kim et al. (2020)).

It is hoped that, in the future, by investigating this fascinating correspondence other scientists may be able to advance their own studies into astronomy and cosmology using the mathematics underlying the double copy to simplify their calculations and even to make new calculations possible that were not possible before. A great deal of interest has already been shown by astronomers and cosmologists in the double copy despite being a primarily quantum field theory based subject as there is already promising work being undertaken in those areas as a result of the double copy.

It is also hoped that a contribution may be made towards the formulation of a Grand Unified Theory of the electroweak and strong nuclear forces or perhaps even lead to a so-called theory of everything in the distant future although this review project will not go far towards achieving that goal. It is simply hoped that by reviewing current knowledge on the classical double copy that others may be inspired to begin their own research into the double copy and its repercussions for the rest of quantum field theory.

3 Prerequisite Mathematical Knowledge and Conventions

3.1 Einstein Notation

In order to form the equations of electromagnetism into the required form to show the double copy relation, we will, firstly, have to formulate the theory of electromagnetism using Einstein notation (Carroll (2019)). We shall begin by introducing the Einstein summation convention:

$$c_i x^i = c_0 x^0 + c_1 x^1 + c_2 x^2 + c_3 x^3 \tag{1}$$

As is shown in Equation 1, when a variable appears as a subscript or a superscript and is not otherwise defined, it represents the summation of those terms where the variable takes values between 0 and 3. These iterations each represent a different component of a metric. For this review project these numbers will either represent:

$$0 = t, 1 = x, 2 = y, 3 = z \tag{2}$$

where t is the time dimension and x,y, and z are orthonormal basis vectors forming a Cartesian basis, or they will represent:

$$0 = t, 1 = r, 2 = \theta, 3 = \phi \tag{3}$$

where t is the time dimension, r is the range from a given point, θ is the polar angle, and ϕ is the azimuthal angle. Typically the coordinates (r,θ,ϕ) are used to define a spherical polar coordinate system. It will be made clear when we are using Einstein notation exactly which of these systems is being used at the time.

3.2 4-vectors

4-vectors are 4 dimensional vectors that usually take the form shown below:

$$j^\mu = (\rho, j_x, j_y, j_z) = (\rho, \vec{j}) \tag{4}$$

where j^μ is a contravariant 4-vector, ρ is the zeroth element of the 4-vector, and j_x, j_y, and j_z are the 1st, 2nd, and 3rd components of that 4-vector which can be simplified to $\vec{j}$ where $\vec{j}$ is a vector comprising the three final components of j^μ (Hartle (2003)).

The covariant 4-vector is defined as follows:

$$j_\mu = (\rho, -j_x, -j_y, -j_z) = (\rho, -\vec{j}) \tag{5}$$

where the covariant vector is denoted by the lowering of the index μ.

An important usage of 4-vectors we shall come across frequently is that of the covariant and contravariant derivative. They are both shown below:

$$\partial_\mu = (\frac{\partial}{\partial t}, \vec{\nabla}) = (\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \Rightarrow \partial^\mu = (\frac{\partial}{\partial t}, -\vec{\nabla}) = (\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}) \tag{6}$$

where the covariant derivative is denoted by ∂_μ and the contravariant derivative is denoted by ∂^μ. Prior knowledge of partial derivatives is assumed.

4 Electromagnetism

4.1 The Electromagnetic Tensor

Now that we have cleared up some fundamental mathematics required for understanding this report we can move on to formulating the equations of electromagnetism in the relativistic form required for formulating the double copy. We shall begin with the electromagnetic tensor shown below:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} = -F_{\nu\mu} \tag{7}$$

where $F_{\mu\nu}$ is the electromagnetic tensor and each element of the tensor involving the letter E relates to the electric field at a point in spacetime, and any point using the letter B relates to the magnetic field at that point in spacetime. As described earlier, the first, second, and third components of these fields are the x, y, and z components forming an orthonormal basis. An alternative way of writing this metric is written below:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} = -F_{\nu\mu} \tag{8}$$

$F_{\mu\nu} = -F_{\nu\mu}$ because of the anti-symmetric nature of the electromagnetic tensor. To be sure of clarity, the meaning of F_{01} is E_1 as the first index corresponds to the row to find the required element, and the second index corresponds to the column. We can also see that the anti-symmetric nature of the electromagnetic tensor is satisfied as follows:

$$F_{01} = -F_{10} = E_1 \tag{9}$$

4.2 Formulating the Maxwell Equations

We shall now formulate the Maxwell equations, the fundamental equations that govern how electric and magnetic fields interact, using only two equations and the electromagnetic tensor. We are going to require the following equation to derive Gauss' law:

$$\partial^\nu F_{\mu\nu} = j_\mu \tag{10}$$

where j_μ is the covariant 4-current (Russakoff (1970)). The equation for the covariant 4-current is shown below:

$$j_\mu = (\rho, -\vec{j}) \tag{11}$$

where ρ is the volume charge density, and $\vec{j}$ is the vector current. When we allow $\mu = 0$ the following two equations can be formulated:

$$\partial^\nu F_{0\nu} = j_0 = \rho \tag{12}$$

$$\partial^\nu F_{0\nu} = \partial^0 F_{00} + \partial^1 F_{01} + \partial^2 F_{02} + \partial^3 F_{03} = \frac{\partial 0}{\partial t} + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{13}$$

By equating equations 12 and 13 we find Gauss' law below:

$$\nabla \cdot \vec{E} = \rho \tag{14}$$

Next we can allow $\mu = 1$ and repeat the same process to recover Ampère's law including the correction later added by James Clerk Maxwell:

$$\partial^\nu F_{1\nu} = j_1 = -j_x \tag{15}$$

$$\partial^\nu F_{1\nu} = \partial^0 F_{10} + \partial^1 F_{11} + \partial^2 F_{12} + \partial^3 F_{13} = -\frac{\partial E_x}{\partial t} - \frac{\partial 0}{\partial x} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \tag{16}$$

By equating equations 15 and 16 we can find equation 17 below:

$$j_x = (\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t})_1 \tag{17}$$

This then produces the x-component of Ampère's law (Russakoff (1970)). A similar procedure can be followed to find the y and z-components of Ampère's law so we can finally produce the entirety of Ampère's law as shown below:

$$\vec{j} = \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} \tag{18}$$

We will need a new equation to derive Gauss's law for magnetism and Faraday's law. It is given below:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{19}$$

If we choose to set $\lambda = 1, \mu = 2$, and $\nu = 3$ we can derive Gauss' law for magnetism:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0 \tag{20}$$

Then after substituting elements of the electromagnetic tensor and the covariant derivative we obtain equation 21:

$$-(\frac{\partial(B_x)}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}) = 0 \tag{21}$$

Finally, we obtain Gauss' law for magnetism:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{22}$$

Next, in order to obtain Faraday's law we simply have to set $\lambda = 0, \mu = 1, \nu = 2$ and repeat the same process:

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = 0 \tag{23}$$

$$-\frac{\partial B_z}{\partial t} - \frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} = 0 \tag{24}$$

Here we have obtained the z-component of Faraday's law but a similar process can be followed to find the x and y-components and, as such we have derived Faraday's law (Russakoff (1970)).

$$(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t})_3 = 0 \Rightarrow \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{25}$$

Deriving the Maxwell equations from the electromagnetic tensor shows that the electromagnetic tensor does indeed represent electromagnetism in Einstein's relativistic notation. We have managed to derive the Maxwell equations without the need to investigate Coulomb's law, without taking the usual integrals, and without using the divergence theorem or Stokes' theorem (Russakoff (1970)).

This also leads to the helpful conclusion that equations 10 and 19 are the Maxwell equations in a more compact form and as such provides the reasoning behind our usage of the electromagnetic tensor.

We can now proceed to explore how we can use the electromagnetic tensor to further investigate the behaviour of electromagnetic fields under relativistic conditions.

4.3 The Magnetic 4-Potential

We can represent the electromagnetic tensor using the following equation:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{26}$$

where A_μ and A_ν represent the 4-potential (Moriyasu (1983)). The forms of the covariant 4-potential and the contravariant 4-potential are shown below:

$$A_\mu = (\phi, -\vec{A}) \Rightarrow A^\mu = (\phi, \vec{A}) \tag{27}$$

where ϕ is the electric potential and $\vec{A}$ is the magnetic vector potential.

Perhaps it would be wise to check the validity of the electromagnetic tensor using this new equation. In order to do this, we will need expressions that relate the 4-potential with the electric field and the magnetic field. The relation between the 4-potential and the electric field is shown below:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{28}$$

where $\vec{\nabla}\phi$ is the gradient of ϕ. The relation between the magnetic vector potential and the magnetic field is shown below:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{29}$$

where $\vec{\nabla} \times \vec{A}$ is the curl of $\vec{A}$.

Now that we have equations 26, 28, and 29 we can finally check the validity of the electromagnetic tensor:

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 = \frac{\partial A_1}{\partial t} - \frac{\partial A_0}{\partial x} = -\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} = E_1 \tag{30}$$

We have successfully found that F_{01} is E_1. This once again supports the validity of the electromagnetic tensor and the validity of equation 26.

4.4 Gauge Invariance

We will now show that the equations of electromagnetism are invariant under a gauge transformation. In the following section, the 4-potential will be written A'^μ and the electromagnetic tensor will be written $F'_{\mu\nu}$ when they have been gauge transformed (Moriyasu (1983)).

Firstly, let us construct a gauge transformed electromagnetic tensor:

$$\partial_\mu(A'^\mu) - \partial_\nu(A'^\mu) = F'_{\mu\nu} \tag{31}$$

The form of the gauge transform takes the form shown below:

$$A'^\mu = A^\mu + \partial^\mu \chi \tag{32}$$

where χ is an arbitrary non-zero function. We will now substitute equation 32 into equation 26 to investigate the behaviour of the gauge transformed electromagnetic tensor:

$$F'_{\mu\nu} = \partial_\mu(A_\nu + \partial_\nu \chi) - \partial_\nu(A_\mu + \partial_\mu \chi) = \partial_\mu A_\nu + \partial_\mu \partial_\nu \chi - \partial_\nu A_\mu - \partial_\mu \partial_\nu \chi = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \tag{33}$$

Since $F'_{\mu\nu} = F_{\mu\nu}$ the electromagnetic tensor is unchanged by the transformation and the equations of electromagnetism remain unchanged (Kim et al. (2020)).

4.5 4-Potential of a Static Point Charge

We can now use the knowledge we have gained to investigate the 4-potential for a static point charge. A point charge is an infinitesimally small point in space with a unit charge and as it is confined to be static there will be no accelerating magnetic fields and the electric and magnetic fields surrounding the point charge will be radial in nature (Kim et al. (2020)).

We start by using the Lorenz Gauge shown below:

$$\partial_\mu A^\mu = 0 \tag{34}$$

We now substitute a contravariant version of the gauge transformed A'^μ found in equation 32 into equation 34 (Kim et al. (2020)).

$$\partial_\mu(A^\mu + \partial^\mu \chi) = 0 \Rightarrow \partial_\mu \partial^\mu \chi = 0 \tag{35}$$

Now recall that the electric potential for a point charge is given by:

$$\phi(r) = \frac{Q}{4\pi r} \tag{36}$$

where Q is the charge of the point charge, and r is the radius between the particle and the point where the electric potential is measured (Kim et al. (2020)). We can now form the 4-potential for a static point charge:

$$A^{\mu} = (\frac{Q}{4\pi r}, \vec{0}) \tag{37}$$

4.6 A Gauge Transformation of the 4-Potential of a Static Point Charge

We now shall search for a gauge transformation that satisfies the equation:

$$A'_{\mu} A'^{\mu} = 0 \tag{38}$$

as it will allow us to take the double copy of a graviton field solution which we will explore in later chapters of this review project. We firstly choose to impose the condition on χ of being a function that only depends on the radius:

$$\chi = \chi(r) \tag{39}$$

Then, we multiply the covariant and contravariant gauge transformed 4-potentials and expand the brackets:

$$(A_{\mu} + \partial^{\mu}\chi)(A^{\mu} + \partial_{\mu}\chi) = A_{\mu}A^{\mu} + A_{\mu}\partial_{\mu}\chi + A^{\mu}\partial^{\mu}\chi + \partial^{\mu}\chi \cdot \partial_{\mu}\chi \tag{40}$$

Next, we substitute the values of the 4-potential for a point charge into the equation to obtain the following:

$$= \frac{Q^2}{(4\pi r)^2} + \frac{Q}{4\pi r}\frac{\partial \chi}{\partial t} - \frac{Q}{4\pi r}\frac{\partial \chi}{\partial t} - (\frac{\partial \chi}{\partial t})^2 - (\frac{\partial \chi}{\partial x})^2 - (\frac{\partial \chi}{\partial y})^2 - (\frac{\partial \chi}{\partial z})^2 \tag{41}$$

We now realise that due to the static nature of the point charge all terms depending on time can be set to 0 and that the remaining terms depending on x, y, and z differentials can be expressed as the squared modulus of the gradient of χ:

$$= \frac{Q^2}{(4\pi r)^2} - |\vec{\nabla}\chi|^2 \tag{42}$$

Now we shall begin to work with spherical polar coordinates rather than Cartesian coordinates. This makes sense as the point charge emits fields radially and as such all points on the surface of a sphere surrounding the particle would have the same field values (Kim et al. (2020)). We begin by expressing the gradient of χ in spherical polar coordinates and rewriting our problem as follows:

$$= \frac{Q^2}{(4\pi r)^2} - (\frac{\partial \chi}{\partial r} + \frac{1}{r}\frac{\partial \chi}{\partial \theta} + \frac{1}{rsin(\theta)}\frac{\partial \chi}{\partial \phi})^2 \tag{43}$$

We can cancel terms in this representation of our problem by remembering that χ depends solely on r as decided in equation 40 and then rearrange the remaining terms to find the following relation:

$$\frac{Q^2}{(4\pi r)^2} = (\frac{\partial \chi}{\partial r})^2 \tag{44}$$

Finally, we can rearrange to find a solution for χ that satisfies our original problem:

$$\chi = \int_R \frac{\pm Q}{4\pi r} dr = \frac{\pm Q ln(R)}{4\pi} + c \tag{45}$$

Now that we have found a suitable gauge transformation for the 4-potential of a static point charge we will be able to create a double copy between a point charge and the Schwarzschild black hole. In order to do this, we will need to explore the theory of general relativity and learn some new mathematics that we shall read into below.

5 General Relativity

5.1 Introduction to General Relativity

General relativity was discovered by Albert Einstein in 1915. It is a theory that describes space and time as one unified concept: spacetime (Woodhouse (2007)). It says that the speed of light is the same for any observer and as such that spontaneity is not absolute and uses matrices called metrics along with mathematics in the form of Einstein notation to describe the ways that matter and energy shape spacetime and are in turn shaped by spacetime. The primary method of describing the trajectory of a particle through spacetime is by using a line element - an infinitesimal line segment relating to a metric in the pseudo-Riemannian manifold that we model spacetime to consist of (Carroll (2019)).

I unfortunately will not be able to explain the entirety of general relativity in this review project but we shall need some essential mathematics from general relativity to continue. As such let us begin by introducing the concept of a metric. The Minkowski metric that defines the way particles interact in a Minkowski spacetime is shown below:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{46}$$

where $\eta_{\mu\nu}$ is the Minkowski metric written using Cartesian coordinates (Wald (2010)). The zeroth component represents the time component, and the first, second, and third components represent the x, y, and z components consecutively. Notice that all off-diagonal components are equal to 0. It is simple to read off each element - μ represents the row number and ν represents the column number so $\eta_{00} = -1$ and $\eta_{11} = \eta_{22} = \eta_{33} = 1$. All other elements are 0.

Another way of depicting this metric is by using its line element. The Minkowski line element is shown below:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{47}$$

where the line element is represented by ds^2. We can now choose to rewrite the Minkowski metric in spherical polar coordinates which will simplify our calculations later considerably:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2(\theta) \end{pmatrix} \tag{48}$$

where we have chosen the zeroth element of the metric to be time, the first element to be the radius r from the origin, the second element to be the polar angle θ, and the third to be the azimuthal angle ϕ (Carroll (2019)). This produces the line element shown below:

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{49}$$

Now we're acquainted with the Minkowski metric we can explore a more complex metric that will prove essential to demonstrating the double copy: the Schwarzschild metric. The Schwarzschild metric is a metric that accurately represents the interactions of a Schwarzschild black hole in a vacuum state (i.e. not at the centre of the black hole) with spacetime (Monteiro, O'Connell and White (2014)). It has some interesting properties that we shall explore when we have learned more about general relativity and have introduced the Einstein field equation. The Schwarzschild metric is shown below:

$$g_{\mu\nu} = \begin{pmatrix} -(1-\frac{2GM}{r}) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2GM}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2(\theta) \end{pmatrix} \tag{50}$$

where $g_{\mu\nu}$ is the Schwarzschild metric, G is Newton's constant of gravitation, and M is the mass of the black hole (Hartle (2003)). The line element for this metric is shown below:

$$ds^2 = -(1-\frac{2GM}{r})dt^2 + (\frac{1}{1-\frac{2GM}{r}})dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{51}$$

This will all prove useful but we are still far from having the mathematics needed to check our double copy against the Einstein field equation. Let us begin by looking at Christoffel symbols:

$$\Gamma^{\nu}_{\mu\lambda} = \frac{1}{2}g^{\nu\alpha}(\partial_\mu g_{\lambda\alpha} + \partial_\lambda g_{\mu\alpha} - \partial_\alpha g_{\mu\lambda}) = \Gamma^{\nu}_{\lambda\mu} \tag{52}$$

where Γ is a Christoffel symbol with indices that must be inserted into the equation to find the value of the symbol, and where α must be summed over for all possible values (meaning t, r, θ, and ϕ) (Woodhouse (2007)). Notice that the ordering of the two bottom indices can be swapped and the resulting two Christoffel symbols will have the same value. $g^{\mu\nu}$ is the contravariant version of the metric and since the inverse of a metric transforms contravariantly we can write the simple relation below:

$$g^{\mu\nu} = (g_{\mu\nu})^{-1} \tag{53}$$

where this indicates the inverse of the covariant metric. There will be 64 Christoffel symbols for a 4×4 metric such as the ones we shall be dealing with. Now that we know how to form Christoffel symbols for a metric we can use them in the Riemann curvature tensor shown below:

$$R^{\lambda}_{\rho\mu\nu} = \partial_{\mu}\Gamma^{\lambda}_{\nu\rho} - \partial_{\nu}\Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} \tag{54}$$

where $R^{\lambda}_{\rho\mu\nu}$ is the Riemann curvature tensor (Wald (2010)). We must again sum over σ with each of the four components we are using. The Riemann curvature tensor is a tensor field for a Riemannian manifold and describes the curvature of spacetime at different points. We shall need it to now find the Ricci tensor below:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = R_{\nu\mu} \tag{55}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor and where λ again must be summed over (Wald (2010)). Notice that the Ricci curvature tensor is a symmetric tensor which reduces the number of Riemann curvature tensors we shall have to calculate later on. The Ricci curvature tensor measures the amount of deviation our metric produces at a given point from Euclidian space (Carroll (2019)). Finally, we can use the Ricci curvature tensor to find the Ricci scalar using the equation below:

$$R = g^{\mu\nu}R_{\mu\nu} \tag{56}$$

where R is the Ricci scalar. We now have all the mathematical knowledge necessary to use the Einstein field equation shown below:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{57}$$

where $T_{\mu\nu}$ is the energy-momentum tensor which describes the distribution of mass and energy, Λ is the cosmological constant which for this report we shall set to 0, and where $G_{\mu\nu}$ is the Einstein tensor shown below:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} \tag{58}$$

Combining these two equations we are left with:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{59}$$

An important point to make here is that for empty space the density of matter and energy is 0 so the value of the energy-momentum tensor $T_{\mu\nu}$ is 0 unless we are at the centre of the black hole (Monteiro, O'Connell and White (2014)). This means that for a vacuum:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{60}$$

and the unique, static and spherically symmetric solution to this equation is the Schwarzschild metric that was shown earlier. If a metric can be shown to satisfy equation 60 then it is the Schwarzschild solution merely written in a different form. We shall now endeavor to do just that.

5.2 The Graviton Field Solution

The graviton is a theoretical particle that would act as a force carrier for gravity. The other fundamental forces each have their own force carrier particles, namely photons for electromagnetism, gluons for the strong nuclear force, and W and Z bosons for the weak nuclear force but no such particle has been discovered for gravity (White (2018)). We shall create a metric incorporating a graviton field into the Minkowski metric for spherical polar coordinates and then show that this corresponds to the Schwarzschild solution by testing this metric using the Einstein field equation.

We shall firstly define a 4-vector k_μ:

$$k_\mu = (1, 1, 0, 0) \tag{61}$$

where the components of the 4-vector are those of spherical polar coordinates. We now construct a known metric for the graviton field $h_{\mu\nu}$:

$$h_{\mu\nu} = \phi k_\mu k_\nu \Rightarrow \phi = \frac{\kappa M}{4\pi r} \tag{62}$$

where κ is a constant. We then add this metric to the Minkowski metric in spherical polar coordinates to form a new metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{63}$$

In matrix form this is written as follows:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2(\theta) \end{pmatrix} + \begin{pmatrix} \frac{\kappa M}{4\pi r} & \frac{\kappa M}{4\pi r} & 0 & 0 \\ \frac{\kappa M}{4\pi r} & \frac{\kappa M}{4\pi r} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{64}$$

and to simplify this appears as such:

$$g_{\mu\nu} = \begin{pmatrix} -1+\frac{\kappa M}{4\pi r} & \frac{\kappa M}{4\pi r} & 0 & 0 \\ \frac{\kappa M}{4\pi r} & 1+\frac{\kappa M}{4\pi r} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2(\theta) \end{pmatrix} \tag{65}$$

Now we find the contravariant metric by taking the inverse of the metric (Carroll (2019)). The result is shown below:

$$g^{\mu\nu} = \begin{pmatrix} -1-\frac{\kappa M}{4\pi r} & \frac{\kappa M}{4\pi r} & 0 & 0 \\ \frac{\kappa M}{4\pi r} & 1-\frac{\kappa M}{4\pi r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 sin^2(\theta)} \end{pmatrix} \tag{66}$$

The line element of this metric has been provided below:

$$ds^2 = (-1+\frac{\kappa M}{4\pi r})dt^2 + (1+\frac{\kappa M}{4\pi r})dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2) + \frac{\kappa M}{2\pi r}dtdr \tag{67}$$

5.3 Christoffel Symbols for the Graviton Field Solution

In order to test our solution with the Einstein field equation, we shall need to calculate all 64 Christoffel symbols for the metric. The result we find is that only nineteen Christoffel symbols are non-zero and of them only fourteen are unique. Let us firstly demonstrate how to find Γ^t_{tt}:

$$\Gamma^t_{tt} = \frac{1}{2}g^{t\alpha}(\partial_t g_{t\alpha} + \partial_t g_{t\alpha} - \partial_\alpha g_{tt}) \tag{68}$$

where α is summed over the the four components t, r, θ, and ϕ. It is important to notice that none of the elements of our metric are functions of t or of ϕ so we can set all terms containing a time derivative or a partial derivative with respect to ϕ to zero and furthermore any terms which are non-diagonal and contain either θ or ϕ can also be set to zero. This leaves us with:

$$\Gamma^t_{tt} = \frac{1}{2}g^{tr}(-\partial_r g_{tt}) = \frac{\kappa^2 M^2}{32\pi^2 r^3} \tag{69}$$

where we have simply substituted in the elements of the covariant and contravariant matrices to obtain our final answer. All other Christoffel symbols have been calculated and all the non-zero Christoffel symbols are shown below:

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{\kappa M}{8\pi r^2} + \frac{\kappa^2 M^2}{32\pi^2 r^3} \qquad \Gamma^t_{rr} = \frac{\kappa M}{4\pi r^2} + \frac{\kappa^2 M^2}{32\pi^2 r^3} \qquad \Gamma^t_{\theta\theta} = -\frac{\kappa M}{4\pi} \tag{70}$$

$$\Gamma^t_{\phi\phi} = -\frac{\kappa M sin^2(\theta)}{4\pi} \qquad \Gamma^r_{rr} = -\frac{\kappa M}{8\pi r^2} - \frac{\kappa^2 M^2}{32\pi^2 r^3} \qquad \Gamma^r_{tr} = \Gamma^r_{rt} = -\frac{\kappa^2 M^2}{32\pi^2 r^3} \tag{71}$$

$$\Gamma^r_{tt} = \frac{\kappa M}{8\pi r^2} - \frac{\kappa^2 M^2}{32\pi^2 r^3} \qquad \Gamma^r_{\theta\theta} = \frac{\kappa M}{4\pi} - r \qquad \Gamma^r_{\phi\phi} = (\frac{\kappa M}{4\pi} - r)sin^2(\theta) \tag{72}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r} \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{cos(\theta)}{sin(\theta)} \qquad \Gamma^\theta_{\phi\phi} = -sin(\theta)cos(\theta) \tag{73}$$

5.4 Ricci Tensor and Scalar For the Graviton Field Solution

Having found the Christoffel symbols for the graviton field solution we can use them to find the Riemann curvature tensor and finally use the elements of the Riemann curvature tensor to find the Ricci curvature tensor (Carroll (2019)).

When we have completed these calculations we find that all of the elements of the Ricci curvature tensor are 0:

$$R_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{74}$$

This of course means that the Ricci scalar is also equal to 0 as it is dependent on the value of the Ricci curvature tensor. This is a remarkable result as it confirms that the graviton field metric we have created does indeed satisfy the Einstein field equations for a vacuum and as such is analogous to the Schwarzschild solution (Monteiro, O'Connell and White (2014)).

6 Kerr-Schild Perturbations

6.1 Understanding Kerr-Schild Perturbations

The equation we used to describe the graviton field shown below is an example of a Kerr-Schild perturbation - a perturbation to a spacetime metric that linearises the Einstein field equations by linearising the Ricci tensor R^{μ}_{ν} (Bah, Dempsey and Weck (2020)).

$$h_{\mu\nu} = \phi k_{\mu} k_{\nu} \tag{75}$$

A general example of a Kerr-Schild perturbation is shown below:

$$h_{\mu\nu} = V l_{\mu} l_{\nu} \tag{76}$$

where V is a scalar field that varies at different points in spacetime and l_{μ} is a null vector with respect to the background spacetime that is being studied. It is clear from this example that for the graviton field we allow the constant V to be the gravitational potential multiplied by some constant κ and we let l_{μ} take the form k_{μ} where $k_{\mu} = (1, 1, 0, 0)$.

We can choose to take a single copy of a Kerr-Schild perturbation - this essentially means we remove the l_{ν} from equation 76 and let the perturbation only consist of one index (Monteiro, O'Connell and White (2014)). This appears generally as so:

$$h_{\mu} = V l_{\mu} \tag{77}$$

If we had approached this from the opposite direction we could instead have multiplied equation 77 by l_{ν} to obtain equation 76. This would be a double copy relationship. Now that we understand what a single and double copy relationship are under the scope of Kerr-Schild perturbations we can explore how we can create a double copy between the graviton field equation and an equation for the 4-potential (Bah, Dempsey and Weck (2020)).

As a side note, a zeroth copy is often used to describe the relationship between an equation such as equation 77 and its scalar field. The scalar field equation that would be created via a zeroth copy is shown below:

$$V = V \tag{78}$$

6.2 The Kerr-Schild 4-Potential Equation

Having found values for χ that allow the covariant and contravariant gauge transformed 4-potentials to be multiplied and equal 0 we can perform the following calculation:

$$A'^{\mu} = (\frac{Q}{4\pi r}, \vec{0}) + (\frac{\partial \chi}{\partial t}, -\vec{\nabla}\chi) \Rightarrow A'^{\mu} = (\frac{Q}{4\pi r}, -\vec{\nabla}\chi) \tag{79}$$

where the value of $\vec{\nabla}\chi$ can be approximated using the following equation:

$$\vec{\nabla}\chi = \frac{\partial \chi}{\partial r}\hat{r} + ... \tag{80}$$

This leaves us with the following value for the gauge transformed 4-potential:

$$A'^{\mu} = (\frac{Q}{4\pi r}, \frac{\pm Q}{4\pi r}\hat{r}) \tag{81}$$

which finally leaves us with this crucial equation for the gauge transformed 4-potential:

$$A'^{\mu} = \phi k^{\mu} \tag{82}$$

where $\phi = \frac{Q}{4\pi r}$. This is a remarkable result as this shows that the gauge transformed 4-potential is the single copy of the graviton field equation, and the graviton field equation is a double copy of the gauge transformed 4-potential (Monteiro, O'Connell and White (2014)).

This could suggest that there is some underlying mathematics that links the two seemingly unrelated theories which could be crucial for understanding the way that these two fundamental forces are linked considering we currently do not have a gauge theory for gravity.

We have shown that a double copy exists for one solution of equation 82 but we should now attempt to show that, due to the properties of Kerr-Schild perturbations, there are infinite solutions for the Maxwell equations that form a double copy.

6.3 Probing the Kerr-Schild Perturbations

Firstly, it is important to say that there will be infinite solutions of the double copy as we can set the constant ϕ to infinitely many values and as such there is an electromagnetic double copy for any graviton field we can create (Monteiro, O'Connell and White (2014)). There are also infinitely many different values that the 4-vector $k_{\mu\nu}$ can take and as such there are an infinite number of solutions for double copies between electromagnetism and gravity. As I described earlier, Kerr-Schild perturbations linearise the Einstein field equation by linearising the Ricci tensor R^{μ}_{ν}. We shall explore this below by using the following two equations:

$$R = \partial_{\mu}\partial_{\nu}(\phi k^{\mu} k^{\nu}) \tag{83}$$

$$R^{\mu}_{\nu} = \frac{1}{2}(\partial^{\mu}\partial_{\alpha}(\phi k^{\alpha} k_{\nu}) + \partial_{\nu}\partial^{\alpha}(\phi k_{\alpha} k^{\mu}) - \partial^{2}(\phi k^{\mu} k_{\nu})) \tag{84}$$

We can use the fact that here $k^a = k_a$ and $\partial^a = \partial_a$ to simplify our mathematics when we calculate R^0_0 to find that:

$$R^0_0 = \frac{1}{2}\vec{\nabla}^2\phi \tag{85}$$

where $\vec{\nabla}^2\phi$ is the Laplacian of ϕ. We then can find the following relation for R^i_0:

$$R^i_0 = \frac{1}{2}\partial_j(\partial^i(\phi k^j) - \partial^j(\phi k^i)) \tag{86}$$

We can then show the following equation for R^i_j:

$$R^i_j = \frac{1}{2}\partial_l[\partial^i(\phi k^l k_j) + \partial_j(\phi k_l k^i) - \partial^l(\phi k^i k_j)] \tag{87}$$

Now let us show that the following relation, that we require to show that our double copy is correct, is true:

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu(\phi k^\nu) - \partial^\nu(\phi k^\mu)) = 0 \tag{88}$$

This is important as it shows the following relation:

$$R^\mu_0 = \partial_\mu F^{\mu\nu} \tag{89}$$

which is the first time we have successfully directly related general relativity and the electromagnetic tensor (Monteiro, O'Connell and White (2014)). We can go even further by writing the following relation:

$$\partial_\alpha(\partial^\alpha(\phi k^i) - \partial^i(\phi k^\alpha)) = 0 \Rightarrow \partial_\alpha F^{\alpha i} = 0 \tag{90}$$

Next we substitute 0 as the value of the index i:

$$\partial_\alpha F^{\alpha 0} = \partial_\alpha(\partial^\alpha(\phi k^0) - \partial^0(\phi k^\alpha)) = \partial_\alpha(\partial^\alpha(\phi)) = \vec{\nabla}^2\phi \tag{91}$$

where we can perform this simplification because $k^0 = 1$ and because the time derivative of ϕk^α is 0. Once again we have found an intriguing relationship between general relativity and the electromagnetic tensor which is shown below:

$$R_0^0 = \partial_\mu F^{\mu 0} \tag{92}$$

This result proves that $A^\mu = \phi k^\mu$ has infinite solutions that satisfy the Maxwell equations and links the Maxwell equations to general relativity (Monteiro, O'Connell and White (2014)). This is a huge result as there is no gauge theory for gravity yet we have managed to show a relation between the two theories of the fundamental forces meanwhile preserving the property of all Kerr-Schild perturbations that they should have infinite solutions.

6.4 Yang-Mills Theory

Whilst we have approached this double copy from the angle of electromagnetism for simplicity this whole relation can be explained using Yang-Mills theory (White (2018)). Yang-Mills theory is a quantum field theory that tracks the interactions of gluons but which, when simplified, yields the Maxwell equations (Jackiw (1980)).

Yang-Mills theory describes the interactions within quark fields which take place within a 3D quark space consisting of red, green, and blue components that we call colours (Donoghue, Golowich and Holstein (2014)). These are not colours taken from the spectrum of light but are merely names applied to these three different forms of charge that a gluon can have.

Gluons are one of the four known force carriers and are emitted when a quark changes its colour charge. In order to conserve colour charge the gluon that is emitted carries some colour charge with it and as such gluons can interact with one another in a way that the force carrier for electromagnetism - the photon - cannot as the photon does not carry charge. This is why the equations for electromagnetism are linear and the equations used in Yang-Mills theory are non-linear - interactions between gluons add an extra level of complexity that makes the equations non-linear (Jackiw (1980)).

Yang-Mills theory was also where the first double copy relationship was noticed in the form of a double copy relationship between scattering amplitudes in gluons and gravity and was discovered before the classical double copy we have investigated was researched (Bern, Carrasco and Johansson (2010)).

Another feature of the Maxwell equations that is important in Yang-Mills theory is the fact that we have shown that there exist gauge transformations within the Maxwell equations. From the perspective of Yang-Mills theory, this means that the electron field must have phase and that we can set the 0 of the phase wherever is useful for calculation (Jackiw (1980)).

Yang-Mills theory produces eight copies of each of the Maxwell equations when the equations of Yang-Mills theory are simplified by imposing the correct conditions and assumptions so creating a double copy between electromagnetism and gravity also creates a link between gravity and the strong nuclear force. For this reason, our research has demonstrated a link between three of the four fundamental forces without needing a gauge theory or a quantum field theory for gravity.

Theories such as the electroweak theory attempt to link the electromagnetic and weak nuclear forces (Bilenky and Hošek (1982)) and many other theories called Grand Unified Theories, for example (Farhi and Susskind (1979)), attempt to link the electroweak force with the strong force. Many other theories such as string theory (Mohaupt (2003)), M-theory, and little Higgs (Schmaltz and Tucker-Smith (2005)) have also attempted to create theories linking the three forces. These theories often rely on unproven physical concepts such as supersymmetry and as such, there is currently no accepted Grand Unified Theory. Theories that attempt to unify all the fundamental forces (including gravity) are often named "theories of everything" but there currently exist no theories that fulfill this challenge. The closest candidates are string theory and M-theory but with both theories making predictions that cannot currently be experimentally verified there is no way to ensure their accuracy (Mohaupt (2003)).

7 Conclusion

We have managed to find a gauge transform of the 4-potential using the electromagnetic field tensor which allowed us to find a Kerr-Schild equation for the gauge transformed 4-potential. We then added a graviton field solution into the Minkowski metric and substituted this metric into the Einstein field equation to show that it satisfied it for a vacuum solution and, as such, was analogous to the solution for a Schwarzschild black hole. We then noticed that the relationship between the 4-potential Kerr-Schild solution and the graviton field we had used was that of a double copy and moved to investigate this by finding relationships between the Maxwell equations of electromagnetism and the values of the Ricci curvature tensor. This showed that the relationship we had discovered between electromagnetism and gravity was not only true but that a solution exists for an infinite number of different graviton field solutions (Monteiro, O'Connell and White (2014)).

This relationship is particularly striking when one considers the fact that there is no known gauge theory for gravity and as such finding a double copy relationship between electromagnetism (and further on with Yang-Mills theory) allows us to relate gravity to these other fundamental forces of nature that do have known gauge theories. With our current lack of understanding with regards to the accelerating expansion of the universe and the lack of a quantum field theory of gravity to investigate the conditions inside the event horizon of a black hole or in the earliest moments after the Big Bang, having a new theory as to a link between these fundamental forces could be vital and could be the first step towards finding a greater theory that underpins the nature of the fundamental forces themselves. Such knowledge could help in better understanding many areas of science involving the fundamental forces including astronomy, cosmology, particle physics, and condensed matter physics as the way that matter interacts with itself is driven by these forces.

The classical double copy has been observed in other situations not discussed in this paper as well. For example, a double copy relationship has been found with the Kerr solution (Monteiro, O'Connell and White (2014)) - the solution found in the empty space surrounding a spherically symmetric, rotating black hole - as well as the Reissner-Nordström solution (Maybee and O'Connell) and the Kerr-Newman metric (Moynihan (2020)) which are both solutions of empty space near a charged black hole with the latter including rotation. The first double copy was noticed between scattering amplitudes in gluons and gravity (Bern, Carrasco and Johansson (2010)) so the research that made this review paper was first conducted into the strong nuclear force and not into the electromagnetic force. The double copy was also found using quantum field theory rather than using the relativistic form that we have used to find the classical double copy.

Current usage of the classical double copy we have derived is in the simplification of calculations used by astronomers and cosmologists particularly in the fairly revolutionary field of gravitational waves as the classical double copy allows for new methods of calculations in many of these fields.

The study of double copy relationships between Kerr-Schild formulations of the fundamental forces is ongoing work that could offer a promising insight into new theoretical physics in the fields of quantum field theory and could help towards finding a theory of everything, all the while providing new methods of calculation for astronomers and cosmologists for their work probing the history and future of the universe and its constituent celestial bodies and formations. It seems that in recent years the field has been supercharged and a gravitational pull towards further progress is unavoidable. For this reason, I believe that the future of the field is bright and look forwards to hearing more about it.

8 References

Bah, I., Dempsey, R. and Weck, P., 2020. Kerr-schild double copy and complex worldlines. *Journal of high energy physics*, 2020(2), pp.1–34.

Bern, Z., Carrasco, J.J.M. and Johansson, H., 2010. Perturbative quantum gravity as a double copy of gauge theory. *Physical review letters*, 105(6), p.061602.

Bilenky, S.M. and Hošek, J., 1982. Glashow-weinberg-salam theory of electroweak interactions and the neutral currents. *Physics reports*, 90(2), pp.73–157.

Carroll, S.M., 2019. *Spacetime and geometry*. Cambridge University Press.

Donoghue, J.F., Golowich, E. and Holstein, B.R., 2014. *Dynamics of the standard model*, vol. 35. Cambridge university press.

Farhi, E. and Susskind, L., 1979. Grand unified theory with heavy color. *Physical review d*, 20(12), p.3404.

Hartle, J.B., 2003. Gravity: an introduction to einstein's general relativity.

Jackiw, R., 1980. Introduction to the yang-mills quantum theory. *Reviews of modern physics*, 52(4), p.661.

Kim, K., Lee, K., Monteiro, R., Nicholson, I. and Veiga, D.P., 2020. The classical double copy of a point charge. *Journal of high energy physics*, 2020(2), pp.1–28.

Maybee, B. and O'Connell, D. The double copy for scattering of reissner-nordström black holes.

Mohaupt, T., 2003. Introduction to string theory. *Quantum gravity*. Springer, pp.173–251.

Monteiro, R., O'Connell, D. and White, C.D., 2014. Black holes and the double copy. *Journal of high energy physics*, 2014(12), p.56.

Moriyasu, K., 1983. *An elementary primer for gauge theory*. World Scientific.

Moynihan, N., 2020. Kerr-newman from minimal coupling. *Journal of high energy physics*, 2020(1), pp.1–21.

Russakoff, G., 1970. A derivation of the macroscopic maxwell equations. *American journal of physics*, 38(10), pp.1188–1195.

Schmaltz, M. and Tucker-Smith, D., 2005. Little higgs theories. *Annu. rev. nucl. part. sci.*, 55, pp.229–270.

Wald, R.M., 2010. *General relativity*. University of Chicago press.

White, C.D., 2018. The double copy: gravity from gluons. *Contemporary physics*, 59(2), pp.109–125.

Woodhouse, N.M.J., 2007. *General relativity*. Springer Science & Business Media.