Magnetostatic considerations for the simulation of permanent magnets

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This technical memo provides the basic theoretical framework required to simulate permanent magnets as equivalent magnetization currents. The cases of cylindrical, ring, block, and prismatic magnets are discussed.

1 Governing equations

We consider a permanent magnet immersed in any medium. Our goal is to obtain the magnetic flux density field provided by such a magnet. This electromagnetic problem is governed by Maxwell equations, which for a static magnetic configuration without surface currents and electric fields become [1]

$$\nabla \cdot \mathbf{B} = 0, \tag{1a}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_e, \tag{1b}$$

where $\mathbf{H}$ [A/m], $\mathbf{B}$ [T], and $\mathbf{M}$ [A/m] are the magnetic, magnetic flux density, and magnetization fields, respectively, and $\mathbf{J}_e$ [A/m^2] is the volume density of electric current. These fields are related through

$$\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M}\right), \tag{2}$$

with $\mu_0 = 4\pi \cdot 10^{-7}$ N/A^2. In a numerical simulation, the magnetization $\mathbf{M}$ is determined by the medium. Permanent magnets are usually modeled with uniform internal $\mathbf{M}$ fields, while ferrofluids exposed to a DC field operate on the basis of the constitutive equation $\mathbf{M} = \chi(H)\mathbf{H}$, where $\chi(H)$ is the magnetic susceptibility [2].

The magnetic boundary conditions of the magnetostatic problem are [1]

$$\mathbf{n} \cdot [\mathbf{B}] = 0, \tag{3a}$$

$$\mathbf{n} \times [\mathbf{H}] = \mathbf{K}_e, \tag{3b}$$

where $\mathbf{n}$ denotes the external normal vector, $\mathbf{K}_e$ [A/m] the electric surface currents, and [-] the difference across the interface. In other words, the normal component of $\mathbf{B}$ is continuous across any interface, and the tangential component of $\mathbf{H}$ jumps according to $\mathbf{K}_e$.

That's it, right? We can just implement our favorite discretization scheme (finite differences, volumes, elements, etc.), express Eqs. 1-3 using an appropriate reference system, convert to matrix form, *et voilà!*. Well, yes. But also, no. Although this is the preferred approach in commercial software like Comsol Multiphysics or ANSYS Maxwell, and although it is relatively easy to implement and operate, sometimes we are interested in *analytical* solutions that result in a deeper understanding of a given problem, are perfectly accurate, or faster to compute. Let's keep working on this...

2 Formulation in terms of the B field

We can reformulate Eqs.1 and 3 and express them in terms of the $\mathbf{B}$ field by making use of Eq. 2, resulting in

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J}_e + \nabla \times \mathbf{M}\right), \end{aligned} \tag{4a}$$

and

$$\begin{aligned} \mathbf{n} \cdot [\mathbf{B}] &= 0, \\ \mathbf{n} \times [\mathbf{B}] &= \mu_0 \left(\mathbf{K}_e + \mathbf{n} \times [\mathbf{M}]\right). \end{aligned} \tag{5a}$$

At this point, it is important to note that the terms

$$\mathbf{J}_m = \nabla \times \mathbf{M} \quad \text{(from Eq. 4a)}, \tag{6a}$$

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$$\mathbf{K}_m = \mathbf{n} \times [\mathbf{M}] \quad \text{(from Eq. 5a)}, \tag{6b}$$

commonly known as *equivalent magnetization currents*, are fully equivalent, from a mathematical perspective, to the volume and surface currents $\mathbf{J}_e$ and $\mathbf{K}_e$. This gives us critical insight into our problem and leads to an interesting conclusion: *any* magnet, independently of its internal magnetization field, can be modeled as a combination of *virtual* volume and surface currents. Because most magnets exhibit uniform magnetization (i.e. $\mathbf{J}_m = \mathbf{0}$), only $\mathbf{K}_m$ is usually required!

3 Magnetic vector potential

The fundamental theorem of vector calculus, also known as **Helmholtz Theorem** or **Helmholtz decomposition/representation**, states that any sufficiently smooth, rapidly decaying vector field in three dimensions can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field. Naturally, this also applies to the vector field $\mathbf{B}$. However, Eq. 1a tells us that $\mathbf{B}$ is solenoidal, implying that it derives from a vector potential and does not have scalar sources. In other words,

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{7}$$

with $\mathbf{A}$ being the *magnetic vector potential*[1]. Adopting the Lorentz gauge, the magnetic vector potential becomes

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}_e(\mathbf{r}) + \mathbf{A}_m(\mathbf{r}) = \frac{\mu_0}{4\pi} \oiint_S \frac{\mathbf{K}_e + \mathbf{K}_m}{|\mathbf{r} - \mathbf{r}'|} dS' + \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_e + \mathbf{J}_m}{|\mathbf{r} - \mathbf{r}'|} dV', \tag{8}$$

where a distinction is made between electric (e) and magnetic (m) parts and where the integrals are defined in the regions where electric and magnetization currents are located. Considering Eq. 6 and making use of some vector calculus identities, the magnetic component is expressed as

$$\mathbf{A}_m(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V_m} \frac{\mathbf{M} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV', \tag{9}$$

with V_m denoting the volumes occupied by magnetized media.

4 Magnetic scalar potential

If we study problems without volume currents ($\mathbf{J}_e = \mathbf{0}$) and magnetized media ($\mathbf{M} = \mathbf{0}$, thus $\mathbf{B} = \mu_0 \mathbf{H}$), then Eq. 4a shows that $\mathbf{B}$ is not only solenoidal but also irrotational. Therefore, we can define a scalar magnetic potential Φ such that

$$\mathbf{B} = -\mu_0 \nabla \Phi_m, \tag{10}$$

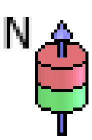
and

$$\nabla^2 \Phi_m = 0. \tag{11}$$

Although restrictive, this formulation is adopted in many numerical simulation modules, as it requires solving just one scalar variable to obtain the $\mathbf{B}$ field. The seemingly arbitrary inclusion of the term $-\mu_0$ establishes a direct analogy with electrostatic problems, where the electric field $\mathbf{E}$ adopts a role analogous to the magnetic field $\mathbf{H}$.

5 Simulation of permanent magnets

5.1 Upright cylindrical magnet

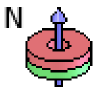


For a vertically magnetized cylindrical magnet with uniform magnetization $\mathbf{M}_m = M_m \mathbf{e}_z$, the equivalent magnetization currents become $\mathbf{J}_m = \mathbf{0}$ in the volume and $\mathbf{K}_m = -M_m \mathbf{e}_\phi$ at the lateral wall. If the magnet has length h, the equivalent electrical system can be modeled as a homogeneous distribution of $N_j \to \infty$ circular loops in the lateral wall with current $M_m h / N_j$.

[1] Note that $\mathbf{A}$ is not uniquely defined: you can always define a new vector potential $\mathbf{A}' = \mathbf{A} + \nabla\phi$ that results in the exact same $\mathbf{B}$ (because $\nabla \times \nabla\phi = 0$). This is known as the *gauge problem*, and has some important implications in numerical simulations. In this technical memo, however, we will adopt the common magnetostatic choice $\nabla \cdot \mathbf{A} = 0$, known as the *Lorentz gauge*.

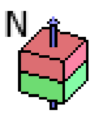
The expressions for the magnetic field produced by a circular current loop can be found in Ref. 3 in Cartesian, Cylindrical, and Spherical coordinates.

5.2 Upright ring magnet



The magnetic field produced by a uniformly magnetized ring magnet with $\mathbf{M}_m = M_m \mathbf{e}_z$ is equivalent to that of a cylindrical magnet with the same external diameter minus a cylindrical magnet with the opposite polarity and an external radius equal to the internal radius of the ring. The equivalent magnetization currents become $\mathbf{J}_m = \mathbf{0}$ in the volume, $\mathbf{K}_m = -M_m \mathbf{e}_\phi$ on the external lateral wall, and $\mathbf{K}_m = +M_m \mathbf{e}_\phi$ on the internal wall. If the magnet has length h, the equivalent electrical system can be modeled as a homogeneous distribution of $N_j \to \infty$ circular loops in the internal and external walls with currents $+M_m h/N_j$ and $-M_m h/N_j$, respectively. The expressions for the magnetic field produced by a circular current loop can be found in Ref. 3 in Cartesian, Cylindrical, and Spherical coordinates.

5.3 Block magnet



For a vertically magnetized block magnet with uniform magnetization $\mathbf{M}_m = M_m \mathbf{e}_y$, the equivalence results in a system of electrical currents with a volume density term $\mathbf{J}_e = \mathbf{0}$ in the volume and a surface density term $\mathbf{K}_e = \mathbf{n} \times \mathbf{M}_m$ on the lateral walls. If the magnet has a height h, the equivalent electrical system can be modeled as a homogeneous distribution of $N_j \to \infty$ wires on the lateral walls with current $M_m h/N_j$.

The magnetic flux density field produced by a finite wire such as that depicted in Fig. 1 is given by

$$\mathbf{B}(r, \phi, z) = \frac{\mu_0 I_w}{4\pi r} \left[\sin(\alpha_2) - \sin(\alpha_1)\right] \mathbf{e}_\phi(\phi) \tag{12}$$

with $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$ being the circumferential reference system with the origin at the center of the wire, L its length, I its current, r the radial distance, and

$$\sin(\alpha_1) = \frac{-\left[L/2 + z\right]}{\sqrt{r^2 + \left[L/2 + z\right]^2}}, \tag{13a}$$

$$\sin(\alpha_2) = \frac{L/2 - z}{\sqrt{r^2 + \left[L/2 - z\right]^2}}. \tag{13b}$$

In other words, the block magnet is modeled as the superposition of rectangular spires located around its magnetization vector.

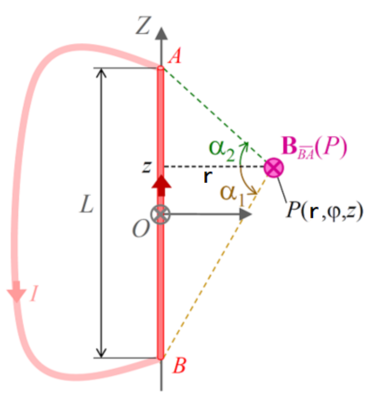
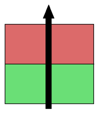


Figure 1: Magnetic field produced by a finite wire

5.4 Infinite 2D prismatic magnet

5.4.1 Solution based on infinite wires



The case of an infinite 2D magnet with uniform magnetization $\mathbf{M}_m = M_m\mathbf{e}_y$ is much simpler than its 3D counterpart. Again, the volume density term $\mathbf{J}_e$ becomes $\mathbf{0}$ in the volume, while surface terms result in $\mathbf{K}_e = -M_m\mathbf{e}_z$ and $\mathbf{K}_e = M_m\mathbf{e}_z$ at the right and left walls, respectively. However, the two-dimensional magnetic field produced by an infinite wire is in this case just

$$\mathbf{B}_w(\mathbf{r}) = \frac{\mu_0 I_w}{2\pi r}\mathbf{e}_\phi(\mathbf{r}), \tag{14}$$

where $\mathbf{r}$ is the position vector from the wire, r its module, I the current of the wire, and $\mathbf{e}_\phi$ the circumferential direction that follows the right-hand rule imposed by the direction of the current vector $\mathbf{I}$. As before, if the magnet has a height h, the equivalent electrical system can be modeled as a homogeneous distribution of $N_j \to \infty$ wires in the lateral wall with current $I_w = M_m h/N_j$.

5.4.2 Solution based on infinite current sheets

Modeling an *infinite* number of wires is computationally expensive and can lead to significant aberrations in close proximity to them. We will therefore consider a current sheet Σ with infinite length and width h. The Cartesian reference system $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ with coordinates (x, y, z) is adopted so that the sheet is infinite along z and its cross-section is contained anywhere in the "x-y" plane. The sheet carries a uniform longitudinal current with surface density $\mathbf{K}_e = K_e\mathbf{e}_z$ that can be split among an infinite number of wire currents along $\mathbf{e}_z$, defined as $\Delta : \{w_x(t), w_y(t)\}$ and with

$$w_x(t) = w_{x,0} + \left(t - \frac{h}{2}\right)\cos\theta, \quad t \in [0, h], \tag{15}$$

$$w_y(t) = w_{y,0} + \left(t - \frac{h}{2}\right)\sin\theta, \quad t \in [0, h], \tag{16}$$

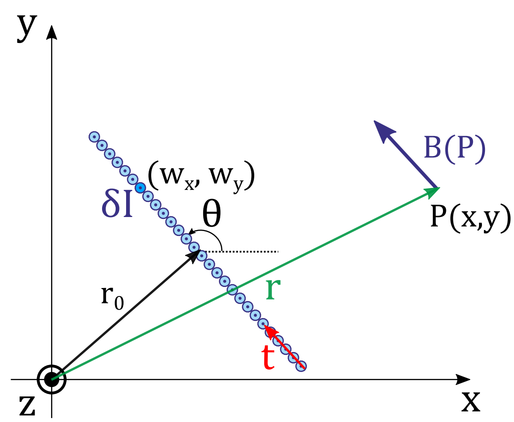


Figure 2: Sheet parameters

where t is the arc parameter along the sheet, $\mathbf{r}_0 = \{w_{x,0}, w_{y,0}\}$ denotes its geometrical center, and θ is its inclination with respect to $\mathbf{e}_x$, as shown in Fig. 2. Each wire carries an infinitesimal current density $\delta I = K_e \mathrm{d}t$. The contribution of each infinitesimal wire to the magnetic field at a point $P(x, y)$ is therefore

$$\mathrm{d}\mathbf{B}_w(x,y,z,t) = \frac{\mu_0 K_e \mathrm{d}t}{2\pi}\left\{-\frac{y - w_y(t)}{[x - w_x(t)]^2 + [y - w_y(t)]^2}\mathbf{e}_x + \frac{x - w_x(t)}{[x - w_x(t)]^2 + [y - w_y(t)]^2}\mathbf{e}_y\right\}, \tag{17}$$

which results in the total magnetic flux density field

$$\begin{aligned}\mathbf{B}(x,y,z) &= \int_{\forall\Delta} \mathbf{B}_w(x,y,z,t)\mathrm{d}t = \\ &= \frac{\mu_0 K_e}{2\pi}\int_0^h \left[-\frac{y - w_y(t)}{(x - w_x(t))^2 + (y - w_y(t))^2}\mathbf{e}_x + \frac{x - w_x(t)}{(x - w_x(t))^2 + (y - w_y(t))^2}\mathbf{e}_y\right]\|\mathbf{r}'(t)\|\mathrm{d}t,\end{aligned} \tag{18}$$

with

$$\|\mathbf{r}'(t)\| = \sqrt{\left(\frac{dw_x}{dt}\right)^2 + \left(\frac{dw_y}{dt}\right)^2} = 1 \tag{19}$$

being the norm of the parametrization described by Eq. 15. Compact solutions to Eq. 18 can be easily found for horizontal and vertical sheets. A more general solution is obtained by solving the (truly cumbersome) full integral with Wolfram Mathematica[2], resulting in

$$B_x(x,y,z) = \frac{\mu_0 K_e}{2\pi} i e^{-i\theta} \left[\log\left(-1 + \frac{4}{2 + \frac{e^{i\theta} h}{-w_{x,0} - i w_{y,0} + x + iy}}\right) - e^{2i\theta} \log\left(1 - \frac{2h}{h + 2e^{i\theta}\left(-w_{x,0} + i w_{y,0} + x - iy\right)}\right)\right], \tag{20a}$$

$$B_y(x,y,z) = -\frac{\mu_0 K_e}{2\pi} e^{-i\theta} \left[\log\left(-1 + \frac{4}{2 + \frac{e^{i\theta} h}{-w_{x,0} - i w_{y,0} + x + iy}}\right) + e^{2i\theta} \log\left(1 - \frac{2h}{h + 2e^{i\theta}\left(-w_{x,0} + i w_{y,0} + x - iy\right)}\right)\right]. \tag{20b}$$

Equation 20 is a valid mathematical solution of the magnetic field induced by a sheet of current, but also one without a clear physical meaning. What follows is a physically intuitive and significantly more compact alternative.

5.4.3 Solution based on infinite current sheets (alternative formulation)

We start by defining a current sheet Σ with infinite length and width h. The coordinate system $\{\xi, \psi, z\}$, adopted in the analysis, is defined together with the orthonormal base $\{\mathbf{e}_\xi, \mathbf{e}_\psi, \mathbf{e}_z\}$ so that $\mathbf{e}_\xi$ is perpendicular to the sheet and Σ is contained in the plane $\xi = 0$ (i.e. the points of the sheet are $P(0, \psi, z)$, with $-(h/2) < \psi < h/2$). The sheet carries a uniform longitudinal current with surface density $\mathbf{K}_e = K_e\, \mathbf{e}_z$ which, as before, can be split among an infinite number of infinite wire currents along $\mathbf{e}_z$, defined as $\Delta : \{\xi = 0; \psi = \upsilon\}$ with $-(h/2) < \upsilon < h/2$, with infinitesimal current density $\delta I = K_e\, \mathrm{d}\upsilon$. The contribution of each of these wire currents to the magnetic field at $P(\xi, \psi, z)$ is, from Eq. 17,

$$\mathrm{d}\mathbf{B} = \frac{\mu_0 K_e\, \mathrm{d}\upsilon}{2\pi} \left[\frac{-(\psi - \upsilon)\, \mathbf{e}_\xi + \xi\, \mathbf{e}_\psi}{\xi^2 + (\psi - \upsilon)^2}\right], \tag{21}$$

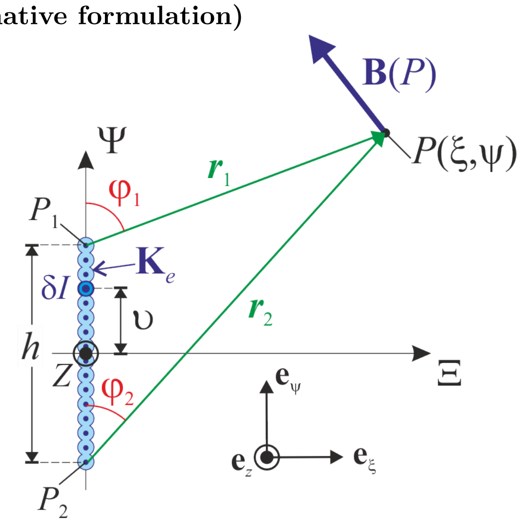


Figure 3: Alternative sheet parameters

and results in the total magnetic flux density field

$$\mathbf{B}(\xi, \psi, z) = \int_{\forall\Delta} \mathrm{d}\mathbf{B} = \frac{\mu_0\, K_e}{2\pi} \left[b_\xi(\xi,\psi)\, \mathbf{e}_\xi + b_\psi(\xi,\psi)\, \mathbf{e}_\psi\right] \tag{22}$$

, where

$$b_\xi(\xi,\psi) = -\int_{-h/2}^{h/2} \frac{(\psi - \upsilon)\, \mathrm{d}\upsilon}{\xi^2 + (\psi - \upsilon)^2} = \frac{1}{2} \ln\left\{\frac{\xi^2 + [\psi - (h/2)]^2}{\xi^2 + [\psi + (h/2)]^2}\right\}, \tag{23a}$$

$$b_\psi(\xi,\psi) = \int_{-h/2}^{h/2} \frac{\xi\, \mathrm{d}\upsilon}{\xi^2 + (\psi - \upsilon)^2} = \arctan\left[\frac{\psi + (h/2)}{\xi}\right] - \arctan\left[\frac{\psi - (h/2)}{\xi}\right], \tag{23b}$$

which is a simple and very elegant solution to the magnetic field produced by a sheet of current in a fixed reference system. We can provide a physical meaning to this solution by defining the top and bottom borders of the sheet as $P_1 = P(0, h/2)$ and $P_2(0 - h/2)$ and describing the relative position of the arbitrary point $P = P(\xi, \psi)$ with respect to them as

$$\overrightarrow{P_1P} \equiv \mathbf{r}_1 = \xi\, \mathbf{e}_\xi + [\psi - (h/2)]\, \mathbf{e}_\psi, \qquad \overrightarrow{P_2P} \equiv \mathbf{r}_2 = \xi\, \mathbf{e}_\xi + [\psi + (h/2)]\, \mathbf{e}_\psi. \tag{24}$$

[2]Georgia Tech students can download and install Wolfram Mathematica for free at https://software.oit.gatech.edu/.

The magnetic flux density field given by Eq. 22 for $P(\xi, \psi, z)$ can now be reformulated as

$$\mathbf{B}(\xi, \psi, z) = \frac{\mu_0 \, K_e}{2\pi} \left[\ln \left(\frac{r_1}{r_2} \right) \mathbf{e}_\xi + (\varphi_1 - \varphi_2) \, \mathbf{e}_\psi \right], \tag{25}$$

where $r_i = |\mathbf{r}_i|$ is the distance between P_i and P and $\varphi_i = \arccos\left(\mathbf{r}_i \cdot \mathbf{e}_\psi / r_i\right)$ is the angle between $\mathbf{r}_i$ and $\mathbf{e}_\psi$, as described in Fig. 3.

OK, we got the solution to the magnetic field produced by the sheet of current and we also gave it a physical meaning. Awesome! However, how can we rotate this solution as we did in Sec. 5.4.2? If you have studied attitude dynamics, you may know that the solution is... the Direct Cosine Matrix (DCM). Following Fig. 4, the reference system "$x - y$" is now defined so that the original plane "$\xi - \psi$" is rotated at an angle $\theta - \pi/2$ with respect to it. Both bases are related through

$$\begin{pmatrix} \mathbf{e}_\xi \\ \mathbf{e}_\psi \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix}, \tag{26}$$

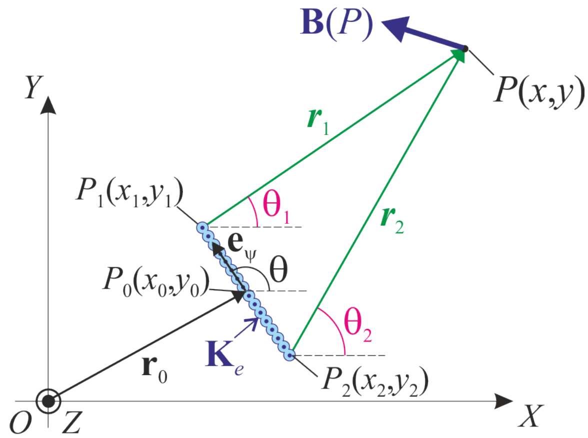


Figure 4: Sheet parameters with rotation.

where the central DCM matrix describes the rotation around z. Adopting the new Cartesian reference system OXY, the magnetic flux density field is now expressed as

$$\mathbf{B}(x, y, z) = \frac{\mu_0 \, K_e}{2\pi} \left[b_x(x, y) \, \mathbf{e}_x + b_y(x, y) \, \mathbf{e}_y \right], \tag{27}$$

where

$$b_x(x, y) = \ln \left(\frac{r_1}{r_2} \right) \sin\theta + (\theta_2 - \theta_1) \cos\theta, \tag{28a}$$

$$b_y(x, y) = -\ln \left(\frac{r_1}{r_2} \right) \cos\theta + (\theta_2 - \theta_1) \sin\theta, \tag{28b}$$

and the new relative angles $\theta_i = \theta - \varphi_i$ between $\mathbf{r}_i \equiv \overrightarrow{P_iP}$ and $\mathbf{e}_x$ have been considered (i.e. $\theta_i = \arccos(\mathbf{r}_i \cdot \mathbf{e}_x / r_i)$). Evaluating all terms in the "x-y" plane results in

$$r_i(x, y) = \sqrt{(x - x_i)^2 + (y - y_i)^2} \, , \quad \text{y} \quad \theta_i(x, y) = \arctan \left[\frac{y - y_i}{x - x_i} \right]. \tag{29}$$

Of course, you can refer the points P_1 and P_2 to the central point of the sheet, $P_0(x_0, y_0)$, through

$$\overrightarrow{OP}_i \equiv \mathbf{r}_0 + (-1)^{i-1} \frac{h}{2} \, \mathbf{e}_\psi \implies \begin{cases} x_i = x_0 + (-1)^{i-1} \dfrac{h}{2} \cos\theta \\ \\ y_i = y_0 + (-1)^{i-1} \dfrac{h}{2} \sin\theta \end{cases} \quad (i = 1, 2), \tag{30}$$

where $\mathbf{r}_0$ is the position vector $\overrightarrow{OP}_0$.

5.4.4 Example: uniformly magnetized prismatic domain

Let's now consider the case of a uniformly magnetized prismatic domain with a rectangular cross-section of width a and height h, shown in Fig.5. The domain has an infinite length along the axis OZ and is arbitrarily oriented in the OXY plane. The barycenter of the domain is located at $P_0(x_0, y_0)$ and its vertices are at $\{P_i(x_i, y_i)\}_{i=1}^4$, resulting in

$$\begin{aligned} \overrightarrow{OP}_1 &\equiv \mathbf{r}_0 - \frac{a}{2} \, \mathbf{e}_\xi + \frac{h}{2} \, \mathbf{e}_\psi \, ; \qquad & \overrightarrow{OP}_2 &\equiv \mathbf{r}_0 - \frac{a}{2} \, \mathbf{e}_\xi - \frac{h}{2} \, \mathbf{e}_\psi \\ \overrightarrow{OP}_3 &\equiv \mathbf{r}_0 + \frac{a}{2} \, \mathbf{e}_\xi + \frac{h}{2} \, \mathbf{e}_\psi \, ; \qquad & \overrightarrow{OP}_4 &\equiv \mathbf{r}_0 + \frac{a}{2} \, \mathbf{e}_\xi - \frac{h}{2} \, \mathbf{e}_\psi \end{aligned} \tag{31}$$

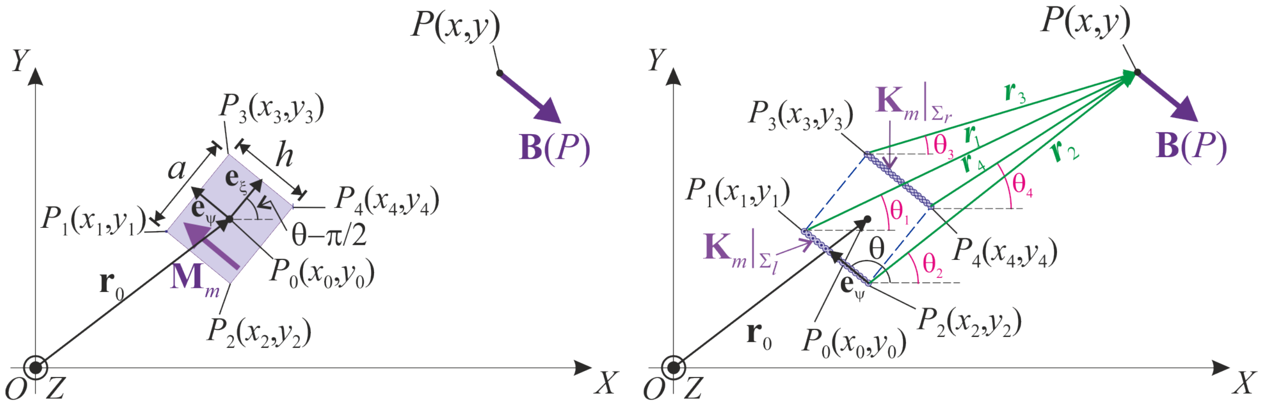


Figure 5: An arbitrarily oriented and uniformly magnetized prismatic magnet is modeled as the combination of two parallel currents.

where $\overrightarrow{OP}_0 \equiv \mathbf{r}_0 = x_0\,\mathbf{e}_x + y_0\,\mathbf{e}_y$, and $\{\mathbf{e}_\xi, \mathbf{e}_\psi\}$ are the orthonormal vectors of the domain-fixed base described in Eq. 26. The uniform imanation of the domain is described as $\mathbf{M}_m = M_m\,\mathbf{e}_\psi = M_m\,(\cos\theta\,\mathbf{e}_x + \sin\theta\,\mathbf{e}_y)$ (i.e. it follows the orientation of segments $\overrightarrow{P_2P_1}$ and $\overrightarrow{P_4P_3}$).

As before, the magnetic flux density field $\mathbf{B}(\mathbf{r})$ of the uniformly magnetized domain is modeled through equivalent magnetization currents $\mathbf{J}_m$ within the domain (volume term) and $\mathbf{K}_m$ at its contour (surface term). Because the magnetization field is uniform, $\mathbf{J}_m = \mathbf{0}$. However, $\mathbf{K}_m = \mathbf{n} \times [\mathbf{M}]$, resulting in the surface current distributions $\mathbf{K}_m = M_m\mathbf{e}_z$ y $\mathbf{K}_m = -M_m\mathbf{e}_z$ at the P_1P_2 and P_3P_4 segments, respectively. Following the results presented in Sec. 5.4.3, the magnetic flux density field produced by the prismatic domain is given by

$$\mathbf{B}(x,y,z) = \frac{\mu_0\,M_m}{2\pi}\left[\ln\left(\frac{r_1\,r_4}{r_2\,r_3}\right)\mathbf{e}_\xi + (\theta_2+\theta_3-\theta_1-\theta_4)\,\mathbf{e}_\psi\right] = \frac{\mu_0\,M_m}{2\pi}\,[b_x(x,y)\,\mathbf{e}_x + b_y(x,y)\,\mathbf{e}_y] \tag{32a}$$

where

$$\begin{aligned} b_x(x,y) &= \ln\left(\frac{r_1\,r_4}{r_2\,r_3}\right)\sin\theta + (\theta_2+\theta_3-\theta_1-\theta_4)\,\cos\theta \\ b_y(x,y) &= -\ln\left(\frac{r_1\,r_4}{r_2\,r_3}\right)\cos\theta + (\theta_2+\theta_3-\theta_1-\theta_4)\,\sin\theta \end{aligned} \tag{32b}$$

with r_i being, again, the distance between $P_i(x_i,y_i)$ and the point of interest $P(x,y)$ and θ_i the angle between $\overrightarrow{P_iP}$ and OX. For $i = 1,2,3,4$,

$$r_i(x,y) = \sqrt{(x-x_i)^2 + (y-y_i)^2}\ , \quad \text{y} \quad \theta_i(x,y) = \arctan\left[\frac{y-y_i}{x-x_i}\right]. \tag{32c}$$

5.5 Verification

Always verify your simulations and analytical estimations with reliable data from an external source. K&J Magnetics offer an interesting online tool to simulate permanent magnets that you may consider for future works.

References

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