Electric and Magnetic Fields from a Circular Coil Using Elliptic Integrals

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ABSTRACT

The focal points of this article are exact expressions for the **E** *field caused by a uniformly charged ring of radius* R *and the* **B** *field caused by a current* I *flowing along such a ring. We first obtained expressions for these two fields by direct application of Coulomb's law and Biot-Savart's law respectively, in terms of a new set of Complete Elliptic Integrals* (K(k),H(k)) *replacing the conventional pair* (K(k),E(k)). *Subsequently we wrote the scalar potential* Φ *and the vector potential* **A** *and re-established the same results by a second route. The new function* H(k) *that replaces* E(k) *is related to the latter by a simple multiplicative factor. We checked our formulas against known approximate formulas by expanding the expressions in power series of* $\frac{R}{r}$.

1 Introduction

Figure 1 shows the geometry we are concerned with. It is a ring of radius R lying on the XY plane. This ring is the source of a static electric field **E** when it is charged uniformly with a line charge density λ, and is the source of a static magnetic field **B** when a constant current I flows around it. Our objective in this article is to obtain exact mathematical expressions for these fields.

Square and circle being the simplest

geometries one may think that the simplest examples of static **E** and **B** fields are provided by charges and currents uniformly distributed over such geometries.[1] However, exact solutions for both these geometries are difficult to find. Jackson[2,3] in both his 2nd and 3rd editions has given an expression for the vector potential **A** due to steady current in a circular coil in a closed form that involves the elliptic integrals $K(k)$ and $E(k)$. However, while writing the **B** field he has either made approximations,[2] or made series expansions of the field containing only two terms.[3]

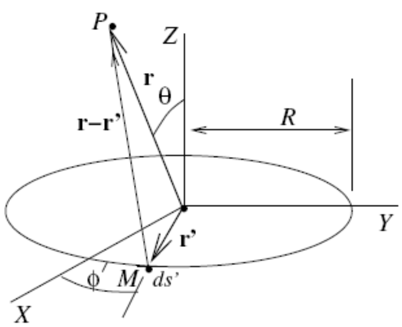

Figure 1. A ring of uniform charge.

In this article we shall use a substitute for $E(k)$, for which we have used the symbol $H(k)$. Although the two functions $H(k)$ and $E(k)$ are related to each other by a simple factor (see Eq. (24)) and are interchangeable, this new candidate $H(k)$, rather than its twin $E(k)$, appears to be most suitable for our job. We shall use the new pair ($K(k)$, $H(k)$) to write exact expressions for the fields **E** and **B** and for the potentials Φ and **A**.

We shall obtain **E** and **B** in two different ways: first by direct application of Coulomb's law and Biot-Savart Law, and then by the other route, viz., $\mathbf{E} = -\nabla\Phi$ and $\mathbf{B} = \nabla\times\mathbf{A}$. We shall then make series expansions of these fields to check our results with the approximate formulas and series expressions written by Jackson.

We adopt spherical coordinate system, and (without loss of generality) take the field point P on the XZ plane. **r** and **r′** are, respectively, the radius vectors of the field point P and the source point M, and $d\mathbf{r}'$ is an infinitesimal segment of the source. Then

$$\mathbf{r}-\mathbf{r}' = (r\sin\theta - R\cos\phi')\mathbf{i} - R\sin\phi'\mathbf{j} + r\cos\theta\mathbf{k}.$$

$$d\mathbf{r}' = R(-\sin\phi'\mathbf{i} + \cos\phi'\mathbf{j})d\phi'. \qquad (1)$$

The **E** field at P due to the circular charge density λ can be obtained directly from Coulomb's law by evaluating the integral

$$\mathbf{E}(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{(r\sin\theta - R\cos\phi')\mathbf{i} - R\sin\phi'\mathbf{j} + r\cos\mathbf{k}}{(r^2+R^2-2rR\sin\theta\cos\phi')^{3/2}}Rd\phi' \qquad (2)$$

Similarly the **B** field at P due to the circular current I can be obtained directly from Biot-Savart's law by evaluating the integral

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint_C \frac{d\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{R\mathbf{k} + r\cos\phi'(\cos\theta\mathbf{i} - \sin\theta\mathbf{k}) + r\cos\theta\sin\phi'\mathbf{j}}{(r^2+R^2-2rR\sin\theta\cos\phi')^{\frac{3}{2}}}Rd\phi' \qquad (3)$$

Alternatively, one may like to compute **E** and **B** from the scalar potential Φ and the vector potential **A**.

$$\Phi(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{Rd\phi'}{\sqrt{r^2+R^2-2rR\sin\theta\cos\phi'}}. \tag{4}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{(-\sin\phi'\mathbf{i}+\cos\phi'\mathbf{j})Rd\phi'}{\sqrt{r^2+R^2-2rR\sin\theta\cos\phi'}}. \tag{5}$$

Our first task will be to evaluate the integrals given in Eqs.(2)-(5) and obtain closed form expressions for these fields. We shall begin with a brief review of the properties of the elliptic integrals relevant to the sequel.

2 Important Identities

The complete elliptic integrals of the *first kind* $K(k)$ and the *second kind* $E(k)$ constitute a pair of functions[4] that are well known for many useful applications in physics and mathematics,[5-7] e.g., time period of a simple pendulum, evaluation of the circumference of an ellipse, analysis of the relativistic planetary orbits, precession of a spinning top, etc. There are standard tables giving values of these functions versus their argument.[8] We shall, however, find it convenient to find a replacement function $H(k)$ for $E(k)$ so that $\{K(k), H(k)\}$, rather than $\{K(k), E(k)\}$, will form the acting pair in this article. These three functions are formally defined as follows.

$$K(k) = \int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\frac{\pi}{2}}\frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}, \tag{6}$$

$$E(k) = \int_0^1\sqrt{\frac{1-k^2t^2}{1-t^2}}dt = \int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\vartheta}\,d\vartheta \tag{7}$$

$$H(k) = \int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)^3}} = \int_0^{\frac{\pi}{2}}\frac{d\vartheta}{(1-k^2\sin^2\vartheta)^{\frac{3}{2}}} \tag{8}$$

In the above k is a real number lying between 0 and l. That is, $0\le k\le 1$. Each one of the above functions can be identified with a hypergeometric series (multiplied by a constant) ${}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};x\right)$ with $b=\frac{1}{2}$, $c=1$, $x=k^2$ common to all the functions and $a=\frac{1}{2}, -\frac{1}{2}$ and $\frac{3}{2}$ for $K(k)$, $E(k)$ and $H(k)$ respectively. The task can be achieved by Binomial expansion of the integrands, and their integration term by term.[4]

$$K(k) = \frac{\pi}{2}\left[{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix};k^2\right)\right]. \quad \text{(a)}$$

$$E(k) = \frac{\pi}{2}\left[{}_2F_1\left(\begin{matrix}-\frac{1}{2},\frac{1}{2}\\ 1\end{matrix};k^2\right)\right]. \quad \text{(b)} \tag{9}$$

$$H(k) = \frac{\pi}{2}\left[{}_2F_1\left(\begin{matrix}\frac{3}{2},\frac{1}{2}\\ 1\end{matrix};k^2\right)\right]. \quad \text{(c)}$$

This identification will enable us to exploit some of the known properties of the hypergeometric series[9] to establish some important identities crucial in our work. For example, invoking Euler transformation

$${}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\,x\right) = (1-x)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a,c-b\\ c\end{matrix};x\right) \tag{10}$$

and setting $a=\frac{1}{2}$; $b=-\frac{1}{2}$; $c=1$; $x=k^2$ would lead to identification of $H(k)$ as $E(k)$ multiplied by a simple factor.

$$E(k)=(1-k^2)H(k). \tag{11}$$

For another illustration we use the contiguous relations

$$x(1-x)\tfrac{dF}{dx}=(c-b)F(b-)+(b-c+a-x)F, \quad \text{(a)}$$

$$x\tfrac{dF}{dx}=b\{F(b+)-F\}, \quad \text{(b)} \quad (12)$$

where F is an abbreviation for ${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right)$ and $F(b+)$, $F(b-)$, for ${}_2F_1\left(\begin{matrix}a,b+1\\c\end{matrix};x\right)$ and ${}_2F_1\left(\begin{matrix}a,b-1\\c\end{matrix};x\right)$ respectively. Setting $x=k^2$, $a=b=\frac{1}{2}$, $c=1$ in (12a), and $x=k^2$, $a=\frac{1}{2}$, $b=-\frac{1}{2}$, $c=1$ in (12b) would lead to the following formulas[3] for the derivatives of $K(k)$ and $E(k)$.

$$\frac{dK(k)}{dk}=\frac{1}{k}\left[\frac{E(k)}{1-k^2}-K(k)\right]$$

$$=\frac{1}{k}[H(k)-K(k)], \quad \text{(a)}$$

$$\frac{dE(k)}{dk}=\frac{1}{k}[E(k)-K(k)]. \quad \text{(b)} \quad (13)$$

3 Evaluation of the Integrals

A look at Eqs. (2)-(5) (and Eq. (21) to follow) would suggest that our main task is evaluation of the integrals

$$X_p=\int_0^{2\pi}\frac{\cos\phi\, d\phi}{(r^2+R^2-2rR\sin\theta\cos\phi)^p}, \text{ (a)}$$

$$\Sigma_p=\int_0^{2\pi}\frac{\sin\phi\, d\phi}{(r^2+R^2-2rR\sin\theta\cos\phi)^p}, \text{ (b)} \quad (14)$$

$$I_p=\int_0^{2\pi}\frac{d\phi}{(r^2+R^2-2rR\sin\theta\cos\phi)^p}, \text{ (c)}$$

with $p=-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$. We set

$$k=\sqrt{\frac{4rR\sin\theta}{r^2+R^2+2rR\sin\theta}}. \tag{15}$$

Note that $k\le 1$, since $0\le\theta\le\pi$. For compactness and economy of space we shall introduce the following variables

$$\rho=\sqrt{r^2+R^2+2rR\sin\theta};$$

$$\xi=\sqrt{r^2+R^2-2rR\sin\theta}. \tag{16}$$

They satisfy the following identities which we shall find useful.

$$k^2\rho^2=4rR\sin\theta; \quad \xi^2=\rho^2(1-k^2). \tag{17}$$

Now we make the substitution $\phi=\pi+2t$. As a result

$$r^2+R^2-2rR\sin\theta\cos\phi=\rho^2(1-k^2\sin^2 t). \tag{18}$$

Using Eqs. (6), (8) and (11) the integrals I_p can now be readily computed.

$$I_{-\frac{1}{2}}=4\rho\int_0^{\frac{\pi}{2}}dt\sqrt{1-k^2\sin^2 t}=4\rho E(k). \text{ (a)}$$

$$I_{\frac{1}{2}}=\frac{4}{\rho}\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-k^2\sin^2 t}}=\frac{4K(k)}{\rho}. \quad \text{(b)} \quad (19)$$

$$I_{\frac{3}{2}}=\frac{4}{\rho^3}\int_0^{\frac{\pi}{2}}\frac{dt}{(1-k^2\sin^2 t)^{3/2}}=\frac{4H(k)}{\rho^3}. \text{ (c)}$$

For evaluation of X_p let us first note that

$$\cos\phi=\frac{(r^2+R^2)-(r^2+R^2-2rR\sin\theta\cos\phi)}{2rR\sin\theta}. \tag{20}$$

Hence

$$X_p = \frac{1}{2rR\sin\theta}[(r^2+R^2)I_p - I_{p-1}]. \qquad (21)$$

Using Eqs.(19) we can now evaluate the above integrals for $p=\frac{1}{2},\frac{3}{2}$, the cases of relevance to us.

$$X_{\frac{1}{2}} = \frac{2}{rR\rho\sin\theta}[(r^2+R^2)K(k) - \rho^2 E(k)]. \quad \text{(a)}$$

$$X_{\frac{3}{2}} = \frac{2}{rR\rho^3\sin\theta}[(r^2+R^2)H(k) - \rho^2 K(k)]. \quad \text{(b)} \quad (22)$$

It is easy to see that

$$\Sigma_p = 0. \qquad (23)$$

To see this transparently all one needs to do is to substitute $\phi \to \pi + x$, so that the integral becomes

$$\Sigma_p = -\int_{-\pi}^{\pi} \frac{\sin x dx}{(r^2 + R^2 + 2rR\sin\theta\cos x)^p}$$

which is zero since the integrand is an antisymmetric function.

4 Expression for the E Field

Equipped with the integration formulas (19)-(23) writing the expressions for the fields becomes an easy task. We shall obtain the r and θ components of the **E** and **B** fields, using the unit vectors $\mathbf{e}_r$, and $\mathbf{e}_\theta$. The appearance of the index $\frac{3}{2}$ in the denominator of Eqs.(2), (3) leading to the integrals (14), and the formula (20) make the pair $\{H(k), K(k)\}$ most suitable for writing expressions for **E** and **B**. Those who prefer the conventional pair $\{E(k), K(k)\}$ may like to convert $H(k)$ into $E(k)$ using the identities (11) and (17), or the transformation rule

$$\frac{H(k)}{\rho^2} = \frac{E(k)}{\xi^2} \qquad (24)$$

It is seen from Eqs.(2) and (14) that

$$\mathbf{E}(\mathbf{r}) = \frac{\lambda R}{4\pi\varepsilon_0}[rI_{\frac{3}{2}}\mathbf{e}_r - RX_{\frac{3}{2}}\mathbf{i} - R\Sigma_{\frac{3}{2}}\mathbf{j}]. \qquad (25)$$

Using Eqs.(19), (22) and (23) we get the following.

$$\mathbf{E}(\mathbf{r}) = \frac{4\lambda R}{4\pi\varepsilon_0\rho^3} \times$$

$$\left[rH(k)\mathbf{e}_r - \frac{1}{2r\sin\theta}\{(r^2+R^2)H(k) - \rho^2 K(k)\mathbf{i}\right]. \qquad (26)$$

Noting that $\mathbf{i}=\cos\theta\mathbf{e}_\theta+\sin\theta\mathbf{e}_r$, we get

$$E_r = \frac{\lambda R}{4\pi\varepsilon_0}\frac{2}{r\rho^3} \times$$

$$[(r^2-R^2)H(k)+\rho^2 K(k)]. \quad \text{(a)}$$

$$E_\theta = -\frac{\lambda R}{4\pi\varepsilon_0}\frac{2\cot\theta}{r\rho^3} \times \qquad (27)$$

$$[(r^2+R^2)H(k)-\rho^2 K(k)]. \quad \text{(b)}$$

5 Expression for the B Field

Noting that $\mathbf{e}_\theta=\cos\theta\mathbf{i}-\sin\theta\mathbf{k}$; $\mathbf{k}=\cos\theta\mathbf{e}_r-\sin\theta\mathbf{e}_\theta$, and using (14) the expression for **B** given in (3) can be rewritten compactly as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}R[R\cos\theta I_{\frac{3}{2}}\mathbf{e}_r +$$

$$(rX_{\frac{3}{2}} - R\sin\theta I_{\frac{3}{2}})\mathbf{e}_\theta + r\cos\theta\Sigma_{\frac{3}{2}}\mathbf{j}]. \qquad (28)$$

Picking up expressions for $I_{\frac{3}{2}}, X_{\frac{3}{2}}, \Sigma_{\frac{3}{2}}$ from (19), (22) and (23), we get

$$B_r = \frac{\mu_0 I}{4\pi}\frac{4R^2\cos\theta}{\rho^3}H(k). \quad \text{(a)}$$

$$B_\theta = \frac{\mu_0 I}{4\pi}\frac{2}{\rho^3\sin\theta} \times \qquad (29)$$

$$[(r^2+R^2-2R^2\sin^2\theta)H(k)-\rho^2 K(k)]. \quad \text{(b)}$$

In a sense Eq.(29) marks the end of this article. We would however do some extra work mainly to confirm that the expressions for the **B** field we have just derived tally with some of the formulas of Jackson.[2,3] The formulas we have in mind are (1) the exact expression Eq.(5.37) for the vector potential **A** in both second and third editions, (2) The approximate formula Eq.(5.40) for **B** in the second edition, (3) the series formula Eq.(5.40) for **B** in the third edition.

6 Obtaining E and B from the Scalar Potential Φ and the Vector Potential A

We shall first evaluate and tabulate some derivatives which will be needed in working out the formulas $\nabla\Phi$ and $\nabla\times\mathbf{A}$. One can evaluate the following derivatives from Eqs.(15) and (I6)

$$\frac{\partial k}{\partial r} = \sqrt{\frac{R\sin\theta}{r}}\,\frac{R^2-r^2}{\rho^3}\,; \quad \frac{\partial\rho}{\partial r} = \frac{r+R\sin\theta}{\rho}\,.$$

$$\frac{\partial k}{\partial\theta} = \frac{2rR\cos\theta}{k\rho^4}\,(R^2+r^2); \quad \frac{\partial\rho}{\partial\theta} = \frac{rR\cos\theta}{\rho}\,. \quad (30)$$

With the help of Eqs.(30), (13) and the first one of the two identities given in (17) it is now easy to establish the following derivatives of $K(k)$.

$$\frac{\partial K(k)}{\partial r} = \frac{dK(k)}{dk}\frac{\partial k}{\partial r} = \frac{R^2-r^2}{2r\rho^2}\,\{H(k)-K(k)\}. \quad \text{(a)}$$

$$\frac{\partial K(k)}{\partial\theta} = \frac{dK(k)}{dk}\frac{\partial k}{\partial\theta} = \frac{(R^2+r^2)\cot\theta}{2\rho^2}\,\{H(k)-K(k)\}. \quad \text{(b)} \qquad (31)$$

To obtain the corresponding derivatives of $E(k)$ we need to replace $H(k)$ on the right sides of the above formulas with $E(k)$, as is obvious from Eq.(13).

$$\frac{\partial E(k)}{\partial r} = \frac{dE(k)}{dk}\frac{\partial k}{\partial r} = \frac{R^2-r^2}{2r\rho^2}\,\{E(k)-K(k)\}. \quad \text{(a)}$$

$$\frac{\partial E(k)}{\partial\theta} = \frac{dE(k)}{dk}\frac{\partial k}{\partial\theta} = \frac{(R^2+r^2)\cot\theta}{2\rho^2}\,\{E(k)-K(k)\}. \quad \text{(b)} \qquad (32)$$

Combining Eqs.(30), (31) and (32), and using the identity (24) to convert $E(k)$ to $H(k)$ when necessary, we get the following formulas to facilitate the next steps in the computations.

$$\frac{\partial}{\partial r}\left(\frac{K(k)}{\rho}\right) = \frac{1}{2r\rho^3}\times [(R^2-r^2)H(k)-\rho^2K(k)], \quad \text{(a)}$$

$$\frac{\partial}{\partial\theta}\left(\frac{K(k)}{\rho}\right) = \frac{\cot\theta}{2\rho^3}\times [(R^2+r^2)H(k)-\rho^2K(k)], \quad \text{(b)}$$

$$\frac{\partial}{\partial r}[\rho E(k)] = \frac{1}{2r\rho}\times [(\xi^2H(k)+(r^2-R^2)K(k)], \quad \text{(c)} \qquad (33)$$

$$\frac{\partial}{\partial\theta}[\rho E(k)] = \frac{\cot\theta}{2\rho}\times [(\xi^2H(k)-(r^2+R^2)K(k)]. \quad \text{(d)}$$

The expression for the scalar potential given in Eq.(4) can now be written, using Eqs.(14c) and (19b) as

$$\Phi = \frac{\lambda R}{4\pi\varepsilon_0}\,\mathrm{I}_{\frac{1}{2}} = \frac{\lambda R}{4\pi\varepsilon_0}\,\frac{4K(k)}{\rho}\,. \qquad (34)$$

The components of **E** can then be obtained from this potential by taking its derivatives with respect to r and θ.

$$E_r = -\frac{\lambda R}{4\pi\varepsilon_0} 4 \frac{\partial}{\partial r}\left(\frac{K(k)}{\rho}\right);$$

$$E_\theta = -\frac{\lambda R}{4\pi\varepsilon_0} \frac{4}{r} \frac{\partial}{\partial \theta}\left(\frac{K(k)}{\rho}\right). \qquad (35)$$

Using the derivative formulas listed in (33) it is now easy to see that the right hand sides of Eqs.(35) will give the same expressions as in (27).

The expression for the vector potential given in Eq.(5) can now be can now be converted, using Eqs.(14c) and (19b) to the following form which is equivalent to the expression given by Jackson[3]

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \frac{2}{r\rho\sin\theta} [(R^2+r^2)K(k) - \rho^2 E(k)]\mathbf{e}_\phi,. \qquad (36)$$

since $\mathbf{i}=\mathbf{e}_\phi$ on the ZX-plane. The r and θ components of **B** are then given as

$$B_r = \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta}\{\sin\theta A_\phi\}$$

$$= \frac{\mu_0 I}{4\pi} \frac{2}{r^2 \sin\theta} \times$$

$$\left[\frac{\partial}{\partial\theta}\left\{(R^2+r^2)\left(\frac{K(k)}{\rho}\right) - \rho E(k)\right\}\right].$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\{rA_\phi\}$$

$$= -\frac{\mu_0 I}{4\pi} \frac{2}{r\sin\theta} \times$$

$$\left[\frac{\partial}{\partial r}\left\{(R^2+r^2)\left(\frac{K(k)}{\rho}\right) - \rho E(k)\right\}\right]. \qquad (37)$$

Collecting the derivatives from (33) the remaining steps can be completed leading to verification of the expressions for the **B** field as given in Eq.(29).

7 Series Expansion of the B Field in Powers of k

Writing $H(k)$ and. $K(k)$ in terms of the hypergeometric series as given in Eqs.(9) makes them particularly suitable for series expansion of the **E** and **B** fields in powers of k^2. We shall illustrate this by taking up the **B** field leaving the other case of **E** as a simple exercise for the reader.

The special function $K(k)$ and its series expansion are well known.[4] This is not the case with $H(k)$ which we have introduced in this article for our convenience. It is easy to write down the two series using their hypergeometric forms given in Eqs.(9).

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty} \beta_n k^{2n} \text{ ; where}$$

$$\beta_0 = 1,\ \beta_n = \left[\frac{(2n-1)!!}{2^n n!}\right]^2 \text{ for } n=1,2,3,\ldots$$

$$H(k) = \frac{\pi}{2}\sum_{n=0}^{\infty} \alpha_n k^{2n} \text{ ; where}$$

$$\alpha_n = (2n+1)\beta_n,\ n=0,1,2,3,\ldots \qquad (38)$$

We shall find it convenient to change the variable from k^2 to

$$x = \frac{k^2}{4} = \frac{rR\sin\theta}{\rho^2}\ ;\ x \le \frac{1}{4} \qquad (39)$$

It now follows that

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty} b_n x^n \text{ ; where}$$

$$b_0 = 1;\ b_n = \left[\frac{(2n-1)!!}{n!}\right]^2 \text{ for } n=1,2,3,\ldots$$

$H(k)= \dfrac{\pi}{2}\sum_{n=0}^{\infty} a_n x^n$; where

$$a_n=(2n+1)b_n \text{ for } n=0,1,2,3,\ldots \tag{40}$$

Going back to Eq.(29a) it is now easy to make a series expansion of B_r:

$$B_r=\frac{\mu_0 I}{4\pi}\frac{\pi R^2\cos\theta}{\rho^3}2\sum_{n=0}^{\infty}a_n x^n \ . \tag{41}$$

Note that $a_0=1$, $a_1=3$, $a_2=\frac{45}{4}$. Therefore,

$$2(a_0+a_1x+a_2x^2)=\frac{2(r^2+R^2)+10rR\sin\theta}{\rho^2}$$

$$+\frac{45(rR\sin\theta)^2}{2\rho^4}=\frac{2(r^2+R^2)+rR\sin\theta}{\rho^2}$$

$$+\frac{3rR\sin\theta\{3(r^2+R^2)+(6+\frac{15}{2})rR\sin\theta\}}{\rho^4}. \tag{42}$$

We have written the series in two different ways to underline the fact that the terms of the series are not unique. This ambiguity can be removed if we specify which one of the three variables r, R, $\sin\theta$ is large compared to the other two. We shall do this exercise in the next section. If we adopt the second line of Eq.(42) then we get the following expansion for B_r.

$$B_r=\frac{\mu_0 I}{4\pi}\frac{\pi R^2\cos\theta}{\rho^5}\Big[\{2(r^2+R^2)+rR\sin\theta\}+$$

$$\frac{3rR\sin\theta}{\rho^2}\left\{3(r^2+R^2)+(6+\tfrac{15}{2})rR\sin\theta\right\}\Big]+\ldots \tag{43}$$

If we retain only the first term inside the square brackets, we get the same approximate expression as given in Ref.2.

We now come to the series expansion of the expression for B_θ as given in Eq.(29b). First note that

$$r^2+R^2-2R^2\sin^2\theta=\rho^2-2rR\sin\theta-2\frac{(rR\sin\theta)^2}{r^2}$$

$$=\rho^2\left[1-2x-2\frac{\rho^2x^2}{r^2}\right]. \tag{44}$$

We now rewrite B_θ of Eq.(29b) in the following way.

$$B_\theta=\frac{\mu_0 I}{4\pi}\frac{2}{\rho^3\sin\theta}\Lambda; \text{ where}$$

$$\Lambda=r^2[H(k)-K(k)-2xH(k)-2\frac{\rho^2x^2}{r^2}H(k)]. \tag{45}$$

Going back to Eq.(38) we can now make a power series expansion of Λ. It is seen from Eq.(40) that $a_0-b_0=0$; $a_1-b_1-2a_0=0$. Hence, with a little manipulation

$$\Lambda=\frac{1}{2}\rho^2x^2\pi\sum_{n=0}^{\infty}c_n x^n \text{ ; where}$$

$$c_n=a_{n+2}-b_{n+2}-2a_{n+1}-2\frac{\rho^2}{r^2}a_n . \tag{46}$$

Using the coefficients given in Eq.(40) it is now simple exercise to prove that

$$c_n=2(2n+1)\left[\frac{(2n+3)(2n+1)}{(n+1)(n+2)}-\frac{\rho^2}{r^2}\right]b_n . \tag{47}$$

Combining Eq.(45) with (46) and using (39) we get the desired series for B_θ.

$$B_\theta=\frac{\mu_0 I}{4\pi}\frac{\pi R^2}{\rho^5}r^2\sin\theta\sum_{n=0}^{\infty}c_n x^n \tag{48}$$

We shall now work out the first two coefficients of the above series with the help of Eq.(47).

$$c_0=3-\frac{2\rho^2}{r^2}=\frac{r^2-2R^2-4rR\sin\theta}{r^2};$$

$$c_1 = 6\left(\frac{5}{2} - \frac{\rho^2}{r^2}\right) = \frac{3[3r^2 - 2R^2 - 4rR\sin\theta]}{r^2} \quad . (49)$$

Hence,

$$B_\theta = \frac{\mu_0 IR^2 \sin\theta}{4\rho^5}\Big[r^2 - 2R^2 - 4rR\sin\theta +$$

$$\frac{3rR\sin\theta}{\rho^2}\{3r^2 - 2R^2 - 4rR\sin\theta\} + ...\Big]$$

$$= \frac{\mu_0 IR^2 \sin\theta}{4\rho^5}\Big[r^2 - 2R^2 - rR\sin\theta +$$

$$\frac{3rR\sin\theta}{\rho^2}\{3r^2 - 2R^2 - 6rR\sin\theta\} + ...\Big]. \quad (50)$$

Again we have written the expansion in two different ways, the first one with direct application of Eqs.(49) and (48), and the second one by a readjustment of terms so that the first order term in the second line becomes identical with the approximate expression given in Ref.2.

8 Expansion for $r > R$

We shall obtain series expansions of B_r and B_θ in the powers of $\frac{R}{r}$ assuming that $r>R$, i.e. for the regions at radial distances larger than the radius of the coil. The other case, viz., $r<R$, in the powers of $\frac{r}{R}$, can be worked out by the reader who will closely follow the current example. We shall begin by constructing a few preliminary series which will serve as the building blocks for our work. We shall limit each series to terms of the order $(\frac{R}{r})^2$. From Eqs.(39) and (16)

$$\frac{1}{\rho^3} = \frac{1}{r^3}\left[1 - 3\frac{R}{r}\sin\theta - \frac{3}{2}(1 - 5\sin^2\theta)\frac{R^2}{r^2} + ...\right]$$

(a)

$$\frac{1}{\rho^5} = \frac{1}{r^5}\left[1 - 5\frac{R}{r}\sin\theta - \frac{5}{2}(1 - 7\sin^2\theta)\frac{R^2}{r^2} + ...\right]$$

(b) (51)

$$x = \frac{R}{r}\sin\theta\left(1 - 2\frac{R}{r}\sin\theta\right) + ...;$$

$$x^2 = \left(\frac{R}{r}\sin\theta\right)^2 + ... \quad (c)$$

We now go back to Eq.(41) for B_r. The required coefficients are written below Eq.(41). Then

$$B_r \approx \frac{\mu_0 I}{4\pi}\frac{2\pi R^2\cos\theta}{r^3}\frac{1}{r^3} \times$$

$$\left[1 - 3\frac{R}{r}\sin\theta - \frac{3}{2}(1 - 5\sin^2\theta)\frac{R^2}{r^2} + ...\right] \times$$

$$\left[1 + 3\frac{R}{r}\sin\theta\left(1 - 2\frac{R}{r}\sin\theta\right) + \frac{45}{4}\left(\frac{R\sin\theta}{r}\right)^2\right].$$

(52)

Simplifying we get

$$B_r = \frac{\mu_0 I}{4\pi}\frac{2\pi R^2\cos\theta}{r^3} \times$$

$$\left[1 + \left(\frac{15}{4}\sin^2\theta - \frac{3}{2}\right)\frac{R^2}{r^2} + ...\right] \quad (53)$$

Series expansion for B_θ is slightly more difficult because the coefficients c_ns in Eq.(48) are not constant terms. Each one of them is quadratic in $\frac{R}{r}$. The first two are already in Eq.(49). We need one more

c_2=2×5 $\left[\frac{7\times5}{3\times4}-\frac{\rho^2}{r^2}\right]\frac{9}{4}=$

$$\frac{45}{4}\left[\frac{23r^2-12R^2-24rR\sin\theta}{6r^2}\right]. \qquad (54)$$

We now evaluate the three relevant terms in the series using Eq.(51b), (49) and (47).

$$\frac{c_0}{\rho^5}=$$

$$\frac{1}{r^5}\left[1-9\frac{R}{r}\sin\theta+\left(\frac{75}{2}\sin^2\theta-\frac{9}{2}\right)\frac{R^2}{r^2}+...\right].$$

$$\frac{c_1 x}{\rho^5}=\frac{1}{r^5}\left[9\frac{R}{r}\sin\theta-75\frac{R^2}{r^2}\sin^2\theta+...\right]. \qquad (55)$$

$$\frac{c_2 x^2}{\rho^5}=\frac{1}{r^5}\left[\frac{23\times15}{8}\frac{R^2}{r^2}\sin^2\theta+...\right].$$

Adding the above three lines we get

$$\frac{1}{\rho^5}\sum_{n=0}^{2}c_n x^n=\frac{1}{r^5}\left[1+\left(\frac{45}{8}\sin^2\theta-\frac{9}{2}\right)\frac{R^2}{r^2}\right]+... \qquad (56)$$

Hence from Eq.(48)

$$B_\theta=\frac{\mu_0 I}{4\pi}\frac{\pi R^2\sin\theta}{r^3}$$

$$\left[1+\left(\frac{45}{8}\sin^2\theta-\frac{9}{2}\right)\frac{R^2}{r^2}+...\right]. \qquad (57)$$

The expressions for B_r, B_θ given in (53) and (57) tally with their counterparts presented by Jackson[3] when the terms written by him are expanded in the power series of $\frac{a}{r}$.

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