# Inhaltsverzeichnis



# 1 Introduction

This document is intended as an extension to the exercise generator:
`https://github.com/TimJ2718/python-exercise-generator`
and as an introduction to linear algebra.

# 2 Determinant

## 2.1 Definition and conclusions

The determinant function $\det(A)$ for a square matrix: $A \in K^{n\times n}$ is defined as the function which maps $K^{n\times n} \to K$ and satifies the following properties:
We write $A = (v_1, ...v_n)$, where $v \in K^{1\times n}$.

- $\det(v_1, ...\lambda \cdot v_i, ...v_n) = \lambda \det(v_1, ...v_i, ...v_n)$
- $\det(v_1, ...v_i + w, ...v_n) = \det(v_1, ...v_i, ...v_n) + \det(v_1, ...w, ...v_n)$
- $v_i = v_j => \det(v_1, ...v_i, ...v_j, ...v_n) = 0$
- $\det(\mathbb{1}) = 1$ (Here $\mathbb{1}$ is the unity matrix)

Important conclusions are:

- $\det(A) = 0$ <=> A does not have a full rank <=> The columns/rows are linear dependent
- $\det(A) = \det(A^T)$
- $\det(A \cdot B) = \det(A) \cdot \det(B)$
- $\det(v_1, ..., v_i, ..., v_j, ...v_n) = \det(v_1, ..., v_i + v_j, ...v_n)$

## 2.2 Calculations

### 2.2.1 Explicit formulas

$$\det\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] = a_{11}a_{22} - a_{21}a_{12}$$

$$\det\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right]$$

$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$

**Examples**

$\det\left[\begin{pmatrix} 2 & -3 \\ 5 & 6 \end{pmatrix}\right] = 2 \cdot 6 - (5 \cdot (-3)) = 12 + 15 = 27$

$\det\left[\begin{pmatrix} 9 & 3 & 1 \\ 2 & 6 & 3 \\ 6 & -1 & -1 \end{pmatrix}\right] = 9 \cdot 6 \cdot (-1) + 3 \cdot 3 \cdot 6 + 1 \cdot 2 \cdot (-1) - 6 \cdot 6 \cdot 1 - (-1) \cdot 3 \cdot 9 - (-1) \cdot 2 \cdot 3 =$
$-54 + 54 - 2 - 36 + 27 + 6 = -5$

### 2.2.2 Blockmatrix

Let be $M \in K^{n \times n}$ and 0 the 0 matrix.
If A and C are quadratic matrices (C does not need to be quadratic) then the determinant of M is given by: $M = \det\left[\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\right] = \det[A] \cdot \det[C]$

$M = \det\left[\begin{pmatrix} A & 0 \\ C & C \end{pmatrix}\right] = \det[A] \cdot \det[C]$

**Example**

$\det\left[\begin{pmatrix} 2 & 3 & 4 & 8 & 2 \\ 6 & 2 & 3 & 9 & 4 \\ 0 & 0 & 9 & 3 & 1 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 6 & -1 & -1 \end{pmatrix}\right] = \det\left[\begin{pmatrix} 2 & 3 \\ 6 & 2 \end{pmatrix}\right] \cdot \det\left[\begin{pmatrix} 9 & 3 & 1 \\ 2 & 6 & 3 \\ 6 & -1 & -1 \end{pmatrix}\right] = -14 \cdot$
$(-5) = 70$

### 2.2.3 Triangular matrix

$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$

There are two kinds of triangular matrices:
A is an upper triangular matrix if all elements under the diagonal are zero: $a_{ij} = 0$ for $i > j$.
A is a lower trinangular matrix if all elements over the diagonal are zeros: $a_{ij} = 0$ for $i < j$.
The determinant of a triangular matrix is given by the product of the diagonal elements: $\det(A) = \prod_{i=1}^{n} a_{ii}$

**Example**

$\det\left[\begin{pmatrix} 2 & 5 & 4 \\ 0 & 3 & 8 \\ 0 & 0 & 4 \end{pmatrix}\right] = 2 \cdot 3 \cdot 4 = 24;\ \det\left[\begin{pmatrix} 2 & 0 & 0 \\ 8 & 3 & 0 \\ 5 & 4 & 4 \end{pmatrix}\right] = 2 \cdot 3 \cdot 4 = 24$

### 2.2.4 Laplace expansion

$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$

Define: $A_{ij}$ as the matrix where row i and the column j is removed:

$$\tilde{A}_{ij} = \begin{pmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{pmatrix}$$

The determinat of A is given by:

$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot \tilde{A}_{ij}$

$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot \tilde{A}_{ij}$

**Examples**

$$\det\left[\begin{pmatrix} 2 & 0 & 2 \\ 8 & 3 & 1 \\ 5 & 4 & 4 \end{pmatrix}\right] = (-1)^{1+1} \cdot 2 \cdot \det\left[\begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}\right] + (-1)^{1+2} \cdot 0 \cdot \det\left[\begin{pmatrix} 8 & 1 \\ 5 & 4 \end{pmatrix}\right] + (-1)^{1+3} \cdot 2 \cdot \det\left[\begin{pmatrix} 8 & 3 \\ 5 & 4 \end{pmatrix}\right]$$

$$\det\left[\begin{pmatrix} 2 & 0 & 2 \\ 8 & 3 & 1 \\ 5 & 4 & 4 \end{pmatrix}\right] = (-1)^{1+2} \cdot 0 \cdot \det\left[\begin{pmatrix} 8 & 1 \\ 5 & 4 \end{pmatrix}\right] + (-1)^{2+2} \cdot 3 \cdot \det\left[\begin{pmatrix} 2 & 2 \\ 5 & 4 \end{pmatrix}\right] + (-1)^{3+2} \cdot 4 \cdot \det\left[\begin{pmatrix} 2 & 2 \\ 8 & 1 \end{pmatrix}\right]$$