Inhaltsverzeichnis

1 Introduction

This document is intended as an extension to the exercise generator:
`https://github.com/TimJ2718/python-exercise-generator`
and as an introduction to linear algebra.

2 Determinant

2.1 Definition and conclusions

The determinant function $\det(A)$ for a square matrix: $A \in K^{n \times n}$ is defined as the function which maps $K^{n \times n} \to K$ and satifies the following properties:
We write $A = (v_1, ...v_n)$, where $v \in K^{1 \times n}$.

- $\det(v_1, ...\lambda \cdot v_i, ...v_n) = \lambda \det(v_1, ...v_i, ...v_n)$
- $\det(v_1, ...v_i + w, ...v_n) = \det(v_1, ...v_i, ...v_n) + \det(v_1, ...w, ...v_n)$
- $v_i = v_j => \det(v_1, ...v_i, ...v_j, ...v_n) = 0$
- $\det(\mathbb{1}) = 1$ (Here $\mathbb{1}$ is the unity matrix)

Important conclusions are:

- $\det(A) = 0$ <=> A does not have a full rank <=> The columns/rows are linear dependent
- $\det(A) = \det(A^T)$
- $\det(A \cdot B) = \det(A) \cdot \det(B)$
- $\det(v_1, ..., v_i, ..., v_j, ...v_n) = \det(v_1, ..., v_i + v_j, ...v_n)$

2.2 Calculations

2.2.1 Explicit formulas

$\det\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] = a_{11}a_{22} - a_{21}a_{12}$

$\det\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right]$

$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$

Examples

$\det\left[\begin{pmatrix} 2 & -3 \\ 5 & 6 \end{pmatrix}\right] = 2 \cdot 6 - (5 \cdot (-3)) = 12 + 15 = 27$

$\det\left[\begin{pmatrix} 9 & 3 & 1 \\ 2 & 6 & 3 \\ 6 & -1 & -1 \end{pmatrix}\right] = 9\cdot6\cdot(-1)+3\cdot3\cdot6+1\cdot2\cdot(-1)-6\cdot6\cdot1-(-1)\cdot3\cdot9-(-1)\cdot2\cdot3 =$
$-54 + 54 - 2 - 36 + 27 + 6 = -5$

2.2.2 Blockmatrix

Let be $M \in K^{n\times n}$ and 0 the 0 matrix.
If A and C are quadratic matrices (C does not need to be quadratic) then the determinant of M is given by: $M = \det\left[\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\right] = \det[A] \cdot \det[C]$

$M = \det\left[\begin{pmatrix} A & 0 \\ C & C \end{pmatrix}\right] = \det[A] \cdot \det[C]$

Example

$\det\left[\begin{pmatrix} 2 & 3 & 4 & 8 & 2 \\ 6 & 2 & 3 & 9 & 4 \\ 0 & 0 & 9 & 3 & 1 \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 6 & -1 & -1 \end{pmatrix}\right] = \det\left[\begin{pmatrix} 2 & 3 \\ 6 & 2 \end{pmatrix}\right] \cdot \det\left[\begin{pmatrix} 9 & 3 & 1 \\ 2 & 6 & 3 \\ 6 & -1 & -1 \end{pmatrix}\right] = -14 \cdot$
$(-5) = 70$

2.2.3 Triangular matrix

$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$

There are two kinds of triangular matrices:
A is an upper triangular matrix if all elements under the diagonal are zero: $a_{ij} = 0$ for $i > j$.
A is a lower trinangular matrix if all elements over the diagonal are zeros: $a_{ij} = 0$ for $i < j$.
The determinant of a triangular matrix is given by the product of the diagonal elements: $\det(A) = \prod_{i=1}^{n} a_{ii}$

Example

$\det\left[\begin{pmatrix} 2 & 5 & 4 \\ 0 & 3 & 8 \\ 0 & 0 & 4 \end{pmatrix}\right] = 2 \cdot 3 \cdot 4 = 24;\ \det\left[\begin{pmatrix} 2 & 0 & 0 \\ 8 & 3 & 0 \\ 5 & 4 & 4 \end{pmatrix}\right] = 2 \cdot 3 \cdot 4 = 24$

2.2.4 Laplace expansion

$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$

Define: A_{ij} as the matrix where row i and the column j is removed:

$\tilde{A}_{ij} = \begin{pmatrix} a_{11} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{n,j-1} & a_{n,j+1} & \dots & a_{nn} \end{pmatrix}$

The determinat of A is given by:

$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot \tilde{A}_{ij}$

$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot \tilde{A}_{ij}$

Examples

$\det\left[\begin{pmatrix} 2 & 0 & 2 \\ 8 & 3 & 1 \\ 5 & 4 & 4 \end{pmatrix}\right] = (-1)^{1+1} \cdot 2 \cdot \det\left[\begin{pmatrix} 3 & 1 \\ 4 & 4 \end{pmatrix}\right] + (-1)^{1+2} \cdot 0 \cdot \det\left[\begin{pmatrix} 8 & 1 \\ 5 & 4 \end{pmatrix}\right] +$
$(-1)^{1+3} \cdot 2 \cdot \det\left[\begin{pmatrix} 8 & 3 \\ 5 & 4 \end{pmatrix}\right]$

$\det\left[\begin{pmatrix} 2 & 0 & 2 \\ 8 & 3 & 1 \\ 5 & 4 & 4 \end{pmatrix}\right] = (-1)^{1+2} \cdot 0 \cdot \det\left[\begin{pmatrix} 8 & 1 \\ 5 & 4 \end{pmatrix}\right] + (-1)^{2+2} \cdot 3 \cdot \det\left[\begin{pmatrix} 2 & 2 \\ 5 & 4 \end{pmatrix}\right] +$
$(-1)^{3+2} \cdot 4 \cdot \det\left[\begin{pmatrix} 2 & 2 \\ 8 & 1 \end{pmatrix}\right]$

3 Eigenvalue problem

3.1 Introduction

A vector v is called eigenvector of the linear map φ if $\varphi(v) = \lambda \cdot v$. Here the scalar λ is called eigenvalue.
For obvious reasons φ must be an endomorphismus and can be expressed as a quadratic matrix.

3.2 Characteristical polynom

Let be: $A \in K^{n\times n}$ and $v \in K^{1\times n}$ and $\lambda \in K$.
$A \cdot V = \lambda V \to (\lambda \cdot \mathbb{1} - A)v = 0$ (Here $\mathbb{1}$ is again the unity matrix.)
This can only be true if $(\lambda \cdot \mathbb{1} - A)$ has no full rank <=> $\det(\lambda \cdot \mathbb{1} - A) = 0$
We define $\chi(\lambda) = \det(\lambda \cdot \mathbb{1} - A)$ as the characterisitcal polynom.

3.3 Eigenvalues

The eigenvalues are given by the roots of the characteristical polynom. The multiplicity of the root is called algebraic multiplicity.

3.4 Eigenvectors

The eigenvectors to the eigenvalue λ can be found by solving the following equation:
$(\lambda_i \cdot \mathbb{1} - A)v = 0$ Every eigenvalue has at least one eigenvector but at most as many as the algebraic multiplicity. The number of linear indepenedent eigenvalues of a eigenvalue is called geometric multiplicity.

3.5 Example

Determine the characteristical polynom, the eigenvalues and the eigenvectors:
$\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$

Chracteristical polynom:
$\chi(\lambda) = \det \begin{pmatrix} \lambda - 1 & -3 \\ 0 & \lambda - 4 \end{pmatrix} = (\lambda - 1) \cdot (\lambda - 4) - 0 \cdot (-3) = \lambda^2 - 5\lambda + 4$

Eigenvalues:
$\chi(\lambda) = 0 <=> (\lambda - 1) \cdot (\lambda - 4) = 0$
$=> \lambda_1 = 1$: Algebraic multiplicity: 1
$=> \lambda_2 = 4$: Algebraic multiplicity: 1

Hint: You can check your result with the following relationship:
The sum of the eigenvalues is equal to the trace of the matrix.
$1 + 4 \stackrel{!}{=} \lambda_1 + \lambda_2$
(Eigenvalues that occur more than once are added multiple times according to their occurance)
(The trace of a matrix is given by the sum of the diagonal elements)

Eigenvectors:
$\lambda_1 = 1$:
$\begin{pmatrix} 0 & -3 \\ 0 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\lambda_2 = 4$:
$\begin{pmatrix} 3 & -3 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \to v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
Hint: You can plug in v and λ in the definition $Av = \lambda v$ and check if your calculation is right.

3.6 Diagonalizability