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Overview of models

Overview of models

The following tables provide an overview of the specified models and their main assumptions, with model extensions/changes highlighted in **bold**. For every model, the observed data are assumed to be complete (i.e., no missing data).

Model	Latent class memberships	Dependent variable	Constant & trend component[1]
1	constant, no explanatory variables	continuous, Normal, iid errors	pooled model, linear
2	constant, no explanatory variables	continuous, Normal, iid errors	pooled model, **quadratic**
3	constant, no explanatory variables	**count**, **Poisson**, iid errors	pooled model, **linear**
4	constant, no explanatory variables	count, Poisson, iid errors	pooled model, **quadratic**

Table 1: Overview of specified models

[1] for each class

General notation

General notation

Latent class (aka, mixture component) c, for $c = 1, ..., C$, where C is the number of classes

Individual n, for $n = 1, ..., N$, where N is the number of individuals

Time period t, for $t = 1, ..., T$, where T is the number of time periods

$\boldsymbol{Y}^{obs}$ is a $N \times T$ matrix representing the observed dependent variable, where *obs* refers to simulated or actual data

$\boldsymbol{X}$ is a matrix of size $N \times T$ representing the explanatory variable (e.g., time periods, starting at zero)

Model 1

Model 1 - latent class memberships

Constant over time periods (i.e., an individual does not switch between classes)

No explanatory variables incorporated to estimate the probability of an individual belonging to a certain class

Model 1 - dependent variable

Continuous data

Normal (aka, Gaussian)

iid errors: independent (aka, uncorrelated) and identically distributed errors over individuals and time periods; however, the standard deviation of the errors is allowed to vary between classes

No missing data

Model 1 - constant and trend component

Constant without interindividual differences within-class (i.e., the constant is allowed to vary between classes but not within classes)

Linear (i.e., non-stationary, deterministic) trend component without interindividual differences within-class (i.e., the trend component is allowed to vary between classes but not within classes)

Based on Koop (2003), the assumption of no interindividual differences within-class regarding the constant and trend component correspond to a pooled model (i.e., a pooled model for each class)

Model 1 - likelihood

Based on Basturk (2010) and Stan Development Team (n.d.),

$$p(\boldsymbol{Y}^{obs}|\boldsymbol{\lambda}, \boldsymbol{M}, \boldsymbol{\sigma}) = \prod_{n=1}^{N} \sum_{c=1}^{C} \lambda_c \prod_{t=1}^{T} Normal(\mu_{c,n,t}, \sigma_c), \tag{1}$$

where $\boldsymbol{\lambda}$ is a row vector of size C representing the mixture proportions, averaged over individuals; thus,

$$0 \leq \lambda_c \leq 1 \quad \text{and} \quad \sum_{c=1}^{C} \lambda_c = 1. \tag{2}$$

Furthermore, $\boldsymbol{M}$ is a C-tuple containing $N \times T$ matrices, and $\boldsymbol{\sigma}$ is a row vector of size C.

Model 1 - likelihood continued

$$\mu_{c,n,t} = \beta_{0,c} + \beta_{1,c}\, x_{n,t}, \tag{3}$$

where $\boldsymbol{\beta}_0$ is a row vector of size C representing the constants. $\boldsymbol{\beta}_1$ is also a row vector of size C, and $\beta_{1,c}\, x_{n,t}$ represents the linear trend component. Furthermore, to make the model identifiable, the order in one or several sets of parameters must be restricted (Stan Development Team, n.d.); e.g.,

$$\beta_{0,c} < \beta_{0,c+1} \tag{4}$$

defines a restriction on $\boldsymbol{\beta}_0$. The number of required restrictions depends on C and $\boldsymbol{Y}^{obs}$.

Model 1 - deduction of likelihood

The latent discrete parameter z_n in $\{1, ..., C\}$ indicates that individual n belongs to class c, with

$$z_n \sim Categorical(\boldsymbol{\lambda}), \tag{5}$$

where $\boldsymbol{z}$ is a column vector of size N. Therefore, the likelihood presented in equation 1 is deduced by marginalizing out z_n:

$$\begin{aligned}
p(\boldsymbol{Y}^{obs}|\boldsymbol{z}, \boldsymbol{\lambda}, \boldsymbol{M}, \boldsymbol{\sigma}) &= \prod_{n=1}^{N} \prod_{c=1}^{C} \left(\lambda_c \prod_{t=1}^{T} p(y_{n,t}^{obs}|z_n, \mu_{c,n,t}, \sigma_c) \right)^{\mathbf{1}(z_n=c)} \\
&= \prod_{n=1}^{N} \sum_{c=1}^{C} \lambda_c \prod_{t=1}^{T} p(y_{n,t}^{obs}|\mu_{c,n,t}, \sigma_c) \\
&= \prod_{n=1}^{N} \sum_{c=1}^{C} \lambda_c \prod_{t=1}^{T} Normal(\mu_{c,n,t}, \sigma_c),
\end{aligned} \tag{6}$$

where $\mathbf{1}(z_n = c)$ defines an indicator function.

Model 1 - log likelihood

Recall the likelihood presented in equation 1:

$$p(\boldsymbol{Y}^{obs}|\boldsymbol{\lambda}, \boldsymbol{M}, \boldsymbol{\sigma}) = \prod_{n=1}^{N} \sum_{c=1}^{C} \lambda_c \prod_{t=1}^{T} Normal(\mu_{c,n,t}, \sigma_c). \tag{1}$$

On the log scale, the likelihood is given by

$$\begin{aligned} &\log p(\boldsymbol{Y}^{obs}|\boldsymbol{\lambda}, \boldsymbol{M}, \boldsymbol{\sigma}) \\ &= \sum_{n=1}^{N} \log \sum_{c=1}^{C} \exp\left(\log \lambda_c + \sum_{t=1}^{T} \log Normal(\mu_{c,n,t}, \sigma_c) \right). \end{aligned} \tag{7}$$

Model 1 - prior

$$\boldsymbol{\lambda} \sim \mathit{Dirichlet}(\boldsymbol{\alpha}), \tag{8}$$

where $\boldsymbol{\alpha}$ is a row vector of size C representing hyperparameters. Furthermore, $\alpha_c = 1$ (i.e., $\boldsymbol{\lambda}$ is assigned a proper flat prior).

$$\beta_{0,c} \sim \mathit{Normal}(\beta_{0,c,\mu}, \beta_{0,c,\sigma}), \tag{9}$$

where $\beta_{0,c,\mu}$ and $\beta_{0,c,\sigma}$ are hyperparameters.

$$\beta_{1,c} \sim \mathit{Normal}(\beta_{1,c,\mu}, \beta_{1,c,\sigma}), \tag{10}$$

where $\beta_{1,c,\mu}$ and $\beta_{1,c,\sigma}$ are hyperparameters.

Model 1 - prior continued

$$\sigma_c \sim \mathit{Normal}(0, \sigma_{c,\sigma})\, \mathbf{1}(\sigma_c > 0), \tag{11}$$

where 0 and $\sigma_{c,\sigma}$ are hyperparameters. The Normal distribution is truncated to the left at zero, modeled via the indicator function $\mathbf{1}(\sigma_c > 0)$.

Model 1 - posterior for z_n

$$Pr(z_n = c | \boldsymbol{y}_n, \boldsymbol{\lambda}, \boldsymbol{M}_n, \boldsymbol{\sigma}) = \frac{\lambda_c \prod_{t=1}^{T} Normal(\mu_{c,n,t}, \sigma_c)}{\sum_{k=1}^{C} \lambda_k \prod_{t=1}^{T} Normal(\mu_{k,n,t}, \sigma_k)}. \tag{12}$$

On the log scale, the posterior for z_n is given by

$$\begin{aligned}
&\log Pr(z_n = c | \boldsymbol{y}_n, \boldsymbol{\lambda}, \boldsymbol{M}_n, \boldsymbol{\sigma}) \\
&= \log \lambda_c + \sum_{t=1}^{T} \log Normal(\mu_{c,n,t}, \sigma_c) \\
&- \log \sum_{k=1}^{C} \exp \left(\log \lambda_k + \sum_{t=1}^{T} \log Normal(\mu_{k,n,t}, \sigma_k) \right).
\end{aligned} \tag{13}$$

Equation 13 corresponds to the softmax function calculated on the log scale (Stan Development Team, n.d.).

Model 3

Model 3 - latent class memberships

Constant over time periods (i.e., an individual does not switch between classes)

No explanatory variables incorporated to estimate the probability of an individual belonging to a certain class

Model 3 - dependent variable

Count data

Poisson

iid errors: independent (aka, uncorrelated) and identically distributed errors over individuals and time periods; however, the standard deviation of the errors is allowed to vary between classes

No missing data

Model 3 - constant and trend component

Constant without interindividual differences within-class (i.e., the constant is allowed to vary between classes but not within classes)

Linear (i.e., non-stationary, deterministic) trend component without interindividual differences within-class (i.e., the trend component is allowed to vary between classes but not within classes)

Based on Koop (2003), the assumption of no interindividual differences within-class regarding the constant and trend component correspond to a pooled model (i.e., a pooled model for each class)

Model 3 - likelihood

Based on Basturk (2010) and Stan Development Team (n.d.),

$$p(\boldsymbol{Y}^{obs}|\boldsymbol{\lambda}, \boldsymbol{\Theta}) = \prod_{n=1}^{N} \sum_{c=1}^{C} \lambda_c \prod_{t=1}^{T} PoissonLog(\theta_{c,n,t}), \tag{14}$$

where $\boldsymbol{\lambda}$ is a row vector of size C representing the mixture proportions, averaged over individuals; thus,

$$0 \leq \lambda_c \leq 1 \quad \text{and} \quad \sum_{c=1}^{C} \lambda_c = 1. \tag{15}$$

Furthermore, $\boldsymbol{\Theta}$ is a C-tuple containing $N \times T$ matrices, and $\theta_{c,n,t}$ represents both the expected value and variance.

Model 3 - likelihood continued

$$\theta_{c,n,t} = \beta_{0,c} + \beta_{1,c}\, x_{n,t}, \tag{16}$$

where $\boldsymbol{\beta}_0$ is a row vector of size C representing the constants. $\boldsymbol{\beta}_1$ is also a row vector of size C, and $\beta_{1,c}\, x_{n,t}$ represents the linear trend component. Furthermore, to make the model identifiable, the order in one or several sets of parameters must be restricted (Stan Development Team, n.d.); e.g.,

$$\beta_{0,c} < \beta_{0,c+1} \tag{17}$$

defines a restriction on $\boldsymbol{\beta}_0$. The number of required restrictions depends on C and $\boldsymbol{Y}^{obs}$.

Model 3 - deduction of likelihood

The latent discrete parameter z_n in $\{1, ..., C\}$ indicates that individual n belongs to class c, with

$$z_n \sim Categorical(\boldsymbol{\lambda}), \tag{18}$$

where $\boldsymbol{z}$ is a column vector of size N. Therefore, the likelihood presented in equation 14 is deduced by marginalizing out z_n:

$$\begin{aligned} p(\boldsymbol{Y}^{obs}|\boldsymbol{z}, \boldsymbol{\lambda}, \boldsymbol{\Theta}) &= \prod_{n=1}^{N}\prod_{c=1}^{C}\left(\lambda_c \prod_{t=1}^{T} p(y_{n,t}^{obs}|z_n, \theta_{c,n,t})\right)^{\mathbf{1}(z_n=c)} \\ &= \prod_{n=1}^{N}\sum_{c=1}^{C}\lambda_c \prod_{t=1}^{T} p(y_{n,t}^{obs}|\theta_{c,n,t}) \\ &= \prod_{n=1}^{N}\sum_{c=1}^{C}\lambda_c \prod_{t=1}^{T} PoissonLog(\theta_{c,n,t}), \end{aligned} \tag{19}$$

where $\mathbf{1}(z_n = c)$ defines an indicator function.

Model 3 - log likelihood

Recall the likelihood presented in equation 14:

$$p(\boldsymbol{Y}^{obs}|\boldsymbol{\lambda}, \boldsymbol{\Theta}) = \prod_{n=1}^{N} \sum_{c=1}^{C} \lambda_c \prod_{t=1}^{T} PoissonLog(\theta_{c,n,t}). \tag{14}$$

On the log scale, the likelihood is given by

$$\begin{aligned} &\log p(\boldsymbol{Y}^{obs}|\boldsymbol{\lambda}, \boldsymbol{\Theta}) \\ &= \sum_{n=1}^{N} \log \sum_{c=1}^{C} \exp\left(\log \lambda_c + \sum_{t=1}^{T} \log PoissonLog(\theta_{c,n,t}) \right). \end{aligned} \tag{20}$$

Model 3 - prior

$$\boldsymbol{\lambda} \sim \textit{Dirichlet}(\boldsymbol{\alpha}), \tag{21}$$

where $\boldsymbol{\alpha}$ is a row vector of size C representing hyperparameters. Furthermore, $\alpha_c = 1$ (i.e., $\boldsymbol{\lambda}$ is assigned a proper flat prior).

$$\beta_{0,c} \sim \textit{Normal}(\beta_{0,c,\mu}, \beta_{0,c,\sigma}), \tag{22}$$

where $\beta_{0,c,\mu}$ and $\beta_{0,c,\sigma}$ are hyperparameters.

$$\beta_{1,c} \sim \textit{Normal}(\beta_{1,c,\mu}, \beta_{1,c,\sigma}), \tag{23}$$

where $\beta_{1,c,\mu}$ and $\beta_{1,c,\sigma}$ are hyperparameters.

Model 3 - posterior for z_n

$$Pr(z_n = c|\boldsymbol{y}_n, \boldsymbol{\lambda}, \boldsymbol{\Theta}_n) = \frac{\lambda_c \prod_{t=1}^{T} PoissonLog(\theta_{c,n,t})}{\sum_{k=1}^{C} \lambda_k \prod_{t=1}^{T} PoissonLog(\theta_{k,n,t})}. \tag{24}$$

On the log scale, the posterior for z_n is given by

$$\begin{aligned} &\log Pr(z_n = c|\boldsymbol{y}_n, \boldsymbol{\lambda}, \boldsymbol{\Theta}_n) \\ &= \log \lambda_c + \sum_{t=1}^{T} \log PoissonLog(\theta_{c,n,t}) \\ &- \log \sum_{k=1}^{C} \exp\left(\log \lambda_k + \sum_{t=1}^{T} \log PoissonLog(\theta_{k,n,t}) \right). \end{aligned} \tag{25}$$

Equation 25 corresponds to the softmax function calculated on the log scale (Stan Development Team, n.d.).

References

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