Maths Problem Set 3

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Ex 4.2

$$D \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 2x \\ 1 \\ 0 \end{bmatrix}$$

So

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is an upper triangular matrix so its eignvalues lie on the diagonal. Therefore the only eigenvalue is 0. It appears three times, so the algebraic multiplicity is 3. The eigenspace is the set of constant functions and has dimension one, which is the geometric multiplicity.

Ex 4.4

(i) We are given that:

$$A = A^H$$
$$\begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix} = \begin{bmatrix} a_1 - b_1 i & a_3 - b_3 i) \\ a_2 - b_2 i & a_4 - b_4 i \end{bmatrix}$$

Therefore we can say that:

$$b_1 = 0$$
$$b_2 = -b_3$$
$$b_4 = 0$$

The characteristic polynomial for matrix A is:

$$P(\lambda) = \lambda^2 - (a_1 + a_4)\lambda + a_1 a_4 - a_2^2 - b_2^2$$

Which has only real routes if:

$$\begin{aligned} B^2 - 4AC &> 0 \\ &= (a_1 + a_4)^2 - 4(a_1 a_4 - a_2^2 - b_2^2]) \\ &= (a_1 - a4)^2 + 4(a_2^2 + b_2^2) \\ &> 0 \end{aligned}$$

(ii) We are given that:

$$A^H = -A$$

$$\begin{bmatrix} a_1 - b_1 i & a_3 - b_3 i) \\ a_2 - b_2 i & a_4 - b_4 i \end{bmatrix} = \begin{bmatrix} -a_1 - b_1 i & -a_2 - b_2 i \\ -a_3 - b_3 i & -a_4 - b_4 i \end{bmatrix}$$

We note that:

$$a_1 = 0 a_3 \qquad = -a_2 a_3 = 0 b_2 \qquad = b_3$$

Again, writing the characteristic polynomial we obtain:

$$P(\lambda) = \lambda^2 - (b_1 + b_4) i\lambda - b_1 b_4 + a_3^2 + b_3^2$$

For only complex routes we require:

$$\begin{aligned} B^2 - 4AC &< 0 \\ &= -(b_1 - b_4)^2 - 4(a_3^2 + b_3^2) \\ &< 0 \end{aligned}$$

Ex 4.6

(i) We want to proof that the diagonal elements of an upper triangular matrix are its eigenvalues. Taking some arbitrary upper triangular matrix, we know that the eigenvalues solve:

$$\begin{aligned} det(\lambda I - A) &= 0 \\ &= \prod_{i=1}^{n} (\lambda - a_{ii}) \\ &= 0 \end{aligned}$$

This is true if $\lambda = a_{ii}$ for some i.
Since λ is generic, the result is proven.

Ex 4.8

(i) S spans V by definition. Therefore S is a basis if its elements are linearly independent. To show that the set S is linearly independent we want to show that:

$$a\cos x + b\sin x + c\cos 2x + d\sin 2x = 0 \ \forall x$$

Implies:

$$a = b = c = d = 0$$

We can proceed by plugging in some values of x, since we know that the expression holds for all x. For $x = 0$

$$\begin{aligned} a + c &= 0 \\ a &= -c \end{aligned}$$

For $x = \pi$

$$\begin{aligned} -a + c &= 0 \\ a &= c \end{aligned}$$

Combining the above two results implies that:

$$\begin{aligned} a &= 0 \\ c &= 0 \end{aligned}$$

For $x = \frac{\pi}{2}$

$$\begin{aligned} b - c &= 0 \\ b &= c \\ b &= 0 \end{aligned}$$

For $x = \frac{\pi}{4}$

$$d = 0$$

Therefore $a = b = c = d = 0$ and the set is linearly independent.

(ii)

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

(iii)

If we split the set in two we can see that we will get two D-invariant subspaces that will be complementary. Namely:

$$Space1 = span\{\cos x, \sin x\}$$
$$Space2 = span\{\cos 2x, \sin 2x\}$$

Ex 4.13

Finding the eigenvalues of A via the characteristic polynomial we get

$$\lambda_1 = 1$$
$$\lambda_2 = 0.4$$

And their corresponding eigenvectors:

$$eig_1 = [1, 0.5]^T$$
$$eig_2 = [1, -1]^T$$

Putting these eigenvectors together, we can create the P Matrix:

$$P = \begin{bmatrix} 1 & 1 \\ 0.5 & -1 \end{bmatrix}$$

Then inverting this we have:

$$P^{-1} = \frac{-2}{3} \begin{bmatrix} -1 & -1 \\ -0.5 & 1 \end{bmatrix}$$

Then we have:

$$P^{-1}AP = D$$

Where D is a diagonal matrix with the eigenvalues along the diagonal.

Ex 4.16

(i) Proposition 4.3.10 says if A and B are similar with $A = P^-1BP$ then $A^k = P^-1B^kP$. From the previous question we have P, B and P^-1.

We can see that:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$$
$$B^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore we can write

$$A^k \to \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

(ii) The answer is the same as above. The Frobenius norm and the infinity norm are topologically equivalent.

(iii) Using proposition 4.3.12, we know that the eigenvalues in question are equal to:

$$f(1) = 3 + 5 + 1^3 = 9f(0.4) = 3 + 2 + 0.4^3 \qquad = 5.064$$

Ex 4.18

If we take the transpose of both sides, then we get:

$$x^T A = \lambda x^T$$
$$A^T x = \lambda x$$

We know that the final equation implies λ is an eigenvalue of A^T. Since A and A^T have the same eigenvalues the result is proven.

Ex 4.20

If A is Hermitian and orthonormally similar to B then there exists an orthonormal matrix U such that:

$$\begin{aligned}
B &= U^H A U \\
B^H &= (U^H A U)^H \\
&= U^H A^H U \\
&= U^H A U \\
&= B
\end{aligned}$$

Ex 4.24

Ignoring the bottom of the fraction (we know it is always real). For hermitian matrices:

$$\begin{aligned}
\langle x, Ax \rangle &= x^H A x \\
&= (x^H A) x \\
&= (A^H x)^H x \\
&= \langle A^H x, x \rangle \\
&= \langle Ax, x \rangle \\
&= \overline{\langle x, Ax \rangle}
\end{aligned}$$

Since $\langle x, Ax \rangle$ equals its own conjugate it must be real.

For hermitian skew matrices we can follow the same procedure:

$$\begin{aligned}
\langle x, Ax \rangle &= x^H A x \\
&= (x^H A) x \\
&= (A^H x)^H x \\
&= \langle A^H x, x \rangle \\
&= \langle -Ax, x \rangle \\
&= \overline{\langle x, -Ax \rangle} \\
&= \overline{-\langle x, Ax \rangle}
\end{aligned}$$

So the inner product must be imaginary.

Ex 4.27

A matrix A is positive definite if for any non-zero column vector x:

$$x^T A x > 0$$

Now lets suppose that x is a column vector containing all zeros and one 1 at row q. We can see that $x^T Ax$ will pick out the diagonal element of A which corresponds to row q, column q. From the above definition of positive definiteness, we know that this must be greater than zero. We can then repeat this exercise for all $q \in n$ and see that all diagonal elements of A must be greater than zero.