

Math Problem Set 2

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Section 1 questions

Ex 3.1

(i)

$$\begin{aligned}\langle x, y \rangle &= \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) \\ &= \frac{1}{4}(\langle x + y, x + y \rangle - \langle x - y, x - y \rangle)\end{aligned}$$

We note we are on a real inner product space so we can write:

$$\begin{aligned}&= \frac{1}{4}(\langle x, x \rangle + \langle y, y \rangle + 2\langle x, y \rangle - \langle x, x \rangle - \langle y, y \rangle + 2\langle x, y \rangle) \\ &= \frac{1}{4}(4\langle x, y \rangle) \\ &= \langle x, y \rangle\end{aligned}$$

(ii)

$$\|x\|^2 + \|y\|^2 = \frac{1}{2}(\|x + y\|^2 + \|x - y\|^2)$$

Again because we in a real space we can write:

$$\begin{aligned}&= \frac{1}{2}(\langle x, x \rangle + \langle y, y \rangle + 2\langle y, x \rangle + \langle x, x \rangle + \langle y, y \rangle - 2\langle y, x \rangle) \\ &= \langle x, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2\end{aligned}$$

Ex 3.2

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x - iy\|^2 - i\|x + iy\|^2)$$

Using the proof from above we can write this as:

$$\begin{aligned}&= \mathcal{R}\langle x, y \rangle + \frac{1}{4}i(\langle x - iy, x - iy \rangle - \langle x + iy, x + iy \rangle) \\ &= \mathcal{R}\langle x, y \rangle + \frac{1}{4}4(\mathcal{I}\langle x, y \rangle) \\ &= \langle x, y \rangle\end{aligned}$$

Ex 3.3

(i)

$$\cos(\theta) = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

Subbing in we have:

$$\begin{aligned}&= \frac{\int_0^1 x^6 dx}{\sqrt{\int_0^1 x^2 dx} \sqrt{\int_0^1 x^{10} dx}} \\ &= \frac{1/7}{\sqrt{1/33}}\end{aligned}$$

Therefore the angle is 34.84 degrees.

(ii)

$$\begin{aligned}\cos(\theta) &= \frac{\int_0^1 x^6 dx}{\sqrt{\int_0^1 x^4 dx} \sqrt{\int_0^1 x^8 dx}} \\ &= \frac{1/7}{\sqrt{1/45}}\end{aligned}$$

Therefore the angle is 16.6 degrees.

Ex 3.8

(i)

A set is orthonormal if the inner products of the combinations of elements of the set satisfy:

- $\langle x_i, x_j \rangle = 1$ if $i = j$
- $\langle x_i, x_j \rangle = 0$ if $i \neq j$

Checking the first condition:

Firstly for $\cos(t)$, $\cos(t)$

$$\begin{aligned}\langle \cos(t), \cos(t) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t)^2 dt \\ &= \frac{1}{\pi} \left[\frac{x}{2} + \frac{\sin(2x)}{4} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} (\pi) \\ &= 1\end{aligned}$$

We can also see that this result will hold for $\cos(2t)$, $\cos(2t)$ as well. (The evaluated sin functions in the integral will still be zero).

Now checking $\sin(t)$, $\sin(t)$, and by virtue of the argument above, $\sin(2t)$, $\sin(2t)$ as well.

$$\begin{aligned}\langle \cos(t), \cos(t) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t)^2 dt \\ &= \frac{1}{\pi} \left[\frac{x}{2} - \frac{1}{4} \sin(2x) \right]_{-\pi}^{\pi} \\ &= 1\end{aligned}$$

Now we need to check the cross terms, and verify that their inner product is zero.

$$\begin{aligned}\langle \cos(t), \sin(t) \rangle &= \frac{1}{\pi} [\sin(t)^2]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

And we note that this also holds for the combinations of $\cos(2t)$, $\sin(t)$ and also $\cos(t)$, $\sin(2t)$.

$$\begin{aligned}\langle \cos(t), \cos(2t) \rangle &= \frac{1}{\pi} \left[\frac{\sin(t)}{2} + \frac{\sin(3t)}{6} \right]_{-\pi}^{\pi} \\ &= 0 \\ \langle \sin(t), \sin(2t) \rangle &= \frac{1}{\pi} \left[\frac{\sin(t)^3}{1.5} \right]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

Therefore the set is orthonormal.

(ii)

$$\begin{aligned}||t||^2 &= \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 dt \\ &= \left[\frac{t^3}{3} \right]_{-\pi}^{\pi} \\ &= \left[\frac{\pi^3}{3} + \frac{\pi^3}{3} \right] \\ &= 2 \frac{\pi^3}{3}\end{aligned}$$

Therefore $||t|| = (\frac{2\pi^3}{3})^{0.5}$

(iii)

Because we are dealing with an orthonormal set we can write:

$$\begin{aligned} Proj_x(cos(3t)) &= \sum_i \langle S_i, cos(3t) \rangle s_i \\ &= \langle cos(t), cos(3t) \rangle cos(t) + \langle cos(2t), cos(3t) \rangle cos(2t) + \langle sin(t), cos(3t) \rangle sin(t) + \langle sin(2t), cos(3t) \rangle sin(2t) \end{aligned}$$

After substituting in the integrals we get

$$= 0$$

i.e. $cos(3t)$ is orthogonal to all the elements in S , as its projection matrix is a zero matrix.

(iv)

$$\begin{aligned} Proj_x(t) &= \sum_i \langle S_i, t \rangle s_i \\ &= \langle cos(t), t \rangle cos(t) + \langle cos(2t), t \rangle cos(2t) + \langle sin(t), t \rangle sin(t) + \langle sin(2t), t \rangle sin(2t) \\ &= 0 + 0 + 2sin(t) - sin(2t) \\ &= 2sin(t) - sin(2t) \end{aligned}$$

Ex 3.9

We use the fact that we can convert the rotation transformation into a matrix in the standard basis, which we call Q . Then, we know that if $Q^T Q = I$ then the transformation is orthonormal.

$$Q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

So

$$Q Q^T = \begin{bmatrix} \cos(\theta)^2 + \sin(\theta)^2 & 0 \\ 0 & \cos(\theta)^2 + \sin(\theta)^2 \end{bmatrix}$$

$$Q Q^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex 3.10

(i)

First we show that if Q is orthonormal then $Q Q^H = I$.

If Q is an orthonormal matrix, then it preserves the inner product of two vectors. i.e.

$$\langle m, n \rangle = \langle Qm, Qn \rangle$$

Which we can rewrite as:

$$m^H n = (Qm)^H (Qn)$$

$$m^H n = m^H (Q^H Q) n$$

Therefore:

$$Q^H Q = I$$

Now we can show that if $Q Q^H = I$, then Q is orthonormal.

$$\text{If } Q Q^H = I$$

Then:

$$\begin{aligned}
\langle Qm, Qn \rangle &= (Qm)^H (Qn) \\
&= m^H Q^H Qn \\
&= \langle m, n \rangle
\end{aligned}$$

(ii)

$$\|Qx\| = \sqrt{\langle Qx, Qx \rangle}$$

By the definition of what a orthonormal matrix is (it preserves the inner product), we can write:

$$\begin{aligned}
&= \sqrt{\langle x, x \rangle} \\
&= \|x\|
\end{aligned}$$

(iii)

If Q is orthonormal we can write:

$$QQ^H = I$$

$$\text{i.e. } Q^H = Q^{-1}$$

Q^H is clearly orthonormal because $(Q^H)^H = Q$, therefore so is Q^{-1} .

(iv)

If Q is orthonormal we know that $G = Q^H Q = I$

For some element of G , we can write that:

$$G_{i,j} = \langle q_i, q_j \rangle$$

Where q_i is the i 'th column of Q .

By the definition of orthonormality, we know that:

$$\langle q_i, q_j \rangle = 1 \text{ if } i = j$$

and

$$\langle q_i, q_j \rangle = 0 \text{ if } i \neq j$$

So we can see that when $i = j$ we are on the diagonal of Q , so clearly $\langle q_i, q_j \rangle = 1$ if $i = j$. And similarly, everywhere else $i \neq j$, and have zero entries, so $\langle q_i, q_j \rangle = 0$ if $i \neq j$.

(v)

We can find a counterexample to show that not all matrices with determinant equal to 1 are orthonormal.

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

We can see that:

$$\det(D) = 1$$

But if we test for orthonormality:

$$DS^H = \begin{bmatrix} 4 & 0 \\ 0 & 0.25 \end{bmatrix} \neq I$$

vi

Checking if the product of the two matrices is an orthonormal matrix:

$$(Q_1 Q_2)(Q_1 Q_2)^H = Q_1 Q_2 Q_2^H Q_1^H$$

Then using the fact that Q_1 and Q_2 are orthonormal we can write:

$$Q_1 Q_2 Q_2^H Q_1^H = Q_1 Q_1^H = I$$

So the product of the matrices is orthonormal.