Math Problem Set 2

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Section 1 questions

Ex 3.1

(i)

$\langle x, y\rangle = \frac{1}{4}(||x+y||^2 - ||x-y||^2)$

$= \frac{1}{4}(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle)$

We note we are on a real inner product space so we can write:

$= \frac{1}{4}(\langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle - \langle x, x\rangle - \langle y, y\rangle + 2\langle x, y\rangle)$

$= \frac{1}{4}(4\langle x, y\rangle)$

$= \langle x, y\rangle$

(ii)

$||x||^2 + ||y||^2 = \frac{1}{2}(||x+y||^2 + ||x-y||^2)$

Again because we in a real space we can write:

$= \frac{1}{2}(\langle x, x\rangle + \langle y, y\rangle + 2\langle y, x\rangle + \langle x, x\rangle + \langle y, y\rangle - 2\langle y, x\rangle)$

$= \langle x, x\rangle + \langle y, y\rangle$

$= ||x||^2 + ||y||^2$

Ex 3.2

$\langle x, y\rangle = \frac{1}{4}(||x+y||^2 - ||x-y||^2 + i||x-iy||^2 - i||x+iy||^2)$

Using the proof from above we can write this as:

$= \mathcal{R}\langle x, y\rangle + \frac{1}{4}i(\langle x-iy, x-iy\rangle - \langle x+iy, x+iy\rangle)$

$= \mathcal{R}\langle x, y\rangle + \frac{1}{4}4(\mathcal{I}\langle x, y\rangle)$

$= \langle x, y\rangle$

Ex 3.3

(i)

$cos(\theta) = \frac{\langle x,y\rangle}{||x||||y||}$

Subbing in we have:

$$= \frac{\int_0^1 x^6 dx}{\sqrt{\int_0^1 x^2 dx}\sqrt{\int_0^1 x^10 dx}}$$

$= \frac{1/7}{\sqrt{1/33}}$

Therefore the angle is 34.84 degrees.

(ii)

$cos(\theta) = \frac{\int_0^1 x^6 dx}{\sqrt{\int_0^1 x^4 dx}\sqrt{\int_0^1 x^8 dx}}$

$= \frac{1/7}{\sqrt{1/45}}$

Therefore the angle is 16.6 degrees.

Ex 3.8

(i)

A set is orthonormal if the inner products of the combinations of elements of the set satisfy:

- $\langle x_i, x_j \rangle = 1$ if $i = j$
- $\langle x_i, x_j \rangle = 0$ if $i \neq j$

Checking the first condition:

Firstly for cos(t), cos(t)

$\langle cos(t), cos(t) \rangle = \frac{1}{\pi} int_{-\pi}^{\pi} cos(t)^2 dt$

$= \frac{1}{\pi}[\frac{x}{2} + \frac{sin(2x)}{4}]_{-\pi}^{\pi}$

$= \frac{1}{\pi}(\pi)$

$= 1$

We can also see that this result will hold for cos(2t), cos(2t) as well. (The evaluated sin functions in the integral will still be zero).

Now checking sin(t), sin(t), and by virtue of the argument above, sin(2t), sin(2t) as well.

$\langle cos(t), cos(t) \rangle = \frac{1}{\pi} int_{-\pi}^{\pi} sin(t)^2 dt$

$= \frac{1}{\pi}[\frac{x}{2} - \frac{1}{4} sin(2x)]_{-\pi}^{\pi}$

$= 1$

Now we need to check the cross terms, and verify that their inner product is zero.

$\langle cos(t), sin(t) \rangle = \frac{1}{\pi}[sin(t)^2]_{-\pi}^{\pi}$

$= 0$

And we note that this also holds for the combinations of cos(2t), sin(t) and also cos(t), sin(2t).

$\langle cos(t), cos(2t) \rangle = \frac{1}{\pi}[\frac{sin(t)}{2} + \frac{sin(3t)}{6}]_{-\pi}^{\pi}$

$= 0$

$\langle sin(t), sin(2t) \rangle = \frac{1}{\pi}[\frac{sin(t)^3}{1.5}]_{-\pi}^{\pi}$

$= 0$

Therefore the set is orthonormal.

(ii)

$||t||^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 dt$

$= [\frac{t^3}{3}]_{-\pi}^{\pi}$

$= [\frac{\pi^3}{3} + \frac{\pi^3}{3}]$

$= 2\frac{\pi^3}{3}$

Therefore $||t|| = (\frac{2\pi^3}{3})^{0.5}$

(iii)

Because we are dealing with an orthnormal set we can write:

$Proj_x(cos(3t)) = \Sigma_i \langle S_i, cos3t \rangle s_i$

$= \langle cos(t), cos(3t) \rangle cos(t) + \langle cos(2t), cos(3t) \rangle cos(2t) + \langle sin(t), cost(3t) \rangle sin(t) + \langle sin(2t), cos(3t) \rangle sin(2t)$

After substituting in the integrals we get

$= 0$

i.e. cos(3t) is orthogonal to all the elements in S, as its projection matrix is a zero matrix.

(iv)

$Proj_x(t) = \Sigma_i \langle S_i, t \rangle s_i$

$= \langle cos(t), t \rangle cos(t) + \langle cos(2t), t \rangle cos(2t) + \langle sin(t), t \rangle sin(t) + \langle sin(2t), t \rangle sin(2t)$

$= 0 + 0 + 2sin(t) - sin(2t)$

$= 2sin(t) - sin(2t)$

Ex 3.9

We use the fact that we can convert the rotation transformation into a matrix in the standard basis, which we call Q. Then, we know that if $Q^T Q = I$ then the transformation is orthonormal.

$$Q = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

So

$$QQ^T = \begin{bmatrix} cos(\theta)^2 + sin(\theta)^2 & 0 \\ 0 & cos(\theta)^2 + sin(\theta)^2 \end{bmatrix}$$

$$QQ^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex 3.10

(i)

First we show that if Q is orthonormal then $QQ^H = I$.

If Q is an orthonormal matrix, then it preserves the inner product of two vectors. i.e.

$\langle m, n \rangle = \langle Qm, Qn \rangle$

Which we can rewrite as:

$m^H n = (Qm)^H (Qn)$

$m^H n = m^H (Q^H Q) n$

Therefore, since this has to hold for all m and n:

$Q^h Q = I$

Now we can show that if $QQ^H = I$, then Q is orthonormal.

If $QQ^H = I$

Then:

$\langle Qm, Qn \rangle = (Qm)^H (Qn)$

$= m^H Q^H Q n$

$= \langle m, n \rangle$

(ii)

$||Qx|| = \sqrt{\langle Qx, Qx \rangle}$

By the definition of what a orthonrmal matrix is (it preserves the inner product), we can write:

$= \sqrt{\langle x, x \rangle}$

$= ||x||$

(iii)

If Q is orthonormal we can write:

$QQ^H = I$

i.e. $Q^H = Q^{-1}$

Q^H is clearly orthonormal because $(Q^H)^H = Q$, therefore so is Q^{-1}.

(iv)

If Q is orthonormal we know that $G = Q^H Q = I$

For some element of G, we can write that:

$G_{i,j} = \langle q_i, q_j \rangle$

Where q_i is the i'th column of Q.

By the definition of orthornomality, we know that:

$\langle q_i, q_j \rangle = 1$ if $i = j$

and

$\langle q_i, q_j \rangle = 0$ if $i \neq j$

So we can see that when $i = j$ we are on the diagonal of Q, so clearly $\langle q_i, q_j \rangle = 1$ if $i = j$. And similarly, everywhere else $i \neq j$, and have zero entries, so $\langle q_i, q_j \rangle = 0$ if $i \neq j$.

(v)

We can find a counterexample to show that not all matrices with determinant equal to 1 are orthonormal.

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

We can see that:

$det(D) = 1$

But if we test for orthonormality:

$$DS^H = \begin{bmatrix} 4 & 0 \\ 0 & 0.25 \end{bmatrix} \neq I$$

vi

Checking if the product of the two matrices is an orthonormal matrix:

$(Q_1 Q_2)(Q_1 Q_2)^H = Q_1 Q_2 Q_2^H Q_1^H$

Then using the fact that Q_1 and Q_2 are orthonormal we can write:

$Q_1 Q_2 Q_2^H Q_1^H = Q_1 Q_1^H = I$

So the product of the matrices is orthonormal.