

Math Problem Set 3

Tim Munday
OSM Boot Camp 2018

July 5, 2018

Ex 4.2

$$D \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 2x \\ 1 \\ 0 \end{bmatrix}$$

So

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is an upper triangular matrix so its eigenvalues lie on the diagonal. Therefore the only eigenvalue is 0. It appears three times, so the algebraic multiplicity is 3. The eigenspace is the set of constant functions and has dimension one, which is the geometric multiplicity.

Ex 4.4

(i) We are given that:

$$A = A^H$$
$$\begin{bmatrix} a_1 + b_1i & a_2 + b_2i \\ a_3 + b_3i & a_4 + b_4i \end{bmatrix} = \begin{bmatrix} a_1 - b_1i & a_3 - b_3i \\ a_2 - b_2i & a_4 - b_4i \end{bmatrix}$$

Therefore we can say that:

$$\begin{aligned} b_1 &= 0 \\ b_2 &= -b_3 \\ b_4 &= 0 \end{aligned}$$

The characteristic polynomial for matrix A is:

$$P(\lambda) = \lambda^2 - (a_1 + a_4)\lambda + a_1a_4 - a_2^2 - b_2^2$$

Which has only real roots if:

$$\begin{aligned} B^2 - 4AC &> 0 \\ &= (a_1 + a_4)^2 - 4(a_1a_4 - a_2^2 - b_2^2) \\ &= (a_1 - a_4)^2 + 4(a_2^2 + b_2^2) \\ &> 0 \end{aligned}$$

(ii) We are given that:

$$\begin{aligned} A^H &= -A \\ \begin{bmatrix} a_1 - b_1i & a_3 - b_3i \\ a_2 - b_2i & a_4 - b_4i \end{bmatrix} &= \begin{bmatrix} -a_1 - b_1i & -a_2 - b_2i \\ -a_3 - b_3i & -a_4 - b_4i \end{bmatrix} \end{aligned}$$

We note that:

$$a_1 = 0a_3 \qquad \qquad \qquad = -a_2a_3 = 0b_2 \qquad \qquad \qquad = b_3$$

Again, writing the characteristic polynomial we obtain:

$$P(\lambda) = \lambda^2 - (b_1 + b_4)i\lambda - b_1b_4 + a_3^2 + b_3^2$$

For only complex roots we require:

$$\begin{aligned} B^2 - 4AC &< 0 \\ &= -(b_1 - b_4)^2 - 4(a_3^2 + b_3^2) \\ &< 0 \end{aligned}$$

Ex 4.6

(i) We want to prove that the diagonal elements of an upper triangular matrix are its eigenvalues. Taking some arbitrary upper triangular matrix, we know that the eigenvalues solve:

$$\begin{aligned} \det(\lambda I - A) &= 0 \\ &= \prod_{i=1}^n (\lambda - a_{ii}) \\ &= 0 \end{aligned}$$

This is true if $\lambda = a_{ii}$ for some i .

Since λ is generic, the result is proven.

Ex 4.8

S spans V by definition. Therefore S is a basis if its elements are linearly independent. To show that the set S is linearly independent we want to show that:

$$a \cos x + b \sin x + c \cos 2x + d \sin 2x = 0 \quad \forall x$$

Implies:

$$a = b = c = d = 0$$

We can proceed by plugging in some values of x, since we know that the expression holds for all x. For $x = 0$

$$\begin{aligned} a + c &= 0 \\ a &= -c \end{aligned}$$

For $x = \pi$

$$\begin{aligned} -a + c &= 0 \\ a &= c \end{aligned}$$

Combining the above two results implies that:

$$\begin{aligned} a &= 0 \\ c &= 0 \end{aligned}$$

For $x = \frac{\pi}{2}$

$$\begin{aligned} b - c &= 0 \\ b &= c \\ b &= 0 \end{aligned}$$

For $x = \frac{\pi}{4}$

$$d = 0$$

Therefore $a = b = c = d = 0$ and the set is linearly independent.

(ii)

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

(iii)

If we split the set in two we can see that we will get two D-invariant subspaces that will be complementary. Namely:

$$Space1 = span\{\cos x, \sin x\}$$

$$Space2 = span\{\cos 2x, \sin 2x\}$$

Ex 4.13

Finding the eigenvalues of A via the characteristic polynomial we get

$$\lambda_1 = 1$$

$$\lambda_2 = 0.4$$

And their corresponding eigenvectors:

$$eig_1 = [1, 0.5]^T$$

$$eig_2 = [1, -1]^T$$

Putting these eigenvectors together, we can create the P Matrix:

$$P = \begin{bmatrix} 1 & 1 \\ 0.5 & -1 \end{bmatrix}$$

Then inverting this we have:

$$P^{-1} = \frac{-2}{3} \begin{bmatrix} -1 & -1 \\ -0.5 & 1 \end{bmatrix}$$

Then we have:

$$P^{-1}AP = D$$

Where D is a diagonal matrix with the eigenvalues along the diagonal.

Ex 4.15

Ex 4.16

Ex 4.18

Ex 4.20

Ex 4.24

Ex 4.25

Ex 4.27

Ex 4.28

Ex 4.31

Ex 4.32

Ex 4.33

Ex 4.36

Ex 4.38