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Lagrange Error

The function for computing the error in a given point x and given some known nodes x_n is:

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)$$

Since we're interested in the maximum error in the interval $[0,1]$ and $n=2$ we need to solve:

$$\frac{\max\limits_{x\in[0,1]}|f^{(2+1)}(\mathrm{x})|}{(2+1)!} * \max_{x\in[0,1]}\left|\prod_{i=0}^{2}(x-x_i)\right|$$

Because in the first part we get the maxima if the numerator is at his maxima (since the denominator is fixed) and we also want to maximise the overall result, therefore the product should also be at his maxima.

First we find f^{n+1}

Since $f=\cos(x) \rightarrow f^{(3)}=\sin(x)$

Then we calculate $(n+1)!=3!=6$

And in the end we can take the product: $\prod_{i=0}^{2}(x-x_i)=(x-0)*(x-0.6)*(x-0.9)$

This leads to

$$\frac{\max\limits_{x\in[0,1]}|sin(\mathrm{x})|}{6}\max_{x\in[0,1]}|(x-0)*(x-0.6)*(x-0.9)|$$

$\max\limits_{x\in[0,1]}|sin(\mathrm{x})|$ **is trivial with $\mathrm{x}=1$ (in our interval sin(x) is monotonically increasing)**

Now we need to find $\max\limits_{x\in[0,1]}|(x-0)*(x-0.6)*(x-0.9)|$

First we derive it in order to find the critical points, which leads to :

$$(x-0.6)(x-0.9)+x*(x-0.9)+x*(x-0.6)=3x^2-3x+0.54=3(x^2-x+0.18)$$

We can then solve for $x^2-x+0.18=0$

$$=\frac{1\pm\sqrt{1-4*0.18}}{2}$$

Which gives us $x_1 \cong 0.764575$ and $x_2 \cong 0.235425$

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To verify which one is the actual maxima we can try all the possibilities (and also adding the bounds of the interval $x_3 = 0 \; and \; x_4 = 1$)

$$(x-0)*(x-0.6)*(x-0.9)$$

$$x = x_1 \rightarrow \; -0.017$$

$$\boldsymbol{x} = \boldsymbol{x}_2 \rightarrow \; 0.057$$

$$x = x_3 \rightarrow \; 0$$

$$x = x_4 \rightarrow \; 0.04$$

Now we can use $\boldsymbol{\xi} = 1$ and $\boldsymbol{x}_2$ in our function and we get

$$\frac{sin(1)}{6}(\,0.235425-0)*(\,0.235425-0.6)*(\,0.235425-0.9) \cong 0.00799966$$

Therefore in the interval $I = [0,1]$ we have $\frac{f^{(3)}(\xi)}{(3)!}\prod_{i=0}^{2}(x-x_i) \leq 0.00799966$

I've tried to calculate the maximum error I get in the interval with a resolution of `0.0000001` and it's half of the result obtained analytically: 0.0040 meaning this is surely a generous upper bound.