Assignment

Exercise sheet 5

Name TN 1: Tim Peinkofer

tim.peinkofer@student.uni-tuebingen.de

Name TN 2: Fabian Kostow

fabian.kostow@student.uni-tuebingen.de

Tutor: Jose Carlos Olvera Meneses

Date: 26. November 2024

Inhaltsverzeichnis

1 Problem I

We need to calculate the Taylor expansion of y_0 until the fourth term

$$
\begin{aligned}
y(x_0+h) = y_1 = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)} + ... \\
y(x_0-h) = y_{-1} = y_0 - hy_0' + \frac{h^2}{2}y_0'' - \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)} + ...
\end{aligned}
$$

now we want to add and subtract the terms, as a result, we get

$$
\begin{aligned}
y_1 + y_{-1} &= \left(y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\
&\quad + \left(y_0 - hy_0' + \frac{h^2}{2}y_0'' - \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\
&= 2y_0 + h^2y_0'' + \frac{h^4}{12}y_0^{(4)}
\end{aligned}
$$

$$
\begin{aligned}
y_1 - y_{-1} &= \left(y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\
&\quad - \left(y_0 - hy_0' + \frac{h^2}{2}y_0'' - \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\
&= 2hy_0' + \frac{h^3}{3}y_0'''.
\end{aligned}
$$

If we compute $y_1 - 2y_0 + y_{-1}$ we get

$$
\begin{aligned}
y_1 - 2y_0 + y_{-1} &= \left(y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\
&\quad - 2y_0 \\
&\quad + \left(y_0 - hy_0' + \frac{h^2}{2}y_0'' - \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\
&= h^2y_0'' + \frac{h^4}{12}y_0^{(4)}
\end{aligned}
$$

Now we can rearrange the terms and get

$$
y_0'' = \frac{y_1 - 2y_0 + y_{-1}}{h^2} - \frac{h^2}{12}y^{(4)} + O(h^3)
$$

The error term $O(h^2)$ is given by

$$
O(h^2) = -\frac{h^2}{12}y^{(4)}
$$

In conclusion, we got the explicit error:

$$
\begin{aligned}
Error &= -\sum_{n=2}^{\infty} \frac{2y^{(2n)}(x_0)}{(2n)!} \cdot h^{2n-2} \\
&= -\frac{h^2}{12} y^{(4)} - \frac{h^4}{60} y^{(6)} - ...
\end{aligned}
$$

In finding the formula for the second derivative, we considered only the terms in the Taylor expansion in the first 5 orders, therefore we only got the error $O(h^2)$ if we consider all terms, we found the error term above. We get this error terms by calculating the follows:

- Calculating all Taylor terms for y
- Combine all Taylor terms after the fifth order based on the formula above.
- Divide all terms with h^2

2 Problem II

For our y_0 we first calculate the Taylor expansion until the seventh term:

$$y(x_0+h) = y_1 = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)} + \frac{h^5}{120}y_0^{(5)} + \frac{h^6}{720}y_0^{(6)}$$

$$y(x_0-h) = y_{-1} = y_0 - hy_0' + \frac{h^2}{2}y_0'' - \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)} - \frac{h^5}{120}y_0^{(5)} + \frac{h^6}{720}y_0^{(6)}$$

$$y(x_0+2h) = y_2 = y_0 + 2hy_0' + 2h^2y_0'' + \frac{4h^3}{3}y_0''' + \frac{2h^4}{3}y_0^{(4)} + \frac{4h^5}{15}y_0^{(5)} + \frac{4h^6}{45}y_0^{(6)}$$

$$y(x_0-2h) = y_{-2} = y_0 - 2hy_0' + 2h^2y_0'' - \frac{4h^3}{3}y_0''' + \frac{2h^4}{3}y_0^{(4)} - \frac{4h^5}{15}y_0^{(5)} + \frac{4h^6}{45}y_0^{(6)}$$

We now combine the four equations the following:

$$y_{-2} - y_2 + 16(y_{-1} - y_1)$$

As a result, we get:

$$\begin{aligned}
&- y_{-2} - y_2 + 16(y_1 + y_{-1}) \\
=&- 2y_0 - 4h^2y_0'' - \frac{4h^4}{3}y_0^{(4)} - + \frac{8h^6}{45}y_0^{(6)} + 16(2y_0 + h^2y_0'' + \frac{h^4}{12}y_0^{(4)} + \frac{2h^6}{720}y_0^{(6)}) \\
=& \, 30y_0 + 12h^2y_0'' - \frac{2h^6}{15}y_0^{(6)}
\end{aligned}$$

If we bring all terms that don't include the second derivative on the left side we get:

$$y_0'' = \frac{-y_{-2} + 16y_1 - 30y_0 + 16y_{-1} - y_2}{12h^2} + \frac{h^4}{90}y_0^{(6)} \tag{1}$$

Where the last term is our failure term:

$$O(h^4) = \frac{h^4}{90}y^{(6)}(\xi)$$

In conclusion, we got the explicit error:

$$
\begin{aligned}
Error &= \sum_{n=3}^{\infty} \frac{8y^{(2n)}(x_0)}{(2n)!} \cdot h^{2n-2} \\
&= \frac{h^4}{90} y^6(x_0) + \frac{h^6}{126} y^8(x_0) + ...
\end{aligned}
$$

Finding the error term is similar to problem I, except that we have combined the equations differently, which also gives us a slightly different formula for the error. We also see that in problem II the first error term starts with $O(h^4)$ instead of $O(h^2)$, but also all error terms of order $O(h^{2n})$ are smaller in this term compared to the first problem. We get this error terms by calculating the follows:

- Calculating all Taylor terms for y
- Combine all Taylor terms after the sixth order based on the formula above.
- Divide all terms with $12h^2$