# Assignment

## Exercise sheet 5

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**Date:** 25. November 2024

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# 1 Problem I

We need to calculate the Taylor expansion of $y_0$ until the fourth term

$$y(x_0+h) = y_1 = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}$$
$$y(x_0-h) = y_{-1} = y_0 - hy_0' + \frac{h^2}{2}y_0'' - \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}$$

now we want to add and subtract the terms, as a result, we get

$$\begin{aligned} y_1 + y_{-1} &= \left(y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\ &\quad + \left(y_0 - hy_0' + \frac{h^2}{2}y_0'' - \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\ &= 2y_0 + h^2y_0'' + \frac{h^4}{12}y_0^{(4)} \end{aligned}$$

$$\begin{aligned} y_1 - y_{-1} &= \left(y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\ &\quad - \left(y_0 - hy_0' + \frac{h^2}{2}y_0'' - \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\ &= 2hy_0' + \frac{h^3}{3}y_0'''. \end{aligned}$$

If we compute $y_1 - 2y_0 + y_{-1}$ we get

$$\begin{aligned} y_1 - 2y_0 + y_{-1} &= \left(y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\ &\quad - 2y_0 \\ &\quad + \left(y_0 - hy_0' + \frac{h^2}{2}y_0'' - \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)}\right) \\ &= h^2y_0'' + \frac{h^4}{12}y_0^{(4)} \end{aligned}$$

Now we can rearrange the terms and get

$$y_0'' = \frac{y_1 - 2y_0 + y_{-1}}{h^2} + \frac{h^2}{12}y^{(4)} + O(h^3)$$

The error term $O(h^2)$ is given by

$$O(h^2) = -\frac{h^2}{12}y^{(4)}$$

We can also write down the explicit error, which is:

$$|y_0'' - \frac{y_1 - 2y_0 + y_{-1}}{h^2}| = \frac{h^2}{12} y^{(4)}$$

Where $\frac{h^2}{12} y^{(4)}$ is our error $\epsilon$. So in this case we get for our explicit error $\epsilon$:

$$\epsilon = |y_0'' - \frac{y_1 - 2y_0 + y_{-1}}{h^2}|$$

# 2 Problem II

For our $y_0$ we first calculate the Taylor expansion until the seventh term:

$$y(x_0+h) = y_1 = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)} + \frac{h^5}{120}y_0^{(5)} + \frac{h^6}{720}y_0^{(6)}$$

$$y(x_0-h) = y_{-1} = y_0 - hy_0' + \frac{h^2}{2}y_0'' - \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{(4)} - \frac{h^5}{120}y_0^{(5)} + \frac{h^6}{720}y_0^{(6)}$$

$$y(x_0+2h) = y_2 = y_0 + 2hy_0' + 2h^2y_0'' + \frac{4h^3}{3}y_0''' + \frac{2h^4}{3}y_0^{(4)} + \frac{4h^5}{15}y_0^{(5)} + \frac{4h^6}{45}y_0^{(6)}$$

$$y(x_0-2h) = y_{-2} = y_0 - 2hy_0' + 2h^2y_0'' - \frac{4h^3}{3}y_0''' + \frac{2h^4}{3}y_0^{(4)} - \frac{4h^5}{15}y_0^{(5)} + \frac{4h^6}{45}y_0^{(6)}$$

We now combine the four equations the following:

$$y_{-2} - y_2 + 16(y_{-1} - y_1)$$

As a result, we get:

$$\begin{aligned}
&- y_{-2} - y_2 + 16(y_1 + y_{-1}) \\
&= - 2y_0 - 4h^2y_0'' - \frac{4h^4}{3}y_0^{(4)} - + \frac{8h^6}{45}y_0^{(6)} + 16(2y_0 + h^2y_0'' + \frac{h^4}{12}y_0^{(4)} + \frac{2h^6}{720}y_0^{(6)}) \\
&= 30y_0 + 12h^2y_0'' - \frac{2h^6}{15}y_0^{(6)}
\end{aligned}$$

If we bring all terms that don't include the second derivative on the left side we get:

$$y_0'' = \frac{-y_{-2} + 16y_1 - 30y_0 + 16y_{-1} - y_2}{12h^2} + \frac{h^4}{90}y_0^{(6)} \tag{1}$$

Where the last term is our failure term:

$$O(h^4) = \frac{h^4}{90}y^{(6)}(\xi)$$

We can also write down the explicit error, which is:

$$|y_0'' - \frac{-y_{-2} + 16y_1 - 30y_0 + 16y_{-1} - y_2}{12h^2}| = \frac{h^4}{90} y_0^{(6)}$$

Where $\frac{h^4}{90} y_0^{(6)}$ is our error $\epsilon$. So in this case we get for our explicit error $\epsilon$:

$$\epsilon = |y_0'' - \frac{y_1 - 2y_0 + y_{-1}}{h^2}|$$