Assignment

Exercise sheet 3

Name TN 1: Tim Peinkofer

tim.peinkofer@student.uni-tuebingen.de

Name TN 2: Fabian Kostow

fabian.kostow@student.uni-tuebingen.de

Tutor: Jose Carlos Olvera Meneses

Date: 18. November 2024

Inhaltsverzeichnis

1 Problem I-B **2**

1.1 $P[2,3]$ 3

1.2 $P[1,4]$ 7

2 Problem 2 **8**

1 Problem I-B

For the Pade-approximation we use the following equation:

$$f(x) - R_n(x) = 0$$

Where $f(x)$ is the function we want to approximate or their McLaurin series. $R_n(x)$ is our approximation in form of a rational function. If we use this equation we get:

$$\frac{(c_0 + c_1 x + ... + c_N x^N) \cdot (1 + b_1 x + ... + b_n x^n) - (a_0 + a_1 x + ... + a_m x^m)}{(1 + b_1 x + ... + b_n x^n)} = 0$$

where $N = m + n$. Because we know the c_i from our function or their McLaurin series. If we write out the above equation, we can calculate our values for the a_i and b_i values. As a result we get:

$$\begin{aligned} a_0 &= c_0 \\ b_1 c_0 + c_1 - a_1 &= 0 \\ b_2 c_0 + b_1 c_1 + c_2 - a_2 &= 0 \\ &... \end{aligned}$$

Let us now use this to calculate the Approximations for $f(x) = e^x$:
We know the McLaurin series of $f(x)$:

$$T(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$$

As a result we can calculate $P[2;3]$ and $P[1;4]$.

$P[m,n]$ has the following form:

$$P[m,n] = \frac{a_0 + a_1 x + ... + a_m x^m}{1 + b_1 x + ... + b_n x^n}$$

So in this case we need six equations to solve for both cases of P.

1.1 $P[2,3]$

We get the following system of equations:

$$\begin{aligned}
a_0 &= 1 \\
b_1 - a_1 &= -1 \\
b_2 + b_1 - a_2 &= -\frac{1}{2} \\
b_3 + b_2 + \frac{1}{2}b_1 &= -\frac{1}{6} \\
b_3 + \frac{1}{2}b_2 + \frac{1}{6}b_1 &= -\frac{1}{24} \\
\frac{1}{2}b_3 + \frac{1}{6}b_2 + \frac{1}{24}b_1 &= -\frac{1}{120}
\end{aligned}$$

To solve this system we will write the equations, excluding the first, as $M \cdot \vec{x} = \vec{b}$ an solve it via Gauss method:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ \frac{1}{2} & 1 & 1 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{2} & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ -\frac{1}{6} \\ -\frac{1}{24} \\ -\frac{1}{120} \end{bmatrix}$$

The code is the following:

```
import numpy as np

# Initialize matrix and vector via numpy
matrix = np.array([[1, 0, 0,-1,0], [1, 1, 0,-0,-1], [0.5, 1, 1,0,0], [1/6, 1/2, 1,0,0], [1/24, 1/6, 1/2,0,0]], dtype=np.float64)
vector = np.array([-1, -0.5, -1/6,-1/24,-1/120], dtype=np.float64).reshape(-1, 1)

print("Original Matrix:") # print the original matrix as a reference
print(matrix)
print("Original Vector:")
print(vector)

# Get dimensions of our matrix for later use
rows, columns = matrix.shape


def gauss(): # Function for gau  elimination

    # Generate a copy of the vector and the matrix for our gau  algorithm
    U_Matrix = np.copy(matrix)
    U_vector = np.copy(vector)


    for i in range(rows - 1):

        if U_Matrix[i][i] == 0:
            for k in range(i + 1, rows):
                if U_Matrix[k][i] != 0:
                    # Swap the rows in both U_Matrix and U_vector if the a_ii component is zero
                    U_Matrix[[i, k]] = U_Matrix[[k, i]]
                    U_vector[[i, k]] = U_vector[[k, i]]
                    break
```

```
        # Continue with elimination if a_ii != 0
        for j in range(i + 1, rows):
            if U_Matrix[i][i] != 0:
                factor = U_Matrix[j][i] / U_Matrix[i][i]
                U_Matrix[j] = U_Matrix[j] - factor * U_Matrix[i]
                U_vector[j] = U_vector[j] - factor * U_vector[i]
    return U_Matrix, U_vector

def solver(mat, vec): # Solver from our first program (a little bit modified)
    x = np.zeros((rows, 1)) # Generate a solution vektor based on the number of rows of our matrix
    for i in range(rows):
        index = rows - i - 1
        b_new = vec[index] / mat[index, index]

        for r in range(index + 1, rows):
            b_new -= mat[index, r] * x[r] / mat[index, index]

        x[index] = b_new
    return x

# Get the triangular matrix and solve the linear equation
m, v = gauss()
solution = solver(m, v)

print("Upper Triangular Matrix:")
print(m)
print("Modified Vector after Gaussian elimination:")
print(v)
print("Solution Vector:")
print(solution)
```

The resulting vector is:

$$\vec{x} = \begin{bmatrix} -\frac{3}{5} \\ \frac{3}{20} \\ -\frac{1}{60} \\ \frac{2}{5} \\ \frac{1}{20} \end{bmatrix}$$

In conclusion we get as approximation:

$$P[2,3] = \frac{1 + \frac{2}{5}x + \frac{1}{20}x^2}{1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3}$$

which is the same as on the excercise sheet.

1.2 $P[1,4]$

We can do the same thing as in the subsection before th get the Pade-approximation $P[1,4]$. For this approximation the system of equations is:

$$\begin{aligned} a_0 &= 1 \\ b_1 - a_1 &= -1 \\ b_2 + b_1 &= -\frac{1}{2} \\ b_3 + b_2 + \frac{1}{2}b_1 &= -\frac{1}{6} \\ b_4 + b_3 + \frac{1}{2}b_2 + \frac{1}{6}b_1 &= -\frac{1}{24} \\ b_4 + \frac{1}{2}b_3 + \frac{1}{6}b_2 + \frac{1}{24}b_1 &= -\frac{1}{120} \end{aligned}$$

If we use our code from above we get:

$$\vec{x} = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{10} \\ -\frac{1}{15} \\ \frac{1}{120} \\ \frac{1}{5} \end{bmatrix}$$

In conclusion we get as approximation:

$$P[2,3] = \frac{1 + \frac{1}{5}x}{1 - \frac{4}{5}x + \frac{3}{10}x^2 - \frac{1}{15}x^3 + \frac{1}{120}x^4}$$

which is also the same as on the excercise sheet.

2 Problem 2

In this problem, we want to calculate a polynomial of Degree three using Newton's method.

$$P_3(x_s) = f_0 + s \cdot \Delta f_0 + \frac{s(s-1)}{2!} \cdot \Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!} \cdot \Delta^3 f_0$$

We get the needed variables from the table in the exercise sheet:

$$\begin{aligned} h &= 2 \\ S &= \frac{x - x_0}{h} = \frac{x-4}{2} \\ f_0 &= 1 \\ \Delta f_0 &= 2 \\ \Delta^2 f_0 &= 3 \\ \Delta^3 f_0 &= 4 \end{aligned}$$

If we plug the values into the equation, we get:

$$P_3(x_s) = 1 + \frac{x-4}{2} \cdot 2 + \frac{x^2 - 10x + 24}{8} \cdot 3 + \frac{x^3 - 18x^2 + 104x - 192}{48} \cdot 4$$

We can rewrite it as:

$$P_3(x_s) = \frac{1}{12} \cdot x^3 - \frac{27}{24} \cdot x^2 + \frac{142}{24} \cdot x^1 - 10$$

As a result we get the equation given in the exercise sheet:

$$P_3(x_s) = \frac{1}{24} \cdot (2x^3 - 27x^2 + 142x - 240)$$