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# Assignment

## Exercise sheet 5

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# 1 Problem I

We need to calculate the Taylor expansion of  $y_0$  until the fourth term

$$\begin{aligned}y(x_0 + h) &= y_1 = y_0 + hy'_0 + \frac{h^2}{2}y''_0 + \frac{h^3}{6}y'''_0 + \frac{h^4}{24}y^{(4)}_0 + \dots \\y(x_0 - h) &= y_{-1} = y_0 - hy'_0 + \frac{h^2}{2}y''_0 - \frac{h^3}{6}y'''_0 + \frac{h^4}{24}y^{(4)}_0 + \dots\end{aligned}$$

now we want to add and subtract the terms, as a result, we get

$$\begin{aligned}y_1 + y_{-1} &= \left( y_0 + hy'_0 + \frac{h^2}{2}y''_0 + \frac{h^3}{6}y'''_0 + \frac{h^4}{24}y^{(4)}_0 \right) \\&\quad + \left( y_0 - hy'_0 + \frac{h^2}{2}y''_0 - \frac{h^3}{6}y'''_0 + \frac{h^4}{24}y^{(4)}_0 \right) \\&= 2y_0 + h^2y''_0 + \frac{h^4}{12}y^{(4)}_0\end{aligned}$$

$$\begin{aligned}y_1 - y_{-1} &= \left( y_0 + hy'_0 + \frac{h^2}{2}y''_0 + \frac{h^3}{6}y'''_0 + \frac{h^4}{24}y^{(4)}_0 \right) \\&\quad - \left( y_0 - hy'_0 + \frac{h^2}{2}y''_0 - \frac{h^3}{6}y'''_0 + \frac{h^4}{24}y^{(4)}_0 \right) \\&= 2hy'_0 + \frac{h^3}{3}y'''_0.\end{aligned}$$

If we compute  $y_1 - 2y_0 + y_{-1}$  we get

$$\begin{aligned}y_1 - 2y_0 + y_{-1} &= \left( y_0 + hy'_0 + \frac{h^2}{2}y''_0 + \frac{h^3}{6}y'''_0 + \frac{h^4}{24}y^{(4)}_0 \right) \\&\quad - 2y_0 \\&\quad + \left( y_0 - hy'_0 + \frac{h^2}{2}y''_0 - \frac{h^3}{6}y'''_0 + \frac{h^4}{24}y^{(4)}_0 \right) \\&= h^2y''_0 + \frac{h^4}{12}y^{(4)}_0\end{aligned}$$

Now we can rearrange the terms and get

$$y''_0 = \frac{y_1 - 2y_0 + y_{-1}}{h^2} - \frac{h^2}{12}y^{(4)}_0 + O(h^3)$$

The error term  $O(h^2)$  is given by

$$O(h^2) = -\frac{h^2}{12}y^{(4)}_0$$

In conclusion we got the explicit error:

$$\begin{aligned} Error &= - \sum_{n=2}^{\infty} \frac{2y^{(2n)}(x_0)}{(2n)!} \cdot h^{2n-2} \\ &= -\frac{h^2}{12}y^{(4)} - \frac{h^4}{60}y^{(6)} - \dots \end{aligned}$$

## 2 Problem II

For our  $y_0$  we first calculate the Taylor expansion until the seventh term:

$$y(x_0 + h) = y_1 = y_0 + hy'_0 + \frac{h^2}{2}y''_0 + \frac{h^3}{6}y'''_0 + \frac{h^4}{24}y^{(4)}_0 + \frac{h^5}{120}y^{(5)}_0 + \frac{h^6}{720}y^{(6)}_0$$

$$y(x_0 - h) = y_{-1} = y_0 - hy'_0 + \frac{h^2}{2}y''_0 - \frac{h^3}{6}y'''_0 + \frac{h^4}{24}y^{(4)}_0 - \frac{h^5}{120}y^{(5)}_0 + \frac{h^6}{720}y^{(6)}_0$$

$$y(x_0 + 2h) = y_2 = y_0 + 2hy'_0 + 2h^2y''_0 + \frac{4h^3}{3}y'''_0 + \frac{2h^4}{3}y^{(4)}_0 + \frac{4h^5}{15}y^{(5)}_0 + \frac{4h^6}{45}y^{(6)}_0$$

$$y(x_0 - 2h) = y_{-2} = y_0 - 2hy'_0 + 2h^2y''_0 - \frac{4h^3}{3}y'''_0 + \frac{2h^4}{3}y^{(4)}_0 - \frac{4h^5}{15}y^{(5)}_0 + \frac{4h^6}{45}y^{(6)}_0$$

We now combine the four equations the following:

$$y_{-2} - y_2 + 16(y_{-1} - y_1)$$

As a result, we get:

$$\begin{aligned} & -y_{-2} - y_2 + 16(y_1 + y_{-1}) \\ &= -2y_0 - 4h^2y''_0 - \frac{4h^4}{3}y^{(4)}_0 - \frac{8h^6}{45}y^{(6)}_0 + 16(2y_0 + h^2y''_0 + \frac{h^4}{12}y^{(4)}_0 + \frac{2h^6}{720}y^{(6)}_0) \\ &= 30y_0 + 12h^2y''_0 - \frac{2h^6}{15}y^{(6)}_0 \end{aligned}$$

If we bring all terms that don't include the second derivative on the left side we get:

$$y''_0 = \frac{-y_{-2} + 16y_1 - 30y_0 + 16y_{-1} - y_2}{12h^2} + \frac{h^4}{90}y^{(6)}_0 \quad (1)$$

Where the last term is our failure term:

$$O(h^4) = \frac{h^4}{90}y^{(6)}(\xi)$$

In conclusion we got the explicit error:

$$\begin{aligned} Error &= \sum_{n=3}^{\infty} \frac{8y^{(2n)}(x_0)}{(2n)!} \cdot h^{2n-2} \\ &= \frac{h^4}{90}y^{(6)}(x_0) + \frac{h^6}{126}y^{(8)}(x_0) + \dots \end{aligned}$$