Formalizing reachability, viability and avoidability in the context of sequential decision problems

Nicola Botta

Outline

- Why formalizing what?
- Minimal goals
- Sequential decision problems
- Reachability, viability and avoidability
- Decision procedures

Why formalizing what?

- *International emissions trading: Good or bad?*, Holtsmark & Sommervoll, 2012: "[. . .] we find that an agreement with international emissions trading leads to increased emissions and reduced efficiency."
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- *Confronting Climate Change: Avoiding the Unmanageable and Managing the Unavoidable*, P. Raven, R. Bierbaum, J. Holdren, UN-Sigma Xi Climate Change Report, 2007.

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Some notion of avoidability is implicit in the WG3_IPCC_AR5_2014 definitions of

- *mitigation*: "A human intervention to reduce the sources or enhance the sinks of greenhouse gases"

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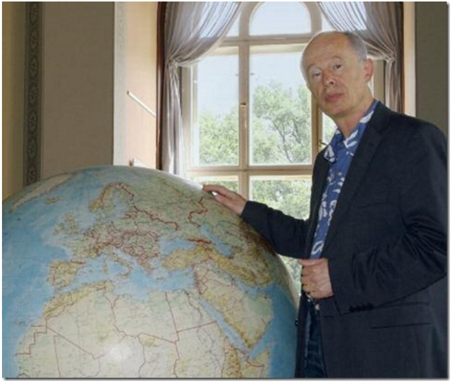
But what does it mean (for atmospheric GHG concentrations) to be *avoidable*?

Why formalizing what?



"Die Rolle der Klimaforschung bleibt weiterhin, die Problemfakten auf den Tisch zu knallen und Optionen für geeignete Lösungswege zu identifizieren."

H.-J. Schellnhuber in *Frankfurter Allgemeine* from 2012-06-19

Why formalizing what?

But how can we produce "hard facts" if the notions used to phrase specific, concrete problems are ambiguous, devoid of precise, well established, meanings?

Minimal goals

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Further questions, goals

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- Explain what it means for future (possibly harmful) states to be avoidable [reachable, viable, ...]
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Further questions, goals

- Can one exploit decidability to derive useful avoidability (levity?) measures?

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- Explain what it means for future (possibly harmful) states to be avoidable [reachable, viable, ...]
- Explain under which conditions is it *decidable* whether future states are avoidable [reachable, viable, ...] or not

Further questions, goals

- Can one exploit decidability to derive useful avoidability (levity?) measures?
- Can one refine decidable notions of viability, avoidability to derive operational notions (measures?) of sustainability, adaptability, resilience?

Sequential decision problems (intuition)

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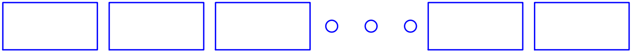
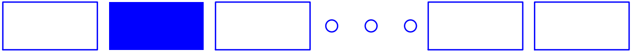


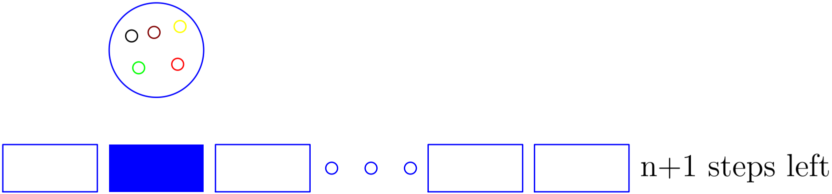
n+1 steps left

You are here. . .

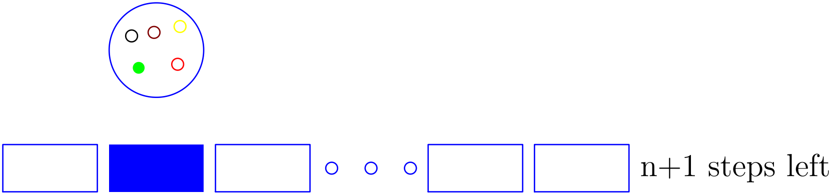


n+1 steps left

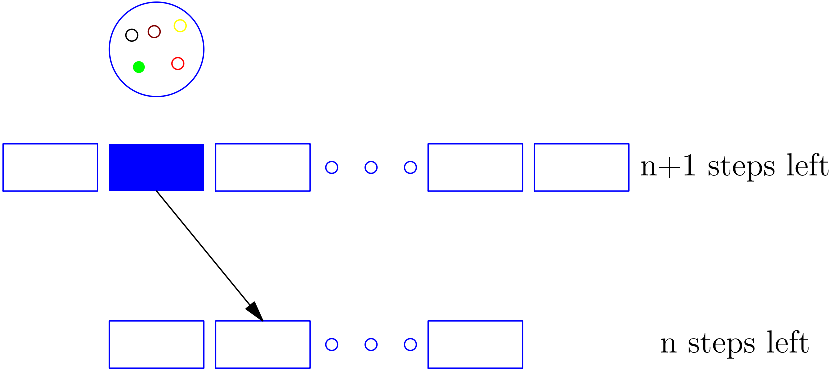
These are your options. . .



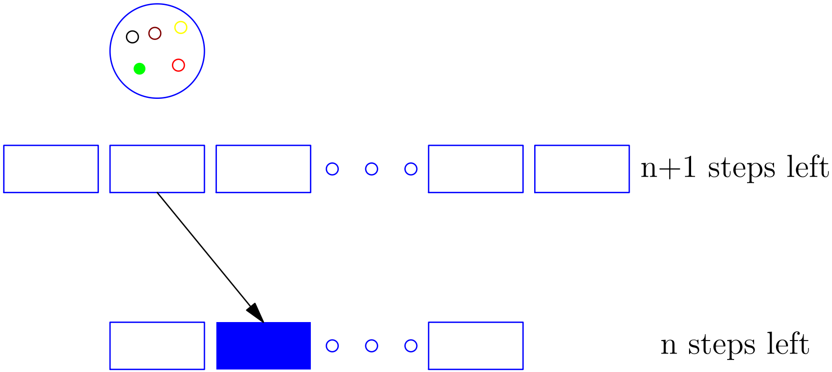
Pick one!



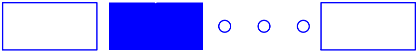
Advance one step...



. . . collect rewards . . .



. . . and go!



n steps left

General sequential decision problems (intuition)

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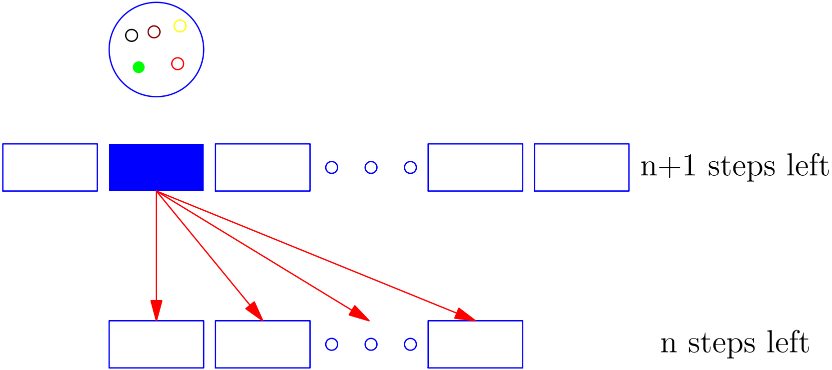
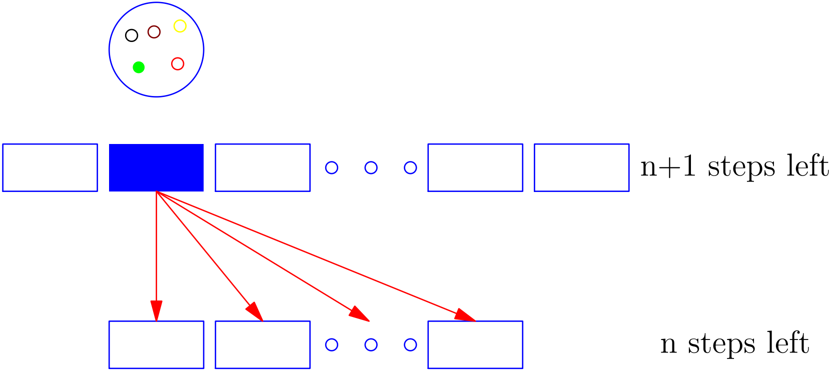


Sequential decision problems (notation)

Idris	Logic
A : $Type$	A is a set
a : A	$a \in A$
P : $Type$	P is a predicate
p : P	p is a proof of P
$FALSE$ (empty type)	False
non-empty type	True
$P \rightarrow Q$	P implies Q
$(a : A ** P\ a)$	there exists an $a \in A$ such that $P\ a$ holds
$(a : A) \rightarrow P\ a$	forall $a \in A$, $P\ a$ holds

Figure: Curry-Howard correspondence relating Idris and logic.

Sequential decision problems (basic ideas)



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At each decision step, a set of possible states:

$$X : (t : \mathbb{N}) \rightarrow Type$$

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At each decision step and for each state, a set of options

$$Y \;:\; (t \;:\; \mathbb{N}) \;\rightarrow\; (x \;:\; X\ t) \;\rightarrow\; \mathit{Type}$$

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A transition function

$$step \; : \; (t \; : \; \mathbb{N}) \; \to \; (x \; : \; X \; t) \; \to \; (y \; : \; Y \; t \; x) \; \to \; M \, (X \, (S \; t))$$

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What about rewards? What are M and S?

Sequential decision problems (uncertainties)

$S\ t$ is just the successor of t:

$$
\begin{array}{l}
\textbf{data}\ \mathbb{N}\ :\ \mathit{Type}\ \textbf{where} \\
\quad Z\ :\ \mathbb{N} \\
\quad S\ :\ \mathbb{N}\ \rightarrow\ \mathbb{N}
\end{array}
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$M\ :\ \mathit{Type}\ \rightarrow\ \mathit{Type}$ represents the uncertainties of the problem:

- deterministic problems: $M = \mathit{Id}$
- non-deterministic problems: $M = \mathit{List}$
- stochastic problems: $M = \mathit{Prob}$

Sequential decision problems (uncertainties)

$S\ t$ is just the successor of t:

data $\mathbb{N}$: *Type* **where**
  Z : $\mathbb{N}$
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M : $Type \rightarrow Type$ represents the uncertainties of the problem:

- deterministic problems: $M = Id$
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- stochastic problems: $M = Prob$

data *Prob* : $Type \rightarrow Type$ **where**
  $mkProb$: $(as : Vect\ n\ a) \rightarrow (ps : Vect\ n\ Float) \rightarrow$
    $sum\ ps = 1.0 \rightarrow Prob\ a$

Sequential decision problems (container monad)

Formally, M is a container monad, that is M is a monad:

$$
\begin{array}{lll}
fmap & : & (a \rightarrow b) \rightarrow M\ a \rightarrow M\ b \\
ret & : & a \rightarrow M\ a \\
bind & : & M\ a \rightarrow (a \rightarrow M\ b) \rightarrow M\ b \\
join & : & M\ (M\ a) \rightarrow M\ a
\end{array}
$$

$$
\begin{array}{lll}
functorSpec1 & : & fmap \circ id = id \\
functorSpec2 & : & fmap\ (f \circ g) = (fmap\ f) \circ (fmap\ g)
\end{array}
$$

$$
\begin{array}{lll}
monadSpec1 & : & (fmap\ f) \circ ret = ret \circ f \\
monadSpec2 & : & bind\ (ret\ a)\ f = f\ a \\
monadSpec3 & : & bind\ ma\ ret = ma \\
monadSpec4 & : & \{f : a \rightarrow M\ b\} \rightarrow \{g : b \rightarrow M\ c\} \rightarrow \\
 & & bind\ (bind\ ma\ f)\ g = bind\ ma\ (\lambda x \Rightarrow bind\ (f\ x)\ g) \\
monadSpec5 & : & join\ mma = bind\ mma\ id
\end{array}
$$

Sequential decision problems (container monad)

and M is a container:

$$\begin{aligned} &Elem &&: a \rightarrow M\ a \rightarrow Type \\ &All &&: (a \rightarrow Type) \rightarrow M\ a \rightarrow Type \end{aligned}$$

$$\begin{aligned} &containerSpec1 &&: a\ `Elem`\ (ret\ a) \\ &containerSpec2 &&: a\ `Elem`\ ma \rightarrow ma\ `Elem`\ mma \rightarrow a\ `Elem`\ (join\ mma) \\ &containerSpec3 &&: All\ p\ ma \rightarrow a\ `Elem`\ ma \rightarrow p\ a \end{aligned}$$

Sequential decision problems (basic ideas)

Thus, a concrete sequential decision problem is defined (up to the rewards) in terms of 4 entities: X, Y, M and $step$

$$X : (t : \mathbb{N}) \rightarrow Type$$

$$Y : (t : \mathbb{N}) \rightarrow (x : X\ t) \rightarrow Type$$

$$step : (t : \mathbb{N}) \rightarrow (x : X\ t) \rightarrow (y : Y\ t\ x) \rightarrow M\ (X\ (S\ t))$$

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We formalize reachability, viability and avoidability in terms of these notions

Reachability and viability (intuition)

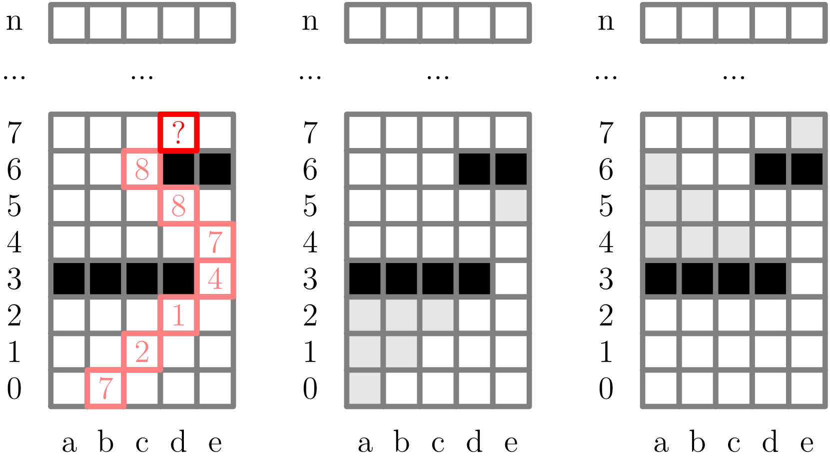

Figure: Possible evolution starting from *b* (left), states with limited viability (middle) and unreachable states (right).

Predecessor relation, reachability and viability

The (possible) predecessor relation:

$$
\begin{aligned}
&Pred\ :\ X\ t\ \to\ X\ (S\ t)\ \to\ Type\\
&Pred\ \{t\}\ x\ x' = (y\ :\ Y\ t\ x\ \ **\ x'\ \text{`}Elem\text{`}\ (step\ t\ x\ y))
\end{aligned}
$$

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\end{aligned}$$

reachability

$$\begin{aligned}
&\mathit{Reachable} \;:\; X\ t' \;\to\; \mathit{Type}\\
&\mathit{Reachable}\ \{t' = Z\}\ x' \quad = \mathit{Unit}\\
&\mathit{Reachable}\ \{t' = S\ t\}\ x' = (x \;:\; X\ t \;**\; (\mathit{Reachable}\ x, x\ \text{`}\mathit{Pred}\text{`}\ x'))
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\end{aligned}$$

and viability

$$\begin{aligned}
&\mathit{Viable} \;:\; (n\ :\ \mathbb{N}) \;\to\; X\ t \;\to\; \mathit{Type}\\
&\mathit{Viable}\ \{t\}\ Z\ x \qquad = \mathit{Unit}\\
&\mathit{Viable}\ \{t\}\ (S\ m)\ x = (y\ :\ Y\ t\ x\ **\ \mathit{All}\ (\mathit{Viable}\ m)\ (\mathit{step}\ t\ x\ y))
\end{aligned}$$

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- We are interested in the avoidability of "possible" future states. Specifically, we are interested in the avoidability of states which are reachable from some given state.
- The notion of avoidability entails the notion of an alternative.

Avoidability

We are interested in the avoidability of states which are reachable from some given state:

$$
\begin{array}{l}
\mathit{ReachableFrom} \;:\; X\ t'' \;\rightarrow\; X\ t \;\rightarrow\; \mathit{Type} \\
\mathit{ReachableFrom}\ \{t'' = Z\}\ \{t\}\ x''\ x \quad = (t = Z, x = x'') \\
\mathit{ReachableFrom}\ \{t'' = S\ t'\}\ \{t\}\ x''\ x = \\
\quad \mathit{Either}\ (t = S\ t', x = x'') \\
\qquad\qquad (x' \;:\; X\ t' \;**\; (x'\ \text{`}\mathit{ReachableFrom}\text{`}\ x, x'\ \text{`}\mathit{Pred}\text{`}\ x''))
\end{array}
$$

where

$$
\textbf{data}\ \mathit{Either}\ a\ b = \mathit{Left}\ a \mid \mathit{Right}\ b
$$

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&\qquad\qquad (x' \;:\; X\ t' \;**\; (x'\ \text{`}\mathit{ReachableFrom}\text{`}\ x, x'\ \text{`}\mathit{Pred}\text{`}\ x''))
\end{aligned}
$$

where

$$\textbf{data}\ \mathit{Either}\ a\ b = \mathit{Left}\ a \mid \mathit{Right}\ b$$

Proof of concept: show that

$$
\begin{aligned}
\mathit{reachableFromLemma} \;:\; & (x'' \;:\; X\ t'') \;\rightarrow\; (x \;:\; X\ t) \;\rightarrow \\
& x''\ \text{`}\mathit{ReachableFrom}\text{`}\ x \;\rightarrow\; t''\ \text{`}\mathit{GTE}\text{`}\ t
\end{aligned}
$$

Avoidability

The notion of avoidability entails the notion of an alternative state x''. This has to fulfill three conditions:

$$
\begin{aligned}
& \mathit{AvoidableFrom} \; : \; (x' \; : \; X\ t') \; \to \; (x \; : \; X\ t) \; \to \\
& \qquad\qquad\qquad\quad x' \,\text{`}\mathit{ReachableFrom}\text{`}\, x \; \to \; (m \; : \; \mathbb{N}) \; \to \; \mathit{Type} \\
& \mathit{AvoidableFrom} \; \{\, t' \,\} \; x' \; x \; r \; m = \\
& \quad (x'' \; : \; X\ t' \; ** \; (x'' \,\text{`}\mathit{ReachableFrom}\text{`}\, x, (\mathit{Viable}\ m\ x'', \mathit{Not}\ (x'' = x'))))
\end{aligned}
$$

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&\mathit{AvoidableFrom}\ \{t'\}\ x'\ x\ r\ m = \\
&\quad (x'' : X\ t' \ {*}{*}\ (x'' \text{ `}\mathit{ReachableFrom}\text{` } x, (\mathit{Viable}\ m\ x'', \mathit{Not}\ (x'' = x'))))
\end{aligned}
$$

Back to the minimal goals: under which conditions are reachability, viability and avoidability decidable?

Decision procedures

For every type (predicate) $P\ :\ Type$, $Not\ P$ is just a synonym for $P\ \rightarrow\ Void$:

$$Not\ :\ Type\ \rightarrow\ Type$$
$$Not\ P = P\ \rightarrow\ Void$$

Decision procedures

For every type (predicate) $P : Type$, $Not\ P$ is just a synonym for $P \rightarrow Void$:

$$
\begin{aligned}
&Not : Type \rightarrow Type \\
&Not\ P = P \rightarrow Void
\end{aligned}
$$

A predicate $P : Type$ is decidable if one can compute either a value $p : P$ or a value of type $Not\ P$:

$$
\begin{aligned}
&Decidable : Type \rightarrow Type \\
&Decidable\ P = Either\ P\ (Not\ P)
\end{aligned}
$$

Decision procedures

Thus, the question is under which conditions one can implement

$$decReachable : (x : X\ t) \rightarrow Decidable\ (Reachable\ x)$$

$$decViable : (n : \mathbb{N}) \rightarrow (x : X\ t) \rightarrow Decidable\ (Viable\ n\ x)$$

$$\begin{aligned} decAvoidableFrom : \; & \{t' : \mathbb{N}\} \rightarrow \{t : \mathbb{N}\} \rightarrow \\ & (x' : X\ t') \rightarrow (x : X\ t) \rightarrow \\ & (r : x'\ \text{`}ReachableFrom\text{`}\ x) \rightarrow (n : \mathbb{N}) \rightarrow \\ & Decidable\ (AvoidableFrom\ \{t'\}\ \{t\}\ x'\ x\ r\ n) \end{aligned}$$

Decision procedures

As one would expect, the conditions

$$fX : (t : \mathbb{N}) \rightarrow Finite\,(X\ t)$$

$$fY : (t : \mathbb{N}) \rightarrow (x : X\ t) \rightarrow Finite\,(Y\ t\ x)$$

$$decElem : \{t : \mathbb{N}\} \rightarrow (x : X\ t) \rightarrow (mx : M\,(X\ t)) \rightarrow Decidable\,(x\ \text{`}Elem\text{`}\ mx)$$

$$decAll : \{t : \mathbb{N}\} \rightarrow (P : X\ t \rightarrow Type) \rightarrow ((x : X\ t) \rightarrow Decidable\,(P\ x)) \rightarrow (mx : M\,(X\ t)) \rightarrow Decidable\,(All\ P\ mx)$$

are sufficient for decidability.

Decision procedures

The key lemma for implementing decision procedures for *Reachable*, *Viable* and *AvoidableFrom* is intuitively obvious

$$
\begin{aligned}
\mathit{finiteDecidableLemma} : \; & \{A : \mathit{Type}\} \to \{P : A \to \mathit{Type}\} \to \\
& \mathit{Finite}\ A \to ((a : A) \to \mathit{Decidable}\,(P\ a)) \to \\
& \mathit{Decidable}\,(a : A \ast\ast\ P\ a)
\end{aligned}
$$

Decision procedures

The key lemma for implementing decision procedures for *Reachable*, *Viable* and *AvoidableFrom* is intuitively obvious

$$\begin{aligned} \mathit{finiteDecidableLemma} \ : \ & \{A : \mathit{Type}\} \rightarrow \{P : A \rightarrow \mathit{Type}\} \rightarrow \\ & \mathit{Finite}\ A \rightarrow ((a : A) \rightarrow \mathit{Decidable}\ (P\ a)) \rightarrow \\ & \mathit{Decidable}\ (a : A \ast\ast P\ a) \end{aligned}$$

but implementing *finiteDecidableLemma* is not trivial!

Decision procedures

With *finiteDecidableLemma*, *fY* and decidability of *Elem* one immediately has decidability of *Pred*

$$\begin{aligned}
&decPred : \{t : \mathbb{N}\} \to (x : X\ t) \to (x' : X\ (S\ t)) \to \\
&\qquad\qquad Decidable\ (x\ `Pred`\ x') \\
&decPred\ \{t\}\ x\ x' = finiteDecidableLemma\ (fY\ t\ x)\ prf\ \mathbf{where} \\
&\quad prf : (y : Y\ t\ x) \to Decidable\ (x'\ `Elem`\ (step\ t\ x\ y)) \\
&\quad prf\ y = decElem\ x'\ (step\ t\ x\ y)
\end{aligned}$$

Decision procedures

and, with

$$decPair \;:\; Decidable\; p \;\to\; Decidable\; q \;\to\; Decidable\; (p, q)$$

decidability of *Reachable*:

$$
\begin{array}{l}
decReachable \;:\; \{t' \;:\; \mathbb{N}\} \;\to\; (x' \;:\; X\; t') \;\to\; Decidable\; (Reachable\; x') \\
decReachable\; \{t' = Z\}\; x' = Left\; () \\
decReachable\; \{t' = S\; t\}\; x' = s1\; \textbf{where} \\
\quad s1 \;:\; Decidable\; (x \;:\; X\; t \; {*}{*}\; (Reachable\; x, x\; \text{`}Pred\text{`}\; x')) \\
\quad s1 = finiteDecidableLemma \\
\qquad\quad (fX\; t) \\
\qquad\quad (\lambda x \Rightarrow decPair\; (decReachable\; x)\; (decPred\; x\; x'))
\end{array}
$$

Decision procedures

Similarly, one implement (prove) decidability of *Viable*:

$$
\begin{aligned}
&\mathit{decViable} : \{t : \mathbb{N}\} \to (n : \mathbb{N}) \to (x : X\ t) \to \mathit{Decidable}\,(\mathit{Viable}\ n\ x)\\
&\mathit{decViable}\,\{t\}\ Z\ x = \mathit{Left}\ ()\\
&\mathit{decViable}\,\{t\}\ (S\ m)\ x = \mathit{s3}\ \textbf{where}\\
&\quad \mathit{s1} \quad : (y : Y\ t\ x) \to \mathit{Decidable}\,(\mathit{All}\,(\mathit{Viable}\ m)\,(\mathit{step}\ t\ x\ y))\\
&\quad \mathit{s1}\ y = \mathit{decAll}\,(\mathit{Viable}\ m)\,(\mathit{decViable}\ m)\,(\mathit{step}\ t\ x\ y)\\
&\quad \mathit{s2} \quad : \mathit{Decidable}\,(y : Y\ t\ x \ast\ast \mathit{All}\,(\mathit{Viable}\ m)\,(\mathit{step}\ t\ x\ y))\\
&\quad \mathit{s2} \quad = \mathit{finiteDecidableLemma}\,(\mathit{fY}\ t\ x)\ \mathit{s1}\\
&\quad \mathit{s3} \quad : \mathit{Decidable}\,(\mathit{Viable}\,(S\ m)\ x)\\
&\quad \mathit{s3} \quad = \mathit{s2}
\end{aligned}
$$

Decision procedures

Similarly, one implement (prove) decidability of *Viable*:

$$
\begin{array}{l}
decViable : \{t : \mathbb{N}\} \to (n : \mathbb{N}) \to (x : X\ t) \to Decidable\ (Viable\ n\ x) \\
decViable\ \{t\}\ Z\ x = Left\ () \\
decViable\ \{t\}\ (S\ m)\ x = s3\ \textbf{where} \\
\quad s1 \quad : (y : Y\ t\ x) \to Decidable\ (All\ (Viable\ m)\ (step\ t\ x\ y)) \\
\quad s1\ y = decAll\ (Viable\ m)\ (decViable\ m)\ (step\ t\ x\ y) \\
\quad s2 \quad : Decidable\ (y : Y\ t\ x\ {*}{*}\ All\ (Viable\ m)\ (step\ t\ x\ y)) \\
\quad s2 \quad = finiteDecidableLemma\ (fY\ t\ x)\ s1 \\
\quad s3 \quad : Decidable\ (Viable\ (S\ m)\ x) \\
\quad s3 \quad = s2
\end{array}
$$

Implementing a decidion procedure for *AvoidableFrom* is a bit more complicated but conceptually equivalent.

Acknowledgments

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These slides:

The code shown: