Formalizing reachability, viability and avoidability in the context of sequential decision problems

Nicola Botta, Cezar Ionescu, Patrik Jansson

Outline

- Why formalizing what?
- Minimal goals
- Sequential decision problems
- Reachability and viability
- Avoidability

Why formalizing what?

- *International emissions trading: Good or bad?*, Holtsmark & Sommervoll, 2012: "[...] we find that an agreement with international emissions trading leads to increased emissions and reduced efficiency."
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- *Confronting Climate Change: Avoiding the Unmanageable and Managing the Unavoidable*, P. Raven, R. Bierbaum, J. Holdren, UN-Sigma Xi Climate Change Report, 2007.

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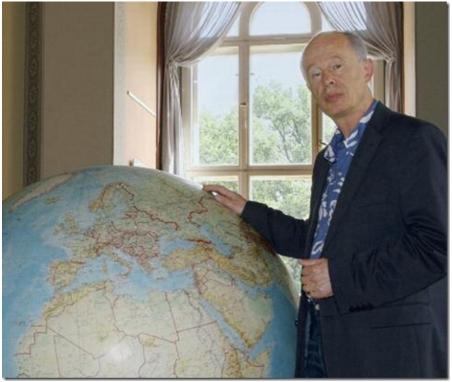
But what does it mean (for atmospheric GHG concentrations) to be *avoidable*?

Why formalizing what?



"Die Rolle der Klimaforschung bleibt weiterhin, die Problemfakten auf den Tisch zu knallen und Optionen für geeignete Lösungswege zu identifizieren."

H.-J. Schellnhuber in *Frankfurter Allgemeine* from 2012-06-19

Why formalizing what?

But how can we produce “hard facts” if the notions used to phrase specific, concrete problems are ambiguous, devoid of precise, well established, meanings?

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Further questions, goals

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Further questions, goals

- Can one exploit decidability to derive useful avoidability (levity?) measures?

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Further questions, goals

- Can one exploit decidability to derive useful avoidability (levity?) measures?
- Can one refine decidable notions of viability, avoidability to derive operational notions (measures?) of sustainability, adaptability, resilience?

Sequential decision problems (intuition)

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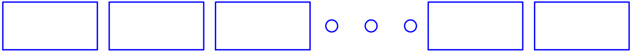
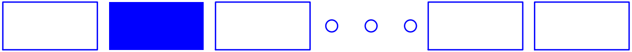


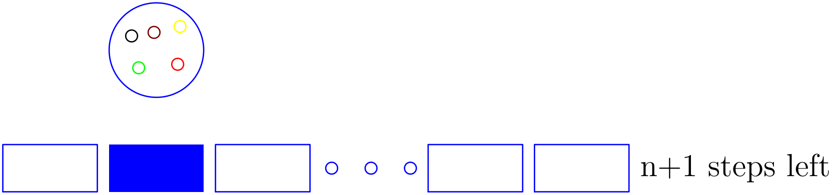
n+1 steps left

You are here. . .

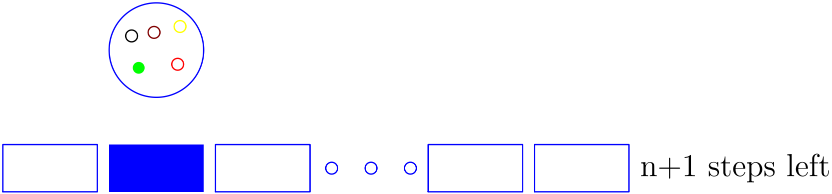


n+1 steps left

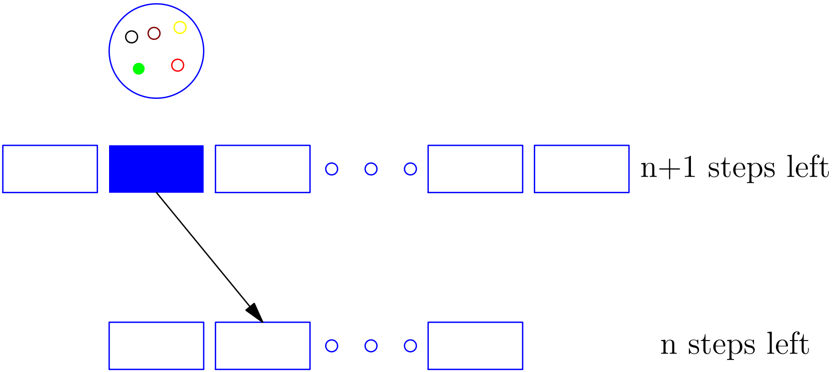
These are your options. . .



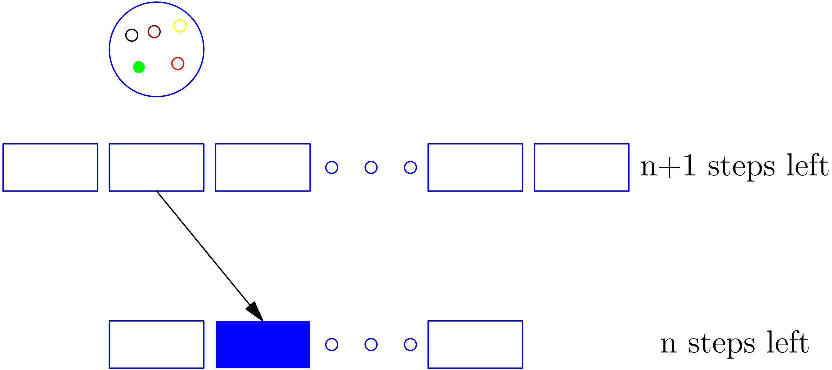
Pick one!



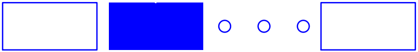
Advance one step. . .



. . . collect rewards . . .



. . . and go!



n steps left

General sequential decision problems (intuition)

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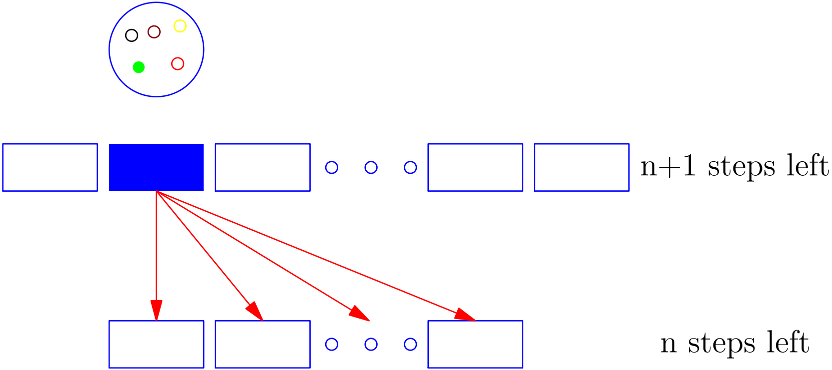
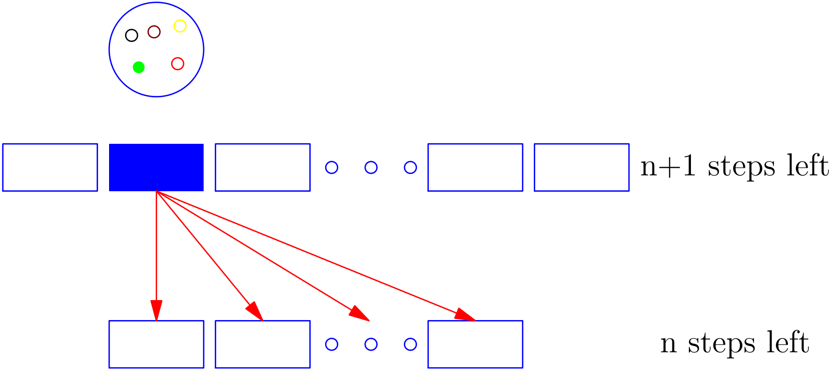


Sequential decision problems (notation)

Idris	Logic
$p : P$	p is a proof of P
$FALSE$ (empty type)	False
non-empty type	True
$P \rightarrow Q$	P implies Q
$\exists \{A\} P$	there exists a wit such that P (wit) holds
$(x : A) \rightarrow P\ x$	forall x of type A, $P\ x$ holds

Figure: Curry-Howard correspondence relating Idris and logic.

Sequential decision problems (basic ideas)



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At each decision step, a set of possible states:

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$$step \ : \ (t \ : \ \mathbb{N}) \ \to \ (x \ : \ X \ t) \ \to \ (y \ : \ Y \ t \ x) \ \to \ M \ (X \ (S \ t))$$

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What about rewards? What are M and S?

Sequential decision problems (uncertainties)

$S\ t$ is just the successor of t:

$$
\begin{array}{l}
\textbf{data}\ \mathbb{N}\ :\ \mathit{Type}\ \textbf{where} \\
\quad Z\ :\ \mathbb{N} \\
\quad S\ :\ \mathbb{N}\ \rightarrow\ \mathbb{N}
\end{array}
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$M\ :\ \mathit{Type}\ \rightarrow\ \mathit{Type}$ represents the uncertainties of the problem:

- deterministic problems: $M = \mathit{Id}$
- non-deterministic problems: $M = \mathit{List}$
- stochastic problems: $M = \mathit{Prob}$

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\begin{array}{l}
\textbf{data}\ \mathit{Prob}\ :\ \mathit{Type} \to \mathit{Type}\ \textbf{where} \\
\quad \mathit{mkProb}\ :\ (\mathit{as}\ :\ \mathit{Vect}\ n\ a) \to (\mathit{ps}\ :\ \mathit{Vect}\ n\ \mathit{Float}) \to \\
\qquad\qquad\quad \mathit{sum}\ \mathit{ps} = 1.0 \to \mathit{Prob}\ a
\end{array}
$$

Sequential decision problems (container monad)

Formally, M is a container monad, that is M is a monad:

$$
\begin{array}{lcl}
\mathit{fmap} & : & (a \to b) \to M\ a \to M\ b \\
\mathit{ret} & : & a \to M\ a \\
\mathit{bind} & : & M\ a \to (a \to M\ b) \to M\ b \\
\mathit{join} & : & M\ (M\ a) \to M\ a
\end{array}
$$

$$
\begin{array}{lcl}
\mathit{functorSpec1} & : & \mathit{fmap} \circ \mathit{id} = \mathit{id} \\
\mathit{functorSpec2} & : & \mathit{fmap}\ (f \circ g) = (\mathit{fmap}\ f) \circ (\mathit{fmap}\ g)
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$$

$$
\begin{array}{lcl}
\mathit{monadSpec1} & : & (\mathit{fmap}\ f) \circ \mathit{ret} = \mathit{ret} \circ f \\
\mathit{monadSpec2} & : & \mathit{bind}\ (\mathit{ret}\ a)\ f = f\ a \\
\mathit{monadSpec3} & : & \mathit{bind}\ \mathit{ma}\ \mathit{ret} = \mathit{ma} \\
\mathit{monadSpec4} & : & \{f : a \to M\ b\} \to \{g : b \to M\ c\} \to \\
 & & \mathit{bind}\ (\mathit{bind}\ \mathit{ma}\ f)\ g = \mathit{bind}\ \mathit{ma}\ (\lambda x \Rightarrow \mathit{bind}\ (f\ x)\ g) \\
\mathit{monadSpec5} & : & \mathit{join}\ \mathit{mma} = \mathit{bind}\ \mathit{mma}\ \mathit{id}
\end{array}
$$

Sequential decision problems (container monad)

and M is a container:

$$\begin{aligned}
&\mathit{Elem} &&: a \rightarrow M\ a \rightarrow \mathit{Type} \\
&\mathit{All} &&: (a \rightarrow \mathit{Type}) \rightarrow M\ a \rightarrow \mathit{Type}
\end{aligned}$$

$$\begin{aligned}
&\mathit{containerSpec1} &&: a\ \text{`}\mathit{Elem}\text{`}\ (\mathit{ret}\ a) \\
&\mathit{containerSpec2} &&: a\ \text{`}\mathit{Elem}\text{`}\ \mathit{ma} \rightarrow \mathit{ma}\ \text{`}\mathit{Elem}\text{`}\ \mathit{mma} \rightarrow a\ \text{`}\mathit{Elem}\text{`}\ (\mathit{join}\ \mathit{mma}) \\
&\mathit{containerSpec3} &&: \mathit{All}\ p\ \mathit{ma} \rightarrow a\ \text{`}\mathit{Elem}\text{`}\ \mathit{ma} \rightarrow p\ a
\end{aligned}$$

Sequential decision problems (basic ideas)

Thus, a concrete sequential decision problem is defined (up to the rewards) in terms of 4 entities: X, Y, M and $step$

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We try to formalize reachability, viability and avoidability in terms of these notions

Reachability and viability (intuition)

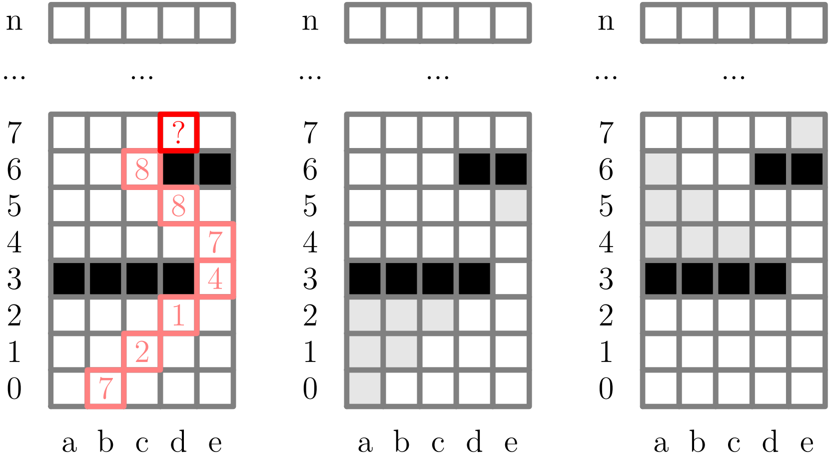

Figure: Possible evolution starting from *b* (left), states with limited viability (middle) and unreachable states (right).

Predecessor relation, reachability and viability

The (possible) predecessor relation:

$$
\begin{array}{l}
\mathit{Pred} \ : \ X \ t \ \rightarrow \ X \ (S \ t) \ \rightarrow \ \mathit{Type} \\
\mathit{Pred} \ \{t\} \ x \ x' = (y \ : \ Y \ t \ x \ ** \ \mathit{Elem} \ x' \ (\mathit{step} \ t \ x \ y))
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\end{aligned}
$$

reachability

$$
\begin{aligned}
&\mathit{Reachable} \;:\; X\;t' \;\rightarrow\; \mathit{Type} \\
&\mathit{Reachable}\;\{t' = Z\}\quad x' = \mathit{Unit} \\
&\mathit{Reachable}\;\{t' = S\;t\}\;x' = (x \;:\; X\;t \;{*}{*}\; (\mathit{Reachable}\;x, \mathit{Pred}\;x\;x'))
\end{aligned}
$$

Predecessor relation, reachability and viability

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and viability

$$\begin{array}{l} Viable\ :(n\ :\ \mathbb{N})\ \to\ X\ t\ \to\ Type \\ Viable\ \{t\}\ \ Z\qquad x = Unit \\ Viable\ \{t\}\ (S\ m)\ x = (y\ :\ Y\ t\ x\ \ast\ast\ All\ (Viable\ m)\ (step\ t\ x\ y)) \end{array}$$

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- We are interested in the avoidability of "possible" future states. Specifically, we are interested in the avoidability of states which are reachable from some given state.
- The notion of avoidability entails the notion of an alternative.

Avoidability

We are interested in the avoidability of states which are reachable from some given state:

$$
\begin{array}{l}
\mathit{ReachableFrom} \; : \; X \; t'' \; \to \; X \; t \; \to \; \mathit{Type} \\
\mathit{ReachableFrom} \, \{ t'' = Z \} \, \{ t \} \; x'' \; x \quad = (t = Z, x = x'') \\
\mathit{ReachableFrom} \, \{ t'' = S \; t' \} \, \{ t \} \; x'' \; x = \\
\quad \mathit{Either} \; (t = S \; t', x = x'') \\
\qquad\qquad (x' \; : \; X \; t' \; {*}{*} \; (\mathit{ReachableFrom} \; x' \; x, \mathit{Pred} \; x' \; x''))
\end{array}
$$

We can show that

$$
\begin{array}{l}
\mathit{reachableFromLemma} \; : (x'' \; : \; X \; t'') \; \to \; (x \; : \; X \; t) \; \to \\
\quad \mathit{ReachableFrom} \; x'' \; x \; \to \; \mathit{GTE} \; t'' \; t
\end{array}
$$

Avoidability

The notion of avoidability entails the notion of an alternative state x''. This has to fulfill three conditions:

$$\begin{aligned}&\mathit{AvoidableFrom} : (x' : X\ t') \to (x : X\ t) \to x'\ \text{`}\mathit{ReachableFrom}\text{`}\ x \to (n\\&\mathit{AvoidableFrom}\ \{t'\}\ x'\ x\ r\ m =\\&\quad (x'' : X\ t'\ **\ (x''\ \text{`}\mathit{ReachableFrom}\text{`}\ x, (\mathit{Viable}\ m\ x'', \mathit{Not}\ (x'' = x'))))\end{aligned}$$

Applications