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A Bayesian Approximation to the Integrated Likelihood Function with Applications in  
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## Abstract

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## CHAPTER 1

**Preface****1.1. Introduction**

The research for my dissertation involves developing a novel algorithm for numerically integrating the likelihood function of a statistical model with respect to a nuisance parameter. This prospectus aims to demonstrate how the algorithm works and explain the appeal of using an integrated likelihood function over other types of pseudolikelihood functions to make inferences about the parameter of interest in a model.

**1.2. Motivation**

The motivation behind my research developed out of the observation that the expression for the integral of a likelihood function follows a form similar to that of the calculation of the marginalizing constant of a posterior distribution.

## CHAPTER 2

**Background****2.1. Assumptions**

Consider a random sample  $\mathbf{x} = (x_1, \dots, x_n)$  drawn from a population. What can we say about the population based on  $\mathbf{x}$ ? Where is its point of central tendency located? Are its values clustered tightly around this point, or are they more diffuse? Are they distributed symmetrically or skewed to one side or the other? Questions like these were the original motivation behind the field of statistical inference, and many of the techniques devised to answer them are still used by statisticians today.

It is important to remember, however, that the real world is messy and no mathematical function will ever perfectly capture the complexities of a population or random process whose properties we wish to infer. To overcome this difficulty, statisticians sacrifice a small amount of accuracy for (hopefully) a large reduction in complexity by imposing additional assumptions on the population of interest. These assumptions are essentially never true in the sense that they are not a flawless representation of reality, but they may nevertheless serve as convenient approximations capable of producing sufficiently accurate answers in their own right. As George Box famously put it, “All models are wrong, but some are useful.”

**2.1.1. Data-Generating Mechanism**

And indeed, in the aggregate these assumptions create what is known as a statistical model. In its most general framework, a statistical model can be formulated as a tuple  $(\mathcal{S}, \mathcal{P})$  where  $\mathcal{S}$  is the set of all possible observations (i.e. the population), and  $\mathcal{P}$  is a set of probability distributions on  $\mathcal{S}$ . The first and most fundamental assumption we make when defining our models is that there exists some unknown mechanism in the population that generates the data we observe from  $\mathcal{S}$ . This mechanism is what induces the “true” probability distribution on  $\mathcal{S}$  though  $\mathcal{P}$  need not contain this distribution, and in practice it seldom does.

### 2.1.2. Parameter Existence

Another assumption found in almost every model is that the set  $\mathcal{P}$  is considered to be *parameterized*. That is, we assume the probability distributions contained in  $\mathcal{P}$  are indexed by a *parameter* that controls their features.<sup>1</sup> This parameter acts like a tuning dial for the population - rotate the dial and certain behaviors of the population (e.g. its location, scale, or shape) will change. Much of statistical inference can be boiled down to figuring out the particular value to which a population's dial has been set. We will denote this assumption as  $\mathcal{P} = \{\mathcal{P}_\theta | \theta \in \Theta\}$ , where  $\theta$  denotes the parameter, and  $\Theta$ , the parameter space, represents the set of all possible values  $\theta$  can take on.<sup>2</sup>

Statisticians also like to assume the parameters in their models can be uniquely identified based on the data they observe. A model is considered *identifiable* if having perfect knowledge of the population enables us to determine the true value of its parameter with absolute certainty.<sup>3</sup> More formally, for any two parameters  $\theta_1$  and  $\theta_2$  in  $\Theta$ , if  $\mathcal{P}_{\theta_1} = \mathcal{P}_{\theta_2}$ , then it must follow that  $\theta_1 = \theta_2$ . A model that is not identifiable could potentially have two or more distinct parameter values that give rise to the same probability distribution. Since we have already assumed  $\mathcal{P}$  is the mechanism generating the data we have observed in the first place, this would make it impossible to determine which value is the “correct” one on the basis of the data alone. Statisticians impose the identifiability criterion on their models as a means of avoiding this undesirable situation.

### 2.1.3. Parameter Space Dimension

The dimension of the parameter space  $\Theta$  is another critical decision statisticians must make when choosing the best model for their research. When  $\Theta \subseteq V$ , where  $V$  is an infinite-dimensional space, the model is said to be a *nonparametric*. The name is a bit of a misnomer in the sense that nonparametric models do not actually lack parameters, but rather they are flexible regarding the exact number and properties of the parameters they do have.

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<sup>1</sup>We will consider the phrases “parameter”, “population parameter”, and “model parameter” all to have the same meaning in this paper and use them interchangeably.

<sup>2</sup> $\theta$  can be and in fact usually is a multi-dimensional vector whose components represent various sub-parameters of the population.

<sup>3</sup>This is of course almost always impossible in practice, but in theory it could be accomplished by obtaining an infinite number of observations from  $\mathcal{S}$  or simply all of its observations if  $|\mathcal{S}|$  is finite.

*Semiparametric* models are those whose parameter spaces have components of both finite and infinite dimensionality. That is,  $\Theta \subseteq \mathbb{R}^k \times V$ , where again  $V$  is an infinite-dimensional space. Usually it is only the finite-dimensional component of the parameter in which we are interested while the infinite-dimensional component is considered a nuisance parameter.

Models for which  $\Theta$  is of finite dimension are called *parametric*. That is,  $\Theta \subseteq \mathbb{R}^k$ , where  $k \in \mathbb{Z}^+$ . This is the most common type of model used by statisticians, with examples including the normal family of distributions as well as the Poisson family. For the purposes of this paper, we will assume all statistical models under discussion are parametric in form.

#### 2.1.4. Regularity

Finally, we will assume that all probability distributions in  $\mathcal{P}$  obey a certain regularity condition. A parametric model  $(\mathcal{S}, \mathcal{P})$  is called *regular* if it satisfies one of the following:

- 1) All of the distributions in  $\mathcal{P}$  are continuous with densities  $p(x; \theta)$ ;
- 2) All of the distributions in  $\mathcal{P}$  are discrete with frequency functions  $p(x; \theta)$  and the set  $\{x : p(x; \theta) > 0\}$  is the same set for all  $\theta$ .

In other words, a regular model is one in which its distributions all have a common support that does not depend on the unknown parameter  $\theta$ .

## 2.2. The Likelihood Function

Once we have chosen a model  $(\mathcal{S}, \mathcal{P})$ , our goal becomes to identify the “true” distribution in  $\mathcal{P}$  or, failing that, the one that best approximates the truth. Since we have assumed our model is parametric and identifiable, this is equivalent to making inferences about the value of the  $k$ -dimensional parameter  $\theta$  indexing the distributions in  $\mathcal{P}$  on the basis of some data we observe. Classically, these inferences come in the form of point estimates, interval estimates, or hypothesis tests though other techniques exist as well. A sensible choice to use as an estimate for the value of  $\theta$  is one which causes the data actually observed to have the highest possible *post-hoc* probability of occurrence out of all possible values in  $\Theta$ .



To formalize this notion, we need some way of analyzing the joint probability of our sample as a function of our parameter  $\theta$ .

### 2.2.1. The Discrete Case

Suppose  $X$  is a discrete random variable with probability mass function  $p(x; \theta)$ . For a single observation  $X = x$ , the *likelihood function* for  $\theta$  is defined as

$$(2.2.1) \quad L(\theta) \equiv L(\theta; x) = p(x; \theta), \quad \theta \in \Theta.$$

That is, when our sample consists only of a single observation, the likelihood function for  $\theta$  is simply equal to  $p(x; \theta)$  itself. However, while  $p(x; \theta)$  is viewed as a function of  $x$  for fixed  $\theta$ , the reverse is actually true for  $L(\theta; x)$ ; we view it as a function of  $\theta$  for fixed  $x$ . The positioning of the arguments  $\theta$  and  $x$  is a reflection of this difference in perspectives.

In this case, we may interpret  $L(\theta)$  as the probability that  $X = x$  given that  $\theta$  is the true parameter value. Crucially, this is *not* equivalent to the inverse probability that  $\theta$  is the true parameter value given  $X = x$ . Though intuitively appealing, this interpretation constitutes a fundamental misunderstanding of what a likelihood function is, and great care must be taken to avoid it.

This definition be extended to include the more common scenario in which the sample consists of multiple observations. For a sample of size  $n$  taken from our sample space  $\mathcal{S}$ , the likelihood is defined as

$$(2.2.2) \quad L(\theta; x_1, \dots, x_n) = p(x_1, \dots, x_n; \theta), \quad \theta \in \Theta.$$

That is, it is equal to the joint probability of the observations  $x_1, \dots, x_n$ , considered as a function of  $\theta$ . When the observations are independent and identically distributed, we can further express the likelihood as

$$(2.2.3) \quad L(\theta; x_1, \dots, x_n) = \prod_{i=1}^n p(x_i; \theta), \quad \theta \in \Theta.$$

### 2.2.2. The Continuous Case

When  $X$  is instead a continuous random variable, the likelihood for  $\theta$  may still be defined as it is in Equation 2.2.1 and Equation 2.2.2. However,  $p(x; \theta)$  has switched from being a probability *mass* function to a probability *density* function over the support of  $X$ . We must therefore forfeit our previous direct interpretation of  $L(\theta)$  as a probability since  $p(x; \theta)$  no longer represents  $\mathbb{P}(X = x|\theta)$ . We may however still think of the likelihood as being proportional to the probability that  $X$  is “close” to the value  $x$ .<sup>4</sup> Specifically, for two different samples  $x_1$  and  $x_2$ , if  $L(\theta; x_1) = c \cdot L(\theta; x_2)$ , where  $c > 1$ , then under this model we may conclude  $X$  is  $c$  times more likely to assume a value closer to  $x_1$  than  $x_2$  given that  $\theta$  is the true value of the parameter.

As in the discrete case, we must also be careful here to avoid using  $L(\theta)$  to make direct statements of probability about  $\theta$ . Indeed, despite our use of one in its definition, the likelihood function is *not* itself a probability density over the parameter  $\theta$  and need not obey the same laws as one.

### 2.2.3. Maximum Likelihood Estimation

Maximum likelihood estimation is one of the most powerful and widespread techniques for obtaining point estimates of model parameters based on some observed data  $x$ . The original intuition behind the method derives from the observation that when faced with a choice between two possible values of a parameter, say  $\theta_1$  and  $\theta_2$ , the sensible choice is the one that makes the data we did observe more probable to have been observed. Fortunately, we have already defined the likelihood function as a means of capturing this probability, which makes expressing this decision rule in terms of it very easy - we simply choose for our estimate the option that produces the higher value of the likelihood function. That is, if  $L(\theta_1; x) > L(\theta_2; x)$ , then  $\theta_1$  is the better estimate of the true parameter value and vice versa.

This can be extended to include as many parameter values as we would like. For  $n$  potential estimates of the true parameter, the best is the one that corresponds to the highest value of the likelihood function based on the observed data  $x$ . Taking this logic to its natural conclusion, the *maximum likelihood estimate* (MLE) of the parameter  $\theta$ , which we will denote by  $\hat{\theta}$  (pronounced “theta hat”), is the one that maximizes

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<sup>4</sup>Here, “close” means that that  $X$  is within a tiny neighborhood of  $x$ .

the value of the likelihood function among all possible choices of  $\theta$  in the parameter space  $\Theta$ . Formally,

$$(2.2.4) \quad \hat{\theta} = \operatorname{argmax}_{\theta \in \Theta} L(\theta; x).$$

There is no singular method for finding the maximum likelihood estimate of a parameter. However, when the likelihood function is differentiable, it is often possible to calculate the MLE analytically using the derivative test for locating the local maxima of a function. In such cases, the MLE can be found by finding the value of  $\theta$  that makes the derivative of the likelihood function with respect to  $\theta$  vanish. A popular technique when finding this value is to take the natural logarithm of the likelihood first. This transformation is common enough that it has its own name - the *log-likelihood function*. Formally, it is defined as

$$(2.2.5) \quad \ell(\theta) \equiv \ell(\theta; x) = \log L(\theta; x), \quad \theta \in \Theta.$$

When working with  $\ell(\theta)$  instead of  $L(\theta)$ , any products in the latter have been transformed into sums in the former, making derivative calculations more tractable while still preserving the argument that corresponds to the global maximum, if it exists, of  $L(\theta)$ .

## CHAPTER 3

## Pseudolikelihood Analysis

## 3.1. Model Parameter Decomposition

## 3.1.1. Introduction

It is often the case that we are not interested in estimating the full parameter  $\theta \in \Theta \subseteq \mathbb{R}^k$ , but rather a different parameter  $\psi$  taking values in a set  $\Psi \subseteq \mathbb{R}^m$ , where  $m < k$ . In such an event, we refer to  $\psi$  as the *parameter of interest*. Since  $\psi$  is of lower dimension than  $\theta$ , it necessarily follows that there is another parameter  $\lambda$ , taking values in a set  $\Lambda \subseteq \mathbb{R}^{k-m}$ , that is made up of whatever is “left over” from the full parameter  $\theta$ .

We refer to  $\lambda$  as the *nuisance parameter* due to its ability to complicate inference regarding the parameter of interest. Despite not being the object of study themselves, nuisance parameters are nevertheless capable of modifying the distributions of our observations and therefore must be accounted for when conducting inference or estimation regarding the parameter of interest. The process by which this is accomplished is often nontrivial and can constitute a significant barrier that must be overcome.

Parameters of interest and nuisance parameters can be broadly classified into two categories, explicit or implicit. For a given statistical model, both types of parameter must occupy the same category - it is not possible for  $\psi$  to be explicit and  $\lambda$  to be implicit, or vice versa. In the following two sections, we will explore each of these cases in more detail.

## 3.1.2. Explicit Parameters

Let us first consider the case in which  $\psi$  and  $\lambda$  are *explicit* parameters. This means that  $\psi$  is a sub-vector of  $\theta$ , so that all the components of  $\psi$  are also components of  $\theta$ . Then there exists a set

$I = \{I_1, \dots, I_m\} \subsetneq \{1, \dots, k\}$  such that

$$(3.1.1) \quad \psi = (\theta_{I_1}, \dots, \theta_{I_m}).$$

It immediately follows that  $\lambda$  is the sub-vector of all components of  $\theta$  that are not part of  $\psi$ . More precisely, if we let  $J = \{J_1, \dots, J_{k-m}\} \subsetneq \{1, \dots, k\}$  such that  $I \cup J = \{1, \dots, k\}$  and  $I \cap J = \emptyset$ , then

$$(3.1.2) \quad \lambda = (\theta_{J_1}, \dots, \theta_{J_{k-m}}).$$

$\theta$  can therefore be decomposed as  $\theta = (\psi, \lambda)$  when  $\psi$  and  $\lambda$  are explicit, provided we shuffle the indices appropriately.

### 3.1.3. Implicit Parameters

Now let us consider the case in which  $\psi$  and  $\lambda$  are *implicit* parameters. This means there exists some function  $\varphi : \Theta \rightarrow \Psi$  for which the parameter of interest can be written as

$$(3.1.3) \quad \psi = \varphi(\theta).$$

As before,  $\Psi$  is still assumed to be a subset of  $\mathbb{R}^m$  where  $m$  is less than  $k$ , the dimension of the full parameter space  $\Theta$ . This reduction in dimension again implies the existence of a nuisance parameter  $\lambda \in \Lambda \subseteq \mathbb{R}^{k-m}$ . However, unlike in the explicit case, a closed form expression for  $\lambda$  in terms of the original components of  $\theta$  need not exist. For this reason, implicit nuisance parameters are in general more difficult to eliminate compared to their explicit counterparts.

Note that when the parameter of interest and nuisance parameter are explicit, it is always possible to define a function  $\varphi$  such that

$$(3.1.4) \quad \varphi(\theta) = (\theta_{I_1}, \dots, \theta_{I_m}) \equiv \psi,$$

where  $\{I_1, \dots, I_m\}$  is defined as above. Hence, the first case is really just a special example of this broader one in which  $\psi = \varphi(\theta)$ . With this understanding in mind, we will use the notation  $\psi = \varphi(\theta)$  to refer to

the parameter of interest in general, only making the distinction between implicitness and explicitness when the difference is relevant to the topic at hand.

## 3.2. Pseudolikelihood Functions

### 3.2.1. Introduction

The natural solution to the hindrance nuisance parameters pose to making inferences on the parameter of interest is to find a method for eliminating them from the model altogether. Since one way of uniquely specifying a model is through its likelihood function, this is equivalent to eliminating the nuisance parameters from the likelihood function itself. The result of this elimination is what is known as a pseudolikelihood function.

In general, a *pseudolikelihood function* for  $\psi$  is defined as being a function of  $\psi$  and the data alone, having properties resembling that of a genuine likelihood function. Suppose  $\psi = \varphi(\theta)$  for some function  $\varphi$  and parameter  $\theta \in \Theta$ . If we let  $\Theta(\psi) = \{\theta \in \Theta : \varphi(\theta) = \psi\}$ , then associated with each  $\psi \in \Psi$  is the set of likelihoods  $\mathcal{L}_\psi = \{L(\theta) : \theta \in \Theta(\psi)\}$ .

Any summary of the values in  $\mathcal{L}_\psi$  that does not depend on  $\lambda$  theoretically constitutes a pseudolikelihood function for  $\psi$ . There exist a variety of methods to obtain this summary but among the most popular are maximization, conditioning, and integration, each with respect to the nuisance parameter. We will explore each of these methods in more detail in the sections to come.

### 3.2.2. The Profile Likelihood

The profile likelihood is the most straightforward method for eliminating a nuisance parameter from a likelihood function.

For example, suppose we are interested in estimating the mean of a random variable  $Y$ , where  $Y \sim N(\mu, \sigma^2)$ . The full model parameter is  $\theta = (\mu, \sigma^2)$  but since we are only interested in estimating the mean, the parameter of interest is  $\psi = \mu$  and the nuisance parameter is  $\lambda = \sigma^2$ .

### 3.2.3. The Conditional Likelihood

### 3.2.4. The Marginal Likelihood

### 3.2.5. The Integrated Likelihood

## 3.3. Asymptotic Properties of Pseudolikelihoods

### 3.3.1. Introduction

### 3.3.2. The Bartlett Identities

The Bartlett identities are a set of equations relating to the expectations of functions of derivatives of a log-likelihood function. A well-specified genuine likelihood function will automatically satisfy each of the Bartlett identities; however, an arbitrary function of  $\theta$  and  $X$  will not. For this reason, the identities act as a litmus test of sorts for determining the validity of a pseudolikelihood as an approximation to the genuine likelihood from which it originated.<sup>1</sup>

Consider the case in which a random variable  $X$  has a probability density  $f$  that depends on a scalar parameter  $\theta$ . Denote the log-likelihood function for  $\theta$  by  $\ell(\theta; x) = \log f(x; \theta)$  and its first derivative with respect to  $\theta$  by  $\ell_\theta(\theta; x) = \frac{\partial}{\partial \theta} \ell(\theta; x)$ . We previously assumed in Section 2.1.4. that all probability distributions for which the results in this paper apply are regular. One consequence of this assumption is that derivatives and integrals of the density functions for these distributions may be interchanged. Now, taking the expectation of  $\ell_\theta(\theta; x)$  gives

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<sup>1</sup>The Bartlett identities offer an alternative way of characterizing the difference between likelihood and pseudolikelihood functions. A genuine likelihood function of  $\theta$  is any nonnegative random function of  $\theta$  for which all of the Bartlett identities hold. A pseudolikelihood of  $\theta$  is any nonnegative random function of  $\theta$  for which at least one of the Bartlett identities does not hold.

$$\begin{aligned}
\mathbb{E}[\ell_\theta(\theta; x)] &= \mathbb{E}\left[\frac{\partial}{\partial\theta}\ell(\theta; x)\right] \\
&= \int_{\mathbb{R}} \left[\frac{\partial}{\partial\theta}\ell(\theta; x)\right] f(x; \theta) dx \\
&= \int_{\mathbb{R}} \left[\frac{\partial}{\partial\theta} \log f(x; \theta)\right] f(x; \theta) dx \\
&= \int_{\mathbb{R}} \frac{\frac{\partial}{\partial\theta} f(x; \theta)}{f(x; \theta)} f(x; \theta) dx \\
&= \int_{\mathbb{R}} \frac{\partial}{\partial\theta} f(x; \theta) dx \\
&= \frac{d}{d\theta} \int_{\mathbb{R}} f(x; \theta) dx && \text{(by regularity of } f\text{)} \\
&= \frac{d}{d\theta} 1 \\
&= 0.
\end{aligned}$$

Therefore,

$$(3.3.1) \quad \mathbb{E}[\ell_\theta(\theta; x)] = 0 \text{ for all } \theta.$$

Equation 3.3.1 is called the first Bartlett identity. In words, it states that the expectation of the first derivative of the log-likelihood function of a statistical model with respect to the model parameter will always be 0. Another name for  $\ell_\theta$  is the *score function*, and any pseudolikelihood that also satisfies the first Bartlett identity is said to be *score-unbiased*.

If we now consider the second derivative of  $\ell(\theta; x)$ , we have



$$\begin{aligned}
\ell_{\theta\theta}(\theta; x) &= \frac{\partial^2}{\partial\theta^2} \ell(\theta; x) \\
&= \frac{\partial}{\partial\theta} \left[ \frac{\partial}{\partial\theta} \ell(\theta; x) \right] \\
&= \frac{\partial}{\partial\theta} \left[ \frac{\partial}{\partial\theta} \log f(x; \theta) \right] \\
&= \frac{\partial}{\partial\theta} \left[ \frac{\frac{\partial}{\partial\theta} f(x; \theta)}{f(x; \theta)} \right] \\
&= \frac{\left[ \frac{\partial^2}{\partial\theta^2} f(x; \theta) \right] f(x; \theta) - \left[ \frac{\partial}{\partial\theta} f(x; \theta) \right] \left[ \frac{\partial}{\partial\theta} f(x; \theta) \right]}{[f(x; \theta)]^2} \\
&= \frac{\frac{\partial^2}{\partial\theta^2} f(x; \theta)}{f(x; \theta)} - \left[ \frac{\frac{\partial}{\partial\theta} f(x; \theta)}{f(x; \theta)} \right]^2 \\
&= \frac{\frac{\partial^2}{\partial\theta^2} f(x; \theta)}{f(x; \theta)} - \left[ \frac{\partial}{\partial\theta} \log f(x; \theta) \right]^2 \\
&= \frac{\frac{\partial^2}{\partial\theta^2} f(x; \theta)}{f(x; \theta)} - [\ell_{\theta}(\theta; x)]^2.
\end{aligned}$$

Rearranging terms and taking expectations yields

$$\begin{aligned}
\mathbb{E}[\ell_{\theta\theta}(\theta; x)] + \mathbb{E}[(\ell_{\theta}(\theta; x))^2] &= \mathbb{E} \left[ \frac{\frac{\partial^2}{\partial\theta^2} f(x; \theta)}{f(x; \theta)} \right] \\
&= \int_{\mathbb{R}} \left[ \frac{\frac{\partial^2}{\partial\theta^2} f(x; \theta)}{f(x; \theta)} \right] f(x; \theta) dx \\
&= \int_{\mathbb{R}} \frac{\partial^2}{\partial\theta^2} f(x; \theta) dx \\
&= \frac{d^2}{d\theta^2} \int_{\mathbb{R}} f(x; \theta) dx \quad (\text{by regularity of } f) \\
&= \frac{d^2}{d\theta^2} 1 \\
&= 0.
\end{aligned}$$

Therefore,

$$(3.3.2) \quad \mathbb{E}[\ell_{\theta\theta}(\theta; x)] + \mathbb{E}[(\ell_{\theta}(\theta; x))^2] = 0 \text{ for all } \theta.$$

Equation 3.3.2 is called the second Bartlett identity. The second term on the left-hand side can be further rewritten as

$$\begin{aligned}\mathbb{E}\left[\left(\ell_{\theta}(\theta; x)\right)^2\right] &= \mathbb{V}[\ell_{\theta}(\theta; x)] + \left(\mathbb{E}[\ell_{\theta}(\theta; x)]\right)^2 \\ &= \mathbb{V}[\ell_{\theta}(\theta; x)].\end{aligned}\quad (\text{by the first Bartlett identity})$$

Another name for this quantity is the *expected information*. It follows from the second Bartlett identity that

$$(3.3.3) \quad \mathbb{E}[-\ell_{\theta\theta}(\theta; x)] = \mathbb{V}[\ell_{\theta}(\theta; x)].$$

The quantity  $-\ell_{\theta\theta}(\theta; x)$  is called the *observed information*. Any pseudolikelihood that satisfies the second Bartlett identity is said to be *information-unbiased*.

It is possible to derive further Bartlett identities by continuing in this manner for an arbitrary number of derivatives of the log-likelihood function, provided that they exist. However, the first two are sufficient for our purposes of evaluating the validity of pseudolikelihoods as approximations to a genuine likelihood so we will not go further. Note that while the above derivations were performed under the assumption that  $\theta$  is a scalar, the Bartlett identities also hold in the case where  $\theta$  is a multi-dimensional vector.

### 3.3.3. Single-Index Asymptotic Theory

Single-index asymptotic theory describes the behavior of pseudolikelihood functions as the sample size ( $n$ ) grows to infinity while the dimension of the nuisance parameter ( $m$ ) remains fixed.

Suppose we have a

### 3.3.4. Two-Index Asymptotics

## CHAPTER 4

## Approximating the Integrated Likelihood Function

### 4.1. The Zero-Score Expectation Parameter

Let  $\psi = \varphi(\theta)$  and  $\lambda$  denote the parameter of interest and nuisance parameter, respectively, for some statistical model  $(\mathcal{S}, \mathcal{P}_\theta)$ . Then the general expression to obtain an integrated likelihood for  $\psi$  may be written as

$$(4.1.1) \quad \bar{L}(\psi) = \int_{\Lambda} L(\psi, \lambda) \pi(\lambda|\psi) d\lambda,$$

where  $\pi(\lambda|\psi)$  is a conditional prior density for  $\lambda$  given  $\psi$ .

Severini (2007) considered the problem of selecting  $\pi(\lambda|\psi)$  such that when the likelihood function is integrated with respect to this density, the result is useful for non-Bayesian inference. To do this, he outlined four properties (see Appendix A) that an integrated likelihood function must satisfy if it is to be of any use. He went on to prove that an integrated likelihood satisfying these properties could be obtained by first constructing a new nuisance parameter  $\phi \in \Phi$  that is unrelated to the parameter of interest (in the sense that its maximum likelihood estimator remains roughly constant for all values of  $\psi$ ) and then choosing a prior density  $\pi(\phi)$  that is independent of  $\psi$ . Once chosen, the desired integrated likelihood function for  $\psi$  is given by

$$(4.1.2) \quad \bar{L}(\psi) = \int_{\Phi} \tilde{L}(\psi, \phi) \pi(\phi) d\phi,$$

where  $\tilde{L}(\psi, \phi)$  is the likelihood function for the model after it has been reparameterized in terms of  $\phi$ . It is important to note that the exact choice of prior density for  $\phi$  is not particularly important; the only restriction we place upon it is that it must not depend on  $\psi$ .

Suppose that we have an explicit parameter of interest and nuisance parameter, so that  $\theta = (\psi, \lambda)$ . Then Severini (2007) defines this new nuisance parameter  $\phi$  as the solution to the equation

$$(4.1.3) \quad \mathbb{E}(\ell_\lambda(\psi, \lambda); \psi_0, \lambda_0) \Big|_{(\psi_0, \lambda_0) = (\hat{\psi}, \phi)} = 0,$$

where  $\ell_\lambda(\psi, \lambda) = \frac{\partial \ell(\psi, \lambda)}{\partial \lambda}$ ,  $\psi_0$  and  $\lambda_0$  denote the true values of  $\psi$  and  $\lambda$ , and  $\hat{\psi}$  is the MLE for  $\psi_0$ . In other words, for a particular value of  $(\psi, \lambda, \hat{\psi})$ , we can find the corresponding value of  $\phi$  by solving for it in Equation 4.1.3.  $\phi$  is called the *zero-score expectation* (ZSE) parameter because it is defined as the value that makes the expectation of the score function (where the derivative is taken with respect to  $\lambda$ ) evaluated at the point  $(\hat{\psi}, \phi)$  equal to zero.

Note that  $\phi$  is a function of the data through  $\hat{\psi}$ . Normally we avoid creating such dependencies in our parameters as it renders them useless for the purpose of parameterizing a statistical model. However, from the perspective of the likelihood function, once the data have been collected they are considered fixed in place and there is no issue with using a quantity such as  $\phi$  that depends on the data to parameterize it.

For a given value of  $(\psi, \phi, \hat{\psi})$ , it is also possible to solve Equation 4.1.3 for  $\lambda$ . This allows us to write Equation 4.1.2 in terms of  $L(\psi, \lambda)$ :

$$(4.1.4) \quad \bar{L}(\psi) = \int_{\Phi} L(\psi, \lambda(\psi, \phi)) \pi(\phi) d\phi.$$

Severini (2018) proved that reparameterizing the nuisance parameter in terms of the ZSE parameter yields the same desirable properties in the subsequent integrated likelihood when  $\psi$  and  $\lambda$  are implicit. Suppose  $\psi = \varphi(\theta)$ , for some function  $\varphi : \Theta \rightarrow \Psi$ , and consider the set of all values of  $\theta$  satisfying  $\varphi(\theta) = \hat{\psi}$ . Call this set  $\Omega_{\hat{\psi}}$  so that

$$(4.1.5) \quad \Omega_{\hat{\psi}} = \{\omega \in \Theta : \varphi(\omega) = \hat{\psi}\}.$$

Elements of  $\Omega_{\hat{\psi}}$  take the form  $(\hat{\psi}, \phi)$ , where  $\phi \in \Lambda$

#### 4.2. Markov Chain Monte Carlo

#### 4.3. The IL Algorithm

## CHAPTER 5

**Applications****5.1. Multinomial Distribution**

Severini (2018) provides an example of the application

**5.2. Standardized Mean Difference**

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## APPENDIX A

**Chapter 3****A.1. Desirable Properties of the Integrated Likelihood****A.1.1. Property 1**

Suppose the likelihood function for a parameter  $\theta$  can be decomposed as the product  $L(\theta) = L_1(\psi)L_2(\lambda)$ . Then the integrated likelihood for  $\psi$  should satisfy

$$\bar{L}(\psi) = L_1(\psi).$$

**A.1.2. Property 2****A.1.3. Property 3****A.1.4. Property 4**

## APPENDIX B

**Chapter 4**