

An Algorithm for Approximating Integrated Likelihood Functions with Applications in
Meta-Analysis

By

Timothy Ruel

Faculty Committee:

Dr. Thomas Severini, Department of Statistics and Data Science

Dr. Elizabeth Tipton, Department of Statistics and Data Science

Dr. Wenxin Jiang, Department of Statistics and Data Science

A Doctoral Dissertation Prospectus Submitted to the Department of Statistics
and Data Science

Northwestern University

Evanston, Illinois

September 18, 2023

Abstract

Table of Contents

Abstract	2
Preface	4
Introduction	4
Motivation	4
Background	5
Assumptions	5

Preface

Introduction

The research for my dissertation involves developing a novel algorithm for numerically integrating the likelihood function of a statistical model with respect to a nuisance parameter. This prospectus aims to demonstrate how the algorithm works and explain the appeal of using an integrated likelihood function over other types of pseudolikelihood functions to make inferences about the parameter of interest in a model.

Motivation

The motivation behind my research developed out of the observation that the expression for the integral of a likelihood function follows a form similar to that of the calculation of the marginalizing constant of a posterior distribution.

Background

Assumptions

Consider a random sample $\mathbf{x} = (x_1, \dots, x_n)$ drawn from a population. What can we say about the population based on \mathbf{x} ? Where is its point of central tendency located? Are its values clustered tightly around this point, or are they more diffuse? Are they distributed symmetrically or skewed to one side or the other? Questions like these were the original motivation behind the field of statistical inference, and many of the techniques devised to answer them are still used by statisticians today.

It is important to remember, however, that the real world is messy and no mathematical function will ever perfectly capture the complexities of a population or random process whose properties we wish to infer. To overcome this difficulty, statisticians sacrifice a small amount of accuracy for (hopefully) a large reduction in complexity by imposing additional assumptions on the population of interest. These assumptions are essentially never true in the sense that they are not a flawless representation of reality, but they may nevertheless serve as convenient approximations capable of producing sufficiently accurate answers in their own right. As George Box famously put it, “All models are wrong, but some are useful.”

And indeed, in the aggregate these assumptions create what is known as a statistical model. In its most general framework, a statistical model can be formulated as a tuple $(\mathcal{S}, \mathcal{P})$ where \mathcal{S} is the set of all possible observations (i.e. the population), and \mathcal{P} is a set of probability distributions on \mathcal{S} . The first and (in this author’s opinion) the most fundamental assumption we make when defining our models is that there exists some unknown mechanism in the population that generates the data we observe from \mathcal{S} . This mechanism is what induces the “true” probability distribution on \mathcal{S} though \mathcal{P} need not contain this distribution, and in practice it seldom does.

Another assumption found in almost every model is that the set \mathcal{P} is considered to be *parameterized*. That is, we assume the probability distributions contained in \mathcal{P} are indexed by a *parameter*¹ that controls their features. This parameter acts like a tuning dial for the population - rotate the dial and certain behaviors of the population (e.g. its location, scale, or shape) will change. Much of statistical inference can be boiled down to figuring out the particular value to which a population's dial has been set. Mathematically, we represent this assumption as $\mathcal{P} = \{\mathcal{P}_\theta \mid \theta \in \Theta\}$, where θ ² denotes the parameter, and Θ , the parameter space, represents the set of all possible values θ can take on.

Statisticians also like to assume the parameters in their models can be uniquely identified based on the data they observe. A model is considered *identifiable* if having perfect knowledge of the population³ enables us to determine the true value of its parameter with absolute certainty. More formally, for any two parameters θ_1 and θ_2 in Θ , if $\mathcal{P}_{\theta_1} = \mathcal{P}_{\theta_2}$, then it must follow that $\theta_1 = \theta_2$. A model that is not identifiable could potentially have two or more distinct parameter values that give rise to the same probability distribution. Since we have already assumed \mathcal{P} is the mechanism generating the data we have observed in the first place, this would make it impossible to determine which value is the “correct” one on the basis of the data alone. Statisticians impose the identifiability criterion on our models as a means of avoiding this undesirable situation.

The dimension of the parameter space Θ is another critical decision statisticians must make when choosing the best model for their research. The most frequent choice is for Θ to be of finite dimension. That is, $\Theta \subseteq \mathbb{R}^k$, where $k \in \mathbb{Z}^+$. Models that satisfy this assumption are said to be *parametric*. Common examples of parametric models include the normal family of distributions as well as the Poisson family.

In contrast, a *nonparametric model* is one in which $\Theta \subseteq I$, where I is an infinite-dimensional space. The name is a bit of a misnomer in the sense that nonparametric models do not actually lack parameters, but rather they are flexible regarding the exact number and properties of the parameters they do have.

¹We will consider the phrases “parameter”, “population parameter”, and “model parameter” all to have the same meaning in this paper and use them interchangeably.

²Note that θ can be and in fact usually is a multi-dimensional vector whose components represent various sub-parameters of the population.

³This is of course almost always impossible in practice, but in theory it could be accomplished by obtaining an infinite number of observations from \mathcal{S} or simply all of its observations if $|\mathcal{S}|$ is finite.

Finally, *semiparametric* models are those whose parameter spaces have components of both finite and infinite dimensionality. That is, $\Theta \subseteq \mathbb{R}^k \times I$, where again I is an infinite-dimensional space. Usually it is only the finite-dimensional component of the parameter in which we are interested while the infinite-dimensional component is considered a nuisance parameter.

In summary, the reader may safely assume that all models to which my research applies consist of a pair $(\mathcal{S}, \mathcal{P})$ such that the probability distributions in \mathcal{P} are parameterized and identifiable and there exists a “true” probability distribution inducing the data-generating process from \mathcal{S} , though this distribution is not necessarily contained in \mathcal{P} . In addition, while estimation of the parameters in nonparametric and semiparametric models is also a major topic of interest in statistical inference, I will restrict my attention in my dissertation solely to parametric models.