MATRIX PRODUCT STATES, GEOMETRY, AND INVARIANT THEORY

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Abstract. Matrix product states play an important role in quantum information theory to represent states of many-body systems. They can be seen as low-dimensional subvarieties of a high-dimensional tensor space. In these notes, we consider two variants: homogeneous matrix product states and uniform matrix product states. Studying the linear spans of these varieties leads to a natural connection with invariant theory of matrices. For homogeneous matrix product states, a classical result on polynomial identities of matrices leads to a formula for the dimension of the linear span, in the case of 2x2 matrices.

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1. Invariant theory of matrices

In this section we give an introduction to the invariant theory of matrices. The exposition is based on the first chapters of [DCP17]. We work over the field of complex numbers.

1.1. **The invariant ring of a group action.** Let V be a finite-dimensional representation of a group G. Then the ring

$$\mathbb{C}[V] = S^\bullet(V^*)$$

of polynomial functions on V comes with an action of G, given by

$$(g \cdot f)(v) = f(g^{-1} \cdot v)$$

for $f \in \mathbb{C}[V]$, $g \in G$, and $v \in V$. The *invariant ring* is defined as

$$\mathbb{C}[V]^G = \{f \in \mathbb{C}[V] \mid g \cdot f = f \quad \forall g \in G\}.$$

A central question in invariant theory is to, for a given G and V, describe this ring, i.e.

(1) Find generators for $\mathbb{C}[V]^G$,

(2) Find the (polynomial) relations between these generators.

Example 1.1. Let $\mathfrak{S}_d$ be the symmetric group of order $d!$, acting on $V = \mathbb{C}^d$ by permuting the coordinates. The invariant ring is the ring

$$\mathbb{C}[x_1, \ldots, x_d]^{\mathfrak{S}_d}$$

of symmetric polynomials. It is known that this ring is generated by the elementary symmetric polynomials $e_1, \dots, e_d$, where

$$e_k := \sum_{i_1 < \dots < i_k} x_{i_1} \cdots x_{i_k}.$$

Furthermore, the e_i are algebraically independent. In other words, the map

$$\begin{aligned} \mathbb{C}[y_1, \dots, y_d] &\xrightarrow{\cong} \mathbb{C}[x_1, \dots, x_d]^{\mathfrak{S}_d} \\ y_i &\mapsto e_i \end{aligned}$$

is an isomorphism of $\mathbb{C}$-algebras.

1.2. **The invariant ring of matrices.** For the rest of this section, we will be concerned with one particular group action. We fix natural numbers m and n, and take $G = GL_m(\mathbb{C})$ and $V = \mathrm{Mat}(m \times m, \mathbb{C})^n$ the space of n-tuples of $m \times m$ matrices. The action of G on V is given by simultaneous conjugation:

$$g \cdot (A_1, \dots, A_n) = (gA_1g^{-1}, \dots, gA_ng^{-1}).$$

We will write

$$\mathbb{C}[V] = \mathbb{C}[x_{ij}^k \mid 1 \leq i, j \leq m; 1 \leq k \leq n] = \mathbb{C}[X_1, \dots, X_n],$$

where $X_k = (x_{ij}^k)_{ij}$ is an $m \times m$ matrix with generic entries. I.e. when we write $f(X_1, \dots, X_n) \in \mathbb{C}[X_1, \dots, X_n]$, this means a polynomial in nm^2 variables. The invariant ring is given by

$$\mathbb{C}[X_1, \dots, X_n]^{GL_m} = \{f \mid f(gX_1g^{-1}, \dots, gX_ng^{-1}) = f(X_1, \dots, X_n) \quad \forall g \in GL_m\}.$$

Here the notation $f(gX_1g^{-1}, \dots, gX_ng^{-1})$ means that we substitute the variable x_{ij}^k with the ij-th entry of the matrix gX_kg^{-1}.

Note that $\mathbb{C}[X_1, \dots, X_n]$ admits an $\mathbb{N}^n$ grading, by putting $\deg(x_{ij}^k)$ equal to the k-th basis vector $e_k = (0, \dots, 1, \dots, 0) \in \mathbb{N}^n$. It is not hard to see that a polynomial is invariant if and only if its graded parts are invariant; hence the invariant ring $\mathbb{C}[X_1, \dots, X_n]^{GL_m}$ inherits the aforementioned grading. The trick to describing this invariant ring is to start with its multilinear part

$$\mathbb{C}[X_1, \dots, X_n]^{GL_m}_{(1,\dots,1)}.$$

1.3. **The multilinear part.** The key ingredient is Schur-Weyl duality, a fundamental result in representation theory. We briefly state it here, and refer the reader to e.g. [FH91, Chapter 6] for a more in-depth discussion.

Theorem 1.2. *Consider the vector space $(\mathbb{C}^m)^{\otimes n}$, equipped with the following actions of the general linear group GL_m and the symmetric group $\mathfrak{S}_n$:*

$$g \cdot (v_1 \otimes \cdots \otimes v_n) = (g \cdot v_1) \otimes \cdots \otimes (g \cdot v_n), \tag{1}$$

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}. \tag{2}$$

These actions commute, meaning that we get a ring homomorphism

$$\mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}_{GL_m}((\mathbb{C}^m)^{\otimes n}). \tag{3}$$

This homomorphism is surjective. If $n \leq m$, it is an isomorphism. If $n > m$, its kernel is given by the two-sided ideal generated by the antisymmetrizer

$$c_{m+1} := \sum_{\sigma \in \mathfrak{S}_{m+1} \subseteq \mathfrak{S}_n} \mathrm{sgn}(\sigma) \cdot \sigma.$$

We will denote this ideal by $I_{m,n}$.

The theorem below describes the multilinear part of our invariant ring as a quotient of the symmetric group algebra $\mathbb{C}[\mathfrak{S}_n]$. We will write elements in $\mathfrak{S}_n$ by cycle notation.

Theorem 1.3. *There is a surjective linear map*

$$\begin{aligned} \mathbb{C}[\mathfrak{S}_n] &\twoheadrightarrow \mathbb{C}[X_1, \dots, X_n]^{GL_m}_{(1,\dots,1)} \\ (i_1 \dots i_k) \cdots (j_1 \dots j_\ell) &\mapsto \operatorname{Tr}(X_{i_1} \cdots X_{i_k}) \cdots \operatorname{Tr}(X_{j_1} \cdots X_{j_\ell}). \end{aligned}$$

If $n \leq m$, *it is an isomorphism. If* $n > m$, *its kernel is given by the two-sided ideal* $I_{m,n}$ *described above.*

Proof sketch. Note that

$$\mathbb{C}[X_1, \dots, X_n]_{(1,\dots,1)} = S^\bullet \left(\operatorname{End}(\mathbb{C}^m)^{\oplus n}\right)^*_{(1,\dots,1)} \cong \left(\operatorname{End}(\mathbb{C}^m)^{\otimes n}\right)^* \cong \operatorname{End}\left((\mathbb{C}^m)^{\otimes n}\right),$$

compatible with the GL_m-actions on both sides. But the ring

$$\operatorname{End}\left((\mathbb{C}^m)^{\otimes n}\right)^{GL_m} = \operatorname{End}_{GL_m}\left((\mathbb{C}^m)^{\otimes n}\right)$$

is described by Schur-Weyl-duality. □

Remark 1.4. Recall the decomposition of $\mathbb{C}[\mathfrak{S}_n]$ into isotypic components:

$$\mathbb{C}[\mathfrak{S}_n] \cong \bigoplus_{\lambda \vdash n} I_\lambda,$$

compatible with both the left and right action of $\mathfrak{S}_n$. Then $I_{m,n}$ is the sum of all I_λ, where λ is a Young diagram with at least $m+1$ rows.

1.4. **The other graded parts.** Fix $(d_1, \dots, d_n)$ and write $d = \sum d_i$. Our goal is to describe the degree $(d_1, \dots, d_n)$ part of our invariant ring. The main idea is that we can use *polarization* to reduce this to the multilinear part.

More precisely: consider the ring $\mathbb{C}[X_1^{(1)}, \dots, X_1^{(d_1)}, \dots, X_n^{(1)}, \dots, X_n^{(d_n)}]$, which is just the ring $\mathbb{C}[X_1, \dots, X_d]$ where we relabeled the $d = \sum d_i$ matrices in a suggestive way. We equip it with the action of the group $\mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_n}$, where $\mathfrak{S}_{d_i}$ acts by permuting the (entries of the) matrices $X_i^{(1)}, \dots, X_i^{(d_i)}$.

Lemma 1.5. *We have an isomorphism of rings*

$$\begin{aligned} \mathbb{C}[X_1^{(1)}, \dots, X_1^{(d_1)}, \dots, X_n^{(1)}, \dots, X_n^{(d_n)}]^{\mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_n}}_{(1,\dots,1)} &\cong \mathbb{C}[X_1, \dots, X_n]_{(d_1,\dots,d_n)} \\ f(X_1^{(1)}, \dots, X_1^{(d_1)}, \dots, X_n^{(1)}, \dots, X_n^{(d_n)}) &\mapsto f(X_1, \dots, X_1, \dots, X_n, \dots, X_n). \end{aligned}$$

This isomorphism is compatible with the GL_m*-action.*

Proof. The inverse map is given by polarization: given $g(X_1, \dots, X_n) \in \mathbb{C}[X_1, \dots, X_n]_{(d_1,\dots,d_n)}$, first substitute each X_i for the sum $\sum_{j=1}^{d_i} X_i^{(j)}$, then take the multilinear part of the obtained polynomial. We leave it to the reader to check that, up to a nonzero scalar, this does define an inverse to the map above. □

Before we can state the main result of this section, we need some additional notation. Let $\mathfrak{S}_d$ be the group of permutations of the set

$$\{(1,1),\ldots,(1,d_1),\ldots,(n,1),\ldots(n,d_n)\}.$$

Let us now consider the subalgebra

$$\mathbb{C}[\mathfrak{S}_d]^{\mathfrak{S}_{d_1}\times\ldots\times\mathfrak{S}_{d_n}} \subseteq \mathbb{C}[\mathfrak{S}_d]$$

where the action of $\mathfrak{S}_{d_1}\times\ldots\times\mathfrak{S}_{d_n} \subset \mathfrak{S}_d$ on $\mathfrak{S}_d$ is given by conjugation. We will write elements of $\mathbb{C}[\mathfrak{S}_d]^{\mathfrak{S}_{d_1}\times\ldots\times\mathfrak{S}_{d_n}}$ as (linear combinations of) "pseudo-permutations" $(i_1\ldots i_k)\ldots(j_1\ldots j_\ell)$, where the number i appears d_i times. By definition, $(i_1\ldots i_k)\ldots(j_1\ldots j_\ell)$ is the image of any element of the form

$$((i_1,\cdot)\ldots(i_k,\cdot))\ldots((j_1,\cdot)\ldots(j_\ell,\cdot))$$

under the symmetrization map.

$$\mathbb{C}[\mathfrak{S}_d] \twoheadrightarrow \mathbb{C}[\mathfrak{S}_d]^{\mathfrak{S}_{d_1}\times\ldots\times\mathfrak{S}_{d_n}}$$

Theorem 1.6. *There is a surjective linear map*

$$\begin{aligned} \mathbb{C}[\mathfrak{S}_d]^{\mathfrak{S}_{d_1}\times\ldots\times\mathfrak{S}_{d_n}} &\twoheadrightarrow \mathbb{C}[X_1,\ldots,X_n]^{GL_m}_{(d_1,\ldots,d_n)} \\ (i_1\ldots i_k)\cdots(j_1\ldots j_\ell) &\mapsto \operatorname{Tr}(X_{i_1}\cdots X_{i_k})\cdots\operatorname{Tr}(X_{j_1}\cdots X_{j_\ell}), \end{aligned} \tag{4}$$

whose kernel is given by

$$I_{m,d}\cap\mathbb{C}[\mathfrak{S}_d]^{\mathfrak{S}_{d_1}\times\ldots\times\mathfrak{S}_{d_n}}.$$

Here the action of $\mathfrak{S}_{d_1}\times\ldots\times\mathfrak{S}_{d_n} \subset \mathfrak{S}_d$ on $\mathfrak{S}_d$ is given by conjugation.

Proof sketch. Combine Lemma 1.5 and Theorem 1.3, noting that in the map

$$\mathbb{C}[\mathfrak{S}_d] \twoheadrightarrow \mathbb{C}[X_1^{(1)},\ldots,X_1^{(d_1)},\ldots,X_n^{(1)},\ldots,X_n^{(d_n)}]_{(1,\ldots,1)},$$

is compatible with the described actions of $\mathfrak{S}_{d_1}\times\ldots\times\mathfrak{S}_{d_n} \subset \mathfrak{S}_d$ (conjugation in $\mathfrak{S}_d$ on the left hand side; permuting matrices on the right hand side). □

From Theorem 1.6 one can deduce that our invariant ring $\mathbb{C}[X_1,\ldots,X_n]^{GL_m}$ is (as a $\mathbb{C}$-algebra) generated by polynomials of the form $\operatorname{Tr}(X_{i_1}\cdots X_{i_k})$. For this reason, it is also called the *trace algebra*. In the future, we will use the notation $R_{m,n} := \mathbb{C}[X_1,\ldots,X_n]^{GL_m}$.

1.5. **Trace relations.** Elements in the kernel of (4) are known as *trace relations*. They describe the polynomial relations that hold between our generators $\operatorname{Tr}(X_{i_1}\cdots X_{i_k})$. Let us illustrate this by an example: take $m=2$ and $n=3$, and consider the degree $(1,1,1)$ part of the invariant ring $R_{2,3}=\mathbb{C}[X_1,X_2,X_3]^{GL_2}$. By Theorem 1.3 this space is spanned by the 6 polynomials

$$\begin{array}{lll} \operatorname{Tr}(X_1)\operatorname{Tr}(X_2)\operatorname{Tr}(X_3), & \operatorname{Tr}(X_1X_2)\operatorname{Tr}(X_3), & \operatorname{Tr}(X_1X_3)\operatorname{Tr}(X_2), \\ \operatorname{Tr}(X_2X_3)\operatorname{Tr}(X_1), & \operatorname{Tr}(X_1X_2X_3), & \operatorname{Tr}(X_1X_3X_2). \end{array}$$

The ideal $I_{2,3}$ turns out to be just the linear span of $c_3=\sum_{\sigma\in\mathfrak{S}_3}\operatorname{sgn}(\sigma)\cdot\sigma$, which corresponds to the trace relation

$$\begin{aligned} &\operatorname{Tr}(X_1)\operatorname{Tr}(X_2)\operatorname{Tr}(X_3)-\operatorname{Tr}(X_1X_2)\operatorname{Tr}(X_3)-\operatorname{Tr}(X_1X_3)\operatorname{Tr}(X_2) \\ &-\operatorname{Tr}(X_2X_3)\operatorname{Tr}(X_1)+\operatorname{Tr}(X_1X_2X_3)+\operatorname{Tr}(X_1X_3X_2)=0. \end{aligned}$$

The reader is invited to verify that the above expression really yields zero, for any 2×2-matrices X_1,X_2,X_3.

2. Polynomial identities of matrices

Definition 2.1. Let $\mathbb{C}\langle x_1,\ldots,x_n\rangle$ be the noncommutative polynomial ring in n variables. An element $f\in\mathbb{C}\langle x_1,\ldots,x_n\rangle$ is a *polynomial identity for $m\times m$ matrices* if it vanishes when we substitute $x_1,\ldots,x_n$ by generic $m\times m$ matrices $X_1,\ldots,X_n$. The quotient of $\mathbb{C}\langle x_1,\ldots,x_n\rangle$ by the ideal of polynomial identities is called the *ring of generic $m\times m$ matrices*. We will denote this ring by $\mathcal{M}_{m,n}$.

The ring of generic matrices is an important example of a *Polynomial Identity (PI) ring*. Both PI-rings and general and rings of generic matrices in particular have been intensively studied since the 80's. We here present just a few concepts and results that will be relevant later. For more, the reader can consult [DF04].

2.1. **The Hilbert series.**
To quantify the number of polynomial identities, one can use the *Hilbert series* of $\mathcal{M}_{m,n}$. Recall that the Hilbert series of a multigraded vector space R is given by

$$H(R;t_1,\ldots,t_n)=\sum_{(d_1,\ldots,d_n)\in\mathbb{N}^n}\dim R_{(d_1,\ldots,d_n)}t_1^{d_1}\cdots t_n^{d_n}.$$

The Hilbert series of $\mathbb{C}\langle x_1,\ldots,x_n\rangle$ is given by

$$H(\mathbb{C}\langle x_1,\ldots,x_n\rangle;t_1,\ldots,t_n)=\frac{1}{1-t_1-\cdots-t_n}.$$

If we also know the Hilbert series of $\mathcal{M}_{m,n}$, we can compute the dimension of the space of polynomial identities of multidegree $(d_1,\ldots,d_n)$ as the coefficient of $t_1^{d_1}\cdots t_n^{d_n}$ in the difference

$$\frac{1}{1-t_1-\cdots-t_n}-H(\mathcal{M}_{m,n};t_1,\ldots,t_n).$$

Similar to the trace relations in the previous sections, one can use polarization to show it is sufficient to restrict to the multilinear part. Note that $\mathcal{M}_{m,n}$ is a quotient of

$$\mathbb{C}\langle x_1,\ldots,x_n\rangle\cong\bigoplus_{d\in\mathbb{N}}(\mathbb{C}^n)^{\otimes d}$$

and as such a representation of GL_n. Let us denote the degree $(1,\ldots,1)$ part of $\mathcal{M}_{m,d}$ by $P_{m,d}$. This is a quotient of

$$\mathbb{C}\langle x_1,\ldots,x_d\rangle_{(1,\ldots,1)},$$

which is equipped with a left $\mathfrak{S}_d$-action via

$$\sigma\cdot(x_{i_1}\cdots x_{i_d})=x_{\sigma(i_1)}\cdots x_{\sigma(i_d)}.$$

Then we have the following theorem, which was independently proved by Berele [Ber82, Theorem 2.7] and Drensky ([Dre81, Remark 1.5] and [Dre84, Lemma 2.3]). The present formulation is based on [DF04, Theorem 2.3.4].

Theorem 2.2. *Suppose the decomposition of $P_{m,d}$ into $\mathfrak{S}_d$-irreps is given by*

$$P_{m,d}=\bigoplus_{\lambda\vdash d}V_\lambda^{\oplus a_\lambda}.$$

Then the decomposition of $\mathcal{M}_{m,n}$ into GL_n-irreps is given by

$$\mathcal{M}_{m,n}=\bigoplus_{d\in\mathbb{N}}\bigoplus_{\substack{\lambda\vdash d\\ \mathrm{len}(\lambda)\le n}}\mathbb{S}^\lambda(\mathbb{C}^n)^{\oplus a_\lambda}.$$

Here for $\lambda \vdash d$ a partition, V_λ denotes the associated Specht module, and $\mathbb{S}^\lambda$ the associated Schur functor.

Corollary 2.3. *The Hilbert series of $\mathcal{M}_{m,n}$ is given by*

$$H(\mathcal{M}_{m,n}; t_1, \dots, t_n) = \sum_{d \in \mathbb{N}} \sum_{\substack{\lambda \vdash d \\ \mathrm{len}(\lambda) \leq n}} a_\lambda s_\lambda(t_1, \dots, t_n),$$

where a_λ are the coefficients from Theorem 2.2, and s_λ denotes the Schur polynomial.

In general, it is a hard task to compute the multiplicities a_λ. However, in these case of 2×2 matrices ($m = 2$) a formula is known [Dre84, For84]:

Theorem 2.4. *For $m = 2$, the coefficients a_λ above are 0 whenever $\mathrm{len}(\lambda) > 4$. If $\mathrm{len}(\lambda) \leq 4$, then a_λ is given by the table below.*

λ	a_λ
$(\lambda_1, 0, 0, 0)$	1
$(\lambda_1, 1, 0, 0)$	λ_1
$(\lambda_1, \lambda_2, 0, 0)$ *with* $\lambda_2 > 1$	$(\lambda_1 - \lambda_2 + 1)\lambda_2$
$(\lambda_1, 1, 1, 0)$	$2\lambda_1 - 1$
$(\lambda_1, 1, 1, 1)$	$\lambda_1 - 1$
$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *not in the cases above*	$(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)$

Combining Theorem 2.4 with Corollary 2.3 yields the Hilbert series of $\mathcal{M}_{2,n}$, for any n. For fixed small n, the obtained expression can be simplified significantly. We present it here in the case $n = 2$.

Theorem 2.5 ([For84])**.** *The Hilbert series of the ring two generic 2×2-matrices is given by*

$$H(\mathcal{M}_{2,2}; t_1, t_2) = \frac{1 - t_1 - t_2 + t_1 t_2 + t_1^2 t_2 + t_1 t_2^2 - t_1^2 t_2^2}{(1 - t_1)^2 (1 - t_2)^2 (1 - t_2 t_2)}.$$

2.2. **Relation to the trace algebra.**

Definition 2.6. By $R^{\mathrm{cyc}}_{m,n} \subseteq R_{m,n}$ we mean the linear subspace spanned by elements of the form $\mathrm{Tr}(g(X_1, \dots, X_n))$, where $g(x_1, \dots, x_n) \in \mathbb{C}\langle x_1, \dots, x_n \rangle$. Furthermore, let $\mathbb{C}[\mathfrak{S}_d]_{\mathrm{cyc}} \subseteq \mathbb{C}[\mathfrak{S}_d]$ be the subspace spanned by permutations consisting of a single cycle of length d.

Note that (4) restricts to a surjective map

$$\begin{aligned} \mathbb{C}[\mathfrak{S}_d]_{\mathrm{cyc}}^{\mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_n}} &\twoheadrightarrow R^{\mathrm{cyc}}_{m,n;(d_1,\dots,d_n)} \\ (i_1 \dots i_d) &\mapsto \mathrm{Tr}(X_{i_1} \cdots X_{i_d}) \end{aligned} \tag{5}$$

whose kernel is given by

$$I_{m,d} \cap \mathbb{C}[\mathfrak{S}_d]_{\mathrm{cyc}}^{\mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_n}}. \tag{6}$$

Example 2.7. $R^{\mathrm{cyc}}_{2,3;(1,1,1)}$ is the linear span of $\mathrm{Tr}(X_1 X_2 X_3)$ and $\mathrm{Tr}(X_1 X_3 X_2)$, and $\mathbb{C}[\mathfrak{S}_d]_{\mathrm{cyc}}$ is the linear span of (123) and (132). In this case, the map (5) is an isomorphism.

Consider the linear map

$$\begin{aligned} T : \mathbb{C}\langle x_1, \dots, x_n\rangle &\to \mathbb{C}[X_1, \dots, X_n, X_{n+1}] \\ f(x_1, \dots, x_n) &\mapsto \mathrm{Tr}(f(X_1, \dots, X_n) \cdot X_{n+1}) \end{aligned}$$

Note that this actually lands in the invariant ring $\mathbb{C}[X_1, \dots, X_n, X_{n+1}]^{GL_m} = R_{m,n+1}$. Moreover, one verifies that

- the kernel of T is the ideal of polynomial identities.
- the image of T is the subspace of $R^{\mathrm{cyc}}_{m,n+1}$ consisting of all trace polynomials that have degree 1 the variable X_{n+1}.

In other words, T induces isomorphisms

$$\mathcal{M}_{m,n;(d_1,\dots,d_n)} \cong R^{\mathrm{cyc}}_{m,n+1;(d_1,\dots,d_n,1)}. \tag{7}$$

Remark 2.8. Combining (7) and (5), we get a surjective linear map

$$\begin{aligned} \mathbb{C}[\mathfrak{S}_{d+1}]_{\mathrm{cyc}}^{\mathfrak{S}_{d_1}\times\dots\times\mathfrak{S}_{d_n}\times\mathfrak{S}_1} &\twoheadrightarrow \mathcal{M}_{m,d;(d_1,\dots,d_n)} \\ (i_1, \dots, i_n, d+1) &\mapsto X_{i_1} \cdots X_{i_n} \end{aligned}$$

whose kernel is given by

$$I_{m,d+1} \cap \mathbb{C}[\mathfrak{S}_{d+1}]_{\mathrm{cyc}}^{\mathfrak{S}_{d_1}\times\dots\times\mathfrak{S}_{d_n}\times\mathfrak{S}_1}. \tag{8}$$

In particular, the space of multilinear polynomial identities in d matrices can be identified with the intersection

$$I_{m,d+1} \cap \mathbb{C}[\mathfrak{S}_{d+1}]_{\mathrm{cyc}}.$$

In other words, we have

$$P_{m,d} \cong \frac{\mathbb{C}[\mathfrak{S}_{d+1}]_{\mathrm{cyc}}}{I_{m,d+1} \cap \mathbb{C}[\mathfrak{S}_{d+1}]_{\mathrm{cyc}}}. \tag{9}$$

Recall that $P_{m,d}$ came equipped with an $\mathfrak{S}_d$-action. Under the isomorphism (9), this action corresponds to the action of $\mathfrak{S}_d \subset \mathfrak{S}_{d+1}$ on $\mathbb{C}[\mathfrak{S}_{d+1}]$ by conjugation.

3. Matrix product states

Matrix product states arise in the context of quantum many-body systems, where they are used to model ground states of certain systems. We will be concerned with two variants which model translation-invariant systems. For the purpose of these notes, we think about them as low-dimensional subvarieties of a high-dimensional tensor space. For more on matrix product states and their geometry, we refer to [PGVWC07, HMOV14, CM14, CMS23].

Definition 3.1. The variety uMPS(m, n, d) of *uniform matrix product states* is the closed image of the map

$$\begin{aligned} (\mathbb{C}^{m\times m})^n &\to (\mathbb{C}^n)^{\otimes d} \\ (A_1, \dots, A_n) &\mapsto \sum_{1\le i_1,\dots,i_d\le n} \mathrm{Tr}(A_{i_1} \cdots A_{i_d})\, e_{i_1} \otimes \cdots \otimes e_{i_d}. \end{aligned} \tag{10}$$

The variety $\mathrm{hMPS}(m,n,d)$ of *homogeneous matrix product* states is the closed image of the map

$$
(11)\qquad \begin{aligned} (\mathbb{C}^{m\times m})^{n+1} &\to (\mathbb{C}^n)^{\otimes d} \\ (X, A_1, \dots, A_n) &\mapsto \sum_{1\le i_1,\dots,i_d\le n} \mathrm{Tr}(XA_{i_1}\cdots A_{i_d})\ e_{i_1}\otimes\cdots\otimes e_{i_d}. \end{aligned}
$$

The goal of this section is to study the linear spaces spanned by these varieties. For previous work on this topic, see [NV18] (hMPS) and [DLMS22] (uMPS).

3.1. **Homogeneous matrix product states.** If we write $e_I := e_{i_1}\otimes\cdots\otimes e_{i_d}$ for a word $I = (i_1,\dots,i_d)$, then the space $(\mathbb{C}^n)^{\otimes d}$ has a basis $\{e_I\}_{I\in[n]^d}$. We will denote the dual basis by $\{x_I\}_{I\in[n]^d}$. The following observation links homogeneous matrix product states to polynomial identities of matrices:

Observation 3.2. *An equation $\sum_I \lambda_I x_I$ vanishes on $\mathrm{hMPS}(m,n,d)$ if and only if the identity $\sum_I \lambda_I \mathrm{Tr}(XA_{i_1}\cdots A_{i_d}) = 0$ holds for all matrices X, A_i, if and only if $\sum_I \lambda_I A_{i_1}\cdots A_{i_d} = 0$ is a polynomial identity of matrices.*

Corollary 3.3. *The linear span of $\mathrm{hMPS}(m,n,d)$ has dimension equal to $\dim \mathcal{M}_{m,n;d}$, the degree d part of the ring of n generic $m\times m$-matrices. In other words, we have*

$$\sum_{d\in\mathbb{N}} \dim\langle \mathrm{hMPS}(m,n,d)\rangle t^d = H(\mathcal{M}_{m,n}; t,\dots,t).$$

Example 3.4. For $m = n = 2$, we have

$$
\begin{aligned}
\sum_{d\in\mathbb{N}} \dim\langle \mathrm{hMPS}(2,2,d)\rangle t^d =& H(\mathcal{M}_{2,2}; t,t) \\
=& \frac{1-2t+t^2+2t^3-t^4}{(1-t)^4(1-t^2)} \\
=& 1+2t+4t^2+8t^3+16t^4+30t^5+53t^6+88t^7+139t^8+210t^9 \\
& +306t^{10}+432t^{11}+594t^{12}+798t^{13}+1051t^{14}+1360t^{15}+\cdots
\end{aligned}
$$

Compare with the top row of Table 1 in [NV18]. For $m = 2$ and larger n, it is possible to write down a similar formula using Theorem 2.4. However, larger m are much more difficult: even for $m = 3$ and $n = 2$ no exact formula is known to the best of my awareness.

3.2. **Uniform matrix product states.** Since traces of matrices are invariant under cyclic permutations, the variety $\mathrm{uMPS}(m,n,d)$ is contained in the linear subspace $\mathrm{Cyc}^d(\mathbb{C}^n) \subseteq (\mathbb{C}^n)^{\otimes d}$ of tensors that are invariant under cyclic permutations. Consider the equivalence relation on $[n]^d$ given by identifying words with their cyclic permutations. The equivalence classes are known as *necklaces* of length d on the alphabet $[n]$; we will denote the set of such necklaces by $N_d([n])$. For a necklace $N\in N_d([n])$, we define $e_N := \sum_{(i_1,\dots,i_d)\in N} e_{i_1}\otimes\cdots\otimes e_{i_d}$. Then $\{e_N\}_{N\in N_d([n])}$ is a basis of $\mathrm{Cyc}^d(\mathbb{C}^n)$. We will denote the dual basis by $\{y_N\}_{N\in N_d([n])}$.

As before, we see that linear equations $\sum_N \lambda_N y_N$ vanishing on $\mathrm{uMPS}(m,n,d)$ correspond to trace relations $\sum_N \lambda_N \mathrm{Tr}(A_{i_1}\cdots A_{i_d}) = 0$, where $(i_1,\dots,i_d)$ is a representative of the necklace N. Hence we find:

Observation 3.5. *The linear span of* $\mathrm{uMPS}(m,n,d)$ *has dimension equal to* $\dim R^{\mathrm{cyc}}_{m,n;d}$. *In other words:*

$$\sum_{d\in\mathbb{N}} \dim\langle \mathrm{uMPS}(m,n,d)\rangle t^d = H(R^{\mathrm{cyc}}_{m,n}; t,\ldots,t).$$

In order to determine the Hilbert series of $R^{\mathrm{cyc}}_{m,n}$, we can try the same approach as for $\mathcal{M}_{m,n}$. In particular, I claim that the analogue of Theorem 2.2 holds. Let us write

$$\widetilde{P}_{m,d} := R^{\mathrm{cyc}}_{m,d;(1,\ldots,1)}.$$

Claim 3.6. *Suppose the decomposition of* $\widetilde{P}_{m,d}$ *into* $\mathfrak{S}_d$*-irreps is given by*

$$\widetilde{P}_{m,d} = \bigoplus_{\lambda\vdash d} V_\lambda^{\oplus b_\lambda}.$$

Then the decomposition of $R^{\mathrm{cyc}}_{m,n}$ *into* GL_n*-irreps is given by*

$$R^{\mathrm{cyc}}_{m,n} = \bigoplus_{d\in\mathbb{N}} \bigoplus_{\substack{\lambda\vdash d\\ \mathrm{len}(\lambda)\le n}} \mathbb{S}^\lambda(\mathbb{C}^n)^{\oplus b_\lambda}.$$

Hence, the Hilbert series of $R^{\mathrm{cyc}}_{m,n}$ *is given by*

$$H(R^{\mathrm{cyc}}_{m,n}; t_1,\ldots,t_n) = \sum_{d\in\mathbb{N}} \sum_{\substack{\lambda\vdash d\\ \mathrm{len}(\lambda)\le n}} b_\lambda s_\lambda(t_1,\ldots,t_n).$$

Proof. Will be done in the next version of these notes. The idea is to use polarization. □

We have

$$\widetilde{P}_{m,d} \cong \frac{\mathbb{C}[\mathfrak{S}_d]_{\mathrm{cyc}}}{I_{m,d}\cap \mathbb{C}[\mathfrak{S}_d]_{\mathrm{cyc}}} \cong P_{m,d-1}.$$

In particular, for $m=2$, Theorem 2.4 gives a decomposition of $\widetilde{P}_{m,d}$ as a representation of the subgroup $\mathfrak{S}_{d-1}\subset\mathfrak{S}_d$. But this is not enough information to deduce the decomposition as an $\mathfrak{S}_d$-representation:

Problem 3.7. *Write down a closed formula for the decomposition of the* $\mathfrak{S}_d$*-representation*

$$\widetilde{P}_{2,d} = R^{\mathrm{cyc}}_{2,d;(1,\ldots,1)} = \frac{\mathbb{C}[\mathfrak{S}_d]_{\mathrm{cyc}}}{I_{2,d}\cap \mathbb{C}[\mathfrak{S}_d]_{\mathrm{cyc}}}.$$

Via Observation 3.5 and Claim 3.6, such a formula would give a formula for the dimension of $\langle \mathrm{uMPS}(2,n,d)\rangle$. In particular, it would solve Conjectures 3.7 and 3.8 in [DLMS22].

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