1 Reelle Zahlen, Euklidische Räume, Komplexe Zahlen

Axioms of Addition:

• **Axioms:**

A1 (Associativity)	$x+(y+z)=(x+y)+z$	$\forall x,y,z\in\mathbb{R}$
A2 (Neutral element)	$x+0=x$	$\forall x\in\mathbb{R}$
A3 (Inverse element)	$x+(-x)=0$	$\forall x\in\mathbb{R}\ \exists(-x)\in\mathbb{R}$
A4 (Commutativity)	$x+y=y+x$	$\forall x,y\in\mathbb{R}$

Axioms of Multiplication:

• **Axioms:**

M1 (Associativity)	$x\cdot(y\cdot z)=(x\cdot y)\cdot z$	$\forall x,y,z\in\mathbb{R}$
M2 (Neutral element)	$x\cdot 1=x$	$\forall x\in\mathbb{R}$
M3 (Inverse element)	$x\cdot x^{-1}=1$	$\forall x\in\mathbb{R}\setminus\{0\}\ \exists x^{-1}\in\mathbb{R}$
M4 (Commutativity)	$x\cdot z=z\cdot x$	$\forall x,z\in\mathbb{R}$

Distributivity (Compatibility of addition and mult.):

• **Axiom:**

D $\quad x\cdot(y+z)=xy+xz=(y+z)\cdot x \qquad \forall x,y,z\in\mathbb{R}$

Order Axioms:

• **Axioms:**

O1 (Reflexivity)	$x\le x$	$\forall x\in\mathbb{R}$
O2 (Transitivity)	$(x\le y\wedge y\le z)\Rightarrow x\le z$	
O3 (Antisymmetry)	$(x\le y\wedge y\le x)\Rightarrow x=y$	
O4 (Totality)	$\forall x,y\in\mathbb{R}:\ x\le y$ or $y\le x$	

Compatibility (Consistency with addition and mult.):

• **Axioms:**

K1	$x\le y\Rightarrow x+z\le y+z$	$\forall x,y,z\in\mathbb{R}$
K2	$x\ge 0,\ y\ge 0\Rightarrow xy\ge 0$	$\forall x,y\in\mathbb{R}$

R 1.1.5:

• The set $\mathbb{Q}$ of rational numbers, equipped with addition, multiplication, and the order relation $\le$, satisfies the above axioms.

V (Completeness axiom):

• The set $\mathbb{R}$ is **order complete**: For any two nonempty sets $A,B\subset\mathbb{R}$ such that

$$a\le b \quad \text{for all } a\in A,\ b\in B,$$

there exists a number $c\in\mathbb{R}$ satisfying

$$a\le c\le b \quad \forall a\in A,\ b\in B.$$

C 1.1.6:

• **The consequences of above axioms are:**

1. Uniqueness of the additive and multiplicative inverse.
2. $0\cdot x=0 \quad \forall x\in\mathbb{R}$.
3. $(-1)\cdot x=-x \quad \forall x\in\mathbb{R}$, in particular $(-1)^2=1$.
4. $y\ge 0 \iff (-y)\le 0$.
5. $y^2\ge 0 \quad \forall y\in\mathbb{R}$, in particular $1=1\cdot 1\ge 0$.
6. If $x\le y$ and $u\le v$, then $x+u\le y+v$.
7. If $0\le x\le y$ and $0\le u$, then $xu\le yu$.

T (Completeness of the real numbers):

• $\mathbb{R}$ is a commutative ordered field that is order-complete.

C 1.1.7 (Archimedean Principle):

• 1. For $x\in\mathbb{R}$ and $y>0$ there exists $n\in\mathbb{N}$ such that

$$ny>x.$$

2. For every $\varepsilon>0$ there exists $n\in\mathbb{N}$ such that

$$\frac{1}{n}<\varepsilon.$$

T 1.1.8:

• For every $t\ge 0$, $t\in\mathbb{R}$, the equation $x^2=t$ has a solution in $\mathbb{R}$.

T (Supremum Infimum):

• 1. Maximum and minimum (if exists) are unique

2. Every non-empty subset bounded below (above) has a unique infimum (supremum).

3. $S=\sup(X)\Leftrightarrow$

$$(\forall x\in X: x\le S)\wedge(\forall\varepsilon>0\ \exists x\in X:\quad x>S-\varepsilon)$$

4. $I=\inf(X)\Leftrightarrow$

$$(\forall x\in X: x\ge I)\wedge(\forall\varepsilon>0\ \exists x\in X:\quad x<I+\varepsilon)$$

D (Maximum):

• Maximum is:

$$\max\{x,y\}:=\begin{cases}x, & \text{if } y\le x,\\ y, & \text{if } x\le y.\end{cases}$$

D (Minimum):

• Minimum is:

$$\min\{x,y\}:=\begin{cases}y, & \text{if } y\le x,\\ x, & \text{if } x\le y.\end{cases}$$

T (Properties of the absolute value):

• Properties:

1. $|x|\ge 0$.
2. $|x+y|\le|x|+|y| \qquad$ (triangle inequality).
3. $|x+y|\ge\big||x|-|y|\big|$ (inv. triangle inequality).

T (Young's inequality):

• For every $\varepsilon>0$ and all $x,y\in\mathbb{R}$ it holds:

$$2|xy|\le\varepsilon x^2+\frac{1}{\varepsilon}y^2.$$

1.1 Vectors and Vector Spaces

D (inner product):

• For $x,y\in\mathbb{R}^n$ the standard inner product is

$$x\cdot y=\langle x,y\rangle=\sum_{i=1}^n x_iy_i$$

D (Euclidean norm):

• The Euclidean norm is

$$\|x\|=\sqrt{\sum_{i=1}^n x_i^2}$$

D (Euclidean distance):

• The Euclidean distance is

$$d(x,y)=\sqrt{\sum_{i=1}^n(x_i-y_i)^2}$$

D (Triangle inequality):

• For all $x,y,z\in\mathbb{R}^n$:

$$\|x-z\|\le\|x-y\|+\|y-z\|$$

1.2 Complex Numbers

D (Complex number):

- A complex number is

$$z = a + ib, \quad a, b \in \mathbb{R}$$

D (set of complex numbers):

- The set of complex numbers is

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$$

D (Terminology of complex numbers):

- Terminology
  - $a = \operatorname{Re}(z)$ — real part
  - $b = \operatorname{Im}(z)$ — imaginary part
  - $i^2 = -1$
  - If $a = 0$, the number is purely imaginary

D (Complex numbers Addition and Subtraction):

- Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

D (Complex numbers Multiplication):

- Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + y_1 x_2)$$

D (Complex conjugate):

- For a complex number $z = x + iy$, the complex conjugate of z is given by

$$\overline{z} = \overline{x + iy} = x - iy.$$

D (Complex numbers Division):

- To divide numbers we use conjugate:

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}$$

D (Properties of the Conjugate):

- Properties are:
  - $z\overline{z} = |z|^2$
  - $z + \overline{z} = 2\operatorname{Re}(z), \quad z - \overline{z} = 2i\operatorname{Im}(z)$
  - $\overline{\overline{z}} = z$
  - $\overline{z \pm w} = \overline{z} \pm \overline{w}$
  - $\overline{zw} = \overline{z}\,\overline{w}$
  - $z = \overline{z}$, if $z \in \mathbb{R}$
  - $\overline{z} = -z$, z is purely imaginary

D (Modulus):

- The magnitude $|z|$ of a complex number $z = a + ib$ is the length of the corresponding vector.

$$|z| = \sqrt{a^2 + b^2}$$

The distance between two complex numbers z1 and z2 is

$$d(z_1, z_2) = |z_2 - z_1|$$

Triangle inequality works also for complex numbers:

$$|z + w| \leq |z| + |w|$$

D (Euler's Formula):

- The exponential function can be represented by a series expansion as

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

If we substitute $x = it$ into this formula, we obtain Euler's formula.

$$e^{it} = \cos t + i \sin t$$

We can write $sin(t)$ and $cos(t)$ using complex exponential functions.

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}$$

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

D (Exponential with Complex Argument):

- We can also allow complex arguments for the exponential function. Then the following rules apply:
  - $e^{z+w} = e^z e^w$
  - $e^{a+ib} = e^a(\cos b + i \sin b)$
  - $e^{z+i2\pi} = e^z$

D (Polar Form):

- A complex number can also be described by specifying the distance from the origin and by specifying a suitable angle (**polar form**): $z = re^{i\varphi}$.
  - $r = |z| \geq 0$ - distance
  - $\varphi \in (-\pi, \pi]$ - polar angle
  - $\varphi = \arg(z) = \operatorname{arc}(z)$

D (Polar to Cartesian conversion):

- Polar to Cartesian conversion is:
  - $x = r\cos\varphi$
  - $y = r\sin\varphi$

D (Cartesian to Polar conversion):

- Cartesian to Polar conversion is:
  - $r = \sqrt{x^2 + y^2}$
  - $\varphi = \arctan \frac{y}{x}$

If $x = 0$:

$$\arg(iy) = \begin{cases} \frac{\pi}{2}, & y > 0 \\ -\frac{\pi}{2}, & y < 0 \end{cases}$$

D (Powers of Complex Numbers):

- The following rules must be observed:
  - $z_1 z_2 = r_1 r_2 e^{i(\varphi_1 + \varphi_2)}$
  - $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)}$
  - $z^n = r^n e^{in\varphi}$

D (Roots from complex numbers):

- For $n \in \mathbb{N}$, the solutions to the equation $z^n = re^{i\varphi}$ are

$$z_k = r^{1/n} e^{i(\varphi/n + 2\pi k/n)}, \quad k = 0, \dots, n-1$$

D (Quadratic Equations):

- For the quadratic equation

$$az^2 + bz + c = 0,$$

in complex numbers the solution is still

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

D (Fundamental Theorem of Algebra):

- Each polynom

$$p(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{C}$$

can be factored into linear factors, i.e. written as

$$p(z) = a_n(z - z_1)(z - z_2)\dots(z - z_n)$$

The numbers z_k are therefore precisely the zero points of $p(z)$ (with **multiplicity**).

C (non-real roots apper in conjugate pairs):

- Non-real roots of a polynomial with **real** coefficients occur in conjugate pairs $z, \overline{z}$.