Two anyons in Hamonic Trap: momentum tail

June 20, 2018

The two fermions wavefunction

$$\Psi_F(x_1, x_2) = \frac{1}{\sqrt{2}} \left[\phi_0(x_1)\phi_1(x_2) - \phi_1(x_1)\phi_0(x_2) \right]$$

where

$$\phi_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} H_n(x) e^{-\frac{x^2}{2}}$$

So

$$\begin{aligned}
\Psi_F(x_1, x_2) &= \frac{1}{\sqrt{2}} \left[\frac{1}{\pi^{1/4}} e^{-\frac{x_1^2}{2}} \frac{1}{\pi^{1/4}\sqrt{2}} 2x_2 e^{-\frac{x_2^2}{2}} - \frac{1}{\pi^{1/4}\sqrt{2}} 2x_1 e^{-\frac{x_1^2}{2}} \frac{1}{\pi^{1/4}} e^{-\frac{x_2^2}{2}} \right] \\
&= \frac{1}{\sqrt{\pi}} e^{-\frac{x_1^2+x_2^2}{2}} \left[x_2 - x_1 \right]
\end{aligned}$$

The hardcore boson wavefunction is

$$\Psi_B(x_1, x_2) = \frac{1}{\sqrt{\pi}} e^{-\frac{x_1^2+x_2^2}{2}} \left| x_2 - x_1 \right|$$

Its momentum distribution from one body density matrix is

$$\begin{aligned}
n_B(p) &= \frac{2}{2\pi} \int dx dx' e^{ip(x-x')} \rho_B(x, x') \\
&= \frac{1}{\pi} \int dx dx' e^{ip(x-x')} \rho_B(x, x') \\
&= \frac{1}{\pi} \int dx dx' \cos(p(x-x')) \rho_B(x, x')
\end{aligned}$$

where

$$\rho_B(x,x') = \int dx_2 \Psi_B^*(x,x_2)\Psi_B(x',x_2)$$

There are ways to calculate $\rho_B(x,x')$ such as using Toplitz matrix, but here for benchmarking the numerical method, we numerically integrate $n_B(p)$ using, first calculate a grid of $\rho_B(x,x')$ then do the summation for the integral.

$$n_B(p) = \frac{1}{\pi}\int dx dx' \cos(p(x-x'))\rho_B(x,x')$$

The result of $n_B(p)p^4$ verses p is in the following figure.

At large p, it tends to constant value. The value is represented by the orange line, which is $\frac{2}{\pi}\sqrt{\frac{2}{\pi}} = 0.5079..$

Here is the analytic derivation of the momentum tail for two hardcore bosons. The momentum distribution can be written as

$$n_B(p) = \frac{1}{\pi}\int dx_2 \left|\int dx\, e^{ipx}\, \Psi_B^*(x,x_2)\right|^2$$

where

$$\int dx\, e^{ipx}\, \Psi_B^*(x,x_2) = \frac{1}{\sqrt{\pi}}\int dx\, e^{ipx} e^{-(x^2+x_2^2)/2}\,|x-x_2| = -\frac{2}{\sqrt{\pi}}\,\frac{1}{p^2}\,e^{-x_2^2+ipx_2}$$

where the last equality is valid in the limit $p \to infty$ and we have used Eq. (4) in PRL **91**, 090401 (2003). Inserting this result into $n_B(p)$, we have

$$\lim_{p\to\infty} n_B(p) = \frac{1}{\pi}\int dx_2 \left|-\frac{2}{\sqrt{\pi}}\,\frac{1}{p^2}\,e^{-x_2^2+ipx_2}\right|^2 = \frac{2}{\pi}\sqrt{\frac{2}{\pi}}\,\frac{1}{p^4}$$

from which we can infer that the Tan contact for two harmonically trapped hardcore bosons is

$$C_B = \frac{2}{\pi}\sqrt{\frac{2}{\pi}}$$

This is consistent with the ground state energy of two bosons in the TG ($g \to \infty$) limit, which reads

$$E_B = E_0 - 2\sqrt{\frac{2}{\pi}}\frac{1}{g}$$

and from the adiabatic sweep theorem, we have (see Eur. Phys. J. Special Topics **226**, 1583 (2017))

$$\frac{dE_B}{d(1/g)} = -2\sqrt{\frac{2}{\pi}} = -\pi C_B$$

Now consider two hardcore anyons. Apart from an overall phase factor, the wavefunction can be written as

$$\Psi_\kappa(x_1, x_2) = e^{-i\frac{\pi\kappa}{2}\epsilon(x_2-x_1)}\,\Psi_B(x_1, x_2)$$

Its momentum distribution is given by

$$n_\kappa(p) = \frac{2}{2\pi}\iint dx dx' e^{ip(x-x')}\rho_\kappa(x, x') = \frac{1}{\pi}\int dx_2 \left|\int dx\, e^{ipx}\,\Psi_\kappa^*(x, x_2)\right|^2$$

Now let us examine the integral

$$\begin{aligned}\int dx\, e^{ipx}\,\Psi_\kappa^*(x, x_2) &= \int dx\, e^{ipx}\, e^{i\frac{\pi\kappa}{2}\epsilon(x_2-x)}\,\Psi_B^*(x, x_2) \\ &= \int dx\, e^{ipx}\,[\cos(\pi\kappa/2) + i\epsilon(x_2 - x)\sin(\pi\kappa/2)]\,\Psi_B^*(x, x_2)\end{aligned}$$

Note that $\epsilon(x_2 - x)\,\Psi_B^*(x, x_2) = \Psi_F(x, x_2)$ is an analytic function, hence in the $p \to \infty$ limit, the sin term above results in an exponentially decaying term. The leading term is then given by the cos term which is simply (using the above result for hardcore bosons):

$$\lim_{p\to\infty}\left[\int dx\, e^{ipx}\,\Psi_\kappa^*(x, x_2)\right] = \cos(\pi\kappa/2)\left(-\frac{2}{\sqrt{\pi}}\frac{1}{p^2}\,e^{-x_2^2+ipx_2}\right)$$

Therefore, we have

$$\lim_{p\to\infty} n_\kappa(p) = \cos^2(\pi\kappa/2)\lim_{p\to\infty} n_B(p) = \cos^2(\pi\kappa/2)\,\frac{2}{\pi}\sqrt{\frac{2}{\pi}}\,\frac{1}{p^4}$$

If we define the momentum tail coefficient as the Tan contact, then

$$C_\kappa = \cos^2(\pi\kappa/2)\,C_B$$

which is consistent with the result obtained in Eur. Phys. J. D **71**, 135 (2017). Note that in that paper, the anyon statistical parameter χ is equal to $(1 - \kappa)$ in our notation.

An anyon gas with interaction strength g has the same energy as a bosonic gas with interaction strength $g' = g/cos(\kappa\pi/2)$. Therefore, in the large g limit, the two harmonically trapped anyons should have energy

$$E_\kappa = E_0 - 2\sqrt{\frac{2}{\pi}}\frac{1}{g'} = E_0 - 2\sqrt{\frac{2}{\pi}}\frac{\cos(\kappa\pi/2)}{g}$$

Therefore, it seems that the adiabatic sweep theorem for anyon gas should read

$$\frac{dE_\kappa}{d(1/g)} = -2\cos(\kappa\pi/2)\sqrt{\frac{2}{\pi}} = -\pi C_\kappa/\cos(\kappa\pi/2)$$