Momentum Distribution Properties of Ultracold Particles

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Research Summary

- We study strongly-interacting, ultracold particles confined in a one-dimensional harmonic trap
- Strongly-interacting 1D systems often exhibit peculiar quantum behavior

Two Central Questions:

- How does the momentum distribution decay for large momentum (e.g., exponentially, inverse power law)?
- What happens to momentum distribution when trapping potential is turned off?

Relevant Factors

- Particle type (e.g., fermion, boson, anyon)
- Particles with spin degree of freedom
- Systems with non-zero temperature
- Number of particles
- Interaction strength

Bosons and Fermions

- In nature, we only observe bosons and fermions
- Distinguishing characteristic is how their wavefunctions behave when two particles are exchanged

$$\psi_B(x_1, x_2) = \psi_B(x_2, x_1)$$
$$\psi_F(x_1, x_2) = -\psi_F(x_2, x_1)$$

- Exchange Statistics: how wave function behaves under particle exchange
- Identical particles $\rightarrow$ exchanging particles can only change ψ by complex phase

Anyons

- As a theoretical idea, we can define another type of particle – the anyon
- Described by its anyonic statistic $\kappa \in [0, 2)$
- Defining characteristic:

$$\psi_\kappa(x_1, x_2) = e^{i\pi\kappa\epsilon(x_1 - x_2)}\psi_\kappa(x_2, x_1)$$

  where $\epsilon(x)$ is the sign function
- $\kappa = 0 \rightarrow e^{i\pi\kappa} = 1 \rightarrow$ Bosons
- $\kappa = 1 \rightarrow e^{i\pi\kappa} = -1 \rightarrow$ Fermions

Hard-Core Particles

- We consider systems 'hard-core' particles: infinite, repulsive contact interactions
- Means that no two of the particles can occupy same position
- If particles are spinless, we can write the Hamiltonian

$$H = \sum_{j=1}^{N} \left[\frac{-1}{2} \frac{\partial^2}{\partial x_j^2} + \frac{1}{2} x_j^2 \right] + g \sum_{j<\ell} \delta(x_j - x_\ell)$$

  where $\hbar = m = \omega = 1$ and in the hard-core limit $g \to \infty$

- Our research will consider hard-core bosons and hard-core anyons

Bose-Fermi Mapping

- Bosons and fermions are very different, but hard-core bosons (HCB) are similar to fermions (e.g., no two can occupy same position)
- M. Girardeau (1960) showed there was a mapping between these two systems
- Procedure for turning N-particle fermion solution into boson solution

$$\psi_B(x_1, \ldots, x_N) = A(x_1, \ldots, x_n)\psi_F(x_1, \ldots, x_N)$$

$$A(x_1, , \ldots, x_N) = \prod_{1 \leq \ell < j \leq N} \epsilon(x_j - x_\ell)$$

Anyon-Fermion Mapping

- Girardeau (2006) showed mapping existed for hard-core anyons (HCA)
- Procedure for turning fermionic solution into bosonic solution

$$\psi_\kappa(x_1, \ldots, x_N) = A_\kappa(x_1, \ldots, x_N)\psi_F(x_1, \ldots, x_N)$$

$$A_\kappa(x_1, \ldots, x_N) = \exp\left(i\pi(1-\kappa) \sum_{1\leq \ell < j \leq N} \theta(x_j - x_\ell) \right)$$

where θ is the heaviside theta function

Methods: One Body Density Matrix

- After obtaining the wave function for our system using the mappings, we study its properties using the One-Body Density Matrix (OBDM)

$$\rho(x,x') = \int_{-\infty}^{\infty} dx_2 \cdots dx_N \psi^*(x,x_2,\ldots,x_N)\psi(x',x_2,\ldots,x_N)$$

- Then, we use this to find an expression for the momentum space density profiles

$$n(p) = \frac{N}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' e^{ip(x-x')}\rho(x,x')$$

- Difficult to solve analytically, even for two hard-core bosons

Methods: Momentum Space Density Profiles

Tan Contact

- Momentum distribution for spinless fermions in a harmonic trap will decay exponentially
- Minguzzi et al. (2002) showed for 2 spinless hard-core bosons

$$\lim_{p\to\infty} n_B(p) = \frac{C_B}{p^4} = \left(\frac{2}{\pi}\right)^{3/2} \frac{1}{p^4}$$

where C_B is the Tan Contact (see Tan (2008)) or momentum tail coefficient

Dynamical Fermionization

- Hard-core bosons and fermions have different momentum distributions
- When trapping potential is turned off for hard-core bosons, Minguzzi et al. (2005) showed that

$$\lim_{t\to\infty} n_B(p;t) = n_F(p)$$

  using a scaling transformation
- This process is known as dynamical fermionization

Tan Contact

- Calculate Tan Contact for two hard-core anyons

$$\lim_{p\to\infty} n_\kappa(p) = \frac{C_\kappa}{p^4} = \cos^2\left(\frac{\pi\kappa}{2}\right)\frac{C_B}{p^4}$$

- And numerically showed dynamical fermionization occurs

Progress: Finite Temperature Tan Contact

- Hao et al. (2017) calculate the Tan Contact for N Hard-core Anyons at finite temperature in grand canonical ensemble
- Their calculation more or less follows Vignolo et al. (2013)

$$\rho_B(x,y) = \sum_{N,\alpha} P_{N,\alpha} N \int_{-\infty}^{\infty} dx_2 \cdots dx_N \psi_{N,\alpha}(x, x_2, \ldots, x_N) \times \psi^*_{N,\alpha}(y, x_2, \ldots, x_N)$$

- N is number of particles and $\alpha = \{\nu_1, \nu_2, \ldots, \nu_N\}$ gives the quantum number for each particle
- $P_{N,\alpha}$ gives the probability that the system has N particles with state α

$$P_{N,\alpha} = \frac{\exp(-\beta(E_{N,\alpha} - \mu N))}{Z}$$

Progress: Finite Temperature Tan Contact

- Density matrix ρ_B can expressed as sum of special functions
- Only first term contributes at high momentum

$$\rho_B^{(j=1)}(x, x') \sim \frac{|x - x'|^3}{3} F(R)$$

where $R = \frac{x+x'}{2}$ can be ignored in Fourier transform

- Use asymptotics of Fourier transforms to evaluate the momentum distribution

$$\lim_{p\to\infty} n_B(p) = \frac{C}{p^4}$$

$$C = \frac{2}{\pi} \int_{\infty}^{\infty} dRF(R)$$

Plans: Two Spinor Case

- The results so far have been with spinless particles
- For two hard-core bosons with spin degree of freedom (called spinors) we ask:
  - What is the total Tan Contact?
  - Will dynamical fermionization occur?
- We will use tools from past research in Pu group:
  - Yang et al. (2015) – simplifies OBDM for strongly-interacting spinor particles by separating into spatial part and spin part
  - Yang et al. (2017) – exploits connection between spinless HCA and spinors to calculate momentum distribution
  - Other unpublished work, Li Yang has shown that the Tan Contact does exist for spinor system

Past Research:

- Calculated Tan Contact for two, spinless hard-core anyons
- Showed dynamical fermionization would occur
- Studied $N-$particle, strongly-intereacting systems with finite temperature

Future Work:

- Use current tools to calculate total Tan Contact for two spinors
- Determine whether dynamical fermionization will occur

References