Warming-Up Exercises Week 1
(to be done in Werkcollege classes)

These are NOT homework exercises, but warming up exercises, to be explained by the TA's in the Werkcollege classes. But note that some of them are harder than the Homework exercises!

1. Consider Epistemic Puzzle no 4 (Muddy Children, as presented Lecture 1-1), but in which we suppose that there are in total **only three** children (named 1,2,3), **exactly two of which have dirty foreheads** (more precisely, *children 1 and 2 are dirty, while child 3 is clean*). Each child can see the foreheads of the others but not his own.

   (a) Represent all the facts, as well as all the children's knowledge, in a Kripke model $\mathbf{M}$ with three agents $\mathcal{A} = \{1, 2, 3\}$ and three atomic propositions $Prop = \{d_1, d_2, d_3\}$ (where d_i means "child i is dirty"). Either draw the model, or else specify formally: the set W of possible worlds; the knowledge relations R_1, R_2, R_3; the valuation. Also, specify which world is the real world, and denote that world by w.

   (b) Is the above model $\mathbf{M}$ an *epistemic* model?

   (c) Check (using the semantics of K_i) that (in the real world w) *everybody knows that at least one of them is dirty*; i.e. show (using the semantics of K_i) that

   $$w \models_{\mathbf{M}} K_1(d_1 \vee d_2 \vee d_3) \wedge K_2(d_1 \vee d_2 \vee d_3) \wedge K_3(d_1 \vee d_2 \vee d_3)$$

   (d) Is it common knowledge (in the real world w) that at least one of them is dirty? Prove your answer formally by encoding it as a sentence in epistemic logic and checking that it is true at the world w in model $\mathbf{M}$ (using the semantics of the common knowledge operator).

   (e) Father makes a *truthful public announcement*: "At least one of you is dirty". Draw (or specify formally) the *updated model* $\mathbf{M}'$ representing the children's knowledge **after** this announcement.

(f) The scenario continues as in last week's slides: in response to Father's first question ("Do you know if you are dirty or not, and if so which?"), children answer truthfully, publicly and simultaneously. What will they answer (in the real world w)? Justify your answer, by writing down one long formal sentence φ_1 (in the language of epistemic logic) that expresses all the children's simultaneous answers to this first question (in the real world w), and checking that this sentence is true at w in the model $\mathbf{M}'$.

(g) Let us interpret children's simultaneous answering to Father's first question as a truthful public announcement $!\varphi_1$ of the sentence you wrote in the previous part. What is the updated model $\mathbf{M}''$ after this new public announcement? (Draw, or else formally specify, this model.)

(h) If Father repeats the same question *now* (after the children's public answering of the first question), what will the children answer now? Justify your answer, by writing down one long formal sentence φ_2 (in the language of epistemic logic) that expresses all the children's simultaneous answers to this second question (at the real world w), and checking that this sentence is true at w in the model $\mathbf{M}''$.

(i) Show that in the next round of questioning, the clean child will answer "I know I am clean". Justify your answer, by drawing the new model $\mathbf{M}'''$ after the previous step (i.e. after all children answered the second question) and checking that

$$w \models_{\mathbf{M}'''} K_3 \neg d_3.$$

2. Prove the validity of the "Veracity of Knowledge"

$$K_a \varphi \Rightarrow \varphi$$

on epistemic models, using the semantical definition of knowledge on these models. (Recall the definition of *epistemic models* from Lecture Notes Week 1).

3. Prove the validity of "Positive Introspection of Knowledge"

$$K_a \varphi \Rightarrow K_a K_a \varphi$$

on epistemic models.

4. Prove the validity of the **left-to-right implication** in (slide 38 of Lecture 1-2) on epistemic models:

$$Ck\varphi \;\Rightarrow\; (\varphi \wedge K_1 Ck\varphi \wedge \ldots \wedge K_n Ck\varphi)$$

5. Prove the validity of the "Induction Axiom" (slide 38 of Lecture 1-2) on epistemic models:

$$Ck(\varphi \Rightarrow K_1\varphi \wedge \ldots \wedge K_n\varphi) \Rightarrow (\varphi \Rightarrow Ck\varphi)$$

6. Prove that the formula

$$(K_a\varphi \wedge K_a\psi \wedge K_b\varphi \wedge K_b\psi) \;\Rightarrow\; Ck(\varphi \wedge \psi)$$

is NOT necessarily valid on (the class of all) epistemic models with only two agents a and b.

HINT: To do last exercise, give a *counterexample*: some specific *epistemic model* **M** having *ONLY two agents* a and b, some specific world w in that model, and some specific formulas φ, ψ (say, some atomic sentences p, q), such that the premise of the above implication is true at w, but the conclusion is false at w.