Mock Final Exam

IMPORTANT NOTE: The total number of points is 100 (in case you solve it perfectly). The exam is **open-book.**

Question 1. (*36 points*) Each of two children, Alice and Bob, has a number written on his/her forehead. It is common knowledge that: (i) the two numbers n_a, n_b belong to the set $\{0, 1, 2, 3\}$, (ii) Alice's number is strictly smaller then Bob's (i.e. $n_a < n_b$), and (iii) each child can see the other's number, but cannot see his/her own number.

1. (*6 points*) *How many possible worlds* are there (that are consistent with the story above)?

2. (*6 points*) *Represent* (draw) the above situation as an *epistemic model* **M**, with two agents (A for Alice, B for Bob) and eight atomic sentences n_a, n_b, with $n \in \{0, 1, 2, 3\}$. The sentence n_a means that Alice's number is n, and similarly for n_b. (Make sure that from your drawing one can easily read off: sthe set of states, the valuations and the accessibility relations.)

3. (*6 points*) Write down a sentence θ that is equivalent to the statement that $n_a + 1 = n_b$ (in the assumption that the numbers belong to the set $\{0, 1, 2, 3\}$), but which uses ONLY the above-mentioned eight atomic sentences n_a, n_b together with any of the usual propositional connectives (negation, conjunction, disjunction, conditional, biconditional).

4. (*6 points*) Alice and Bob are now asked the following question: "Do you know whether or not the two numbers are in immediate succession (i.e. $n_a + 1 = n_b$), and if so then which of the two ?" They have to answer publicly, truthfully and simultaneously. Each of their answers can be: (a) I don't know, (b) I know that $n_a + 1 = n_b$, or (c) I know that $n_a + 1 \neq n_b$.

   Let us suppose that in fact *they both answer "I don't know"*. Write down a sentence ϕ in epistemic logic that expresses the *conjunction of their simultaneous answers.*

NOTE: Your sentence should use ONLY the above-mentioned eight atomic sentences n_a, n_b and the operators of epistemic logic (i.e. the logical connectives of propositional logic and the knowledge modalities). So you CANNOT use sentences such as " $n_a + 1 = n_b$ ". But instead, you may use as an auxiliary notation the abbreviation θ for the sentence written in the previous part (so that your answer doesn't get too long).

5. (*6 points*) Interpreting the above simultaneous answers "I don't know" as a truthful public announcement $!\phi$ of the sentence written in the previous part, *represent (draw) the updated model* $\mathbf{M}^{!\phi}$ *after this public announcement.*

6. (*6 points*) Suppose now that, *after* the first round of answering "I don't know", they are asked again the same question and they *both answer again "I don't know"*. **What are the numbers** n_a and n_b? **Justify** your answer by *updating again* the model obtained in the previous part with the truthful public announcement $!\phi$ (of the same sentence ϕ).

Question 2. (*36 points*)

1. (*6 points*) Consider the following scenario:

   Bob has two friends Alice and Eve. Bob's birthday is this weekend, and in fact he is planning to throw a birthday party. He hasn't yet told anyone about the party. It is common knowledge that: Alice and Eve consider it possible that Bob might throw a party (since they know it's his birthday), but they do not know for sure that there will be a party. And it is also of course common knowledge that Bob knows whether he'll throw a party or not.

   Represent (draw) this situation as a Kripke model $\mathbf{M}$, which captures all the above information, using one *atomic sentence* p (denoting the fact that Bob will throw a party), three accessibility relations R_a, R_b and R_e, denoting respectively Alice's, Bob's and Eve's, uncertainty relations; and valuations (labeling), indicating which atomic sentences hold at which states. Specify on your drawing what is the *actual (real) state* of the system.

2. (*6 points*) Suppose now the following action happens: Bob sends an email to Alice to invite her to the party, but he does **not** invite Eve. In the meantime, Eve does not know (and *does not even suspect*) that this is going on: she has no clue about the invitation. Alice on the other hand believes that Bob invited all his friends ((i.e. both Alice and Eve), and moreover that this fact is common knowledge: so she thinks that Bob's invitation was in fact a "public invitation" (to all his friends). In fact, Alice doesn't even consider it possible that Bob didn't invite Eve as well. Moreover, we assume that *Bob knows all this*: he knows that Even doesn't suspect anything and he knows Alice believes this to be a public invitation.

   Represent (draw) this complex action using an *event model* $\mathbf{\Sigma}$ with *three actions*. (Make clear what are the preconditions of each actions, and what are each agent's relations R_a, R_b, R_c representing his/her beliefs about what is going on. Specify on your drawing what is the *real) action* of the system.) *Is this an epistemic model, a doxastic model or none of the two*?

3. (*6 points*) *Represent (draw) a model* $\mathbf{M}'$ for the situation *after* the action described in the previous part, by computing the update product $\mathbf{M}' = \mathbf{M} \otimes \mathbf{\Sigma}$. Specify which is *the real world* of your new model.

4. (*6 points*) Suppose that after the above action, Eve accidently finds out about the party (e.g. because she sees Bob shopping for the party), without Alice or Bob suspecting that she finds out. (Moreover, Eve knows that they don't suspect that she found out.) *Draw an event model* $\mathbf{\Sigma}'$ *representing this action.*

5. (*6 points*) Represent (draw) a model of the situation after the action in the previous part, by computing the update product $\mathbf{M}'' = \mathbf{M}' \otimes \mathbf{\Sigma}'$.

6. (*6 points*) At the end of the above scenario, is it true that *all* agents (Alice, Bob and Eve) know that there will be a party? Is the *party common knowledge*? Briefly justify your answers by referring to properties holding at worlds in the model $\mathbf{M}''$ drawn in the last part.

Question 3 (*28 points*)

1. (*14 points*) Show that Negative Introspection of Beliefs

$$(\neg B_a \phi) \Rightarrow B_a \neg B_a \phi$$

   is valid on (the class of) all *doxastic models*.

   HINT: You need to use the Euclideanness condition (from the definition of doxastic models).

2. (*14 points*) Show that the sentence

$$(\neg \phi) \Rightarrow B_a \neg B_a \phi$$

   is NOT valid on (the class of) all *doxastic models*.

   HINT: Give a counterexample (by presenting some world, in a doxastic model, at which a sentence of the form $(\neg p) \Rightarrow B_a \neg B_a p$ is FALSE, i.e. $\neg p$ is TRUE but $B_a \neg B_a p$ is FALSE at this world.)