Exam Computational Logic 2015-16

IMPORTANT NOTE: The total number of points is 100. The exam is **open-book** (but no electronic devices are allowed).

Question 1 (*10 points*)

1. (*5 points*) Show that the sentence

   $$K_a(p \wedge q) \Rightarrow (K_a p \wedge K_a q)$$

   is **valid** on (the class of) all **epistemic (**$S5$**)** models.

2. (*5 points*) Show that the sentence

   $$K_a(p \vee q) \Rightarrow (K_a p \vee K_a q)$$

   is **NOT valid** on (the class of) **epistemic** ($S5$) models.

   HINT: For part 2, give a **counterexample**: some epistemic model containing a world w such that $w \models K_a(p \vee q)$, but $w \not\models (K_a p \vee K_a q)$.

Question 2. (*48 points*) Each of two people, Mr. Sum and Ms. Product, is each given an envelope containing a sheet of paper with a number written on it. It is **common knowledge** that: *both numbers belong to the set* $\{1, 2, 3, 4\}$*; NONE of the children can see any of the two numbers; Mr. Sum was told **the sum** (and only the sum) of the two numbers: so he knows their sum; and Ms. Product was told **the product** (and only the product) of the two numbers: so she knows their product.*

1. (*6 points*) **How many possible worlds** are there (that are consistent with the story above)?

2. (*6 points*) **Represent** (draw) the above situation as an **epistemic model** $\mathbf{M}_1$, with *two agents* (S for Mr. Sum, P for Ms. Product), *two variables* n_S (denoting the number in Mr. Sum's envelope) and n_P (denoting the number in Ms. Product's envelope), and *eight atomic sentences* of the form $n_i = c$ (with $i \in \{S, P\}$ and $c \in \{1, 2, 3, 4\}$).

   NOTE: It's an epistemic model, so all relations are reflexive and symmetric. To keep the drawing simple, you **can just skip all the loops and the arrow directions, simply drawing (undirected) lines** between worlds, **lines labeled by the agents** S or P.

3. (*6 points*) Let θ be the sentence "*At least one of the two numbers is equal to* 1". **Encode this sentence in propositional logic** (using ONLY the above-mentioned eight atomic sentences $n_i = c$ and the usual propositional connectives $\neg$, $\wedge$, $\vee$).

4. (*6 points*) Mr. Sum is asked the following question: "*Do you know if any of the two numbers is equal to* 1?" He can answer either: (1) *I know that **(at least)** one of the numbers equals* 1, or (2) *I know that **none** of the two numbers is equal to* 1, or (3) *I don't know.*

   Write down sentences ϕ_1, ϕ_2, ϕ_3 encoding in epistemic logic each of Mr. Sum's three possible answers. Also, **for each of the three sentences ϕ_1, ϕ_2, ϕ_3, list ALL the worlds in the model $\mathbf{M}_1$ satisfying that sentence**.

   HINT: To keep it short, your sentence can use as abbreviation the symbol θ for the sentence encoded in the previous part (and in addition any epistemic logic connectives $\neg$, $\wedge$, $\vee$, K_S, K_P).

5. (*6 points*) Let us suppose that in fact Mr. Sum answers ϕ_3, i.e.: "*I don't know*". (We assume he answers truthfully and publicly, based only on his knowledge, without any guesses or any cheating.)

   Represent (draw) the updated model $\mathbf{M}_2 = \mathbf{M}_1{}^{!\phi_3}$ after this truthful public announcement $!\phi_3$.

6. (*6 points*) **After hearing Mr. Sum's answer**, Ms. Product is asked the **same question**: "*Do you know if any of the two numbers is equal to* 1?". (As before, she has three possible answers, e.g. *I know that at least one of the numbers is equal to* 1, etc.)

   Write down sentences ψ_1, ψ_2, ψ_3 (in the same language as above) encoding each of Ms. Product's three possible answers. Also, **for each of the three sentences ψ_1, ψ_2, ψ_3, list ALL the worlds in the LAST model $\mathbf{M}_2$ satisfying that sentence**.

7. (*6 points*) Suppose that in fact Ms. Product answers ψ_3, i.e.: "*I don't know*". (As before, she has to answer truthfully and publicly, based only on her knowledge.)

   Represent (draw) the updated model $\mathbf{M}_3 = \mathbf{M}_2{}^{!\psi_3}$ after this new public announcement $!\psi_3$.

8. (*6 points*) **After hearing Ms. Product's answer**, Mr. Sum is asked the **same question**. (As before, he has to answer truthfully, and his possible answers can again be encoded as ϕ_1, ϕ_2, ϕ_3.)

   But suppose that this time he answers ϕ_2, i.e.: "*I know that* ***none*** *of the two numbers is equal to* 1 ".

   Represent (draw) the updated model $\mathbf{M}_4 = \mathbf{M}_3{}^{!\phi_2}$ after this new truthful public announcement $!\phi_2$, and **use this model to answer the following question**: *What are two numbers?*

Question 3. (*42 points*) There are four agents: Alice, Bob, Charles and Eve (the evil outsider). A coin is on the table, and it is **common knowledge that Alice, and ONLY Alice, can see the upper face of the coin**. Let's suppose that, ***in reality, the coin lies Heads up***.

1. (*6 points*) **Represent** (draw) this situation as a **state model** $\mathbf{M}_0$. (Use *atomic sentences* $\mathbf{H}, \mathbf{T}$, where $\mathbf{H}$ means that the coin lies Heads up, $\mathbf{T}$ means that the coin lies Tails up. Draw the accessibility relations R_a, R_b, R_c and R_e for each agent, indicate which atomic sentences hold at which states, and mark the "real" world with a star.)

2. (*6 points*) Alice sends a private email message to Bob, with BCC (Blind Carbon Copy) to Charles, saying "*The coin lies Heads up*". It is common knowledge that Alice never lies, and that the message is sent over a secure channel and is guaranteed to instantly reach (and be read by) its recipients.

   (Recall how BCC works: being in BCC, Charles sees the email, including the fact that it was sent to Bob: but Bob *cannot* see the BCC, so he doesn't know that Charles also got this email!) Moreover, we assume that (it is common knowledge that) *Bob believes the email to be a completely private communication* between Alice and himself (so he doesn't even suspect that the email was sent with BCC). Finally, the evil *Eve doesn't suspect any of this* happening.

   Represent (draw) this action using an **event model $\mathbf{\Sigma}$** with **3 actions**. (Specify the actions' preconditions, draw relations R_a, R_b, R_c, R_e representing agents' beliefs about what is going on, and specify which action is the "real" one.)

3. (*6 points*) **Draw a state model $\mathbf{M}_1$** for the situation **after** the action described in part 2, by computing the update product $\mathbf{M}_1 = \mathbf{M}_0 \otimes \mathbf{\Sigma}$.

4. (*6 points*) Let us now consider an **alternative scenario**: **everything goes as in part 2** (Alice attempting to privately send to Bob with BCC to Charles the same true message "The coin lies Heads up"), **EXCEPT that** in fact the **evil outsider Eve hacks the communication channel**. So, *unknown, and unsuspected, by anybody else*, Eve reads the message (including the name of the sender, the receiver, and of the BCC recipient). Nothing else changes (-so the message, while secretely read by Eve, is still delivered to all its recipients).

   Represent this action using an **event model $\mathbf{\Sigma}'$** with **4 actions**.

5. (*6 points*) Assuming we *start again from the initial situation* (in model $\mathbf{M}_0$), **draw a model $\mathbf{M}'_1$** for the situation **after** the action described in the previous part, by computing the update product $\mathbf{M}'_1 = \mathbf{M}_0 \otimes \mathbf{\Sigma}'$.

6. (*6 points*) Finally, let us now consider **yet another alternative scenario**: **everything goes as in part 4** (Alice attempting to privately send the message "The coin lies Heads up" to Bob with BCC to Charles, while Eve in fact hacks the communication channel and secretely reads the message), **EXCEPT that** in addition **Eve blocks the message from being delivered to any of its recipients**. (So Bob and Charles *don't get* any message, and don't even suspect this communication was attempted. Alice of course alsp *doesn't suspect* that her message was hacked and blocked.)

   Represent this action using an **event model $\mathbf{\Sigma}''$** with 4 actions.

7. (*6 points*) Assuming we start again from the *initial situation* (in model $\mathbf{M}_0$), **draw a model $\mathbf{M}''_1$** for the situation **after** the action described in the previous part, by computing the update product $\mathbf{M}''_1 = \mathbf{M}_0 \otimes \mathbf{\Sigma}''$.