Warming-Up Exercises Week 2
(to be done in Werkcollege classes)

These are NOT homework exercises, but warming up exercises, to be explained by the TA's in the Werkcollege classes. But note that some of them are harder than the Homework exercises!

Question 1. In this question, we assume that email communication is completely secure and reliable: messages instantly arrive at destination and they are instantly read by the recipients, with no possibility of any messages getting lost, missed, unread or intercepted by other agents.

Consider the following scenario: there is a coin lying on a table under a cover, and we have three agents named Albert, Bob and Claire. It is common knowledge that none of them sees (knows) if the coin lies Heads up or Tails up. Let us also assume that in reality (unknown to the agents) the coin lies Heads up

1. *Represent* (draw) this situation as a Kripke model $\mathbf{M}_1$, using *atomic sentences* H (Heads up), T (Tails up). Draw the accessibility relations R_a, R_b and R_c for each agent, indicate which atomic sentences hold at which states, and specify which possible world is the "real" one.

2. Suppose now the following action happens: in front of everybody, Albert takes a very quick glance at the coin under the cover. He may have succeeded to see which face is up, but it is also possible that he didn't see this (-since this was a very quick glance!). So (it is common knowledge that) Bob and Claire don't know if Albert succeeded or not to see the face. However, in reality Albert does succeed to see (correctly) that the coin lies Heads up.

   Represent (draw) this action using an *event model* $\mathbf{\Sigma}_1$ with 3 actions: the "real" action (in which Albert succeeds to see the coin lying Heads up); an alternative action in which Albert succeeds to see the face but the coin lies Tails up; and the action in which Albert fails to see the face of the coin (so that essentially nothing really happens in this case). Specify the actions' preconditions, draw relations R_a, R_b, R_c representing agents' beliefs about what is going on, and specify which action is the "real" one.

3. *Represent (draw) a model* $\mathbf{M}_2$ for the situation *after* the action described in the previous part, by computing the update product $\mathbf{M}_2 = \mathbf{M}_1 \otimes \mathbf{\Sigma}_1$. Specify which world is *the "real" one.*

4. Suppose now that, *after the previous action*, the following happens: Albert sends a secret email from his mobile phone to Bob, telling him "I (Albert) know that the coin lies Heads up". Claire is not sent any email, and she doesn't suspect anything: (it is common knowledge that) she thinks that nothing happened (i.e. that no email was sent).

   Represent (draw) this action using an *event model* $\mathbf{\Sigma}_2$. Specify the actions' preconditions, draw relations R_a, R_b, R_c representing agents' beliefs about what is going on, and specify which action is the "real" one.

5. Represent (draw) a model $\mathbf{M}_3$ for the situation *after* the action described in the previous part, by computing the update product $\mathbf{M}_3 = \mathbf{M}_2 \otimes \mathbf{\Sigma}_2$. Specify which world is the "real" one.

6. Let us now consider an *alternative scenario*: instead of the action described in the previous two parts, the following happens. After seeing the face of the coin, Albert does send an email to Bob saying "I (Albert) know that the coin lies Heads up", but the message is sent with BCC (Blind Carbon Copy) to Claire. Recall how BCC works: by listing Claire in BCC, Albert makes sure that she also gets to see the email, including the fact that it was sent to Bob: but Bob obviously *cannot* see the BCC, so he doesn't know that Claire also got this email! Moreover, let us suppose that (it is common knowledge that) Bob *does not even suspect* that this email was sent with BCC: so Bob believes the email to be a completely private communication between Albert and himself (unknown and unsuspected by Claire).

   Represent (draw) this action using an *event model* $\mathbf{\Sigma}'_2$. Specify the actions' preconditions, draw relations R_a, R_b, R_c representing agents' beliefs about what is going on, and specify which action is the "real" one.

7. Assuming we start just after Albert saw the face of the coin (in model $\mathbf{M}_2$), *represent (draw) a model* $\mathbf{M}'_3$ for the situation *after* the action described in the previous part, by computing the update product $\mathbf{M}'_3 = \mathbf{M}_2 \otimes \mathbf{\Sigma}'_2$. Specify which world is *the "real" one.*

8. Finally, let us now consider yet *another alternative scenario*: instead of the action described in the previous two parts, the following happens. After seeing the face of the coin, Albert does send an email to Bob saying "I (Albert) know that the coin lies Heads up", but now he lists Claire in the CC (Carbon Copy) for this email. (Recall how CC works:

everybody on the CC list, as well as the sender and the recipient of the message, see the message, including the CC list itself; so everybody knows who else got the message.)

Represent (draw) this action using an *event model* $\mathbf{\Sigma}_2''$. Specify the actions' preconditions, draw relations R_a, R_b, R_c representing agents' beliefs about what is going on, and specify which action is the "real" one.)

9. If we start again just after Albert saw the face of the coin (in model $\mathbf{M}_2$), what is the model $\mathbf{M}_3'' = \mathbf{M}_2 \otimes \mathbf{\Sigma}_2''$ representing the situation after this last action?

Question 2.

1. As in the previous question, we assume that email communication is instant, completely secure and reliable. Consider the following scenario:

   Bob has two friends Alice and Eve. Bob's is planning to throw a birthday party. He hasn't yet told anyone about the party. It is common knowledge that: Alice and Eve consider it possible that Bob might throw a party (since they know it's his birthday), but they do not know for sure that there will be a party. And it is also of course common knowledge that Bob knows whether he'll throw a party or not.

   Represent (draw) this situation as a Kripke model $\mathbf{M}$, which captures all the above information, using one *atomic sentence* p (denoting the fact that Bob will throw a party), three accessibility relations R_a, R_b and R_e, denoting respectively Alice's, Bob's and Eve's, uncertainty relations; and valuations (labeling), indicating which atomic sentences hold at which states. Specify on your drawing what is the *actual (real) state* of the system.

2. Suppose now the following action happens: Bob calls Alice to invite her to the party. Alice believes that Bob invited all his friends ((i.e. both Alice and Eve), and moreover she believes that this fact (that he invites both of them) is *common knowledge*. (So Alice doesn't even consider it possible that Bob didn't invite Eve as well.) However, in reality Bob does **not** invite Eve. And in fact, Eve does not know (and *does not even suspect*) that all this is going on: she has no clue about the invitation. Moreover, we assume that *Bob knows all this*: he knows that Even doesn't suspect anything and he also knows that Alice believes that he invited bothof them.

   Represent (draw) this complex action using an *event model* $\mathbf{\Sigma}$ with *three actions*. (Make clear what are the preconditions of each actions, and what are each agent's relations R_a, R_b, R_c representing his/her beliefs about what is going on. Specify on your drawing what is the *real) action* of the system.) *Is this an epistemic model, a doxastic model or none of the two*?

3. *Represent (draw) a model* $\mathbf{M}'$ for the situation *after* the action described in the previous part, by computing the update product $\mathbf{M}' = \mathbf{M} \otimes \mathbf{\Sigma}$. Specify which is *the real world* of your new model.

4. Suppose that after the above action, Eve accidently finds out about the party (e.g. because she sees Bob shopping for the party), without anybody suspecting that she finds out. *Draw an event model* $\mathbf{\Sigma}'$ *representing this action.*

5. Represent (draw) a model of the situation after the action in the previous part, by computing the update product $\mathbf{M}'' = \mathbf{M}' \otimes \mathbf{\Sigma}'$.

6. At the end of the above scenario, is it true that *all* agents (Alice, Bob and Eve) know that there will be a party? Is the *party common knowledge*? Briefly justify your answers by referring to properties holding at worlds in the model $\mathbf{M}''$ drawn in the last part.

Question 3

1. Show that Negative Introspection of Beliefs

$$(\neg B_a \phi) \Rightarrow B_a \neg B_a \phi$$

   is valid on (the class of) all *doxastic models.*

   HINT: You need to use the Euclideanness condition (from the definition of doxastic models).

2. Show that the sentence

$$(\neg \phi) \Rightarrow B_a \neg B_a \phi$$

   is NOT valid on (the class of) all *doxastic models.*

   HINT: Give a counterexample (by presenting some world, in a doxastic model, at which a sentence of the form $(\neg p) \Rightarrow B_a \neg B_a p$ is FALSE, i.e. $\neg p$ is TRUE but $B_a \neg B_a p$ is FALSE at this world.)

Question 4.

1. Each of two children, Alice and Bob, is given a sheet of paper with a natural number written on it. It is common knowledge that the two numbers n_A, n_B belong to the set $\{0, 1, 2, 3\}$, that they are *distinct* ($n_A \neq n_B$) and that *each of the two children can see only the number written on his/her sheet of paper.* Let us assume that, in fact, Alice's number is 2 and Bob's number is 3. (So $n_A = 2, n_B = 3$.)

   Represent the above situation by an *epistemic model* $(\mathbf{S}, s)$, with two agents (A for Alice, B for Bob) and eight atomic sentences k_A, k_B, with $k \in \{0, 1, 2, 3\}$. The sentence k_A means that Alice's number is k, and similarly for k_B. Clearly specify the set of states, the valuations and the accessibility relations, and point out what is the real state.

2. Alice and Bob are now asked to answer publicly, truthfully and simultaneously the following question: "Is your number even or odd?"

   Assuming that the children do what they are asked, represent this simultaneous announcement as an epistemic action in an action model $\mathbf{\Sigma}$. (Do not use any other atomic sentences but the ones listed above!) Clearly specify the set of actions, the preconditions and the accessibility relations, and point out what is the real action.

3. *Represent (by a pointed state model)* the epistemic situation *after* the above announcement, by computing the *update product* $\mathbf{S} \otimes \mathbf{\Sigma}$ of the two models. Clearly specify the set of states, the valuations and the accessibility relations, and point out what is the real state.

4. *Does either of the children know now (after the announcement) the other's number? Justify your answer.*

5. If, instead of assuming the numbers to be 2 and 3, we assume that the numbers *have the same parity* (i.e. are both odd or both even), would this make any difference to the answer to the question in the previous part? *Justify your answer, by looking at the possible output-states in this case (after the Alice and Bob truthfully answer the above question).*

6. We now make the same assumption about the numbers as in parts 1 and 2 above (i.e. $n_A = 2, n_B = 3$), but we assume that in fact the two children are *lying* when answering the question: so Alice will say her number is odd, and Bob will say his number is even. *It is common knowledge that neither of the children suspects the other is lying*: they're both convinced that the other is telling the truth.

Represent this "simultaneous lying" announcement, as an epistemic action in a (pointed) action model $\mathbf{\Sigma}'$.

7. *Represent (as a pointed state model)* the epistemic situation *after the "simultaneous lying" announcement* from the previous part, by computing the *update product* $\mathbf{S} \otimes \mathbf{\Sigma}'$.

Question 5. Suppose there are two children, Alice and Bob, who are perfect logicians. Alice has a natural number $n_a \in \{0, 1, 2, \ldots\}$ written on her forehead and Bob has a natural number $n_b \in N = \{0, 1, 2, \ldots\}$ written on his forehead. It is common knowledge that: (a) **each of them can see the other's number, but neither of them can see his/her number**; (b) one of the two numbers is the immediate successor of the other (i.e. either $n_a = n_b + 1$ or $n_b = n_a + 1$).

1. Draw, or represent mathematically, the initial epistemic model for this situation (with two agents a and b).

   HINT: Note that the situation is dual to the story in this week's slides: in that story the children could see their own number, but not the other's number; while here we assume the other way around! So the model will be different from the one drawn on the slide 10 of this Tuesday's lecture; more precisely, the possible worlds will be the same, but the accessibility relations will be different (encoding the different assumptions about who can see what).

2. Are there any possible worlds in your model in which Alice knows her number? If so, list all those worlds.

   Same question for Bob: list all the worlds (if any) in which Bob knows his number.

3. Let us **suppose for the moment that Alice's number is** 1 **and Bob's number is** 2. The Father asks them (once) "Do you know your own number?". The two are supposed to answer truthfully, publicly, simultaneously and independently (without any other communication). **What will they answer**?

4. In the same assumptions as in the previous part ($n_a = 1, n_b = 2$), let us interpret the children's simultaneous answers to Father's first question as one big (truthful) public announcement. Draw, or represent mathematically, the epistemic model of the situation **after** this announcement (be deleting all the worlds that are incompatible with their answers).

5. In the same assumptions as in the previous part ($n_a = 1, n_b = 2$), let us suppose that after the children have answered Father's first question

(as above), Father **repeats the same question**: "Do you know your own number?". **What will the children answer now**?

6. Finally, let us drop the assumption that $n_a = 1, n_b = 2$: we only know (like the children themselves) that one of the numbers is the immediate successor of the other, but **we don't known the numbers**. Instead, we are told the following: Father keeps repeating the same question "Do you know your number?" over and over and over again. In the first round, both children answer (truthfully, publicly and simultaneously) "I don't know my number"; in the second round, they both answer again "I don't know my number"; in the third round, they both answer again "I don't know my number". Finally, in the **fourth round** (Alice still answers ' don't know" but) Bob answers "Now I know my number". **What were the numbers**?

Question 6.

Prove the validity on epistemic models of the formula:

$$K_a \varphi \Rightarrow D\varphi.$$