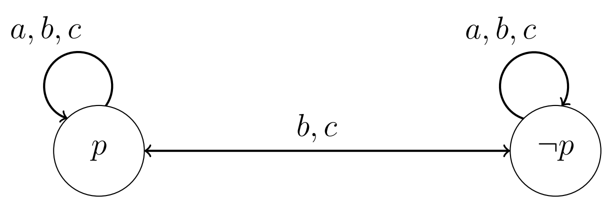
Warming up Week 3

November 16, 2018

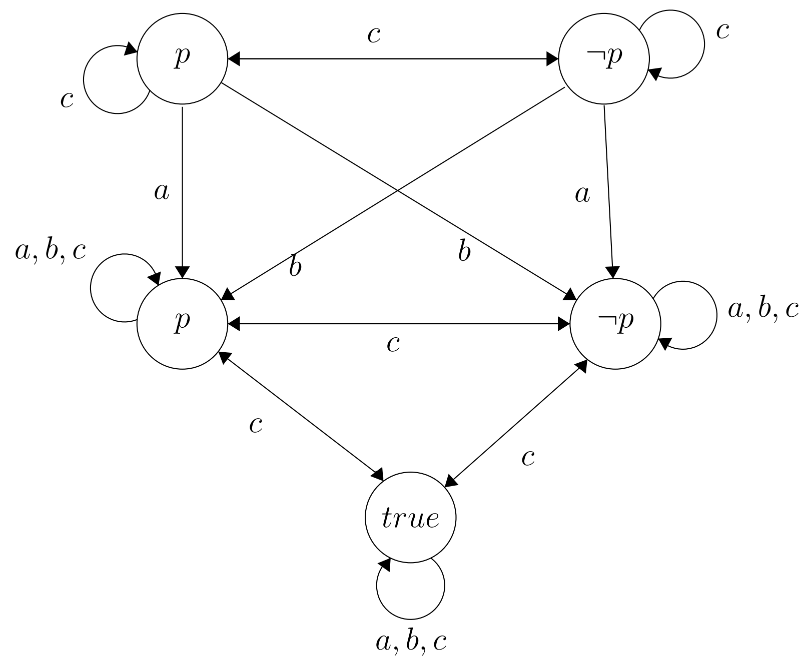
Recall the story of slides 23-24 of Week 3, Hoorcollege 2:

Agent A sends a message to agent B. This message is either p or $\neg p$ and A knows that message. Agent C intercepts the message, but he can't read it (so it doesn't know whether the message is p or $\neg p$). What he does however is to modify the content of the message (so if the message is p, it becomes $\neg p$, and if the message was $\neg p$, it is now p). B receives the message and announces to A that he got the message. Neither A nor B suspects that C could have intercepted the message. They think that C thinks that either both A and B know the content of the message, or that they both don't know.

The initial model is the following model.



The event model is the following graph.



Exercise slide p.23-24.

(a) Show that $[\alpha]\Box_A\Box_B p$.

Hint. Use the Knowledge-Action axiom to push $[\alpha]$ inside the modalities. Use also that $(\phi \Rightarrow \phi) \iff \mathit{True}$ and $(\phi \Rightarrow \mathit{True}) \iff \mathit{True}$ and $\Box_B \mathit{True} \iff \mathit{True}$.

(b) Derive that $[\alpha](\Box_a\Box_B p \vee \Box_A\Box_B\neg p)$.

Hint. You can use the validity $[\alpha]\phi \Rightarrow [\alpha](\phi \vee \psi)$.

(c) Similarly to (a), show that $[\beta]\Box_A\Box_B\neg p$.

(d) Derive that $[\beta](\Box_a\Box_B p \vee \Box_A\Box_B\neg p)$.

(e) Using (b) and (d), show that $[\alpha]\Box_C(\Box_A\Box_B p \vee \Box_A\Box_B\neg p)$.

Hint. Use that $\mathit{True} \wedge \mathit{True} \iff \mathit{True}$.

Proof of (a). We have

$$
\begin{aligned}
[\alpha]\Box_A\Box_B p &= (p \Rightarrow \Box_A[\alpha']\Box_B p) && (1)\\
&= (p \Rightarrow \Box_A(p \Rightarrow \Box_B[\alpha']p)) && (2)\\
&= (p \Rightarrow \Box_A(p \Rightarrow \Box_B(p \Rightarrow p))) && (3)\\
&= (p \Rightarrow \Box_A(p \Rightarrow \Box_B \mathit{True})) && (4)\\
&= (p \Rightarrow \Box_A(p \Rightarrow \mathit{True})) && (5)\\
&= (p \Rightarrow \Box_A \mathit{True}) && (6)\\
&= (p \Rightarrow \mathit{True}) && (7)\\
&= \mathit{True} && (8)
\end{aligned}
$$

Here, (1) follows from the Knowledge-Action axiom and the fact that the only a-successor of α in the event model is α' (and the precondition of α' is p). Equivalence (2) follows from Knowledge-Action axiom and the fact that the only b-successor of α' in the event model is α' (and the precondition of α' is p). For (3), it follows from the Atomic Permanence axiom that

$$[\alpha']p = p \Rightarrow p.$$

For (4), we use the fact that $(p \Rightarrow p) \iff \mathit{True}$. (5) follows from the hint $\Box_B \mathit{True} \iff \mathit{True}$. Equivalence (6) is obtained using the hint $(\phi \Rightarrow \mathit{True}) \iff \mathit{True}$. Then, for (7), we use the hint $\Box_A \mathit{True} \iff \mathit{True}$. Finally, equivalence (8) is obtained using the hint $(\phi \Rightarrow \mathit{True}) \iff \mathit{True}$.

Proof of (b). By (a), we have $[\alpha]\Box_A\Box_B p$ is valid. So it follows from the hint that $[\alpha](\Box_a\Box_B p \vee \Box_A\Box_B\neg p)$ is also valid.

Proof of (c). This is very similar to (a). We have

$$
\begin{aligned}
[\beta]\Box_A\Box_B\neg p &= (\neg p \Rightarrow \Box_A[\beta']\Box_B\neg p) && (9)\\
&= (\neg p \Rightarrow \Box_A(\neg p \Rightarrow \Box_B[\beta']\neg p)) && (10)\\
&= (\neg p \Rightarrow \Box_A(\neg p \Rightarrow \Box_B(\neg p \Rightarrow \neg p))) && (11)\\
&= (\neg p \Rightarrow \Box_A(\neg p \Rightarrow \Box_B \mathit{True})) && (12)\\
&= (\neg p \Rightarrow \Box_A(\neg p \Rightarrow \mathit{True})) && (13)\\
&= (\neg p \Rightarrow \Box_A \mathit{True}) && (14)\\
&= (\neg p \Rightarrow \mathit{True}) && (15)\\
&= \mathit{True} && (16)
\end{aligned}
$$

Here, (9) follows from the Knowledge-Action axiom and the fact that the only a-successor of β in the event model is β' (and the precondition of β' is $\neg p$). Equivalence (10) follows from Knowledge-Action axiom and the fact that the only b-successor of β' in the event model is β' (and the precondition of β' is $\neg p$). For (11), it follows from the Atomic Permanence axiom that

$$[\beta']\neg p = \neg p \Rightarrow \neg p.$$

For (12), we use the fact that $(p \Rightarrow p) \iff \mathit{True}$. (13) follows from the hint $\Box_B \mathit{True} \iff \mathit{True}$. Equivalence (14) is obtained using the hint $(\phi \Rightarrow \mathit{True}) \iff \mathit{True}$. Then, for (15), we use the hint $\Box_A \mathit{True} \iff \mathit{True}$. Finally, equivalence (16) is obtained using the hint $(\phi \Rightarrow \mathit{True}) \iff \mathit{True}$.

Proof of (d). By (c), we have $[\beta]\Box_A\Box_B\neg p$ is valid. So it follows from the hint that $[\beta](\Box_a\Box_B p \vee \Box_A\Box_B\neg p)$ is also valid.

Proof of (e). We have

$$\begin{aligned}
[\alpha]\Box_C(\Box_A\Box_B p \vee \Box_A\Box_B\neg p) &= (p \Rightarrow [\alpha](\Box_a\Box_B p \vee \Box_A\Box_B\neg p)) \wedge && (17)\\
&\quad (p \Rightarrow \Box_C[\beta](\Box_a\Box_B p \vee \Box_A\Box_B\neg p)) && (18)\\
&= (p \Rightarrow \Box_C \mathit{True}) \wedge (p \Rightarrow \Box_C \mathit{True}) && (19)\\
&= (p \Rightarrow \mathit{True}) \wedge (p \Rightarrow \mathit{True}) && (20)\\
&= \mathit{True} \wedge \mathit{True} && (21)\\
&= \mathit{True} && (22)
\end{aligned}$$

Here, (18) follows from the Knowledge-Action axiom and the fact that the only c-successors of α in the event model are α and β. For (19), we use the facts that $[\alpha](\Box_a\Box_B p \vee \Box_A\Box_B\neg p)$ is equivalent to *True* (from (a)) and that $[\beta](\Box_a\Box_B p \vee \Box_A\Box_B\neg p)$ is equivalent to *True* (from (b)).

For (20), we use the fact that $\Box_C \mathit{True} \iff \mathit{True}$. For (21), we use the hint that $\phi \Rightarrow \mathit{True}$ is equivalent to *True*. For (21), we use the hint that $\mathit{True} \wedge \mathit{True}$ is equivalent to *True*.