Mid-Term Exam Computational Logic 2017-18

IMPORTANT NOTE: The total number of points is 100. The exam is **open-book**. Please don't forget to **write your full name on each page in capital letters**!

Question 1 (*20 points*)

1. (*10 points*) Show that the sentence

$$B_a B_a p \Rightarrow B_a p$$

   is **valid** on (the class of) all **doxastic** models.

   HINT: Use Euclideanness! (And notice that Euclideannes holds also when the second and the third world are the same!)

   NOTE: Transitivity will NOT help you (-although it is true in doxastic models). And you CANNOT use reflexivity! (Why?)

2. (*10 points*) Define the "true belief" operator as an abbreviation by:

$$Tb_a\phi \ := \ (\phi \wedge B_a\phi)$$

   (So a sentence ϕ is "true belief for agent a " iff it is both true and believed by a.) Show that the sentence

$$B_a Tb_a p \Rightarrow Tb_a p$$

   is **NOT valid** on (the class of) **doxastic** models.

   HINT: For part 2, give a **counterexample**: some doxastic model containing a world w such that $w \models B_a Tb_a p$, but $w \not\models Tb_a p$.

Question 2. (*34 points*) Two children, S ("Mr. Sum") and P ("Ms. Product"), are told that two numbers secret x and y are printed inside a sheet of paper hidden in an envelope. It is **common knowledge** that: *both numbers belong to the set* $\{2, 3, 4, 5, 6\}$*; the first number is strictly smaller than the second (i.e.* $x < y$*); NONE of the children can see any of the two numbers; but Mr. Sum was told* ***the sum*** *(and only the sum) of the two numbers (-so he knows their sum); and Ms. Product was told* ***the product*** *(and only the product) of the two numbers (-so she knows their product)*; and *the children always tell the truth.*

1. (*4 points*) **How many possible worlds** are there (that are consistent with the story above)?

2. (*6 points*) **Represent** (draw) the above situation as an **epistemic model $\mathbf{M}_1$**, which possible worlds represented as pairs of numbers (i, j) (where this is the world in which $x = i$ and $y = j$), with $i, j \in \{2, 3, 4, 5, 6\}$. For any of the *two variables* variables x and y, consider (only) atomic sentences of the form $Odd(x)$ or $Odd(y)$, saying that "*the value of x is an odd number*" and respectively "*the value of y is an odd number*". Write down the **valuation** for these atomic sentences $Odd(x)$ and $Odd(y)$ (i.e. specify: in which worlds is each of them true?)

   NOTE: This an *epistemic model*, so all relations are reflexive, symmetric and transitive. To keep the drawing simple, you **can skip in this exercise all the loops and the arrow directions, simply drawing (undirected) lines labeled by agents S or P.**

3. (*6 points*) Mr. Sum is asked "Do you know the numbers?", and he answers loud: "*No, I don't know the numbers*". Encode **this answer as a sentence ϕ in a formal language**, using the operator for numerical knowledge.

4. (*6 points*) Since children must tell the truth, we consider the previous answer to be a truthful public announcement $!\phi$. **Represent (draw) the updated model $\mathbf{M}_2 = \mathbf{M}_1{}^{!\phi}$** after this announcement.

5. (*6 points*) After the previous announcement, Ms. Product is asked the same question, and she also answers loud: "*No, I don't know the numbers*". As before, **write this answer as a sentence ψ in a formal language, and represent (draw) the updated model $\mathbf{M}_3 = \mathbf{M}_2{}^{!\psi}$** after this new truthful public announcement.

6. (*6 points*) After the announcement in the previous part, Mr. Sum says: "*But NOW I know the numbers, and moreover (just to give you a hint, I can tell you that) x is an odd number*!" As before, **write this answer as a sentence θ in a formal language**, **represent (draw) the updated model $\mathbf{M}_4 = \mathbf{M}_3{}^{!\theta}$** after this new truthful public announcement, and **use this model to answer the question**: ***What are two numbers?***

Question 3. (*46 points*) There are three agents: Alice, Bernard and Cheryl. Each of the them receives a "bit" (i.e. a number from the set $\{0, 1\}$), **known only to him or her**. But it is *common knowledge* that: **at least two of the three bits have values equal to** 0; **our agents always tell the truth**; and **agents can send emails over fully secure communication channels** (no hacking!) and **all their messages are instantly delivered**.

1. (*4 points*) Let us use three variables x_a, x_b, x_c to denote the three bits. **Represent** (draw) the above situation as an **epistemic (state) model** $\mathbf{S}_1$, with *three agents* a, b, c. You may denote the worlds as triplets (x_a, x_b, x_c) of the three values. As atomic sentences, consider only statements of the form $x_a = i$, $x_b = i$ or $x_c = i$ (with $i \in \{0, 1\}$).

2. (*6 points*) Cheryl asks Alice loud (and public) "*What is your bit*?", and Alice answers equally loud "*I am replying to you by private email*" (so **everybody, including Bernard, can hear them**). At the same time (since she never lies), she **indeed sends a private email to Cheryl, telling her bit** x_a.

   Represent **(draw) this action, using an *event model*** $\mathbf{\Sigma}$ with 2 actions (for the two possible answers "$x_a = 1$ " and "$x_a = 0$ ").

   HINT: This is a "*fair-game announcement*" (similar to the "legal private viewing" example).

3. (*6 points*) *Draw a state model* $\mathbf{S}_2$ for the situation *after* the action described in the previous part, by computing the **update product** $\mathbf{S}_2 = \mathbf{S}_1 \otimes \mathbf{\Sigma}$.

4. (*6 points*) Alice asks Cheryl loud (and public) "*Do you know Bob's bit, and if so what is it*?". Cheryl answers equally loud "*I am replying to you by private email*" (so **everybody, including Bernard, can hear them**). As before, since Cheryl never lies, she simultaneously **sends a private email to Alice** (-telling her one of the following: either that *she, Cheryl, knows that Bob's bit is* 1, or that *she knows that Bob's bit is* 0, or that *she doesn't know*).

   Represent **(draw) this action, using an *event model*** $\mathbf{\Sigma}'$ with 3 actions (corresponding to the three possible answers).

   HINT: This is again a "*fair-game announcement*" (similar to the "legal private viewing" example), but with three actions.

5. (*6 points*) *Draw a state model* $\mathbf{S}_3$ for the situation *after* the action described in the previous part, by computing the **update product** $\mathbf{S}_3 = \mathbf{S}_2 \otimes \mathbf{\Sigma}'$.

6. (*6 points*) Bob **sends to Cheryl a supposedly secret message, but with BBC (Blind Carbon Copy) to Alice**, saying "*I know Alice's bit*" (without telling what is the value!). Obviously, Cheryl cannot see that BCC, so she *believes this is a fully private communication* from Bob (and so that Alice doesn't suspect anything). But of course Alice, being in BCC, also gets the message (and both her and Bob know this).

   Represent **(draw) this action, using an *event model*** $\mathbf{\Sigma}''$ with 3 actions (one of which is the *skip* action in which nothing happens).

7. (*6 points*) *Draw a state model* $\mathbf{S}_4$ for the situation *after* the action described in the previous part, by computing the **update product** $\mathbf{S}_4 = \mathbf{S}_3 \otimes \mathbf{\Sigma}''$.

8. (*6 points*) **What are the bits of each agent**?

   HINT: The fact that the announcement in part 7 is **truthful** (since we assumed nobody ever lies) should give you the necessary information to find the three bits.