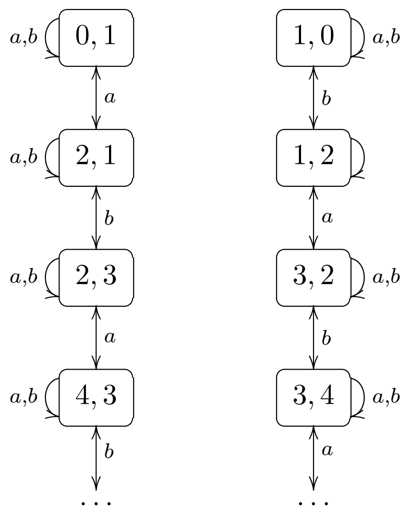
# Answers to Questions 5 and 6

**Question 5**: Suppose there are two children, Alice and Bob, who are perfect logicians. Alice has a natural number $n_a \in \{0, 1, 2, \ldots\}$ written on her forehead and Bob has a natural number $n_b \in N = \{0, 1, 2, \ldots\}$ written on his forehead. It is common knowledge that: (a) **each of them can see the other's number, but neither of them can see his/her number**; (b) one of the two numbers is the immediate successor of the other (i.e. either $n_a = n_b + 1$ or $n_b = n_a + 1$).

1. Draw, or represent mathematically, the initial epistemic model for this situation (with two agents $a$ and $b$).

   ANSWER:

2. Are there any possible worlds in your model in which Alice knows her number? If so, list all those worlds.

   Same question for Bob: list all the worlds (if any) in which Bob knows his number.

   ANSWER: Yes, Alice knows her number **only in world** $(1, 0)$: there are no uncertainty arrows for Alice going from $(1, 0)$ to any other world, so if the world is $(1, 0)$ then she knows it (hence she knows her number). In all the other worlds, she doesn't know.

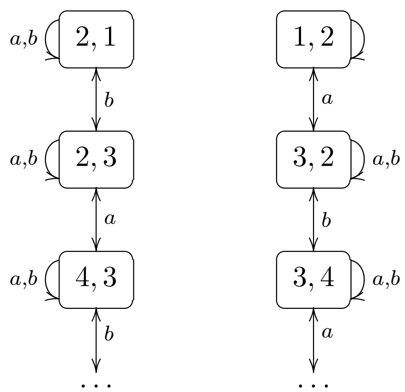
   Bob knows his number **(only) in world** $(0, 1)$, for similar reasons.

3. Let us **suppose for the moment that Alice's number is** 1 **and Bob's number is** 2. The Father asks them (once) "Do you know your own number?". The two are supposed to answer truthfully, publicly, simultaneously and independently (without any other communication). **What will they answer**?

   ANSWER: They both answer "I don't know my number". Indeed, in world $(1, 2)$ none of them knows his/her number: they both have uncertainty arrows going to other worlds, in which their own number is different.

4. In the same assumptions as in the previous part ($n_a = 1, n_b = 2$), let us interpret the children's simultaneous answers to Father's first question as one big (truthful) public announcement. Draw, or represent mathematically, the epistemic model of the situation **after** this announcement (be deleting all the worlds that are incompatible with their answers).

   ANSWER: The updated model is obtained by deleting the worlds in which one or the other knows his/her number, i.e. deleting $(1, 0)$ and $(0, 1)$:

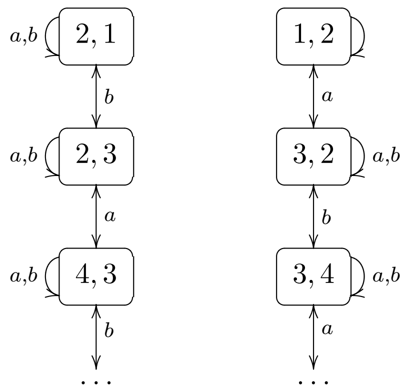



5. In the same assumptions as in the previous part ($n_a = 1, n_b = 2$), let us suppose that after the children have answered Father's first question (as above), Father **repeats the same question**: "Do you know your own number?". **What will the children answer now**?

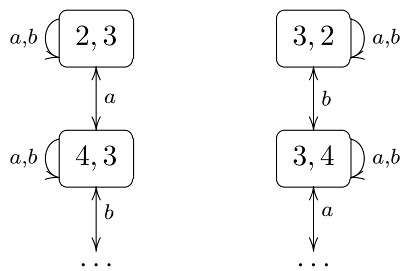
   ANSWER: Now Bob will answer "I know my number" (while Alice still answers "I don't know"). This is because in the updated model (from previous part), Bob knows his number in world $(1, 2)$ (since he has no longer any uncertainty arrows going from $(1, 2)$ to any other worlds), while Alice still doesn't know (since she can't distinguish between $(1, 2)$ and $(3, 2)$).

6. Finally, let us drop the assumption that $n_a = 1, n_b = 2$: we only know (like the children themselves) that one of the numbers is the immediate successor of the other, but **we don't known the numbers**. Instead, we are told the following: Father keeps repeating the same question "Do you know your number?" over and over and over again. In the first round, both children answer (truthfully, publicly and simultaneously) "I don't know my number"; in the second round, they both answer again "I don't know my number"; in the third round, they both answer again "I don't know my number". Finally, in the **fourth round** (Alice still answers ' don't know" but) Bob answers "Now I know my number". **What were the numbers**?

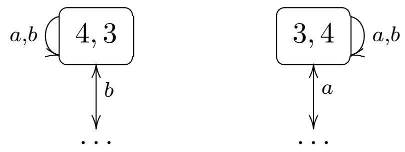
   ANSWER: Alice's number is 3 and Bob's number is 4. To show this, we need to do three deletions corresponding to the three rounds of answering "I don't know". In the first round, $(1, 0)$ and $(0, 1)$ are deleted obtaining as before:




   After the second round of "I don't know", worlds $(1, 2)$ and $(2, 1)$ are deleted, obtaining:




   After the third round of "I don't know", worlds $(2, 3)$ and $(3, 2)$ are deleted, obtaining:




Finally, we are told that in the fourth round Bob answers "I know my number". The only world in the last model above in which Bob knows his number is $(3, 4)$ (since this is the only world having no outgoing $b$-arrows pointing to any other world). So $(3, 4)$ must be the real world: hence, Alice's number is 3 and Bob's number is 4.

**Question 6**. Prove the validity on epistemic models of the formula:

$$K_a\varphi \Rightarrow D\varphi.$$

Let $\mathbf{M} = (W, R_a, \ldots, \nu)$ be any model and let $w \in W$ be any world satisfying

$$(1) \qquad w \models_{\mathbf{M}} K_a\varphi.$$

Let now $w'$ be **ANY world such that** $(w, w') \in R_1 \cap \ldots \cap R_n$. Then in particular we have that $(w, w') \in R_a$ (since $a \in \{1, \ldots, n\}$), so using (1) and the semantics of $K_a$ we conclude that

$$w' \models_{\mathbf{M}} \varphi.$$

This is for EVERY world $w'$ with $(w, w') \in R_1 \cap \ldots \cap R_n$, so (by the semantics of $D$), we obtain the desired conclusion:

$$w \models_{\mathbf{M}} D\varphi.$$