Discrete and Algorithmic Geometry

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Sheet 4

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Software

The **integral Gale complexity** of a polytope $P \subset \mathbb{R}^d$ with n vertices is

$$\operatorname{igc}(P) \;=\; \min\{\|G\|_\infty : G \subset \mathbb{Z}^e \text{ is a Gale diagram of } P\},$$

where $e = n-d-1$, $\|\mathcal{A}\|_\infty = \max\{\|v\|_\infty : v \in \mathcal{A}\}$ and $\|v\|_\infty = \max\{|v_i|\}$ for $v = (v_1, \dots, v_e)$.

While the existence of nonrational polytopes shows that $\operatorname{igc}(P) = \infty$ is possible (since $\min \emptyset = \infty$), here we are concerned with the following problem:

Problem G. *For $e \in \mathbb{Z}_{\geq 0}$ and $n, m \in \mathbb{Z}_{>0}$, determine*

$$q(e,n,m) \;=\; \#\left\{ \begin{array}{c} G \subset \mathbb{Z}^e : G \textit{ is a Gale diagram of a polytope} \\ \textit{with } n \textit{ vertices and } \operatorname{igc}(G) = m \end{array} \right\} \Big/ \textit{combinatorial equivalence.}$$

For example, $q(0,n,0) = 1$ and $q(1,n,m) = q(1,n,1)$ for all $m, n \geq 1$.

In more down-to-earth terms, we want to solve the following problem:

Problem G*. *Enumerate, up to combinatorial equivalence, all balanced configurations $\mathcal{V}$ of n vectors in $\mathbb{Z}^e$ whose coordinates are all at most m in absolute value, such that*

(1) *the maximum m is achieved by some $v \in \mathcal{V}$,*
(2) *and such that no hyperplane spanned by $e-1$ of the vectors strictly separates exactly one vector from the others.*

For this, recall

- that a vector configuration $\mathcal{V} = (v_1, \dots, v_n)$ is **balanced** if $\sum_i v_i = 0$;
- that no hyperplane defined by $e-1$ elements of $\mathcal{V}$ separates exactly one vector from the others iff the Gale dual of $\mathcal{V}$ is in convex position;
- and that two vector configurations are **combinatorially equivalent** if they define the same oriented matroid.

Your job is to write a function in the `polymake` framework that calculates $q(e,n,m)$. Some considerations to keep in mind:

- Correctness is more important than efficiency, but efficiency is supremely important.
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