Discrete and Algorithmic Geometry

Julian Pfeifle, UPC, 2016

Sheet 1

due on Monday, November 13, 2016

Reading

(1) Read Lectures 0,1,2 from Ziegler's *Lectures on Polytopes.*

(2) Read Sections 5.1, 5.2, 5.3 from Matoušek's *Lectures on Discrete Geometry.*

Writing

The *Minkowski sum* of two polytopes $P, Q \subset \mathbb{R}^d$ is the set

$$P + Q \;=\; \{p + q : p \in P,\, q \in Q\}.$$

The *Newton polytope* of a polynomial $f(x) = \sum_{i=1}^m c_i\, x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_d^{a_{id}} \in \mathbb{R}[x_1, x_2, \dots, x_d]$ is

$$\operatorname{New}(f) \;=\; \operatorname{conv}\big\{(a_{11}, a_{12}, \dots, a_{1d}), \dots, (a_{m1}, a_{m2}, \dots, a_{md})\big\} \;\subset\; \mathbb{R}^d.$$

(a) Prove that $P + Q \subset \mathbb{R}^d$ is a convex polytope. How does $P + Q$ change when we translate P and Q in $\mathbb{R}^d$?

(b) How does the Minkowski sum New(f)+New(g) (a geometric object) relate to the algebraic objects f and g?

(c) Let $f(x, y) = c_0 + c_1 x + c_2 xy + c_3 y$ and $g(x, y) = d_0 + d_1 x^2 y + d_2 xy^2$ be two polynomials with $c_i \in \mathbb{R}$ and $d_i \in \mathbb{R}$ for all i, and let $P = \operatorname{New}(f)$ and $Q = \operatorname{New}(g)$ be their Newton polygons. Calculate the *mixed area*

$$M(P, Q) \;=\; \operatorname{area}(P + Q) - \operatorname{area}(P) - \operatorname{area}(Q).$$

(d) Subdivide $P + Q$ into five pieces: a translate of P, a translate of Q, and three parallelograms, and relate $M(P, Q)$ to this subdivision.

Bernstein's Theorem asserts that for any generic polynomials f and g in two variables, the number of solutions in $(\mathbb{C}^\times)^2$ of the system $f(x, y) = 0$, $g(x, y) = 0$ equals the mixed area $M\big(\operatorname{New}(f), \operatorname{New}(g)\big)$. Here $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. In fact, this theorem applies to *Laurent polynomials*, which means that the exponents can be arbitrary integers, possibly negative.

(e) Use Bernstein's Theorem to conclude the geometric fact that for any *lattice polygons* P, Q (i.e., all vertices have integer coordinates), the mixed area $M(P, Q)$ is an integer.

(f) Use Bernstein's Theorem to conclude *Bezout's Theorem*: Two general plane algebraic curves of degree d respectively e meet in $d \cdot e$ points. (*Hint:* Consider the Newton polytopes of two general bivariate polynomials of degree d and e, respectively.) How does this relate to the polynomials f and g in parts (c) and (d)? (Be sure you understand —or look up— the difference between a *generic* and a *general* polynomial.)