4 A Tool Box

The definitions of triangulation and subdivision given in Chapter 2 are nice for theoretic reasoning, but they present a fundamental problem for computations. How can one check Definition 2.3.1 on a computer? Let $\mathbf{A}$ be a point configuration in $\mathbb{R}^m$, with set of labels J. Recall that a collection $\mathscr{S}$ of subsets of J is a *polyhedral subdivision* of $\mathbf{A}$ if it satisfies the following conditions:

(CP) If $C \in \mathscr{S}$ and $F \leq C$ then $F \in \mathscr{S}$ as well. (Closure Property)

(UP) $\bigcup_{C\in\mathscr{S}} \operatorname{conv}_{\mathbf{A}}(C) \supseteq \operatorname{conv}_{\mathbf{A}}(J)$. (Union Property)

(IP) If $C \neq C'$ are two cells in $\mathscr{S}$ then $\operatorname{relint}_{\mathbf{A}}(C) \cap \operatorname{relint}_{\mathbf{A}}(C') = \emptyset$. (Intersection Property)

For instance, how to prove that a given family of convex sets covers another convex set (Property (UP)? It is a non-trivial computational challenge, one that we have to solve in practice. A key goal of this chapter is to develop the computational tools to computational test conditions (CP), (UP) and (IP).

In this chapter, we give an overview of the most frequently used tools and concepts for manipulating and investigating triangulations and polyhedral subdivisions. We begin by developing combinatorial tools and concepts that lead to a fully algorithmic definition of a triangulation. The foundations are the oriented matroid notions of *circuits, cocircuits, and Gale transforms*. A more detailed introduction into these concepts can be found in [334, Chapter 6] and the comprehensive books [53, 59]. After a thorough look at some natural constructions, like deletion and contraction, we encounter the need to deal with vector configurations. We describe special regular triangulations in Section 4.3, such as pulling and placing triangulations. We discuss what happens to triangulations and polyhedral subdivisions during those processes. Finally, we present several equivalent characterizations of flips and of polyhedral subdivisions that may come in handy in different contexts. Some of them are aimed for use in computer programs for the enumeration of all triangulations of a point set (see, e. g., [260]).

4.1 Combinatorics of configurations

The basic idea in this section is that, in order to compute or characterize triangulations, the fundamental primitives needed are vectors of signs arising from affine/linear functionals, dependences, and determinants. To give a uniform treatment of them we introduce the following notation.

Definition 4.1.1. A *signature* on a finite set J is a partition of J into three subsets, V_-, V_0 and V_+. A signature is called *positive* if V_- is empty, and *negative* if V_+ is empty. Given a signature V_+, V_0, V_- on J, the set $V = V_- \cup V_+$ is called its *support*.

If $J = \{1, \dots, n\}$, then we can think of a signature as a vector of length n with entries in $\{-1, 0, +1\}$, and call it a *sign vector*. Reciprocally, every vector in $\mathbb{R}^n$ (or, more formally, every map $J \to \mathbb{R}$) induces a signature, consisting of the preimages of $(-\infty, 0)$, $\{0\}$, and $(0, \infty)$. An equivalent, and sometimes more convenient, way of representing signatures is as the ordered pair (V_+, V_-). This will be our preferred representation.

Definition 4.1.2. Let (V_+, V_-) and (U_+, U_-) be two signatures on a set J. We say that (V_+, V_-) and (U_+, U_-) are *conformal* if $V_+ \cap U_- = V_- \cap U_+ = \emptyset$.

(+,+,0,−,0,−,+,−,−)
(+,0,0,−,+,−,+,0,−)

(−,−,+,+,+,0)
(−,+,−,0,−,−)

Figure 4.1: Two pairs of sign vectors. The top pair is conformal and the second one is not.

Put differently, (V_+, V_-) and (U_+, U_-) are conformal if the *coordinate-wise* product of their representations as vectors does not have negative elements.

The *conformal sum* of two conformal signatures (V_+, V_-) and (U_+, U_-) is the signature $(V_+ \cup U_+, V_- \cup U_-)$. Observe this is not a signature if (V_+, V_-) and (U_+, U_-) are not conformal, since then the unions of positive and negative parts will not be disjoint. This is a reflection of the fact that the signature associated to a sum $v + \omega$ of two vectors $v, \omega \in \mathbb{R}^n$ can only be deduced from the signatures of v and ω if they are conformal.

4.1.1 Dependences, circuits, and the intersection property

Definition 4.1.3 (Dependence signatures)**.** Let $\mathbf{A} = (\mathbf{p}_i)_{i \in J}$ be a vector configuration with label set J. Let $\sum_{i \in J} \lambda_i \mathbf{p}_i = 0$ be an (affine or linear) dependence. That is, $(\lambda_i)_{i \in J}$ is a vector in the kernel of the matrix representing $\mathbf{A}$. The signature of this vector is called a *dependence signature* of $\mathbf{A}$.

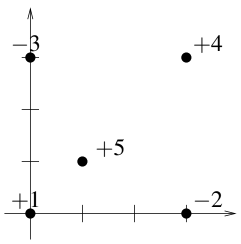

Figure 4.2: A dependence (affine) on a set of five points.

Example 4.1.4 (Five points in the plane)**.** We use the five planar point configuration of Example 2.2.9. Our choices of homogeneous coordinates and label set are

$$\begin{array}{cc} & \begin{array}{ccccc} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} \end{array} \\ \mathbf{A} = & \begin{pmatrix} 0 & 3 & 0 & 3 & 1 \\ 0 & 0 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \end{array}$$

We also display a picture of the points with a different dependent signatures that is not a circuit (see Figure 4.2). The full list of circuits for the configuration is given, in short notation, by $(23,45), (123,5), (14,5), (14,23)$.

One interesting property of dependence signatures is that the relative interiors of their positive and negative parts intersect.

Lemma 4.1.5. *Let V_+ and V_- be two disjoint subsets of the label set J of a vector configuration $\mathbf{A}$. Then the following are equivalent:*

(i) (V_+, V_-) *is a dependence signature on* $\mathbf{A}$.

(ii) $\operatorname{relint}(V_+) \cap \operatorname{relint}(V_-) \neq \emptyset$.

This lemma is true regardless of whether V_+ or V_- are empty. If V_+ is empty, then its relative interior is considered to be the zero cone $\{0\}$, which lies in the relative interior of a non-empty V_- if and only if there is a strictly positive linear dependence among its elements. In this case, $(\emptyset, V_-)$ is a dependence signature, as well as $(\emptyset, \emptyset)$.

Proof. The statement that (V_+, V_-) is the dependence signature for the dependence $(\lambda_i)_{i\in J}$ is equivalent to the following with $\lambda := \sum_{i\in V_+} \lambda_i = \sum_{i\in V_-} -\lambda_i$:

$$\sum_{i\in V_+} \lambda_i \mathbf{p}_i + \sum_{i\in V_-} \lambda_i \mathbf{p}_i = 0 \iff \sum_{i\in V_+} \lambda_i \mathbf{p}_i = \sum_{i\in V_-} -\lambda_i \mathbf{p}_i$$
$$\iff \sum_{i\in V_+} \frac{\lambda_i}{\lambda} \mathbf{p}_i = \sum_{i\in V_-} -\frac{\lambda_i}{\lambda} \mathbf{p}_i$$

Since all λ_i are non-zero, the left hand side and the right hand side are convex combinations in the relative interiors of both V_+ and V_-, respectively. □

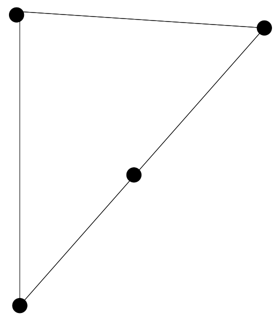

Figure 4.3: A dependence of four points which is not a circuit.

An example of this is the case of a circuit Z, of $\mathbf{A}$, i.e., a minimal dependent subconfiguration. Then, there is a unique dependence with support in Z, hence there is a unique dependence signature (Z_+, Z_-) with support Z (up to exchanging Z_+ and Z_-). These minimal dependence signatures get a special treatment.

Definition 4.1.6 (Oriented Circuit)**.** A minimally dependent subconfiguration Z of $\mathbf{A}$ is called a *circuit in* $\mathbf{A}$. A dependence signature with support on Z is also called a *circuit signature* or an *oriented circuit* of $\mathbf{A}$.

Lemma 4.1.7 (Uniqueness of Circuit Signatures)**.** *Let Z be a circuit in a point configuration $\mathbf{A}$ with signature (Z_+, Z_-). Then (Z_-, Z_+) is the only other possible circuit signature with support Z.*

Proof. There is only one dependence equation among the elements of Z (modulo a scalar factor) because if there were two, they could be used to get another one with smaller support, hence Z would not be minimally dependent. If the scalar factor is positive then it does not change the signature; if it is negative it exchanges the positive and negative parts of it. □

Remark 4.1.8. In concrete examples, it is easy to compute the signature of a circuit in matrix form. Let $\mathbf{p}_1, \ldots, \mathbf{p}_k$ be vectors minimally dependent, so that the matrix $\mathbf{Z}$ having them as columns has rank $k-1$. The unique dependence is the unique (modulo a scalar constant) element in the kernel of this matrix. To compute it, assume without loss of generality that the number of coordinates in your points is exactly $k-1$ (If there are more coordinates, just choose a subset of $k-1$ independent rows in your matrix) so that deleting the i-th column gives a square matrix. Let ω_i denote the determinant of the square matrix obtained deleting the i-th column. Then, the vector

$$\omega = (\omega_1, -\omega_2, \ldots, (-1)^k \omega_k)$$

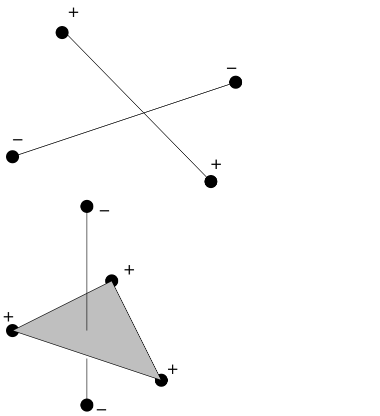

Figure 4.4: Two circuits represented as Radon partitions.

is in the kernel, as the following standard argument shows:

Consider the square $k \times k$ matrix obtained repeating the j-th row, for any j. Since it has a repeated row, this matrix is clearly singular. But its determinant, developed at the repeated row, is simply the scalar product of that row and the vector ω that we just defined. Hence, ω is orthogonal to every row of $\mathbf{Z}$.

One interesting property of circuits is that the relative interiors of the positive and negative parts intersect in a unique point. More interesting is the fact that this property essentially characterizes circuits.

Lemma 4.1.9. *Let Z_+ and Z_- be two non-empty subsets of the label set J of a vector configuration $\mathbf{A}$. Then the following are equivalent:*

(i) (Z_+, Z_-) *is a circuit signature.*

(ii) Z_+ *and* Z_- *are independent and* $\operatorname{relint}(Z_+) \cap \operatorname{relint}(Z_-)$ *is a single point (for a point configuration) or a ray (for a vector configuration).*

Remark 4.1.10. Here Z_+ and Z_- are required to be non-empty because in a non-acyclic configuration there are *positive circuits*, with signature $(Z_+, \emptyset)$, to which this lemma does not apply: Z_+ is not independent and $\operatorname{relint}(Z_+) \cap \operatorname{relint}(\emptyset)$ is not a ray.

Proof. For (i)⇒(ii), the uniqueness of the dependence, Lemma 4.1.7, implies that the intersection of the relative interiors is a unique point or ray. The fact that Z_+ is not empty implies that Z_- is independent, and vice versa.

For the other implication, let Z_+ and Z_- be independent and $\operatorname{relint}(Z_+) \cap \operatorname{relint}(Z_-)$ be a single point or ray. Lemma 4.1.5 says that (Z_+, Z_-) is a dependence signature. Let λ be the vector of coefficients in this dependence. We need to show that there is no proper subset of $Z_+ \cup Z_-$ supporting another dependence signature. If there was one, let μ be the vector of coefficients in this new dependence, and let $\lambda' = \lambda + \varepsilon\mu$ with ε sufficiently small. Since the support of μ is strictly contained in that of λ, λ' has the same signature Z as λ. But then we have that the following are points (or rays) in the relative interior of both Z_+ and Z_-:

$$\mathbf{x} = \sum_{i \in Z_+} \lambda_i \mathbf{p}_i = -\sum_{i \in Z_-} \lambda_i \mathbf{p}_i, \qquad \mathbf{y} = \sum_{i \in Z_+} \lambda'_i \mathbf{p}_i = -\sum_{i \in Z_-} \lambda'_i \mathbf{p}_i.$$

By assumption, $\mathbf{x}$ and $\mathbf{y}$ must be the same point (for a point configuration) or lie in the same ray (for a vector configuration). But, then, the fact that Z_+ and Z_- are independent implies that λ' and λ are proportional, which contradicts the construction of λ' $\square$

There is an important consequence of this characterization of interior intersection: we have an algorithm to tell whether or not a point is in the convex hull or the relative interior of a subconfiguration.

Lemma 4.1.11 (Circuits and convex hulls)**.** *Let $i \in J$ and $B \subset J$. Let $\mathbf{p}_i$ be the vector labeled by i. Then,*

- $\mathbf{p}_i \in \mathrm{conv}_{\mathbf{A}}(B)$ *if and only if there is a circuit* $(Z_+, \{i\})$ *with* $Z_+ \subseteq B$.
- $\mathbf{p}_i \in \mathrm{relint}_{\mathbf{A}}(B)$ *if and only if* $(B, \{i\})$ *is a dependence signature.*

In order to provide another application of circuits, we continue this section with a closer look at the exact relation between circuit signatures and other dependence signatures.

Lemma 4.1.12. *For every dependence signature* (V_+, V_-) *with non-empty support there is a circuit* (Z_+, Z_-) *with* $Z_+ \subseteq V_+$ *and* $Z_- \subseteq V_-$.

Proof. The proof proceeds by contradiction. Suppose $V_+ \cup V_-$ is dependent but does not contain a circuit as claimed. Assume that the cardinality of $V_+ \cup V_-$ is smallest possible. Since (V_+, V_-) is not a circuit, there are at least two independent dependences $(\lambda_i)_{i\in J}$ and $(\mu_i)_{i\in J}$ with support contained in $V_+ \cup V_-$. We let $(\lambda_i)_{i\in J}$ be one with signature (V_+, V_-). Then, for every $\varepsilon \in \mathbb{R}$, $(\lambda_i + \varepsilon\mu_i)_{i\in J}$ is also a dependence. Let ε_0 be the biggest (necessarily negative) value for which there is an i with $\lambda_i \neq 0$ and $\lambda_i + \varepsilon\mu_i = 0$. For every bigger ε we have that $(\lambda_i + \varepsilon\mu_i)_{i\in J}$ has signature (V_+, V_-), while for ε_0 it has signature (W_+, W_-) with $W_- \subseteq V_-$, $W_+ \subseteq V_+$, and with one of the containments strict. We reach a contradiction with the minimality of (V_+, V_-). $\square$

The following corollary implies that the set of circuit signatures and the set of all dependence signatures can be recovered from one another. Observe also that the circuits in the statement must necessarily be conformal, since V_+ and V_- are disjoint.

Corollary 4.1.13. *Every dependence signature is a conformal sum of signed circuits. That is, for every dependence signature* (V_+, V_-) *with non-empty support there is a finite collection of signed circuits* (Z^i_+, Z^i_-), $i = 1, \dots, k$, *with* $V_+ = \cup_i Z^i_+$ *and* $V_- = \cup_i Z^i_-$.

Proof. This follows easily from the previous lemma, by induction on the cardinality of $V_+ \cup V_-$. Indeed, let $(\lambda_i)_{i\in J}$ be a dependence giving the signature (V_+, V_-) and let $(\mu_i)_{i\in J}$ be one giving a circuit (Z_+, Z_-) with $Z_+ \subseteq V_+$ and $Z_- \subseteq V_-$.

As in the previous proof, for every $\varepsilon \in \mathbb{R}$, $(\lambda_i + \varepsilon\mu_i)_{i\in J}$ is also a dependence and a ε exists for which this dependence has signature (W_+, W_-) with $W_- \subseteq V_-$, $W_+ \subseteq V_+$, and with one of the containments strict. We apply the inductive hypothesis to (W_+, W_-) and add (Z_+, Z_-) to the set of circuits obtained for it. $\square$

Finally, we can present the promised most useful application of Lemma 4.1.9: whenever two cells intersect improperly there is a circuit as a certificate for this intersection:

Theorem 4.1.14. *Let* $\mathscr{S}$ *be a family of subsets of the label set* J *of a configuration* $\mathbf{A}$. *Assume that* $\mathscr{S}$ *satisfies the closure property (CP). Then, the following two properties are equivalent:*

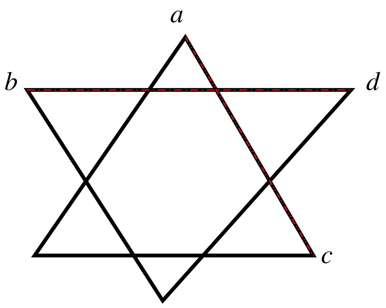

Figure 4.5: A dependence with six points contains a circuit with four points a,b,c,d.

(IP) *If $B \neq B'$ are two cells in $\mathscr{S}$ then* $\mathrm{relint}(B) \cap \mathrm{relint}(B') = \emptyset$. *(Intersection Property)*

(CiP) *For each circuit (Z_+, Z_-) such that $Z_+ \subseteq B \in \mathscr{S}$ for some B, either there is no cell in $\mathscr{S}$ containing Z_- or every cell containing Z_+ contains Z_- too. (Circuit Property)*

Proof. For (IP)⇒(CiP), assume that $Z_+ \subseteq B \in \mathscr{S}$ and $Z_- \subseteq B' \in \mathscr{S}$. Our goal is to prove that $Z_- \subseteq B$ (and $Z_+ \subseteq B'$, but this is the same proof).

Let $F = \mathrm{carrier}(Z_+, B)$ and $F' = \mathrm{carrier}(Z_-, B')$ be the carrier faces of Z_+ and Z_- in B and B' respectively. By Lemmas 4.1.5 and 2.1.23,

$$\emptyset \neq \mathrm{relint}(Z_+) \cap \mathrm{relint}(Z_-) \subseteq \mathrm{relint}(F) \cap \mathrm{relint}(F')$$

and then by (IP) $F = F'$. So, $Z_- \subseteq F' \subseteq B$.

For the converse, suppose (CiP) and let $B, B' \in \mathscr{S}$ be different cells such that $\mathrm{relint}(B) \cap \mathrm{relint}(B') \neq \emptyset$. If $B \cap B' = \emptyset$, then Lemma 4.1.5 implies that (B, B') is a dependence signature. If $B \cap B' \neq \emptyset$, our condition implies that there is an expression

$$\sum_{i \in B} \lambda_i \mathbf{p}_i = \sum_{j \in B'} \mu_j \mathbf{p}_j,$$

with all λ's and μ's positive. Subtracting $\sum_{i \in B \cap B'} \min\{\lambda_i, \mu_i\} \mathbf{p}_i$ on both sides we conclude that there are $V \subseteq B$ and $V' \subseteq B'$ disjoint and not both empty such that (V, V') is a dependence signature. Lemma 4.1.12 then gives the circuit we are looking for. □

Before continuing observe that the circuit property admits a simpler statement if we are interested only in the case where all cells of $\mathscr{S}$ are independent (for example, if we want to check whether $\mathscr{S}$ is a triangulation): no independent cell can contain a circuit, and every subset of an independent cell is a cell (by the closure property) so the circuit property can be relaxed for triangulations:

Theorem 4.1.15. *Let $\mathscr{T}$ be a family of independent subsets of the label set J of a configuration* $\mathbf{A}$ *satisfying (CP). Then, the following two properties are equivalent:*

(CiP) *For each circuit (Z_+, Z_-) such that $Z_+ \subseteq B \in \mathscr{T}$ for some B, either there is no cell in $\mathscr{T}$ containing Z_- or every cell containing Z_+ contains Z_-, too. (Circuit Property)*

(TriangCiP) *There is no circuit (Z_+, Z_-) with $Z_+, Z_- \in \mathscr{T}$ (Circuit property for triangulations).*

In particular, if we want to check whether or not two cells B, B' intersect properly or not, we simply need to look at all circuits of $\mathbf{A}$ that are not completely contained in at least one of B and B' and check whether one of them has its positive part completely in B and negative part completely in B', or vice versa. This is a substantial progress in checking the intersection property (IP). Let us briefly think about how many circuits there are in a point configuration with n points in dimension d. Each spanning $(d+2)$-element contains a unique circuit and every circuit can be extended to a spanning $(d+2)$-element. Hence, if worse comes to worst, then every $(d+2)$-element subconfiguration is a circuit, and we have $\binom{n}{d+2}$ of them. This happens if and only if the configuration is in general position.

4.1.2 Evaluations, cocircuits, and the union property

In general it is an extremely difficult task to check whether a collection of simplices covers a polytope. Since (IP) is now easy to check (as we have learned in the previous section) we only need to check (UP) for collections that already satisfy (IP) and (CP). Our starting point is the following result:

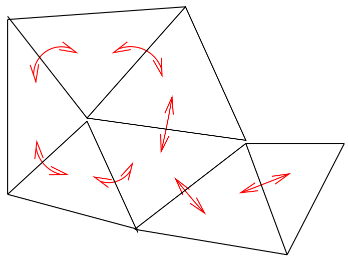

Figure 4.6: The pseudomanifold property.

Lemma 4.1.16. *Let $\mathscr{S}$ be a family of subsets of the label set J of a configuration $\mathbf{A}$. Assume that $\mathscr{S}$ satisfies the closure and intersection properties (CP) and (IP), and that all maximal elements in $\mathscr{S}$ have full dimension. Then, the following properties are equivalent:*

(UP) $\bigcup_{B\in\mathscr{S}} \operatorname{conv}(B) \supseteq \operatorname{conv}(\mathbf{A})$ *(Union Property).*

(MaxMP) *For each facet F of a maximal cell B in $\mathscr{S}$, either F is contained in a facet of $\mathbf{A}$ or there is another maximal cell B' in $\mathscr{S}$ that contains F as a facet (Pseudo-Manifold Property, see Figure 4.6).*

The "Max" in the abbreviation of the Pseudo-Manifold Property is there to remind us that this property needs to be checked only for maximal cells in a subdivision.

Proof. Assume (UP) and let F be a facet of a maximal cell B in $\mathscr{S}$ that is not contained in a facet of $\mathbf{A}$. Pick a point $\mathbf{x}$ in the relative interior of F and another point $\mathbf{y}$ in general position, very close to $\mathbf{x}$ away from $\operatorname{conv}(B)$, i.e., so that $\operatorname{conv}(F)$ separates $\mathbf{y}$ from $\operatorname{relint} B$. Since F is not in a facet of $\mathbf{A}$, it is possible to do so with $\mathbf{y}$ still in the interior of $\operatorname{conv}(\mathbf{A})$.

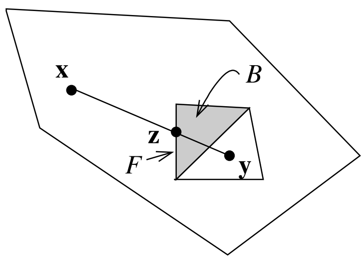

Figure 4.7: Proof of Lemma 4.1.16.

Since (UP) holds there is a B' in $\mathscr{S}$ with $\mathbf{y} \in \operatorname{conv}(B')$, which must be different from B. Since $\mathbf{y}$ is in general position, $\mathbf{y}$ must be in the relative interior of B' and B' must be full-dimensional. Since $\mathbf{y}$ is very close to $\mathbf{x}$, each point on the open segment $(\mathbf{x},\mathbf{y})$ must be also in $\operatorname{relint}(B')$, so that $\mathbf{x} \in \operatorname{conv}(B')$. By (IP) and (CP), the carriers of $\mathbf{x}$ in B' and in F must be the same, and the latter is F itself. So, $F < B'$, as desired.

Now assume that (MaxMP) holds. Let $\mathbf{x} \in \operatorname{conv}(\mathbf{A})$ and, for the sake of contradiction, assume that there is no cell in $\mathscr{S}$ with $\mathbf{x}$ in its convex hull. Pick a point $\mathbf{y}$ in general position, meaning that no hyperplane contains a dependent subset of $\mathbf{A} \cup \{\mathbf{x},\mathbf{y}\}$. For this it suffices to pick $\mathbf{y}$ outside all of the (finite set of) hyperplanes spanned by points of $\mathbf{A} \cup \{\mathbf{x}\}$. Suppose also that $\mathbf{y}$ does lie in the convex hull of some cell of $\mathscr{S}$. Consider the segment $[\mathbf{x},\mathbf{y}]$ and let $\mathbf{z} \in [\mathbf{x},\mathbf{y}]$ be the point closest to $\mathbf{x}$ and that is in the convex hull of some cell of $\mathscr{S}$. This point exists because each $\operatorname{conv}(B)$ intersects $[\mathbf{x},\mathbf{y}]$ in either a point or a segment. Let B be a full-dimensional cell with $\mathbf{z} \in \operatorname{conv}(B)$. The general position assumption on $\mathbf{y}$ implies that the carrier of $\mathbf{z}$ in B is a facet F of B. Since $\mathbf{x}$ and $\mathbf{y}$ lie on opposite sides of it, F is not contained in a facet of $\mathbf{A}$. Hence, (MaxMP) implies that there is a second maximal cell B' with $F < B'$ and in particular $z \in \operatorname{conv}(B')$. The intersection property (IP) then implies that $\operatorname{conv}(B)$ and $\operatorname{conv}(B')$ lie on opposite sides of $\operatorname{conv}(F)$, which contradicts our assumption that no point in $[\mathbf{x},\mathbf{z})$ lies in the convex hull of a cell of $\mathscr{S}$. $\square$

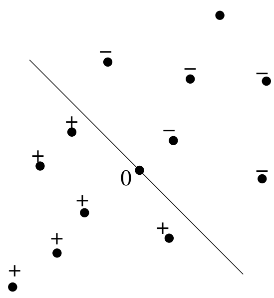

Figure 4.8: An evaluation signature.

For triangulations, Condition (MaxMP) is already a "combinatorial property", since the facets of a simplicial d-cell are just its d-subsets. Thus, for triangulations, (MaxMP) is easy to check computationally, so that Lemma 4.1.16 implies that *we can check whether a set of cells is a triangulation*. This is great news! Still, for general subdivisions we need to elaborate on the property. The key idea is to rewrite it in the language of signatures. Since faces are defined by linear functionals, the following seems the right tool for it:

Definition 4.1.17 (Evaluation signatures)**.** Let $\mathbf{A}$ be a vector configuration with label set J. An *evaluation signature* on $\mathbf{A}$ is the signature of the vector $(\operatorname{sign}(\psi(p_i)) : i \in J)$ where $\psi : J \to \mathbb{R}$ is a linear functional.

To mimic the situation with dependence signatures, we give a special name to the evaluation signatures with minimal support. The reason for the name chosen will be apparent in the next section.

Definition 4.1.18 (Cocircuits)**.** A *cocircuit signature* is an evaluation signature with minimal support. Its support is a *cocircuit*.

Equivalently, the above definitions can be given in the sometimes more intuitive language of *oriented hyperplanes*:

Definition 4.1.19. For a linear functional $\psi \in (\mathbb{R}^d)^*$, let the *oriented hyperplane* $\mathbf{H}^\psi$ *defined by* ψ be given by $\mathbf{H} = \{ x \in \mathbb{R}^d : \psi(x) = 0 \}$.

The *positive open halfspace* $\mathbf{H}_+^\psi$ *induced by* $\mathbf{H}^\psi$ is defined as $\mathbf{H}_+^\psi := \{ x \in \mathbb{R}^d : \psi(x) > 0 \}$. Similarly, the *negative open halfspace* $\mathbf{H}_-^\psi$ *induced by* $\mathbf{H}^\psi$ is defined as $\mathbf{H}_-^\psi := \{ x \in \mathbb{R}^d : \psi(x) < 0 \}$.

The closures of the open halfspaces are denoted by $\overline{\mathbf{H}_+^\psi}$ and $\overline{\mathbf{H}_-^\psi}$, respectively.

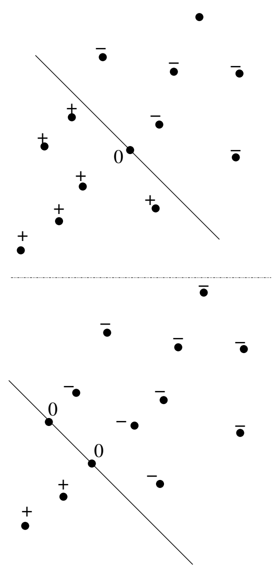

Figure 4.9: On top, an evaluation signature, but not a cocircuit. Below an honest cocircuit.

Example 4.1.20 (Continuation of Example 4.1.4)**.** Now we list all covectors of our running example of five points in the plane (see Example 4.1.4) $(245,\emptyset)$, $(1,34)$, $(125,\emptyset)$, $(2,3)$, $(15,4)$, $(345,\emptyset)$, $(135,\emptyset)$, $(1,24)$. As always, the "negative" of each of these cocircuits is also a cocircuit.

Lemma 4.1.21. *Let* $\mathbf{A}$ *be a full-dimensional configuration. Let* V_+ *and* V_- *be disjoint subsets of the label set of* $\mathbf{A}$*, not both empty. Then:*

(i) (V_+, V_-) *is an evaluation signature if and only if there is an oriented hyperplane* $\mathbf{H}$ *such that*

$$V_+ = \{i : \mathbf{p}_i \in \mathbf{H}_+\}, \qquad V_- = \{i : \mathbf{p}_i \in \mathbf{H}_-\}.$$

(ii) (V_+, V_-) *is a cocircuit signature if, moreover,* $\mathbf{H}$ *is spanned by a subset of elements of* $\mathbf{A}$*. Equivalently, if* $\mathbf{H}$ *is unique.*

Proof. Part (i) follows from the fact that every hyperplane is the zero set of a (non-zero) linear functional ψ, and the two half-spaces defined by $\mathbf{H}$ are $\psi^{-1}(0,\infty)$ and $\psi^{-1}(-\infty,0)$.

For part (ii), it is clear that uniqueness is equivalent to $\mathbf{H}$ being spanned by $J\setminus(V_+\cup V_-)$. If $\mathbf{H}$ is not spanned we can "slightly rotate it" without changing its associated signature, and two different hyperplanes $\mathbf{H}_1$ and $\mathbf{H}_2$ exist, then the elements in $J\setminus(V_+\cup V_-)$ span, at most, their intersection $\mathbf{H}_1\cap\mathbf{H}_2$.

So, let us prove that these two properties are equivalent to being a cocircuit. If ψ_1 and ψ_2 define two different hyperplanes producing the signature (V_+,V_-), then any linear combination $\alpha\psi_1+\beta\psi_2$ has signature with support contained in $V_+\cup V_-$. Choosing α and β appropriately we can make the functional be zero in any particular element, in particular obtaining an evaluation signature with support strictly contained in $V_+\cup V_-$.

Conversely, if $\mathbf{H}$ is not spanned by elements of $\mathbf{A}$, let L be the sub-linear space of $\mathbf{H}$ spanned by $\mathbf{H}\cap\mathbf{A}$. As before, we can rotate $\mathbf{H}$ around L and get a hyperplane whose signature is zero on any particular element, in particular one with support strictly contained in $V_+\cup V_-$. □

Remark 4.1.22 (Oriented hyperplanes for point configurations). As usual, $\mathbf{A}$ is here thought of either as a vector configuration or as a point configuration in homogeneous coordinates. In both cases, by a hyperplane we mean a linear hyperplane (one containing the origin). It $\mathbf{A}$ is a d-dimensional point configuration and you prefer to think of it as embedded in $\mathbb{R}^d$, then in this lemma (and elsewhere) you need to consider *affine* hyperplanes.

Corollary 4.1.23. *Let V_0 be a subset of the label set of a configuration* $\mathbf{A}$. *Then,*

(i) V_0 is the zero set (that is, the complement of the support) of an evaluation signature if and only if it equals the intersection of $\mathbf{A}$ *with a subspace.*

(ii) V_0 is the complement of a cocircuit if and only if it is not spanning and is maximal with this property.

Proof. For Part (i), the zero set of an evaluation signature is, by definition, the intersection of $\mathbf{A}$ with a hyperplane. For the converse, if V_0 equals the intersection of $\mathbf{A}$ with a subspace $\mathbf{L}$, let $\mathbf{H}$ be a hyperplane containing $\mathbf{L}$ and "sufficiently generic". Then $\mathbf{L}\cap\mathbf{A}=\mathbf{H}\cap\mathbf{A}$.

For Part (ii), already Part (i) implies that if V_0 is the complement of a cocircuit then it is not spanning. By minimality, of cocircuits, V_0 must span a hyperplane. The converse is clear. □

Important for us is the following relation between evaluation signatures and faces of a configuration. Simply put: faces are the same as non-negative evaluation circuits:

Lemma 4.1.24. *Let (V_+,V_-) be an evaluation signature of* $\mathbf{A}$*, with zero set $V_0=J\setminus(V_+\cup V_-)$. Then the following are equivalent:*

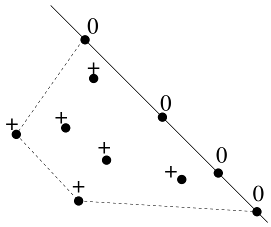

Figure 4.10: A positive cocircuit and the face it determines on the convex hull.

(i) Either $V_+ = \emptyset$ or $V_- = \emptyset$.

(ii) V_0 is a face of $\mathbf{A}$.

If (V_+, V_-) is a cocircuit,

Proof. This follow directly from the properties of supporting hyperplanes of faces. □

The following characterization of facets of point configurations will directly go into one of the combinatorial characterizations in Section 4.5.2.

Lemma 4.1.25. *Let (Z_+, Z_-) be a signed cocircuit of* $\mathbf{A}$. *Then the following are equivalent:*

(i) Z_0 is a facet of $\mathbf{A}$.

(ii) Either $Z_+ = \emptyset$ or $Z_- = \emptyset$.

Proof. Minimal evaluation signatures correspond to maximal zero-sets, thus to maximal proper faces, i.e., facets. □

The following three statements say that the relation of cocircuits to evaluation signatures is the same one as we had for circuits and dependence signatures:

Lemma 4.1.26 (Uniqueness of Cocircuit Signatures)**.** *Let Z be a cocircuit in a point configuration* $\mathbf{A}$ *with signature (Z_+, Z_-). Then (Z_-, Z_+) is the only other possible circuit signature on Z.*

Proof. By the previous Lemma, there is a unique hyperplane containing Z_0, and any two linear functionals vanishing on that hyperplane have to be multiples of each other. This implies that the induced signature are either identical or opposite. □

Lemma 4.1.27 (Conformal decomposition of evaluations)**.** *For every evaluation signature (V_+, V_-) with non-empty support there is a cocircuit (Z_+, Z_-) with $Z_+ \subseteq V_+$ and $Z_- \subseteq V_-$.*

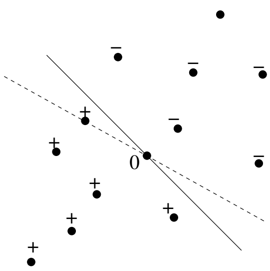

Figure 4.11: An evaluation and its conformally contained cocircuit (in dotted line).

Proof. If the codimension of V_0 is more than 1, the orthogonal complement of V_0 in $\mathbb{R}^m$ contains at least one two-dimensional subspace that contains ψ. Now, turn around ψ in this subspace. This will generate a family of hyperplanes that covers $\mathbb{R}^m$. Now, we perform this process continuously, starting at ψ, and we stop at the first time when one or more new points are contained in the moving hyperplane. We remove this point from V, and either the resulting covector is already a cocircuit or we repeat the process. See Figure 4.11. □

As with circuits, this has the following corollary, whose proof we omit.

Corollary 4.1.28 (Conformal decomposition of evaluations)**.** *For every evaluation signature (V_+, V_-) with non-empty support there is a collection of signed cocircuits (Z^i_+, Z^i_-), $i = 1, \ldots, k$, with $V_+ = \cup_i Z^i_+$ and $V_- = \cup_i Z^i_-$.*

In particular, cocircuits and evaluation signatures carry the same information on the configuration $\mathbf{A}$.

The punch-line of our study of circuits was that we can use them to check for improper intersections (Property (IP) of subdivisions). Now similarly, in order to check (UP), we can utilize cocircuits, which we show next. Variations on this are given in Section 4.5.2. In order to check (UP) for $\mathscr{S}$, in Lemma 4.1.16 we provided Condition (MaxMP). If $\mathscr{S}$ is of pure dimension and satisfies (IP) and (CP) already, then condition (UP) is fulfilled if every interior facet of a maximal cell is a facet of some other maximal cell in $\mathscr{S}$. Because of Lemma 4.1.25, we can now check this by looking at cocircuits of $\mathbf{A}$. Because of the importance of this observation we formulate it as a theorem.

Theorem 4.1.29. *Let $\mathscr{S}$ be a set of cells in a point configuration $\mathbf{A}$ labeled by J satisfying (CP) and (IP). Then the following are equivalent:*

(UP) $\bigcup_{B\in\mathscr{S}} \operatorname{conv}(B) \supseteq \operatorname{conv}(\mathbf{A})$ *(Union Property)*

(CoP) For each d-cell $B \in \mathscr{S}$ and for all cocircuits Z^* that are positive cocircuits on B but not positive on J there is another d-cell $B' \in \mathscr{S}$ with $B \cap B' = Z^*_0 \cap B$ (Cocircuit Property).

Again, the situation is simpler for triangulations since all facets of d-cells are obtained by deleting one single label from the cell.

Theorem 4.1.30. *Let $\mathscr{T}$ be a set of independent subsets in a point configuration $\mathbf{A}$ labeled by J, satisfying (CP) and (IP). Then the following are equivalent:*

(UP) $\bigcup_{B\in\mathscr{T}} \operatorname{conv}(B) \supseteq \operatorname{conv}(\mathbf{A})$ *(Union Property)*

(TriangCoP) For every d-subset F of a d-cell $B \in \mathscr{T}$, if one cocircuit Z^* spanned by F in $\mathbf{A}$ is neither positive nor negative on J, then there is another d-cell $B' \in \mathscr{T}$ with $B \cap B' = Z^*_0 \cap B$. (Cocircuit Property for Triangulations).

And how many cocircuits are there? Well, as many as hyperplanes spanned by subsets of elements of $\mathbf{A}$. If worse comes to worst, each subconfiguration of d elements spans a different hyperplane, which makes up for $\binom{n}{d}$ different cocircuits (up to sign reversal) in a point configuration consisting of n elements in dimension d.

Let us summarize the findings of this and the previous section in terms of a fully combinatorial characterization of polyhedral subdivisions. Here is a characterization that is crucial for the investigations in Chapter 6. It follows directly from Theorems 4.1.14 and 4.1.29.

Theorem 4.1.31. *A set $\mathscr{S}$ of d-dimensional subconfigurations of a point configuration $\mathbf{A}$ in $\mathbb{R}^d$ labeled by J is the set of maximal cells of a polyhedral subdivision of $\mathbf{A}$ if and only if it satisfies the following two conditions:*

(CoP) For each d-cell $B \in \mathscr{S}$ and for all cocircuits Z^* that are positive cocircuits on B but not positive on J there is another d-cell $B' \in \mathscr{S}$ with $B \cap B' = Z^*_0 \cap B$ (Cocircuit Property).

(CiP) *For each circuit (Z_+, Z_-) such that $Z_+ \subseteq B \in \mathscr{S}$ for some B, either there is no cell in $\mathscr{S}$ containing Z_- or every cell containing Z_+ contains Z_- too. (Circuit Property)*

□

If we restrict ourselves to triangulations, then things become easier (see Theorems 4.1.15 and 4.1.30).

Corollary 4.1.32. *A set $\mathscr{T}$ of d-dimensional independent subconfigurations of a point configuration $\mathbf{A}$ in $\mathbb{R}^d$ is the set of maximal cells of a triangulation of $\mathbf{A}$ if and only if it satisfies the following two conditions:*

(TriangCoP) *For every d-subset F of a d-cell $B \in \mathscr{T}$, if one cocircuit Z^* spanned by F in $\mathbf{A}$ is neither positive nor negative on J, then there is another d-cell $B' \in \mathscr{T}$ with $B \cap B' = Z_0^* \cap B$. (Cocircuit Property for Triangulations).*

(TriangCiP) *For each pair of cells $B, B' \in \mathscr{T}$, there is no signed circuit (Z_+, Z_-) in $\mathbf{A}$ with $Z_+ \subseteq B$ and $Z_- \subseteq B'$ (Circuit Property for Triangulations).* □

4.1.3 Gale transforms and the duality between circuits and cocircuits

In this section we introduce the reader to a rather useful construction and duality notion used very frequently in discrete geometry, the *Gale transform*. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a vector configuration of rank k (for example, a point configuration, in a homogenized version) with n elements.

Definition 4.1.33. We call *linear dependences* the vectors of coefficients of dependences on $\mathbf{A}$. A *linear evaluation* is the restriction to $\mathbf{A}$ of a linear function $\mathbb{R}^m \to \mathbb{R}$.

In other words, linear dependences and linear evaluations are the vectors whose signatures define dependence signatures and evaluation signatures. These two objects are closely related.

Lemma 4.1.34. *Linear dependences and linear evaluations of $\mathbf{A}$ form two orthogonal complementary linear subspaces of $\mathbb{R}^n$, of ranks $n-k$ and k respectively.*

Proof. Linear evaluations are the row span of the matrix $\mathbf{A}$ (because every linear functional is a linear combination of the coordinate functionals and vice versa) and linear dependences are, by definition, the orthogonal complement of it. □

Definition 4.1.35. A *Gale transform* of a configuration $\mathbf{A}$ is a configuration $\mathbf{B}$ such that the linear dependences of $\mathbf{A}$ are the linear evaluations on $\mathbf{B}$ and vice versa. The set of all Gale transforms of $\mathbf{A}$ is denoted by $\mathrm{Gale}(\mathbf{A})$. Nevertheless, in the future we will often abuse notation and denote by $\mathrm{Gale}(\mathbf{A})$ an explicitly or implicitly chosen Gale transform for $\mathbf{A}$.

Lemma 4.1.36. *Every configuration $\mathbf{A}$ has at least one Gale transform.*

Proof. Compute the orthogonal complement of the row span of $\mathbf{A}$ (that is, the kernel of $\mathbf{A}$) and take any basis of it as the rows of a matrix for the configuration $\mathbf{B}$. Moreover, all Gale transforms of $\mathbf{A}$ are linearly isomorphic vector configurations. $\square$

Example 4.1.37. Continuing the running Example 4.1.4, from the given coordinates we can easily calculate the affine dependences among the five points as the kernel of the matrix

$$\mathbf{A} = \begin{array}{c} \begin{matrix} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} \end{matrix} \\ \begin{pmatrix} 0 & 3 & 0 & 3 & 1 \\ 0 & 0 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

This gives, for instance, a Gale transform

$$\begin{array}{c} \begin{matrix} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & -3 \\ 2 & 0 & 0 & 1 & -3 \end{pmatrix} \end{array}.$$

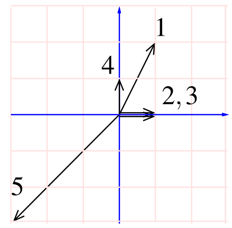

Figure 4.12: A Gale transform for the five-point planar configuration.

These five vectors are represented in Figure 4.12. Of course, a basis for the kernel is not unique and thus the Gale transform is not unique either. An alternative would be

$$\begin{array}{c} \begin{matrix} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} \end{matrix} \\ \begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix} \end{array}.$$

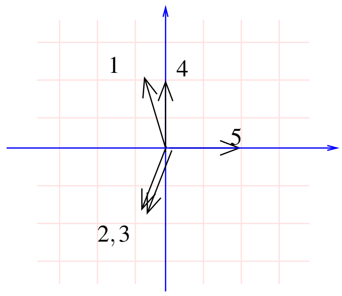

Figure 4.13: Another Gale transform for Example 4.1.4.

Precisely this Gale transform was used for drawing Figures 2.33 and 2.34: the horizontal and vertical unit vectors represent, respectively, the height vectors ω_5 and ω_4 Gale dual to the elements 5 and 4 of $\mathbf{A}$, respectively.

Observe that in the definition of a Gale transform, the two configurations are implicitly assumed to be labeled by the same set, so that we can use the same symbol to refer to subsets of elements in both. Moreover, there are the following relations among the properties of the same subset in both configurations. See also Figures 4.14, 4.15, and 4.16:

Lemma 4.1.38. *Let* $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$. *Then:*

(i) The dependence signatures of $\mathbf{A}$ *are the evaluation signatures of* $\mathbf{B}$*, and vice versa.*

(ii) The circuit signatures of $\mathbf{A}$ *are the cocircuits of any* $\mathbf{B}$*, and vice versa.*

(iii) The faces of $\mathbf{A}$ *are the complements of positive dependence signatures of any* $\mathbf{B}$*, and vice versa.*

(iv) The independent subconfigurations of $\mathbf{A}$ *are the complements of the full-dimensional subconfigurations of any* $\mathbf{B}$*, and vice versa.*

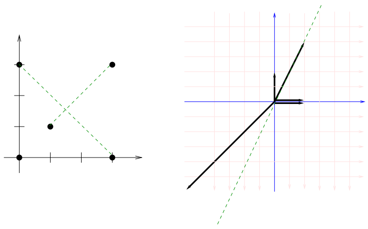

Figure 4.14: Circuits in $\mathbf{A}$ correspond to cocircuits in $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$.

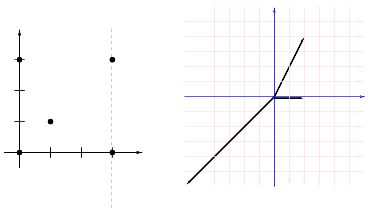

Figure 4.15: Cocircuits in $\mathbf{A}$ correspond to circuits in $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$.

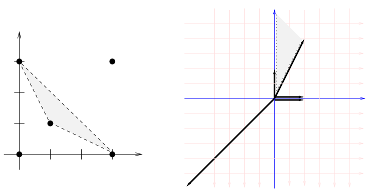

Figure 4.16: Independent sets in $\mathbf{A}$ correspond to complements of independent sets in $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$.

Proof. Part (i) is straightforward from the definition, and the rest of properties follow easily from what we said in the previous sections: minimal dependences are circuits while evaluations with minimal support are cocircuits; faces are the complements of positive evaluation signatures, and circuits [cocircuits] are minimal dependent sets [complements of maximal non-spanning sets]. □

Why are Gale transforms interesting for studying subdivisions and triangulations? The main reason has to do with the fact that two height vectors $\omega, \omega' : \mathbf{A} \to \mathbb{R}$ whose difference is an affine [a linear] function on the coordinates of the point [vector] configuration, i.e., a linear evaluation, produce the same regular subdivision (more precisely, they produce lifted configurations that are linearly equivalent and with the same notion of "lower" face). This will be discussed in detail in Section 5.4.1. However, there is one fact we wish to reveal right away: Heights that produce subdivisions for vector configurations can be characterized very nicely using a Gale transform.

Theorem 4.1.39. *Let* $\mathbf{A}$ *be a vector configuration with Gale transform* $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$. *Let* $\omega : J \to \mathbb{R}$ *be a height vector. Then, the following conditions are equivalent:*

1. ω *produces a regular subdivision of* $\mathbf{A}$. *That is, the lifted configuration* $\mathbf{A}^\omega$ *has lower faces.*
2. *For every non-negative linear dependence* λ *of* $\mathbf{A}$ *we have* $\omega \cdot \lambda \geq 0$.
3. $\omega \mathbf{B}^T \in \mathrm{cone}(\mathbf{B})$.
4. ω *is, modulo addition of a linear evaluation to it, non-negative.*

Proof. The implication from (1) to (2) is by contradiction. Suppose that there is a linear dependence λ with $\omega \cdot \lambda = c < 0$. This implies that the vector $(0, \cdots, 0, c)$ is in $\mathrm{cone}(\mathbf{A}^\omega)$, so the whole ray in the negative direction of the lifting coordinate is in the cone. This implies the cone cannot have a lower face.

For the implication from (2) to (3) contradiction works too: if $\omega \mathbf{B}^T \notin \mathrm{cone}(\mathbf{B})$ then, by Farkas lemma, there is a linear functional ϕ that separates $\omega \mathbf{B}^T$ from $\mathrm{cone}(\mathbf{B})$. That is, $\phi(\omega \mathbf{B}^T) < 0$ but ϕ is non-negative on every element of $\mathbf{B}$. Let λ be the evaluation vector corresponding to that functional. By Gale duality, λ is a dependence in $\mathbf{A}$, and by construction it is non-negative. Its scalar product with ω equals $\phi(\omega)$. This is a contradiction.

For the implication from (3) to (1), observe first that:

$$\begin{aligned}
\omega \mathbf{B}^T \in \mathrm{cone}(\mathbf{B}) &\iff \omega \mathbf{B}^T = \mu \mathbf{B}^T \text{ for some } \mu \in \mathbb{R}^n, \mu \geq 0 \\
&\iff \omega \mathbf{B}^T - \mu \mathbf{B}^T = 0 \text{ for some } \mu \in \mathbb{R}^n, \mu \geq 0 \\
&\iff (\omega - \mu) \mathbf{B}^T = 0 \text{ for some } \mu \in \mathbb{R}^n, \mu \geq 0 \\
&\iff (\omega - \mu) \in \ker(\mathbf{B}^T) \text{ for some } \mu \in \mathbb{R}^n, \mu \geq 0
\end{aligned}$$

This means, there is a nonnegative $\mu : J \to \mathbb{R}$ for which $(\omega - \mu)$ is a dependence of $\mathbf{B}$. By the definition of a Gale transform, $(\omega - \mu)$ is a linear evaluation of $\mathbf{A}$. Thus:

$$\omega \mathbf{B}^T \in \mathrm{cone}(\mathbf{B}) \iff (\omega - \mu) = c^T \mathbf{A} \text{ for some } \mu \in \mathbb{R}^n, \mu \geq 0$$

In words: a height vector maps via $\mathbf{B}$ into the conical hull of $\mathbf{B}$ if and only if ω plus some linear height function is a nonnegative height function, i.e., ω produces the same set of lower faces as some nonnegative height function. That nonnegative height functions induce all regular subdivisions was already proven in Lemma 2.5.11.

Finally (3) is clearly equivalent to (4).

□

That is:

> The space of "allowable" height functions to obtain a regular subdivision of a vector configuration $\mathbf{A}$ with $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$ is naturally identified with $\mathrm{cone}(\mathbf{B})$.

Now, if $\mathbf{B}$ is not totally cyclic, the above identification sends some particular heights functions to lie in the boundary of $\mathrm{cone}(\mathbf{B})$. What is special about the regular subdivisions defined by them? The following result gives the answer. Its proof, similar to that of Theorem 4.1.39, is left to the reader.

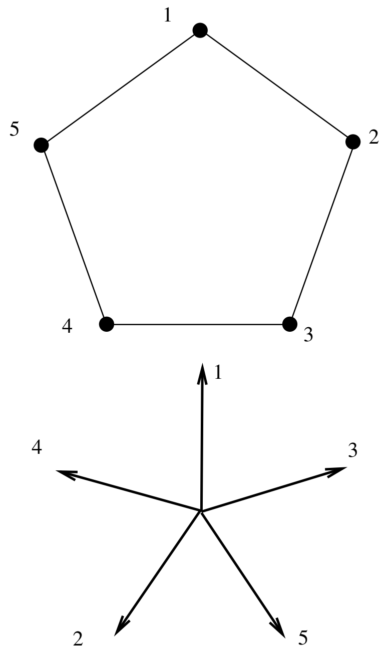

Figure 4.17: Five vertices of a regular pentagon and a Gale transform of it.

Theorem 4.1.40. *Let* $\mathbf{A}$ *be a vector configuration with Gale transform* $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$. *Let* $\omega : J \to \mathbb{R}$ *be a height vector. Then, the following conditions are equivalent:*

1. ω *produces a regular subdivision of* $\mathbf{A}$ *with all its cells acyclic. That is, the lifted configuration* $\mathbf{A}^\omega$ *has lower faces and they are all acyclic.*

2. *For every non-negative linear dependence* λ *of* $\mathbf{A}$ *we have* $\omega \cdot \lambda > 0$.

3. $\omega \mathbf{B}^T \in \mathrm{relint}(\mathbf{B})$.

4. ω *is, modulo addition of a linear evaluation to it, strictly positive.*

Example 4.1.41 (The regular pentagon)**.** Let $\mathbf{C}_5$ be the vertex set of a regular pentagon. We leave it to the reader to check that one of its Gale transform is precisely the configuration of five vectors in the plane that appeared in Example 2.5.10 (but be careful with the bijection between the points in one and the vectors in the transform; it is not the one you would expect). The five rays in the Gale transform correspond to the heights ω_i, which produce a subdivision of a pentagon into a quadrilateral and a triangle. The cone in between the two rays, represents the set of heights that produce a specific triangulation of the pentagon, with the ray between two adjacent cones corresponding to a flip.

We can also look at this example backwards. The regular subdivisions of the configuration dual to $\mathbf{C}_5$ were all depicted in Figure 2.56. The two bottom rows in the figure are the "acyclic" subdivisions appearing in

Theorem 4.1.40, which somehow correspond to choices of ω lying in the interior of $\operatorname{conv}\mathbf{C}_5$. The three on the top row, reproduced in Figure 4.19 correspond to choices in $\partial \operatorname{conv}\mathbf{C}_5$. More precisely, the first one corresponds to the zero height vector (observe that $\mathbf{C}_5$ has to be understood as a vector configuration, and $\operatorname{conv}\mathbf{C}_5$ as a pointed cone, to take full advantage of Gale duality), the second one to a choice in one of the vertices of $\operatorname{conv}\mathbf{C}_5$, and the third one to a choice in the relative interior of an edge.

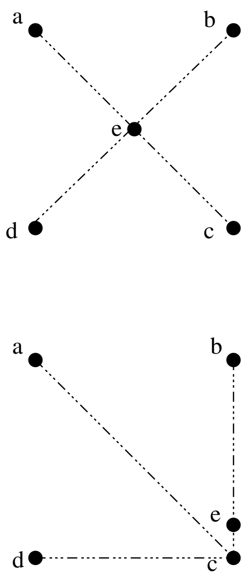

Figure 4.18: These two point configurations have different oriented matroids.

Example 4.1.42. Four vectors in general position We now turn our attention to the configuration of Example 2.5.8:

$$\begin{array}{cccc} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} \end{array} \qquad\qquad \begin{array}{cccc} \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} \end{array}$$
$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}, \qquad \operatorname{Gale}(\mathbf{A}) \ni \mathbf{B} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The indicated Gale transform is (a homogenized version of) a point configuration consisting on four points along a line. Its decomposition into the four points and the three open intervals is in bijection with the seven non-trivial subdivisions of $\mathbf{A}$.

We have seen that combinatorial structures of a point configuration carry all the information relevant to their subdivisions, except for regularity. That means, we can consider any two point configurations $\mathbf{A}$ and $\mathbf{A}'$ equivalent if and only if they have identical sets of signed circuits and, thus, cocircuits. The following is common language in combinatorial geometry. See [53] for a full exposition of the theory of oriented matroids:

Definition 4.1.43 (Oriented Matroid)**.** Two (point or vector) configurations are *combinatorially equivalent* if they have (up to relabeling) identical sets of circuits (and thus identical sets of cocircuits, identical dependence signatures, and identical evaluation signatures).

This is an equivalence relation on the set of all (point and vector) configurations. The equivalence class of a configuration with respect to this equivalence relation is called its *oriented matroid*.

One can give a more general definition of oriented matroid (or orientable matroids), one without any reference to a point or a vector configuration. One can specify various set of axioms that describe how a set of, e.g., circuits should look like (see the authoritative reference [53] for details). Within this more general framework, some oriented matroids may not be given by point or vector configuration with the same sets of circuits. Such oriented matroids are called *non-realizable* oriented matroids. In this book we are only directly interested in realizable oriented matroids, thus we can get away with the definition above. See Figure 4.18 for an example of two distinct realizable oriented matroids.

That oriented matroids are crucial to the study of triangulations and subdivisions of point configurations follows from the following corollary of Theorem 4.1.31.

Corollary 4.1.44. *Two combinatorially equivalent configurations have the same (up to relabeling) poset of polyhedral subdivisions, in particular the same set and graph of triangulations.*

It is important to observe, however, that regularity of triangulations and subdivisions *cannot* be deduced from the oriented matroid alone. It is an exercise for the reader to think of an example!

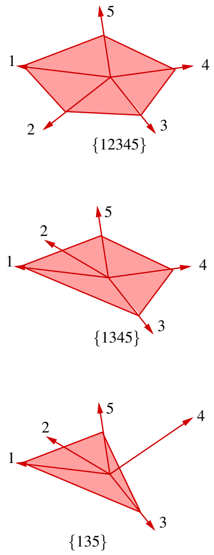

Figure 4.19: The three regular subdivisions of the Gale transform of $\mathbf{C}_5$ containing some non-acyclic cells. They correspond to lifting heights "lying" in the boundary of $\operatorname{conv} \mathbf{C}_5$.

4.2 Manipulating vector configurations

We now define various operations on *point* or *vector* configurations and study their behavior with respect to triangulations and subdivisions. We begin with *joins and products*, both very easy operations on the level of point configurations. Their easiest examples are, respectively, pyramids and prisms. For both joins and products there are easy and natural ways to produce canonical subdivisions of the new configuration from the original subdivisions of the operands. If we are after triangulations, however, joins turn out to be much more natural than products because they preserve affine independence. Later on we discuss *Deletion and contraction* operations, which provide proof techniques in combinatorics because they maintain complementary information of an object. We conclude with the operation of *one-point suspension*.

4.2.1 Pyramids and joins

Consider a d-dimensional point configuration $\mathbf{A}$ embedded into $\mathbb{R}^m$. Put a point $\mathbf{p}$ outside the hyperplane spanned by $\mathbf{A}$. The resulting new point configuration is *the pyramid of* $\mathbf{A}$ *with apex* $\mathbf{p}$. Since it really does not matter (as we will see) where we put the point $\mathbf{p}$, there is a standard model of the pyramid construction that we will use. More specifically:

Definition 4.2.1 (Pyramid)**.** Let $\mathbf{A}$ be a point configuration labeled by J. Then

$$\operatorname{pyram}(\mathbf{A}) := \begin{array}{c} \begin{array}{cc} J & * \end{array} \\ \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ 0 & 1 \end{pmatrix} \end{array}$$

is *the standard pyramid over* $\mathbf{A}$.

Each point configuration combinatorially equivalent to the standard pyramid over $\mathbf{A}$ is called a *pyramid over* $\mathbf{A}$. The new point $\mathbf{p}_* = \binom{\mathbf{0}}{1}$ is called the *apex* of the pyramid, the old point configuration, augmented with a row of zeroes, is called the *base* of the pyramid. By abuse of notation, we will denote the base in the pyramid again by $\mathbf{A}$.

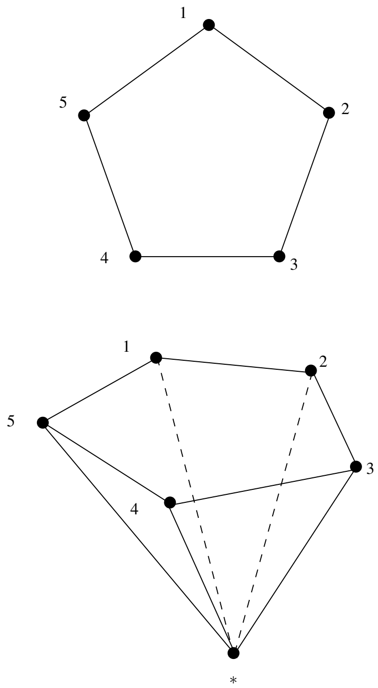

Figure 4.20: Five vertices of a regular pentagon and a pyramid over it.

What does the set of all subdivisions of a pyramid look like? Let $\mathscr{S}^*$ be a subdivision of $\operatorname{pyram}(\mathbf{A})$. Since $\mathbf{A}$ is a face of $\operatorname{pyram}(\mathbf{A})$, it certainly restricts to a subdivision $\mathscr{S}$ of $\mathbf{A}$. This subdivision is obtained by removing all cells in $\mathscr{S}^*$ that contain $*$.

Now let $\mathscr{S}$ be a subdivision of $\mathbf{A}$. No circuit in $\operatorname{pyram}(\mathbf{A})$ can contain $*$, since $\mathbf{p}_*$ is the only point with a one in the last row. Therefore, any set of cells that intersects properly in the base will intersect properly after the addition of $*$ to each cell, by the equivalence of (IP) and (CiP). Thus, adding a cell $C \cup \{*\}$ for each cell $C \in \mathscr{S}$ yields a collection of cells in $\operatorname{pyram}(\mathbf{A})$ that satisfies (CiP) and, obviously, (CP).

Moreover, each point $\mathbf{x}$ in the convex hull of the pyramid induces a unique point $\mathbf{y}$ in the base by taking the intersection of the line through the apex and $\mathbf{x}$ with the base. Since $\mathscr{S}$ satisfies (UP), there is a cell C having $\mathbf{y}$ in its convex hull. Therefore, $\mathbf{p}_*$ and $\mathbf{y}$, and hence by convexity also $\mathbf{x}$, are contained in the convex hull of $C \cup \{*\}$. Therefore, adding a cell $C \cup \{*\}$ for each cell $C \in \mathscr{S}$ yields a collection of cells that satisfies (UP) as well.

That means, the set of maximal cells that is obtained by adding $*$ to every maximal cell of a subdivision of $\mathbf{A}$ is the set of maximal cells of a subdivision of $\mathrm{pyram}(\mathbf{A})$. Conversely, removing $*$ from every cell in a subdivision of $\mathrm{pyram}(\mathbf{A})$. This motivates the following definition:

Definition 4.2.2 (Pyramid over a Subdivision)**.** Let $\mathscr{S}$ be a subdivision of a point configuration $\mathbf{A}$ labeled by J. Then the collection of cells

$$\mathrm{pyram}(\mathscr{S}) := \mathscr{S} \cup \{\, C \cup \{*\} : C \in \mathscr{S} \,\}$$

is the *pyramid over* $\mathscr{S}$, which is a subdivision of $\mathrm{pyram}(\mathbf{A})$.

The considerations above can be summarized now as follows.

Observation 4.2.3. *For all subdivisions $\mathscr{S}$ of $\mathbf{A}$, the set* $\mathrm{pyram}(\mathscr{S})$ *is a subdivision of* $\mathrm{pyram}(\mathbf{A})$*, and it is regular if and only if $\mathscr{S}$ is regular. Moreover, all subdivisions of* $\mathrm{pyram}(\mathbf{A})$ *arise in this way.* □

The pyramid construction has a natural generalization. The underlying geometric operation is called the *join*, and it is a binary operation on two complexes that reside in skew affine subspaces. We give again a standard model of this operation for point configurations:

Definition 4.2.4 (Join)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be point configurations labeled by J and K, respectively. Then

$$\mathbf{A} * \mathbf{B} := \begin{array}{c} \begin{array}{cc} J \times \{1\} & K \times \{2\} \end{array} \\ \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \end{array}$$

is *the standard join of* $\mathbf{A}$ *and* $\mathbf{B}$, where a label $(j,1)$ or $(k,2)$ is sometimes written j_1 or k_2, respectively. If J and K are disjoint, then the join can be labeled by $J \cup K$ in the obvious way.

A point configuration combinatorially equivalent to the standard join of $\mathbf{A}$ and $\mathbf{B}$ is called a *join of* $\mathbf{A}$ *and* $\mathbf{B}$. The natural embeddings of $\mathbf{A}$ and $\mathbf{B}$ into $\mathbf{A} * \mathbf{B}$ will, by abuse of notation, again be denoted by $\mathbf{A}$ and $\mathbf{B}$, respectively.

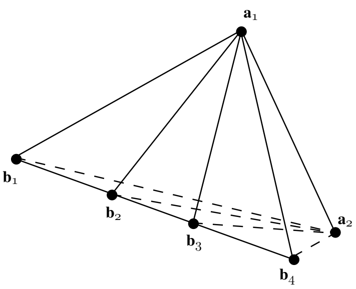

Figure 4.21: The join of two subdivisions is a subdivision of the join.

Example 4.2.5. In Figure 4.21 we represent the join of two subdivided line segments, $\{a_1, a_2\}$ and $\{b_1, b_2, b_3, b_4\}$, and one of their resulting triangulations.

Of course, the pyramid is a special case of a join. The interesting thing about joins is that, similar to the pyramid, the set of all subdivisions of a join is completely determined by the sets of subdivisions of the operands. The construction for triangulations is similar to what is usually called the *simplicial join* of simplicial complexes.

Definition 4.2.6 (Join of Subdivisions)**.** Let $\mathscr{S}$ and $\mathscr{R}$ be subdivisions of $\mathbf{A}$ and $\mathbf{B}$, respectively. Then, the *join of* $\mathscr{S}$ *and* $\mathscr{R}$ is the collection of cells

$$\mathscr{S} * \mathscr{R} := \{ (C \times \{1\}) \cup (B \times \{2\}) : C \in \mathscr{S}, B \in \mathscr{R} \}.$$

Again, if $\mathbf{A}$ and $\mathbf{B}$ are labeled by disjoint label sets J and K then the labeling of the cells need not be modified.

The main property of join subdivisions is that they are the only subdivisions of the join.

Theorem 4.2.7. *The join of any two subdivisions of point configurations* $\mathbf{A}$ *and* $\mathbf{B}$ *is a subdivision of* $\mathbf{A} * \mathbf{B}$*. Moreover, every subdivision of* $\mathbf{A} * \mathbf{B}$ *arises in this way.*

Proof. Since both $\mathbf{A}$ and $\mathbf{B}$ are not full-dimensional, there can be no circuit of $\mathbf{A} * \mathbf{B}$ containing points of both pieces. Thus, the join of two subdivisions satisfies condition (CiP) (see Theorem 4.1.14) if and only if both operands satisfy (CiP). Property (CP) for the join of two subdivisions is also equivalent to (CP) for both operands, by construction.

Each point $\mathbf{x}$ in $\operatorname{conv}(\mathbf{A} * \mathbf{B})$ has a representation as a convex combination of a point $\mathbf{u}$ in $\operatorname{conv}_{\mathbf{A}*\mathbf{B}}(J)$ and a point $\mathbf{v} \in \operatorname{conv}_{\mathbf{A}*\mathbf{B}}(K)$. Therefore, if $\mathscr{S}$ and $\mathscr{R}$ are subdivisions of $\mathbf{A}$ and $\mathbf{B}$, respectively, then $\mathbf{u}$ lies in the convex hull of a cell C of $\mathscr{S}$ and $\mathbf{v}$ lies in the convex hull of a cell B of $\mathscr{R}$. This implies that $\mathbf{x}$ lies in the convex hull of $C \cup B$, which is a cell of $\mathscr{S} * \mathscr{R}$. Thus, (UP) is satisfied for the join subdivision.

In summary, each join of valid subdivisions is a subdivision of the join. Now let $\mathscr{S}$ be any subdivision of $\mathbf{A} * \mathbf{B}$. This subdivision induces, by Lemma 2.3.4(iv), subdivisions of both $\mathbf{A}$ and $\mathbf{B}$ because both configurations are faces of $\mathbf{A} * \mathbf{B}$. Moreover, each cell in $\mathscr{S}$ is exactly the union of the cells in the two restrictions. But that means that it is the join of the two restrictions, which proves the second claim. $\square$

In summary, the composition of join and restriction to the operands as well as the composition of restriction to the operands and join yields the identity. Obviously both maps are order preserving.

Corollary 4.2.8. *There is a bijection of refinement posets between* $\operatorname{Subdivs}(\mathbf{A} * \mathbf{B})$ *and* $\operatorname{Subdivs}(\mathbf{A}) \times \operatorname{Subdivs}(\mathbf{B})$*. This bijection is order preserving with respect to the refinement order.*

4.2.2 Prisms and products

Consider a d-dimensional point configuration $\mathbf{A}$ embedded into $\mathbb{R}^{d+1}$ at "height" zero. If we put a parallel copy at height one, then we obtain what is called the *prism* over $\mathbf{A}$. On the level of convex hulls the prism is the product with an interval. As in the previous section, we want to give a standard model of a prism.

Definition 4.2.9 (Prism)**.** Let $\mathbf{A}$ be a point configuration labeled by J. Then

$$\operatorname{prism}(\mathbf{A}) := \begin{array}{c} \begin{array}{cc} J \times \{1\} & J \times \{2\} \end{array} \\ \begin{pmatrix} \mathbf{A} & \mathbf{A} \\ 1 & 0 \end{pmatrix} \end{array}$$

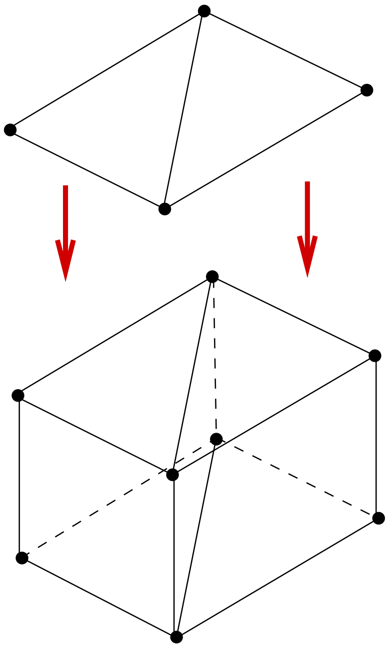

Figure 4.22: The prism over a square (a cube) is subdivided. The subdivision is obtained from a triangulation of the square.

is *the standard prism over* $\mathbf{A}$.

Each point configuration combinatorially equivalent to the standard prism over $\mathbf{A}$ is called a *prism over* $\mathbf{A}$.

There are two direct observations: first, every cell in a subdivision $\mathscr{S}$ of $\mathbf{A}$ labels a subconfiguration over which we may take the prism. For a maximal cell in $\mathscr{S}$ this prism is a full-dimensional subconfiguration in the prism. However, a prism over an independent cell does not yield an independent subconfiguration in the prism. So, the prism over a subdivision that we define below cannot be expected to give all possible subdivisions of the prism; it does not even yield a single triangulation.

Definition 4.2.10 (Prism of a Subdivision)**.** Let $\mathscr{S}$ be a subdivision of a point configuration $\mathbf{A}$ that is labeled by J. Then the collection of cells

$$\operatorname{prism}(\mathscr{S}) := \bigcup_{C\in\mathscr{S}} \{C\times\{1\}, C\times\{2\}, C\times\{1,2\}\}$$

of all prisms over cells in $\mathscr{S}$ is the *prism over* $\mathscr{S}$.

Our considerations can be summarized as follows (see Figure 4.22)

Observation 4.2.11. *For every subdivision* $\mathscr{S}$ *of* $\mathbf{A}$ *the collection of cells* $\operatorname{prism}(\mathscr{S})$ *forms a subdivision of* $\operatorname{prism}(\mathbf{A})$. □

Prisms can be generalized by products in a similar way as pyramids by joins. The Cartesian product of Euclidean point sets is a standard construction in geometry. For completeness we give our standard representation of a product now for labeled point configurations.

Definition 4.2.12 (Product)**.** Let $\mathbf{A} = (\mathbf{p}_j)_{j\in J}$ and $\mathbf{B} = (\mathbf{q}_k)_{k\in K}$ be point configurations labeled by J and K, respectively. Then

$$\mathbf{A}\times\mathbf{B} := \begin{array}{c} (j,k) \\ \begin{pmatrix} \mathbf{p}_j \\ \mathbf{q}_k \end{pmatrix}_{j\in J, k\in K} \end{array}$$

is *the standard product of* $\mathbf{A}$ *and* $\mathbf{B}$.

Each point configuration combinatorially equivalent to the standard product of $\mathbf{A}$ and $\mathbf{B}$ is called a *product of* $\mathbf{A}$ *and* $\mathbf{B}$.

As before, we define an operation on subdivisions that yields subdivisions of the product. It is no surprise that this is the product of cells.

Definition 4.2.13 (Product of Subdivisions)**.** Let $\mathscr{S}$ and $\mathscr{R}$ be subdivisions of $\mathbf{A}$ and $\mathbf{B}$, respectively. Then, the *product of* $\mathscr{S}$ *and* $\mathscr{R}$ is the collection of cells

$$\mathscr{S}\times\mathscr{R} := \{C\times B : C\in\mathscr{S}, B\in\mathscr{R}\}.$$

The admittedly obvious result reads as follows.

Remark 4.2.14. The product of any two subdivisions of point configurations $\mathbf{A}$ and $\mathbf{B}$ is a subdivision of $\mathbf{A}\times\mathbf{B}$. Not every subdivision of $\mathbf{A}\times\mathbf{B}$ arises in this way. In particular, whenever both $\mathbf{A}$ and $\mathbf{B}$ have more than one point then the product of two subdivisions is never a triangulation.

Any subdivision of $\mathbf{A}\times\mathbf{B}$ yields a subdivision of $\mathbf{A}$ for every extreme point $\mathbf{q}$ of $\mathbf{B}$ via the restriction to the face $\mathbf{A}\times\mathbf{q}$ of $\mathbf{A}\times\mathbf{B}$. Similarly we get a subdivision of $\mathbf{B}$ for every extreme point of $\mathbf{A}$.

This means that whereas subdivisions of a join of two point configurations is completely determined by the subdivisions of the two operands; forming the product may result in a much more interesting set of subdivisions. The fact that triangulations of cubes (products of segments) are far from being understood today is partly explained by this pathology. Another example of how products yield rich structures can be found in Chapter 7.3. This is in contrast to other situations where products only lead to trivial objects.

4.2.3 Deletion

Definition 4.2.15. Let $\mathbf{A}$ be a configuration with label set J and let $i\in J$ be one of its elements. The deletion of i in $\mathbf{A}$ is the subconfiguration obtained removing the corresponding element from the labeled set $\mathbf{A}$ (that is, removing the corresponding column from the matrix of $\mathbf{A}$). Slightly abusing notation, we denote it $\mathbf{A}\setminus i$.

Thus, deleting a point means just "forgetting it". Not every subdivision or triangulation of $\mathbf{A}$ is compatible with one of $\mathbf{A}\setminus i$, as the following example shows.

Example 4.2.16. Consider Example 3.6.15, the configuration consisting of the six vertices of a Schönhardt polyhedron, the important non-convex polyhedron of Example 3.6.1, together with an exterior point along its axis of symmetry. We use again the coordinates

$$\begin{array}{c} \begin{array}{ccccccc} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{r} \end{array} \\ \mathbf{A}=\begin{pmatrix} 1-\varepsilon & 0 & \varepsilon & 1 & 0 & 0 & 1/3 \\ \varepsilon & 1-\varepsilon & 0 & 0 & 1 & 0 & 1/3 \\ 0 & \varepsilon & 1-\varepsilon & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 10 \end{pmatrix}, \end{array}$$

where $\varepsilon>0$ is small. The left picture in Figure 4.23 shows the profile of this point set. The set is almost equal to its convex hull with some "cavities". Consider again the triangulation $\mathscr{T}$ of $\mathbf{A}$ consisting of the ten tetrahedra:

$$\begin{array}{ccc} \{\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2\}, & \{\mathbf{q}_2,\mathbf{q}_3,\mathbf{p}_2,\mathbf{p}_3\}, & \{\mathbf{q}_3,\mathbf{q}_1,\mathbf{p}_3,\mathbf{p}_1\}, \\ \{\mathbf{r},\mathbf{q}_1,\mathbf{p}_1,\mathbf{p}_2\}, & \{\mathbf{r},\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_2\}, & \{\mathbf{r},\mathbf{q}_2,\mathbf{p}_2,\mathbf{p}_3\}, \\ \{\mathbf{r},\mathbf{q}_2,\mathbf{q}_3,\mathbf{p}_3\}, & \{\mathbf{r},\mathbf{q}_3,\mathbf{p}_3,\mathbf{p}_1\}, & \{\mathbf{r},\mathbf{q}_3,\mathbf{q}_1,\mathbf{p}_1\}, \\ & \{\mathbf{r},\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\}. & \end{array}$$

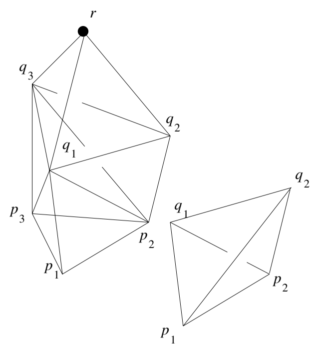

Figure 4.23: Triangulation $\mathscr{T}$ shown with a detached tetrahedron $\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2$.

See Figure 4.23 for a view of the triangulation where the tetrahedron $\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2$ has been detached. If we delete point r and all the simplices of $\mathscr{T}$ using it, we are left with only the tetrahedra $\mathbf{p}_1\mathbf{p}_2\mathbf{q}_1\mathbf{q}_2$, $\mathbf{p}_2\mathbf{p}_3\mathbf{q}_2\mathbf{q}_3$ and

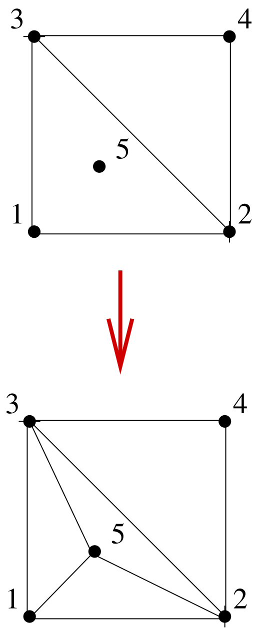

Figure 4.24: A regular triangulation of $\mathbf{A} \setminus 5$ extends to a regular triangulation of $\mathbf{A}$.

$\mathbf{p}_1\mathbf{p}_3\mathbf{q}_1\mathbf{q}_3$. The resulting object comes from twisting a prism, thus these tetrahedra are almost flat as they were rectangles before the twisting. The non-convex subregion of $\mathrm{conv}(\mathbf{A})$ that is not covered by these tetrahedra is the *Schönhardt polyhedron*. As we discussed in Chapter 3, it is famous for not having any triangulation (unless you allow yourself to insert additional vertices). That is to say:

> There is no triangulation of $\mathbf{A} \setminus r$ that extends the subcomplex of $\mathscr{T}$ not using r.

Thus the deletion will in general only yield a canonical partial subdivision that may not be a subset of any full subdivision. This is a big difference to the two-dimensional case where all partial subdivisions can be completed.

The reader probably has noticed the similarity of Figures 4.23 and 2.23. This indicates that non-regularity plays a role in Example 4.2.16. This is indeed the case, as the following lemma shows:

Lemma 4.2.17. *If $\mathscr{S}$ is a regular subdivision of $\mathbf{A}$ and $i \in \mathbf{A}$ is one of its points (meaning that there is a cell in $\mathscr{S}$ with i in it), then*

- *there is a subdivision of $\mathbf{A} \setminus i$ that uses all the cells of $\mathscr{S}$ that do not contain i.*
- *If $\mathscr{S}$ is a regular triangulation, the same statement holds with a regular triangulation of $\mathbf{A} \setminus i$.*

Proof. If ω is a height vector that produces $\mathscr{S}$, forgetting the entry of ω corresponding to point i gives a height vector that produces the desired regular subdivision of $\mathbf{A}$. If $\mathscr{S}$ is a triangulation, assume ω sufficiently generic and this gives a triangulation of $\mathbf{A} \setminus i$. Details are left to the reader. $\square$

Observe, however, that, even in the case of regular subdivisions the deletion process is not unique: different choices of ω in the proof of the lemma may give different subdivisions of $\mathbf{A} \setminus i$ even if they give the same one in $\mathbf{A}$. For an extreme example, suppose that $\mathbf{A}$ is in general position in the plane and that $\mathscr{S}$ is the regular triangulation obtained by "pulling" i in the trivial subdivision of $\mathbf{A}$. Then, *all* the subdivisions of $\mathbf{A} \setminus i$ will do the job.

Please observe that in some special situations the assumption of regularity in Lemma 4.2.17 is not necessary. For instance, in dimension two, the extended validity of Lemma 4.2.17 actually plays a crucial role in the proof of Theorem 3.3.6. Similarly, for *cyclic polytopes*, Section 6.1.5 we will discuss a canonical way to obtain a triangulation of the deletion from any triangulation of the whole cyclic point configuration.

Example 4.2.18. Looking again at Example 4.1.4, we can compute all circuits and cocircuits of $\mathbf{A} \setminus 5$. The only circuit remaining is $(14, 23)$. While the cocircuits are now $(24, \emptyset), (12, \emptyset), (2, 3), (1, 4), (34, \emptyset), (13, \emptyset)$.

4.2.4 Contraction

By contracting $\mathbf{A}$ at a point or vector $\mathbf{p}_i \in \mathbf{A}$ we essentially mean "projecting towards $\mathbf{p}_i$". We begin with a very general definition.

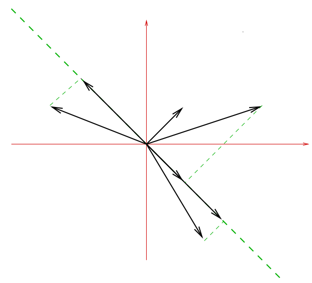

Figure 4.25: The contraction at point i consists of 3 collinear vectors.

Definition 4.2.19. Let $\mathbf{A} \subset \mathbb{R}^m$ be a vector configuration (or a point configuration, but in homogeneous coordinates). Let i be one of its elements, and $\mathbf{p}_i$ be the corresponding column of $\mathbf{A}$, assumed not to be zero.

Let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-1}$ be any linear surjective map that sends $\mathbf{p}_i$ to zero. We call the configuration

$$\mathbf{A}/i := (\pi(\mathbf{p}_j))_{j \in J \setminus \{i\}}.$$

the *contraction of* $\mathbf{A}$ *at* i, and denote it by $\mathbf{A}/i$ (see Figure 4.25 for an example of a contraction). That is, in matrix notation, $\mathbf{A}/i$ equals $\pi\mathbf{A}$, with the (now zero) column corresponding to i deleted.

The properties for the contraction operation are sort of opposite to those in the deletion case. (Almost) every subdivision of $\mathbf{A}$ produces a subdivision of $\mathbf{A}/i$ by taking the link at i (observe that the bottom picture in Figure 4.23 is actually an example of this operation), while only regular subdivisions of $\mathbf{A}/i$ are guaranteed to be links of some subdivision of $\mathbf{A}$ in general.

Lemma 4.2.20. *Let $\mathscr{S}$ be a subdivision of $\mathbf{A}$ that uses the element $i \in J$ (meaning that there is a cell $B \in \mathscr{S}$ with $i \in B \in \mathscr{S}$). Then:*

1. $\mathrm{link}_{\mathscr{S}}(i)$ *is a subdivision of* $\mathbf{A}/i$.
2. *If $\mathscr{S}$ is regular then* $\mathrm{link}_{\mathscr{S}}(i)$ *is regular.*
3. *If $\mathscr{S}$ is a triangulation then* $\mathrm{link}_{\mathscr{S}}(i)$ *is a triangulation.*

Proof. Again, all statements are easy. For the regularity, the same height function that gives $\mathscr{S}$ on $\mathbf{A}$ gives $\mathrm{link}_{\mathscr{S}}(i)$ in $\mathbf{A}/i$, as long as we take $\omega(i) = 0$ in the former. $\square$

Corollary 4.2.21. *The operation "take the link at point i" is a well-defined and order-preserving map from the subposet of* $\mathrm{Subdivs}(\mathbf{A})$ *consisting of subdivisions that use i to the poset* $\mathrm{Subdivs}(\mathbf{A}/i)$.

Observe that if i is extremal (that is, a vertex of $\mathrm{conv}(\mathbf{A})$ and not repeated as an element of $\mathbf{A}$) then the subposet referred to in the statement is the whole $\mathrm{Subdivs}(\mathbf{A})$.

Corollary 4.2.22. *Let i be an element of $\mathbf{A}$, then every flip between two triangulations $\mathscr{T}_1$ and $\mathscr{T}_2$ does one and only one of the following things:*

1. *Makes the element i appear or disappear.*
2. *Preserves the link at i, or*
3. *Produces a flip between the two triangulations* $\mathrm{link}_{\mathscr{T}_1}(i)$ *and* $\mathrm{link}_{\mathscr{T}_2}(i)$ *of* $\mathbf{A}/i$.

Proof. We assume that $\mathrm{link}_{\mathscr{T}_1}(i) \neq \mathrm{link}_{\mathscr{T}_2}(i)$ and that none of the two is empty, or otherwise we are clearly in one (and only one) of the first two cases. By the previous corollary, $\mathrm{link}_{\mathscr{T}_1}(i)$ and $\mathrm{link}_{\mathscr{T}_2}(i)$ are two different triangulations of $\mathbf{A}/i$. Now, let $\mathscr{T}$ be the "flip", that is, the subdivision of $\mathbf{A}$ that is only refined by $\mathscr{T}_1$ and $\mathscr{T}_2$. Again by the previous corollary, $\mathrm{link}_{\mathscr{T}}(i)$ is a subdivision of $\mathbf{A}/i$ only refined by $\mathrm{link}_{\mathscr{T}_1}(i)$ and $\mathrm{link}_{\mathscr{T}_2}(i)$, so it is a flip between them. ☐

Example 4.2.23 (Example 3.6.16 continued)**.** We wish to illustrate how to use the contraction to count flips. Let us explicitly compute the flips in the triangulation of Example 3.6.16. Recall that our point set and triangulation were:

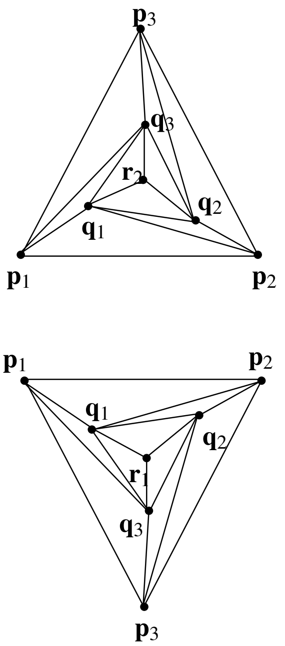

Figure 4.26: The links of $\mathbf{r}_1$ (bottom) and $\mathbf{r}_2$ (top).

$$\mathbf{A} = \begin{array}{c} \begin{array}{cccccccc} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{r}_1 & \mathbf{r}_2 \end{array} \\ \begin{pmatrix} 4-\varepsilon & 0 & \varepsilon & 2 & 1 & 1 & 4/3 & 4/3 \\ \varepsilon & 4-\varepsilon & 0 & 1 & 2 & 1 & 4/3 & 4/3 \\ 0 & \varepsilon & 4-\varepsilon & 1 & 1 & 2 & 4/3 & 4/3 \\ 0 & 0 & 0 & 1 & 1 & 1 & -10 & 10 \end{pmatrix} \end{array}.$$

and

$$\mathscr{T} = \left\{ \begin{array}{l} \{\mathbf{r}_1,\mathbf{r}_2,\mathbf{q}_1,\mathbf{q}_2\},\ \{\mathbf{r}_1,\mathbf{r}_2,\mathbf{q}_1,\mathbf{q}_3\},\ \{\mathbf{r}_1,\mathbf{r}_2,\mathbf{q}_2,\mathbf{q}_3\}, \\ \{\mathbf{r}_1,\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_2\},\ \{\mathbf{r}_1,\mathbf{q}_1,\mathbf{p}_1,\mathbf{p}_2\},\ \{\mathbf{r}_2,\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_2\},\ \{\mathbf{r}_2,\mathbf{q}_1,\mathbf{p}_1,\mathbf{p}_2\} \\ \{\mathbf{r}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{p}_3\},\ \{\mathbf{r}_1,\mathbf{q}_2,\mathbf{p}_2,\mathbf{p}_3\},\ \{\mathbf{r}_2,\mathbf{q}_2,\mathbf{q}_3,\mathbf{p}_3\},\ \{\mathbf{r}_2,\mathbf{q}_2,\mathbf{p}_2,\mathbf{p}_3\} \\ \{\mathbf{r}_1,\mathbf{q}_3,\mathbf{q}_1,\mathbf{p}_1\},\ \{\mathbf{r}_1,\mathbf{q}_3,\mathbf{p}_3,\mathbf{p}_1\},\ \{\mathbf{r}_2,\mathbf{q}_3,\mathbf{q}_1,\mathbf{p}_1\},\ \{\mathbf{r}_2,\mathbf{q}_3,\mathbf{p}_3,\mathbf{p}_1\} \end{array} \right\}.$$

Now, every flip removes at least a tetrahedron of the list. Since every tetrahedron uses one (or both) of $\mathbf{r}_1$ and $\mathbf{r}_2$, every flip will appear as a flip in one (or both) of the two links $\mathrm{link}_{\mathscr{T}}(r_1)$, $\mathrm{link}_{\mathscr{T}}(r_2)$. To draw the links we recall the observation that the six points $\mathbf{p}_i$ and $\mathbf{q}_i$ project vertically to a perturbation of "the mother of all examples" with one of its two parallel triangles slightly rotated. Since $\mathbf{r}_1$ and $\mathbf{r}_2$ are relatively far along the axis of this projection, the same is true for the contractions at them. That is, the contractions look like the top and bottom parts of Figures 4.26 respectively. There we see that each of the links has exactly four flips: one that removes the central point and three edge flips on the three sides. We only need to check which of them are contractions of flips in $\mathscr{T}$. For this let us list the circuits involved in these flips (remark: each circuit C in $\mathbf{A}/i$ gives either C or $C \cup i$ as an unoriented circuit in $\mathbf{A}$. Moreover, the former can only happen if $\mathbf{A}$ is not in general position). The list contains seven instead of eight circuits because the first one gives a flip in both links:

$$\begin{array}{ll} (\ \{\mathbf{r}_1,\mathbf{r}_2\}\ ,\ \{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3\}\ ), & \\ (\ \{\mathbf{q}_1,\mathbf{p}_2\}\ ,\ \{\mathbf{r}_2,\mathbf{p}_1,\mathbf{q}_2\}\ ), & (\ \{\mathbf{r}_1,\mathbf{q}_1,\mathbf{p}_2\}\ ,\ \{\mathbf{p}_1,\mathbf{q}_2\}\ ), \\ (\ \{\mathbf{q}_2,\mathbf{p}_3\}\ ,\ \{\mathbf{r}_2,\mathbf{p}_2,\mathbf{q}_3\}\ ), & (\ \{\mathbf{r}_1,\mathbf{q}_2,\mathbf{p}_3\}\ ,\ \{\mathbf{p}_2,\mathbf{q}_3\}\ ), \\ (\ \{\mathbf{q}_3,\mathbf{p}_1\}\ ,\ \{\mathbf{r}_2,\mathbf{p}_2,\mathbf{q}_1\}\ ), & (\ \{\mathbf{r}_1,\mathbf{q}_3,\mathbf{p}_1\}\ ,\ \{\mathbf{p}_2,\mathbf{q}_1\}\ ). \end{array}$$

The first and second column differs only in that $\mathbf{r}_1$ appears on the positive part of the circuit while $\mathbf{r}_2$ lies on the negative part. This reflects the fact that the triangles $\{\mathbf{q}_i,\mathbf{q}_{i+1},\mathbf{p}_{i+1}\}$ and $\{\mathbf{q}_i,\mathbf{p}_i,\mathbf{p}_{i+1}\}$ in their common link are folded convex when seen from $\mathbf{r}_2$ and concave from $\mathbf{r}_1$. This difference makes the circuits in the right column not support flips in $\mathscr{T}$, since the almost-flat tetraahedra $\{\mathbf{q}_i, \mathbf{q}_{i+1}, \mathbf{p}_i, \mathbf{p}_{i+1}\}$ would need to be in $\mathscr{T}$ for that. So, as we claimed in Example 3.6.16, only the four circuits in the left column support flips.

Lemma 4.2.24. *If $\mathscr{S}$ is a regular subdivision of $\mathbf{A}/i$, then there is a regular subdivision of $\mathbf{A}$ that has $\mathscr{S}$ as its link at i.*

Proof. Consider the height function ω that defines the regular subdivision $\mathscr{S}$ of $\mathbf{A}/i$. Lift the configuration $\mathbf{A} \setminus i \subset \mathbb{R}^d$ to $\mathbb{R}^{d+1}$ using exactly the same heights. Finally, lift the point i to a point i^0 at height zero. We claim that the regular subdivision we produced from these values has its link at i equal to $\mathscr{S}$. Indeed, a face $\mathbf{F}$ in the link of i must be supported by a hyperplane $\mathbf{H}$ with normal vector f. Now, for each point $b \in \mathbf{F}$ its lifting, b^ω is projected, by the projection orthogonal to i, on its lift $\pi(b)^\omega$. While f is still projected onto itself because f is perpendicular to i^0. Thus the normal vectors inducing supporting hyperplanes are all still supporting hyperplanes in the orthogonal projection. $\square$

To show that regularity is needed in this result, we use essentially the same example as in the case of the deletion.

Example 4.2.25. Let $\mathbf{A}$ be the point configuration of Example 4.2.23 (which is the continuation of Example 3.6.16). We delete the point $\mathbf{r}_1$ from the configuration

$$\mathbf{B} = \mathbf{A} \setminus \mathbf{r_1} = \begin{array}{c} \begin{array}{ccccccc} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{r}_2 \end{array} \\ \begin{pmatrix} 4-\varepsilon & 0 & \varepsilon & 2 & 1 & 1 & 4/3 \\ \varepsilon & 4-\varepsilon & 0 & 1 & 2 & 1 & 4/3 \\ 0 & \varepsilon & 4-\varepsilon & 1 & 1 & 2 & 4/3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 10 \end{pmatrix} \end{array}$$

The first six points of $\mathbf{B}$ define the vertices of the prism obtained by truncating, at point $(2,2,2)$, the triangle-based pyramid that results as the convex hull of the first six points above and $(2,2,2)$. Since point r_2 is further above this apex, the contraction of this configuration is exactly "the mother of all examples", that is, the vertex set of two concentric triangles in a plane. We consider the following simplices with vertices among the points of $\mathbf{B} \setminus \mathbf{r_2}$:

$$\mathscr{T} = \{\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3, \mathbf{p}_1\mathbf{p}_2\mathbf{q}_1, \mathbf{p}_2\mathbf{q}_1\mathbf{q}_2, \mathbf{p}_2\mathbf{p}_3\mathbf{q}_2, \mathbf{p}_3\mathbf{q}_2\mathbf{q}_3, \mathbf{p}_1\mathbf{p}_3\mathbf{q}_3, \mathbf{p}_1\mathbf{q}_1\mathbf{q}_3\}.$$

If this was the correct link of a certain triangulation (or subdivision) $\mathscr{S}$ of $\mathbf{B}$, $\mathscr{S}$ would consist of $\mathscr{T} * \mathbf{r}_2$ plus a certain triangulation (or subdivision) of the prism $\operatorname{conv}(\{\mathbf{B} \setminus \mathbf{r}_2\}$ with the property of using the three diagonals $\mathbf{p}_2\mathbf{q}_1$, $\mathbf{p}_3\mathbf{q}_2$ and $\mathbf{p}_1\mathbf{q}_3$ of the prism. But the prism does not have any triangulation using this cyclic set of diagonals. Hence:

> There is no triangulation of $\mathbf{B}$ that extends the triangulation $\mathscr{T}$ of the link at $\mathbf{r}_2$.

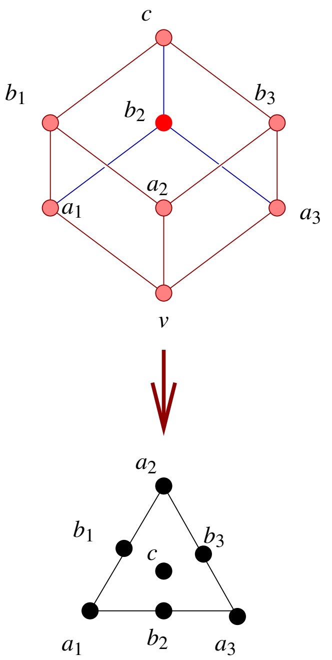

Figure 4.27: The homogeneous contraction of vertex $\mathbf{v}$ of a regular cube.

To better appreciate the concept of contraction it is worth looking at a related construction.

Definition 4.2.26. Let $\mathbf{A} = (\mathbf{p}_j)_{j\in J}$ be a point configuration in $\mathbb{R}^d$ and let i be one of its extreme elements. That is, assume that $\{i\}$ is a face of $\mathbf{A}$.

Let $\mathbf{H}$ be any hyperplane that separates $\mathbf{p}_i$ from the rest of the configuration. Let $\pi : J \setminus \{i\} \to \mathbf{H}$ be the central projection that sends each element $j \in J \setminus \{i\}$ to $\mathbf{H}$. That is, geometrically, $\pi(j) = \mathbf{H} \cap [\mathbf{p}_i\mathbf{p}_j]$, for each j; but recall that $\mathbf{A}$ may have repeated points, which will still be repeated by π.

We call the *homogeneous contraction of* $\mathbf{A}$ *at* i the resulting point configuration

$$\mathbf{A}//i := (\pi(j))_{j\in J\setminus\{i\}}\,.$$

Remark 4.2.27. $\mathbf{A}//i$ has one less element and rank one less than $\mathbf{A}$. Even if $\mathbf{A}$ does not have multiple points, $\mathbf{A}//i$ may have them (for instance, points of $\mathbf{A}$ lying on the same ray from i).

Example 4.2.28. If we do the homogeneous contraction of the point configuration formed by the vertices of a regular cube at any one of its vertices, say vertex $\mathbf{v}$ in Figure 4.27, we will obtain a point configuration bounded by a triangle, with a point in its center representing the point antipodal to $\mathbf{v}$ and three more vertices along the edges representing opposite vertices of the cube to $\mathbf{v}$ along each of the three facets containing $\mathbf{v}$.

Strictly speaking the definition of homogeneous contraction is not unique, since the point configuration obtained depends on the choice of the hyperplane $\mathbf{H}$. But different choices produce configurations which are projectively equivalent and, in particular, have the same subdivisions and triangulations (See Appendix 2.6 of [334] for a description of projective transformations and projective equivalence). Also, readers familiar with polytope theory should keep in mind the homogeneous contraction is a generalization of the *vertex figures* (see Lecture 2 in [334]).

Now we finally explain what the homogeneous contraction has to do with the contraction we defined at the beginning.

Remark 4.2.29. Assume that $\mathbf{A}$ is configuration of points in $\mathbb{R}^d$, given in homogeneous coordinates as an $d+1\times n$ matrix via the addition of a constant row of 1's to the coordinates of the points. Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the projection with matrix

$$\pi = \begin{pmatrix} \mathbf{I} & -\mathbf{p}_i \end{pmatrix},$$

where $\mathbf{I}$ is the identity $d\times d$ matrix and $\mathbf{p}_i$ is the point at which we want to take the contraction, without its last—homogenization—coordinate.

The contraction $\mathbf{A}/i := \pi\mathbf{A}$ has as columns

$$\left\{\,\mathbf{p}_j - \mathbf{p}_i \,:\, j \in J\setminus\{i\}\,\right\},$$

where $\mathbf{p}_j$ denotes the j-th point of $\mathbf{A}$. Then, the homogeneous contraction of $\mathbf{A}$, considered as a homogeneous configuration, is obtained from the linear contraction $\mathbf{A}/i$ by just scaling each column by a positive scalar λ_i in such a way that the scaled points $\lambda_i(\mathbf{p}_j - \mathbf{p}_i)$ are coplanar.

Finally, there is a rather useful duality relation between the deletion and contraction operations:

Lemma 4.2.30. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *be a vector configuration of rank* k *with* n *elements. Suppose* $\mathbf{p}$ *is a vector of the configuration, then* $\mathrm{Gale}(\mathbf{A/p})$ *is equal to* $\mathrm{Gale}(\mathbf{A}) \setminus \mathbf{p}$.

Proof. By construction, any Gale transform of $\mathbf{A}$ consists of the rows of a matrix $\mathbf{B}$, such that $\mathbf{AB} = 0$, while the contraction of $\mathbf{p}$ is given by a matrix π projecting $\mathbf{A}$ where $\mathbf{p}$ is projected to the origin. This means that $\mathbf{B}$ minus the row corresponding to $\mathbf{p}$ is still a Gale transform for $\pi\mathbf{A}$. □

Corollary 4.2.31. *The circuits (cocircuits) of the contraction* $\mathbf{A}/\mathbf{p}$ *are the cocircuits (circuits) of any element of* $\mathrm{Gale}((\mathrm{A}))\setminus\mathbf{p}$.

4.2.5 One-point suspension

In this section we will meet another operation that behaves quite well with respect to the space of all subdivisions.

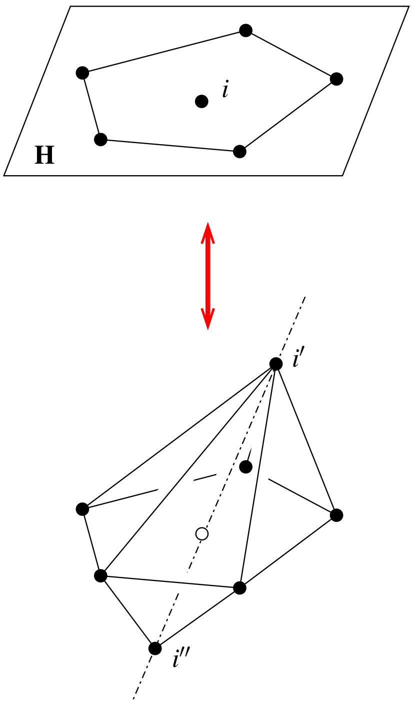

Figure 4.28: The one-point suspension construction.

Definition 4.2.32. Let $\mathbf{A}$ be a configuration in $\mathbb{R}^m$ with label set J. Let $i \in J$ be an element, whose corresponding column is $\mathbf{p}_i$.

Then the *standard one-point suspension* $\mathbf{A}\updownarrow^i_i$ *of* $\mathbf{A}$ *over* i is given as

$$\mathbf{A}\updownarrow^i_i := \begin{array}{c} \begin{array}{ccc} J\setminus i & i_1 & i_2 \end{array} \\ \begin{pmatrix} \mathbf{A}\setminus\mathbf{p}_i & \mathbf{p}_i & \mathbf{p}_i \\ \mathbf{0} & 1 & -1 \end{pmatrix} \end{array}$$

Any point configuration combinatorially equivalent to the standard one-point suspension of $\mathbf{A}$ will be called a *one-point suspension of* $\mathbf{A}$.

The general picture for a one-point suspension is as follows: Suppose that $\mathbf{A}$ lies in a linear hyperplane $\mathbf{H}$. Let $\mathbf{q}$ and $\mathbf{r}$ be vectors outside $\mathbf{H}$ with

$$\mathbf{p}_i \in \mathrm{relint}(\{\mathbf{q},\mathbf{r}\}).$$

The *one-point suspension* of $\mathbf{A}$ over the element i, denoted by $\mathbf{A}\updownarrow^i_i$ is the following configuration:

$$\mathbf{A}\updownarrow^i_i := \begin{array}{c} \begin{array}{ccc} J\setminus i & i_1 & i_2 \end{array} \\ \begin{pmatrix} \mathbf{A}\setminus\mathbf{p}_i & \mathbf{q} & \mathbf{r} \end{pmatrix} \end{array}$$

See Figure 4.28 for an example. Observe that in the general definition i need not be an interior point. The assumption that $\mathbf{A}$ lies in a hyperplane is not relevant, since we can always embed $\mathbb{R}^d$ as a hyperplane in $\mathbb{R}^{d+1}$.

This operation is in a sense an inverse of the contraction: contracting either of the two new points i_1 or i_2 in $\mathbf{A}\updownarrow^i_i$ we recover exactly the configuration $\mathbf{A}$ (check it!). Even if the operation seems too specific and perhaps less natural than both deletion and contraction, it has the following very nice feature:

Theorem 4.2.33. *The posets* $\mathrm{Subdivs}(\mathbf{A})$ *and* $\mathrm{Subdivs}(\mathbf{A}\updownarrow^i_i)$ *are isomorphic. The isomorphism sends triangulations to triangulations, lexicographic ones to lexicographic ones, and regular subdivisions to regular subdivisions.*

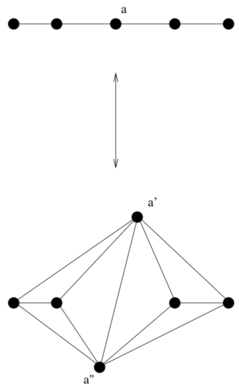

Figure 4.29: The one-point suspension construction.

Proof. The isomorphism sends each subdivision $\mathscr{S}$ of $\mathbf{A}$ to the subdivision $\mathscr{S}\updownarrow_i^i$ consisting of the following cells:

- The cells $B \cup \{i_1\}$ and $B \cup \{i_2\}$ for every cell $B \in \mathscr{S}$ with $i \notin B$.
- The cell $B \cup \{i_1, i_2\}$ for each cell $B \cup \{i\} \in \mathscr{S}$ containing i.

Said differently: $\mathscr{S}\updownarrow_i^i$ consists of the link of i in $\mathscr{S}$ joined to the segment $\{i_1, i_2\}$, together with the anti-star of i in $\mathscr{S}$ joined to the vertices $\{i_1\}$ and $\{i_2\}$. See Figure 4.29 for an example. It is easy, and left to the reader, to check that $\mathscr{S}\updownarrow_i^i$ is a polyhedral subdivision of $\mathbf{A}\updownarrow_i^i$. That this process induces a bijection, preserving order in both directions, follows from the fact that taking the link at i_1 in a subdivision of $\mathbf{A}\updownarrow_i^i$ can always be done and is the inverse operation.

That the bijection preserves regularity can be easily proved if we restrict our attention to height functions on $\mathbf{A}$ and $\mathbf{A}\updownarrow_i^i$ that give height zero to the three special points i, i_1 and i_2. In this way the spaces of height functions on both configurations are identified to one another, and the height functions producing a regular subdivision $\mathscr{S}$ on $\mathbf{A}$ will produce the regular subdivision $\mathscr{S}\updownarrow_i^i$ on $\mathbf{A}\updownarrow_i^i$.

Finally, that triangulations are sent to triangulations is straightforward from the definition of $\mathscr{S}\updownarrow_i^i$. $\square$

In fact, there is an easier way to see that the one-point suspension does not change the refinement poset of *regular* subdivisions. This, however, requires tools from Chapter 5 that utilize the Gale transformation (see Section 4.1.3). At this point, we prepare this by investigating what a Gale transform of the one-point suspension $\mathbf{A}\updownarrow_i^i$ of $\mathbf{A}$ at point $i \in J$ looks like. The result will be strikingly easy.

The following considerations provide an excellent example for computing Gale transforms without actually calculating: We solely use Lemma 4.1.38 in our reasoning. First of all, the one-point suspension contains two versions i_1 and i_2 of point i with the following property (you prove this in Exercise 4.11):

- Whenever i is in a dependence vector V of $\mathbf{A}$ with $i \in V_+$, then there is a dependence vector $V\updownarrow_i^i$ in $\mathbf{A}\updownarrow_i^i$ with $i_1, i_2 \in \left(V\updownarrow_i^i\right)_+$.
- There are no other dependence vectors containing i_1 or i_2.

That means, by Lemma 4.1.38, that in any Gale transform of $\mathbf{A}\updownarrow_i^i$, any cocircuit with i_1 or i_2 in its zero-set has both i_1 and i_1 in its zero-set. In other words, every hyperplane that contains one of i_1 or i_2 contains both. That means, there is no hyperplane separating i_1 and i_2. This is only possible when i_1 and i_2 are parallel vectors in the Gale transform. Therefore, we can assume, without loss of generality on oriented matroid level, that i_1 and i_2 are identical elements.

We summarize:

Proposition 4.2.34. *A Gale transform of the one-point suspension $\mathbf{A}\updownarrow_i^i$ is up to combinatorial equivalence obtained by doubling element i in a Gale transform of $\mathbf{A}$ and labeling the repeated element by i_1 and i_2, resp.* $\square$

In Exercise 5.21 you will apply this together with the tools of Chapter 5 to obtain an almost obvious proof of the bijection in Theorem 4.2.33 in the regular case.

As said at the beginning, although in our examples so far we apply the one point suspension to an interior point, it can be applied to any point in the configuration. See Figure 4.30.

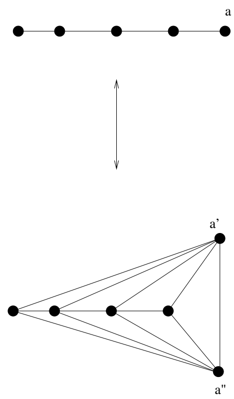

Figure 4.30: A one-point suspension of an extreme point.

Moreover, iterating the construction gives rise to the following nice examples, the details of which will be subject of Exercise 4.12:

- If the one-point suspension is applied three times to the configuration consisting on three copies of the same point (one time on each of the original points), the result is the vertex set of an octahedron. More generally, if it is applied d times to d copies of the same point, it gives the configuration $\{\pm\mathbf{e}_1,\ldots,\pm\mathbf{e}_d\}$ in $\mathbb{R}^d$. This is the vertex set of the so-called *cross-polytope* of dimension d. In the exercise you will—among other things—show that the (regular) cross-polytope has exactly d triangulations, each using a different diameter (edge joining two opposite vertices) of it.

- If it is applied to the interior point in the configuration of Example 4.1.4 it produces a non-regular octahedron. Hence, this non-regular octahedron has four triangulations (as opposed to the regular octahedron, which has three). There are octahedra with six triangulations, but no more; see Section 5.5.

- If it is applied to an independent configuration, it produces an independent configuration of one more dimension. No surprise, since both have exactly one triangulation. Similarly, when applied to a circuit it produces a circuit of one dimension more.

More interestingly, when the one-point suspension is applied to a non-extreme element of an acyclic configuration, it produces a configuration with one less non-extreme element (since i_1 and i_2 are extreme elements of $\mathbf{A}\updownarrow^i_i$). In particular, the one-point suspension applied to all the non-extreme points of an arbitrary point configuration $\mathbf{A}$ one by one produces a configuration in convex position (the vertex set of a polytope) with exactly the same set of triangulations and subdivisions as $\mathbf{A}$. That is to say:

Theorem 4.2.35. *Let $\mathbf{A}$ be an arbitrary point configuration, of dimension d and with n elements. Let v be the number of elements of $\mathbf{A}$ that are extremal. Then, there is a point configuration $\widetilde{\mathbf{A}}$ in convex position, of dimension $d+n-v$ and with $2n-v$ elements such that* $\mathrm{Subdivs}(\mathbf{A})$ *and* $\mathrm{Subdivs}(\widetilde{\mathbf{A}})$ *are isomorphic as posets. The bijection between them preserves regularity.*

Put differently: Unless you are interested in a fixed dimension, the study of subdivisions of point configurations in convex position cannot be considered simpler than the same for arbitrary point configurations.

Observe that a point configuration in convex position cannot have repeated points (see Remark 2.1.21). Also, that in convex position the "bullet-proof" definition of subdivision via labeled cells is equivalent to the more geometric one of Section 2.2. Thus, Theorem 4.2.35 is a new justification for the conceptual approach used in this book: Without this approach, sets in non-convex position would give rise to new, more complicated, spaces of subdivisions.

4.3 Generating polyhedral subdivisions

The concept of regular subdivision allows us to easily construct triangulations of any (point or vector) configuration: Given a configuration $\mathbf{A}$ with label set J, choose a random height function $\omega : J \to \mathbb{R}$ and compute the convex hull of the lifted point set to extract from it the regular subdivision $\mathscr{S}(\mathbf{A}, \omega)$. Typically, the subdivision will be a triangulation (See Part 1 of Lemma 2.3.15). If it is not, you can still "perturb" the height vector ω (as in part 3 of the same Lemma). In this section we revisit two combinatorial ways of constructing triangulations. We saw already particular cases in Chapter 3 but here we present their general definition valid in all dimensions:

4.3.1 The placing (or pushing) triangulation

Let $\mathbf{A}$ be a configuration of dimension d and $i \in J$ be one of its elements. In these conditions, we say that a face F of $J \setminus \{i\}$ is *visible* from i if there is a linear functional which is zero on F, positive on i and negative on the rest of J. Put differently, visible faces are those that have a supporting hyperplane $\mathbf{H}$ separating the point $\mathbf{p}_i$ labeled by i from relint$(J \setminus i)$. We say that a set $B \subset J \setminus i$ is visible from i if it is contained in some face visible from i. Since faces of visible faces are visible, this is equivalent to saying that the carrier of B is visible from i. Another characterization more related to our definition of subdivisions is:

Lemma 4.3.1. *Let F be a face of $J \setminus i$ and let point $\mathbf{x}$ be in the relative interior of F. Then F is visible from a point $\mathbf{p}_i$ if and only if the segment $[\mathbf{x}, \mathbf{p}_i]$ intersects* $\operatorname{conv}(J \setminus i)$ *only at* $\mathbf{x}$.

Observe in particular that if $\mathbf{p}_i \in \operatorname{conv}(J \setminus i)$ then no face is considered visible. That is, when we say "visible" we mean "externally visible".

Proof. For Part (i). When F is visible from i then, for any point $\mathbf{x}$ in its relative interior the line segment $[\mathbf{x}, \mathbf{p}_i]$ is also separated by a supporting hyperplane, thus the one-point intersection condition is necessary. For the converse, let us proceed by contradiction, suppose F is not visible from i, then consider the intersection $\mathbf{M}$ of all supporting half-spaces whose hyperplane supports F must contain the point $\mathbf{p}_i$, but then any line segment $[\mathbf{x}, \mathbf{p}_i]$ must cross the interior of $\operatorname{conv}(J \setminus i)$, a contradiction.

Part (ii) follows easily from Part (i) because the intersection

$$\operatorname{relint}(\operatorname{conv}(B \cup \{i\})) \cap \operatorname{conv}(J \setminus i) = \emptyset.$$

is non-empty precisely when for some $\mathbf{x}$, the line segment $[\mathbf{x}, \mathbf{p}_i]$ intersects $\operatorname{conv}(J \setminus i)$ at a point other than $\mathbf{x}$. $\square$

Lemma 4.3.2. *Let $\mathbf{A} \setminus i$ denote the configuration obtained deleting the element i from $\mathbf{A}$ (that is, deleting the column labeled i, if $\mathbf{A}$ is represented as a matrix). Let $\mathscr{T}$ be a subdivision of $\mathbf{A} \setminus i$.*

Then, the following is a subdivision of $\mathbf{A}$ and it is the only one that extends (i.e., contains) $\mathscr{T}$.

$$\mathscr{T}' := \mathscr{T} \cup \{ B \cup \{i\} : B \in \mathscr{T} \text{ and is visible from } i \}.$$

Observe that another description is that the new subdivision $\mathscr{T}'$ is equal to $\mathscr{T}$ together with the join of i and the subcomplex of $\mathscr{T}$ visible from i. Observe also that if $\mathscr{T}$ is a triangulation then $\mathscr{T}'$ is a triangulation too.

Proof. If i lies in $\mathrm{conv}(J \setminus i)$ then $\mathscr{T}$ is already a subdivision of $\mathbf{A}$ and we cannot add anything to it without violating the proper intersection property of subdivisions. If $J \setminus i$ has less rank than i then *every* subdivision of $\mathbf{A}$ is obtained by joining a subdivision of $\mathbf{A} \setminus i$ to i, by Lemma 2.2.2, and the result is also true.

In the general case, it is intuitively clear that $\mathscr{T}'$ is indeed a subdivision of $\mathbf{A}$, but we postpone a formal proof of it until Section 4.3.4 (Lemma 4.3.10), where this statement is generalized. What is easy to proof is that no other subdivision satisfies the required properties. Indeed, any such subdivision must consist of all of $\mathscr{T}$ plus some sets of the form $B \cup \{i\}$. The closure property implies that the set $B \subseteq J \setminus \{i\}$ is then a simplex in $\mathscr{T}$, and for $\mathscr{T}'$ to satisfy the intersection property, B must be visible from i, by Lemma 4.3.1. $\square$

Definition 4.3.3 (Placing, or pushing, subdivision)**.** The subdivision $\mathscr{T}'$ of the previous lemma is said to be obtained by *placing* i in the subdivision $\mathscr{T}$ of $\mathbf{A} \setminus i$. The triangulation of $\mathbf{A}$ obtained placing its points one by one in a certain order is called the *placing*, or *pushing* triangulation of $\mathbf{A}$ for that order. (The starting step is the unique "triangulation of the first point").

The insertion ordering of the points in the placing process can be given by some geometric property (e.g., a "sweep-hyperplane ordering" is the ordering by a linear functional) or it may be completely arbitrary. Figure 4.31 shows an example of the process.

Lemma 4.3.4. *Let $\mathbf{A}$ be a configuration, with labels $J = \{1, \ldots, n\}$. There is a constant $c_0 > 0$ such that the pushing triangulation of $\mathbf{A}$ (for the ordering given by the labels) equals the regular triangulation obtained by taking any height vector $\omega : J \to \mathbb{R}$ with $\omega(i+1) > \omega(i)c_0 > 0$ for all i.*

Loosely speaking, this statement can be rephrased as "the pushing triangulation is the regular triangulation obtained lifting the points in a certain order with each newly lifted point sufficiently higher than the previous ones".

Proof. Let $\mathscr{T}_2$ be a regular subdivision of $\mathbf{A}$ obtained as in the statement, for a certain constant. We will present a particular c that makes $\mathscr{T}_2$ equal to the placing triangulation. We proceed by induction on the number of vertices. The statement holds for dimension many vertices. By the inductive

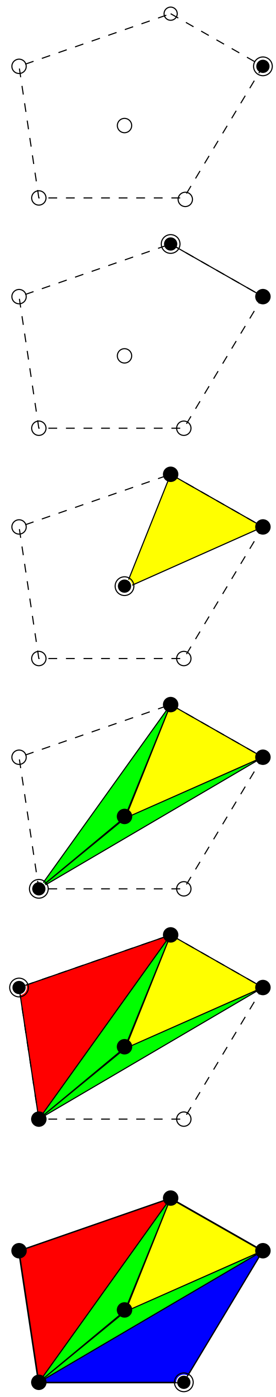

Figure 4.31: A placing triangulation. At each step we mark the new point being placed.

hypothesis on the number of vertices, we assume that there is a constant $c_0 > 0$ such that if $c > c_0$ then the placing triangulation $\mathscr{T}$ of $\mathbf{A} \setminus \mathbf{p}_n$ equals the regular triangulation for the heights $(\omega(1), \dots, \omega(n-1))$ that satisfy $\omega(i+1) > \omega(i)c > 0$ for all i.

For each maximal simplex $B \in \mathscr{T}$, we claim that there is a constant c_B such that if $c > c_B$ then B is a simplex in $\mathscr{T}_2$. This implies that for any c bigger than c_0 and the maximum of all the c_B's, the triangulation $\mathscr{T}_2$ extends $\mathscr{T}$. This indeed proves that $\mathscr{T}_2$ is the placing triangulation, by the uniqueness part in Lemma 4.3.2.

To prove the claim, consider a particular $B = \{i_1, \dots, i_k\}$. Since B is maximal, it is a basis of the linear span of $\mathbf{A}$ (or an affine basis of the affine span if $\mathbf{A}$ is a *point* configuration). Write $\mathbf{p}_n$ as a linear combination of B, that is, write $\mathbf{p}_n = \sum \lambda_k \mathbf{p}_{i_k}$. The hyperplane containing the lifted simplex B^ω contains the point $(\mathbf{p}_n, \sum \lambda_k \omega(k))$. It is enough to choose $c_B > \sum |\lambda_k|$. This suffices, since then

$$\sum \lambda_k \omega(k) \leq \sum |\lambda_k| \omega(n)/c_B \leq \omega(n).$$

□

Lemma 4.3.5. *The placing of i in subdivisions of $\mathbf{A} \setminus i$ is a well-defined and order-preserving map from* $\mathrm{Subdivs}(\mathbf{A} \setminus i)$ *to* $\mathrm{Subdivs}(\mathbf{A})$*, with the following properties:*

1. *Placing i in a triangulation produces a triangulation.*
2. *Placing i in a regular subdivision produces a regular subdivision.*
3. *Placing i in a non-regular triangulation produces a non-regular triangulation.*

We recall that a map $f : (P, \leq) \to (Q, \leq)$ between posets is order preserving if $a \leq b$ implies $f(a) \leq f(b)$.

Proof. All the statements are easy, and left to the reader. For the part about regular subdivisions see perhaps the proof of Lemma 4.3.4. For the part about non-regular ones, observe that if heights exist for the triangulation after placing the point i, then the same heights (restricted to the rest of points) work before the placing. □

However, as strange as it may seem, there is no (natural) map in the other direction. See again Example 4.2.16.

4.3.2 The pulling triangulation

We now define the "opposite" triangulation to a placing triangulation. One way of doing it is just take the opposite lifting heights. That is, we take heights with $\omega(i+1) < c\omega(i) < 0$, with c positive and big. Remember, however, that negative lifting heights are only allowed for point configurations (or, equivalently, for acyclic vector configurations). So, we assume in this subsection that $\mathbf{A}$ is acyclic.

Lemma 4.3.6. *Let* $\mathbf{A}$ *be an acyclic configuration, labeled by* $J = \{1, \ldots, n\}$.

1. *There is a constant* c_0 *such that every height vector* $\omega : J \to \mathbb{R}$ *satisfying* $\omega(i+1) < c_0\omega(i) < 0$ *for all* i *produces a triangulation* $\mathscr{T}$, *and the triangulation is unique.*
2. *In this triangulation, every maximal simplex contains* n.
3. *For each facet* F *of* J, *let* $\mathscr{T}_F$ *denote the triangulation obtained in this way for the restricted configuration* $\mathbf{A}|_F$ *(and with the restricted lifting). Then, the simplices of* $\mathscr{T}$ *are precisely the ones obtained joining* n *to all the simplices in all the* $\mathscr{T}_F$*'s such that* $n \notin F$.

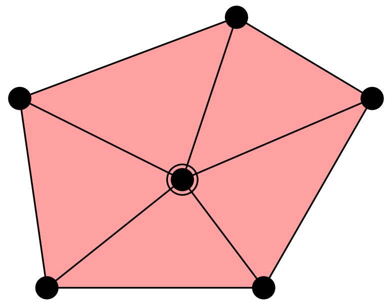

Figure 4.32: A pulling triangulation.

Proof. The proof uses induction on the dimension of the point configuration. In particular, we assume parts (1) and (2) hold for the triangulations of the facets of J and use this to prove the three statements for $\mathbf{A}$ itself.

Actually, the first and third statements follow from the second one. For statement (3), we proved in Lemma 2.3.15 that the restriction to a face F of $\operatorname{conv}(\mathbf{A})$ of a regular triangulation of $\mathbf{A}$ equals the regular triangulation of $\mathbf{A}_F$ obtained with the same heights. In particular, if (2) holds, there is only one possibility for the triangulation $\mathscr{T}$, namely the one stated in part (3). This also proves part (1).

To prove part (2), we need to prove that for any particular potential full-dimensional simplex $B \subseteq J \setminus \{n\}$ there is a constant c_B such that for any lifting with $c > c_B$ the point n is lifted below the hyperplane passing through the lifted vertices of B. This prevents B from being in the regular triangulation produced by the lifting. Taking c_0 to be greater than the maximum of the c_B's (which are a finite set) any triangulation obtained satisfies part (2). If c_0 is also greater than the constant needed for obtaining the desired triangulation $\mathscr{T}_F$ on every facet, then the triangulation also satisfies part (3).

It remains to prove the claim for each B, but this is exactly the same computation we did in Lemma 4.3.4 except now the sign of the lifting heights is reversed and we get the opposite conclusion. □

Definition 4.3.7 (Pulling triangulation)**.** The *pulling triangulation* of an acyclic point configuration $\mathbf{A}$, with respect to a given ordering of the labels, is the regular triangulation obtained by taking heights $\omega(i+1) < c_0\omega(i) < 0$ for every i and for a positive and *sufficiently big* positive constant c_0.

The combinatorial description of the pulling triangulation as the join of the last point to "the pulling triangulations on the faces of $\mathbf{A}$ that do not contain the last point" gives another reason why the triangulation is not well-defined for non-acyclic configurations. In a non-acyclic configuration, there may not be "any face that does not contain the last point". For example, if $\operatorname{conv}(J) = \mathbb{R}^d$ (that is, if $\mathbf{A}$ is totally cyclic), then J itself is the only face of J!

Finally, we can make two simple, but useful, observations about the pulling triangulation of a point set are: (1) It always uses the last point as a vertex, but no other interior point, and (2) If the point set is in general position (more generally, if there is no affinely dependent subset of **A** contained in a facet of $\mathrm{conv}(J)$) then the pulling triangulation depends only on which is the last point and not on the rest of the ordering.

4.3.3 Lexicographic triangulations

Lexicographic triangulations are the combination of pushing and pulling triangulations. We again assume the point set to be ordered, but additionally we prescribe a direction (a sign) to lift each point.

Observe that a more explicit way of describing height functions that produce the pushing and the pulling triangulation is $\omega(i) = c^i$ for the pushing and $\omega(i) = -c^i$ for the pulling, where in both cases c is positive and sufficiently big. As in the case of the pulling triangulation, we here assume that our configurations are acyclic.

Definition 4.3.8 (Lexicographic triangulation)**.** Let **A** be an acyclic configuration with label set $J = \{1, \ldots, n\}$.

The *lexicographic triangulation* of **A** for the given ordering and for a choice of signs $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, +1\}^n$ is the regular triangulation $\mathscr{S}(\mathbf{A}, \omega)$ obtained by taking $\omega_i = \varepsilon_i c^i$ for every i, and for a positive and sufficiently big positive constant c.

The proof that this triangulation is well-defined, in the sense that there is a c_0 such that the regular triangulation $\mathscr{S}(\mathbf{A}, \omega)$ is the same for all $c > c_0$, as well as a description of this triangulation, can be obtained by combining the proofs of Lemmas 4.3.4 and 4.3.6. Indeed:

Lemma 4.3.9. *Under the conditions of Definition 4.3.8:*

- *If* $\varepsilon_n = +1$*, then the lexicographic triangulation of* **A** *equals the triangulation obtained as the union of the lexicographic triangulation of* $\mathbf{A} \setminus n$ *and the simplices joining* n *to the (triangulated) faces of* $\mathbf{A} \setminus i$ *visible from it.*

- *If* $\varepsilon_n = -1$*, then the lexicographic triangulation of* **A** *equals the unique triangulation in which every maximal simplex contains the last point* n *and which, restricted to each proper face* F *of* **A***, coincides with the lexicographic triangulation of that face.*

Proof. We can proceed by induction on n. Clearly the statements are true for $n = 1$.

If $\varepsilon_n = +1$, the same proof of Lemma 4.3.2 shows that there is a c_n such that for every $c > c_n$ the regular triangulation $\mathscr{S}(\mathbf{A}, \omega)$ contains a triangulation of $\mathbf{A} \setminus n$. This triangulation must be the regular triangulation of $\mathbf{A} \setminus n$ for the restricted height function (which we assume to be the same for all sufficiently big c by inductive hypothesis). By Lemma 4.3.4, $\mathscr{S}(\mathbf{A}, \omega)$ is as stated.

If $\varepsilon_n = -1$, then there is a c_n such that for every $c > c_n$ the regular triangulation in question has all maximal simplices containing n. Since restricted to each facet of $\mathrm{conv}(\mathbf{A})$ the triangulation must coincide with the lexicographic triangulation of that facet, we get the statement by induction. □

As before, these descriptions tell us how to recursively construct the lexicographic triangulation without knowing exactly how big the height constant c needs to be. In the next section we come back to lexicographic triangulations and give a different description of them which will explain the reason for the names "pushing" and "pulling" that we used. The intuitive image is that in the lifted point set, in a first approximation all points can be considered "almost coplanar" except for the last point n which has been "pushed up" or "pulled down". See Figure 4.33.

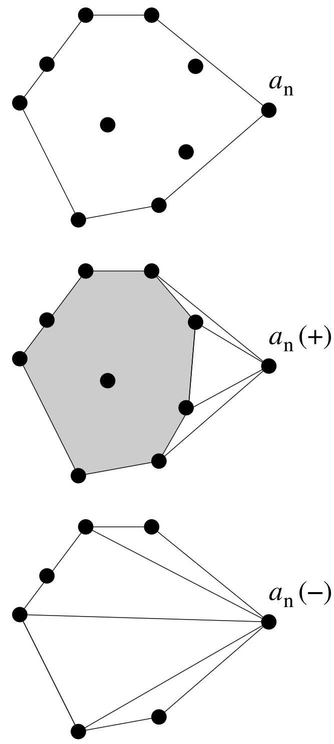

Figure 4.33: Pushing (middle) and pulling (bottom) a point in a point set (top).

4.3.4 Pushing and pulling refinements

We know that the simplest height vector, the all-zero function, produces the simplest regular subdivision, the trivial one. What subdivisions are produced by the next simplest height vectors, which are zero in all but one of the points?

Lemma 4.3.10. *Let $i \in J$ be an element of our configuration and let $\omega : J \to \mathbb{R}$ be a height vector with $\omega(j) = 0$ for all $j \in J \setminus \{i\}$. Let $\mathscr{S} = \mathscr{S}(\mathbf{A}, \omega)$ be the regular subdivision produced by ω:*

- *If $\omega(i) > 0$, then*

$$\mathscr{S} = \{F : F \leq J \setminus i\} \cup \{F \cup \{i\} : F \leq J \setminus \{i\} \text{ and } F \text{ is visible from } i\}.$$

- *If $\omega(i) < 0$ (and we assume that $\mathbf{A}$ is acyclic), then*

$$\mathscr{S} = \{\emptyset\} \cup \{F \cup \{i\} : a \notin F \leq \mathbf{A}\}.$$

Proof. (i) Suppose $\omega(i) > 0$. Then $J \setminus \{i\}$ is clearly a cell in $\mathscr{S}$ (projection of the horizontal lower face of $\mathbf{A}^{\omega}$).

If $i \in \operatorname{conv}(J \setminus i)$ then $J \setminus i$ together with all its faces is already a subdivision of $\mathbf{A}$. The statement holds because in this case no face of $J \setminus i$ is visible.

If $i \notin \operatorname{conv}(J) \setminus i$, that is, if i is extremal in $\mathbf{A}$, then let F be a face of $J \setminus i$ visible from i, and let $\mathbf{H}$ be a supporting hyperplane of F, not containing i. The hyperplane of $\mathbb{R}^{d+1}$ containing $\mathbf{H} \times \{0\}$ and the lifted point $\mathbf{p}_a^{\omega}$ contains the lift of $F \cup \{i\}$ and leaves the rest of $\mathbf{A}^{\omega}$ above. Hence, $F \cup \{i\}$ is also a cell in $\mathscr{S}$. To see that this is the full list of cells in $\mathscr{S}$, it suffices to show that the relative interiors of these cells already cover $\operatorname{conv}(J)$, and observe that Property (IP) prevents any other cell to appear, or otherwise some point would be covered by two relative interiors.

Clearly, $\operatorname{conv}(J \setminus i)$ is already covered by the relative interior of $J \setminus i$ and its faces. So, let $\mathbf{x} \in \operatorname{conv}(J)$ and assume $\mathbf{x} \notin \operatorname{conv}(J \setminus i)$, and that $\mathbf{x} \neq \mathbf{p}_i$.

Consider the ray, or half-line, $\mathbf{l}$ starting at $\mathbf{p}_i$ and passing through $\mathbf{x}$. This ray must intersect the polytope $\operatorname{conv} J \setminus i$ because otherwise $\mathbf{x}$ cannot be written as a convex combination of i and a point in $\operatorname{conv} J \setminus i$. Let $\mathbf{x}'$ be the first point where the ray $\mathbf{l}$ hits $\operatorname{conv} J \setminus i$ and let F be the carrier face of point

$\mathbf{x}'$ in $\operatorname{conv} J \setminus i$. By Lemma 4.3.1 F is visible from $\mathbf{p}_i$. By construction $\mathbf{x}$ is in the convex hull of $F \cup i$.

(ii) If $\omega(i) < 0$, then let F be a face of $\mathbf{A}$ not containing i. Since F must be a proper face, let $\mathbf{H}$ be a supporting hyperplane. Again, the hyperplane of $\mathbb{R}^{d+1}$ containing $\mathbf{H} \times \{0\}$ and i^ω contains $F^\omega \cup \{i^\omega\}$ and leaves the rest of $\mathbf{A}^\omega$ above. Hence, $F \cup \{i\}$ is a cell in $\mathscr{S}$. We claim that this (together with the empty face!) is the full list of cells in $\mathscr{S}$. That is:

$$T = \{\emptyset\} \cup \{F \cup \{i\} : i \notin F < \mathbf{A}\}.$$

The proof that the list is complete is left to the reader. □

The regular subdivisions described in these two paragraphs will be called, respectively, *pushing* (respectively, *pulling*) of i in the trivial subdivision of $\mathbf{A}$. The idea is that we start with (a height vector that produces) the trivial subdivision and then "push" i up or "pull" it down slightly.

Using the notion of *regular refinement* introduced in Definition 2.3.17 and studied in Lemma 2.3.16 we can apply the pushing or pulling to subdivisions other than the trivial one. Recall that the regular refinement of a subdivision $\mathscr{S}$ with respect to a height function $\omega : J \to \mathbb{R}$ is the polyhedral subdivision

$$\mathscr{S}_\omega := \cup_{B \in \mathscr{S}} \mathscr{S}(\mathbf{A}|_B, \omega).$$

Definition 4.3.11. Let $\mathscr{S}$ be a subdivision of a point configuration $\mathbf{A}$. Let $i \in J$ be an element in the label set. Let $\omega : J \to \mathbb{R}$ be a height function that is zero on $J \setminus \{i\}$ and ± 1 at i. We call the subdivision $\mathscr{S}_\omega$ the

1. *Pushing refinement of* i in $\mathscr{S}$ if $\omega_i = +1$, or the
2. *Pulling refinement of* i in $\mathscr{S}$ if $\omega_i = -1$, and every cell of $\mathscr{S}$ that contains i is acyclic.

Observe that the pulling refinement is not well-defined if i lies in some non-acyclic cell of $\mathscr{S}$.

Lemma 4.3.12. *Let $\mathscr{S}$ be a polyhedral subdivision of* $\mathbf{A}$ *and let $i \in$* $\mathbf{A}$*. Then, the pushing (respectively, pulling) of i in $\mathscr{S}$ is the subdivision of* $\mathbf{A}$ *obtained pushing (respectively, pulling) i in each cell B of $\mathscr{S}$.*

In particular, the pushing and pulling refinements of a regular subdivision are regular.

Proof. The first part is just Lemma 2.3.16.

For the regularity, observe that the pushing or pulling of a single element in a subdivision $\mathscr{S}$ produces, by Lemma 4.3.10, the regular refinement $\mathscr{S}_\omega$, for the height function ω specified there. By Lemma 2.3.16, the regular refinement of a regular subdivision is regular. □

We now relate the pushing-pulling operation to the pushing and pulling triangulations (more generally, to the lexicographic triangulations) introduced in Sections 4.3.1, 4.3.2 and 4.3.3. We recall that in the more general form, there is a lexicographic triangulation associated to each ordering of the points of $\mathbf{A}$ and choice of one sign $+$ or $-$ for each point.

Proposition 4.3.13. *The lexicographic triangulation of* **A** *for a given ordering and string of signs is the one obtained starting with the trivial subdivision and pushing or pulling the points, as indicated by their signs (positive for pushing, negative for pulling), in the reverse order.*

Proof. We proof this by induction on the number n of points in **A**. The base case $n = 1$ is trivial and for the inductive step we distinguish the cases where the sign given to the last element n is positive or negative.

- If the sign is positive, the lexicographic triangulation is, by definition, the only one that extends the lexicographic triangulation of $\mathbf{A} \setminus n$ for the given ordering and signs. By inductive hypothesis, the latter is obtained from the trivial subdivision of $\mathbf{A} \setminus n$ by the corresponding reverse sequence of pushings and pullings on $\mathbf{A} \setminus n$. On the other hand, the sequence of pullings and pushings applied to **A** starting with n, first creates the cell $\mathbf{A} \setminus n$ (among others) and then refines it in the lexicographic manner. Hence, it extends the lexicographic triangulation of $\mathbf{A} \setminus n$.
- If the sign is negative the lexicographic triangulation, by definition, refines the subdivision $\mathscr{S}$ obtained from the trivial one by pulling n, and agrees with the lexicographic triangulation (for the restricted ordering and signs) on each facet F not containing n. By inductive hypothesis, the lexicographic triangulation on F is the one obtained by pushing and pulling the points in these facets in the restricted (reverse) orderings, and by Lemma 4.3.12 it agrees with the subdivision obtained pushing and pulling directly in $F \cup \{n\}$ (observe that F is also a facet of this). Hence, the two triangulations coincide.

□

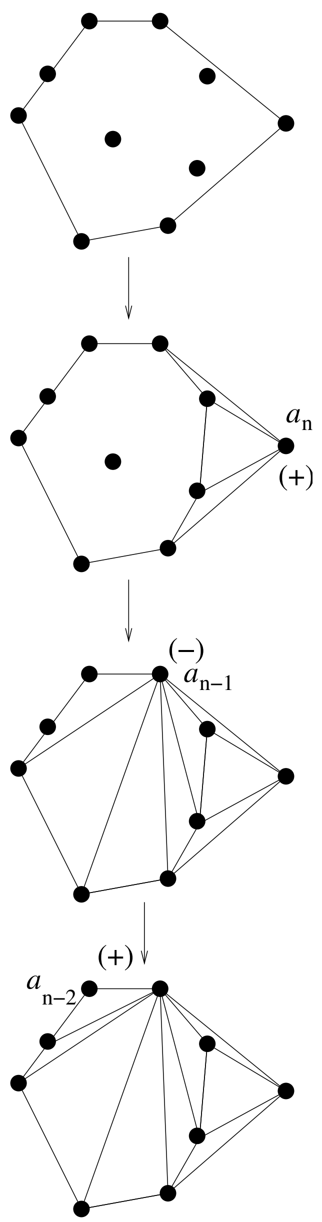

Figure 4.34: A lexicographic triangulation obtained by first pushing the point n, then pulling $n - 1$ and finally pushing $n - 2$. You have to imagine you are looking at the lifted point set from below.

4.4 Two equivalent characterizations of flips

The definition of a flip we gave in Section 2.4 is conceptually easy. However, in proofs and in computer calculations it is not very useful: Given a triangulation $\mathscr{T}$, in order to find a flip, we need to find an almost triangulation that $\mathscr{T}$ refines. We have seen that signed circuits played a fundamental role in combinatorial characterizations of subdivisions; here they are useful again for a characterization of flips. This is no surprise, since circuit signatures were already used to specify the two possible triangulations of a corank-1 configuration. And this was the starting idea for flips in the first place.

4.4.1 Characterization of flips via circuits

The closure Property (CP) of triangulations can be rephrased saying that a triangulation is an (abstract) simplicial complex. In particular, we can apply constructions on simplicial complexes to them like the link and the join (see Sections 2.6.1 and 4.2.1).

When we defined flips in Section 2.4 we saw that any corank-1 configuration that is not minimal is a pyramid over a corank-1 configuration with one