Discrete and Algorithmic Geometry

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Sheet 4

due on Monday, December 22, 2014

The **integral Gale complexity** of a polytope $P \subset \mathbb{R}^d$ with n vertices is

$$\operatorname{igc}(P) \;=\; \min\{\|G\|_\infty : G \subset \mathbb{Z}^e \text{ is a Gale diagram of } P\},$$

where $e = n-d-1$, $\|\mathcal{A}\|_\infty = \max\{\|v\|_\infty : v \in \mathcal{A}\}$ and $\|v\|_\infty = \max\{|v_i|\}$ for $v = (v_1, \dots, v_e)$.

While the existence of nonrational polytopes shows that $\operatorname{igc}(P) = \infty$ is possible (since $\min \emptyset = \infty$), here we are concerned with the following problem:

Problem G. *For $e \in \mathbb{Z}_{\geq 0}$ and $n, m \in \mathbb{Z}_{>0}$, determine*

$$q(e,n,m) \;=\; \#\left\{ \begin{matrix} G \subset \mathbb{Z}^e : G \textit{ is a Gale diagram of a polytope} \\ \textit{with } n \textit{ vertices and } \operatorname{igc}(G) = m \end{matrix} \right\} \Big/ \textit{combinatorial equivalence.}$$

For example, $q(0,n,0) = 1$ and $q(1,n,m) = q(1,n,1)$ for all $m, n \geq 1$. In more down-to-earth terms, we want to solve the following problem:

Problem G*. *Enumerate, up to combinatorial equivalence, all balanced configurations $\mathcal{V}$ of n vectors in $\mathbb{Z}^e$ whose coordinates are all at most m in absolute value, such that*

(1) *the maximum m is achieved by some $v \in \mathcal{V}$,*
(2) *and such that no hyperplane spanned by $e-1$ of the vectors strictly separates exactly one vector from the others.*

For this, recall that a vector configuration $\mathcal{V} = (v_1, \dots, v_n)$ is **balanced** if $\sum_i v_i = 0$; that no hyperplane defined by $e-1$ elements of $\mathcal{V}$ separates exactly one vector from the others iff the Gale dual of $\mathcal{V}$ is in convex position; and that two vector configurations are **combinatorially equivalent** if they define the same oriented matroid.

Your job is to write a function in the `polymake` framework that calculates $q(e,n,m)$. Some considerations to keep in mind:

- Correctness is more important than efficiency, but efficiency is supremely important.
- Your code should be correct in all dimensions: no cutting corners by assuming $e = 2$!
- Your code should be able to calculate at least $q(2,5,2)$ and $q(2,6,2)$.

You will need to think carefully about several independent aspects:

(1) How do you iterate over all vector configurations?
(2) Once you have generated a new configuration, how do you test whether you have already seen a combinatorially isomorphic copy?

My recommendation is to first implement quick-and-dirty solutions to these to test your code for correctness, then iteratively optimize. The choice of data structures will be very important, as well as the decision whether to call `polymake` methods on objects or not. Some methods make your work almost trivial (`P.callPolymakeMethod("LATTICE_POINTS")`, `P.give("VERTICES_IN_FACETS")`, `graph::isomorphic(VIF1, VIF2)`), but some of these are quite time-consuming if you call them often as they have to pass through the perl interface.

Logistics

Please put your code into a directory `exercises/sheet4/code/your_name_here` in the repository; further instructions are in the README file there.

You will find a directory `reports/` with three subdirectories corresponding to Tuesday, December 16; Friday, December 19; and Monday, December 22. Please put a short text file detailing your progress into each of these directories at the appropriate date.

The more often you commit and pull your code, the more often I will be able to give you feedback!

At https://github.com/julian-upc/discrete-geometry/wiki/2014-Sheet-4 there is a wiki where we can discuss any arising questions.

... and beyond

This could turn into a research project, or even a paper! Can you construct a family of rational polytopes with e fixed, but with unbounded integral Gale complexity? Or is

$$\mathrm{igc}(e) \;:=\; \max\left\{ \mathrm{igc}(P) : P \text{ rational},\ \dim \mathrm{Gale}(P) = e \right\}$$

always bounded? If so, how does the bound depend on e?