QF 620 - Stochastic Modelling in Finance
Project Report

Group 19

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PART I (ANALYTICAL OPTION FORMULAE)

The first four pricing models we will be going over are as follows; Black-Scholes model, Bachelier Model, Black Model (1976) and Displaced Diffusion (alternatively, Displaced Black Scholes).

The Black-Scholes formula for a call option is given by:

$$C(S,K,r,\sigma,T) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

$$d_1 = \frac{\log\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

Similarly, the Black-Scholes formula for a put option is given by:

$$P(S,K,r,\sigma,T) = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$

By put-call parity, which states that:

$$C(S,K,r,\sigma,T) - P(S,K,r,\sigma,T) = S - Ke^{-rT}$$

we know that call and put option should be worth the same amount when $K = Se^{rT}$.

In Additional Example 5 Question 16, we explain why is it that a call option struck at $K + \Delta K$ is worth more than a put option struck at $K - \Delta K$. We can numerically verify this using our option pricing formula here.

After completing the pricing of vanilla options using the Black-Scholes model, we next derive the pricing of 4 digital options using the Black-Scholes model.

Black-Scholes model: $dS_t = rS_t d_t + \sigma S_t dw_t^*$

Black-Scholes	Call	Put
Cash or Nothing $(1s_t > K)$	$C = e^{-rT} \cdot N(d_2)$	$P = e^{-rT} \cdot N(-d_2)$
Asset or Nothing $(s_{ts_t} > K)$	$C = S \cdot N(d_1)$	$P = S \cdot N(-d_1)$

Here, we have:

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Next, we derive the pricing formulas of the Bachelier model, Black7 model, and Displaced-diffusion model for the vanilla option and the four digital options respectively.

Bachelier model: $dFt = \sigma_{ba} W_t^*$

Unlike the Black-Scholes model, which assumes that the stock price follows a geometric Brownian motion leading to a log-normal distribution, the Bachelier model assumes that the price of the underlying asset follows a Brownian motion with a normal distribution.

In the Bachelier model, the option's value is calculated using the current price of the asset, the strike price, the time to expiration, the volatility of the underlying asset, and the risk-free interest rate.

Bachelier	Call	Put
Vanilla	$C = e^{-rT}\left[(S-K)N(c) + \sigma S\sqrt{T}\phi(c)\right]$	$P = e^{-rT}\left[(K-S)N(-c) + \sigma S\sqrt{T}\phi(-c)\right]$
Cash or Nothing $(1s_t > K)$	$C = e^{-rT}N(c)$	$P = e^{-rT}N(-c)$
Asset or Nothing $(s_{ts_t} > K)$	$C = e^{-rT}\left[SN(c) + \sigma S\sqrt{T}\phi(c)\right]$	$P = e^{-rT}\left[SN(-c) - \sigma S\sqrt{T}\phi(-c)\right]$

Here, we have:

$$c = \frac{S-K}{\sigma S\sqrt{T}}$$

And $\phi(c)$ is the probability density function of the standard normal distribution.

Black76 model: $dF_t = \sigma F_t W_t^*$

In the Black model, the price of a European option on a futures contract is determined by considering the current futures price instead of the spot price of the underlying asset.

The model uses the futures price, the strike price, the time to expiration, the volatility of the underlying asset, and the risk-free interest rate to calculate the option's value.

Black76	Call	Put
Vanilla	$C = e^{-rT}[FN(d_1) - KN(d_2)]$	$P = e^{-rT}[KN(-d_2) - FN(-d_1)]$
Cash or Nothing $(1s_t > K)$	$C = e^{-rT}N(d_2)$	$P = e^{-rT}N(-d_2)$
Asset or Nothing $(s_{ts_t} > K)$	$C = Fe^{-rT}N(d_1)$	$P = Fe^{-rT}N(-d_1)$

Here, we have:

$$d_1 = \frac{ln\left(\frac{F}{K}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}\ , \qquad F = Se^{rT}$$

Displaced Diffusion Model: $dF_t = \sigma[\beta F_t - (1-\beta)F_0]\, dW_t$

In the Displaced Diffusion Model, the underlying asset price S is replaced with$(S + \beta \cdot d)$, where β is the displacement factor.

The valuation formula for a call or put option in this model uses the modified price of the underlying asset in the Black-Scholes or Black model framework. It still employs the same inputs: the strike price, the time to expiration, the volatility of the underlying asset (which might be adjusted for the displacement), and the risk-free interest rate.

Displaced Diffusion Model	Call	Put
Vanilla	$C = e^{-rT}\left[\frac{F}{\beta}N(c_1) - \left(\frac{(1-\beta)F}{\beta} + K\right)N(c_2)\right]$	$P = e^{-rT}\left[\left(\frac{(1-\beta)F}{\beta} + K\right)N(-c_2) - \frac{F}{\beta}N(-c_1)\right]$
Cash or Nothing $(1s_t > K)$	$C = e^{-rT}N(c_2)$	$P = e^{-rT}N(-c_2)$
Asset or Nothing $(s_{ts_t} > K)$	$C = e^{-rT}\left[\frac{F}{\beta}N(c_1) - \frac{(1-\beta)F}{\beta}N(c_2)\right]$	$P = e^{-rT}\left[\frac{F}{\beta}N(-c_1) - \frac{(1-\beta)F}{\beta}N(-c_2)\right]$

Here, we have:

$$c_1 = \frac{ln\left(\frac{F}{F + \beta(K - F)}\right) + \frac{(\beta\sigma)^2}{2}T}{\beta\sigma\sqrt{T}} \qquad c_2 = c_1 - \beta\sigma\sqrt{T}$$

PART II (MEDEL CALIBRATION)

Here we calibrated the DD Model and SABR model using the Option price of SPX and SPY. At first we need to invert the volatility from each respective model to Black-Scholes implied volatility.

$$Market\ Price - Black\ Sholes(S, T, r, Vol, T) = 0$$

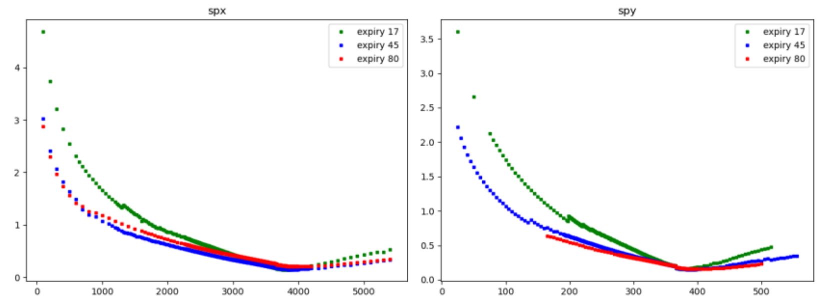
Setting beta in the DD model to zero to get a Normal model, then try to match the market price by varying the volatility parameter in the zero beta DD model. Thus, the volatility that gives the zero-beta DD model the same valuation as the market is defined as the normal model vol.

$$Market\ Price - Displaced\ diffusion(S, T, r, Vol, T, \beta = 0) = 0$$

We can now numerically optimize for the parameters to approach observed implied volatility as much as possible. The implied volatility target is those reported by Black-Scholes model. We also assume the early exercise premium should be negligible for model calibrations.

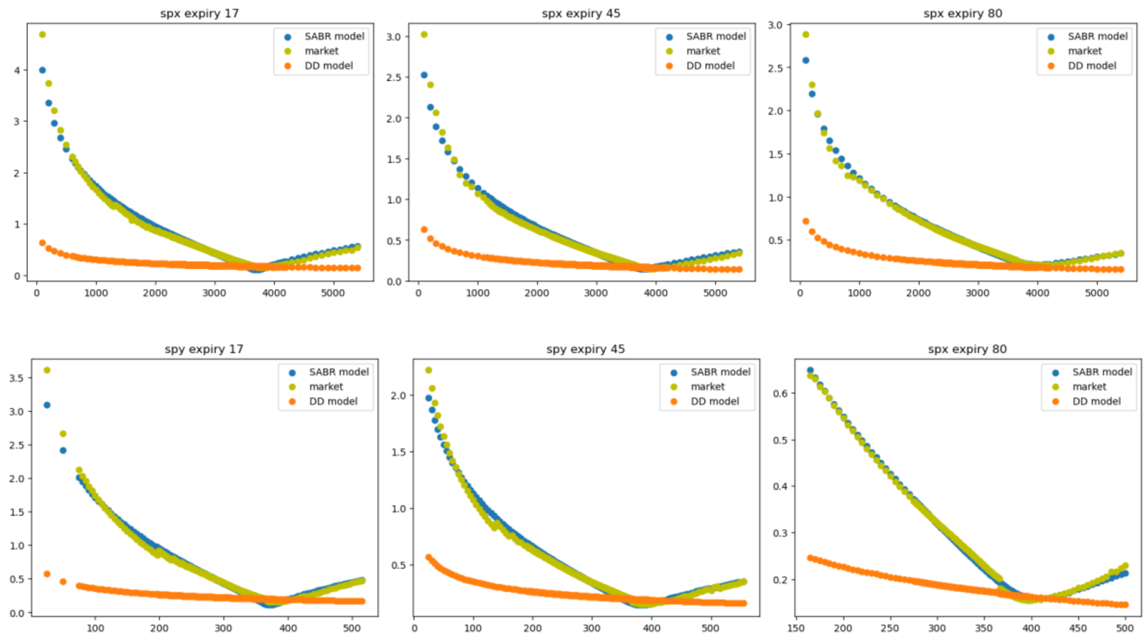
Market expectations

The discrepancies between different maturities may indicate how short-term and long-term market volatilities are perceived differently by market participants.



Volatility skew

By these charts, we can assess the calibration quality of the DD and SABR models. A good fit would see the model's implied volatilities closely tracking the market observed data.

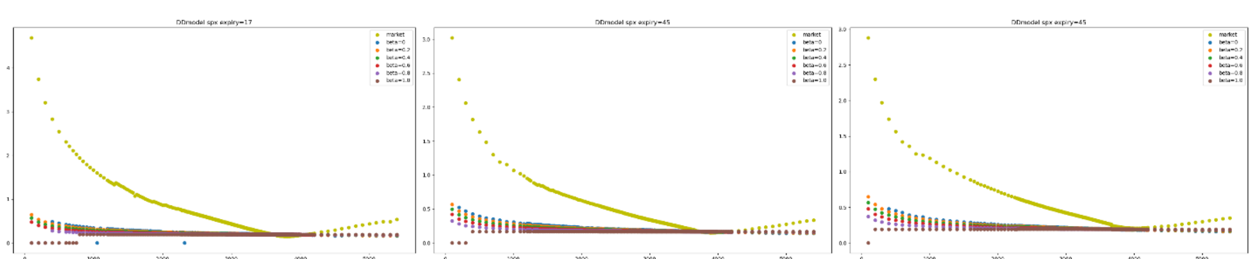


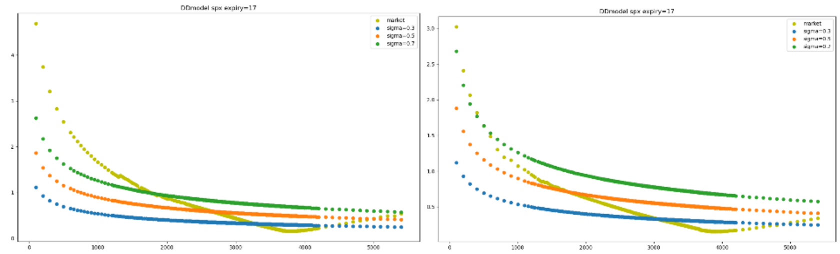
Displace Diffusion implied volatility curve

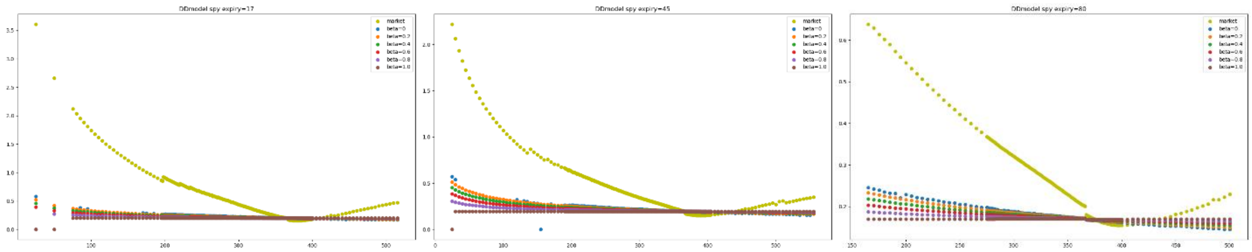
These images are crucial for demonstrating the sensitivity of the DD model to the β parameter and for selecting the appropriate β to match the market's implied volatility structure.

A lower β value tends to produce a flatter volatility curve, which may understate the risk for ITM and OTM options. A higher β value brings the model's implied volatility closer to the market, especially around ATM options, suggesting a more accurate risk assessment in those areas.

They also highlight the challenges of modeling implied volatility across different strikes and maturities.







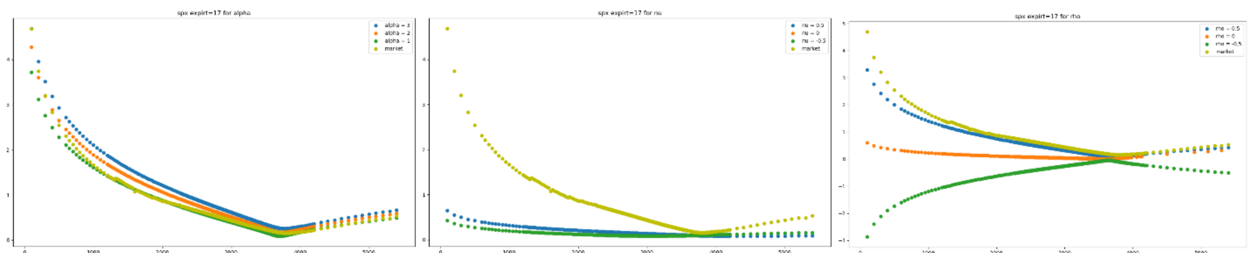
SABR implied volatility curve

In the case of SABR model, there are three degree of freedom. Thus, the calibrated volatility smile will be much closer to the market. Following the same method as the DD model, the parameters α, ρ, ν are then determined by minimizing the sum of squared error terms.

$$d\sigma_t = v\sigma_t \, dW_t^{\sigma} \qquad \sigma_T = \sigma_0 e^{-1/2v^2T + vW_T^{\alpha}}$$

$$dF_t = \alpha_t F_t^{\beta} \, dW_t^F \qquad d\alpha = v\alpha_t \, dW_t^{\alpha} \qquad dW_t^F dW_t^{\alpha} = \rho \, dt$$

These charts are valuable for understanding how each SABR model parameter influences the shape of the implied volatility smile and for calibrating the model to fit the observed market data accurately.



Result:

Calibrated parameters using SABR Model

Index	*Maturity*	*Calibrated α*	*Calibrated β*	*Calibrated ρ*	*Calibrated ν*
SPX	17	1.212	0.7	-0.301	5.460
SPX	45	1.817	0.7	-0.404	2.790
SPX	80	2.140	0.7	-0.575	1.842
SPY	17	0.665	0.7	-0.412	5.250
SPY	45	0.908	0.7	-0.489	2.729
SPY	80	1.121	0.7	-0.633	1.742

The SABR model, known for its extensive flexibility, effectively represents the volatility skew.

In the displaced diffusion model, β adjusts the blend between normal and lognormal distributions, shaping the implied volatility smile. In the SABR model, ρ, the correlation parameter, affects skewness; a negative ρ raises volatility for declining stock prices, influencing OTM put prices, while a positive ρ does the opposite. In other words, when ρ is positive, it implies increased volatility with positive returns, a scenario less commonly seen than its inverse. Conversely, a negative ρ leads to negative skewness, associating high volatility with market downturns, thereby valuing out-of-the-money puts more than calls.

The vol of vol parameter, ν, affects kurtosis; an increase in ν leads to fatter tails, impacting prices for OTM options. A high ν value results in greater volatility variation, introducing more pronounced concavity. This increased volatility uncertainty boosts the implied volatility for far out-of-the-money options, reflecting their role as a hedge against unexpected spikes in volatility.

These parameters crucially shape the volatility smile, with β affecting its curvature, ρ its asymmetry, and ν the tail thickness.

PART III (STATIC REPLICATION)

For any twice-differentiable payoff $h(S_T)$, Breeden-Litzenberger states that:

$$V_0 = e^{-rT}h(F) + \underbrace{\int_0^F h''(K)P(K)dK}_{\text{put integral}} + \underbrace{\int_F^\infty h''(K)C(K)dK}_{\text{call integral}}$$

To implement the Carr-Madan static replication formula in Python, we need to be able to handle numerical integration for the put/call integrals. Then following the Payoff function $S_T^{\frac{1}{3}} + 1.5 \times log(S_T) + 10$ and "Model-free" integrated variance $\sigma_{MF}^2 T = E\left[\int_0^T \sigma_t^2 d_t\right]$, we can test our static replication implementation with:

$$\mathbb{E}[\int_0^T \sigma_t^2 dt] = 2e^{rT}(\int_0^F \frac{P(K)}{K^2}dK + \int_F^\infty \frac{C(K)}{K^2}dK)$$

Since we are using Black76 model in the above, we can cross-check our implementation with the integrated variance based on Black76 model, since under Black76 (or Black-Scholes) model, volatility is deterministic. Hence we have:

$$\mathbb{E}[\int_0^T \sigma_t^2 dt] \quad = \int_0^T \sigma^2 dt \ = \sigma^2 T$$

We also calculate the average of the implied volatilities of a call and a put option by different models. This averaged implied volatility value σ can be interpreted as a balanced measure of market expectations about the volatility of the underlying asset for the specified time period (until the options' expiration). It's a way to get a more comprehensive view of the market's volatility expectations, considering both types of options.

$$\sigma_{Ave} = \frac{\sigma_{Call} + \sigma_{put}}{2}$$

For Black-Scholes model, we have:

$$\sigma_{call} = S\Phi(d_1) - Ke^{-rT}\Phi(d_2) - Market\ Price\ ,\ \sigma_{put} = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1) - Market\ Price$$

Here: $d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ $d_2 = d_1 - \sigma\sqrt{T}$

Once tested, we should replace Black-Scholes with SABR model in order to accurately capture the implied volatility smile/skew in the option market. It suggests that while the Black-Scholes model might provide a baseline price, it doesn't capture the complexities of the actual market where implied volatilities vary for different strikes and maturities. The SABR model is more nuanced and can adapt to these market characteristics, thus providing a more accurate price for exotic options or instruments where the volatility smile or skew is significant.

Black-Scholes model: $S_T = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T}$, $E^*\left[\int_0^T \sigma_t^2\, dt\right] = \sigma_{LN-ATM}^2 T$

Using Black-Scholes model that we have developed in Part I, we can easily calculate the expected integrated variance for both SPX and SPY, which is 0.004214 for SPX and 0.00421 for SPY.

The price P of the derivative under the Black-Scholes Model is given by:

$$Price_{BSM} = \left(S^{\frac{1}{3}}e^{\frac{1}{3}rT-\frac{1}{9}\Sigma^2 T} + 1.5\left(\log(S) + rT - \frac{1}{2}\Sigma^2 T\right) + 10\right)e^{-rT}$$

Then we can calculate the SPX option price which is 37.714138 and SPY option price which is 25.999444. And we also find the $\sigma_{Ave-BSM}$ of SPX is 0.184873 and for SPY the number is 0.184785.

Bachelier Model: $S_T = S_0 + \sigma W_T = S_0\left(1 + \frac{\sigma}{S_0}W_T\right) \approx S_0 e^{\frac{\sigma}{S_0}W_T}$, $E^*\left[\int_0^T \sigma_t^2\, dt\right] \approx 2rT$

Using Bachelier Model we can easily calculate the Bachelier Model Integrated Variance. For SPX, the integrated variance is 0.00424, and for SPY, it is 0.004236, which are similar with the result we got from Black-Scholes model. The volatility (σ) to use in this model are also implied by market data but reflects the characteristics of the Bachelier model.

For the Bachelier Model, the code uses a numerical integration method(‘quad’) to integrate over the payoff function multiplied by the density of a standard normal distribution. The formula would be:

Let $\Sigma_{BACH} = \sqrt{\frac{V_{arBACH}}{T}}$ and $f(x)$ be the payoff function, then:

$$Price_{BACH} = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}[\left(S + \Sigma_{BACH}\sqrt{T}x\right)^{\frac{1}{3}} + 1.5\log\left(S + \Sigma_{BACH}\sqrt{T}x\right) + 10]e^{-\frac{x^2}{2}}\,dx$$

From this Bachelier Model Derivative Pricing Model, we can calculate that the Bachelier Model Derivative Pricing is 37.704847 for SPX, which is 25.995122 for SPY. Then we calculate the $\sigma_{Ave-BCH}$ for both of the two indexes. For SPX, the $\sigma_{Ave-BCH}$ is 677.060201, which is 67.631935 for SPY.

Here we can see there is a difference for $\sigma_{Ave-BCH}$ between SPX and SPY. The significant difference could reflect varying market conditions or characteristics of different assets. This discrepancy might indicate different levels of volatility expected in the markets or assets they represent, or it could suggest that the Bachelier model fits one set of market conditions better than the other. Additionally, it could also signify different market sentiments or expectations for the future movements of these assets. The key to interpreting this difference lies in understanding the specific context of each set of parameters and the market environment they represent.

Static-replication of European payoff:

The SABR model adjusts the volatility used in the pricing formulas by replacing Σ_{BSM} and Σ_{BACH} with Σ_{SABR}, here $\Sigma_{SABR} = \sqrt{\frac{V_{arSABR}}{T}}$.

For the Black-Scholes model under SABR volatility:

$$Price_{BSM-SABR} = S^{\frac{1}{3}} e^{\left(\frac{1}{3}rT - \frac{1}{9}\Sigma_{SABR}^2 T\right)} + 1.5\left(\log(S) + rT - \frac{1}{2}\Sigma_{SABR}^2 T\right) + 10$$

For the Bachelier model under SABR volatility:

$$Price_{BACH-SABR} = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} [\left(S + \Sigma_{SABR}\sqrt{T}x\right)^{\frac{1}{3}} + 1.5\log\left(S + \Sigma_{SABR}\sqrt{T}x\right) + 10]e^{-\frac{x^2}{2}}\,dx$$

Here we substituting SABR model σ into Black-Scholes Model. For SPX, we find that the expected integrated variance is 0.006337, and the exotic option price is 37.700407. For SPY, we find that the expected integrated variance is 0.006017, and the exotic option price is 25.992673.

	BSM		BACH		Sta-Repl	
	σ	$Price$	σ	$Price$	σ	$Price$
SPX	0.004214	37.714138	0.00424	37.704847	0.006337	37.700407
SPY	0.00421	25.999444	0.004236	25.995122	0.006017	25.992673

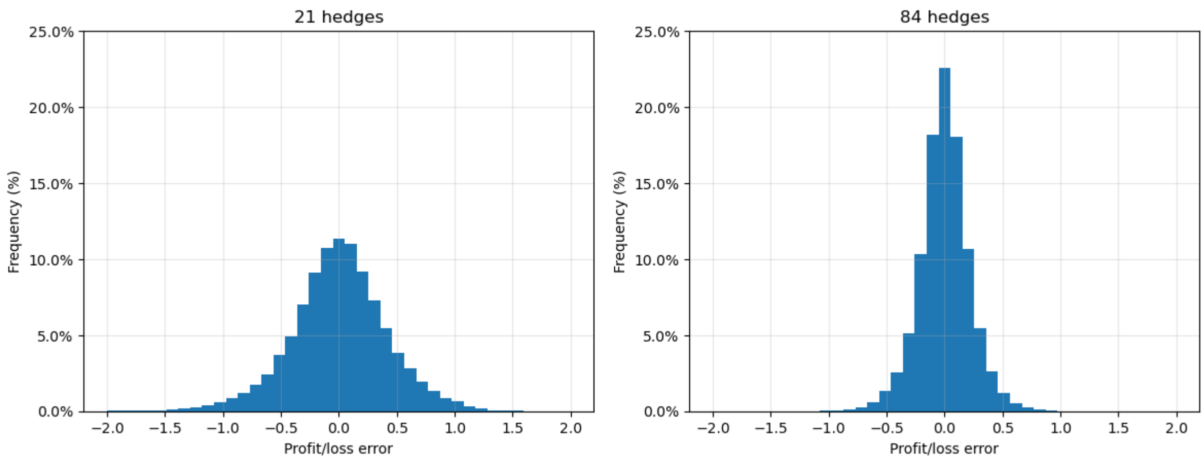
PART IV (DYNAMIC HEDGING)

Dynamic hedging is a strategy that involves frequently adjusting the position in the underlying asset (in this case, a stock) to hedge against the option's price movement. The goal is to be delta-neutral: the portfolio's value is not affected by small changes in the stock price. We have the dynamic hedging strategy for an option is:

$$C_t = \phi_t S_t - \psi_t B_t$$

The hedging error is the discrepancy between the actual payoff of the option at maturity and the value of the hedged portfolio. This error can arise due to factors like the discrete nature of real-world hedging (as opposed to the continuous adjustment assumed in the Black-Scholes model), gaps in the volatility assumption, transaction costs, or slippage. The hedging error is calculated from:

$$Hedging\ error = C_t(replicated\ position) + C_0(\,V_c\ at\ time\ 0) - final\ call\ option\ payoff$$



Hedges	Mean P&L	Standard Deviation of P&L	Standard Deviation of P&L(%) of option premium
21	0.000402	0.423640	16.864190
84	-0.001341	0.217363	8.652736

The histogram 21 hedges reveals that the hedging error range for 21 instances of hedging, or once daily, spans roughly from -1.5 to 1.5. In contrast, for 84 instances of hedging, equivalent to four times per trading day, the range is narrower, between -1 and 1.

We explore the impact of limiting Black-Scholes model's continuous hedging by only allowing discrete interval rebalancing in option replication. Using Monte Carlo simulations for a single European-style option, we analyze the replication error under these constraints. Our findings suggest a rule-of-thumb: replication error is typically proportional to the option's vega(its volatility sensitivity) and the uncertainty in its observed volatility.

$$\sigma_{Error} = \sqrt{\frac{\pi}{4}}(\kappa)\frac{\sigma}{\sqrt{N}}$$

Here κ is the options vega.

The sensitivity of the option price to volatility according to the standard Black-Scholes model, assessed at the initial spot value and the trading date.

$$\kappa = \frac{dC}{d\sigma}(t = 0, S = S_0) = \frac{s_0\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}$$

Our results reveal three key insights: Firstly, with accurate volatility, rates, and dividends, discrete hedging yields an average final profit/loss of zero, showing no directional bias. Secondly, more frequent hedging significantly reduces P&L's standard deviation; for instance, increasing hedging frequency fourfold cuts the standard deviation of replication error by half. Lastly, the replication error's final distribution approximates a normal distribution, validating the use of standard deviation for risk assessment. Despite a 16% standard deviation in daily rehedging, larger hedging errors, both positive and negative, are normally distributed possibilities.