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A dynamic epistemic framework for reasoning about conformant probabilistic plans

Yanjun Li [a], Barteld Kooi [b], Yanjing Wang [c,*]

[a] College of Philosophy, Nankai University, 300350 Tianjin, China
[b] Faculty of Philosophy, University of Groningen, 9712 GL Groningen, the Netherlands
[c] Department of Philosophy, Peking University, 100871 Beijing, China

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ABSTRACT

In this paper, we introduce a probabilistic dynamic epistemic logical framework that can be applied for reasoning and verifying conformant probabilistic plans in a single agent setting. In conformant probabilistic planning (CPP), we are looking for a linear plan such that the probability of achieving the goal after executing the plan is no less than a given threshold probability δ. Our logical framework can trace the change of the belief state of the agent during the execution of the plan and verify the conformant plans. Moreover, with this logic, we can enrich the CPP framework by formulating the goal as a formula in our language with action modalities and probabilistic beliefs. As for the main technical results, we provide a complete axiomatization of the logic and show the decidability of its validity problem.

1. Introduction

1.1. Background of automated planning

Automated planning is a branch of artificial intelligence concerned with devising a plan, which might be a strategy or an action sequence, to achieve the agent's goals. Automated planning technology is widely applied in a variety of areas, ranging from controlling the operations of spacecraft to playing the game of bridge [1]. Classical planning which is the simplest form of automated planning deals with the problem of finding a linear action sequence in a deterministic transition system such that executing the plan in the initial state will achieve the goal (see e.g., [1]). There are two important simplifying assumptions for classical planning: *determinacy of actions* and *full observability*. Full observability indicates one has complete knowledge about the system and the state in which the system starts.

Conformant planning generalizes classical planning by relaxing these two restrictions, namely that it allows lack of knowledge of one's state in the system (and no ability to observe where one is located) and it allows actions to be non-deterministic. The former means that the set of initial states is no longer necessarily a singleton (and corresponds to the agent's initial uncertainty) but also means one cannot attain certainty regarding one's whereabouts based on observation during plan execution. A conformant plan is an action sequence to guarantee the agent's arrival at one of the goal states no matter which initial state the plan starts from and no matter how the (non-deterministic) plan is executed [1]. Since

* Corresponding author.
E-mail addresses: lyjlogic@nankai.edu.cn (Y. Li), b.p.kooi@rug.nl (B. Kooi), y.wang@pku.edu.cn (Y. Wang).

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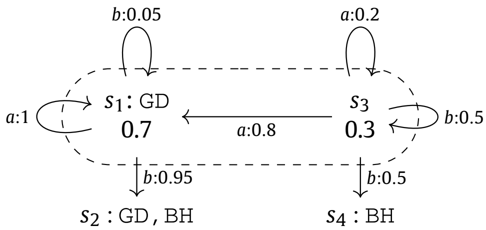
conformant planning brings out agent's uncertainty, therefore, besides the traditional AI approaches, we can also take an epistemic-logical perspective on conformant planning.

Conformant probabilistic planning (CPP) is a significant generalization of conformant planning. The demands that conformant planning puts on the solution may be too strong, in the sense that a solution has to be found in *all* cases. It may be the case that a solution in this sense is impossible while there may still be some plan that leads to a goal state with very high probability. In CPP, we are looking for a linear plan such that the probability of achieving the goal after executing the plan is no less than a given threshold probability. The model for CPP includes a probability distribution over initial states and the probability that a certain action will lead to a certain successor state.

Let us consider the following toy example of CPP.[1] Take a robot whose gripper is possibly wet. The gripper needs to hold a block, but gripping a block while the gripper is slippery is more difficult than when it is dry. This can be modeled in a transition system with probabilities as below where the bubble denotes the initial uncertainty about whether the gripper is dry.



There are two propositions: GD stands for gripper-dry and BH for block-held, and two actions: a stands for drying and b for picking up. Please note that, in the model graph above (as well in all the following model graphs), only positive propositions will be mentioned on states (e.g., BH, GD) and that if a proposition isn't mentioned it could be assumed to be false. We model the initial belief state by a probability distribution $\mathcal{B}$ over the states of the system, which assigns the following probabilities: $\mathcal{B}(s_1) = 0.7$ and $\mathcal{B}(s_3) = 0.3$. The action a dries a dry gripper with probability 1, but make a wet gripper dry with probability 0.8. The action b picks up the block with probability 0.95 if the gripper is dry and with probability 0.5 if the gripper is wet. It is impossible to find a plan in this example that will guarantee that after executing the plan the robot will hold the block. However, for practical purposes it may be enough to find a plan to hold the block, which succeeds at least 90% of the time.

CPP is well studied in the AI literature (see, e.g., [2–6]). The probability of achieving a goal state t by executing a plan $\pi = a_1 \cdots a_n$ is calculated by the following way (cf. [4]):

$$\mu_\pi(t) = \sum_{\{s_0\cdots s_n \mid \forall 1\le i\le n: s_{i-1}\xrightarrow{a_i} s_i, s_n = t\}} \mathcal{B}(s_0) \times Pr(s_0, a_1, s_1) \times \cdots \times Pr(s_{n-1}, a_n, s_n)$$

where $Pr(s_{i-1}, a_i, s_i)$ denotes the probability of reaching s_i after executing a_i at s_{i-1}. For example, in the Slippery Gripper example, the probability of achieving s_2 by executing ab is the following:

$$\begin{aligned}
&\mu_{ab}(s_2)\\
&= \mathcal{B}(s_1) \times Pr(s_1, a, s_1) \times Pr(s_1, b, s_2) + \mathcal{B}(s_3) \times Pr(s_3, a, s_1) \times Pr(s_1, b, s_2)\\
&= 0.7 \times 1 \times 0.95 + 0.3 \times 0.8 \times 0.95\\
&= 0.665 + 0.228 = 0.893
\end{aligned}$$

To achieve a higher probability of holding the block, the robot has to (try to) dry the gripper twice first.

1.2. Motivation

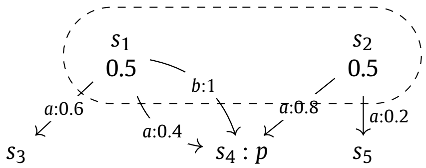
To capture the probabilistic belief change in CPP formally, we propose a *single-agent* dynamic epistemic framework for reasoning about conformant probabilistic plans over probabilistic transition systems. To layout a very general logical foundation, we chose to deviate from the *dynamic epistemic logic* (DEL) approach by considering the *probabilistic transition systems* as our models with both probabilistic labeled transitions and a probability distribution of the initial states. The graph for the previously mentioned Slippery Gripper is an illustration of such a model. This model also is a natural probabilistic extension of the models used by Wang and Li in [7] for a non-probabilistic dynamic epistemic framework over transition systems with initial uncertainty. However, the probabilistic case is much more complicated than the non-probabilistic one

[1] It is a variant of the Slippery Gripper example discussed in [2,3].

handled in [7]. In order to handle the reasoning behind CPP, we not only need to track the probabilistic belief updates but also need to handle the probability of the executability of plans.

Moreover, the logical framework presented in this paper could make a contribution to CPP. First of all, by using a powerful logical language we can express more planning goals, such as epistemic goals, by formulas. The logical language can also help us to distinguish different conformant probabilistic plans for the same goal. Our proof system can help to capture the essential reasoning of CPP, where the axioms can reveal the underlying assumptions behind the probabilistic updates. Finally, the proof system also gives us a (syntactic) way to compute final probability after actions in terms of initial ones. We use the following toy example to illustrate our ideas:

Example 1. Consider the scenario where a patient is in a very critical condition in the ICU. The doctor has to decide quickly what to do and there is no second chance to change. Now the cause for the patient's condition is still uncertain, it could be s_1 or s_2. If the actual cause is s_1 then treatment b would simply save the life of the patient for sure (p). However, if the cause is s_2 then b is not (successfully) executable, but the trial of b may waste the precious time. On the other hand, treatment a will always be executable but it has uncertain outcome depending on the actual cause: if s_2 is the case then the success rate is 80%, while otherwise the success rate is only 40%. The situation is depicted by the following model $\mathcal{N}$ where the bubble denotes the initial uncertainty with a half–half (subjective) probability distribution over s_1 and s_2:



Now, as the doctor, which action should you choose?

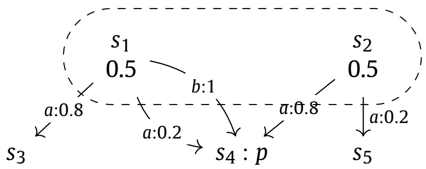
We can calculate the probability of a and b in successfully achieving the goal p:

$$\mu_b(p) = 0.5 \times 1 + 0 = 0.5 \qquad \mu_a(p) = 0.5 \times 0.8 + 0.5 \times 0.4 = 0.6$$

As a rational doctor, we may choose a instead of b to save the patient's life.

However, the situation can be more subtle. Let us lower the success rate of doing a given s_1 a little bit and consider the scenario depicted by the following model $\mathcal{M}$:

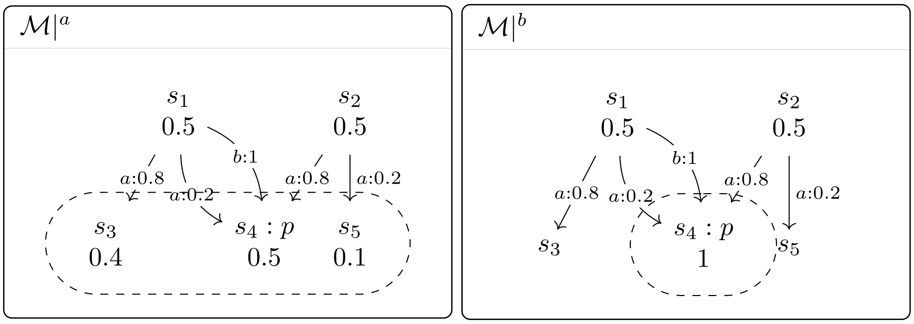
Example 2.



Now we have:

$$\mu_b(p) = 0.5 \times 1 + 0 = 0.5 = 0.5 \times 0.8 + 0.5 \times 0.2 = \mu_a(p)$$

In CPP, we are indifferent between a and b. However, intuitively we still have some reason to prefer a over b: doing b means if the cause is s_2 then we have totally *no* chance to save the patient while using a we always have *some* chance to save the patient no matter what the cause is. These can be made explicit in our language by $B_\epsilon \langle b \rangle p = 0.5$ and $B_\epsilon \langle a \rangle p = 1$ respectively (B_ϵ can be viewed as a belief operator). On the other hand, there is also some reason for us to prefer b over a: the probability of p given that a is successfully executed, which is 0.5, is lower than the probability of p given that b is successfully executed, which is 1. These could be explicitly expressed in our language by $[a]B_\epsilon p = 0.5$ and $[b]B_\epsilon p = 1$ respectively. Such conditional probability can be computed by considering the updated models w.r.t. the given actions. Recall that the core idea of dynamic epistemic logic is to treat actions as updates of models. Intuitively, after successfully executing a given action, the agent carries the bubble forward along the transitions labeled by this action and then computes the new probability distribution within the new bubble accordingly. In our framework executing a and b will give us the following models $\mathcal{M}|^a$ and $\mathcal{M}|^b$ (to be defined precisely in our framework):



Therefore although $\mu_a(p) = \mu_b(p)$, there are finner decompositions of the probability of success which may affect the decision. As we will see, our language can make all these explicit and rigor.

Furthermore, the preference to a over b may also be explained by having an epistemic goal: making sure that (eventually) the agent knows the probability of p is no less than 0.5. This can be expressed by a formula $\varphi = B_\epsilon p \geq 0.5$ in our language. Now, if we recompute the probability in our framework w.r.t. this goal, we have $\mu_a(\varphi) = 1$ but $\mu_b(\varphi) = 0.5$. This is yet another reason to prefer a over b.

We will provide a complete axiomatization of this logic (Section 4). The axiomatization system reveals the implicit assumptions about probabilistic belief updates in CPP. Axiom `ITSP` indicates that in CPP we assume that the agent knows that he believes what he believes and that he knows that he does not believe what he does not believe. Axiom `CP` indicates that we assume that the agent has perfect recall. The formula of perfect recall intuitively means that the agent never forgets: if the agent knows that a will make φ true then after executing a he does know φ. What is more, based on this axiomatization system, we show that each formula can be equivalently transformed to a formula in which each probability modality B_π will never occur in the scope of the action modalities $[a]$ or other probabilities modalities $B_{\pi'}$.[2] From the perspective of planning, it means that checking an epistemic formula after a plan execution equals to checking a different epistemic formula at the initial model (see Proposition 34). For example, in Example 2, the formula $[a]B_\epsilon p = 0.5$ is equivalent to $B_a p = 0.5 B_\epsilon \langle a \rangle \top$. For the formula $[a]B_\epsilon p = 0.5$, we need to check the agent's belief state in the updated model where the agent's belief state is updated after doing a, while, for the formula $B_a p = 0.5 B_\epsilon \langle a \rangle \top$, we only need to check it in the initial belief state. This will become clear after we introduce the details of our framework.

2. Related work

2.1. DEL approach

Dynamic Epistemic Logic (DEL) was invented to model the change of knowledge and belief due to informational events (cf. e.g., [8]). Its core idea is to treat actions (or events) as updates on the epistemic models where uncertainty is encoded by equivalence classes of states of the system. For example, when proposition p is publicly announced by a trustworthy source, we delete the states that do not satisfy p in our current equivalence class. Recent years have witnessed a growing interest in using DEL to handle planning with uncertainty (cf. e.g., [9–16]). In conformant planning, to make sure a goal is achieved eventually by doing a plan, it is crucial to track the transitions of belief states during the execution of the plan. This is the place where DEL can play a role because DEL could track the belief change step by step. For example, in single-agent DEL, after the action of announcing p, the initial uncertain set will be updated to be a subset where p is true in each state. Compared to the traditional AI approach, DEL can make the reasoning behind planning more explicit. Moreover, the rich language of DEL can express epistemic goals naturally. This could make a real contribution to CPP. As it is shown in Example 2, the probability of a and b in successfully achieving the goal p is the same, but we can still make further distinctions in favor of a or b as we mentioned.

Since the traditional CPP is single agent, here we only focus on the single-agent case of DEL approach. The epistemic model of single-agent DEL is an uncertainty set, therefore, to handle the reasoning behind probabilistic planning, we do need to extend DEL with probabilities, such as the work on Probabilistic DEL (PDEL) (cf. [17–19]). However, there has not been much work connecting PDEL with automated planning, in particular CPP. We would like to make the first step in this paper. Our logical framework is inspired by the ideas of PDEL, but it is not the standard formalism of PDEL. The most relevant DEL-style work to ours is the PDEL framework proposed and studied in [17]. There are two significant differences in models and the logical language respectively. First of all, our models are probabilistic transition systems instead of probabilistic epistemic models for PDEL which do not have actions explicitly in the model. In order to express the CPP requirement in terms of μ_π (cf. the introduction), we do need the information of probabilities of actions (transitions) in the model. μ_π is

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[2] B_π roughly captures the probabilistic belief of the agent after action sequence π has occurred.

not only a probability purely about possible current states but also a probability about states weighed by the probability of π being successfully executed (and in that sense it is closer to a prior probability), namely:

$$\mu_\pi(t) = Pr(\pi \text{ is successfully executed}) \cdot Pr(t \text{ is reached given } \pi \text{ is executed}).$$

Correspondingly, in the language we have a modality $B_\pi \varphi \geq q$ which generalizes the one in [17] by allowing a sequence of actions π (note that here π could be ϵ) as an index for each modality to express the probability of reaching a certain goal by π. In PDEL we can only express the probability of a certain proposition being true in the current state, namely, $B_\epsilon \varphi \geq q$.

2.2. Reduction and regression in planning

DEL approaches also often come with a proof system using the so-called reduction axioms which syntactically relate the belief after an action to the belief before an action, and can recursively eliminate the action modality. For example, in [17], the formula $[!p]B\varphi = 0.5$ (which means that the agent's belief degree of φ is 0.5 given the action that p being true is successfully announced) could be equivalently reduced to $B(p \wedge (p \to \varphi)) = 0.5Bp$. It is often shown that any formula can be equivalently reduced to a formula without the action modality. Therefore checking an epistemic goal after a sequence of actions can be reduced to checking some purely epistemic formula at the initial model. The reduction axioms often induce a way of eliminating action modalities in the language. From the perspective of planning, we may use this technique to turn the problem of checking a formula after a plan execution into the problem of checking a different formula at the initial model. There is a clearly natural connection to the regression method in planning using *Situation Calculus* (see, e.g., [20,21]), which has been addressed in the DEL literature as well (see, e.g., [22]). However, in our work, unlike the usual DEL-style logics, we cannot eliminate the action modality completely. Nevertheless, we still have a way to reduce an arbitrary formula to a simpler formula without non-trivial nested modalities (see Proposition 34). As mentioned before, the formula $[a]B_\epsilon p = 0.5$ in Example 2 can be transformed to $B_a p = 0.5B_\epsilon\langle a\rangle\top$. We can show this syntactically (in the proof system), which also leaves the possibility to design a regression method in CPP.

2.3. Compilation to classical planning

Although we have said that there are limitations of classical planning, there is an important line of work trying to transform conformant planning problems and even conformant probabilistic planning problems into classical planning problems w.r.t. different subsets of the original initial uncertainty set with high enough probability (see, e.g., [23,6]). The fully faithful transformations usually requires high computational costs, but we can find some sound but incomplete transformations which can work well in practice. This also raises a natural question to us: whether we can also do without the probability and use the previous non-probabilistic logical framework in [24] for CPP. However, a closer look shows that the transformation of [6] relies on the assumption that the actions are deterministic, but here we do not want to make this assumption. Note that although [23] shows that how to do without non-determinacy in the non-probability setting, it is still unclear to us how can we eliminate nondeterminacy in the probabilistic setting in general. Nevertheless, we can actually show, both semantically and in terms of formal proof in our system (see Proposition 16), that if the actions are deterministic, we can indeed do the corresponding reduction described in [6].

2.4. On logics over MDP and POMDP

Our model is quite similar to *Partially Observable Markov Decision Processes* (POMDP), see e.g., [25,26]. However, our framework implicitly assumes that the agent does not have any observational power when executing the plan, as in non-probabilistic conformant planning. Planning with POMDP is closer to contingent planning where some observational power is assumed. The goal of planning using POMDP is usually to optimize the reward (or cost) in the finite or infinite horizon by some policy which may run forever. On the other hand, the notion of a plan for us is a finite sequence of actions and the goal is a proposition. There are also recent papers discussing the POMDP-based policy synthesis w.r.t. goals expressed by temporal logics (see, e.g., [27]), similar to the discussions in the context of MDP with temporal logic goals [28–30]. Note that our language focuses on the epistemic aspect instead of the temporal aspect, and it allows us to express probabilistic epistemic goals.

Our contributions can be summarized as below:

- We propose a dynamic epistemic framework for reasoning about conformant probabilistic plans over probabilistic transition systems such that:
  - it captures the step-by-step probabilistic belief change formally.
  - we can verify standard conformant probabilistic plans formally, which is beyond the existing PDEL framework.
  - we can express the planning goal by formulas in a very rich language and formalize various subtle probabilistic criteria for conformant plans, which cannot be done in the standard CPP approaches without a powerful logical language.
- We give a sound and complete proof system of the logic such that

  - it reveals implicit assumptions about probabilistic belief updates in CPP.
  - checking an epistemic formula in the updated belief state w.r.t. a plan is equivalent to checking a formula in the initial belief state.
- We show that the logic is decidable, which can facilitate an automated theorem prover for CPP in the future.

Note that, in this paper, we *cannot* yet automatically generate plans (the plan existence problem) within our logical framework. We do need a richer language as in [24] to turn planning into a model checking problem. We leave this and the comparison with existing CPP tools for future work.

The rest of this paper is organized as follows. Section 3 introduces the language and semantics, and also defines conformant probabilistic planning in terms of our logic framework. Section 4 presents the axiomatics of this logic and proves its soundness. Section 5 proves the completeness of the axiomatics. Section 6 shows that the logic is decidable. The last section concludes with some future directions.

3. Language and semantics

In this section we introduce the language of our logic. We build our language based on action modalities and linear inequalities of weighed probabilistic terms inspired by [31,17]. A probabilistic term expresses the probability that a sequence of actions reaches a certain set of states. To keep things simple in this paper we focus on the single-agent case. This language differs from the usual languages of PDEL, and one of the most important differences is that probabilistic expressions here are indexed by a sequence of actions, while in single-agent PDEL probabilistic expressions have no index or can be equally seen as having only index ϵ.

Definition 3 *(Language).* Let a countable set of propositional variables $\mathtt{P}$ and a finite set of actions $\mathtt{A}$ be given. The language $\mathscr{L}$ is defined as the following BNF:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid [a]\varphi \mid q_1 B_{\pi_1}\varphi_1 + \cdots + q_n B_{\pi_n}\varphi_n \geq q$$

where $p \in \mathtt{P}$, $\pi_i \in \mathtt{A}^*$, i.e. a finite string (possible empty) of actions and $q, q_i \in \mathbb{Q}$ for each $1 \leq i \leq n$.

Besides the usual abbreviations, we have the following.

$$\begin{array}{lcl}
\sum_{i=1}^n q_i B_{\pi_i}\varphi_i \geq q & := & q_1 B_{\pi_1}\varphi_1 + \cdots + q_n B_{\pi_n}\varphi_n \geq q \\
q_1 B_{\pi_1}\varphi_1 \geq q_2 B_{\pi_2}\varphi_2 & := & q_1 B_{\pi_1}\varphi_1 + (-q_2) B_{\pi_2}\varphi_2 \geq 0 \\
\sum_{i=1}^n q_i B_{\pi_i}\varphi_i \leq q & := & \sum_{i=1}^n (-q_i) B_{\pi_i}\varphi_i \geq (-q) \\
\sum_{i=1}^n q_i B_{\pi_i}\varphi_i < q & := & \neg(\sum_{i=1}^n q_i B_{\pi_i}\varphi_i \geq q) \\
\sum_{i=1}^n q_i B_{\pi_i}\varphi_i > q & := & \neg(\sum_{i=1}^n q_i B_{\pi_i}\varphi_i \leq q) \\
\sum_{i=1}^n q_i B_{\pi_i}\varphi_i = q & := & (\sum_{i=1}^n q_i B_{\pi_i}\varphi_i \geq q) \wedge (\sum_{i=1}^n q_i B_{\pi_i}\varphi_i \leq q) \\
B_\pi\varphi = B_{\pi'}\varphi' & := & 1 B_\pi\varphi + (-1) B_{\pi'}\varphi' = 0 \\
K\varphi & := & B_\epsilon\varphi = 1 \\
\hat{K}\varphi & := & \neg K \neg\varphi \\
\langle a\rangle\varphi & := & \neg[a]\neg\varphi \\
(\!|a|\!)\varphi & := & [a]\varphi \wedge \langle a\rangle\varphi \\
(\!|a_1 \cdots a_n|\!)\varphi & := & (\!|a_1|\!) \cdots (\!|a_n|\!)\varphi \\
[a_1 \cdots a_n]\varphi & := & [a_1] \cdots [a_n]\varphi
\end{array}$$

Let us explain how to read the formulas of the language. Propositional variables such as p express basic properties of a world, such as “the coin landed heads”. Then we have standard negation and conjunction. We read formulas of the form $[a]\varphi$ as “after all executions of action a it is the case that φ”. In order to read linear inequality formulas of the form $q_1 B_{\pi_1}\varphi_1 + \cdots + q_n B_{\pi_n}\varphi_n \geq q$, we first explain how to read $B_\pi\varphi$. The essential idea is that it represents the probability of getting φ using π. More precisely, it consists of two parts: the probability of φ given the successful execution of π, and the probability of the successful execution of π. Roughly speaking, $B_\pi\varphi$ is $Pr(\varphi \mid ex(\pi)) \cdot Pr(ex(\pi))$ where $ex(\pi)$ means that π can be successfully executed. As it will become more clear after introducing the semantics, $Pr(\varphi \mid ex(\pi))$ will be calculated in the updated model given the execution of π. In other words, $B_\pi\varphi$ is the non-normalized probability of φ in the model you get by executing π. If π is not executable, $B_\pi\varphi$ should be zero, and if it is executable it is the probability of φ in the updated model multiplied by the probability of the executability of π.

The language is interpreted on models which are in a sense (discrete) probabilistic transition systems with initial uncertainty.[3] There are two kinds of probabilistic elements in these models. There is a prior probability distribution representing the initial uncertainty of the agent, and there is a probability function which indicates for each state and each action that

[3] You can also view them as simplified Partially Observable Markov Decision Processes (POMDP) without rewards and observations.

can be executed at that state which probability one has of reaching some other state. Note that, as in the models of non-probabilistic conformant planning, some actions may be not executable on some states.[4] The initial uncertainty is a subset of the set of all states.

Definition 4 *(Model).* A model $\mathcal{M}$ is a tuple $\langle \mathcal{S}^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}}, Pr^{\mathcal{M}}, \mathcal{U}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}}, \mathcal{V}^{\mathcal{M}} \rangle$ such that

- $\mathcal{S}^{\mathcal{M}} \neq \emptyset$, a finite set of states[5];
- $\mathcal{R}^{\mathcal{M}} \subseteq \mathcal{S}^{\mathcal{M}} \times \mathtt{A} \times \mathcal{S}^{\mathcal{M}}$, a non-deterministic execution relation for each action;
- $Pr : \mathcal{R}^{\mathcal{M}} \to \mathbb{Q}^{+}$, a probability function expressing the probability that an action will lead to another state, such that for each $a \in \mathtt{A}$ it holds that $\sum_{t \in \mathcal{R}^{\mathcal{M}}_a(s)} Pr^{\mathcal{M}}(s, a, t) = 1$;
- $\mathcal{U}^{\mathcal{M}}$, a non-empty subset of $\mathcal{S}^{\mathcal{M}}$, consisting of those states that the agent considers possible;
- $\mathcal{B}^{\mathcal{M}} : \mathcal{U}^{\mathcal{M}} \to \mathbb{Q}^{+}$, a probability function expressing the subjective probability of the agent such that $\sum_{s' \in \mathcal{U}^{\mathcal{M}}} \mathcal{B}^{\mathcal{M}}(s') = 1$;
- $\mathcal{V}^{\mathcal{M}} : \mathtt{P} \to \mathcal{P}(\mathcal{S}^{\mathcal{M}})$, a valuation function indicating for each propositional variable in which set of worlds it holds.

For any $s \in \mathcal{U}^{\mathcal{M}}$, $(\mathcal{M}, s)$ is a pointed model.

Given $\mathcal{M}$, $(s, a, t) \in \mathcal{R}^{\mathcal{M}}$ is also denoted as $s \xrightarrow{a} t$, $(s, t) \in \mathcal{R}^{\mathcal{M}}_a$ or $t \in \mathcal{R}^{\mathcal{M}}_a(s)$.

Before we provide the semantics, we first provide the notions needed to define how models are updated by executing a sequence of actions, since we need those models to interpret actions and probabilistic statements. First we define the semantic structure that is associated with a sequence of actions, called the set of execution paths.

Definition 5. Given $\mathcal{M}$, $\pi = a_1 \cdots a_n$, we call $s_0 \cdots s_n$ an execution path of π on $\mathcal{M}$ if $s_0 \in \mathcal{U}^{\mathcal{M}}$ and $s_{i-1} \xrightarrow{a_i} s_i$ for each $1 \le i \le n$. The set of execution paths of π on $\mathcal{M}$ is denoted as $EP_{\mathcal{M}}(\pi)$.

After executing a sequence π the probability the agent assigns to the states of the model changes. Let $\mathcal{U}^{\mathcal{M}}|^{a}$ be the set $\{t \in \mathcal{S}^{\mathcal{M}} \mid s \xrightarrow{a} t \text{ for some } s \in \mathcal{U}^{\mathcal{M}}\}$, and $\mathcal{U}^{\mathcal{M}}|^{\pi} = \mathcal{U}^{\mathcal{M}}|^{a_1} \cdots |^{a_n}$ where $\pi = a_1 \cdots a_n$. We use the following auxiliary notion to update this probability.

Definition 6. Given $\mathcal{M}$ and $\pi = a_1 \cdots a_n \in \mathtt{A}^*$, the function $\mu^{\mathcal{M}}_{\pi} : \mathcal{U}^{\mathcal{M}}|^{\pi} \to \mathbb{Q}$ is defined as follows: for each $t \in \mathcal{U}^{\mathcal{M}}|^{\pi}$,

$$\mu^{\mathcal{M}}_{\pi}(t) = \sum_{\{s_0 \cdots s_n \in EP_{\mathcal{M}}(\pi) \mid s_n = t\}} (\mathcal{B}^{\mathcal{M}}(s_0) \times \Pi^{n}_{i=1} Pr^{\mathcal{M}}(s_{i-1}, a_i, s_i))$$

Given $T \subseteq \mathcal{U}^{\mathcal{M}}|^{\pi}$ and π, let $\mu^{\mathcal{M}}_{\pi}(T) = \sum_{t \in T} \mu^{\mathcal{M}}_{\pi}(t)$, especially, $\mu^{\mathcal{M}}_{\pi}(\emptyset) = 0$.

Remark 1. Similar to the forward algorithm for computing the probability of a particular observable sequence in Hidden Markov Models (cf. e.g., [32]), we can also compute $\mu^{\mathcal{M}}_{\pi}(t)$ recursively by computing $\mu^{\mathcal{M}}_{\pi'}(t')$ for all the initial segments π' of π and the relevant states t'.

The updated probability of the agent applies to a possibly updated set of states that the agent considers possible. Now we define the updated probability of the agent.

Definition 7. Given $\mathcal{M}$ and $\pi = a_1 \cdots a_n \mathtt{A}^*$ such that $\mathcal{U}^{\mathcal{M}}|^{\pi} \neq \emptyset$, function $\mathcal{B}^{\mathcal{M}}|^{\pi} : \mathcal{U}^{\mathcal{M}}|^{\pi} \to \mathbb{Q}^{+}$ is defined as follows: for each $t \in \mathcal{U}^{\mathcal{M}}|^{\pi}$,

$$\mathcal{B}^{\mathcal{M}}|^{\pi}(t) = \frac{\mu^{\mathcal{M}}_{\pi}(t)}{\mu^{\mathcal{M}}_{\pi}(\mathcal{U}^{\mathcal{M}}|^{\pi})}$$

Note that in this definition both the numerator and the denominator are non-zero given the way we set things up. Note that by assuming that $\mathcal{U}^{\mathcal{M}}|^{\pi}$ is non-empty it follows that $EP_{\mathcal{M}}(\pi)$ is non-empty. Since we assumed that the probability functions in the model only assign positive probabilities and t is in $\mathcal{U}^{\mathcal{M}}|^{\pi}$ both numerator and denominator are non-zero. More formally:

Proposition 8. *Given $\mathcal{M}$, $\pi = a_1 \cdots a_n$ and $\mathcal{U}|^{\pi} \neq \emptyset$, we have that $\mathcal{B}^{\mathcal{M}}|^{\pi}$ is a probability function from $\mathcal{U}^{\mathcal{M}}|^{\pi}$ to $\mathbb{Q}^{+}$ and $\sum_{t \in \mathcal{U}^{\mathcal{M}}|^{\pi}} \mathcal{B}^{\mathcal{M}}|^{\pi}(t) = 1$.*

---

4 An alternative approach is to introduce dump states which are the results of *executing* non-executable actions.

5 The restriction to a finite set of states is to make the presentation simper. We could easily remove this restriction and use σ-algebras and fully general probability theory, but this will only distract from the issues we are exploring in this paper.

Given all these definitions, it is now easy to define the updated model.

Definition 9. Given model $\mathcal{M} = \langle \mathcal{S}^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}}, Pr^{\mathcal{M}}, \mathcal{U}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}}, \mathcal{V}^{\mathcal{M}} \rangle$ and $\mathcal{U}^{\mathcal{M}}|^{\pi} \neq \emptyset$, model $\mathcal{M}|^{\pi}$ is defined as $\langle \mathcal{S}^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}}, Pr^{\mathcal{M}}, \mathcal{U}^{\mathcal{M}}|^{\pi}, \mathcal{B}^{\mathcal{M}}|^{\pi}, \mathcal{V}^{\mathcal{M}} \rangle$.

We use this definition of an updated model in the semantics of actions and the linear inequalities of probabilities. The rest of the semantics is far more straightforward.

Definition 10 *(Semantics).* Given pointed model $\mathcal{M}, s$, the truth relation is defined as follows:

$$\begin{array}{rcl}
\mathcal{M}, s \vDash p & \iff & s \in \mathcal{V}^{\mathcal{M}}(p) \\
\mathcal{M}, s \vDash \neg\varphi & \iff & \mathcal{M}, s \nvDash \varphi \\
\mathcal{M}, s \vDash \varphi \wedge \psi & \iff & \mathcal{M}, s \vDash \varphi \text{ and } \mathcal{M}, s \vDash \psi \\
\mathcal{M}, s \vDash [a]\varphi & \iff & \text{for all } s' : s \xrightarrow{a} s' \text{ implies } \mathcal{M}|^{a}, s' \vDash \varphi \\
\mathcal{M}, s \vDash \sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \geq q & \iff & \sum_{i=1}^{n} q_i \mu_{\pi_i}^{\mathcal{M}}([\![\varphi_i]\!]^{\mathcal{M}|^{\pi_i}}) \geq q
\end{array}$$

where $[\![\varphi]\!]^{\mathcal{M}|^{\pi_i}} = \{s \in \mathcal{U}^{\mathcal{M}|^{\pi_i}} \mid \mathcal{M}|^{\pi_i}, s \vDash \varphi\}$.

Remark 2. Note that if $\mathcal{M}, s$ is a pointed model, i.e., $s \in \mathcal{U}^{\mathcal{M}}$, and $s \xrightarrow{a} s'$ then $\mathcal{M}|^{a}, s'$ is also a pointed model, i.e., $s' \in \mathcal{U}^{\mathcal{M}}|^{a} = \mathcal{U}^{\mathcal{M}|^{a}}$.

Note that we use $\mu_{\pi}^{\mathcal{M}}$ to define the updated probabilistic distribution of states in the updated model. $\mu_{\pi}^{\mathcal{M}}$ itself is not normalized.

Proposition 11. *$\mu_{\pi}^{\mathcal{M}}$ is a non-normalized probability function, and it has the following properties:*

(1) $\mu_{\pi}^{\mathcal{M}}([\![\varphi]\!]^{\mathcal{M}|^{\pi}}) \geq 0$
(2) $\mu_{\pi}^{\mathcal{M}}([\![\varphi]\!]^{\mathcal{M}|^{\pi}}) + \mu_{\pi}^{\mathcal{M}}([\![\neg\varphi]\!]^{\mathcal{M}|^{\pi}}) = \mu_{\pi}^{\mathcal{M}}([\![\top]\!]^{\mathcal{M}|^{\pi}})$
(3) $\mu_{\epsilon}^{\mathcal{M}}([\![\top]\!]^{\mathcal{M}}) = 1$
(4) $\mu_{\pi a}^{\mathcal{M}}([\![\top]\!]^{\mathcal{M}|^{\pi a}}) = \mu_{\pi}^{\mathcal{M}}([\![\langle a\rangle\top]\!]^{\mathcal{M}|^{\pi}})$

Proof. Since $[\![\varphi]\!]^{\mathcal{M}|^{\pi}} \subseteq \mathcal{U}^{\mathcal{M}}|^{\pi}$, (1) is obvious by Definition 6. Since $[\![\neg\varphi]\!]^{\mathcal{M}|^{\pi}} = \mathcal{U}^{\mathcal{M}}|^{\pi} \setminus [\![\varphi]\!]^{\mathcal{M}|^{\pi}}$ and $[\![\top]\!]^{\mathcal{M}|^{\pi}} = \mathcal{U}^{\mathcal{M}}|^{\pi}$, (2) is obvious by Definition 6. Since $\mu_{\epsilon}^{\mathcal{M}} = \mathcal{B}^{\mathcal{M}}$, (3) is obvious. For (4), let $\pi = a_1 \cdots a_n$ and $a_{n+1} = a$ then we have the following:

$$\begin{aligned}
&\mu_{a_1\cdots a_{n+1}}^{\mathcal{M}}([\![\top]\!]^{\mathcal{M}|^{a_1\cdots a_{n+1}}}) \\
=&\mu_{a_1\cdots a_{n+1}}^{\mathcal{M}}(\mathcal{U}^{\mathcal{M}}|^{a_1\cdots a_{n+1}}) \\
=&\sum_{s_0\cdots s_{n+1}\in EP_{\mathcal{M}}(a_1\cdots a_{n+1})} (\mathcal{B}^{\mathcal{M}}(s_0) \times \Pi_{i=1}^{n+1} Pr^{\mathcal{M}}(s_{i-1}, a_i, s_i)) \\
=&\sum_{\{s_0\cdots s_n\in EP_{\mathcal{M}}(a_1\cdots a_n) \mid \exists t: t\in\mathcal{R}_a^{\mathcal{M}}(s_n)\}} \\
&(\mathcal{B}^{\mathcal{M}}(s_0) \times \Pi_{i=1}^{n} Pr^{\mathcal{M}}(s_{i-1}, a_i, s_i) \times (\sum_{t\in\mathcal{R}_a^{\mathcal{M}}(s_n)} Pr^{\mathcal{M}}(s_n, a, t))) \\
=&\sum_{\{s_0\cdots s_n\in EP_{\mathcal{M}}(\pi) \mid \exists t: t\in\mathcal{R}_a^{\mathcal{M}}(s_n)\}} (\mathcal{B}^{\mathcal{M}}(s_0) \times \Pi_{i=1}^{n} Pr^{\mathcal{M}}(s_{i-1}, a_i, s_i)) \\
=&\sum_{\{s_0\cdots s_n\in EP_{\mathcal{M}}(a_1\cdots a_n) \mid s_n\in[\![\langle a\rangle\top]\!]^{\mathcal{M}|^{a_1\cdots a_n}}\}} (\mathcal{B}^{\mathcal{M}}(s_0) \times \Pi_{i=1}^{n} Pr^{\mathcal{M}}(s_{i-1}, a_i, s_i)) \\
=&\mu_{a_1\cdots a_n}^{\mathcal{M}}([\![\langle a\rangle\top]\!]^{\mathcal{M}|^{a_1\cdots a_n}}) \quad \square
\end{aligned}$$

Proposition 12. *Given $\pi = a_1 \cdots a_n$, we have that $\mu_{\pi}^{\mathcal{M}}([\![\top]\!]^{\mathcal{M}|^{\pi}}) = 1$ if and only if $\mathcal{M}, s \vDash K(\!|\pi|\!)\top$.*

Proof. Let $\pi_{(i)} = a_1 \cdots a_i$ for each $1 \leq i \leq n$ and $\pi_{(0)} = \epsilon$ then it is easy to show that $\mathcal{M}, s \vDash K(\!|\pi|\!)\top$ if and only if $\mathcal{M}|^{\pi_{(i)}}, v \vDash \langle a_{i+1}\rangle\top$ for each $0 \leq i < n$ and each $v \in \mathcal{U}^{\mathcal{M}}|^{\pi_{(i)}}$.

From left to right: It follows by Proposition 11 that for each $0 \le i < n$, we have

$$\mu^{\mathcal{M}}_{\pi(i+1)}([\![\top]\!]^{\mathcal{M}|^{\pi(i+1)}}) = \mu^{\mathcal{M}}_{\pi(i)}([\![\langle a_{i+1}\rangle\top]\!]^{\mathcal{M}|^{\pi(i)}}) \le \mu^{\mathcal{M}}_{\pi(i)}([\![\top]\!]^{\mathcal{M}|^{\pi(i)}}).$$

Since $\mu^{\mathcal{M}}_{\pi}([\![\top]\!]^{\mathcal{M}|^{\pi}}) = 1$ and $\mu^{\mathcal{M}}_{\epsilon}([\![\top]\!]^{\mathcal{M}}) = 1$, it follows that $\mu^{\mathcal{M}}_{\pi(i)}([\![\langle a_{i+1}\rangle\top]\!]^{\mathcal{M}|^{\pi(i)}}) = \mu^{\mathcal{M}}_{\pi(i)}([\![\top]\!]^{\mathcal{M}|^{\pi(i)}})$ for each $0 \le i < n$. Assume that $\mathcal{M}, s \nvDash K(\!|\pi|\!)\top$ then it follows that there are $0 \le j < n$ and $v \in \mathcal{U}^{\mathcal{M}}|^{\pi(j)}$ such that $\mathcal{M}, v \nvDash \langle a_{j+1}\rangle\top$. Since $v \in \mathcal{U}^{\mathcal{M}}|^{\pi(j)}$, it follows by Definition 6 that $\mu^{\mathcal{M}}_{\pi(j)} > 0$. Thus, we have $\mu^{\mathcal{M}}_{\pi(j)}([\![\langle a_{j+1}\rangle\top]\!]^{\mathcal{M}|^{\pi(j)}}) < \mu^{\mathcal{M}}_{\pi(j)}([\![\top]\!]^{\mathcal{M}|^{\pi(j)}})$. Contradiction. Therefore, we have $\mathcal{M}, s \vDash K(\!|\pi|\!)\top$.

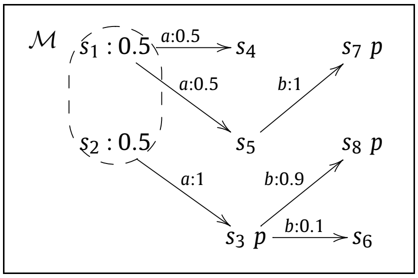
From right to left: We will show that $\mu^{\mathcal{M}}_{\pi(i)}([\![\top]\!]^{\mathcal{M}|^{\pi(i)}}) = 1$ for each $0 \le i \le n$ by induction on i. It is obvious if $i = 0$. If $i = j+1$ where $0 \le j < n$, $\mu^{\mathcal{M}}_{\pi(j+1)} = \mu^{\mathcal{M}}_{\pi(j)a_{j+1}}$. It follows by Proposition 11, we have the following:

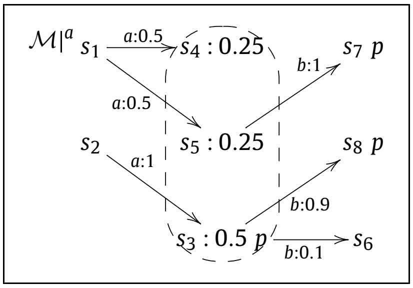
$$\mu^{\mathcal{M}}_{\pi(j)a_{j+1}}([\![\top]\!]^{\mathcal{M}|^{\pi(j)a_{j+1}}}) = \mu^{\mathcal{M}}_{\pi(j)}([\![\langle a_{j+1}\rangle\top]\!]^{\mathcal{M}|^{\pi(j)}})$$
$$\mu^{\mathcal{M}}_{\pi(j)}([\![\langle a_{j+1}\rangle\top]\!]^{\mathcal{M}|^{\pi(j)}}) + \mu^{\mathcal{M}}_{\pi(j)}([\![\neg\langle a_{j+1}\rangle\top]\!]^{\mathcal{M}|^{\pi(j)}}) = \mu^{\mathcal{M}}_{\pi(j)}([\![\top]\!]^{\mathcal{M}|^{\pi(j)}})$$

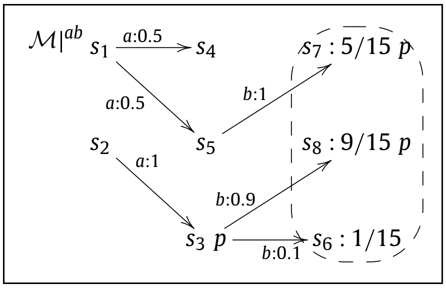
It follows by IH that $\mu^{\mathcal{M}}_{\pi(j)}([\![\top]\!]^{\mathcal{M}|^{\pi(j)}}) = 1$. Therefore, we only need to show that $\mu^{\mathcal{M}}_{\pi(j)}([\![\neg\langle a_{j+1}\rangle\top]\!]^{\mathcal{M}|^{\pi(j)}}) = 0$. Since $\mathcal{M}, s \vDash K(\!|\pi|\!)\top$, namely, $\mathcal{M}|^{\pi(i)}, v \vDash \langle a_{i+1}\rangle\top$ for each $0 \le i < n$ and each $v \in \mathcal{U}^{\mathcal{M}}|^{\pi(i)}$, we have $[\![\neg\langle a_{j+1}\rangle\top]\!]^{\mathcal{M}|^{\pi(j)}} = \emptyset$. Thus, $\mu^{\mathcal{M}}_{\pi(j)}([\![\neg\langle a_{j+1}\rangle\top]\!]^{\mathcal{M}|^{\pi(j)}}) = 0$. □

Recall that $[a]\varphi$ means that φ holds after executing a, and $\mu^{\mathcal{M}}_{\pi_i}([\![\varphi_i]\!]^{\mathcal{M}|^{\pi_i}})$ is the probability of reaching φ_i by executing π_i.[6] We will show how the semantics works by working through an example.

Example 13. Again, consider the scenario where a patient is in a very critical condition but the cause is uncertain (s_1 or s_2). The doctor has to decide in one-step what to do. The left-hand-side model below depicts the effects of taking the pills a or b. According to the instructions of the medicine, b can only be taken after taking a. The effect of a on s_1 is uncertain, but if it ends up at s_5 then taking the pill b will save the patient (p). On the other hand, it might also cause some allergy (s_4) and then taking b is no longer an option. If the actual cause is s_2 then a will work but you might not see it immediately and then taking b might cancel the effect of a (s_6) or have no side-effect at all (s_8).







We can verify the following:

1. $\mathcal{M}, s_1 \vDash B_\epsilon\langle a\rangle\top = 1$
2. $\mathcal{M}, s_1 \vDash [a]B_\epsilon p = 0.5$
3. $\mathcal{M}, s_1 \vDash B_a p = 0.5$
4. $\mathcal{M}, s_1 \vDash B_\epsilon(\!|a|\!)(\!|b|\!)\top = 0.5$
5. $\mathcal{M}, s_1 \vDash [ab]B_\epsilon p > 0.6$
6. $\mathcal{M}, s_1 \vDash B_{ab} p = 0.7$

[6] Note that $q_1 B_{\pi_1}\varphi_1 + \cdots + q_n B_{\pi_n}\varphi_n \ge q$ formulas cannot distinguish states in the same uncertainty set but $[a]\varphi$ formulas can, thus $[a]$ cannot be eliminated.

1. $B_\epsilon\langle a\rangle\top = 1$ says that the agent knows that a is executable, i.e., action a is executable in the set $\mathcal{U} = \{s_1, s_2\}$, indicated by the bubble.
2. $[a]B_\epsilon p = 0.5$ says that after executing action a, the agent assigns probability 0.5 to p, i.e., the probability of $\{s_3\}$ in $\mathcal{M}|^a$ is 0.5.
3. $B_a p = 0.5$ says that the agent assigns probability 0.5 to ending up in a p-state by successfully executing a, i.e., the probability of successfully executing a, which is 1, times the probability of p after doing a, which is 0.5.
4. $B_\epsilon(\!|a|\!)(\!|b|\!)\top = 0.5$ expresses that initially the sequence of actions ab is applicable with probability 0.5, because $(\!|a|\!)(\!|b|\!)\top$ is only true at s_2.
5. $[ab]B_\epsilon p > 0.6$ says that after each successful execution of the sequence ab the probability of p is more than 0.6, because p is true in both s_7 and s_8 and the probability of $\{s_7, s_8\}$ in $\mathcal{M}|^{ab}$ is $14/15 > 0.6$.
6. $B_{ab}p = 0.7$ expresses that the probability of ending up in a p-state by successfully executing ab is 0.7, i.e., $\mu_{ab}^{\mathcal{M}}(\{s_7, s_8\}) = 0.5 * 0.5 * 1 + 0.5 * 1 * 0.9 = 0.7$. In contrast to $[ab]B_\epsilon p > 0.6$, we now also take into account the executability of ab.

We will use this logic as a tool to develop a framework for probabilistic conformant planning, which we can now define in a precise way.

Definition 14 *(Conformant probabilistic planning).* Given a model $\mathcal{M}, s$, a goal formula $\varphi \in \mathscr{L}$, and a threshold δ, probabilistic conformant planning for φ over $\mathcal{M}, s$ w.r.t. δ is to find a linear plan $\pi \in \mathtt{A}^*$ such that $\mathcal{M}, s \vDash B_\pi\varphi \geq \delta$, where π is called a *solution* to the probabilistic planning problem.

According to the above definition, to verify that π is a solution is to model check $B_\pi\varphi \geq \delta$ on the pointed model. In the above example, according to item 6, if $\delta \leq 0.7$ then ab is a solution to the probabilistic planning problem for p over $\mathcal{M}, s_1$ w.r.t. δ.

Let us come back to the examples in the introduction to demonstrate the use of our logical language. In the Slippery Gripper example, we can verify that $B_{ab}(\mathtt{BH}) < 0.9$ holds thus ab is not a good plan if $\delta = 0.9$, but $B_{aab}(\mathtt{BH}) > 0.9$ holds thus aab is a good plan. As for Example 1, $B_b(p) < 0.6 \wedge B_a(p) = 0.6$ thus a looks like a better plan. On the other hand, in Example 2, $B_b(p) = B_a(p)$ holds. However, we can still differentiate the two by verifying the following:

- $B_\epsilon\langle a\rangle p = 1$ but $B_\epsilon\langle b\rangle p = 0.5$, i.e., the agent knows that there is always a chance to save the patient by doing a, compared to doing b. It may give preference to a over b.
- On the other hand, $[a](B_\epsilon p = 0.5)$ but $[b](B_\epsilon p = 1)$, i.e., given b is successfully executed the agent know for sure the patient is saved while the agent is not so sure given a is successfully executed. It may give preference to b over a.
- Further more, if we care more about the epistemic goal: I believe p more than $\neg p$, then $B_a(B_\epsilon(p) > 0.5) = 1$ but $B_b(B_\epsilon(p) > 0.5) = 0.5$, which may give preference to a over b.

To justify our semantics, we first connect it with non-probabilistic conformant planning.

Proposition 15. *Given $\mathcal{M}, s$ and φ, if $\delta = 1$ then the probabilistic conformant planning problem for a non-probabilistic φ over $\mathcal{M}, s$ w.r.t. δ is a standard conformant planning problem for φ over $\mathcal{M}, s$ where the probabilities over the states and transitions do not matter, i.e. $\mathcal{M}, s \vDash B_\pi\varphi = 1 \iff \mathcal{M}, s \vDash K(\!|\pi|\!)\varphi$.*

Proof. We only need to show that $\mu_\pi^{\mathcal{M}}([\![\varphi]\!]^{\mathcal{M}|^\pi}) = 1$ if and only if $\mathcal{M}, s \vDash K(\!|\pi|\!)\varphi$.

From left to right: Since $[\![\varphi]\!]^{\mathcal{M}|^\pi} \subseteq [\![\top]\!]^{\mathcal{M}|^\pi}$, it follows that $\mu_\pi^{\mathcal{M}}([\![\top]\!]^{\mathcal{M}|^\pi}) \geq 1$. It follows by Proposition 11 that $\mu_\pi^{\mathcal{M}}([\![\top]\!]^{\mathcal{M}|^\pi}) \leq \mu_\epsilon^{\mathcal{M}}([\![\top]\!]^{\mathcal{M}}) = 1$. Therefore, $\mu_\pi^{\mathcal{M}}([\![\top]\!]^{\mathcal{M}|^\pi}) = 1$. It follows by Proposition 12 that $\mathcal{M}, s \vDash K(\!|\pi|\!)\top$. Thus, we only need to show that $[\![\neg\varphi]\!]^{\mathcal{M}|^\pi} = \emptyset$. Since $\mu_\pi^{\mathcal{M}}([\![\varphi]\!]^{\mathcal{M}|^\pi}) = \mu_\pi^{\mathcal{M}}([\![\top]\!]^{\mathcal{M}|^\pi}) = 1$, it follows by Proposition 11 that $\mu_\pi^{\mathcal{M}}([\![\neg\varphi]\!]^{\mathcal{M}|^\pi}) = 0$. Therefore, $[\![\neg\varphi]\!]^{\mathcal{M}|^\pi} = \emptyset$.

From right to left: Since $\mathcal{M}, s \vDash K(\!|\pi|\!)\varphi$, we have that $\mathcal{M}, s \vDash K(\!|\pi|\!)\top$ and $[\![\neg\varphi]\!]^{\mathcal{M}|^\pi} = \emptyset$. It follows by Proposition 12 that $\mu_\pi^{\mathcal{M}}([\![\top]\!]^{\mathcal{M}|^\pi}) = 1$. Since $[\![\neg\varphi]\!]^{\mathcal{M}|^\pi} = \emptyset$, we have $\mu_\pi^{\mathcal{M}}([\![\neg\varphi]\!]^{\mathcal{M}|^\pi}) = 0$. It follows by Proposition 11 that $\mu_\pi^{\mathcal{M}}([\![\varphi]\!]^{\mathcal{M}|^\pi}) = 1$. □

Another degenerated case of conformant probabilistic planning is discussed in [6] under the assumption that the actions are deterministic. It is shown in [6] that the probabilistic planning problem can then be reduced to a non-probabilistic one. The reduction relies on the fact that, over deterministic models, the probability of reaching a φ-world by a sequence π can be reduced to the probability of the truth of $\langle\pi\rangle\varphi$ at the initial state, which is proved formally in our framework below.

Proposition 16. *Given $\pi = a_1 \cdots a_n$, $\mu_\pi^{\mathcal{M}}([\![\varphi]\!]^{\mathcal{M}|^\pi}) = \mu_\epsilon^{\mathcal{M}}([\![\langle\pi\rangle\varphi]\!]^{\mathcal{M}})$ if all actions a_i ($1 \leq i \leq n$) are deterministic.*

Proof. If $\mathcal{U}|^\pi = \emptyset$, this follows that $[\![\varphi]\!]^{\mathcal{M}|^\pi} = \emptyset$ and $[\![\langle\pi\rangle\varphi]\!]^{\mathcal{M}} = \emptyset$. Thus, it is obvious that $\mu_\pi^{\mathcal{M}}([\![\varphi]\!]^{\mathcal{M}|^\pi}) = \mu_\epsilon^{\mathcal{M}}([\![\langle\pi\rangle\varphi]\!]^{\mathcal{M}})$.

Table 1
System $\mathbb{SCPP}$.

AXIOMS	
	All instances of propositional tautologies
	All instances of linear inequality axioms
DIST(a)	$[a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi)$
T	$K\varphi \to \varphi$
Nonneg(π)	$B_\pi \varphi \geq 0$
PRTR(ϵ)	$K\top$
PRF(πa)	$B_{\pi a}\varphi \leq B_\pi \langle a\rangle\varphi$
PRFEQ(πa)	$B_\pi(\langle a\rangle\varphi \wedge \langle a\rangle\neg\varphi) = 0 \leftrightarrow B_{\pi a}\varphi = B_\pi\langle a\rangle\varphi$
Add(π)	$B_\pi(\varphi \wedge \psi) + B_\pi(\varphi \wedge \neg\psi) = B_\pi\varphi$
ITSP	$(\sum_{i=1}^n q_i B_{\pi\pi_i'}\varphi_i \bowtie q B_\pi \top) \to B_\pi(\sum_{i=1}^n q_i B_{\pi_i'}\varphi_i \bowtie q) = B_\pi \top$
CP	$\langle a\rangle\top \to ([a](\sum_{i=1}^n q_i B_{\pi_i}\varphi_i \bowtie q) \leftrightarrow \sum_{i=1}^n q_i B_{a\pi_i}\varphi_i \bowtie q B_a \top)$
DET	$\langle a\rangle\varphi \to [a]\varphi$ where φ is a probability formula
RULES	
MP	From $\varphi \to \psi$ and φ, infer ψ
GEN	From φ, infer $[a]\varphi$
Equivalence	From $\varphi \leftrightarrow \psi$, infer $B_\pi\varphi = B_\pi\psi$

If $\mathcal{U}|^\pi \neq \emptyset$, we have that

$$\mu_\pi^{\mathcal{M}}([\![\varphi]\!]^{\mathcal{M}|^\pi}) = \sum_{\{s_0\cdots s_n \in EP_{\mathcal{M}}(\pi) | s_n \in [\![\varphi]\!]^{\mathcal{M}|^\pi}\}} \mathcal{B}(s_0) \times \Pi_{i=1}^n Pr(s_{i-1}, a_i, s_i).$$

Since each a_i $(1 \leq i \leq n)$ is deterministic, this follows that $Pr(s_{i-1}, a_i, s_i) = 1$ for all $1 \leq i \leq n$. Thus, we have

$$\mu_\pi^{\mathcal{M}}([\![\varphi]\!]^{\mathcal{M}|^\pi}) = \sum_{\{s_0\cdots s_n \in EP_{\mathcal{M}}(\pi) | s_n \in [\![\varphi]\!]^{\mathcal{M}|^\pi}\}} \mathcal{B}(s_0).$$

We know that $\mathcal{M}, s_0 \vDash \langle\pi\rangle\varphi$ if and only if there is $s_0 \cdots s_n \in EP_{\mathcal{M}}(\pi)$ such that $\mathcal{M}|^\pi, s_n \vDash \varphi$. Therefore, we have that

$$\sum_{\{s_0\cdots s_n \in EP_{\mathcal{M}}(\pi) | s_n \in [\![\varphi]\!]^{\mathcal{M}|^\pi}\}} \mathcal{B}(s_0) = \sum_{s_0 \in [\![\langle\pi\rangle\varphi]\!]^{\mathcal{M}}} \mathcal{B}(s_0).$$

Thus, we have $\mu_\pi^{\mathcal{M}}([\![\varphi]\!]^{\mathcal{M}|^\pi}) = \mu_\epsilon^{\mathcal{M}}([\![\langle\pi\rangle\varphi]\!]^{\mathcal{M}})$. $\square$

4. Axiomatization

In this section we provide a Hilbert-style proof system for the logic presented above. A proof consists of a sequence of formulas such that each formula is either an instance of an axiom or it can be obtained by applying one of the rules to formulas occurring earlier in the sequence.

Definition 17. Let $\bowtie$ be one of $\leq, =, \geq, <, >$, and $\overline{\bowtie}$ be the negation of $\bowtie$. System $\mathbb{SCPP}$ is defined in Table 1, where the linear inequality axioms can be found in Definition 57 of Appendix J.

Let us explain how the above axioms are to be read. We only focus on those involving probability. Axiom T expresses that truths are assigned positive probability. This is because the empty sequence is always executable and we defined pointed models such that the state is always in $\mathcal{U}^{\mathcal{M}}$, so it will always receive positive probability.

Axiom Nonneg(π) expresses that any formula receives a non-negative probability (since negative probabilities don't make sense).

Axiom PRTR(ϵ) expresses that the set of states that the agent considers possible is assigned probability 1.

Axiom PRF(πa) expresses that the probability of those πa-execution paths leading to φ-states, is less than or equal to the probability of those π-execution paths leading to states where a can lead to a φ-state. This is because executing π may lead to a state where executing a may lead to a φ-state, but executing a could also lead to a non-φ-state.

Axiom PRFEQ(πa) expresses the condition under which the above probabilities are equal. This is the case if either all a-paths in π-reachable states lead to φ-states or if all a-paths in π-reachable states lead to non-φ-states, or in other words whenever the probability that executing a can lead to a φ-state and can lead to a non-φ-state is zero.

Axiom Add(π) expresses that probabilities are additive.

Axiom ITSP can be viewed as the combination of the introspection axioms 4 and 5 in epistemic logic. Note that two simple forms of ITSP are:

$$B_\epsilon\varphi \geq q \to B_\epsilon(B_\epsilon\varphi \geq q) = 1 \qquad \text{you know that you believe what you believe}$$
$$\neg(B_\epsilon\varphi \geq q) \to B_\epsilon(B_\epsilon\varphi < q) = 1 \qquad \text{you know that you don't believe what you don't believe}$$

Axiom CP is essentially the definition of the update using normalization, given that a is executable. A simple form of CP is the following.

$$\langle a\rangle\top \to ([a](B_\epsilon\varphi \bowtie q) \leftrightarrow B_a\varphi \bowtie q B_a\top)$$

Note that DET is not valid for arbitrary ψ. It is crucial that a is not deterministic for basic facts.

As a simple but non-trivial example of derivation in our proof system, we can show that if the models are deterministic then the probability of reaching φ using π is the same as the probability of $\langle\pi\rangle\varphi$ initially, which was proved semantically by Proposition 16.

Proposition 18. *In the system $\mathbb{SCPP}^+$ which extends $\mathbb{SCPP}$ with the unrestricted deterministic axiom $\langle a\rangle\varphi \to [a]\varphi$, we have $\vdash_{\mathbb{SCPP}^+} B_\pi\varphi = B_\epsilon\langle\pi\rangle\varphi$.*

Proof. *Since* $\vdash \langle a\rangle\varphi \to [a]\varphi$ for all actions a and all formulas φ, this follows that $\vdash \langle a\rangle\varphi \wedge \langle a\rangle\neg\varphi \leftrightarrow \bot$. Since $\vdash B_\pi\bot = 0$, this follows that $\vdash B_\pi(\langle a\rangle\varphi \wedge \langle a\rangle\neg\varphi) = 0$. Then it follows by Axiom PRFEQ(πa), and Rule Equivalence that $\vdash B_{\pi a}\varphi = B_\pi\langle a\rangle\varphi$. By induction on π, we can prove that $\vdash B_\pi\varphi = B_\epsilon\langle\pi\rangle\varphi$. $\square$

For the rest of this section, we will prove that the axiomatization is sound. Given that the logic is built on well-understood modal logic, we will not show that the usual modal axioms and rules are sound. Also, the part of the axiomatization concerned with linear inequalities is well-understood and we do not show the soundness of the part either. Instead, we will focus on the axioms and rules that deal with the interplay between action and probability.

We will leave the soundness proofs of Nonneg(π) and PRTR(ϵ) to the reader. They are rather straightforward. In order to prove the soundness of PRF(πa), we first prove two auxiliary propositions. The first is about the relation between probabilities in a model after an action and probabilities preceding the action.

Proposition 19. *Given model $\mathcal{M}$ and $\mathcal{U}^{\mathcal{M}}|^\pi \neq \emptyset$, we have $\mu_{\pi'}^{\mathcal{M}|^\pi}(t) = \mu_{\pi\pi'}^{\mathcal{M}}(t) / \mu_\pi^{\mathcal{M}}(\mathcal{U}^{\mathcal{M}}|^\pi)$ for each $t \in \mathcal{S}^{\mathcal{M}}$.*

Proof.

$$\begin{aligned}
&\mu_{\pi'}^{\mathcal{M}|^\pi}(t)\\
=& \sum_{\{s_0\cdots s_n \in EP_{\mathcal{M}|^\pi}(\pi') | s_n = t\}} (\mathcal{B}^{\mathcal{M}|^\pi}(s_0) \times \Pi_{i=1}^n Pr^{\mathcal{M}|^\pi}(s_{i-1}, a_i, s_i))\\
=& \ 1/\mu_\pi^{\mathcal{M}}(\mathcal{U}^{\mathcal{M}}|^\pi) \sum_{\{s_0\cdots s_n \in EP_{\mathcal{M}|^\pi}(\pi') | s_n = t\}} (\mu_\pi^{\mathcal{M}}(s_0) \times \Pi_{i=1}^n Pr^{\mathcal{M}|^\pi}(s_{i-1}, a_i, s_i))\\
=& \ 1/\mu_\pi^{\mathcal{M}}(\mathcal{U}^{\mathcal{M}}|^\pi) \sum_{\{s_0\cdots s_n \in EP_{\mathcal{M}|^\pi}(\pi') | s_n = t\}} \Bigg(\sum_{\{s'_0\cdots s'_m \in EP_{\mathcal{M}}(\pi) | s'_m = s_0\}}\\
&\quad \mathcal{B}^{\mathcal{M}}(s_0) \times \Pi_{i=1}^n Pr^{\mathcal{M}}(s'_{i-1}, a'_i, s'_i) \times \Pi_{i=1}^n Pr^{\mathcal{M}|^\pi}(s_{i-1}, a_i, s_i)\Bigg)\\
=& \ 1/\mu_\pi^{\mathcal{M}}(\mathcal{U}^{\mathcal{M}}|^\pi) \sum_{\{u_0\cdots u_{m+n} \in EP_{\mathcal{M}}(\pi\pi') | u_{m+n} = t\}} (\mu_\pi^{\mathcal{M}}(u_0) \times \Pi_{i=1}^{m+n} Pr^{\mathcal{M}}(s_{i-1}, a_i, s_i))\\
=& \ \mu_{\pi\pi'}^{\mathcal{M}}(t)/\mu_\pi^{\mathcal{M}}(\mathcal{U}^{\mathcal{M}}|^\pi) \quad \square
\end{aligned}$$

Using this proposition we can prove the second auxiliary proposition that expresses that updating a model with a composed action is the same as updating the model sequentially, first with the one component of the action, then the other component.

Proposition 20. *Given model $\mathcal{M}$ and $\mathcal{U}^{\mathcal{M}}|^{\pi\pi'} \neq \emptyset$, we have $\mathcal{M}|^\pi|^{\pi'} = \mathcal{M}|^{\pi\pi'}$.*

Proof. We only need to show that $\mathcal{B}^{\mathcal{M}|^\pi|^{\pi'}}(t) = \mathcal{B}^{\mathcal{M}|^{\pi\pi'}}(t)$ for each $t \in \mathcal{U}^{\mathcal{M}}|^{\pi\pi'}$.

$$\begin{aligned}
\mathcal{B}^{\mathcal{M}|^\pi|^{\pi'}}(t) &= \frac{\mu_{\pi'}^{\mathcal{M}|^\pi}(t)}{\mu_{\pi'}^{\mathcal{M}|\pi}(\mathcal{U}^{\mathcal{M}}|^{\pi\pi'})}\\
&= \frac{\mu_{\pi\pi'}^{\mathcal{M}}(t)/\mu_\pi^{\mathcal{M}}(\mathcal{U}^{\mathcal{M}}|^\pi}{\sum_{s\in\mathcal{U}^{\mathcal{M}}|^{\pi\pi'}} \mu_{\pi\pi'}^{\mathcal{M}}(s)/\mu_\pi^{\mathcal{M}}(\mathcal{U}^{\mathcal{M}}|^\pi)} \qquad \text{by Proposition 19}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu^{\mathcal{M}}_{\pi\pi'}(t)}{\sum_{s\in\mathcal{U}^{\mathcal{M}}|^{\pi\pi'}} \mu^{\mathcal{M}}_{\pi\pi'}(s)} \\
&= \mathcal{B}^{\mathcal{M}|^{\pi\pi'}}(t) \quad \square
\end{aligned}
$$

Using this proposition we can show the soundness of $\mathtt{PRF}(\pi a)$.

Proposition 21. $\vDash B_{\pi a}\varphi \leq B_{\pi}\langle a\rangle\varphi$.

Proof. Given model $\mathcal{M}$, we only need to show that $\mu^{\mathcal{M}}_{\pi a}([\![\varphi]\!]^{\mathcal{M}|^{\pi a}}) \leq \mu^{\mathcal{M}}_{\pi}([\![\langle a\rangle\varphi]\!]^{\mathcal{M}|^{\pi}})$. If $\mathcal{U}^{\mathcal{M}}|^{\pi a} = \emptyset$, $\mu^{\mathcal{M}}_{\pi a}([\![\varphi]\!]^{\mathcal{M}|^{\pi a}}) = 0$, since $\mu^{\mathcal{M}}_{\pi}([\![\langle a\rangle\varphi]\!]^{\mathcal{M}|^{\pi}})$, it is obvious. If $\mathcal{U}^{\mathcal{M}}|^{\pi a} \neq \emptyset$, we have that for each $t \in [\![\varphi]\!]^{\mathcal{M}|^{\pi a}} \subseteq \mathcal{U}^{\mathcal{M}}|^{\pi a}$, there exists $s \in \mathcal{U}^{\mathcal{M}}|^{\pi}$ such that $s \xrightarrow{a} t$. Moreover, it follows by Definition 6 that for each $t \in \mathcal{U}^{\mathcal{M}}|^{\pi a}$,

$$
\mu^{\mathcal{M}}_{\pi a}(t) = \sum_{\{s\in\mathcal{U}^{\mathcal{M}}|^{\pi} \mid s\xrightarrow{a} t\}} \mu^{\mathcal{M}}_{\pi}(s) \times Pr^{\mathcal{M}}(s,a,t)
$$

We then have the following:

$$
\begin{aligned}
& \mu^{\mathcal{M}}_{\pi a}([\![\varphi]\!]^{\mathcal{M}|^{\pi a}}) \\
= \; & \textstyle\sum_{t\in[\![\varphi]\!]^{\mathcal{M}|^{\pi a}}} \mu^{\mathcal{M}}_{\pi a}(t) \\
= \; & \textstyle\sum_{t\in[\![\varphi]\!]^{\mathcal{M}|^{\pi a}}} \left(\sum_{\{s\in\mathcal{U}^{\mathcal{M}}|^{\pi} \mid s\xrightarrow{a} t\}} \mu^{\mathcal{M}}_{\pi}(s) \times Pr^{\mathcal{M}}(s,a,t)\right) \\
= \; & \textstyle\sum_{\{s\in\mathcal{U}^{\mathcal{M}}|^{\pi} \;\mid\; \exists t\in[\![\varphi]\!]^{\mathcal{M}|^{\pi a}} : s\xrightarrow{a} t\}} \mu^{\mathcal{M}}_{\pi}(s) \times \left(\sum_{t\in([\![\varphi]\!]^{\mathcal{M}|^{\pi a}} \cap \mathcal{R}^{\mathcal{M}}_{a}(s))} Pr^{\mathcal{M}}(s,a,t)\right) \\
\leq \; & \textstyle\sum_{\{s\in\mathcal{U}^{\mathcal{M}}|^{\pi} \;\mid\; \exists t\in[\![\varphi]\!]^{\mathcal{M}|^{\pi a}} : s\xrightarrow{a} t\}} \mu^{\mathcal{M}}_{\pi}(s) \\
= \; & \textstyle\sum_{\{s\in\mathcal{U}^{\mathcal{M}}|^{\pi} \;\mid\; \exists t\in[\![\varphi]\!]^{\mathcal{M}|^{\pi}|^{a}} : s\xrightarrow{a} t\}} \mu^{\mathcal{M}}_{\pi}(s) \qquad \text{by Proposition 20} \\
= \; & \textstyle\sum_{s\in[\![\langle a\rangle\varphi]\!]^{\mathcal{M}|^{\pi}}} \mu^{\mathcal{M}}_{\pi}(s) \\
= \; & \mu^{\mathcal{M}}_{\pi}([\![\langle a\rangle\varphi]\!]^{\mathcal{M}|^{\pi}}) \quad \square
\end{aligned}
$$

The soundness of this axiom is used in the proof of the soundness of $\mathtt{PRFEQ}(\pi a)$.

Proposition 22. $\vDash B_{\pi}(\langle a\rangle\varphi \wedge \langle a\rangle\neg\varphi) = 0 \leftrightarrow B_{\pi a}\varphi = B_{\pi}\langle a\rangle\varphi$.

Proof. Given a pointed model $\mathcal{M}, s$, we only need to show that $\mathcal{M}, s \vDash B_{\pi}(\langle a\rangle\varphi \wedge \langle a\rangle\neg\varphi) = 0$ iff $\mathcal{M}, s \vDash B_{\pi a}\varphi = B_{\pi}\langle a\rangle\varphi$. It is obvious if $\mathcal{U}^{\mathcal{M}}|^{\pi a} = \emptyset$. Next, we only focus on the case of $\mathcal{U}^{\mathcal{M}}|^{\pi a} \neq \emptyset$. We have the following:

$$
\begin{aligned}
& \mathcal{M}, s \nvDash B_{\pi a}\varphi = B_{\pi}\langle a\rangle\varphi \\
\Leftrightarrow\; & \mu^{\mathcal{M}}_{\pi a}([\![\varphi]\!]^{\mathcal{M}|^{\pi a}}) < \mu^{\mathcal{M}}_{\pi}([\![\langle a\rangle\varphi]\!]^{\mathcal{M}|^{\pi}}) \qquad \text{by Proposition 21} \\
\Leftrightarrow\; & \text{there exists } s' \in \mathcal{U}^{\mathcal{M}}|^{\pi} \text{ such that } \mathcal{M}|^{\pi a}, t' \vDash \varphi \text{ for some } t' \in \mathcal{R}^{\mathcal{M}}_{a}(s) \text{ and} \\
& \left(\sum_{t\in([\![\varphi]\!]^{\mathcal{M}^{\pi a}} \cap \mathcal{R}^{\mathcal{M}}_{a}(s))} Pr^{\mathcal{M}}(s,a,t)\right) < 1 \quad \text{by the proof of Proposition 21} \\
\Leftrightarrow\; & \text{there exists } s' \in \mathcal{U}^{\mathcal{M}}|^{\pi} \text{ such that } \mathcal{M}|^{\pi a}, t' \vDash \varphi \text{ for some } t' \in \mathcal{R}^{\mathcal{M}}_{a}(s) \text{ and} \\
& \mathcal{M}|^{\pi a}, t \nvDash \varphi \text{ for some } t \in \mathcal{R}^{\mathcal{M}}_{a}(s) \\
\Leftrightarrow\; & \text{there exists } s' \in \mathcal{U}^{\mathcal{M}}|^{\pi} \text{ such that } \mathcal{M}|^{\pi}, s \vDash \langle a\rangle\varphi \wedge \langle a\rangle\neg\varphi \\
\Leftrightarrow\; & \mu^{\mathcal{M}}_{\pi}([\![\langle a\rangle\varphi \wedge \langle a\rangle\neg\varphi]\!]^{\mathcal{M}|^{\pi}}) > 0 \\
\Leftrightarrow\; & \mathcal{M}, s \nvDash B_{\pi}(\langle a\rangle\varphi \wedge \langle a\rangle\neg\varphi) = 0 \quad \square
\end{aligned}
$$

We will leave the proof of the soundness of $\mathtt{Add}(\pi)$ to the reader. The next axiom for which we prove soundness is $\mathtt{ITSP}$. This axiom is a scheme for many different formulas. We abstract from the (in)equality expressed. We call the axiom introspection because it is closely related to the usual axioms 4 and 5 in epistemic logic that express positive and negative introspection respectively. Since an inequality is a negation, this scheme captures both positive and negative introspection in our probabilistic setting.

Proposition 23. $\vDash (\sum_{i=1}^{n} q_i B_{\pi\pi_i'}\varphi_i \bowtie q B_\pi \top) \to B_\pi(\sum_{i=1}^{n} q_i B_{\pi_i'}\varphi_i \bowtie q) = B_\pi \top$.

Proof. If $\mathcal{M}, s \vDash \sum_{i=1}^{n} q_i B_{\pi\pi_i'}\varphi_i \bowtie q B_\pi \top$, namely, $\sum_{i=1}^{n} q_i \cdot \mu^{\mathcal{M}}_{\pi\pi'}([\![\varphi_i]\!]^{\mathcal{M}|^{\pi\pi'}}) \bowtie q \cdot \mu^{\mathcal{M}}_{\pi}([\![\top]\!]^{\mathcal{M}|^{\pi}})$, we need to show $\mathcal{M}, s \vDash B_\pi(\sum_{i=1}^{n} q_i B_{\pi_i'}\varphi_i \bowtie q) = B_\pi \top$, namely, $\mu^{\mathcal{M}}_{\pi}([\![\sum_{i=1}^{n} q_i B_{\pi_i'}\varphi_i \bowtie q]\!]^{\mathcal{M}|^{\pi}}) = \mu^{\mathcal{M}}_{\pi}([\![\top]\!]^{\mathcal{M}|^{\pi}})$. If $\mathcal{U}^{\mathcal{M}}|^{\pi\pi'} = \emptyset$, it is obvious.

Next, we focus on the situation of $\mathcal{U}^{\mathcal{M}}|^{\pi\pi'} \neq \emptyset$. To show $\mu^{\mathcal{M}}_{\pi}([\![\sum_{i=1}^{n} q_i B_{\pi_i'}\varphi_i \bowtie q]\!]^{\mathcal{M}|^{\pi}}) = \mu^{\mathcal{M}}_{\pi}([\![\top]\!]^{\mathcal{M}|^{\pi}})$, we only need to show that $[\![\sum_{i=1}^{n} q_i B_{\pi_i'}\varphi_i \bowtie q]\!]^{\mathcal{M}|^{\pi}} = \mathcal{U}^{\mathcal{M}}|^{\pi}$. By semantics, we only need to show $\sum_{i=1}^{n} q_i \cdot \mu^{\mathcal{M}|^{\pi}}_{\pi_i'}([\![\varphi_i]\!]^{\mathcal{M}|^{\pi}|^{\pi_i'}}) \bowtie q$. Since $\mathcal{U}^{\mathcal{M}}|^{\pi\pi'} \neq \emptyset$, it follows by Propositions 20 and 19 that $\mu^{\mathcal{M}|^{\pi}}_{\pi'}([\![\varphi_i]\!]^{\mathcal{M}|^{\pi}|^{\pi'}}) = \mu^{\mathcal{M}}_{\pi\pi'}([\![\varphi_i]\!]^{\mathcal{M}|^{\pi\pi'}}) / \mu^{\mathcal{M}}_{\pi}(\mathcal{U}^{\mathcal{M}}|^{\pi})$. Therefore, we have $\sum_{i=1}^{n} q_i \cdot \mu^{\mathcal{M}|^{\pi}}_{\pi_i'}([\![\varphi_i]\!]^{\mathcal{M}|^{\pi}|^{\pi_i'}}) \bowtie q$ if and only if $\sum_{i=1}^{n} q_i \cdot \mu^{\mathcal{M}}_{\pi\pi'}([\![\varphi_i]\!]^{\mathcal{M}|^{\pi\pi'}}) \bowtie q \cdot \mu^{\mathcal{M}}_{\pi}([\![\top]\!]^{\mathcal{M}|^{\pi}})$. $\square$

The last axiom for which we prove soundness is CP, an axiom about conditional probability. It expresses the relation between prior and posterior probability in our setting.

Proposition 24. $\vDash \langle a\rangle\top \to ([a](\sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \bowtie q) \leftrightarrow \sum_{i=1}^{n} q_i B_{a\pi_i}\varphi_i \bowtie q B_a \top)$.

Proof. Given a pointed $\mathcal{M}, s \vDash$ and $s \xrightarrow{a} t$ for some $t \in \mathcal{S}$, we need to show that $\mathcal{M}, s \vDash [a](\sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \bowtie q) \leftrightarrow \sum_{i=1}^{n} q_i B_{a\pi_i}\varphi_i \bowtie q B_a \top$. Since $\mathcal{M}, s \vDash [a](\sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \bowtie q)$ if and only if $\mathcal{M}|^a, t \vDash \sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \bowtie q$. Thus, we only need to show $\mathcal{M}|^a, t \vDash \sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \bowtie q$ if and only if $\mathcal{M}, s \vDash \sum_{i=1}^{n} q_i B_{a\pi_i}\varphi_i \bowtie q B_a \top$. It is obvious if $\mathcal{U}^{\mathcal{M}}|^{a\pi_i} = \emptyset$ for all $1 \le i \le n$. Next we only focus on the case of $\mathcal{U}^{\mathcal{M}}|^{a\pi_i} \neq \emptyset$ for all $1 \le i \le n$.

$$\begin{aligned}
& \mathcal{M}|^a, t \vDash \textstyle\sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \bowtie q \\
\Leftrightarrow\ & \textstyle\sum_{i=1}^{n} q_i \mu^{\mathcal{M}|^a}_{\pi_i}([\![\varphi_i]\!]^{\mathcal{M}|^a|^{\pi_i}}) \bowtie q \\
\Leftrightarrow\ & \textstyle\sum_{i=1}^{n} q_i \mu^{\mathcal{M}|^a}_{\pi_i}([\![\varphi_i]\!]^{\mathcal{M}|^{a\pi_i}}) \bowtie q \quad \text{by Proposition 20} \\
\Leftrightarrow\ & \textstyle\sum_{i=1}^{n} q_i / \mu^{\mathcal{M}}_{a}(\mathcal{U}^{\mathcal{M}}|^a) \cdot \mu^{\mathcal{M}}_{a\pi_i}([\![\varphi_i]\!]^{\mathcal{M}|^{a\pi_i}}) \bowtie q \quad \text{by Proposition 19} \\
\Leftrightarrow\ & \textstyle\sum_{i=1}^{n} q_i \mu^{\mathcal{M}}_{a\pi_i}([\![\varphi_i]\!]^{\mathcal{M}|^{a\pi_i}}) \bowtie q \mu^{\mathcal{M}}_{a}(\mathcal{U}^{\mathcal{M}}|^a) \quad \text{due to } \mu^{\mathcal{M}}_{a}(\mathcal{U}^{\mathcal{M}}|^a) > 0 \\
\Leftrightarrow\ & \mathcal{M}, s \vDash \textstyle\sum_{i=1}^{n} q_i B_{a\pi_i}\varphi_i \bowtie q B_a \top \quad \square
\end{aligned}$$

Based on the propositions above, the soundness lemma can be proven by induction on the length of the proof. We will leave the proof to the reader.

Theorem 25 *(Soundness). For each formula φ, $\vdash \varphi$ implies $\vDash \varphi$.*

5. Completeness

In this section we show that the axiomatization presented above is complete with respect to the semantics we presented earlier. One important strategy that has been employed to prove completeness for dynamic epistemic logic is to use reduction axioms (see for instance [33]). Reduction axioms are a way of relating what is the case after an action (e.g., an announcement) to what is the case before the action. Thus we can recursively eliminate all the action modalities. Unfortunately, as in the case of [7], here we cannot eliminate the action modalities completely. For example, we cannot reduce $[a]p$ to a propositional formula, since unlike the standard DEL approach, the truth values of p at the a-successors are not fully determined by the current state (see [34] for an in-depth discussion on the use and failure of such reductions). However, we will try to transform the language $\mathscr{L}$ to its fragment $\mathscr{L}^0$ in which formula's nesting degree is 0 and prove completeness with respect to $\mathscr{L}^0$. As we will see below, in our case a formula with degree 0 can still have action modalities.

5.1. A fragment $\mathscr{L}^0$ of the whole language

Definition 26 *(Nesting degree).* Nesting degree of a formula or an item is defined as follows.

$$\begin{aligned}
d(p) &= 0 \\
d(\neg\varphi) &= d(\varphi) \\
d(\varphi \wedge \psi) &= max\{d(\varphi), d(\psi)\} \\
d([a]\varphi) &= \begin{cases} 1 + d(\varphi) & \text{if a probability term occurs in } \varphi \\ 0 & \text{else} \end{cases} \\
d(\textstyle\sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \ge q) &= max\{d(B_{\pi_i}\varphi_i) \mid 1 \le i \le n\}
\end{aligned}$$

$$d(B_\pi\varphi) \quad = \quad \begin{cases} 1+d(\varphi) & \text{if a probability term occurs in } \varphi \\ 0 & \text{else} \end{cases}$$

We use $\mathscr{L}^0$ to denote the formula set $\{\varphi \in \mathscr{L} \mid d(\varphi) = 0\}$.

Nesting degree captures the depth of the probability modality B_π in the scope of modalities $[a]$ or other probability modalities $B_{\pi'}$. For example, $d([a]B_\pi p \geq q) = 1$, $d(B_{\pi_1}(B_{\pi_2} \geq q_2) \geq q_1) = 1$, and $d(B_\pi[a]p \geq q) = 0$. For each formula $\varphi \in \mathscr{L}^0$, each probability modality B_π in φ will never occur in the scope of the modalities $[a]$ or other probability modalities $B_{\pi'}$. Next we will show that each formula $\varphi \in \mathscr{L}$ can be equivalently transformed to a formula $\varphi' \in \mathscr{L}^0$. To do that, we need the following auxiliary notion.

Definition 27 *(Conjunctive normal form).* A formula φ is in conjunctive normal form if it is a conjunction of disjunctions of 'literals', where a 'literal' is a formula in the form of p, $\neg p$, $[a]\psi$, $\neg[a]\psi$, $\sum_{i=1}^n q_i B_{\pi_i}\psi_i \geq q$ or $\neg(\sum_{i=1}^n q_i B_{\pi_i}\psi_i \geq q)$, where ψ and ψ_i are also in conjunctive normal form.

The following proposition means that the replacement rule is admissible in our axiomatization system. The replacement rule plays a fundamental role in reducing a formula $\varphi \in \mathscr{L}$ to a formula of conjunctive normal form and then to a formula $\varphi' \in \mathscr{L}^0$.

Proposition 28. *If* $\vdash \psi \leftrightarrow \chi$ *then* $\vdash \varphi \leftrightarrow \varphi(\psi/\chi)$.

Proof. Please find the proof in Appendix A. □

With the replacement rule, the following proposition can be proved in a similar process as in propositional logic.

Proposition 29. *For each formula* φ*, there exists a formula* φ' *such that* $\vdash \varphi \leftrightarrow \varphi'$ *and* φ' *is in conjunctive normal form.*

Proof. Please find the proof in Appendix B. □

By Proposition 29, we only need to show that each formula φ of conjunctive normal form can be equivalently transformed to be a formula $\varphi' \in \mathscr{L}^0$. The key is to show that literals with nesting degree 1 of the form $[a]\psi$ or $\sum_{i=1}^n q_i B_{\pi_i}\psi_i \geq q$ can be equivalently transformed to be formulas in $\mathscr{L}^0$. Next we will deal with these two cases respectively.

To show that each literal $[a]\varphi$ with $d([a]\varphi) = 1$ can be equivalently transformed to be a formula in $\mathscr{L}^0$, we firstly show the following proposition.

Proposition 30. `ConP`: $\vdash [a](\sum_{i=1}^n q_i B_{\pi_i}\varphi_i \bowtie q \vee \psi) \leftrightarrow (\sum_{i=1}^n q_i B_{a\pi_i}\varphi_i \bowtie q B_a\top) \vee [a]\psi$.

Proof. Please find the proof in Appendix C. □

Now we are ready to show that each literal $[a]\psi$ with $d([a]\psi) = 1$ can be equivalently transformed to a formula in $\mathscr{L}^0$.

Proposition 31. *Given* $[a]\psi$ *and* $d([a]\psi) = 1$*, there exists a formula* φ *such that* $d(\varphi) = 0$ *and* $\vdash \varphi \leftrightarrow [a]\psi$.

Proof. Please find the proof in Appendix D. □

Next we will transform the literal $\varphi := \sum_{i=1}^n q_i B_{\pi_i}\psi_i \bowtie q$ with $d(\varphi) = 1$ to be a formula in $\mathscr{L}^0$. Firstly, we need the following proposition.

Proposition 32. *Let* $T := \sum_{j=1}^m q_j B_{\pi_j}\psi_j$, $\delta_0 := \sum_{i=1}^n q_i' B_{\pi_i'}\varphi_i \bowtie_1 q$*, and* $\delta_1 := \sum_{i=1}^n q_i' B_{\pi_i'}\varphi_i \bowtie_1 q B_\pi\top$*. We have* $\vdash q_0 B_\pi((\delta_0 \vee \psi) \wedge \chi) + T \geq q \leftrightarrow (\delta_1 \wedge (q_0 B_\pi \chi + T \bowtie_2 q)) \vee (\neg\delta_1 \wedge (q_0 B_\pi(\psi \wedge \chi) + T \bowtie_2 q))$.

Proof. Please find the proof in Appendix E. □

Now we are ready to show that each probability literal φ with $d(\varphi) = 1$ can be equivalently transformed to be a formula in $\mathscr{L}^0$.

Proposition 33. *Given* $\varphi := \sum_{i=1}^n q_i B_{\pi_i}\varphi_i \geq q$ *and* $d(\varphi) = 1$*, there exists a formula* φ' *such that* $d(\varphi') = 0$ *and* $\vdash \varphi \leftrightarrow \varphi'$.

Proof. Please find the proof in Appendix F. □

Finally we will show the following proposition that each formula $\varphi \in \mathscr{L}$ can be equivalently transformed to be a formula $\varphi' \in \mathscr{L}^0$.

Proposition 34. *For each formula $\varphi \in \mathscr{L}$, we can effectively compute a formula φ' such that $\vdash \varphi \leftrightarrow \varphi'$ and $d(\varphi') = 0$.*

Proof. We prove it by induction on $d(\varphi)$. It is obvious if $d(\varphi) = 0$. If $d(\varphi) = n + 1$, by Proposition 29, we assume φ is in conjunctive normal form. It follows by Proposition 28 that we only need to show that for each literal ψ in φ there exists a formula ψ' such that $\vdash \psi \leftrightarrow \psi'$ and $d(\psi') = 0$. By IH, it is straightforward if $d(\psi) = n$. If $d(\psi) = n + 1$, ψ is in the form of $[a]\psi'$, $\neg[a]\psi'$, $\sum_{i=1}^{n} q_i B_{\pi_i} \psi'_i \geq q$ or $\neg \sum_{i=1}^{n} q_i B_{\pi_i} \psi'_i \geq q$. We only focus on the case of $[a]\psi'$ and $\sum_{i=1}^{n} q_i B_{\pi_i} \psi'_i \geq q$; the other cases are similar.

If $\psi := [a]\psi'$ and $d([a]\psi') = n + 1$, it follows that $d(\psi') = n$. By IH, it follows that there exists a formula χ' such that $\vdash \psi' \leftrightarrow \chi'$ and $d(\chi') = 0$. Thus, we have $\vdash \psi \leftrightarrow [a]\chi'$ and $d([a]\chi') \leq 1$. If $d([a]\chi') = 1$, it follows by Proposition 31 that there exists a formula χ such that $\vdash \chi \leftrightarrow [a]\chi'$ and $d(\chi) = 0$. It follows that $\vdash \psi \leftrightarrow \chi$.

If $\psi := \sum_{i=1}^{n} q_i B_{\pi_i} \psi'_i \geq q$ and $d(\psi) = n + 1$, it follows that $d(\psi'_i) \leq n$ for all $1 \leq i \leq n$. By IH, it follows that for each ψ'_i there exists a formula χ'_i such that $\vdash \psi'_i \leftrightarrow \chi'_i$ and $d(\chi'_i) = 0$. It follows that $\vdash \psi \leftrightarrow \sum_{i=1}^{n} q_i B_{\pi_i} \chi'_i \geq q$ and $d(\sum_{i=1}^{n} q_i B_{\pi_i} \chi'_i \geq q) \leq 1$. If $d(\sum_{i=1}^{n} q_i B_{\pi_i} \chi'_i \geq q) = 1$, it follows by Proposition 33 that there exists a formula χ such that $\vdash \sum_{i=1}^{n} q_i B_{\pi_i} \chi'_i \geq q \leftrightarrow \chi$ and $d(\chi) = 0$. It follows that $\vdash \psi \leftrightarrow \chi$. □

Remark 3. The above results tell us that instead of checking φ in the final situation, we can check another formula without nested actions in the initial situation. It also provides the possibility of a regression method in the spirit of the regression in situation calculus, e.g., [20,21].

5.2. Nonstandard models

We have shown that each formula in $\mathscr{L}$ can be equivalently transformed to be a formula in $\mathscr{L}^0$. To show the completeness, we only need to show that each consistent formula $\varphi \in \mathscr{L}^0$ is satisfiable. Our strategy is that firstly we define a notion of nonstandard model and show that if $\varphi \in \mathscr{L}^0$ is satisfiable in nonstandard models then it is also satisfiable in standard models. Secondly, we construct a canonical nonstandard model with respect to $\varphi \in \mathscr{L}^0$ and show that φ is satisfiable in the canonical nonstandard model.

Definition 35 *(Nonstandard model).* A nonstandard model $\mathfrak{M}$ is a tuple $\langle \mathcal{S}^{\mathfrak{M}}, \mathcal{R}^{\mathfrak{M}}, \mathcal{U}^{\mathfrak{M}}, \{\mu_\pi^{\mathfrak{M}} \mid \pi \in \mathtt{A}^*\}, \mathcal{V}^{\mathfrak{M}} \rangle$ such that

- $\mathcal{S}^{\mathfrak{M}}$ is a non-empty finite set of states,
- $\mathcal{R}^{\mathfrak{M}} \subseteq \mathcal{S}^{\mathfrak{M}} \times \mathtt{A} \times \mathcal{S}^{\mathfrak{M}}$,
- $\mathcal{U}^{\mathfrak{M}}$ is a non-empty subset of $\mathcal{S}^{\mathfrak{M}}$,
- $\mu_\pi^{\mathfrak{M}} : \mathcal{U}^{\mathfrak{M}}|^{\pi} \to [0, 1]$ is a function such that
  - $\mu_\epsilon^{\mathfrak{M}}(\mathcal{U}^{\mathfrak{M}}) = 1$ and $\mu_\epsilon^{\mathfrak{M}}(s) > 0$ for each $s \in \mathcal{U}^{\mathfrak{M}}$;
  - $\mu_{\pi a}^{\mathfrak{M}}(\mathcal{U}^{\mathfrak{M}}|^{\pi a}) = \mu_\pi^{\mathfrak{M}}(\{s \in \mathcal{U}^{\mathfrak{M}}|^{\pi} \mid \mathcal{R}_a^{\mathfrak{M}}(s) \neq \emptyset\})$ and $\mu_{\pi a}^{\mathfrak{M}}(s) > 0$ for each $s \in \mathcal{U}^{\mathfrak{M}}|^{\pi a}$;
  - $\mu_{\pi a}^{\mathfrak{M}}(E) \leq \mu_\pi^{\mathfrak{M}}(\{s \in \mathcal{U}^{\mathfrak{M}}|^{\pi} \mid \exists t \in E : s \xrightarrow{a} t\})$ for each $E \subseteq \mathcal{U}^{\mathfrak{M}}|^{\pi a}$;
  - $\mu_{\pi a}^{\mathfrak{M}}(E) < \mu_\pi^{\mathfrak{M}}(\{s \in \mathcal{U}^{\mathfrak{M}}|^{\pi} \mid \exists t \in E : s \xrightarrow{a} t\})$ for each $E \subseteq \mathcal{U}^{\mathfrak{M}}|^{\pi a}$ such that $\mathcal{R}_a^{\mathfrak{M}}(s) \cap E \neq \emptyset$ and $\mathcal{R}_a^{\mathfrak{M}}(s) \setminus E \neq \emptyset$ for some $s \in \mathcal{U}^{\mathfrak{M}}|^{\pi}$,
- $\mathcal{V}^{\mathfrak{M}} : \mathtt{P} \to \mathcal{P}(\mathcal{S}^{\mathfrak{M}})$.

From now on, we call models and semantics defined in Section 3 as *standard model* and *standard semantics*. A nonstandard model is almost the same as a standard model except probability functions. First, there are no transition probabilities in nonstandard models. Second, the functions μ_π of nonstandard models intuitively are the same as the functions of standard models defined in Definition 6. The requirements of the functions $\mu_\pi^{\mathfrak{M}}$ of nonstandard models make sure that there exists transition probabilities such that $\mu_\pi^{\mathfrak{M}}$ can be calculated in the way shown in Definition 6.

Definition 36 *(Nonstandard semantics).* Given a nonstandard model $\mathfrak{M}$, a state $s \in \mathcal{S}^{\mathfrak{M}}$ and a formula φ with $d(\varphi) = 0$, the truth relation is defined as follows:

$$
\begin{array}{rcl}
\mathfrak{M}, s \Vdash p & \iff & s \in V^{\mathfrak{M}}(p) \\
\mathfrak{M}, s \Vdash \neg\varphi & \iff & \mathfrak{M}, s \nVdash \varphi \\
\mathfrak{M}, s \Vdash (\varphi \wedge \psi) & \iff & \mathfrak{M}, s \Vdash \varphi \text{ and } \mathfrak{M}, s \Vdash \psi \\
\mathfrak{M}, s \Vdash [a]\varphi & \iff & \text{for all } s' : s \xrightarrow{a} s' \text{ implies } \mathfrak{M}, s' \Vdash \varphi \\
\mathfrak{M}, s \Vdash \sum_{i=1}^{n} q_i B_{\pi_i} \varphi_i \geq q & \iff & \sum_{i=1}^{n} q_i \mu_{\pi_i}^{\mathfrak{M}}([\![\varphi_i]\!]_{\pi_i}^{\mathfrak{M}}) \geq q
\end{array}
$$

where $[\![\varphi]\!]_{\pi_i}^{\mathfrak{M}} = \{s \in \mathcal{U}^{\mathfrak{M}}|^{\pi_i} \mid \mathfrak{M}, s \Vdash \varphi\}$.

Please note that in nonstandard semantics we care only about formulas in $\mathscr{L}^0$.

The following proposition shows that if $\varphi \in \mathscr{L}^0$ is satisfiable in nonstandard models then it is also satisfiable in standard models.

Proposition 37. *Given a nonstandard model $\mathfrak{M}$ and $\mathfrak{M}, u \Vdash \varphi$ where $u \in \mathcal{U}^{\mathfrak{M}}$ and $d(\varphi) = 0$, there exists a standard pointed model that satisfies φ.*

Proof. The proof is quite involved and can be found in Appendix K. $\square$

Given a consistent formula $\varphi \in \mathscr{L}^0$, next we will construct a canonical nonstandard model with respect to φ and show that φ is satisfiable in this nonstandard model. The canonical model will be built by levels, and the number of its levels will be bounded by the modal depth of φ. The notion of modal depth is defined in the following.

Definition 38 *(Modal depth).* The modal depth of a formula is defined inductively as follows.

$$\begin{array}{rcl} md(p) & = & 0 \\ md(\neg\varphi) & = & md(\varphi) \\ md(\varphi \wedge \psi) & = & max\{md(\varphi), md(\psi)\} \\ md([a]\varphi) & = & 1 + md(\varphi) \\ md(\sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \geq q) & = & max\{|\pi_i| + md(\varphi_i) \mid 1 \leq i \leq n\} \end{array}$$

where $|\pi_i|$ is the length of the sequence π_i.

Here are some notions before we construct the canonical nonstandard model for φ. We use $\mathbb{A}|_\varphi$ to denote the set of actions occurring in φ, and $(\mathbb{A}|_\varphi)^n$ to denote the set of sequences whose length is no bigger than n and whose actions are in $\mathbb{A}|_\varphi$. If s is a finite set of formulas, we use φ_s to denote $\bigwedge_{\psi\in s}\psi$. Let $\sim\psi = \chi$ if $\psi = \neg\chi$, otherwise, $\sim\psi = \neg\psi$. It is obvious that $\vdash \neg\psi \leftrightarrow \sim\psi$. We use $sub^+(\varphi)$ to denote the set $Sub(\varphi) \cup \{\sim\psi \mid \psi \in Sub(\varphi)\}$, where $Sub(\varphi)$ is the set of all subformulas of φ. Let $md(\varphi) = h$, and let $\mathscr{L}^{\texttt{B-Free}}$ be the set of formula that no probability formulas occurring in it. Next we will define sets of maximal consistent sets for each $k \leq h$.

Definition 39. For each $0 \leq k \leq h$, Γ_k^φ and $Atom_k^\varphi$ are defined as follows.

- $k = h$
  - $\Gamma_h^\varphi = \{\psi \in sub^+(\varphi) \mid md(\psi) = 0 \text{ and } \psi \in \mathscr{L}^{\texttt{B-Free}}\}$;
  - $Atom_h^\varphi = \{(s, h) \mid s \text{ is a maximal consistent subset of } \Gamma_h^\varphi\}$;
- $k < h$ but $k > 0$
  - $\Gamma_k^\varphi = \{\psi \in sub^+(\varphi) \mid md(\psi) \leq h - k \text{ and } \psi \in \mathscr{L}^{\texttt{B-Free}}\} \cup \{sub^+(\langle a\rangle\varphi_s) \mid a \in (\mathbb{A}|_\varphi), (s, k+1) \in Atom_{k+1}^\varphi\}$;
  - $Atom_k^\varphi = \{(s, k) \mid s \text{ is a maximal consistent subset of } \Gamma_k^\varphi\}$;
- $k = 0$
  - $\Gamma_0^\varphi = sub^+(\varphi) \cup \{sub^+(B_\epsilon\langle\pi\rangle\varphi_s > 0) \mid (s, j) \in Atom_j^\varphi \text{ for some } 1 \leq j \leq h \text{ and } \pi \in (\mathbb{A}|_\varphi)^j\} \cup \{sub^+(B_\epsilon(\psi_1 \wedge \cdots \wedge \psi_j) \geq 0) \mid \psi_1, \cdots, \psi_j \in sub^+(\varphi) \cap \mathscr{L}^{\texttt{B-Free}}\}$;
  - $Atom_0^\varphi = \{(s, 0) \mid s \text{ a maximal consistent subset of } \Gamma_0^\varphi\}$.

From the definition above, we can see that $Atom_k^\varphi$ is the set of all maximal consistent subsets of Γ_k^φ. Γ_h^φ is the set of all propositional letters in $sub^+(\varphi)$. For each formula $\psi \in \Gamma_k^\varphi$, we have $md(\psi) \leq k$. Probability formulas only occur in Γ_0^φ.

Since φ is consistent and $\varphi \in \Gamma_0^\varphi$, it follows by Lindenbaum's lemma that there exist $(u, 0) \in Atom_0^\varphi$ such that $\varphi \in u$. Next we will construct a canonical nonstandard model with respect to φ and u and will show that φ is satisfiable in it.

Definition 40 *(Canonical nonstandard model).* The canonical nonstandard model $\mathfrak{M}_u^\varphi$ w.r.t φ and u is defined as

- $\mathcal{S}^{\mathfrak{M}_u^\varphi} = \{(s, k) \in Atom_k^\varphi \mid 0 \leq k \leq md(\varphi)\}$
- $\mathcal{R}^{\mathfrak{M}_u^\varphi} = \{((s, k), a, (t, k+1)) \mid \varphi_s \wedge \langle a\rangle\varphi_t \text{ is consistent}, a \in \mathbb{A}|_\varphi\}$
- $\mathcal{U}^{\mathfrak{M}_u^\varphi} = \{(s, 0) \in Atom_0^\varphi \mid s \text{ and } u \text{ contain the same probability formulas}\}$
- $\mathcal{V}^{\mathfrak{M}_u^\varphi}(p) = \{(s, k) \mid p \in s\}$ for each $p \in sub^+(\varphi)$
- $\mu_\pi^{\mathfrak{M}_u^\varphi}$ will be defined later

By induction on k, it is easy to show that all Γ_k^φ and all $Atom_k^\varphi$ are finite. Since each $Atom_k^\varphi$ is the set of all maximally consistent subset of Γ_k^φ, we have the following three propositions.

Proposition 41. *For each $0 \le k \le h$, we have $\vdash \bigvee_{s \in Atom^{\varphi}_{k}} \varphi_s$.*

Proposition 42. *For each $\psi \in \Gamma^{\varphi}_{k}$, we have $\vdash \psi \leftrightarrow \bigvee_{\{s \in Atom^{\varphi}_{k} \mid \psi \in s\}} \varphi_s$.*

Proposition 43. *Let $B_\pi \psi$ occur in some formula in $sub^{+}(\varphi)$, then we have $\vdash B_\pi \psi = \sum_{\{(s,|\pi|) \in Atom^{\varphi}_{|\pi|} \mid \psi \in s\}} B_\pi \varphi_s$.*

Before we show the truth lemma, we need to show two things: the *existence lemma* for formulas of the form $\langle a \rangle \psi$, and a proper definition of the functions $\mu_\pi^{\mathfrak{M}^{\varphi}_{u}}$. The following proposition is the "existence lemma" for formulas of the form $\langle a \rangle \psi$.

Proposition 44. *For each $(s,k) \in \mathcal{S}^{\mathfrak{M}^{\varphi}_{u}}$ and $\langle a \rangle \psi \in sub^{+}(\varphi)$, $\langle a \rangle \psi \in s$ iff there exists a state $(t, k+1) \in \mathcal{S}^{\mathfrak{M}^{\varphi}_{u}}$ such that $\psi \in t$ and $(s,k) \xrightarrow{a} (t,k+1)$.*

Proof. Please find the proof in Appendix G. $\square$

Next we will deal with the functions $\mu_\pi^{\mathfrak{M}^{\varphi}_{u}}$. By Definition 35, we know that these functions should satisfy some conditions to make that $\mathfrak{M}^{\varphi}_{u}$ is a properly defined nonstandard model. Our strategy is to show that such functions $\mu_\pi^{\mathfrak{M}^{\varphi}_{u}}$ does exist due to the completeness of linear inequality logic. By Definition 6, we know that $\mu_\pi^{\mathfrak{M}^{\varphi}_{u}}$ is defined on $\mathcal{U}|^{\pi}$. We firstly show that $\mu_\pi^{\mathfrak{M}^{\varphi}_{u}}(t) > 0$ for each $t \in \mathcal{U}|^{\pi}$ and that $\mu_\pi^{\mathfrak{M}^{\varphi}_{u}}(t) = 0$ if $t \notin \mathcal{U}|^{\pi}$. The following proposition will guarantee that $\mu_\pi^{\mathfrak{M}^{\varphi}_{u}}(t) > 0$ for each $t \in \mathcal{U}|^{\pi}$.

Proposition 45. *Given $(s,k) \in \mathcal{S}^{\mathfrak{M}^{\varphi}_{u}}$ and $\pi \in (\mathbb{A}|_{\varphi})^{k}$, $(s,k) \in \mathcal{U}^{\mathfrak{M}^{\varphi}_{u}}|^{\pi}$ implies $\vdash \varphi_u \to B_\pi \varphi_s > 0$.*

Proof. Please find the proof in Appendix H. $\square$

To show that $\mu_\pi^{\mathfrak{M}^{\varphi}_{u}}(t) = 0$ if $t \notin \mathcal{U}|^{\pi}$, we need the following two auxiliary propositions.

Proposition 46. *If $B_\epsilon \psi > 0 \in u$ then there exists $(s,0) \in \mathcal{U}^{\mathfrak{M}^{\varphi}_{u}}$ such that $\psi \in s$.*

Proof. Please find the proof in Appendix I. $\square$

Proposition 47. *If $\langle \pi \rangle \psi \in s$ for some $(s,0) \in \mathcal{S}^{\mathfrak{M}^{\varphi}_{u}}$, there exists $(t,|\pi|) \in \mathcal{S}^{\mathfrak{M}^{\varphi}_{u}}$ such that $(s,0) \xrightarrow{\pi} (t,|\pi|)$ and ψ is consistent with t.*

Proof. We prove it by induction on π. It is obvious if $\pi := \epsilon$. If it is πa, it follows by induction on π that there exists $(s',|\pi|) \in \mathcal{S}^{\mathfrak{M}^{\varphi}_{u}}$ such that $(s,0) \xrightarrow{\pi} (s',|\pi|)$ and $\langle a \rangle \psi$ is consistent with s'. Next, we only need to show that there exists $(t,|\pi a|) \in \mathcal{S}^{\mathfrak{M}^{\varphi}_{u}}$ such that $(s',|\pi|) \xrightarrow{a} (t,|\pi a|)$ and ψ is consistent with t.

We construct an appropriate $(t,|\pi a|) \in Atom^{\varphi}_{|\pi a|}$ by forcing choices. Enumerate the formulas in $\Gamma^{\varphi}_{|\pi a|}$ as $\chi_1, \cdots, \chi_m$. Define D_0 to be $\{\psi\}$ then $\varphi_{s'} \wedge \langle a \rangle \varphi_{D_0}$ is consistent. Suppose as an inductive hypothesis that D_j is defined such that $\varphi_{s'} \wedge \langle a \rangle \varphi_{D_j}$ is consistent where $0 \le j \le m$. Therefore, either for $D' = D_j \cup \{\chi_{j+1}\}$ or for $D' = D_j \cup \{\neg \chi_{j+1}\}$ we have that $\varphi_{s'} \wedge \langle a \rangle \varphi_{D'}$ is consistent. Choose D_{j+1} to this consistent expansion, and let t be $D_m \cap \Gamma^{\varphi}_{|\pi a|}$. Thus, we have $(t,|\pi a|) \in Atom^{\varphi}_{|\pi a|}$, $\varphi_{s'} \wedge \langle a \rangle \varphi_t$ is consistent and t is consistent with ψ. Therefore, we have $(s',|\pi|) \xrightarrow{a} (t,|\pi a|)$ and $(s,0) \xrightarrow{\pi a} (t,|\pi a|)$. $\square$

The following proposition will guarantee that $\mu_\pi^{\mathfrak{M}^{\varphi}_{u}}(t) = 0$ if $t \notin \mathcal{U}|^{\pi}$.

Proposition 48. *Given $(s,k) \in \mathcal{S}^{\mathfrak{M}^{\varphi}_{u}}$ and $\pi \in (\mathbb{A}|_{\varphi})^{k}$, $(s,k) \notin \mathcal{U}^{\mathfrak{M}^{\varphi}_{u}}|^{\pi}$ implies $\vdash \varphi_u \to B_\pi \varphi_s = 0$.*

Proof. For the case of $k = 0$ and $\pi := \epsilon$, without loss of generality, assume that $\neg \psi \in u$ and $\psi \in s$ for some probability formula in Γ^{φ}_{0}. Let $\chi := \wedge(s \setminus \{\psi\})$. By Axioms PRTR($\epsilon$) and Add($\epsilon$), it follows that $\vdash B_\epsilon \varphi_s > 0 \leftrightarrow \psi \wedge B_\epsilon \chi > 0$. Therefore, we have $\vdash \varphi_u \wedge B_\epsilon \varphi_s > 0 \to \bot$, and consequently $\vdash \varphi_u \to B_\epsilon \varphi_s \le 0$. It follows by Axiom Nonneg(ϵ) that $\vdash \varphi_u \to B_\epsilon \varphi_s = 0$.

For the case of $k+1$ and πa, by Axiom Nonneg(ϵ), we only need to show $\vdash \varphi_u \to B_{\pi a} \varphi_s \le 0$. If $\varphi_u \wedge B_{\pi a} \varphi_s > 0$ is consistent, it follows by Axiom PRTR(ϵ) that $\varphi_u \wedge B_\epsilon \langle \pi a \rangle \varphi_s > 0$ is consistent. Since $B_\epsilon \langle \pi a \rangle \varphi_s > 0 \in \Gamma^{\varphi}_{0}$, it follows that $B_\epsilon \langle \pi a \rangle \varphi_s > 0 \in u$. It follows by Proposition 46 that $\langle \pi a \rangle \varphi_s \in w$ for some $(w,0) \in \mathcal{U}^{\mathfrak{M}^{\varphi}_{u}}$. By Proposition 47 that there exists $(v,k+1) \in \mathcal{S}^{\mathfrak{M}^{\varphi}_{u}}$ such that $(w,0) \xrightarrow{\pi a} (v,k+1)$ and φ_s is consistent with v. This means that $s = v$, and then $(s,k+1) \in \mathcal{U}^{\mathfrak{M}^{\varphi}_{u}}|\pi a$. This is contradictory with our assumption. Therefore, $\varphi_u \wedge B_{\pi a} \varphi_s > 0$ is not consistent, and consequently $\vdash \varphi_u \to B_{\pi a} \varphi_s = 0$. $\square$

Now we are ready to show that there exist functions $\mu_\pi^{\mathfrak{M}_u^\varphi}$ that are properly defined.

Proposition 49. *There exist functions $\mu_\pi^{\mathfrak{M}_u^\varphi}$ where $\pi \in \mathbb{A}^*$ such that $\mathfrak{M}_u^\varphi$ is a nonstandard model and that $\sum_{i=1}^n q_i B_{\pi_i}\psi_i \geq q \in u$ iff $\sum_{i=1}^n q_i \mu_{\pi_i}^{\mathfrak{M}_u^\varphi}(D_i) \geq q$ where $D_i = \{(s, |\pi_i|) \in \mathcal{U}^{\mathfrak{M}_u^\varphi}|^{\pi_i} \mid \psi_i \in s\}$ for each $\sum_{i=1}^n q_i B_{\pi_i}\psi_i \geq q \in sub^+(\varphi)$.*

Proof. We only need to focus on the case of $\pi \in \bigcup_{0\leq k\leq md(\varphi)}(\mathbb{A}|_\varphi)^k$.

Firstly, it follows from Proposition 41 that $\vdash \top \leftrightarrow \bigvee_{(s,0)\in Atom_0^\varphi}\varphi_s$. By Axioms $\mathtt{PRTR}(\epsilon)$ and $\mathtt{Add}(\epsilon)$ and Rules, it follows that

$$\vdash \sum_{(s,0)\in Atom_0^\varphi} B_\epsilon \varphi_s = 1 \tag{1}$$

By Proposition 45, for each $(s, 0) \in \mathcal{U}^{\mathfrak{M}_u^\varphi}$, we have

$$u \vdash B_\epsilon \varphi_s > 0 \tag{2}$$

By Proposition 48, for each $(s, 0) \notin \mathcal{U}^{\mathfrak{M}_u^\varphi}$, we have

$$u \vdash B_\epsilon \varphi_s = 0 \tag{3}$$

Secondly, it follows by Proposition 41 and 42 that $\vdash \top \leftrightarrow \bigvee_{(s,|\pi a|)\in Atom_{|\pi a|}^\varphi}\varphi_s$ and $\vdash \langle a\rangle\top \leftrightarrow \bigvee_{\{(s,|\pi|)\in Atom_{|\pi|}^\varphi \mid \langle a\rangle\top\in s\}}\varphi_s$. By Axioms $\mathtt{PRTR}(\epsilon)$ and $\mathtt{Add}(\epsilon)$ and Rules, it follows that

$$\vdash \sum_{(s,|\pi a|)\in Atom_{|\pi a|}^\varphi} B_{\pi a}\varphi_s = \sum_{\{(s,|\pi|)\in Atom_{|\pi|}^\varphi \mid \langle a\rangle\top\in s\}} B_\pi \varphi_s \tag{4}$$

By Proposition 45, for each $(s, |\pi a|) \in \mathcal{U}^{\mathfrak{M}_u^\varphi}|^{\pi a}$, we have

$$u \vdash B_{\pi a}\varphi_s > 0 \tag{5}$$

By Proposition 45, for each $(s, |\pi a|) \notin \mathcal{U}^{\mathfrak{M}_u^\varphi}|^{\pi a}$, we have

$$u \vdash B_{\pi a}\varphi_s = 0 \tag{6}$$

Thirdly, for each set $E \subseteq \mathcal{U}^{\mathfrak{M}_u^\varphi}|^{\pi a}$, it follows by Axiom $\mathtt{Add}(\pi a)$ that $\vdash B_{\pi a}\bigvee_{(t,|\pi a|)\in E}\varphi_t = \sum_{(t,|\pi a|)\in E} B_{\pi a}\varphi_t$. For each $(t, |\pi a|) \in E$, it follows by Proposition 42 that $\vdash \langle a\rangle\varphi_t \leftrightarrow \bigvee_{\{(s,|\pi|)\in Atom_{|\pi|}^\varphi \mid \langle a\rangle\varphi_t\in s\}}\varphi_s$. What is more, since $\vdash \langle a\rangle(\bigvee_{(t,|\pi a|)\in E}\varphi_t) \leftrightarrow \bigvee_{(t,|\pi a|)\in E}\langle a\rangle\varphi_t$, we have that

$$\vdash \langle a\rangle(\bigvee_{(t,|\pi a|)\in E}\varphi_t) \leftrightarrow \bigvee_{\{(s,|\pi|)\in Atom_{|\pi|}^\varphi \mid \exists(t,|\pi a|)\in E:\langle a\rangle\varphi_t\in s\}}\varphi_s.$$

By Axiom $\mathtt{Add}(\pi)$, we have

$$\vdash B_\pi\langle a\rangle(\bigvee_{(t,|\pi a|)\in E}\varphi_t) = \sum_{\{(s,|\pi|)\in Atom_{|\pi|}^\varphi \mid \exists(t,|\pi a|)\in E:\langle a\rangle\varphi_t\in s\}} B_\pi\varphi_s.$$

By Axiom $\mathtt{PRTR}(\pi)$, we have $\vdash B_{\pi a}\bigvee_{(t,|\pi a|)\in E}\varphi_t \leq B_\pi\langle a\rangle(\bigvee_{(t,|\pi a|)\in E}\varphi_t)$. Therefore, we have

$$\vdash \sum_{(t,|\pi a|)\in E} B_{\pi a}\varphi_t \leq \sum_{\{(s,|\pi|)\in Atom_{|\pi|}^\varphi \mid \exists(t,|\pi a|)\in E:\langle a\rangle\varphi_t\in s\}} B_\pi\varphi_s \tag{7}$$

Moreover, for each set $E \subseteq \mathcal{U}^{\mathfrak{M}_u^\varphi}|^{\pi a}$, if there exists $(s, |\pi|) \in \mathcal{U}^{\mathfrak{M}_u^\varphi}|^\pi$ such that $\mathcal{R}^{\mathfrak{M}_u^\varphi}(s, |\pi|) \cap E \neq \emptyset$ and $\mathcal{R}^{\mathfrak{M}_u^\varphi}(s, |\pi|) \setminus E \neq \emptyset$, namely $(t, |\pi a|) \in E$ and $(t', |\pi a|) \notin E$ for some $(t, |\pi a|), (t, |\pi a|) \in \mathcal{R}^{\mathfrak{M}_u^\varphi}(s, |\pi|)$, it follows that $\vdash \varphi_t \to \varphi_E$ (let $\varphi_E := \bigvee_{(t,|\pi a|)\in E}\varphi_t$) and $\vdash \varphi_{t'} \to \neg\varphi_E$. Therefore, we have $\vdash \langle a\rangle\varphi_t \wedge \langle a\rangle\varphi_{t'} \to \langle a\rangle\varphi_E \wedge \langle a\rangle\neg\varphi_E$. Since $\vdash \varphi_s \to \langle a\rangle\varphi_t \wedge \langle a\rangle\varphi_{t'}$, it follows that $\vdash \varphi_s \to \langle a\rangle\varphi_E \wedge \langle a\rangle\neg\varphi_E$. Therefore, we have $\vdash B_\pi\varphi_s \leq B_\pi(\langle a\rangle\varphi_E \wedge \langle a\rangle\neg\varphi_E)$. It follows by Proposition 45 that $u \vdash B_\pi\langle a\rangle\varphi_E \wedge \langle a\rangle\neg\varphi_E > 0$. Thus, by Axiom $\mathtt{PRTR}(\pi)$, we have $u \vdash B_{\pi a}\varphi_E < B_\pi\langle a\rangle\varphi_E$, namely $u \vdash B_{\pi a}\bigvee_{t\in E}\varphi_t < B_\pi\bigvee_{t\in E}\langle a\rangle\varphi_t$. Therefore, we have

$$u \vdash \sum_{(t,|\pi a|)\in E} B_{\pi a}\varphi_t < \sum_{\{(s,|\pi|)\in\mathcal{U}^{\mathfrak{M}_u^\varphi}|^\pi \mid \exists(t,|\pi a|)\in E:\langle a\rangle\varphi_t\in s\}} B_\pi\varphi_s \tag{8}$$

Furthermore, for each $\chi := \sum_{i=1}^{n} q_i B_{\pi_i}\psi_i \geq q \in sub^+(\varphi)$, if $\chi \in u$, it follows by Proposition 43 that

$$u \vdash \sum_{i=1}^{n}(q_i \cdot \sum_{\{(s,|\pi_i|)\in Atom^{\varphi}_{|\pi_i|} \mid \psi_i \in s\}} B_{\pi_i}\varphi_s) \geq q \tag{9}$$

If $\chi \notin u$, we have

$$u \vdash \sum_{i=1}^{n}(q_i \cdot \sum_{\{(s,|\pi_i|)\in Atom^{\varphi}_{|\pi_i|} \mid \psi_i \in s\}} B_{\pi_i}\varphi_s) < q \tag{10}$$

Finally, we now can construct a set of linear inequalities by replacing $B_\pi\varphi_s$ in formulas of (1) to (10) by variables $x_{\pi(s,|\pi|)}$, which represents $\mu_\pi^{\mathfrak{M}_u^\varphi}(s,|\pi|)$. Since u is consistent, it follows that this set of linear inequalities is also consistent. By completeness of linear inequality system, this inequality set has a solution. We define $\mu_\pi^{\mathfrak{M}_u^\varphi}$ by assigning $\mu_\pi^{\mathfrak{M}_u^\varphi}(s,|\pi|)$ the value of $x_{\pi(s,|\pi|)}$. It follows by (1) to (8) that $\mathfrak{M}_u^\varphi$ is a nonstandard model. For each $\chi := \sum_{i=1}^{n} q_i B_{\pi_i}\psi_i \geq q \in sub^+(\varphi)$, it follows by (9) and (10) that $\chi \in u$ iff $\sum_{i=1}^{n} q_i \mu_{\pi_i}^{\mathfrak{M}_u^\varphi}(\{(s,|\pi_i|) \in Atom^{\varphi}_{|\pi_i|} \mid \psi_i \in s\}) \geq q$. By (3) and (6), we have that $\mu_{\pi_i}^{\mathfrak{M}_u^\varphi}(s,|\pi|) = 0$ for each $(s,|\pi_i|) \notin \mathcal{U}^{\mathfrak{M}_u^\varphi}|^{\pi_i}$. Therefore, we have $\mu_{\pi_i}^{\mathfrak{M}_u^\varphi}(\{(s,|\pi_i|) \in Atom^{\varphi}_{|\pi_i|} \mid \psi_i \in s\}) = \mu_{\pi_i}^{\mathfrak{M}_u^\varphi}(D_i)$. □

Now we are ready to show the truth lemma.

Lemma 50 *(Truth lemma). For each $0 \leq k \leq h$, each $(s,k) \in \mathcal{S}^{\mathfrak{M}_u^\varphi}$ and each $\psi \in sub^+(\varphi) \cap \Gamma_k^\varphi$, we have $\mathfrak{M}_u^\varphi,(s,k) \Vdash \psi$ iff $\psi \in s$.*

Proof. If $h > 0$, we prove it by induction on k. For the case of $k = h$, each formula $\psi \in sub^+(\varphi) \cap \Gamma_h^\varphi$ is a boolean formula. Therefore, by induction on ψ, it is easy to show that $\mathfrak{M}_u^\varphi,(s,h) \Vdash \psi$ iff $\psi \in s$.

With the induction hypothesis that $\mathfrak{M}_u^\varphi,(s,k) \Vdash \psi$ iff $\psi \in s$ for each $k \leq h$, each $(s,k) \in \mathcal{S}^{\mathfrak{M}_u^\varphi}$ and each $\psi \in sub^+(\varphi) \cap \Gamma_k^\varphi$, we will show that $\mathfrak{M}_u^\varphi,(s,k-1) \Vdash \psi$ iff $\psi \in s$ for each $(s,k-1) \in \mathcal{S}^{\mathfrak{M}_u^\varphi}$ and each $\psi \in sub^+(\varphi) \cap \Gamma_{k-1}^\varphi$. We prove this by induction on ψ. The boolean cases are easy by IH. For the case of $\langle a\rangle\psi$, due to Proposition 44, the result can be shown by a standard process [35]. For the case of $\psi := \sum_{i=1}^{n} q_i B_{\pi_i}\psi_i \geq q$, by Definition 39, we know that $k-1 = 0$. Then we need to show that $\mathfrak{M}_u^\varphi,(s,0) \Vdash \sum_{i=1}^{n} q_i B_{\pi_i}\psi_i \geq q$ iff $\sum_{i=1}^{n} q_i B_{\pi_i}\psi_i \geq q \in s$. We have the following:

$$\begin{aligned}
& \mathfrak{M}_u^\varphi,(s,0) \Vdash \sum_{i=1}^{n} q_i B_{\pi_i}\psi_i \geq q \\
\iff & \sum_{i=1}^{n} q_i \mu_{\pi_i}^{\mathfrak{M}_u^\varphi} [\![\psi_i]\!]_{\pi_i}^{\mathfrak{M}_u^\varphi} \geq q \\
& \text{where } [\![\psi_i]\!]_{\pi_i}^{\mathfrak{M}_u^\varphi} = \{(t,k) \in \mathcal{U}^{\mathfrak{M}_u^\varphi}|^{\pi_i} \mid \mathfrak{M}_u^\varphi,(t,k) \vDash \psi_i\} \\
& \text{(Please note that by induction on } \pi \text{ it is easy to show that } (t,k) \in \mathcal{U}^{\mathfrak{M}_u^\varphi}|^{\pi} \text{ implies } k = |\pi|) \\
\iff & \sum_{i=1}^{n} q_i \mu_{\pi_i}^{\mathfrak{M}_u^\varphi} D_i \geq q \\
& \text{where } D_i = \{(t,|\pi_i|) \in \mathcal{U}^{\mathfrak{M}_u^\varphi}|^{\pi_i} \mid \psi_i \in t\} && \text{(by IH, we have } [\![\psi_i]\!]_{\pi_i}^{\mathfrak{M}_u^\varphi} = D_i) \\
\iff & \sum_{i=1}^{n} q_i B_{\pi_i}\psi_i \geq q \in u && \text{(by Proposition 49)} \\
\iff & \sum_{i=1}^{n} q_i B_{\pi_i}\psi_i \geq q \in s && \text{(by } (s,0) \in \mathcal{U}^{\mathfrak{M}_u^\varphi})
\end{aligned}$$

If $h = 0$, by induction on ψ, for the similar process, we will also have $\mathfrak{M}_u^\varphi,(s,0) \Vdash \psi$ iff $\psi \in s$. □

Theorem 51 *(Completeness). For each formula φ, $\vDash \varphi$ implies $\vdash \varphi$.*

Proof. We only need to show that if $\neg\varphi$ is consistent then there exists a pointed model $\mathcal{M}, s$ such that $\mathcal{M}, s \vDash \neg\varphi$. If $\neg\varphi$ is consistent, it follows by Proposition 34 that there exists a formula $\neg\varphi'$ such that $\vdash \neg\varphi \leftrightarrow \neg\varphi'$ and $d(\neg\varphi') = 0$. Please note that $\neg\varphi'$ is also consistent. It follows by Lindenbaum's lemma that there exists $(u,0) \in Atom_0^{\neg\varphi'}$ such that $\neg\varphi' \in u$. Thus, we can construct a canonical model $\mathfrak{M}_u^{\varphi}$ based on $\neg\varphi'$ and u. It follows by Lemma 50 that $\mathfrak{M},(u,0) \Vdash \neg\varphi'$. By Proposition 37, we have that there exists a pointed model $\mathcal{M}, s$ such that $\mathcal{M}, s \vDash \neg\varphi'$. Since $\vdash \neg\varphi \leftrightarrow \neg\varphi'$, it follows by Theorem 25 that $\mathcal{M}, s \vDash \neg\varphi$. □

6. Decidability

This section will show that the problem whether a formula $\varphi \in \mathscr{L}$ is satisfiable in standard models is decidable. First, we show that the problem whether a formula $\varphi \in \mathscr{L}^0$ is satisfiable in nonstandard models is decidable. Second, we show that a formula $\varphi \in \mathscr{L}^0$ is satisfiable in standard models if and only if it is satisfiable in nonstandard models. Since each $\varphi \in \mathscr{L}$ can be transformed to be a formula $\varphi' \in \mathscr{L}^0$, thus the decidability of φ' in nonstandard models will lead to the decidability of φ in standard models.

Given $\varphi \in \mathscr{L}^0$, we use $|\varphi|$ to denote the length of φ and use $f(|\varphi|)$ to denote the size of the canonical nonstandard model. From the construction of the canonical nonstandard model, we can see that $f(|\varphi|)$ is a finite number bounded by $O(md(\varphi) \cdot 2^{|\varphi| \cdot md(\varphi)})$. We use $||\varphi||$ to denote the length of the longest coefficients that appear in φ.

Next we will show that the problem whether $\varphi \in \mathscr{L}^0$ is satisfiable in nonstandard models is decidable. By Lemma 50, we know that if $\varphi \in \mathscr{L}^0$ is satisfiable then it is satisfiable in a finite model with size $f(|\varphi|)$. Please note that this does not imply $\mathscr{L}^0$'s decidability in nonstandard models because there might be infinitely many different probability functions on a finite bounded domain. Therefore, to show $\mathscr{L}^0$ is decidable in nonstandard models, the key is to show that the size of the probability functions that are shown in Proposition 49 are bounded.

Proposition 52. *If $\varphi \in \mathscr{L}^0$ is satisfiable in nonstandard models then it is also satisfiable in a nonstandard model with at most $f(|\varphi|)$ states where the value assigned to each state by $\mu_\pi^{\mathfrak{M}}$ is a rational number with size of $O(r||\varphi|| + r\log r)$, where $r = O(|\varphi|^{|\varphi|} + 2^{f(|\varphi|)})$.*

Proof. If $\varphi \in \mathscr{L}^0$ is satisfiable in nonstandard models, it follows by Proposition 37 that φ is satisfiable in standard models. By the soundness of $\mathbb{SCPP}$ with respect to standard models, we have that φ is consistent. As it is shown in the proof of completeness, φ is satisfiable in the canonical nonstandard model whose size is $f(|\varphi|)$. Next, we will show that for each $t \in \mathcal{U}^{\mathfrak{M}_u^{\varphi}}|^{\pi}$, $\mu_\pi^{\mathfrak{M}_u^{\varphi}}(t)$ is a rational number whose size can be bounded by $O(r||\varphi|| + r\log r)$. Please note that we only need to care about the action sequence $\pi \in (\mathbb{A}|_{\varphi})^{md(\varphi)}$.

In the proof of Proposition 49, we know that the value of $\mu_\pi^{\mathfrak{M}_u^{\varphi}}(t)$ is determined by the system of linear inequalities listed by (1)–(10) in the proof of Proposition 49. Next, we will show how many linear inequalities are listed by (1)–(10).

By (9) and (10), for each χ of the form $\sum_{i=1}^m q_i\psi_i \geq q$ and $\chi \in sub^+(\varphi)$, there is a corresponding linear inequality. Therefore, (9) and (10) list at most $|\varphi|$ linear inequalities into the system.

(1)–(3) are the requirements that the function $\mu_\epsilon^{\mathfrak{M}_u^{\varphi}}$ needs to satisfy. They list 5 linear inequalities into the system. Please note that the linear inequality $x_1 + \cdots x_k = q$ is two inequalities in the system, that is, $x_1 + \cdots x_k \geq q$ and $(-1)x_1 + \cdots (-1)x_k \leq -q$.

(4)–(8) are the requirements that the function $\mu_{\pi a}^{\mathfrak{M}_u^{\varphi}}$ needs to satisfy for each $\pi a \in (\mathbb{A}|_{\varphi})^{md(\varphi)}$. Given $\pi a \in (\mathbb{A}|_{\varphi})^{md(\varphi)}$, (4)–(6) list 5 linear inequalities. (7)–(8) list 2 linear inequalities for each $E \subseteq \mathcal{U}^{\mathfrak{M}_u^{\varphi}}|^{\pi a}$. Since $\mathcal{U}^{\mathfrak{M}_u^{\varphi}}|^{\pi a} \subseteq \mathcal{S}^{\mathfrak{M}_u^{\varphi}}$ and the size of $\mathcal{S}^{\mathfrak{M}_u^{\varphi}}$ is $f(|\varphi|)$, there are at most $2^{f(|\varphi|)}$ such subset E. Therefore, (4)–(8) list at most $5 + 2 \times 2^{f(|\varphi|)}$ linear inequalities for each $\pi a \in (\mathbb{A}|_{\varphi})^{md(\varphi)}$. Since there are at most $|\varphi|^{|\varphi|}$sequences in $(\mathbb{A}|_{\varphi})^{md(\varphi)}$, thus (4)–(8) list at most $|\varphi|^{|\varphi|}(5 + 2 \times 2^{f(|\varphi|)})$ linear inequalities in the system.

Therefore, (1)–(10) list at most $|\varphi| + 5 + |\varphi|^{|\varphi|}(5 + 2 \times 2^{f(|\varphi|)})$ linear inequalities in the system of linear inequalities. Since $|\varphi| + 5 + |\varphi|^{|\varphi|}(5 + 2 \times 2^{f(|\varphi|)}) \leq r$, there are at most r linear inequalities in the system. It follows by the lemma 4.10 in [31] that there exists a probability function $\mu_\pi^{\mathfrak{M}}$ such that the value assigned to each state by $\mu_\pi^{\mathfrak{M}}$ is a rational number with size of $O(r||\varphi|| + r\log r)$. □

The following proposition follows immediately.

Proposition 53. *Given $\varphi \in \mathscr{L}^0$, the problem whether φ is satisfiable in nonstandard models is decidable.*

Proposition 37 has shown that if $\mathscr{L}^0$ is satisfiable in nonstandard models then it is satisfiable in standard models. To reduce the satisfiability of $\mathscr{L}^0$ in standard models to the satisfiability of $\mathscr{L}^0$ in nonstandard models, we still need to show that if $\mathscr{L}^0$ is satisfiable in standard models then it is satisfiable in nonstandard models.

Proposition 54. *If $\varphi \in \mathscr{L}^0$ is satisfiable in standard models then φ is satisfiable in nonstandard models.*

Proof. The proof can be found in Appendix L. □

With Propositions 37 and 54, we have the following proposition.

Proposition 55. *$\varphi \in \mathscr{L}^0$ is satisfiable in standard models if and only if φ is satisfiable in nonstandard models.*

Now, we are ready to show the decidability in standard models.

Theorem 56 *(Decidability). Given $\varphi \in \mathscr{L}$, the problem whether φ is satisfiable in standard models is decidable.*

Proof. Assume the nesting degree of φ is $d(\varphi) = k$. It is obvious that $k \leq |\varphi|$. Let $\varphi_1, \cdots, \varphi_i$ where $i \leq k$ is the subformulas of φ such that $d(\varphi_j) = 1$ for all $1 \leq j \leq i$. The proofs of Proposition 31 and Proposition 33 supply procedures to transform each φ_j to a formula φ'_j such that $\vdash \varphi_j \leftrightarrow \varphi'_j$ and $d(\varphi'_j) = 0$. Since the length of each φ_j is finite, the procedures can be terminated in a finite number of steps. We can then obtain the formula φ' by replacing each ψ_j with ψ'_j. It follows that $d(\varphi') = k - 1$. By Proposition 28, we have $\vdash \varphi \leftrightarrow \varphi'$. If $k - 1 > 0$, we do the same procedure for φ'. Therefore, we can obtain a formula ψ in a finite number of steps such that $\vdash \varphi \leftrightarrow \psi$ and $d(\psi) = 0$.

It follows by the soundness that φ is satisfiable in standard models if and only if ψ is satisfiable in standard models. Since $\psi \in \mathscr{L}^0$, it follows by Proposition 54 that ψ is satisfiable in standard models if and only if it is satisfiable in nonstandard models. By Proposition 53, the problem whether ψ is satisfiable in nonstandard models is decidable. Therefore, the problem whether φ is satisfiable in standard models is decidable. □

7. Conclusion

In this paper we developed a logical framework for conformant probabilistic planning. As we argued, this approach differs from existing approaches to conformant probabilistic planning by focusing on a logical language with which to specify plans. Rather than thinking of goals of plans as subsets of the set of nodes of a probabilistic transition system, our framework allows one to think of the goal as a formula, which may be more convenient when we formulate goals that are probabilistic in nature.

The particular logic we developed allows for reasoning about conformant plans and their probabilistic consequences. We provided an intuitive semantics, which makes it clear how probabilities change as actions take place. We also provided a complete axiomatization of the logic, which shows it is rather well-behaved for a logic that deals with conformant probabilistic planning.

As for future work, the first thing to do is find a way to actually do planning using our framework, like the previous work in the non-probabilistic setting [24]. For now, only plan verification can be captured by model checking within our framework. We need to extend the language with some program operator as in [24]. We also hope to expand this work to the multi-agent setting, where different agents may have different prior probability distributions about the current state of the transition system.

Another direction for future research is to handle contingent planning. For this, we need to generalize the framework by defining the logical language directly over POMDP with explicit observations. For probabilistic *contingent planning* based on POMDP, the plan is usually a policy mapping belief states into actions (cf. [36]). Therefore, to deal with the reasoning in POMDP planning, we also need to expand the language and semantics to talk about policies.

Furthermore, with possible implementation in mind, future research will include determining the complexity of algorithms for model checking and planning, which will make a comparison with other AI approaches to planning (with or without probability), such as the complexity results in [37,38]. The ultimate goal is to implement our framework such that we can compare with existing tools in AI based on plan generation time or plan quality.

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Appendix A. Proof of Proposition 28

Proof. We prove it by induction on φ. We only focus on the case of $\sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \geq q$; the other cases are straightforward.

If $\varphi := \sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \geq q$, it follows by IH that $\vdash \varphi_i \leftrightarrow \varphi_i(\psi/\chi)$ for each $1 \leq i \leq n$. By the Equivalence rule in Table 1, we have that $B_{\pi_i}\varphi_i = B_{\pi_i}\varphi_i(\psi/\chi)$. It follows by linear inequality logic that $\vdash \varphi \leftrightarrow \sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i(\psi/\chi) \geq q$. □

Appendix B. Proof of Proposition 29

Proof. Firstly, with the theorems that $\vdash \neg(\varphi \wedge \psi) \leftrightarrow \neg\varphi \vee \neg\psi$, $\vdash \neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi$, and $\vdash \neg\neg\varphi \leftrightarrow \varphi$, we can push negations down. Therefore, we can assume that φ is of the form that its negation subformulas are negative literals, namely, $\neg p$, $\neg[a]\psi$, or $\neg(\sum_{i=1}^{n} q_i B_{\pi_i}\psi_i \geq q)$. Then we can define a translation function t as: $t(p) = p$; $t(\neg\varphi) = \neg t(\varphi)$; $t([a]\varphi) = [a]t(\varphi)$; $t(\sum_{i=1}^{n} q_i B_{\pi_i}\psi_i \geq q) = \sum_{i=1}^{n} q_i B_{\pi_i} t(\psi_i) \geq q$; $t(\psi \wedge \chi) = t(\psi) \wedge t(\chi)$; and $t(\psi \vee \chi)$ is defined as follows.

1. $t(\psi \vee \chi) = t(\psi) \vee t(\chi)$ if ψ and χ are literals.
2. If $\psi = \psi_1 \wedge \cdots \wedge \psi_n$ then

$$t(\psi \vee \chi) = (t(\psi_1) \vee t(\chi)) \wedge \cdots \wedge (t(\psi_n) \vee t(\chi))$$

If χ is not a literal but of the form $\chi_1 \wedge \cdots \wedge \chi_k$, we take one more step to replace each $t(\psi_i) \vee t(\chi)$ with $(t(\psi_i) \vee t(\chi_1)) \wedge \cdots \wedge (t(\psi_i) \vee t(\chi_k))$.

By induction on φ, it can be shown that $\vdash \varphi \leftrightarrow t(\varphi)$ and that $t(\varphi)$ is in conjunctive normal form. □

Appendix C. Proof of Proposition 30

Proof. To make the proof shorter, let $\chi := \sum_{i=1}^{n} q_i B_{\pi_i}\varphi_i \bowtie q$ and $\chi' := \sum_{i=1}^{n} q_i B_{a\pi_i}\varphi_i \bowtie q$.

$\Rightarrow$

(1) $\vdash [a](\chi \vee \psi) \wedge [a]\neg\chi \to [a]\psi$ by normal modal logic
(2) $\vdash \langle a\rangle\chi \to \langle a\rangle\top \wedge [a]\chi$ by normal modal logic and Axiom DET
(3) $\vdash \langle a\rangle\top \wedge [a]\chi \to \chi'$ by Axiom CP
(4) $\vdash \langle a\rangle\chi \to \chi'$ by (2), (3)
(5) $\vdash \neg\chi' \to [a]\neg\chi$ by (4)
(6) $\vdash [a](\chi \vee \psi) \wedge \neg\chi' \to [a]\psi$ by (1) and (5)
(7) $\vdash [a](\chi \vee \psi) \to \chi' \vee [a]\psi$ by (7)

$\Leftarrow$

(1) $\vdash [a]\psi \to [a](\chi \vee \psi)$ by normal modal logic
(2) $\vdash \chi' \to (\langle a\rangle\top \to [a]\chi)$ by Axiom CP
(3) $\vdash \chi' \to [a]\bot \vee [a]\chi$ by (2)
(4) $\vdash [a]\bot \to [a]\chi$ by normal modal logic
(5) $\vdash \chi' \to [a]\chi$ by (3) and (4)
(6) $\vdash \chi' \to [a](\chi \vee \psi)$ by (5)
(7) $\vdash \chi' \vee [a]\psi \to [a](\chi \vee \psi)$ by (1) and (6) □

Appendix D. Proof of Proposition 31

Proof. Since $d([a]\psi) = 1$, ψ cannot be in the form of $[b]\chi$ or $\neg[b]\chi$. By Proposition 29, we assume ψ is in conjunctive normal form and $[a]\psi := [a](\psi_1 \vee \cdots \vee \psi_n \vee \psi')$ where $d([a]\psi') = 0$ and for all $1 \leq i \leq n$, $\psi_i := \sum_{j=1}^{i_n} q_{i_j} B_{\pi_{i_j}} \chi_{i_j} \geq q_i$ and for all $1 \leq j \leq i_n$, $d(B_{\pi_{i_j}} \chi_{i_j}) = 0$. By induction on n, we will show that there exists a formula φ with $d(\varphi) = 0$ such that $\vdash [a]\psi \leftrightarrow \varphi$.

If $n = 1$, $[a]\psi := [a](\sum_{j=1}^{m} q_j B_{\pi_j}\chi_j \geq q \vee \psi')$. Let $\varphi := (\sum_{j=1}^{m} q_j B_{a\pi_j}\chi_j \geq q B_a\top) \vee [a]\psi'$ then we have $d(\varphi) = 0$. It follows by Axiom ConP that $\vdash [a]\psi \leftrightarrow \varphi$. If $[a]\psi := [a](\psi_1 \vee \cdots \vee \psi_{n+1} \vee \psi')$ where $\psi_i := \sum_{j=1}^{i_n} q_{i_j} B_{\pi_{i_j}} \chi_{i_j} \geq q_i$ for each $1 \leq i \leq n+1$. Let $\varphi' := (\sum_{j=1}^{1_n} q_{1_j} B_{a\pi_{1_j}} \chi_{1_j} \geq q_1 B_a\top) \vee [a](\psi_2 \vee \cdots \vee \psi_{n+1} \vee \psi')$ then we have $d(\sum_{j=1}^{1_n} q_{1_j} B_{a\pi_{1_j}} \chi_{1_j} \geq q_1 B_a\top) = 0$. It follows by Axiom ConP that $\vdash [a]\psi \leftrightarrow \varphi'$. By induction on n, it follows that there exists a formula φ'' such that $d(\varphi'' = 0)$ and $\vdash \varphi'' \leftrightarrow [a](\psi_2 \vee \cdots \vee \psi_{n+1} \vee \psi')$. Let $\varphi := (\sum_{j=1}^{1_n} q_{1_j} B_{a\pi_{1_j}} \chi_{1_j} \geq q_1 B_a\top) \vee \varphi''$ then we have $d(\varphi) = 0$. It follows by Proposition 28 that $\vdash \varphi \leftrightarrow \varphi'$. Since $\vdash [a]\psi \leftrightarrow \varphi'$, it follows that $\vdash [a]\psi \leftrightarrow \varphi$. □

Appendix E. Proof of Proposition 32

Proof. (1) $\vdash \delta_1 \to B_\pi\delta_0 = B_\pi\top$ by Axiom ITSP
(2) $\vdash B_\pi\delta_0 = B_\pi\top \to B_\pi((\delta_0 \vee \psi) \wedge \chi) = B_\pi\chi$ by probability logic
(3) $\vdash \delta_1 \to B_\pi((\delta_0 \vee \psi) \wedge \chi) = B_\pi\chi$ by (1) and (2)
(4) $\vdash \neg\delta_1 \to B_\pi\neg\delta_0 = B_\pi\top$ by Axiom ITSP
(5) $\vdash B_\pi\neg\delta_0 = B_\pi\top \to B_\pi\delta_0 = 0$ by probability logic
(6) $\vdash \neg\delta_1 \to B_\pi\delta_0 = 0$ by (4) and (5)
(7) $\vdash B_\pi\delta_0 = 0 \to B_\pi((\delta_0 \vee \psi) \wedge \chi) = B_\pi(\psi \wedge \chi)$ by probability logic

(8) $\vdash \neg\delta_1 \to B_\pi((\delta_0 \vee \psi) \wedge \chi) = B_\pi(\psi \wedge \chi)$ by (6) and (7)
(9) $\vdash q_0 B_\pi((\delta_0 \vee \psi) \wedge \chi) + T \bowtie_2 q \leftrightarrow (\delta_1 \wedge (q_0 B_\pi \chi + T \bowtie_2 q)) \vee (\neg\delta_1 \wedge (q_0 B_\pi(\psi \wedge \chi) + T \bowtie_2 q))$ by (3), (8) and linear inequality logic $\square$

Appendix F. Proof of Proposition 33

Proof. By Proposition 29, we assume that each φ_i $(1 \le i \le n)$ is in conjunctive normal form. Since $d(\varphi) = 1$, it follows that at least one φ_i has a probability literal for some $1 \le i \le n$. Assume that φ_1 has a probability literal, namely $\varphi_1 := (\sum_{j=1}^m B_{\pi'_j}\chi'_j \bowtie q' \vee \chi'') \wedge \chi'''$. Then φ is in the form of $q_1 B_{\pi_1}((\sum_{j=1}^m B_{\pi'_j}\chi'_j \bowtie q' \vee \chi'') \wedge \chi''') + \sum_{i=2}^n q_i B_{\pi_i}\varphi_i \ge q$ where $d(B_{\pi'_j}\chi'_j) = 0$ for all $1 \le j \le m$. Let k be the number of occurrences of probability literals in $\varphi_1, \cdots, \varphi_n$. We prove it by induction on k.

If $k = 1$, it follows that $d(B_\pi(\chi'' \wedge \chi''')) = 0$ and $d(B_{\pi_i}\varphi_i) = 0$ for all $2 \le i \le n$. Let $\psi_1 := q_1 B_{\pi_1}\chi''' + \sum_{i=2}^n B_{\pi_i}\varphi_i \ge q$ and $\psi_2 := q_1 B_{\pi_1}(\chi'' \wedge \chi''') + \sum_{i=2}^n B_{\pi_i}\chi_i \ge q$. It follows that $d(\psi_1) = d(\psi_2) = 0$. Let $\varphi' := ((\sum_{j=1}^m B_{\pi\pi'_j}\chi'_j \bowtie q' B_\pi \top) \wedge \psi_1) \vee (\neg(\sum_{j=1}^m B_{\pi\pi'_j}\chi'_j \bowtie q' B_\pi \top) \wedge \psi_2)$. Since $d(\sum_{j=1}^m B_{\pi\pi'_j}\chi'_j \bowtie q' B_\pi \top) = 0$, it follows that $d(\varphi') = 0$. It follows by Proposition 32 that $\vdash \varphi \leftrightarrow \varphi'$.

If $k = h + 1$ and $h > 0$, Let $\psi_1 := q_1 B_{\pi_1}\chi''' + \sum_{i=2}^n B_{\pi_i}\varphi_i \ge q$ and $\psi_2 := q_1 B_{\pi_1}(\chi'' \wedge \chi''') + \sum_{i=2}^n B_{\pi_i}\chi_i \ge q$. It follows that $d(\psi_2) = 1$ and the number of occurrences of probability literals in $\chi'' \wedge \chi''', \varphi_2, \cdots, \varphi_n$ is h. It follows by IH that there exists a formula ψ'_2 such that $d(\psi'_2) = 0$ and $\vdash \psi_2 \leftrightarrow \psi'_2$. If $d(\psi_1) = 1$, it follows that the number of occurrences of probability literals in $\chi''', \varphi_2, \cdots, \varphi_n$ is less than or equal to h. It follows by IH that there exists a formula ψ'_1 such that $d(\psi'_1) = 0$ and $\vdash \psi_1 \leftrightarrow \psi'_1$. Let $\varphi'' := ((\sum_{j=1}^m B_{\pi\pi'_j}\chi'_j \bowtie q' B_\pi \top) \wedge \psi_1) \vee (\neg(\sum_{j=1}^m B_{\pi\pi'_j}\chi'_j \bowtie q' B_\pi \top) \wedge \psi_2)$ and $\varphi' := ((\sum_{j=1}^m B_{\pi\pi'_j}\chi'_j \bowtie q' B_\pi \top) \wedge \psi'_1) \vee (\neg(\sum_{j=1}^m B_{\pi\pi'_j}\chi'_j \bowtie q' B_\pi \top) \wedge \psi'_2)$. Since $d(\sum_{j=1}^m B_{\pi\pi'_j}\chi'_j \bowtie q' B_\pi \top) = 0$, we have $d(\varphi') = 0$. It follows by Proposition 32 that $\vdash \varphi \leftrightarrow \varphi''$. It follows by Proposition 28 that $\vdash \varphi' \leftrightarrow \varphi''$. Therefore, we have $\vdash \varphi \leftrightarrow \varphi'$. $\square$

Appendix G. Proof of Proposition 44

Proof. We leave the proof of left-to-right to the reader. Please note that it follows by the definition that $k < h$ since $\langle a \rangle \psi \in s$ and $s \in Atom^\varphi_k$. Assume that $\langle a \rangle \psi \in s$ and that there does not exist $(t, k+1) \in \mathcal{S}^{\mathfrak{M}^\varphi_u}$ such that $\psi \in t$ and $(s, k) \xrightarrow{a} (t, k+1)$. It follows that for all $t \in Atom^\varphi_{k+1}$: if $\psi \in t$ then $\varphi_s \vdash [a]\neg\varphi_t$. Let $t_1, \cdots, t_n$ be all the sets in $Atom^\varphi_{k+1}$ such that ψ is a member of them. It follows by Proposition 42 that $\vdash \psi \leftrightarrow \varphi_{t_1} \vee \cdots \vee \varphi_{t_n}$. Moreover, since $\vdash \varphi_s \to ([a]\neg\varphi_{t_1} \wedge \cdots \wedge [a]\varphi_{t_n})$, it is easy to show that $\vdash \varphi_s \to [a]\neg\psi$. This is in contradiction with $\langle a \rangle \psi \in s$ and the assumption that s is consistent. Therefore, we have shown if $\langle a \rangle \psi \in s$ then there exists $(t, k+1) \in \mathcal{S}^{\mathfrak{M}^\varphi_u}$ such that $\psi \in t$ and $(s, k) \xrightarrow{a} (t, k+1)$. $\square$

Appendix H. Proof of Proposition 45

Proof. For the case of $k = 0$ and $\pi := \epsilon$, let $D \subseteq s$ be the set of all probability formulas in s, and let $\psi := \wedge D$ and $\chi := \wedge(s \setminus D)$. By Axioms $\mathtt{PRTR}(\epsilon)$ and $\mathtt{Add}(\epsilon)$, it follows that $\vdash B_\epsilon \varphi_s \le 0 \leftrightarrow \neg\psi \vee B_\epsilon \chi \le 0$. Since $(s, 0) \in \mathcal{U}^{\mathfrak{M}^\varphi_u}$, it follows that $\vdash \varphi_u \to \psi$. Thus, we have $\vdash \varphi_u \wedge B_\epsilon \varphi_s \le 0 \to B_\epsilon \chi \le 0$. Since $\vdash \varphi_s \to \chi$, it follows by Axiom $\mathtt{T}$ that $\vdash \varphi_s \to B_\epsilon \chi > 0$. By Definition 39, it follows that $B_\epsilon \chi > 0 \in \Gamma^\varphi_0$. Thus, we have $B_\epsilon \chi > 0 \in s$, and consequently $B_\epsilon \chi > 0 \in u$. Therefore, we have $\vdash \varphi_u \wedge B_\epsilon \varphi_s \le 0 \to \bot$, and consequently $\vdash \varphi_u \to B_\epsilon \varphi_s > 0$.

For the case of $k+1$ and πa, it follows by $(s, k+1) \in \mathcal{U}^{\mathfrak{M}^\varphi_u}|^{\pi a}$ that there exists $(w, 0) \in \mathcal{U}^{\mathfrak{M}^\varphi_u}$ such that $(w, 0) \xrightarrow{\pi a} (s, k+1)$. Since $\vdash B_\pi \langle a \rangle \psi > 0 \to B_{\pi a}\psi > 0$, by induction on π, it can be shown that $\vdash B_\epsilon \langle \pi a \rangle \psi > 0 \to B_{\pi a}\psi > 0$. Since w and u share the same probability formulas and $B_\epsilon \langle \pi a \rangle \varphi_s > 0 \in \Gamma^\varphi_0$, by Axioms $\mathtt{T}$, we only need to show that $\langle \pi a \rangle \varphi_s \in w$. Next we will show it by induction on π. It is obvious for the case of a. For the case of πa, we have that $(w, 0) \xrightarrow{\pi} (w', k) \xrightarrow{a} (s, k+1)$ for some $(w', k) \in \mathcal{S}^{\mathfrak{M}^\varphi_u}$. It follows by induction on π that $\langle \pi \rangle \varphi_{w'} \in w$. Moreover, since $\langle a \rangle \varphi_s \in w'$, we have $\vdash \varphi_w \to \langle \pi \rangle \varphi_{w'}$ and $\vdash \varphi_{w'} \to \langle a \rangle \varphi_s$. Therefore, we have $\vdash \varphi_w \to \langle \pi a \rangle \varphi_s$, and consequently $\langle \pi a \rangle \varphi_s \in w$. $\square$

Appendix I. Proof of Proposition 46

Proof. Let D be the set of all the probability formulas in u. We then only need to show that $D \cup \{\psi\}$ is consistent. If it is not, we have $\vdash \varphi_D \to \neg\psi$. It follows by Axiom $\mathtt{PRTR}(\epsilon)$ that $B_\epsilon(\neg\varphi_D \vee \neg\psi) = 1$. Since $\neg\varphi_D$ is a boolean composition of probability formulas, it can be shown that $\vdash \neg\varphi_D \vee B_\epsilon \neg\psi = 1$. Since $\vdash \varphi_u \to \varphi_D$, we have $\vdash \varphi_u \to B_\epsilon \neg\psi = 1$. By Axioms $\mathtt{PRTR}(\epsilon)$ and $\mathtt{Add}(\epsilon)$, it follows that $\vdash \varphi_u \to B_\epsilon \psi = 0$. This is contradictory with $B_\epsilon \psi > 0 \in u$. Therefore, $D \cup \{\psi\}$ is consistent. $\square$

Table J.2
Linear inequality axioms.

Identity	$t \geq t$
0 terms	$\sum_{i=1}^{n} q_i t_i \geq q \leftrightarrow \sum_{i=1}^{n} q_i t_i + 0t' \geq q$
Permutation	$\sum_{i=1}^{n} q_i t_i \geq q \rightarrow \sum_{i=1}^{n} q_{k_i} t_{k_i} \geq q$ where $k_1, \cdots k_n$ is a permutation of $1, \cdots n$.
Addition	$(\sum_{i=1}^{n} q_i t_i \geq q) \wedge (\sum_{i=1}^{n} q'_i t_i \geq q') \rightarrow \sum_{i=1}^{n} (q_i + q'_i) t_i \geq q + q'$
Multiplication	$\sum_{i=1}^{n} q_i t_i \geq q \leftrightarrow \sum_{i=1}^{n} d q_i t_i \geq dq$ (d is a positive rational)
Dichotomy	$(t \geq q) \vee (t \leq q)$
Monotonicity	$(t \geq q) \rightarrow (t > q')$ where $q > q'$

Appendix J. Logic of linear inequalities

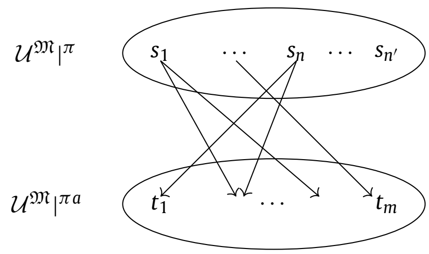
Definition 57 *(Linear inequality axioms).* Let $t_1, \cdots t_n$ be terms and q and q' be rationals. The axioms of linear inequality logic (see [31]) are presented in Table J.2

Appendix K. Proof of Proposition 37

The only differences between standard models and nonstandard models are probabilities. Recall Definition 6, and we know that in standard models, μ_a is calculated by the probability $\mathcal{B}$ and the probability Pr_a. Since μ_ϵ in nonstandard models is the same as $\mathcal{B}$ in standard models, the first claim is to show that there exist such functions Pr_a that Pr_a is a probability distribution and that μ_a in nonstandard models coincides with μ_a which is calculated by this Pr_a and μ_ϵ. The idea of the proof is that we list a set of inequalities based on the probability μ_a in nonstandard models and the conditions that Pr_a needs to satisfy, and then we show the inequality set is satisfiable.

Claim 1. *Define probability functions $Pr^{\mathfrak{M}}_{\pi a} : \mathcal{R}^{\mathfrak{M}}_a|_{\mathcal{U}^{\mathfrak{M}}|^{\pi} \times \mathcal{U}^{\mathfrak{M}}|^{\pi a}} \rightarrow \mathbb{Q}^+$ such that $\sum_{t \in \mathcal{R}^{\mathfrak{M}}_a(s)} Pr^{\mathfrak{M}}_{\pi a}(s,t) = 1$ for each $s \in \mathcal{U}^{\mathfrak{M}}|^{\pi}$ where a is executable at s, and that $\sum_{\{s \in \mathcal{U}^{\mathfrak{M}}|^{\pi} | t \in \mathcal{R}^{\mathfrak{M}}_a(s)\}} \mu^{\mathfrak{M}}_{\pi}(s) \cdot Pr^{\mathfrak{M}}_{\pi a}(s,t) = \mu^{\mathfrak{M}}_{\pi a}(t)$ for each $t \in \mathcal{U}^{\mathfrak{M}}|^{\pi a}$.*

Proof of Claim 1. If s is the current state after doing π and $s \xrightarrow{a} t$, $Pr^{\mathfrak{M}}_{\pi a}(s,t)$ represents the probability of reaching t by continuing to do a in s. Let $\mathcal{U}^{\mathfrak{M}}|^{\pi} = \{s_1, \cdots, s_n, \cdots, s_{n'}\}$ such that action a is executable at each $1 \leq i \leq n$ and unexecutable at each $n < i \leq n'$. Let $\mathcal{U}^{\mathfrak{M}}|^{\pi a} = \{t_1, \cdots, t_m\}$, then the accessibility relation of a on $\mathcal{U}^{\mathfrak{M}}|^{\pi} \times \mathcal{U}^{\mathfrak{M}}|^{\pi a}$ can be roughly depicted as follows.



We now describe a set of linear inequalities over variables of the form $x_{(i,j)}$ for $(s_i, t_j) \in \mathcal{R}^{\mathfrak{M}}_a|_{\mathcal{U}^{\mathfrak{M}}|^{\pi} \times \mathcal{U}^{\mathfrak{M}}|^{\pi a}}$. We can think of $x_{(i,j)}$ as representing $\mu^{\mathfrak{M}}_{\pi}(s_i) \cdot Pr^{\mathfrak{M}}_{\pi a}(s_i, t_j)$. For each $(s_i, t_j) \in \mathcal{R}^{\mathfrak{M}}_a|_{\mathcal{U}^{\mathfrak{M}}|^{\pi} \times \mathcal{U}^{\mathfrak{M}}|^{\pi a}}$, to make sure $Pr^{\mathfrak{M}}_{\pi a}(s_i, t_j) > 0$, we only need to request that

$$x_{(i,j)} > 0. \tag{K.1}$$

For each $1 \leq i \leq n$, to make sure $\sum_{t_j \in \mathcal{R}^{\mathfrak{M}}_a(s_i)} Pr^{\mathfrak{M}}_{\pi a}(s_i, t_j) = 1$, we only need to request that

$$\sum_{t_j \in \mathcal{R}^{\mathfrak{M}}_a(s_i)} x_{(i,j)} = \mu^{\mathfrak{M}}_{\pi}(s_i). \tag{K.2}$$

For each $1 \leq j \leq m$, to make sure $\sum_{\{s_i \in \mathcal{U}^{\mathfrak{M}}|^{\pi} | t_j \in \mathcal{R}^{\mathfrak{M}}_a(s_i)\}} \mu^{\mathfrak{M}}_{\pi}(s_i) \cdot Pr^{\mathfrak{M}}_{\pi a}(s_i, t_j) = \mu^{\mathfrak{M}}_{\pi a}(t_j)$, we only need to request that

$$\sum_{\{s_i \in \mathcal{U}^{\mathfrak{M}}|^{\pi} | t_j \in \mathcal{R}_a^{\mathfrak{M}}(s_i)\}} x_{(i,j)} = \mu_{\pi a}^{\mathfrak{M}}(t_j). \tag{K.3}$$

Next, we only need to show that the set, S, of linear inequalities described in (K.1) to (K.3) has a solution. By the solvability theorem [39]:

Let S be a set of linear inequalities. If the inequality $0x_1 + \cdots + 0x_n > 0$ is not a legal linear combination of the inequalities of S, then S is solvable/satisfiable.

We only need to show that

$$\sum_{(s_i,t_j) \in \mathcal{R}_a^{\mathfrak{M}}|_{\mathcal{U}^{\mathfrak{M}}|^{\pi} \times \mathcal{U}^{\mathfrak{M}}|^{\pi a}}} 0x_{(i,j)} > 0 \tag{K.4}$$

is not a possible legal linear combination of S. (For the definition of *legal liner combination* please see [39].) If possible, let $\mu_{\pi}^{\mathfrak{M}}(s_i) = a_i$ and $\mu_{\pi a}^{\mathfrak{M}}(t_j) = b_j$ then there exists a scheme (see [39]) of S as shown in Table K.3 such that

$$u_{(i',j')} > 0 \text{ for some } (s_{i'}, t_{j'}) \in \mathcal{R}_a^{\mathfrak{M}}|_{\mathcal{U}^{\mathfrak{M}}|^{\pi} \times \mathcal{U}^{\mathfrak{M}}|^{\pi a}}; \tag{K.5}$$

$$d_{(i,j)} = u_{(i,j)} + r_i + w_j = 0 \text{ for each } (s_i, t_j) \in \mathcal{R}_a^{\mathfrak{M}}|_{\mathcal{U}^{\mathfrak{M}}|^{\pi} \times \mathcal{U}^{\mathfrak{M}}|^{\pi a}}; \tag{K.6}$$

$$d = -u_0 + r_1 a_1 + \cdots + r_n a_n + w_1 b_1 + \cdots + w_m b_m = 0 \tag{K.7}$$

where $r_i = r'_{2i-1} - r'_{2i}$ and $w_j = w'_{2j-1} - w'_{2j}$ for all $1 \le i \le n$ and $1 \le j \le m$.

Let a group G be a minimal subset of $\mathcal{U}^{\mathfrak{M}}|^{\pi a}$ such that for each $t \in G$ if $s \xrightarrow{a} t$ and $s \xrightarrow{a} t'$ for some $s \in \mathcal{U}^{\mathfrak{M}}|^{\pi}$ then $t' \in G$. It is obvious that $\mathcal{U}^{\mathfrak{M}}|^{\pi a}$ can be divided into several such groups. Let one of these groups be $\{t_1, \cdots, t_h\}$, and we will write it as $\{1, \cdots, h\}$ for abbreviation. For each $j \in G$, let $D_j = \{1 \le i \le n \mid (s_i, t_j) \in \mathcal{R}_a^{\mathfrak{M}}|_{\mathcal{U}^{\mathfrak{M}}|^{\pi} \times \mathcal{U}^{\mathfrak{M}}|^{\pi a}}\}$, which is the set of all the numbers i such that $s_i \in \mathcal{U}^{\mathfrak{M}}|^{\pi}$ and $s_i \xrightarrow{a} t_j$. For each $i \in D_j$, since $d_{(i,j)} = 0$ and $u_{(i,j)} \ge 0$, it follows that $w_j \le -r_i$. Given $j \in G$, we use r_{w_j} to denote the maximal number in $\{r_i \mid i \in D_j\}$. Since $w_j \le -r_i$ for all $i \in D_j$, it follows that $w_j \le -r_{w_j}$. Without loss of generality, we assume that $r_{w_1} \le \cdots \le r_{w_h}$. We use $D_{1,j}$ as an abbreviation for $D_1 \cup \cdots \cup D_j$. It follows by Definition 35 that $\sum_{i \in D_{1,h}} a_i = b_1 + \cdots + b_h$ and that $\sum_{i \in D_{1,k}} a_i > b_1 + \cdots + b_k$ for each $k < h$. We then have the following:

$$\begin{aligned}
&\textstyle\sum_{i \in D_{1,h}} r_i a_i + \sum_{j=1}^{h} w_j b_j \\
&\le \textstyle\sum_{i \in D_{1,h}} r_i a_i + \sum_{j=1}^{h} -r_{w_j} b_j \\
&\le r_{w_1}(\textstyle\sum_{i \in D_1} a_i - b_1) + \sum_{i \in D_{1,h} \setminus D_1} r_i a_i + \sum_{j=2}^{h} -r_{w_j} b_j \\
&\le r_{w_2}(\textstyle\sum_{i \in D_{1,2}} a_i - b_1 - b_2) + \sum_{i \in D_{1,h} \setminus D_{1,2}} r_i a_i + \sum_{j=3}^{m} -r_{w_j} b_j \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{because of (K.9)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\le r_{w_h} \cdot (\textstyle\sum_{i \in D_{1,h}} a_i + \sum_{j=1}^{h} -b_j) \\
&= 0
\end{aligned} \tag{K.8}$$

For each $1 \le k \le h$, we have the following.

$$\begin{aligned}
&r_{w_k}(\textstyle\sum_{i \in D_{1,k}} a_i + \sum_{j=1}^{k} -b_j) + \sum_{i \in D1,h \setminus D_{1,k}} r_i a_i + \sum_{j=k+1}^{h} -r_{w_j} b_j \\
&\le r_{w_k}(\textstyle\sum_{i \in D_{1,k}} a_i + \sum_{j=1}^{k} -b_j) + r_{w_{k+1}} \sum_{i \in D_{k+1} \setminus D_{1,k}} a_i + \sum_{i \in D_{1,h} \setminus D_{1,(k+1)}} r_i a_i + \sum_{j=k+1}^{h} -r_{w_j} b_j \\
&\le r_{w_{k+1}}(\textstyle\sum_{i \in D_{1,k}} a_i + \sum_{j=1}^{k} -b_j) + r_{w_{k+1}} \sum_{i \in D_{k+1} \setminus D_{1,k}} a_i + \sum_{i \in D_{1,h} \setminus D_{1,(k+1)}} r_i a_i + \sum_{j=k+1}^{m} -r_{w_j} b_j \\
&= r_{w_{k+1}}(\textstyle\sum_{i \in D_{1,(k+1)}} a_i + \sum_{j=1}^{k+1} -b_j) + \sum_{i \in D_{1,h} \setminus D_{1,(k+1)}} r_i a_i + \sum_{j=k+2}^{h} -r_{w_j} b_j
\end{aligned} \tag{K.9}$$

Because of the property of group, it follows that if G and G' are two different groups, and $t \in G$, $t' \in G'$ then $D_t \cap D_{t'} = \emptyset$. Assuming $\mathcal{U}^{\mathfrak{M}}|^{\pi a}$ is divided into l groups, it follows that

$$d = -u_0 + \sum_{1 \le k \le l} (\sum_{i \in D_{G_k}} r_i a_i + \sum_{j \in G_k} w_j b_j) \tag{K.10}$$

Since $d = 0$, $-u_0 \le 0$ and $\sum_{i \in D_G} r_i a_i + \sum_{j \in G} w_j b_j \le 0$ for each group G, it follows that $u_0 = 0$ and $\sum_{i \in D_G} r_i a_i + \sum_{j \in G} w_j b_j = 0$. It follows that each inequality of (K.8) or (K.9) equals 0, especially,

Table K.3
Scheme.

$u_0 \geq 0$:	$\sum_{(s_i,t_j)\in\mathcal{R}_a^{\mathfrak{M}}\vert_{\mathcal{U}^{\mathfrak{M}}\vert^{\pi}\times\mathcal{U}^{\mathfrak{M}}\vert^{\pi a}}} 0x_{(i,j)}$	> -1
$u_{(1,1)} \geq 0$:	$x_{(1,1)}$	> 0
$\cdots$	$\cdots$	$\cdots$
$u_{(i,j)} \geq 0$:	$x_{(i,j)}$	> 0
$r'_1 \geq 0$:	$\sum_{t_j\in\mathcal{R}_a^{\mathfrak{M}}(s_1)} x_{(1,j)}$	$\geq a_1$
$r'_2 \geq 0$:	$\sum_{t_j\in\mathcal{R}_a^{\mathfrak{M}}(s_1)} -x_{(1,j)}$	$\geq -a_1$
$\cdots$	$\cdots$	$\cdots$
$r'_{2n-1} \geq 0$:	$\sum_{t_j\in\mathcal{R}_a^{\mathfrak{M}}(s_n)} x_{(n,j)}$	$\geq a_n$
$r'_{2n} \geq 0$:	$\sum_{t_j\in\mathcal{R}_a^{\mathfrak{M}}(s_n)} -x_{(n,j)}$	$\geq -a_n$
$w'_1 \geq 0$:	$\sum_{\{s_i\in\mathcal{U}^{\mathfrak{M}}\vert^{\pi}\vert t_1\in\mathcal{R}_a^{\mathfrak{M}}(s_i)\}} x_{(i,1)}$	$\geq b_1$
$w'_2 \geq 0$:	$\sum_{\{s_i\in\mathcal{U}^{\mathfrak{M}}\vert^{\pi}\vert t_1\in\mathcal{R}_a^{\mathfrak{M}}(s_i)\}} -x_{(i,1)}$	$\geq -b_1$
$\cdots$	$\cdots$	$\cdots$
$w'_{2m-1} \geq 0$:	$\sum_{\{s_i\in\mathcal{U}^{\mathfrak{M}}\vert^{\pi}\vert t_m\in\mathcal{R}_a^{\mathfrak{M}}(s_i)\}} x_{(i,m)}$	$\geq b_m$
$w'_2 m \geq 0$:	$\sum_{\{s_i\in\mathcal{U}^{\mathfrak{M}}\vert^{\pi}\vert t_m\in\mathcal{R}_a^{\mathfrak{M}}(s_i)\}} -x_{(i,m)}$	$\geq -b_m$
	$d_{(1,1)}x_{(1,1)} + \cdots + d_{(i,j)}x_{(i,j)}$	$> d$

(Scheme is a systematical way to get a logical consequence of a set of inequalities. Please see [39].)

$$\sum_{i\in D_G} r_i a_i + \sum_{j\in G} w_j b_j = \sum_{i\in D_G} r_i a_i + \sum_{j\in G} -r_{w_j} b_j = 0. \tag{K.11}$$

Moreover, by (K.9), we have that for each $1 \leq k < h$,

$$r_{w_k}(\sum_{i\in D_{1,k}} a_i + \sum_{j=1}^{k} -b_j) = r_{w_{k+1}}(\sum_{i\in D_{1,k}} a_i + \sum_{j=1}^{k} -b_j), \tag{K.12}$$

$$\sum_{i\in D_{k+1}\setminus D_{1,k}} r_i a_i = r_{w_{k+1}} \sum_{i\in D_{k+1}\setminus D_{1,k}} a_i. \tag{K.13}$$

Since $\sum_{i\in D_{1,k}} a_i + \sum_{j=1}^{k} -b_j > 0$, it follows by (K.12) that $r_{w_k} = r_{w_{k+1}}$ for each $1 \leq k < h$. Next, we will show that for each $1 \leq k \leq h$, namely $t_k \in G$, $i \in D_k$ implies $r_i = r_{w_k}$. For the case of $k = 1$, it is obvious from (K.8). For the case of $k+1$, if $i \in D_{k+1} \setminus D_{1,k}$, it is obvious from (K.13). If $i \notin D_{k+1} \setminus D_{1,k}$, it follows by IH that $r_i = r_{w_{k'}}$ for some $k' \leq k$. Since $r_{w_k} = r_{w_{k+1}}$ for all $1 \leq k < h$, it follows that r, it follows that $r_i = r_{w_{k+1}}$.

By (K.5), we have known that $u_{(i',j')} > 0$. Since $r_i = r_{i'}$ and $d_{(i,j')} = u_{(i,j')} + r_i + w_{j'} = 0$ for all $i \in D_{j'}$, it follows that $u_{(i,j')} = u_{i',j'}$ for all $i \in D_{j'}$. Since $u_{(i',j')} > 0$, it follows that $r_{w_{j'}} + w_{j'} < 0$, namely $w_{j'} < -r_{w_{j'}}$. Thus, for the group G such that $j' \in G$, we have the following

$$\sum_{i\in D_G} r_i a_i + \sum_{j\in G} w_j b_j < \sum_{i\in D_G} r_i a_i + \sum_{j\in G} -r_{w_j} b_j.$$

This is contradictory with (K.11). Therefore, (K.4) cannot be a legal linear combination of S, and it follows by the solvability theorem that S has a solution.

Therefore, we define function $Pr_{\pi a}^{\mathfrak{M}}$ on $\mathcal{R}_a^{\mathfrak{M}}\vert_{\mathcal{U}^{\mathfrak{M}}\vert^{\pi}\times\mathcal{U}^{\mathfrak{M}}\vert^{\pi a}}$ as $Pr_{\pi a}^{\mathfrak{M}}(s_i, t_j) = x_{(i,j)}/\mu_{\pi}^{\mathfrak{M}}(s_i)$ for each $(s_i, t_j) \in \mathcal{R}_a^{\mathfrak{M}}\vert_{\mathcal{U}^{\mathfrak{M}}\vert^{\pi}\times\mathcal{U}^{\mathfrak{M}}\vert^{\pi a}}$. It follows from (K.1) to (K.3) that $Pr_{\pi a}^{\mathfrak{M}} : \mathcal{R}_a^{\mathfrak{M}}\vert_{\mathcal{U}^{\mathfrak{M}}\vert^{\pi}\times\mathcal{U}^{\mathfrak{M}}\vert^{\pi a}} \to \mathbb{Q}^+$ such that $\sum_{t\in\mathcal{R}_a^{\mathfrak{M}}(s)} Pr_{\pi a}^{\mathfrak{M}}(s,t) = 1$ for each $s \in \mathcal{U}^{\mathfrak{M}}\vert^{\pi}$ where a is executable at s, and that $\sum_{\{s\in\mathcal{U}^{\mathfrak{M}}\vert^{\pi}\vert t\in\mathcal{R}_a^{\mathfrak{M}}(s)\}} \mu_{\pi}^{\mathfrak{M}}(s) \cdot Pr_{\pi a}^{\mathfrak{M}}(s,t) = \mu_{\pi a}^{\mathfrak{M}}(t)$ for each $t \in \mathcal{U}^{\mathfrak{M}}\vert^{\pi a}$. $\square$

Please recall that an execution path $\sigma \in EP_{\mathfrak{M}}(a_1 \cdots a_n)$ is an alternating sequence of states and actions, $s_0 a_1 \cdots s_n$, where $s_0 \in \mathcal{U}^{\mathfrak{M}}$ and $s_{i-1} \xrightarrow{a_i} s_i$ for each $1 \leq i \leq n$. Given $\sigma := s_0 a_1 \cdots s_n$, we use $T(\sigma)$ to denote the last state s_n and $\rho(\sigma)$ to the action sequence $a_1 \cdots a_n$. Given $t \in \mathcal{U}^{\mathfrak{M}}\vert^{\pi}$, let $[\sigma]_t^{\pi} = \{\sigma \in EP_{\mathfrak{M}}(\pi) \mid T(\sigma) = t\}$. Next, we construct a standard model based on the execution paths of $\mathfrak{M}$.

Claim 2. *Construct a standard model $\mathfrak{M}^{\bullet}$ such that $\mu_{\pi}^{\mathfrak{M}^{\bullet}}([\sigma]_t^{\pi}) = \mu_{\pi}^{\mathfrak{M}}(t)$ where $\pi \in \bigcup_{0\leq k\leq md(\varphi)} (\mathbb{A}\vert_{\varphi})^k$ and $t \in \mathcal{U}^{\mathfrak{M}}\vert^{\pi}$.*

Proof of Claim 2. We define the standard model $\mathfrak{M}^{\bullet}$ as follows.

$$
\begin{array}{lcl}
\mathcal{S}^{\mathfrak{M}^\bullet} & = & \{\sigma \in EP_{\mathfrak{M}}(\pi) \mid \pi \in \bigcup_{0\le k\le md(\varphi)}(\mathbb{A}|_\varphi)^k\} \\
\mathcal{R}^{\mathfrak{M}^\bullet} & = & \{(\sigma, a, \sigma') \mid a \in \mathbb{A}|_\varphi, \sigma' = \sigma a t\} \\
Pr^{\mathfrak{M}^\varphi_u}(\sigma, a, \sigma a t) & = & Pr^{\mathfrak{M}}_{\rho(\sigma)a}(T(\sigma), t) \\
\mathcal{U}^{\mathfrak{M}^\bullet} & = & \mathcal{U}^{\mathfrak{M}} \\
\mathcal{B}^{\mathfrak{M}^\bullet} & = & \mu^{\mathfrak{M}}_{\epsilon} \\
\mathcal{V}^{\mathfrak{M}^\bullet}(p) & = & \{\sigma \mid T(\sigma) \in \mathcal{V}^{\mathfrak{M}}(p)\} \text{ for each } p \in sub^+(\varphi)
\end{array}
$$

By the definition above, it is obvious that $\sigma \in \mathcal{U}^{\mathfrak{M}^\bullet}|^\pi$ iff $T(\sigma) \in \mathcal{U}^{\mathfrak{M}}|^\pi$ for each $\sigma \in \mathcal{S}^{\mathfrak{M}^\bullet}$. By induction on π we will show that $\mu^{\mathfrak{M}^\bullet}_\pi([\sigma]^\pi_t) = \mu^{\mathfrak{M}}_\pi(t)$. It is obvious for the case of ϵ. For the case of πa, we have the following.

$$
\begin{aligned}
\mu^{\mathfrak{M}^\bullet}_{\pi a}([\sigma a t]^{\pi a}_t) &= \mu^{\mathfrak{M}^\bullet}_{\pi a}(\{\sigma' a t \mid s \in \mathcal{U}^{\mathfrak{M}}|^\pi, t \in \mathcal{R}^{\mathfrak{M}}_a(s), \sigma' \in [\sigma]^\pi_s\}) \\
&= \sum_{\{s \in \mathcal{U}^{\mathfrak{M}}|^\pi \mid t \in \mathcal{R}^{\mathfrak{M}}_a(s)\}} \sum_{\sigma' \in [\sigma]^\pi_s} \mu^{\mathfrak{M}^\bullet}_{\pi a}(\sigma' a t) \\
&= \sum_{\{s \in \mathcal{U}^{\mathfrak{M}}|^\pi \mid t \in \mathcal{R}^{\mathfrak{M}}_a(s)\}} \sum_{\sigma' \in [\sigma]^\pi_s} \mu^{\mathfrak{M}^\bullet}_{\pi}(\sigma') \cdot Pr^{\mathfrak{M}^\bullet}(\sigma', a, \sigma' a t) \\
&= \sum_{\{s \in \mathcal{U}^{\mathfrak{M}}|^\pi \mid t \in \mathcal{R}^{\mathfrak{M}}_a(s)\}} \sum_{\sigma' \in [\sigma]^\pi_s} \mu^{\mathfrak{M}^\bullet}_{\pi}(\sigma') \cdot Pr^{\mathfrak{M}}_{\pi a}(s, t) \\
&= \sum_{\{s \in \mathcal{U}^{\mathfrak{M}}|^\pi \mid t \in \mathcal{R}^{\mathfrak{M}}_a(s)\}} Pr^{\mathfrak{M}}_{\pi a}(s, t) \sum_{\sigma' \in [\sigma]^\pi_s} \mu^{\mathfrak{M}^\bullet}_{\pi}(\sigma') \\
&= \sum_{\{s \in \mathcal{U}^{\mathfrak{M}}|^\pi \mid t \in \mathcal{R}^{\mathfrak{M}}_a(s)\}} Pr^{\mathfrak{M}}_{\pi a}(s, t) \cdot \mu^{\mathfrak{M}}_{\pi}(s) && \text{(by IH)} \\
&= \mu^{\mathfrak{M}}_{\pi a}(t) && \text{(by Claim 1)}
\end{aligned}
$$

Therefore, we have shown that $\mu^{\mathfrak{M}^\bullet}_\pi([\sigma]^\pi_t) = \mu^{\mathfrak{M}}_\pi(t)$ for each $t \in \mathcal{U}^{\mathfrak{M}}|^\pi$. □

Claim 3. *$\mathfrak{M}^\bullet, \sigma \vDash \psi$ iff $\mathfrak{M}, T(\sigma) \Vdash \psi$ for each $\sigma \in \mathcal{S}^{\mathfrak{M}^\bullet}$ and $\psi \in sub^+(\varphi)$ such that no probability formula occurs in ψ.*

Proof of Claim 3. By the definition of $\mathfrak{M}^\bullet$ in Claim 2, it is obvious. □

Claim 4. *$\mathfrak{M}^\bullet, s \vDash \psi$ iff $\mathfrak{M}, s \Vdash \psi$ for each $s \in \mathcal{U}^{\mathfrak{M}^\bullet}$ and $\psi \in sub^+(\varphi)$.*

Proof of Claim 4. We prove it by induction on ψ. We only focus on the cases of $[a]\psi$ and $\sum_{i=1}^n q_i B_{\pi_i}\psi_i \ge q$; the other cases are straightforward.

For the case of $[a]\psi$, since $d(\varphi) = 0$ and $[a]\psi \in sub^+(\varphi)$, it follows that there is no probability formula occurring in $[a]\psi$. It follows by Claim 3 that $\mathfrak{M}^\bullet, s \vDash [a]\psi$ iff $\mathfrak{M}, s \Vdash [a]\psi$.

For the case of $\psi := \sum_{i=1}^n q_i B_{\pi_i}\psi_i \ge q$, we only need to show $\mu^{\mathfrak{M}}_{\pi_i}([\![\psi_i]\!]^{\mathfrak{M}}_{\pi_i}) = \mu^{\mathfrak{M}^\bullet}_{\pi_i}([\![\psi_i]\!]^{\mathfrak{M}^\bullet|^{\pi_i}})$. Please note that there is no probability formula occurring in ψ_i since $d(\varphi) = 0$. We have the following.

$$
\begin{aligned}
\mu^{\mathfrak{M}}_{\pi_i}([\![\psi_i]\!]^{\mathfrak{M}}_{\pi_i}) &= \sum_{\{t \in \mathcal{U}^{\mathfrak{M}}|^{\pi_i} \mid \mathfrak{M}, t \Vdash \psi_i\}} \mu^{\mathfrak{M}}_{\pi_i}(t) \\
&= \sum_{\{t \in \mathcal{U}^{\mathfrak{M}}|^{\pi_i} \mid \mathfrak{M}, t \Vdash \psi_i\}} \mu^{\mathfrak{M}}_{\pi_i}([\sigma]^{\pi_i}_t) && \text{(by Claim 2)} \\
&= \sum_{t \in \mathcal{U}^{\mathfrak{M}}|^{\pi_i}} \sum_{\{\sigma' \in [\sigma]^{\pi_i}_t \mid \mathfrak{M}, t \Vdash \psi_i\}} \mu^{\mathfrak{M}}_{\pi_i}(\sigma') \\
&= \sum_{t \in \mathcal{U}^{\mathfrak{M}}|^{\pi_i}} \sum_{\{\sigma' \in [\sigma]^{\pi_i}_t \mid \mathfrak{M}^\bullet, \sigma' \vDash \psi_i\}} \mu^{\mathfrak{M}}_{\pi_i}(\sigma') && \text{(by Claim 3)} \\
&= \sum_{\{\sigma \in \mathcal{U}^{\mathfrak{M}^\bullet}|^{\pi_i} \mid \mathfrak{M}^\bullet, \sigma \vDash \psi_i\}} \mu^{\mathfrak{M}^\bullet}_{\pi_i}(\sigma) \\
&= \mu^{\mathfrak{M}^\bullet}_{\pi_i}([\![\psi_i]\!]^{\mathfrak{M}^\bullet|^{\pi_i}}) \quad \square
\end{aligned}
$$

Appendix L. Proof of Proposition 54

Proof. Given a standard model $\mathcal{M} = \langle \mathcal{S}^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}}, Pr^{\mathcal{M}}, \mathcal{U}^{\mathcal{M}}, \mathcal{B}^{\mathcal{M}}, \mathcal{V}^{\mathcal{M}} \rangle$ with $\mathcal{M}, s \vDash \varphi$ for some $s \in \mathcal{U}^{\mathcal{M}}$, we define the nonstandard model $\mathcal{M}^\bullet$ as $\mathcal{S}^{\mathcal{M}^\bullet} = \mathcal{S}^{\mathcal{M}}$; $\mathcal{R}^{\mathcal{M}^\bullet} = \mathcal{R}^{\mathcal{M}}$; $\mathcal{U}^{\mathcal{M}^\bullet} = \mathcal{U}^{\mathcal{M}}$; $\mathcal{V}^{\mathcal{M}^\bullet} = \mathcal{V}^{\mathcal{M}}$; $\mu_\pi^{\mathcal{M}^\bullet} = \mu_\pi^{\mathcal{M}}$. Please note that $\mu_\pi^{\mathcal{M}}$ is defined in Definition 6.

First, we need to show that $\mathcal{M}^\bullet$ is indeed a nonstandard model. We need to show the following claim.

Claim 5.

1. *$\mu_\epsilon^{\mathcal{M}^\bullet}(\mathcal{U}^{\mathcal{M}^\bullet}) = 1$ and $\mu_\epsilon^{\mathcal{M}^\bullet}(s) > 0$ for each $s \in \mathcal{U}^{\mathcal{M}^\bullet}$;*
2. *$\mu_{\pi a}^{\mathcal{M}^\bullet}(\mathcal{U}^{\mathcal{M}^\bullet}|^{\pi a}) = \mu_\pi^{\mathcal{M}^\bullet}(\{s \in \mathcal{U}^{\mathcal{M}^\bullet}|^{\pi} \mid \mathcal{R}_a^{\mathcal{M}^\bullet}(s) \neq \emptyset\})$ and $\mu_{\pi a}^{\mathcal{M}^\bullet}(t) > 0$ for each $t \in \mathcal{U}^{\mathcal{M}^\bullet}|^{\pi a}$;*
3. *$\mu_{\pi a}^{\mathcal{M}^\bullet}(E) \leq \mu_\pi^{\mathcal{M}^\bullet}(\{s \in \mathcal{U}^{\mathcal{M}^\bullet}|^{\pi} \mid \exists t \in E : s \xrightarrow{a} t\})$ for each $E \subseteq \mathcal{U}^{\mathcal{M}^\bullet}|^{\pi a}$;*
4. *$\mu_{\pi a}^{\mathcal{M}^\bullet}(E) < \mu_\pi^{\mathcal{M}^\bullet}(\{s \in \mathcal{U}^{\mathcal{M}^\bullet}|^{\pi} \mid \exists t \in E : s \xrightarrow{a} t\})$ for each $E \subseteq \mathcal{U}^{\mathcal{M}^\bullet}|^{\pi a}$ such that $\mathcal{R}_a^{\mathcal{M}^\bullet}(s) \cap E \neq \emptyset$ and $\mathcal{R}_a^{\mathcal{M}^\bullet}(s) \setminus E \neq \emptyset$ for some $s \in \mathcal{U}^{\mathcal{M}^\bullet}|^{\pi}$.*

Proof of Claim 5. 1. Since $\mu_\epsilon^{\mathcal{M}^\bullet} = \mathcal{B}^{\mathcal{M}}$, this is obvious.

2. First, we show $\mu_{\pi a}^{\mathcal{M}^\bullet}(t) > 0$ given $t \in \mathcal{U}^{\mathcal{M}^\bullet}|^{\pi a}$. Since $t \in \mathcal{U}^{\mathcal{M}^\bullet}|^{\pi a} = \mathcal{U}^{\mathcal{M}}|^{\pi a}$, it follows that there is a sequence $s_0 a_1 \cdots s_n$ such that $s_0 \in \mathcal{U}^{\mathcal{M}}$, $s_i \xrightarrow{a_{i+1}} s_{i+1}$ for all $0 \leq i < n$, and $s_n = t$. It follows that $\mathcal{B}^{\mathcal{M}}(s_0) > 0$, $Pr^{\mathcal{M}}(s_i, a_{i+1}, s_{i+1}) > 0$ for all $0 \leq i < n$. It follows by Definition 6 that $\mu_{\pi a}^{\mathcal{M}}(t) \geq \mathcal{B}^{\mathcal{M}}(s_0) \times \Pi_{i=1}^n Pr^{\mathcal{M}}(s_{i-1}, a_i, s_i)$. Since $\mathcal{B}^{\mathcal{M}}(s_0) \times \Pi_{i=1}^n Pr^{\mathcal{M}}(s_{i-1}, a_i, s_i) > 0$ and $\mu_{\pi a}^{\mathcal{M}^\bullet} = \mu_{\pi a}^{\mathcal{M}}$, it follows that $\mu_{\pi a}^{\mathcal{M}^\bullet}(t) > 0$.
   Second, let $D = \{s \in \mathcal{U}^{\mathcal{M}^\bullet}|^{\pi} \mid \mathcal{R}_a^{\mathcal{M}^\bullet}(s) \neq \emptyset\}$ then we will show $\mu_{\pi a}^{\mathcal{M}^\bullet}(\mathcal{U}^{\mathcal{M}^\bullet}|^{\pi a}) = \mu_\pi^{\mathcal{M}^\bullet}(D)$. By the definition, we only need to show $\mu_{\pi a}^{\mathcal{M}}(\mathcal{U}^{\mathcal{M}}|^{\pi a}) = \mu_\pi^{\mathcal{M}}(D')$ where $D' = \{s \in \mathcal{U}^{\mathcal{M}}|^{\pi} \mid \mathcal{R}_a^{\mathcal{M}}(s) \neq \emptyset\}$. If $\mathcal{U}^{\mathcal{M}}|^{\pi a} = \emptyset$, it is obvious. If $\mathcal{U}^{\mathcal{M}}|^{\pi a} \neq \emptyset$, it follows that $\mathcal{U}^{\mathcal{M}}|^{\pi a} = [\![\top]\!]^{\mathcal{M}|^{\pi a}}$ and $D' = [\![\langle a \rangle \top]\!]^{\mathcal{M}|^{\pi}}$. By Proposition 11, it follows that $\mu_{\pi a}^{\mathcal{M}}(\mathcal{U}^{\mathcal{M}}|^{\pi a}) = \mu_\pi^{\mathcal{M}}(D')$.
3. We only need to show that $\mu_{\pi a}^{\mathcal{M}}(E) \leq \mu_\pi^{\mathcal{M}}(\{s \in \mathcal{U}^{\mathcal{M}}|^{\pi} \mid \exists t \in E : s \xrightarrow{a} t\})$ for each $E \subseteq \mathcal{U}^{\mathcal{M}}|^{\pi a}$. Given $E \subseteq \mathcal{U}^{\mathcal{M}}|^{\pi a}$, let $D = \{s \in \mathcal{U}^{\mathcal{M}}|^{\pi} \mid \exists t \in E : s \xrightarrow{a} t\}$. If $E = \emptyset$, it is obvious. If $E \neq \emptyset$, for each $t \in E$, there exists $s \in D$ such that $s \xrightarrow{a} t$. Moreover, it follows by Definition 6 that for each $t \in E$,

$$\mu_{\pi a}^{\mathcal{M}}(t) = \sum_{\{s \in D \mid s \xrightarrow{a} t\}} \mu_\pi^{\mathcal{M}}(s) \times Pr^{\mathcal{M}}(s, a, t)$$

   We then have the following:

$$\begin{aligned} & \mu_{\pi a}^{\mathcal{M}}(E) \\ = \; & \textstyle\sum_{t \in E} \mu_{\pi a}^{\mathcal{M}}(t) \\ = \; & \textstyle\sum_{t \in E} \big(\sum_{\{s \in D \mid s \xrightarrow{a} t\}} \mu_\pi^{\mathcal{M}}(s) \times Pr^{\mathcal{M}}(s, a, t) \big) \\ = \; & \textstyle\sum_{s \in D} \mu_\pi^{\mathcal{M}}(s) \times \big(\sum_{t \in (E \cap \mathcal{R}_a^{\mathcal{M}}(s))} Pr^{\mathcal{M}}(s, a, t) \big) \\ \leq \; & \textstyle\sum_{s \in D} \mu_\pi^{\mathcal{M}}(s) \qquad \text{since } 0 < \sum_{t \in (E \cap \mathcal{R}_a^{\mathcal{M}}(s))} Pr^{\mathcal{M}}(s, a, t) \leq 1 \\ = \; & \mu_\pi^{\mathcal{M}}(D) \end{aligned}$$

4. Given $u \in \mathcal{U}^{\mathcal{M}}|^{\pi}$ and $E \subseteq \mathcal{U}^{\mathcal{M}}|^{\pi a}$, there are $v, v' \in \mathcal{R}_a^{\mathcal{M}}(u)$ such that $v \in E$ and $v' \notin E$. We need to show $\mu_{\pi a}^{\mathcal{M}}(E) < \mu_\pi^{\mathcal{M}}(D)$ where $D = \{s \in \mathcal{U}^{\mathcal{M}}|^{\pi} \mid \exists t \in E : s \xrightarrow{a} t\}$. In 3. above, we have shown that $\mu_{\pi a}^{\mathcal{M}}(E) \leq \mu_\pi^{\mathcal{M}}(D)$ since for each $s \in D$:

$$\mu_\pi^{\mathcal{M}}(s) \times \Big(\sum_{t \in (E \cap \mathcal{R}_a^{\mathcal{M}}(s))} Pr^{\mathcal{M}}(s, a, t) \Big) \leq \mu_\pi^{\mathcal{M}}(s)$$

   which is due to

$$0 < \sum_{t \in (E \cap \mathcal{R}_a^{\mathcal{M}}(s))} Pr^{\mathcal{M}}(s, a, t) \leq 1.$$

   However, since there are $v, v' \in \mathcal{R}_a^{\mathcal{M}}(u)$ such that $v \in E$ and $v' \notin E$, thus we have

$$0 < \sum_{t \in (E \cap \mathcal{R}_a^{\mathcal{M}}(u))} Pr^{\mathcal{M}}(u, a, t) < 1.$$

Therefore, we have

$$\mu_\pi^{\mathcal{M}}(u) \times \Big(\sum_{t \in (E \cap \mathcal{R}_a^{\mathcal{M}}(u))} Pr^{\mathcal{M}}(u,a,t) \Big) < \mu_\pi^{\mathcal{M}}(u).$$

Since $u \in D$, it follows that $\mu_{\pi a}^{\mathcal{M}}(E) < \mu_\pi^{\mathcal{M}}(D)$. □

Second, by induction on the formula ψ, it is easy to show that $\mathcal{M}^\bullet, t \Vdash \psi$ if and only if $\mathcal{M}|^\pi, t \vDash \psi$ for each $\psi \in \mathscr{L}^{\texttt{B-Free}}$, each $t \in \mathcal{S}^{\mathcal{M}}$, and each $\pi \in \mathbb{A}^*$ with $t \in \mathcal{U}^{\mathcal{M}}|^\pi$.

Third, we will show $\mathcal{M}^\bullet, u \Vdash \psi$ if and only if $\mathcal{M}, u \vDash \psi$ for each $\psi \in \mathscr{L}^0$ and each $u \in \mathcal{U}^{\mathcal{M}}$. We prove it by induction on ψ. Please note that $\psi \in \mathscr{L}^0$. Due to the second step, here we only focus on the case of $\sum_{i=1}^n q_i B_{\pi_i} \psi_i \geq q$. Since $\mu_{\pi_i}^{\mathcal{M}} = \mu_{\pi_i}^{\mathcal{M}^\bullet}$, we only need to show $[\![\psi_i]\!]^{\mathcal{M}|^{\pi_i}} = [\![\psi_i]\!]_{\pi_i}^{\mathcal{M}^\bullet}$. Since $\psi_i \in \mathscr{L}^{\texttt{B-Free}}$, it follows by the second step that $[\![\psi_i]\!]^{\mathcal{M}|^{\pi_i}} = [\![\psi_i]\!]_{\pi_i}^{\mathcal{M}^\bullet}$. □

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