Dynamic Local Regret for Non-convex Online Forecasting

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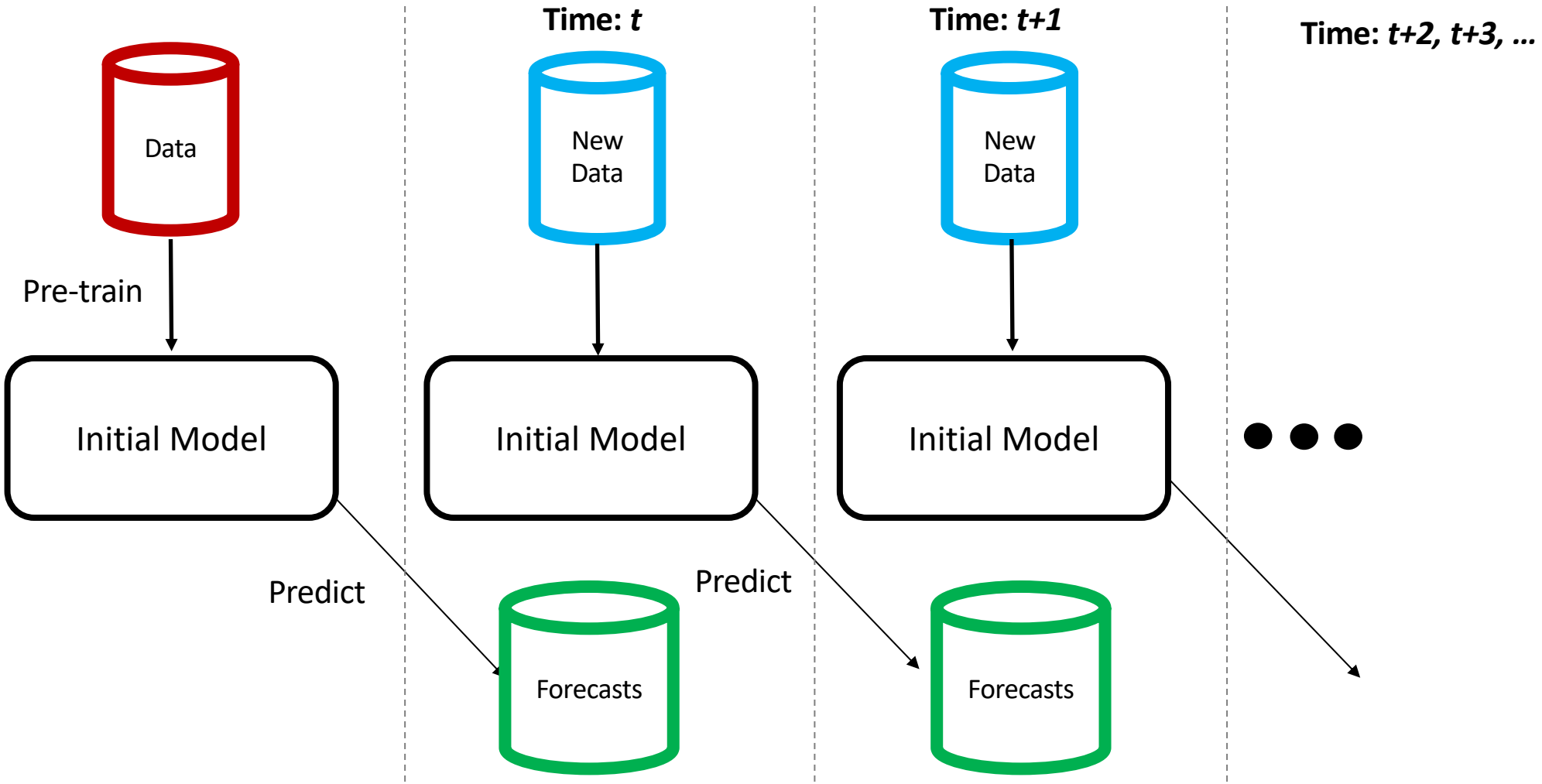
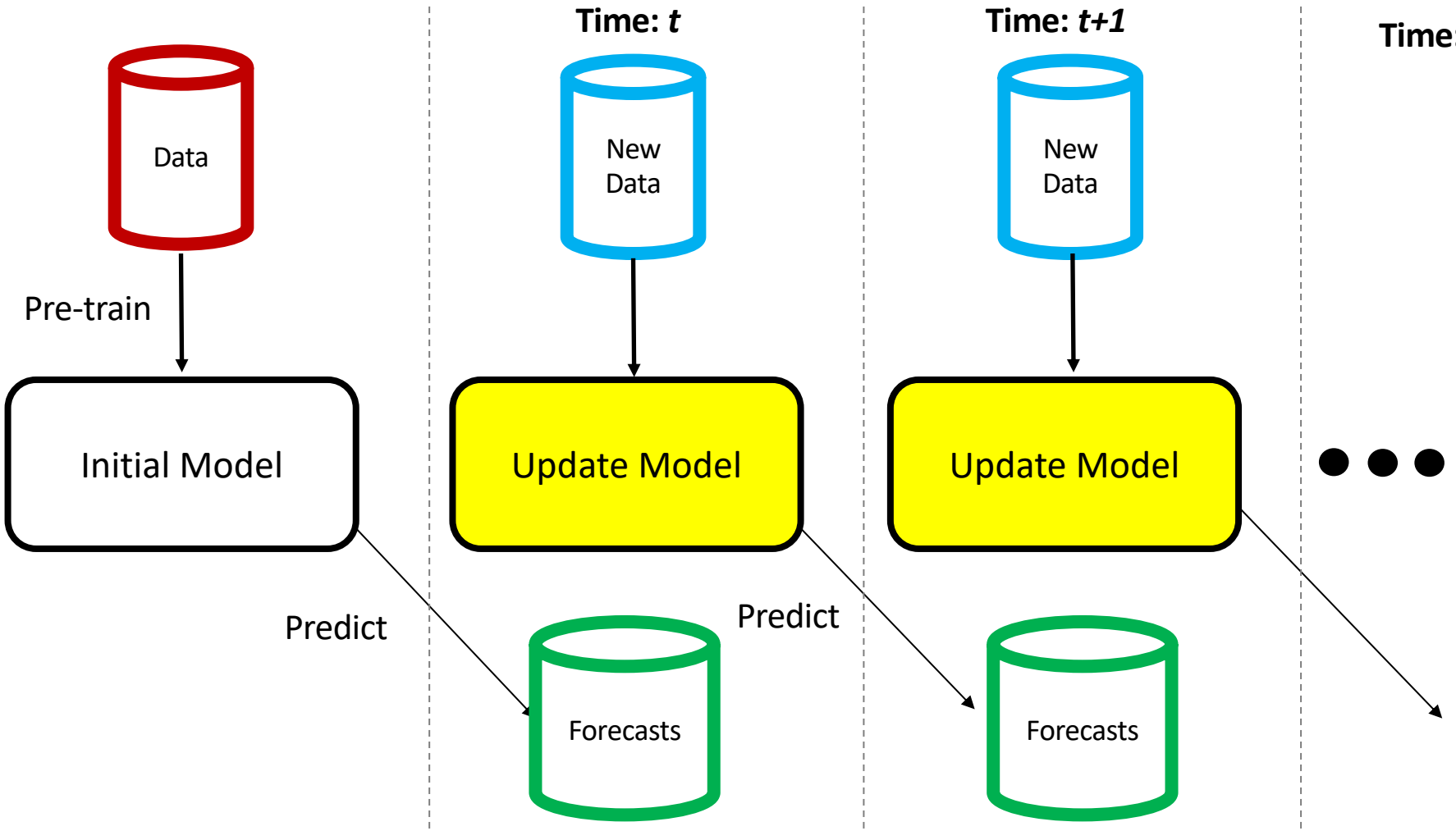
Consider a Forecasting Problem with Frozen Model

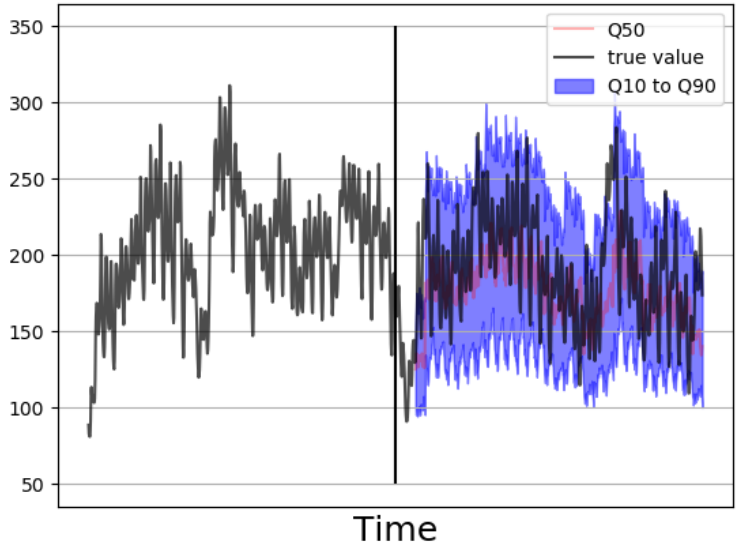


Ideally, we should update our model as fresh data is observed



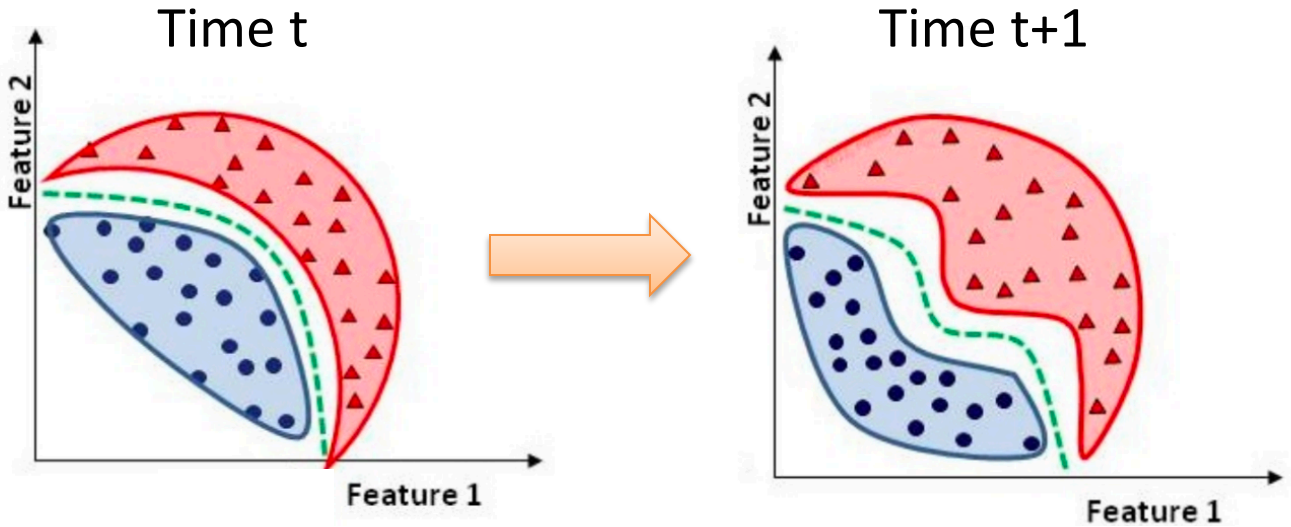
Forecasting Overview

- **Time Series Applications:** Financial market analysis, inventory planning, prediction of weather, earthquake forecasting, etc.
- **Forecasting**: the task of predicting future outcomes based on previous observations



- **Concept Drift:** The underlying relationship between inputs and outputs can change over time.

- **Solution 1**: Re-train model → Computationally Expensive
- **Solution 2:** Update model → Online Learning!



Hoens et al, 2011

Problem Settings: Online Learning

- A collection of T consecutive time points: $t \in \mathcal{T} = \{1, \cdots, T\}$
- Non-convex loss functions at each time t:

$$f_1, \cdots, f_T : \mathcal{K} \to \mathbb{R}$$

- The loss function using the observed data at time t using the model parameters x_t: $f_t(x_t)$
- The performance of Online Learning algorithms is commonly evaluated by the regret:

$$R(T) \triangleq \sum_{t=1}^{T} f_t(x_t) - \min_{x \in \mathcal{K}} \sum_{t=1}^{T} f_t(x)$$

- However, this definition is not appropriate for nonconvex problems due to NP-hardness.

Static Local Regret

- It is common to use the magnitude of the gradient to analyze convergence for nonconvex problems.
- Hazan et al. 2017 introduced a local regret measure:

$$SLR_w(T) \triangleq \sum_{t=1}^{T} \|\nabla F_{t,w}(x_t)\|^2 \quad \text{where} \quad \mathcal{K} = \mathbb{R}^d$$

$$F_{t,w} \triangleq \frac{1}{w} \sum_{i=0}^{w-1} f_{t-i}(x_t)$$

- **Problem**: Assumes static best model and effectively evaluates today's forecasts on yesterday's loss functions.

Proposed Dynamic Local Regret

- Static Local Regret definition assumes a static best model.
- Consider the first order Taylor's series expansion of cumulative loss.

$$\sum_{t=1}^{T} f_t(x_t + u) \approx \sum_{t=1}^{T} f_t(x_t) + \sum_{t=1}^{T} \langle u, \nabla f_t(x_t) \rangle$$

- If the updates $\{x_1, \ldots, x_T\}$ are *well-calibrated*, then perturbing x_t by any u cannot substantially reduce the cumulative loss.
- **Lemma**: For all x_s, the following equality holds: $\sup_{\|u\|=1} \sum_{s=t-w+1}^{t} \langle u, \nabla f_s(x_s) \rangle = \left\| \sum_{s=t-w+1}^{t} \nabla f_s(x_s) \right\|$
- **Proposed Dynamic Local Regret**: We address both non-convexity and dynamic environment in a regret framework.

$$DLR_w(T) \triangleq \sum_{t=1}^{T} \|\nabla S_{t,w,\alpha}(x_t)\|^2$$

$$S_{t,w,\alpha}(x_t) \triangleq \frac{1}{W} \sum_{i=0}^{w-1} \alpha^i f_{t-i}(x_{t-i})$$

$$W \triangleq \sum_{i=0}^{w-1} \alpha^i$$

Motivation via a Toy Example

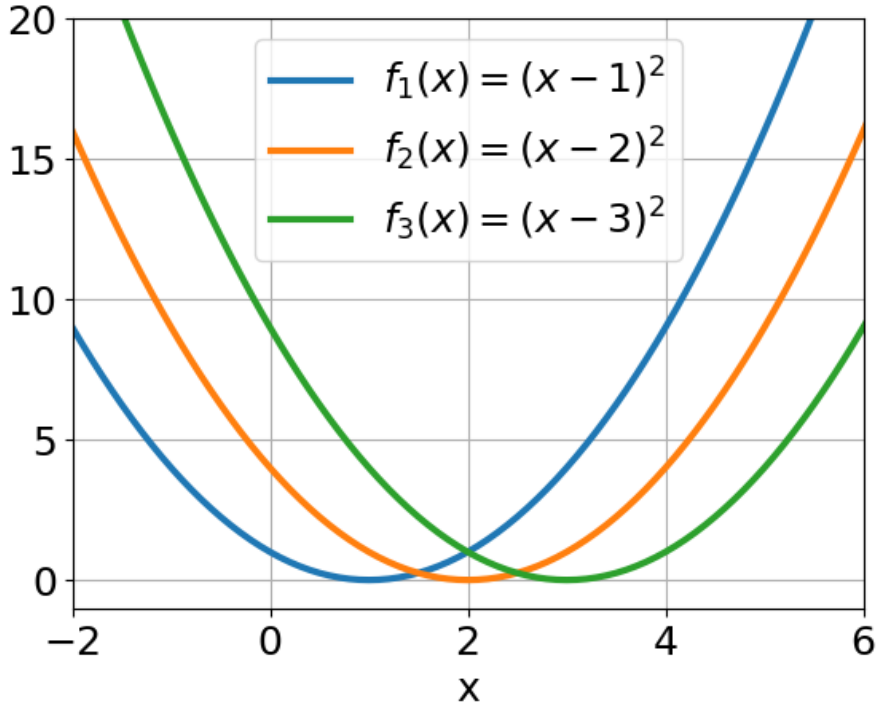
- Consider an online learning problem with concept drift with T=3.
- Oracle Policy: $x_1 = 1, x_2 = 2, x_3 = 3$.
- Stale Policy: $x_1 = 1, x_2 = 1.5, x_3 = 2$.
- Recall the formulation of local regrets

Hazan et al

$$SLR_3(3) = \left\| \frac{\nabla f_3(x_3) + \nabla f_2(x_3) + \nabla f_1(x_3)}{3} \right\|^2 + \left\| \frac{\nabla f_2(x_2) + \nabla f_1(x_2)}{3} \right\|^2 + \left\| \frac{\nabla f_1(x_1)}{3} \right\|^2$$

Ours

$$DLR_3(3) = \left\| \frac{\nabla f_3(x_3) + \nabla f_2(x_2) + \nabla f_1(x_1)}{3} \right\|^2 + \left\| \frac{\nabla f_2(x_2) + \nabla f_1(x_1)}{3} \right\|^2 + \left\| \frac{\nabla f_1(x_1)}{3} \right\|^2$$



- Let's compute loss and regret for the two policies

Regret	Oracle Policy	Stale Policy	Decision
Cumulative Loss	0	$5/4$	Oracle policy is better
Standard Regret	-2	-3/8	Oracle policy is better
Static Local Regret (Hazan et al.)	$40/9$	$4/9$	Stale policy is better
Dynamic Local Regret (Ours)	0	$10/9$	Oracle policy is better

Time-smoothed SGD Algorithms

Hazan et al, 2017

Algorithm 1 Static Time-Smoothed Stochastic Gradient Descent (STS-SGD)

Require: window size $w \geq 1$, learning rate $\eta > 0$, Set $x_1 \in \mathbb{R}^n$ arbitrarily

1: **for** $t = 1, \cdots, T$ **do**
2: Predict x_t. Observe the cost function $f_t : \mathbb{R}^b \to \mathbb{R}$
3: Update $x_{t+1} = x_t - \frac{\eta}{w} \sum_{i=0}^{w-1} \hat{\nabla} f_{t-i}(x_t)$.
4: **end for**

Guarantees upper bound

$$\frac{T}{w} \times \text{Constant}$$

according to the Static Local Regret

Ours

Algorithm 2 Dynamic Exponentially Time-Smoothed Stochastic Gradient Descent (DTS-SGD)

Require: window size $w \geq 1$, learning rate $\eta > 0$, exponential smoothing parameter $\alpha \to 1^-$ (means that α approaches 1 from the left), normalization parameter $W \triangleq \sum_{i=0}^{w-1} \alpha^i$, Set $x_1 \in \mathbb{R}^n$ arbitrarily

1: **for** $t = 1, \cdots, T$ **do**
2: Predict x_t. Observe the cost function $f_t : \mathbb{R}^b \to \mathbb{R}$
3: Update $x_{t+1} = x_t - \frac{\eta}{W} \sum_{i=0}^{w-1} \alpha^i \hat{\nabla} f_{t-i}(x_{t-i})$.
4: **end for**

Guarantees upper bound

$$\frac{T}{W} \times \text{Constant}$$

according to our dynamic local regret

Quantile Loss

- Standard **mean squared error** summarizes average relationship between inputs and outputs
- The resulting forecast will be a **point forecast** which is the most likely outcome
- Many forecasting applications require richer information that a point forecast
- **Quantile loss** minimizes a sum with asymmetric penalties for overprediction and underprediction
- The quantile loss for a given quantile q between true value y and forecast value $\hat{y}$:

$$L_q(y, \hat{y}) = q \max(y - \hat{y}, 0) + (1 - q) \max(\hat{y} - y, 0)$$

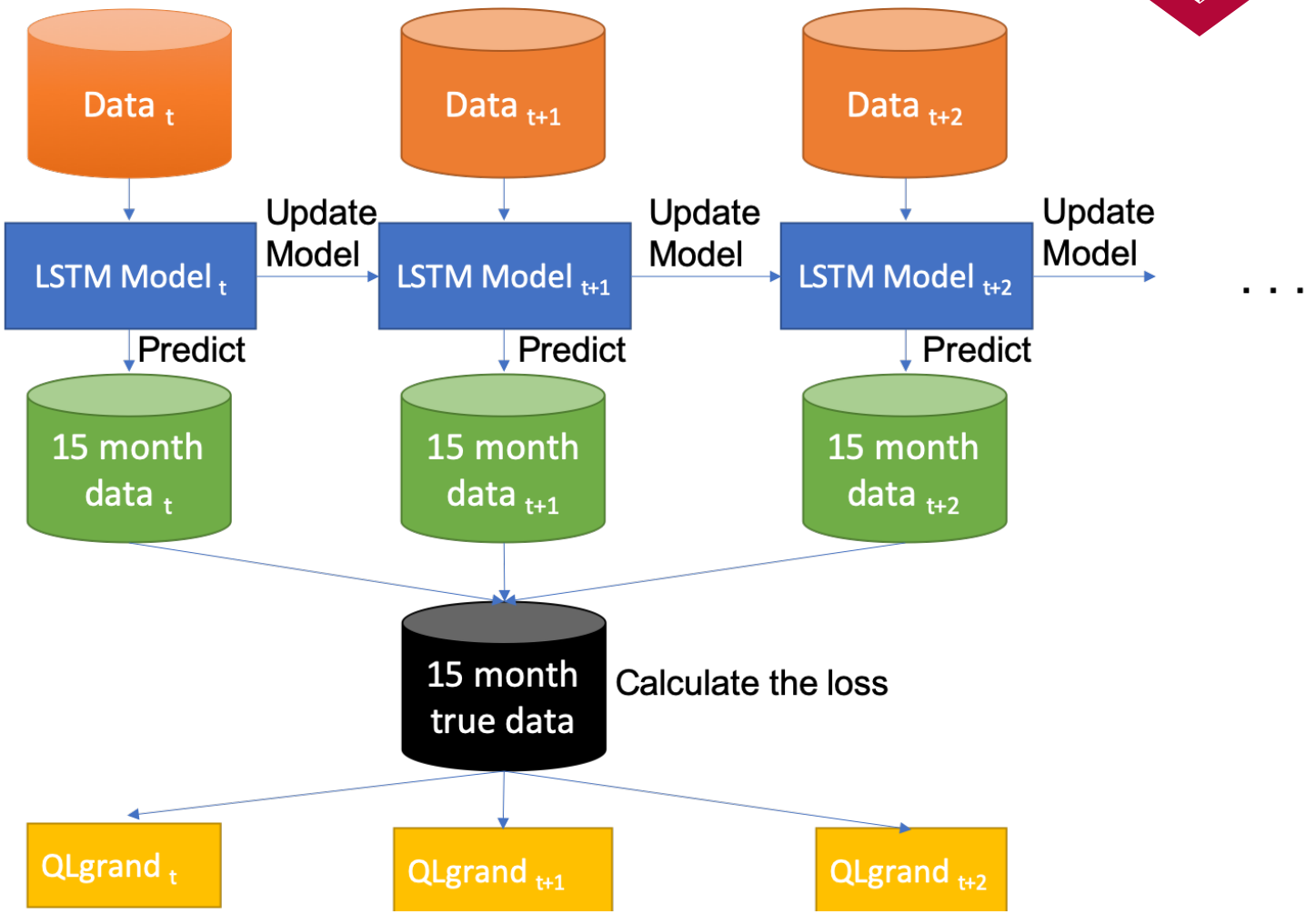
- Typically, in forecasting we may be interested in multiple quantiles and horizons where the total loss function becomes

$$\sum_t \sum_k \sum_q L_q(y_{t+k}, \hat{y}^q_{t+k})$$

t: Forecast creation date; k: horizon; q: quantiles

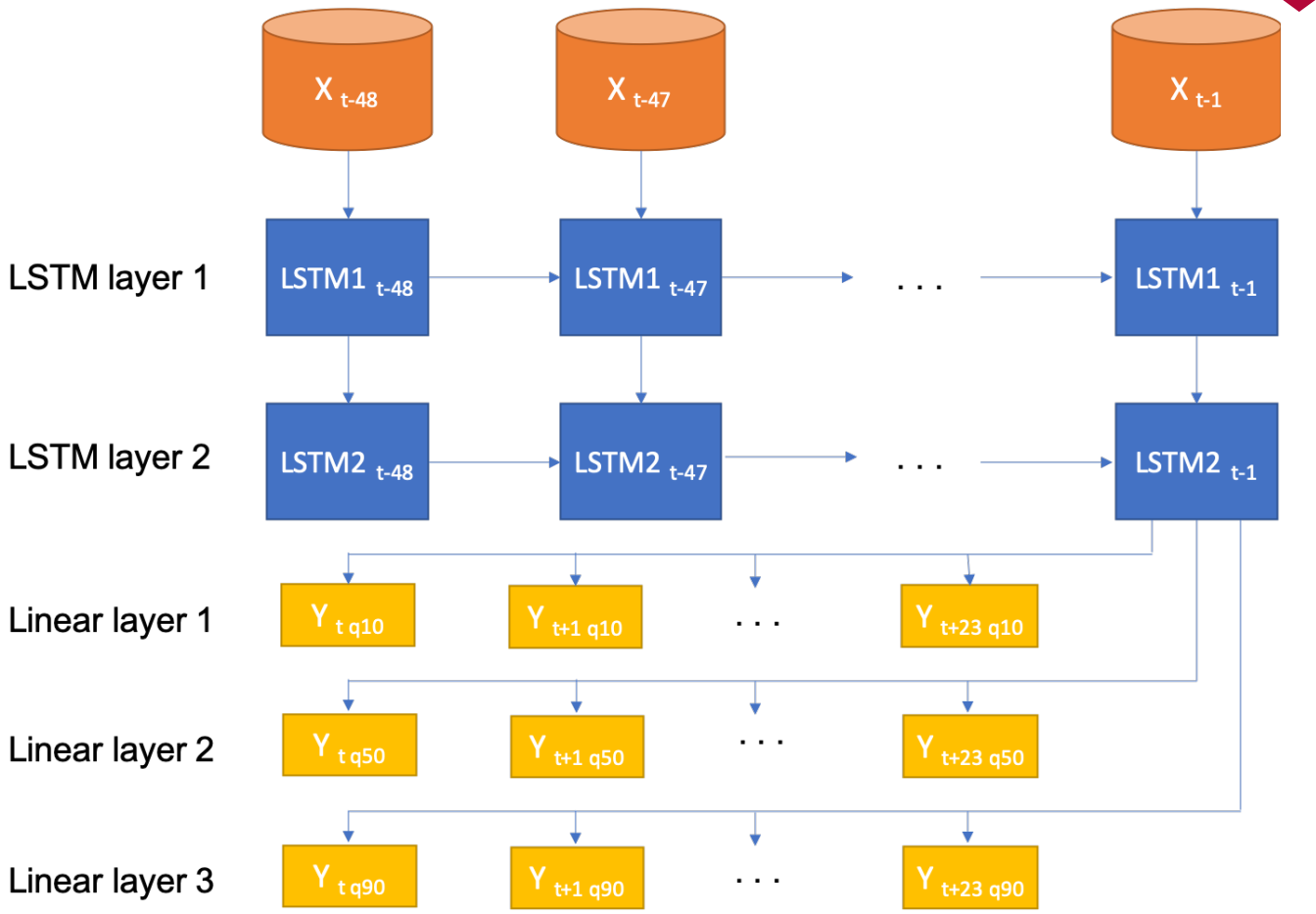
Flowchart of Our Experiments

- Public GEFCom2014 data [Barta et al. 2017]
- Hourly Electric Load values:
  - **Training**: January 2005 – September 2010
  - **Test**: October 2010 – December 2012
- **Metric**: Average quantile loss of 15 months in test data



LSTM Model

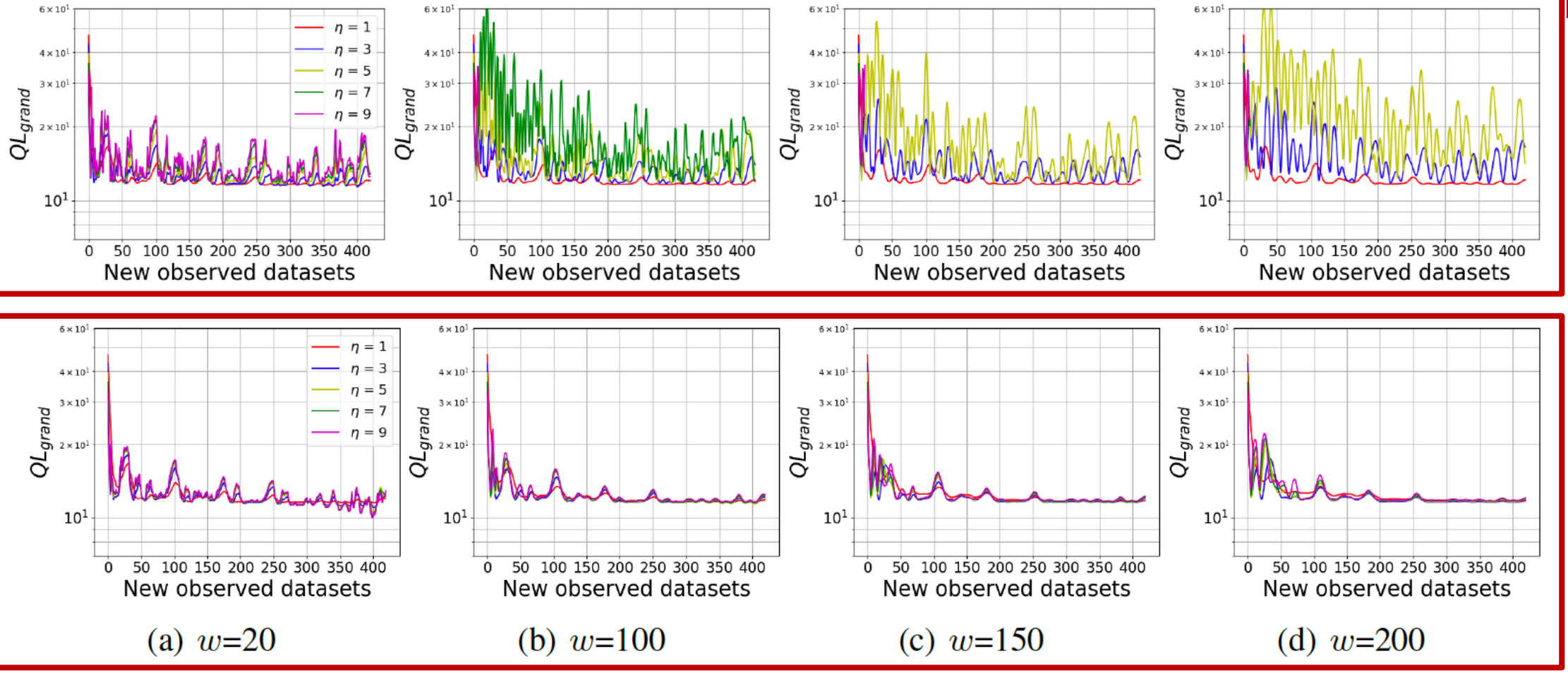
- LSTMs are special kind of RNN [Hochreiter and Schmidhuber, 1997]
- Multi-step LSTM is used to forecast multiple quantiles
- **Metric**: Average quantile loss of 15 months in test data



Experimental Results – Stability against window size

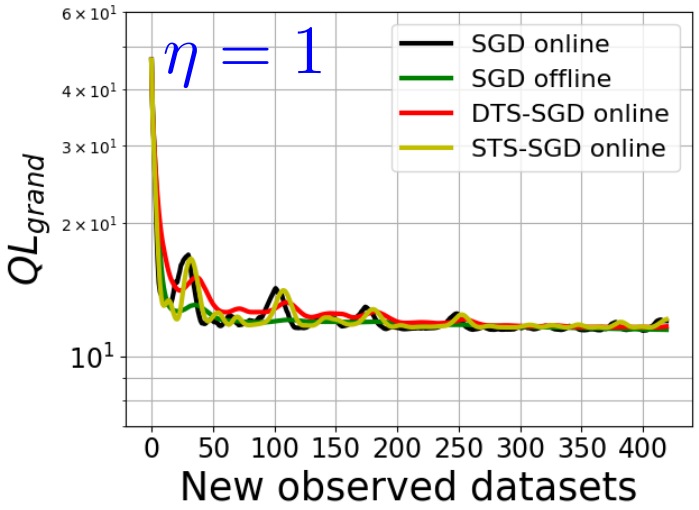
STS-SGD (Hazan et al. 2017)

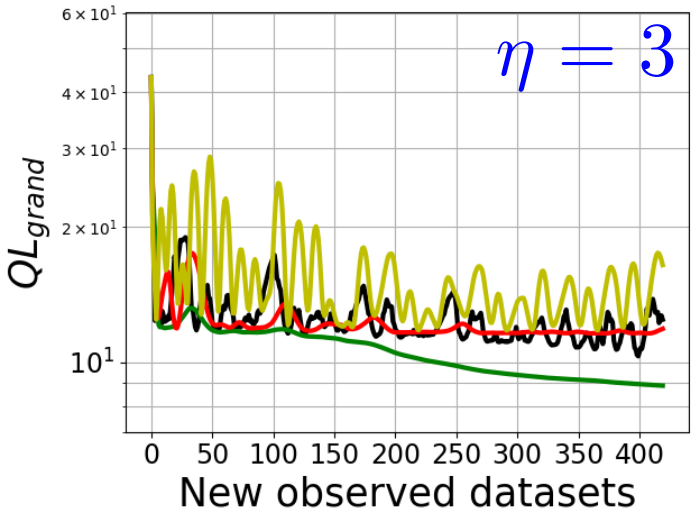
DTS-SGD (ours)

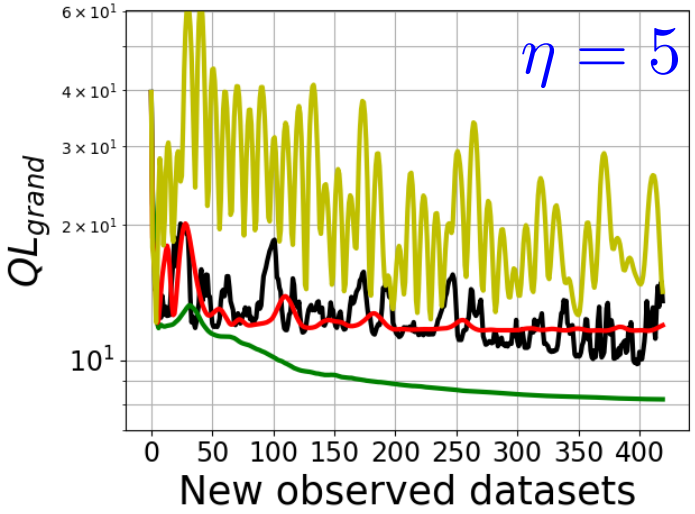


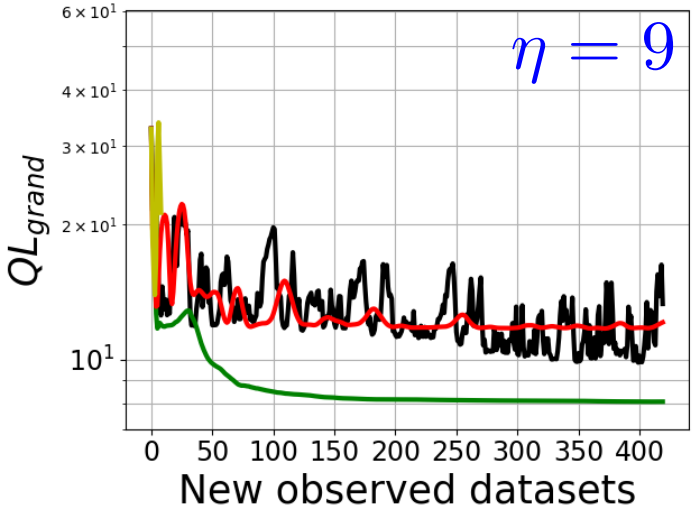
- The learning curves for DTS-SGD stay more stable than STS-SGD against different window sizes and learning rates.

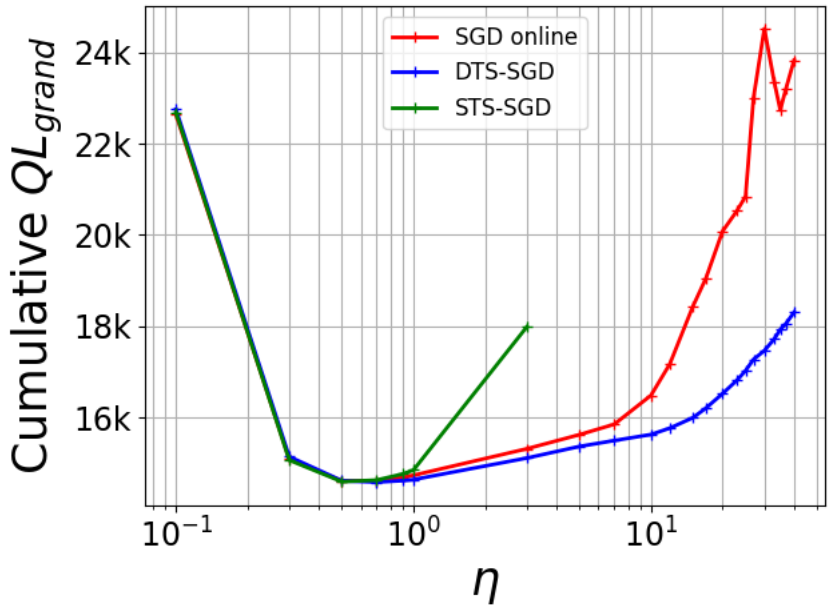
Experimental Results – Stability against learning rate







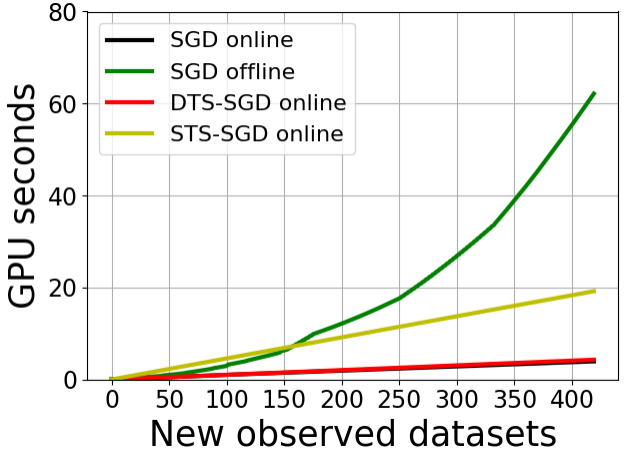




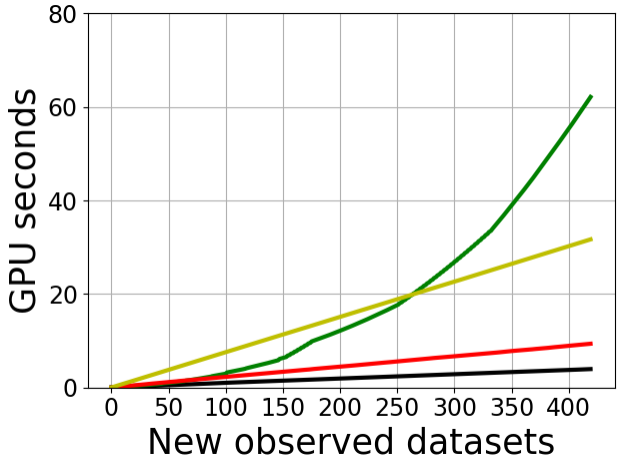
- Our DTS-SGD is less sensitive to η than SGD online and STS-SGD and performs well for a wider range of η.

Experimental Results – Computation Time

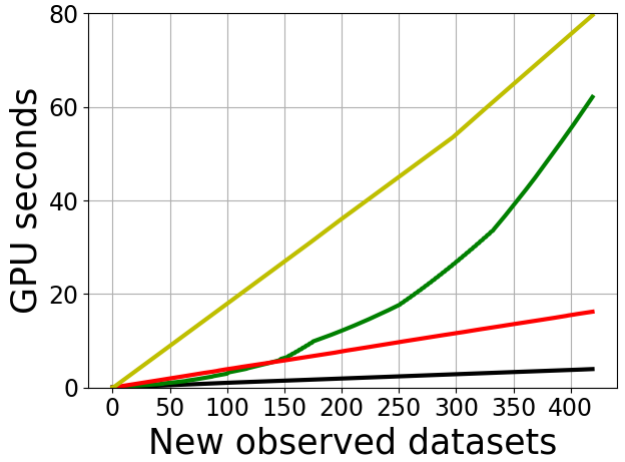
$w = 10$



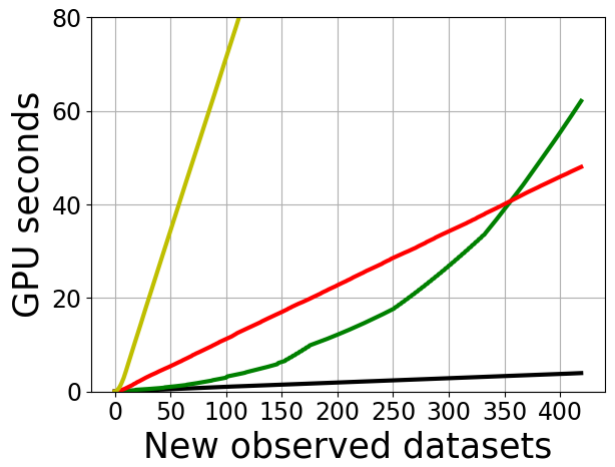
$w = 20$



$w = 50$



$w = 200$



- Computation time for STS-SGD and DTS-SGD increases as w increases.
- Our DTS-SGD is more efficient than the SGD offline even for large w.

Conclusion

- We introduced a novel local regret for forecasting problems with non-convex models
- We introduced an update rule for our regret: dynamic exponentially time-smoothed SGD update
- Our update rule is sublinear according our proposed regret
- Our experimental results show that our PTS-SGD is
  - not sensitive to the learning rate
  - computationally efficient

Thank You!

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