Report for exercise 4 from group F

Tasks addressed:	5
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Source code:	https://github.com/Chuck00027/MLCMS-GroupF/tree/main/Exercise4

The work on tasks was divided in the following way:

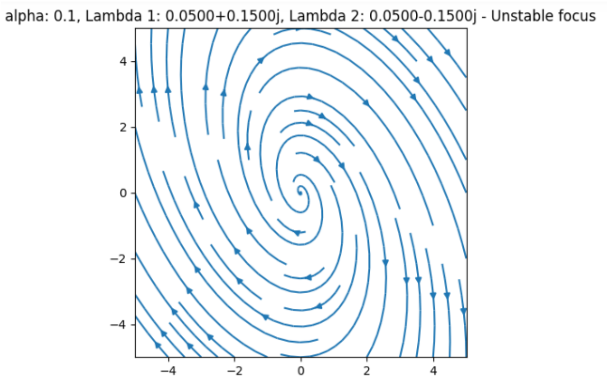
Pemba Sherpa (03760783) **Project lead**	Task 1	0%
	Task 2	100%
	Task 3	0%
	Task 4	0%
	Task 5	0%
Hao Chen (03764817)	Task 1	50%
	Task 2	0%
	Task 3	50%
	Task 4	50%
	Task 5	0%
Yang Cheng (03765398)	Task 1	50%
	Task 2	0%
	Task 3	50%
	Task 4	50%
	Task 5	0%
Jianzhe Liu (03751196)	Task 1	0%
	Task 2	0%
	Task 3	0%
	Task 4	0%
	Task 5	100%

Report on task TASK 1, Vector fields, orbits, and visualization

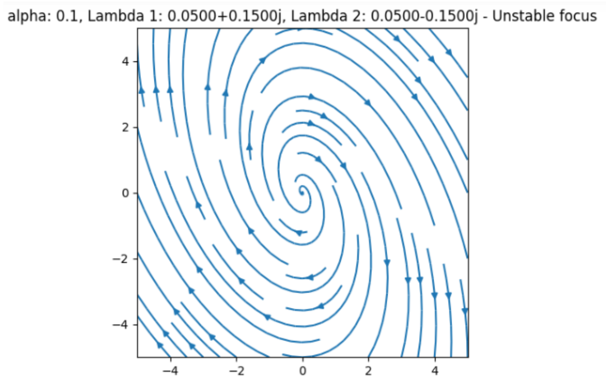
Short description of the setup in the report? Which parametrized matrix did you choose? In this task, we explore how the dynamic behavior of a 2D linear system can be changed by changing its parameters. We chose a parameterized matrix of the form:

$$A = \begin{bmatrix} \alpha & \alpha \\ -\frac{1}{4} & 0 \end{bmatrix}$$

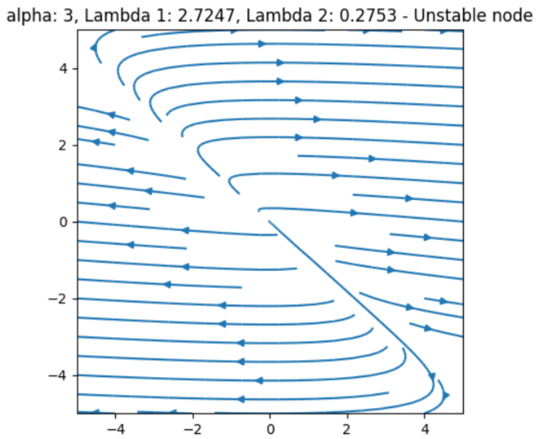
Construct a figure similar to Fig. 2.5 in the book of Kuznetsov and specify the value of the parameter for each of your phase portraits? In the parameterized matrix A, α is our variable parameter, and we change the value of α to study how the dynamic behavior of the system changes as the parameter changes. Our goal is to construct a diagram similar to Figure 2.5 in Kuznetsov's book, which shows how the phase image of the system changes by changing the value of α. We chose 3 different α values, namely 0.1, -0.5, 3 , to generate 3 different types of phase images: For α=0.1, we get an unstable focus, as shown in figure 1a. For α=-0.5, we get an unstable saddle, as shown in figure 1b. For α=3 we get an an unstable node, as shown in figure 1c. In the context of dynamical systems, the terms "unstable focus", "unstable saddle", and "unstable node" play pivotal roles. An "unstable focus" refers to an equilibrium point from which trajectories spiral outwards, indicative of an unstable system that deviates from equilibrium over time. The "unstable saddle" represents an equilibrium point that, while stable in one direction, is unstable in another, leading to the system diverging away from this equilibrium along certain trajectories. An "unstable node" describes an equilibrium point from which trajectories radiate outward along straight lines, suggesting system instability.These unstable points are crucial as they help in understanding the system's response to perturbations, indicating where rapid changes in the system state may occur and offering valuable insights for prediction and control of system behaviors.



(a) Phase portrait of an unstable focus



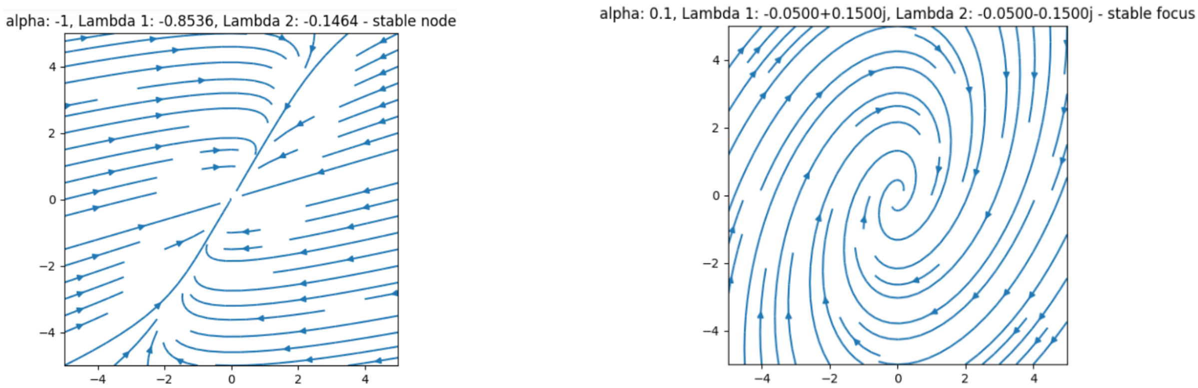
(b) Phase portrait of an unstable saddle



(c) Phase portrait of an unstable node

Figure 1: Phase portraits of dynamical systems

We found that we cannot get a stable focus and a stable node by setting the two elements of the first row of the parameter matrix to the same number, because if we do, we cannot get complex eigenvalues with negative real parts, which in turn is a necessary condition for obtaining a stable focus, likewise, we cannot get two real negative eigenvalues, which in turn is a necessary condition for obtaining a stable node. So we set $\alpha1$=-0.1, $\alpha2$=0.1, so that we can get complex eigenvalues with real part of -0.05 and imaginary part of ±0.15, so we can get a stable focus,as shown in figure 2a. At the same time, we set $\alpha1$=-1, $\alpha2$=0.5, so that we can get two real negative eigenvalues, so we can get a stable node, as shown in figure 2b.



(a) Phase portrait of a stable node

(b) Phase portrait of a stable focus

Figure 2: Phase portraits of dynamical systems

Are these systems topologically equivalent? Why, or why not (no formal proof necessary)? Now, let's talk about the topological equivalence of these five phase portraits. Although all five systems are derived from the same parameterized matrix (except for the last two which differ a bit), they are not topologically equivalent. Topological equivalence means that there is a continuous one-to-one mapping that transforms the orbit and behavior of one system into the orbit and behavior of another system. However, the stability and the attractive or repulsive nature of the orbitals are significantly different among these five systems and cannot be interconverted by continuous one-to-one mapping.

Report on task TASK 2, Common bifurcations in nonlinear systems

Two simple quadratic functions with a single parameter α is given for this task.

1. $f(x) = \alpha - x^2$, shown in figure 3
2. $f(x) = \alpha - 2x^2 - 3$, shown in figure 4

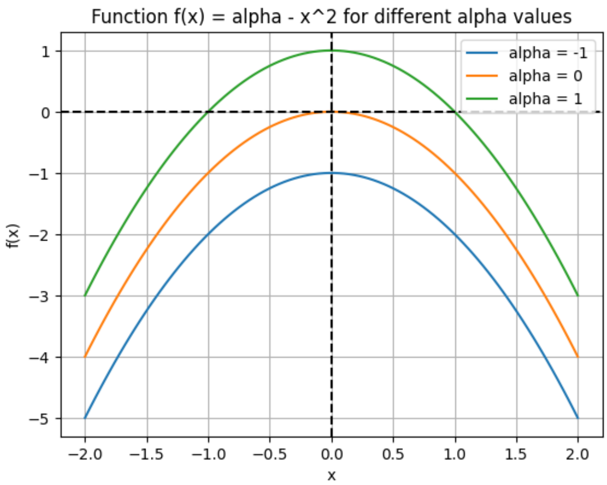
For the function 1, the fixed points are given by $x_* = \pm\sqrt{\alpha}$. Two real fixed points exist when $\alpha > 0$ and no real fixed points exist when $\alpha < 0$. By looking at the phase portrait, we can identify the stability of the fixed points. In figure 3.(e), we can see that there is convergence (stability) on the positive x values and divergence(unstable) at the negative x values. This type of bifurcation is called saddle node and is seen commonly in quadratic functions. At $\alpha = 0$, equilibrium occurs and as the α value goes higher the equilibrium point splits into two different stable and unstable fixed points.

For the function 2, the fixed points are given by $x_* = \pm\sqrt{(\alpha - 3)/2}$. The real fixed points only exist when $\alpha > 3$ and no real fixed points exist when $\alpha < 3$. By looking at the phase portrait, we can identify the stability of the fixed points when $\alpha > 3$. In figure 4.(e), we can see that there is convergence (stability) on the positive x values and divergence(unstable) at the negative x values. This is also a saddle node bifurcation. At $\alpha = 3$, equilibrium occurs and as the α value goes higher the equilibrium point splits into two different stable and unstable fixed points.

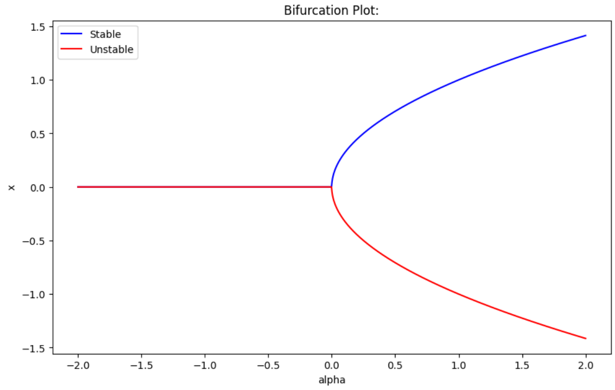
Both function 1 and 2 are quadratic functions with the 2nd function shifted -3 units vertically and steeper as well due to the coefficient 2 in front of x^2. This can also be noticed by using the quadratic normal form $f(x) = ax^2 + bx + c$. For both, the coefficient b = 0 with c denoting $\alpha + -const$. If we only look at x^2 and $2x^2$, they both can represent the same function but at different x values.

A simple python class was created for each action: to plot 1) the bifurcation and 2) phase portraits of the given functions. Since the functions were simple, only two python libraries, numpy and matplotlib, was necessary. The dynamical system f(x) was defined inside each classes within a function which is then called during runtime.

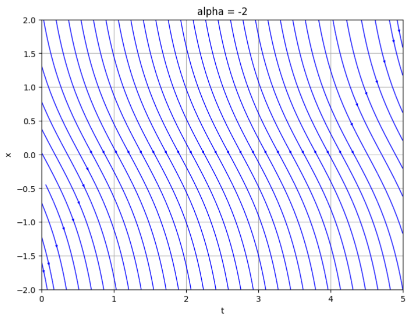
Bifurcation is a great way to find qualitative change over the long-time solution in a nonlinear differential equation by adding small change. The above functions and their respective bifurcation plots shows that how small change in the coefficient of the quadractic function resulted in the location of stable points.



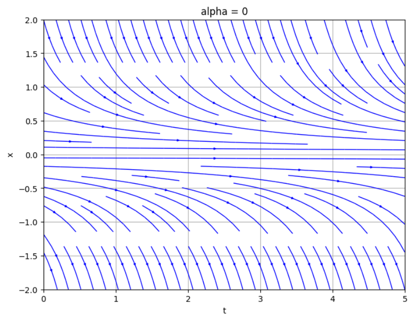
(a) Plot of dx/dt at different alpha values



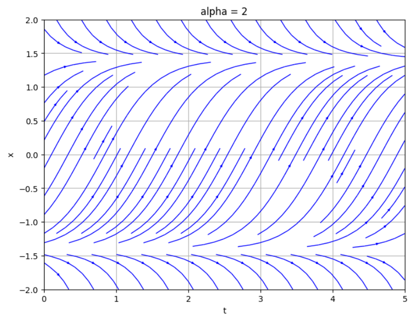
(b) Bifurcation plot of function



(c) Phase portrait at α=-2



(d) Phase portrait at α=0



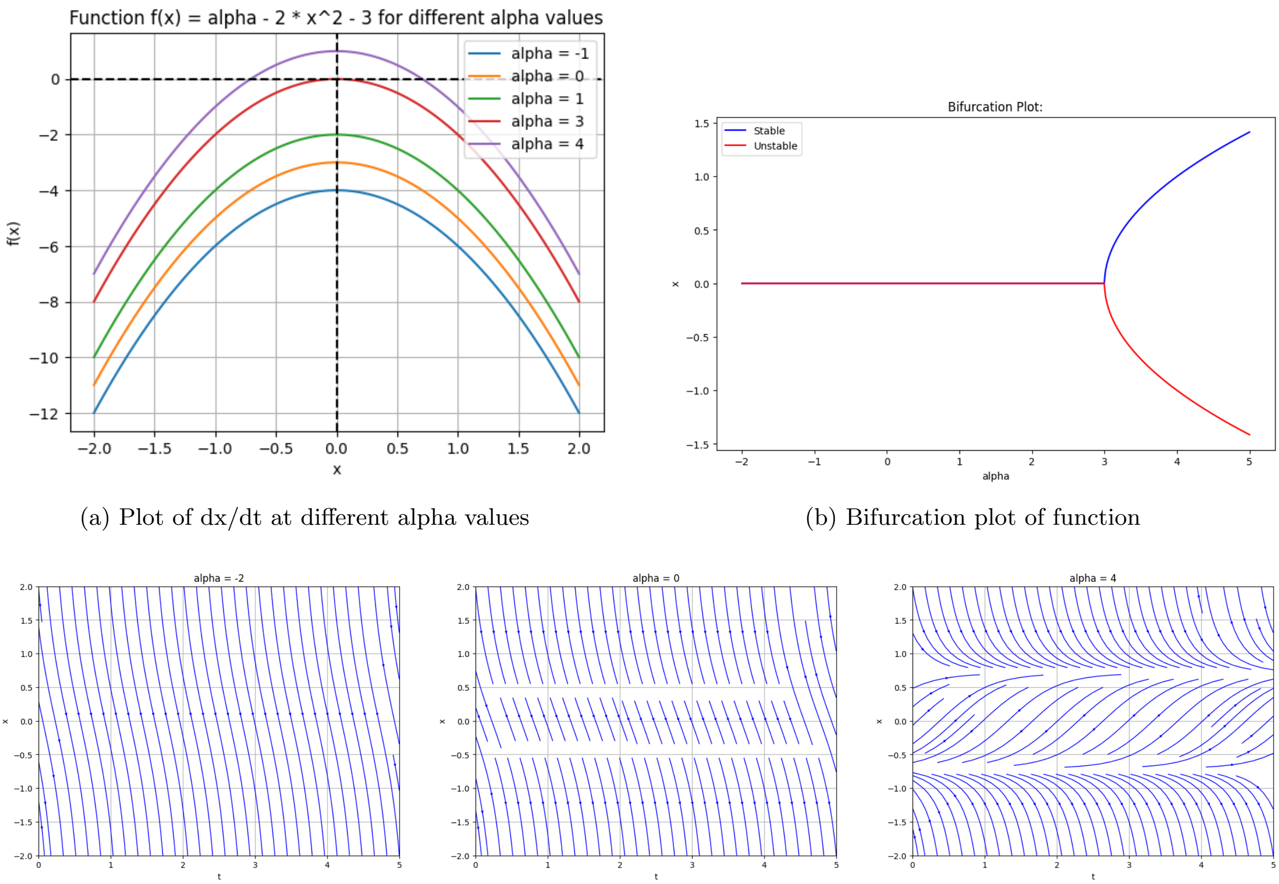
(e) Phase portrait at α=2

Figure 3: Plots for function $f(x) = \alpha - x^2$

Report on task TASK 3, Bifurcations in higher dimensions

Short description of the setup in the report? In this task, we investigate a dynamic system, the behavior of which is influenced by a parameter α. We explore the dynamics of the system by altering the value of α and observing phase portraits of the system. Specifically, we focus on behaviors at α equals to -1, 0, and 1, corresponding to a stable steady state, bifurcation point, and limit cycle of the system respectively.In the latter part of the experiment, we chose α to be 1 and computed and plotted the orbits of the system starting from two different points. This aids in understanding how the system state evolves over time. Lastly, we employed a function to visualize the cusp bifurcation of the system. It's a special type of bifurcation named after its shape where the bifurcation surface appears as a cusp in three-dimensional space.

Visualize the bifurcation of the system by plotting three phase diagrams? Through observation of phase portraits, we clearly witness a change in the dynamics of the system with varying α. When α=-1,as shown in figure 5a, we can observe that there is a stable steady state in the system, and all trajectories will converge to this steady state. We will see that all streamlines eventually point to a central point. When α=0, as shown in figure 5b, we see what is called a "bifurcation" of the system. A bifurcation point refers to a particular value of a parameter at which the system undergoes a qualitative change in behavior. Such points are important because they signal where small alterations can cause drastic shifts in system dynamics.In this case, the original stable state (at α=-1) is no longer stable. At this point, we do not observe additional stable states emerging, but instead see the behavior of the system transition from a stable point to a cycle around a trajectory. In this case, we observe that the trajectory of the system does not converge to a specific point, but



(a) Plot of dx/dt at different alpha values

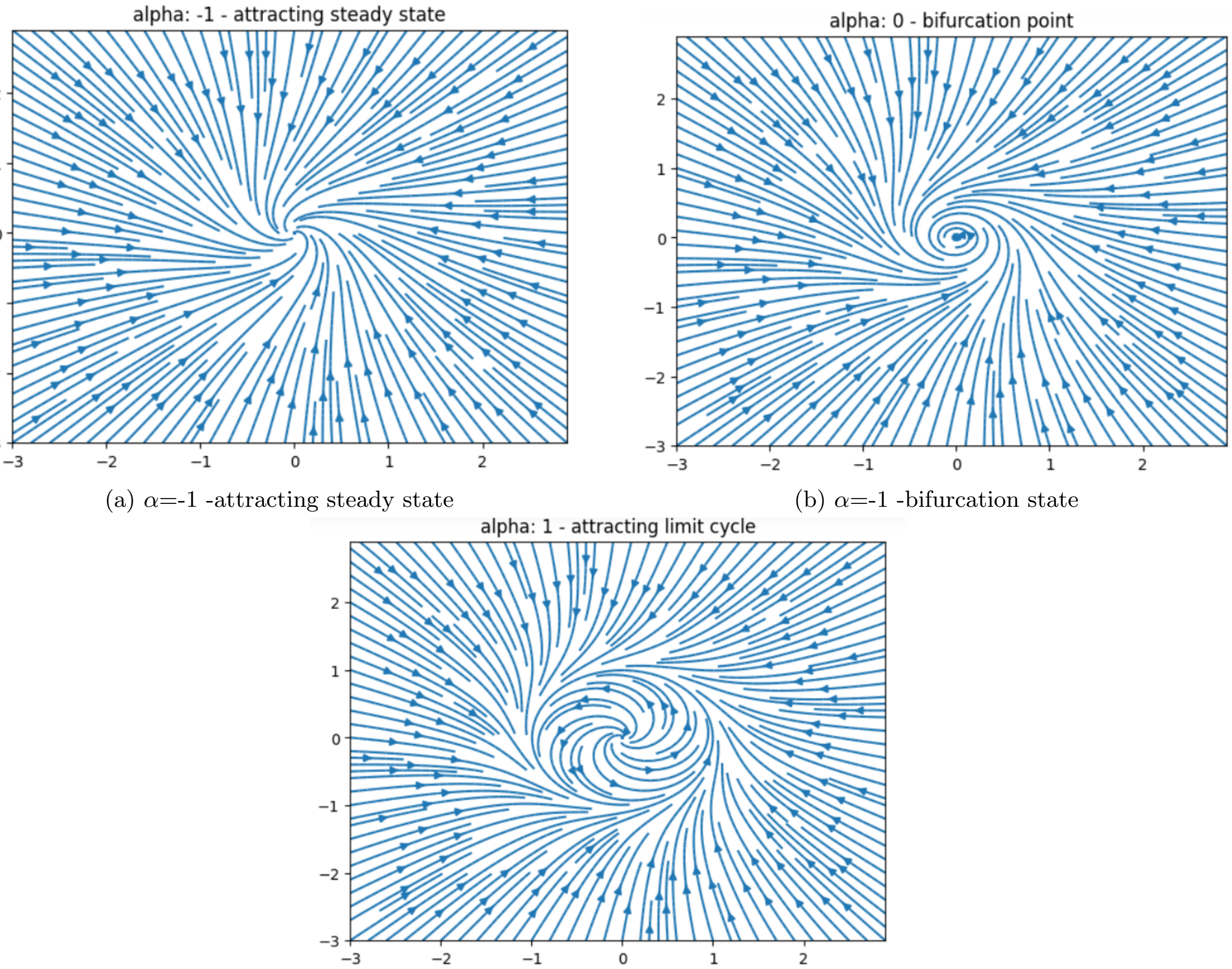
(b) Bifurcation plot of function

(c) Phase portrait at α=-2

(d) Phase portrait at α=0

(e) Phase portrait at α=4

Figure 4: Plots for function $f(x) = \alpha - 2 * x^2 - 3$



(a) α=-1 -attracting steady state

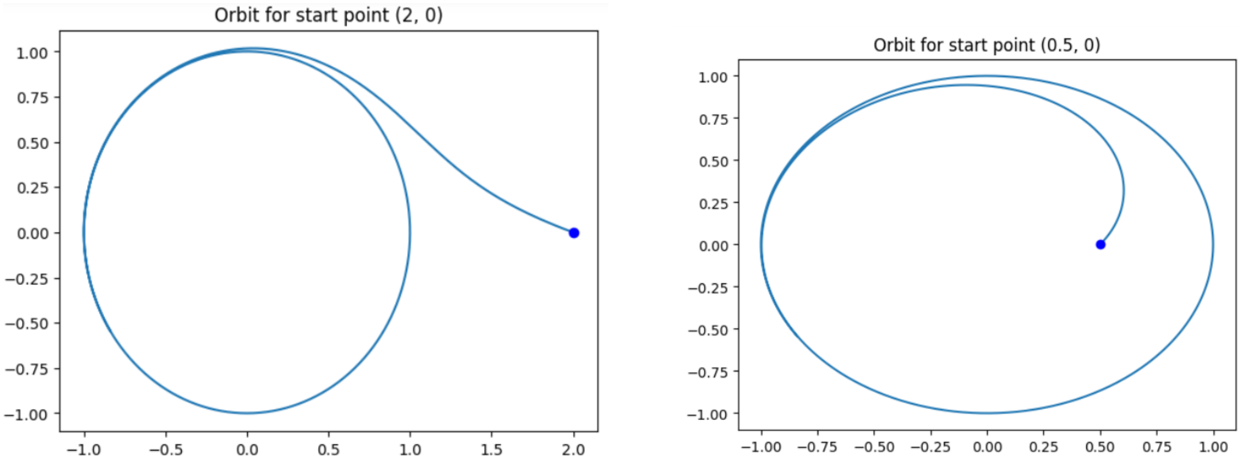
(b) α=-1 -bifurcation state

(c) α=1 -atrracting limit cycle

Figure 5: Phase portraits of dynamical systems

starts to loop around a hollow region.When α=1, as shown in figure 5c, the system exhibits an attractive limit cycle. A limit cycle is essentially a closed trajectory in the phase plane, indicating a periodic behavior of the system. The presence of limit cycles in a system signifies potential oscillations. In this case, no matter which initial state we choose, the state of the system will eventually converge to a closed trajectory. In the image, we can see that all trajectories converge towards a closed circular trajectory.

For $\alpha = 1$, numerically compute and visualize two orbits of the system (8)? Further, we plotted two orbits at α equals to 1 corresponding to the system state starting from two different points, as shown in figure 6a and 6b. We observe these orbits converging to the same limit cycle (although they look different due to proportions, they are actually the same), confirming the attracting nature of the system.



(a) orbits at α=1, start point at (2,0)

(b) orbits at α=1, start point at (0.5,0)

Figure 6: Orbits at α=1, starting from different points

Visualize the bifurcation surface of the cusp bifurcation (9) in a 3D plot? In our task, we are asked to visualize the bifurcation surface of the cusp bifurcation in a 3D plot with two parameters $\alpha 1$ and $\alpha 2$. The third dimension is the steady state x. We need to depict the points (x, $\alpha 1$, $\alpha 2$) where the rate of change of x, equals zero. Our code accomplishes this task by creating a scatter plot in 3D, as shown in figure 7a. We uniformly sample values for x and $\alpha 2$ and calculate the corresponding $\alpha 1$ values that make the derivative equal to zero. This is done using the formula: $\alpha 1 = -\alpha 2 \cdot x + x^3$.The 3D plot we create visualizes the bifurcation surface of the cusp bifurcation. In this plot, $\alpha 1$ and $\alpha 2$ form the base plane and x forms the third dimension. The points in our plot are colored based on the $\alpha 2$ value, providing a visual representation of the relationship between $\alpha 1$,$\alpha 2$, and x where the rate of change of x equals zero. We also generate a 2D plot which is a projection of the 3D plot onto the $\alpha 1$-$\alpha 2$ plane,as shown in figure 7b. We color the points in this plot based on the number of solutions for each pair of $\alpha 1$ and $\alpha 2$ values - red for multiple solutions and blue for a single solution.

Why is it called cusp bifurcation? As for why this is called a cusp bifurcation,I think it comes from the unique shape of the bifurcation surface. In the 3D plot, there is a region where the surface folds over itself, forming a 'cusp' or pointed shape. This is seen as a region where certain pairs of $\alpha 1$ and $\alpha 2$ yield multiple x values (steady states), and these pairs are represented as red points in the 2D plot. This 'cusp' or fold is a characteristic feature of this type of bifurcation, and hence the name 'cusp bifurcation'.

Report on task TASK 4, Chaotic dynamics

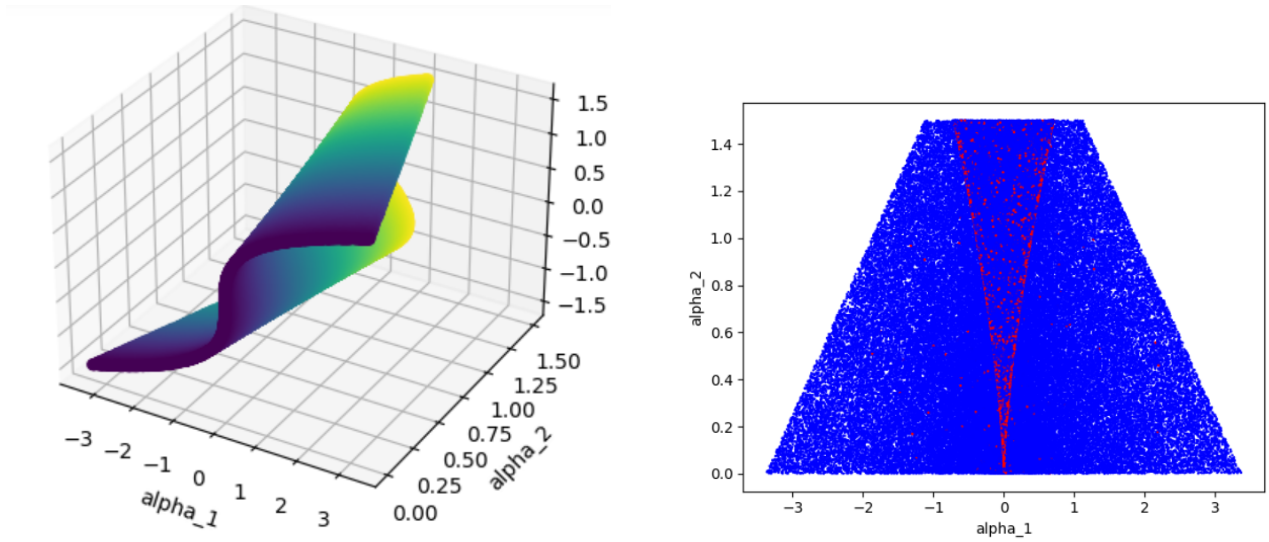
Short description of the setup in the report

The setup of the report involved analyzing the behavior of a system by varying the parameter $\boldsymbol{r}$ within a certain range. The system's behavior was visualized using a bifurcation diagram, which displayed the relationship between the parameter $\boldsymbol{r}$ and the system's qualitative dynamics. The analysis focused on observing the emergence of stable equilibrium points, the appearance of bifurcation points, the splitting of branches, and the onset of chaotic behavior as $\boldsymbol{r}$ increased. The aim was to understand the system's behavior and identify the changes in its dynamics as $\boldsymbol{r}$ varied.

Part 1

The logistic map

$$x_{n+1} = rx_n(1 - x_n)$$



(a) 3D-graph of cusp bifurcation (b) 2D-graph of cusp bifurcation

Figure 7: Cusp bifurcation

embodies a polynomial mapping, also known as a recurrence relation. It serves as a classic illustration of how remarkably intricate and chaotic dynamics can emerge from the humblest nonlinear equations.

Cobweb plots can be a useful tool for analyzing the behavior of iterative maps, such as the logistic map. They provide a visual representation of the iterations and can help in understanding the dynamics of the system. We will also use Cobweb plots to analyse.

Vary r from 0 to 2. Which bifurcations occur? At which numerical values do you find steady states of the system?

When varying the parameter $\boldsymbol{r}$ from 0 to 2 in the logistic map, the system exhibits various bifurcations and the emergence of steady states. Initially, at $\boldsymbol{r} = 0$, the only steady state is at $\boldsymbol{x} = 0$, which remains stable as $\boldsymbol{r}$ increases up to 1. At $\boldsymbol{r} = 1$, a bifurcation occurs(shown in Figure 12), leading to the appearance of a new stable steady state at $\boldsymbol{x}$ = 1 - $1/\boldsymbol{r}$,, independent of the initial population

When $0 < \boldsymbol{r} < 1$,As shown in Figure 8, the logistic map exhibits stable behavior. The Cobweb plot shows convergence to a single fixed point, which is the long-term behavior of the system. As we iterate the map, the points will gradually approach and settle at this fixed point.

When $1 < \boldsymbol{r} < 2$,As shown in Figure 9,the logistic map displays periodic behavior. The Cobweb plot reveals cycles or oscillations between multiple points. As we iterate the map, the points will follow a repeating pattern, visiting different values in a cyclic manner. The Cobweb plot will depict the progression of points along these cycles.

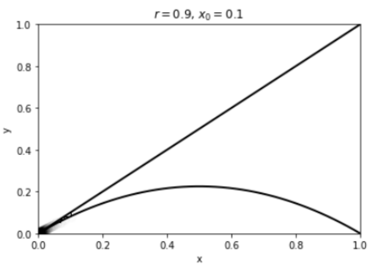


Figure 8: r=0.9, x_0=0.1

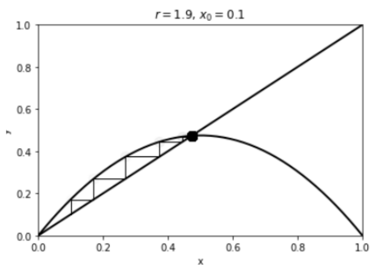


Figure 9: r=1.9, x_0=0.1

Now vary r from 2 to 4. What happens? Describe the behavior.

For r values between 2 and 3, the logistic map exhibits convergence towards a stable equilibrium point. The state approaches a particular value from both sides, displaying decaying oscillations. As r approaches 3, the convergence becomes slower, indicating a more gradual approach to the equilibrium state.

As r continues to increase, approximately in the range (3, 3.4], the population undergoes permanent oscillations between two values. These values are dependent on the specific r value chosen. Moreover, as r increases, the distance between these two limit values also increases. This behavior is notable on the bifurcation plot 12, where we can observe two branches.

Moving further, for r values approximately in the range (3.4, 3.5], the stable value of the oscillation splits into four branches. With each increase in r, the number of branches doubles, resulting in eight branches, sixteen branches, and so on. The length of the intervals between the branches becomes shorter as r increases.The trend of increasing complexity and the absence of regular patterns persist as r further increases .

When $2 < \boldsymbol{r} < 3$,As shown in Figure 10,the logistic map displays samiliar periodic behavior as in the range of $1 < \boldsymbol{r} < 2$.But first will fluctuate around that value for some time.

When $\boldsymbol{r} > 3$,As shown in Figure 10,the logistic map enters a chaotic regime. The Cobweb plot exhibits a highly intricate and unpredictable pattern. The points will no longer follow recognizable cycles or converge to fixed points. Instead, they will exhibit sensitive dependence on initial conditions, leading to a random-like behavior with no apparent order.

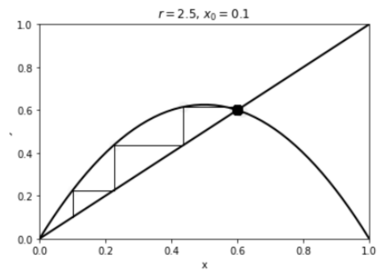


Figure 10: r=2.5, x_0=0.1

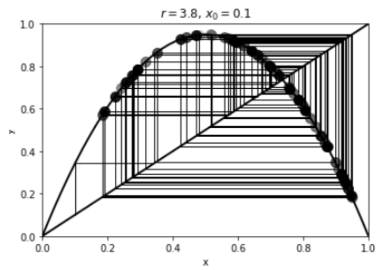


Figure 11: r=3.8, x_0=0.1

In summary, by varying r from 2 to 3 in the logistic map, we observe a transition from convergence towards a stable equilibrium with decaying oscillations to the emergence of permanent oscillations between two values. As r increases further, the bifurcation plot reveals an increasing number of branches, indicating a more intricate dynamics.

Plot a bifurcation diagram for r between 0 and 4

In the bifurcation diagram for the logistic map with parameter $\boldsymbol{r}$ ranging from 0 to 4, the diagram shows the behavior of the system as $\boldsymbol{r}$ changes. The vertical axis represents the system state $\boldsymbol{x}$, which ranges from 0 to 1.

As shown in the Figure 12,as $\boldsymbol{r}$ increased from 0 to 1, the bifurcation diagram revealed a smooth and continuous transition in the system's behavior. The system exhibited stable equilibrium points, and there were no significant qualitative changes observed.

However, as $\boldsymbol{r}$ increased further, dynamics started to emerge. The bifurcation diagram displayed various bifurcation points where the system underwent qualitative changes in behavior. These bifurcation points were characterized by the appearance of new branches or regions of oscillations.

In the range of $\boldsymbol{r}$ from approximately 2 to 3, the bifurcation diagram showed the emergence of periodic oscillations and the splitting of branches. As $\boldsymbol{r}$ continued to increase, the bifurcation diagram exhibited more complex behavior with the appearance of multiple branches and regions of chaotic dynamics.

At around $\boldsymbol{r} = 3.5$, the bifurcation diagram indicated the onset of chaotic behavior. The system exhibited sensitive dependence on initial conditions, and its long-term behavior became unpredictable and highly intricate.

Part 2

Visualize a single trajectory of the Lorenz system starting in this step,we need to visualize a single trajectory of the Lorenz system starting at the initial condition x0 = (10, 10, 10).The Lorenz system is a well-known example of a chaotic dynamical system, which is defined as :

$$\frac{dx}{dt} = \sigma(y - x) \tag{1}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{2}$$

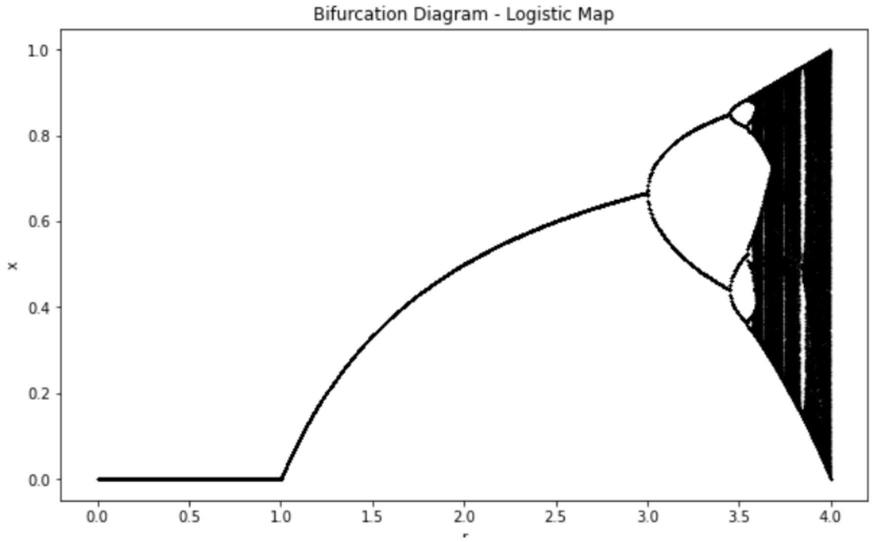


Figure 12: Bifurcation diagram with x and r

$$\frac{dz}{dt} = xy - \beta z \tag{3}$$

The system exhibits interesting dynamics characterized by sensitive dependence on initial conditions, a strange attractor, and chaotic behavior.By numerically solving the Lorenz equations using the provided initial condition (10, 10, 10), we obtain a trajectory that represents the evolution of the system over time. The trajectory is plotted in Figure 13, with time on the x-axis and the corresponding values of x, y, and z on the y-axis.

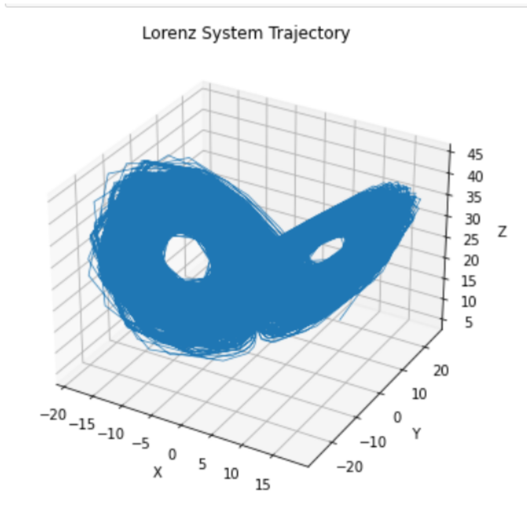


Figure 13: Lorenz System Trajectory

When visualizing the trajectory of the Lorenz system, we immediately notice the characteristic butterfly-shaped attractor that defines its behavior. As the trajectory evolves, it explores various regions of the attractor, displaying intricate and seemingly random patterns. This behavior demonstrates the system's sensitivity to initial conditions, where even minor changes in the starting point can result in vastly different trajectories over time. Such sensitivity highlights the chaotic nature of the Lorenz system, where small perturbations can have a profound impact on the long-term evolution of the system.

Test initial condition dependence by plotting another trajectory In this part, we examine the influence of initial conditions on the trajectory of the Lorenz system by plotting another trajectory. As the request, we slightly modify the initial condition of the system by adding a small perturbation. Specifically, we set the initial condition to

$$\hat{x_0} = (10 + 10^{-8}, 10, 10)$$

By solving the Lorenz system with the modified initial condition, we generate a new trajectory. Visualizing this trajectory14, we observe that it closely resembles the original trajectory for a certain period of time. However, as time progresses, the two trajectories start to diverge, with the differences becoming more pronounced.

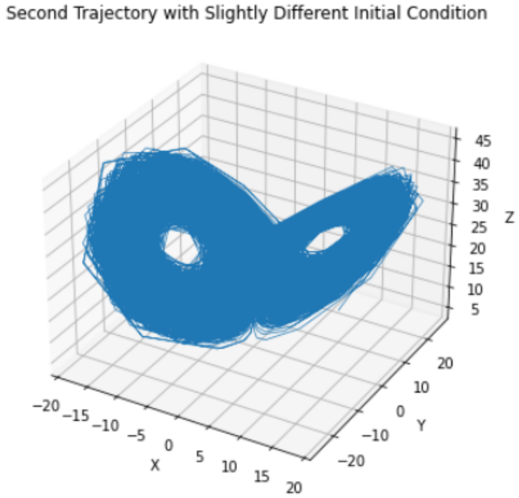


Figure 14: Second trajectory with slightly different initial condition

This behavior demonstrates the system's sensitivity to initial conditions. Even a minuscule change in the initial state can lead to significant divergence in the long run, resulting in entirely different trajectories. The test trajectory serves as a clear example of this sensitivity, highlighting the chaotic nature of the Lorenz system.

At what time is the difference between the points on the trajectory larger than 1? To determine the time at which the difference exceeds 1, we compute the difference between the two trajectories at each time step. Specifically, we calculate the Euclidean distance between the corresponding points on the trajectories and store these differences in an array.

By plotting the difference as a function of time(As shown in Figure 15 and Figure 16), we observe how it evolves over the simulation period. Initially, the difference remains small, indicating that the trajectories are relatively close. However, as time progresses, the difference starts to increase, reflecting the system's sensitivity to initial conditions and the amplification of small perturbations.

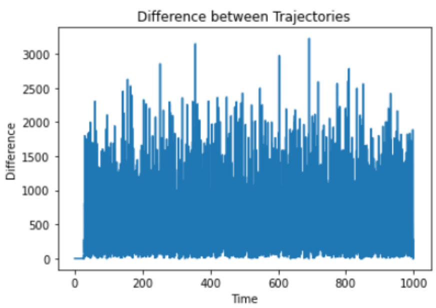


Figure 15: Difference between Trajectories

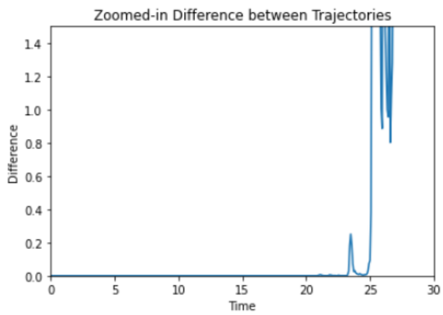


Figure 16: Zoomed-in Difference between Trajectories

We analyze the plot to identify the time at which the difference exceeds 1. We iterate through the difference array and find the first index where the value exceeds the threshold. This index corresponds to the time step at which the difference becomes larger than 1.

The obtained time 25.154737343278846 seconds indicates the point in the trajectory at which the divergence between the two trajectories reaches a significant magnitude. This result demonstrates the chaotic behavior of the Lorenz system, where even tiny differences in initial conditions can lead to substantial deviations in the long-term evolution.

Change to $\rho = 0.5$ and again compute and plot the two trajectories.

We analyze whether a bifurcation occurs by changing the value of ρ to 0.5 in the Lorenz system and computing and plotting the trajectories for the modified system.

To determine if there is a bifurcation, we need to compare the trajectories of the modified system with the original system. The corresponding trajectories are shown in Figure 17.According to the result shown in Figure 17, the trajectories are identical to each other.No bifurcation happens when $\rho = 0.5$.In this case, with $\rho = 0.5$, the system did not exhibit a distinct bifurcation. The trajectories showed smooth and continuous behavior without any evident branching or sudden shifts in stability.

But the trajectories are quite different from the previous.We observed that there is a significant change in the behavior of the system compared to the previous setting of $\rho = 28$. The trajectories now exhibit a different pattern and dynamics.When we set ρ to 0.5, the system displayed continuous and smooth behavior, indicating a different dynamical regime compared to the previous setting of $\rho = 28$.

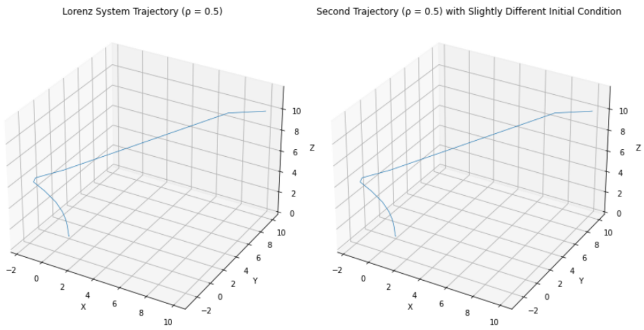


Figure 17: Second trajectory with slightly different initial condition

Report on task TASK 5, Bifurcations in crowd dynamics

Part1 & Part 2

In these two parts, we need to first complete the downloaded version of *sir_Model.py* and *sir_bed_model-unfinished.ipynb*, then try to run them. The given differential equations are shown followings:

$$\frac{dS}{dt} = A - \delta S - \frac{\beta SI}{S+I+R} \tag{4}$$

$$\frac{dI}{dt} = -(\delta + \upsilon)I - \mu(b, I)I + \frac{\beta SI}{S+I+R} \tag{5}$$

$$\frac{dR}{dt} = \mu(b, I)I - \delta R \tag{6}$$

So all we need to do is to replace those in the method "model" in *sir_Model.py* with the given equations (as shown in 18). In this way, the returned values will be correct.

```python
def model(t, y, mu0, mu1, beta, A, d, nu, b):
    """
    SIR model including hospitalization and natural death.

    Parameters:
    -----------
    mu0
        Minimum recovery rate
    mu1
        Maximum recovery rate
    beta
        average number of adequate contacts per unit time with infectious individuals
    A
        recruitment rate of susceptibles (e.g. birth rate)
    d
        natural death rate
    nu
        disease induced death rate
    b
        hospital beds per 10,000 persons
    """
    S,I,R = y[:]
    m = mu(b, I, mu0, mu1)


    # fill in the blank with the given differential equations
    dSdt = A - d * S - (beta * S * I) / (S + I + R)
    dIdt = - (d + nu) * I - m * I + (beta * S * I) / (S + I + R)
    dRdt = m * I - d * R

    return [dSdt, dIdt, dRdt]
```

Figure 18: completed version of model code

Another job we need to finish here is to optimize the structure of these files. In the given file *sir_bed_model-unfinished.ipynb*, some plotting functions are mixed with parameter settings and loop commands, so we move them into *sir_Model.py*, name them as **def plot_SIR_behaviour** and **def plot_SIR_trajectories**.

Finally, with completed the reconstructed files, we can successfully run them and get its results (shown in 19)

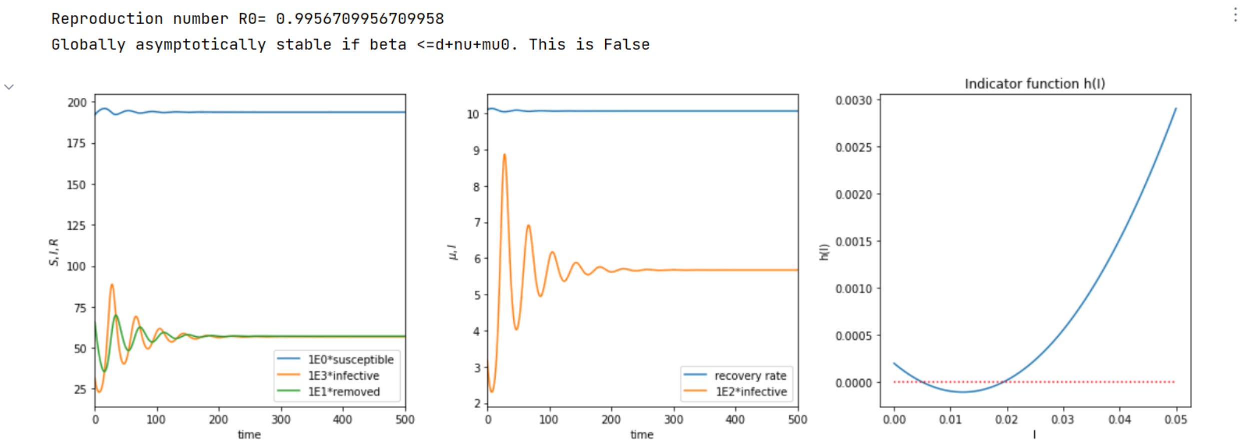


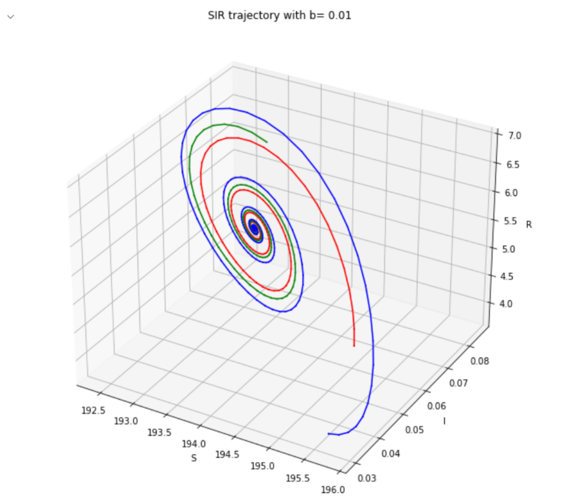
Figure 19: SIR variables over time, recovery rate against infective variable and indicator function

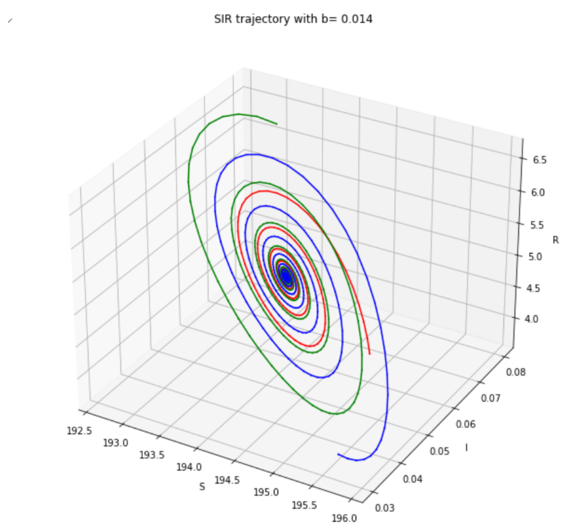
Part 3

Since the basic parameters are already set, in this part, we will only change the parameter **b** in order to see a special bifurcation. The range of **b** is [0.01, 0.03], while the changing step should be small enough (around 0.001). Considering at most nine plots are allowed to be shown here, we have decided to set the step to be 0.002, so that we have 11 plots in total, and make it easy for us to choose.

There are in total 3 starting points given, as shown below:

- SIR point 1: [195.3 0.052 4.40], with red trajectory.
- SIR point 2: [195.7 0.030 3.92], with blue trajectory.
- SIR point 3: [193.0 0.080 6.21], with green trajectory.





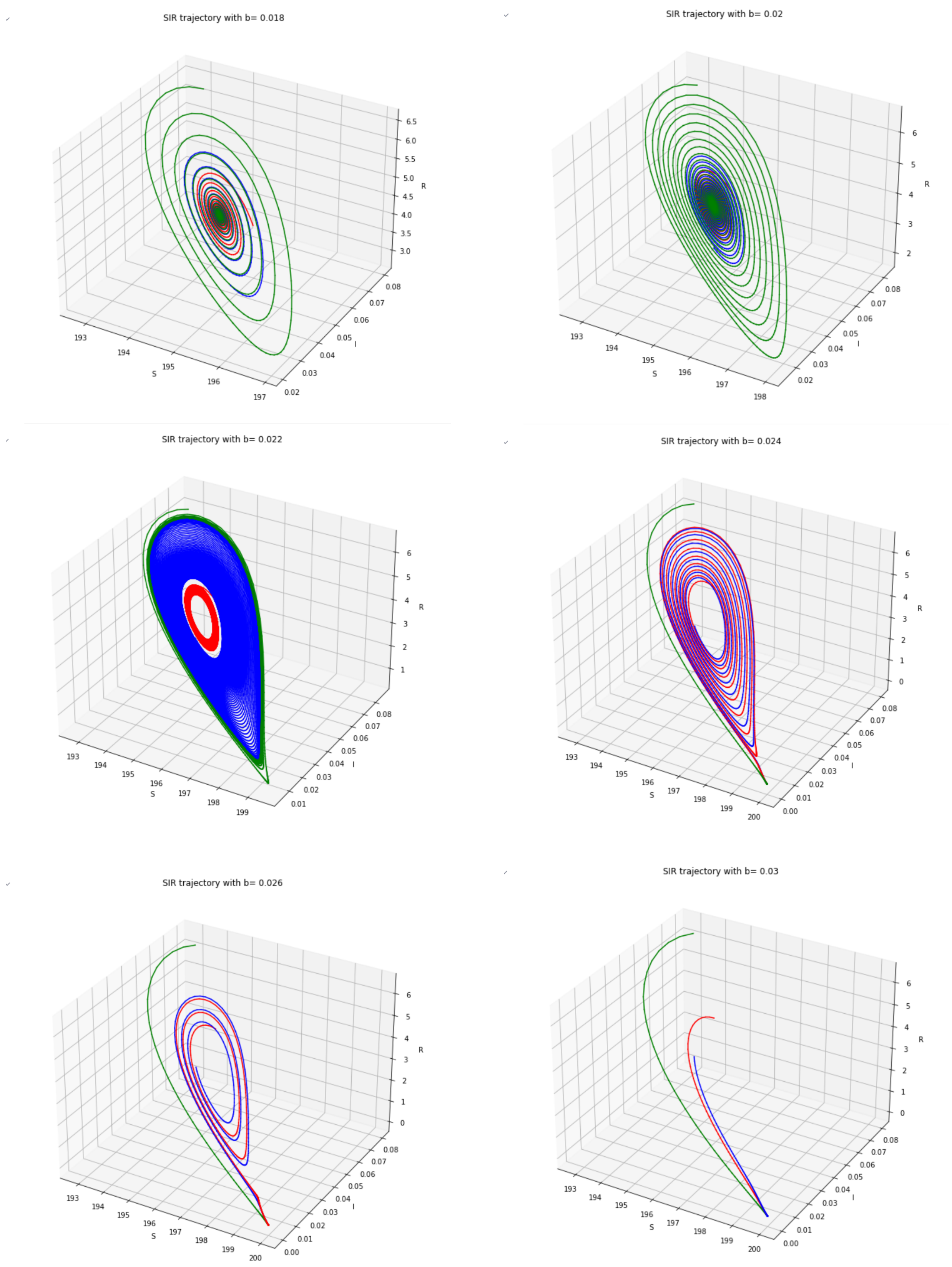


Figure 21: Trajectory of SIR with different value of b

By changing the parameter b, the results can be divided into three stages:

- when b $<$ 0.022: all three curves show the same steady state, surrounding around one point. And as the value of the parameter b increases, the stable focus around that point is getting more obvious.
- when b = 0.022: the bifurcation point shows up at this value of b. There is a weak stable point attracting the red curve while the other two curves being attracted to a stable limit cycle.
- when b $>$ 0.022: the system converges again around the point [200 0 0], which is exactly the disease free equilibrium point E_0. (Here with brief explanation, E_0 means dS, dI, dR and I should be 0. From it we get R = 0 and S = A/d, in our case, A = 20 and d = 0.1), that is:

  $E_0 = (A/d, 0, 0) = (20/0.1, 0, 0) = (200, 0, 0)$

Part 4

The phenomenon that is happening is called Hopf Bifurcation, with a exact value b = 0.022. From this points, the system is going from a local steady state to a limit cycle. From the given reading material of Kuznetsov [1], the normal form of a Hopf Bifurcationand can be written as followings:

$$\begin{aligned} \dot{x}_1 &= ax_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + ax_2 - x_2(x_1^2 + x_2^2) \end{aligned} \tag{7}$$

Part 5

Here in part 5, it is asked to analyse the reproduction rate $\mathbb{R}_0$.

$$\mathbb{R}_0 = \frac{\beta}{d + \upsilon + \mu_1} \tag{8}$$

Again, we can find more information about it in the given reading material of Chunhua Shan and Huaiping Zhu [2]. Here again we repeat the meanings of those variables used to represent the reproduction rate (they are given once in the question description):

- β: represents the average number of adequate contacts per unit time with infectious individuals.
- d: represents the per capita natural death rate.
- υ: represents the per capita disease-induced death rate.
- μ_1: represents the maximum recovery rate based on the number of available beds.

This reproduction rate indicates the extend of the infection that is happening among the population. Shortly to say, $\mathbb{R}_0$ determines how many people can one invective person infect. Combining with the variable β, it can be concluded that when β increases, $\mathbb{R}_0$ will also increase, thus more people can be infected by one, and when β decreases, same trend goes on $\mathbb{R}_0$ and less people will get infected by one.

Part 6

The theorem that "the disease free equilibrium E_0 = (A/d, 0, 0) at R0 $<$ 1 is an attracting node" first tells a truth that whenever the system state reaches this free equilibrium point, it will no longer change. Then for those values of the SIR variables that are near to this point (A/d, 0, 0), they will rapidly and finally reach there. With R0 $<$ 1, meaning the number of infected people is decreasing, and I = 0, meaning the final state doesn't exist infected people, this stable point shows that the virus die out.

But just as we have discussed in part 3, for $b > 0.022$, the system can surely converge to E_0, but when $b < 0.022$, it is still attracted by another local attractive point.(considering point (193, 0.08, 6.21) can still be called as a set of near values)

References

[1] Yuri A. Kuznetsov. Elements of Applied Bifurcation Theory. Springer New York, 2004.

[2] Chunhua Shan and Huaiping Zhu. Bifurcations and complex dynamics of an SIR model with the impact of the number of hospital beds. Journal of Differential Equations, 257(5):1662–1688, September 2014.