I.1 Multiplication Ax using Columns of A

- Matrix-vector Multiplication
- Column Space
- rank

Ex. ①. Multiply A times x using rows
②. Multiply using columns

$$A=\begin{bmatrix}2&3\\2&4\\3&7\end{bmatrix} \quad x=\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

①. $$\begin{bmatrix}2&3\\2&4\\3&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2x_1+3x_2\\2x_1+4x_2\\3x_1+7x_2\end{bmatrix}$$ ⇒ INNER PRODUCT of rows with x

②. $$\begin{bmatrix}2&3\\2&4\\3&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\begin{bmatrix}2\\2\\3\end{bmatrix}+x_2\begin{bmatrix}3\\4\\7\end{bmatrix}$$ ⇒ combination of columns a_1 and a_2

"INNER PRODUCT" ⇔ Dot Product
→ use for computing, not understanding

row·column = $(2,3)\cdot(x_1,x_2)=2x_1+3x_2$

Higher level: Ax is a "linear combination of a_1 and a_2".

Linear Combination

①. Multiply a_1 and a_2 by SCALARS x_1 and x_2
②. Add vectors $x_1a_1+x_2a_2=Ax$

Ax is a Linear combination of the columns of A

Column Space

- all combinations of the columns
- for all $x_1, x_2 \in \mathbb{R}$

Space includes Ax for all vectors x → infinitely many vectors Ax

Ex

$$\begin{bmatrix}2&3\\2&4\\3&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\underbrace{x_1\begin{bmatrix}2\\2\\3\end{bmatrix}+x_2\begin{bmatrix}3\\4\\7\end{bmatrix}}_{\text{Some vector in 3D space} - \mathbb{R}^3}$$

vectors with three complex components lie in $\mathbb{C}^3$.

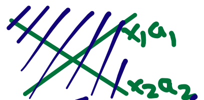
Key Qs:

All combinations $Ax=x_1a_1+x_2a_2$ produce what part of the full 3D space?

Ex

$$\begin{bmatrix}2&3\\2&4\\3&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\begin{bmatrix}2\\2\\3\end{bmatrix}+x_2\begin{bmatrix}3\\4\\7\end{bmatrix}$$

vectors $\begin{bmatrix}2\\2\\3\end{bmatrix}$ and $\begin{bmatrix}3\\4\\7\end{bmatrix}$ produce a plane.



⇒ INFINITE plane containing the two lines

Sum of any vector on one line plus any vector on the other line

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Def (column Space)

The Combinations of the columns fill out the column space of A.

$b=(b_1,b_2,b_3)$ is in the column space of A exactly when $Ax=b$ has a solution (x_1,x_2)

Notation: $C(A)$

Solution x: shows how to express b in as a combination of the columns.

Ex.

$b=\begin{bmatrix}1\\1\\1\end{bmatrix}$ is not in $C(A)$

$A=\begin{bmatrix}2&2\\2&4\\3&7\end{bmatrix}$. why?

$$\begin{bmatrix}2x_1+2x_2\\2x_1+4x_2\\3x_1+7x_2\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$ is unsolvable.

$$\left.\begin{aligned}2x_1+2x_2&=1\\2x_1+4x_2&=1\end{aligned}\right\} \quad x_1=\tfrac{1}{2},\ x_2=0$$

But. $3(\frac{1}{2})+7(0)\neq 1$

Ex.
what are the column spaces of
$$A_2=\begin{bmatrix}2&3&5\\2&4&6\\3&7&10\end{bmatrix}$$ and
$$A_3=\begin{bmatrix}2&3&1\\2&4&1\\3&7&1\end{bmatrix}$$

Independent Columns & Rank of A

- find a basis of the column space of A
- factor A into C times R
- prove the first great theorem of L.A.

Goal: create a matrix C whose columns comes directly from matrix A, excluding the redundant ones. (i.e. any column that is a combination of previous columns)

C has r columns ($r \le n$), they will be a "basis" for column space of A. (basic columns)

$$A_2 = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 7 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 6 \\ 10 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$$

So $\begin{bmatrix} 5 \\ 6 \\ 10 \end{bmatrix}$ is dependent and ADDS NOTHING NEW.

$C(A_2)$ = plane formed by vectors $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$

$$A_3 = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$$

$C(A_3)$ = the whole 3D space $\mathbb{R}^3$

ALL possible column spaces inside $\mathbb{R}^3$:
subspaces of $\mathbb{R}^3$

- The zero vector $(0, 0, 0)$
- The line of all vectors $x_1 a_1$
- The plane of all vectors $x_1 a_1 + x_2 a_2$
- The whole $\mathbb{R}^3$ with all vectors $x_1 a_1 + x_2 a_2 + x_3 a_3$

we need a_1, a_2, a_3 to be independent. the only combination that gives the zero vector is $0a_1 + 0a_2 + 0a_3$

In linear Algebra:
Def (Invertible matrix)

- Three independent columns in $\mathbb{R}^3$ produce an invertible matrix: $AA^{-1} = A^{-1}A = I$
- $Ax = 0$ requires $x = (0, 0, 0)$. Then $Ax = b$ has exactly one solution $x = A^{-1}b$.

For an n by n invertible matrix, the combinations fill its column space: all of $\mathbb{R}^3$!

Def (Basis)

The basis for a subspace is a full set of independent vectors: All vectors in the space are combinations of the basis vectors.

Ex

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}_{n=3} \quad \text{Find } C(A).$$

$$\begin{bmatrix} 8 \\ 6 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}_{r=2} \qquad r \le n \checkmark$$

Ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}_{n=3} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}_{r=3}$$

Ex

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}_{n=3} \quad C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{r=1}$$

Def (r)

r is the RANK of A. It counts independent columns.

Dimension of the column space of both A and C (same space)

Def (rank)

The rank of a matrix is the dimension of its column space.
(rank(A) = dim(im(A)))

Factorization of A

Matrix C connects to A by a third matrix R.

$$\underset{m\times n}{A} = \underset{m\times r}{C}\,\underset{r\times n}{R}$$

Ex.

C — independent columns

R — rref(A) (without 0 rows)

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 8 \\ 6 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \underset{r=1}{\begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}} = \underset{r=1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}\underset{r=1}{\begin{bmatrix} 1 & 2 & 5 \end{bmatrix}}$$

Column rank = Row rank.

Fact: The rank Theorem

independent columns = # independent rows

matrix R has r rows. Multiply by C takes combination of those rows.

Two ways to look at CR:

$$\begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} 2 \\ 2 \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{} & & \end{bmatrix}$$

$1+0$

combination of the columns

$$\begin{bmatrix} \boxed{1 \quad 3} \\ 1 \quad 2 \\ 0 \quad 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \boxed{} \\ \\ \end{bmatrix}$$

1, 3

Rows of R are independent, and they form the Row Space of A.

- Column space and row space both have dim = r
- r basis vectors are: ①. Columns of C
  ②. Rows of R

Ex.

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \boxed{1 \quad 0 \quad 2} \\ \boxed{0 \quad 1 \quad 2} \end{bmatrix}$$

Independent.
(1's and 0's)
→ No row is a combination of the other rows.

"SVD"
↳ big factorization for Data Science.

- C → r orthogonal columns
- R → r orthogonal rows.