ACM 104 Homework #1

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① $AB = C$ where $c_{ij} = \sum_{k=1}^{p} a_{ik} \cdot b_{kj}$ by def. of matrix multiplication.

Observe that $c_j = \sum_{k=1}^{p} a^k \cdot b_{kj}$ hence $C = \left[c_1 \; c_2 \; \dots \; c_n \right]$

$$= \left[\sum_{k=1}^{p} a^k \cdot b_{k1} \; \dots \; \sum_{k=1}^{p} a^k \cdot b_{kn} \right]$$

Instead of having summation for every column,
we can rewrite C as sum of matrices, by def. of matrix addition:

$$C = \sum_{k=1}^{p} \left[a^k \cdot b_{k1} \; \cdots \; a^k \cdot b_{kn} \right]$$

$$= \sum_{k=1}^{p} a^k \cdot b_k \quad \text{by def. of matrix multiplication}$$

QED

② Let $A \in M_{n \times n}$ be a strictly upper triangular matrix.

Then, by def. of strictly up. triang. matrices:

$$a_{ij} = \begin{cases} 0 & \text{if } i \geq j \\ \mathbb{R} & \text{if } i < j \end{cases}$$

Claim: If A is strictly upper triangular,
then for A^k, $k \geq 1$:

$$a_{ij}^k = \begin{cases} 0 & \text{if } i \geq j-k+1 \\ \mathbb{R} & \text{if } i < j-k+1 \end{cases}$$

Clearly, for $k = n$ $i \geq j-k+1$ will always be true,
hence A is nilpotent.

Note: if A already has 0 diagonals above $i=j$, k can be less than n and still show nilpotency.

Proof:

Show that for A^k, $i \geq j-k+1 \Rightarrow a_{ij}=0$.

Base case: $k=1$, trivial, claim holds by definition of A.

Inductive hypothesis (I.H.): Assume claim holds for $k=p$,
i.e. A^p, $i \geq j-p+1 \Rightarrow a_{ij}=0$

Induction step: Given IH, prove that claim holds for $k=p+1$.
Let $A^p = B$
Then: $A^{p+1} = A \cdot A^p = A \cdot B = C$ where $c_{ij} = \sum_{r=1}^{n} a_{ir} \cdot b_{rj}$

Observe that:

1) $a_{ir}=0$ for $r \leq i$, by def of A
2) $b_{rj}=0$ for $r \geq j-p+1$, by I.H.

Hence for $i \geq j-p$:

~~a) when i>j-p, cij=0 because for any~~

For any $r \in \{1, \dots, n\}$ $c_{ij}=0$ because either 1 or 2 will be true.

In the worst case, $i=j-p$; 1 and 2 do not overlap but still result in $c_{ij}=0$ because r, i, j are integers and 1 and 2 cover the whole domain $\{1, \dots, n\}$.

QED

Conclusion: Claim holds for any $k \geq 1$.

③ $P_n = I_n$

$$L_n = \begin{bmatrix} 1 & & & & 0 \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & -\left(\frac{n-1}{n}\right) & 1 \end{bmatrix} \Rightarrow \text{for all } i=j+1 \Rightarrow l_{ij} = -\left(\frac{i-1}{i}\right)$$

$i=j \Rightarrow l_{ij}=1$

otherwise $l_{ij}=0$

$$U_n = \begin{bmatrix} \frac{2}{1} & -1 & & & 0 \\ & \frac{3}{2} & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ 0 & & & & \frac{n+1}{n} \end{bmatrix} \Rightarrow \text{for all } i=j \Rightarrow u_{ij} = \frac{j+1}{j}$$

$i=j-1 \Rightarrow u_{ij}=-1$

otherwise $u_{ij}=0$

Demonstration:

$$P_n A_n = L_n U_n$$

$P_n = I_n \Rightarrow A_n = L_n U_n$

Let $L_n U_n = C_n$

Then $c_{ij} = \sum_{k=1}^{n} l_{ik} \cdot u_{kj}$

case $i=j=1 \Rightarrow C_{11} = \sum_{k=1}^{n} l_{i1} \cdot u_{ij} = 2$ (trivial)

case $i=j\neq 1 \Rightarrow$ Note that the 2 non-zero values in each vector overlap, so we get:

$$c_{ij} = -\left(\frac{i-1}{i}\right)\cdot(-1) + (1)\cdot\left(\frac{j+1}{j}\right)$$

$$= +1 - \cancel{\frac{1}{i}} + 1 + \cancel{\frac{1}{j}} \quad \text{since } i=j$$

$= 2 \Rightarrow C_n$ has 2s on the diagonal case $i=j-1 \Rightarrow$ Note that overlap only occurs on one non-zero value:

$$c_{ij} = (-1)\cdot 1 = -1 \Rightarrow -1\text{s on super-diagonal in } C_n$$

case $i=j+1 \Rightarrow$ Overlap only occurs on one value:

$$c_{ij} = -\left(\frac{i-1}{i}\right)\left(\frac{j+1}{j}\right)$$

$$= -\left(\frac{\cancel{j}}{\cancel{j+1}}\right)\left(\frac{\cancel{j+1}}{\cancel{j}}\right) \quad \text{since } i=j+1$$

$$= -1 \Rightarrow -1\text{s on sub-diagonal in } C_n$$

For $i > j+1$ and $i < j-1$ there is no ~~overl~~ overlap so $c_{ij} = 0$.

$\therefore C_n = A_n \Rightarrow P_n A_n = L_n U_n$ Q.E.D

Let P be a permutation matrix.

(4) a) Permutation matrix represents a set of elementary row operations of type 2. Each of these operations swaps rows of the matrix its applied to, hence to undo its effect the permutation matrix can be applied again, hence $P = P^{-1}$

$$\text{e.g. } P = \begin{array}{cc} \begin{matrix} \quad & \quad & i & \quad & j & \quad \end{matrix} & \\ \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ \cdots & \cdots & 0 & \cdots & 1 & \cdots \\ & & & 1 & & \\ \cdots & \cdots & 1 & \cdots & 0 & \cdots \\ & & & & & 1 \end{bmatrix} & \begin{matrix} \\ \\ i \\ \\ j \\ \\ \end{matrix} \end{array}$$

swapping rows i and j

Clearly, $P = P^T$

Since $P = P^{-1} = P^T$, P is orthogonal.

b) No, counter example:

$$\text{Let } A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Clearly, $A = A^T$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Hence $A = A^T = A^{-1}$ but A is not a permutation matrix as it has values not equal to one or zero.

⑤ Let $A \in M_{n \times n}$ be some matrix.

Construct S such that:

$$s_{ij} = \begin{cases} a_{ij} & \text{if } i = j \\ \dfrac{a_{ij} + a_{ji}}{2} & \text{if } i \neq j \end{cases}$$

Construct J such that:

$$j_{ij} = \begin{cases} 0 & \text{if } i = j \\ \dfrac{a_{ij} - a_{ji}}{2} & \text{if } i \neq j \end{cases}$$

Note that S is symmetric because it takes the average of a_{ij} and a_{ji}, which will always be the same, i.e. $s_{ij} = s_{ji}$.
Note also that J is skew-symmetric because it uses the difference of the value and the average for every pair a_{ij} and a_{ji}, i.e. $j_{ij} = -j_{ji}$

Let $C = S + J$, then
$$c_i = \begin{cases} a_{ij} & \text{if } i = j \\ \dfrac{a_{ij} + a_{ji}}{2} + \dfrac{a_{ij} - a_{ji}}{2} = \dfrac{2a_{ij}}{2} = a_{ij} & \text{if } i \neq j \end{cases}$$

Hence $C = A$ QED

6 See attached plot and solution file.

7 a) Note that the difference between any 2 adjacent columns is: $\left.\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right\} n$

Hence 2 ~~...~~ columns from ~~...~~ A can be used to express other columns:

Example formula: $a_1 + k(a_2 - a_1)$ where $k \in \{0, \dots, n-1\}$

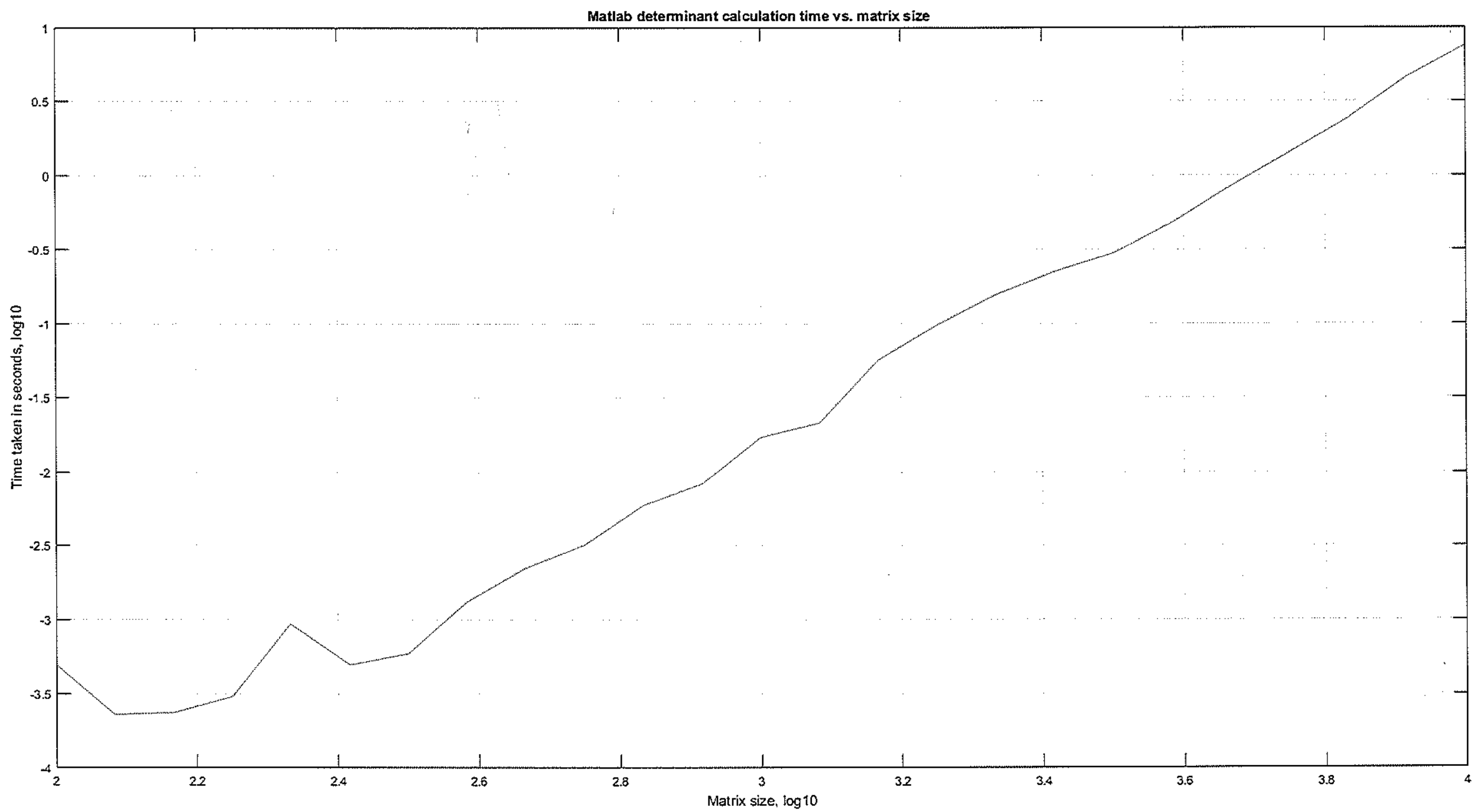
$(= (1-k)a_1 + ka_2)$

Hence any 2 columns can be used to span the span of all columns in A, hence $\text{rank}(A) = 2$

b) See attached solution file for matlab code.

The solution is $x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0.01 \end{bmatrix}$, where the first $n-1$ entries are 0.

Hence $x_{n-1} = 0$ and $\underbrace{x_n = 0.01}_{\text{The only non-zero component.}}$,


Matlab determinant calculation time vs. matrix size
Time taken in seconds, log10
Matrix size, log10
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
2
2.2
2.4
2.6
2.8
3
3.2
3.4
3.6
3.8
4