ACM104 Problem Set #4 Solutions

Timur Kuzhagaliyev

November 3, 2017

Problem 1

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 0 \\ 5 \\ 6 \\ 8 \end{bmatrix}$$

$$\begin{aligned} p(x) = \|Ax - b\|^2 &= x^T(A^TA)x - 2\ x^T(A^Tb) + \|b\|^2 \\ &= x^TKx - 2xf + c \end{aligned}$$

$$K = A^T \cdot A = \begin{bmatrix} 1 & 0 & 1 & -3 \\ 2 & -2 & 5 & 1 \\ -1 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 & -5 \\ 4 & 34 & -12 \\ -5 & -12 & 12 \end{bmatrix}$$

$$f = A^Tb = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} -18 \\ 28 \\ 17 \end{bmatrix}$$

First we need to check if the matrix K is positive definite. We know that matrix is positive definite if all of its principal minors are positive. Indeed, $\det A_1 = 11 > 0$; $\det A_2 = 11 \cdot 34 - 4^2 = 18 > 0$; $\det A_3 = 11 \cdot 264 - 4 \cdot (-12) - 5 \cdot 122 = 2904 + 48 - 610 = 2342 > 0$; Therefore K is positive definite and there exists a global minimizer $x^* = K^{-1}f$.

$$
\begin{aligned}
x^* = K^{-1}f = Kf &= \begin{bmatrix} 11 & 4 & -5 \\ 4 & 34 & -12 \\ -5 & -12 & 12 \end{bmatrix} \begin{bmatrix} -18 \\ 28 \\ 17 \end{bmatrix} \\
&= \frac{1}{2342} \begin{bmatrix} 264 & 12 & 122 \\ 12 & 107 & 112 \\ 122 & 112 & 358 \end{bmatrix} \begin{bmatrix} -18 \\ 28 \\ 17 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \quad \text{(solution)} \\
\text{LSE} = \sqrt{\|b\|^2 - b^T A x^*} &= \sqrt{\sqrt{125}^2 - 125} \\
&= 0
\end{aligned}
$$

The least squares error is 0, so in this case x^* is the exact solution.

Problem 2

WIP

Problem 3

$$
y = f(x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 \quad \text{and} \quad \beta^* = (\beta_0^*, \beta_1^*, \beta_2^*, \beta_3^*)^T
$$

a) Deriving a system of normal equations on β^*:

$$
\begin{aligned}
r_i &= y^{(i)} - f(x_1^{(i)}, x_2^{(i)}) \\
&= y^{(i)} - \begin{bmatrix} 1 & x_1^{(i)} & x_2^{(i)} & x_1^{(i)} x_2^{(i)} \end{bmatrix} \begin{bmatrix} \beta_0^* \\ \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{bmatrix}
\end{aligned}
\quad \Rightarrow \quad
r = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ \vdots \\ y^{(m)} \end{bmatrix} - \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} & x_1^{(1)} x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & x_1^{(2)} x_2^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_1^{(m)} & x_2^{(m)} & x_1^{(m)} x_2^{(m)} \end{bmatrix} \begin{bmatrix} \beta_0^* \\ \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{bmatrix}
$$

Problem 4

Problem 5