ACM104 Problem Set #2 Solutions

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Problem 1

a) In this case, W is not a subspace of V. Consider 2 matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

Note that $A, B \in V$ since $\det A = 1 * 0 - 1 * 0 = 0$ and $\det B = 0 * 2 - 1 * 0 = 0$. Consider the sum:

$$A + B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$\det(A + B) = 1 * 2 - 1 * 1 = 1 \neq 0$ hence $A + B \notin W$, violating the subspace property.

b) W is a subspace of V as its closed under matrix addition and multiplication by a scalar. Zero vector is the zero matrix, as $\operatorname{tr} 0 = \sum_{i=1}^{n} 0 = 0$. Proof:

$$A, B \in W \quad \operatorname{tr}(A + B) = \operatorname{tr}A + \operatorname{tr}B = 0 + 0 = 0 \quad \Rightarrow \quad A + B \in W$$

$$A \in W \;\; \alpha \in \mathbb{R} \quad \operatorname{tr}(\alpha \cdot A) = \sum_{i=1}^{n} \alpha \cdot a_{ii} = \alpha \sum_{i=1}^{n} \cdot a_{ii} = \alpha \cdot 0 = 0 \quad \Rightarrow \quad \alpha \cdot A \in W$$

$A, B \in W \;\; \alpha, \beta \in \mathbb{R}$

$$\operatorname{tr}(\alpha A + \beta B) = \sum_{i=1}^{n} \alpha \cdot a_{ii} + \sum_{i=1}^{n} \beta \cdot b_{ii} = \alpha \sum_{i=1}^{n} \cdot a_{ii} + \beta \sum_{i=1}^{n} \cdot b_{ii} = \alpha \cdot 0 + \beta \cdot 0 = 0 \quad \Rightarrow \quad \alpha A + \beta B \in W$$

c) W is not a subspace. Let $g(x) = 2 - 1.5x$, then $g(x) \in W$ since $g(0)g(1) = 2 * 0.5 = 1$. Now $h(x) = \alpha g(x)$ with some $\alpha \in \mathbb{R},\ \alpha \neq 1$. Observe that $h(0)h(1) = \alpha g(0) \alpha g(1) = \alpha^2 \cdot 1 = \alpha^2 \neq 1$. Therefore $h(x) \notin W$, violating subspace property.

d) W is a subspace of V since it's closed under addition and scalar multiplication. Zero vector is $f(x) = 0$, clearly $f(\frac{1}{2}) = 0$ and $\int_0^1 f(t)\ dt = \int_0^1 0\ dt = 0$. Proof:

$$f(x), g(x) \in W \quad \int_0^1 (f(t) + g(t))dt = \int_0^1 f(t)dt + \int_0^1 g(t)dt = f(0.5) + g(0.5) \quad \Rightarrow \quad f(x) + g(x) \in W$$

$$f(x) \in W \ \alpha \in \mathbb{R} \quad \int_0^1 \alpha \cdot f(t)dt = \alpha \int_0^1 f(t)dt = \alpha \cdot f(0.5) \quad \Rightarrow \quad \alpha \cdot f(x) \in W$$

$f(x), g(x) \in W\ \alpha, \beta \in \mathbb{R}$

$$\begin{aligned}\int_0^1 (\alpha \cdot f(t) + \beta \cdot g(t))dt &= \int_0^1 \alpha \cdot f(t)dt + \int_0^1 \beta \cdot g(t)dt \\ &= \alpha \int_0^1 f(t)dt + \beta \int_0^1 g(t)dt \\ &= \alpha \cdot f(0.5) + \beta \cdot g(0.5) \quad \Rightarrow \quad \alpha \cdot f(t) + \beta \cdot g(t) \in W\end{aligned}$$

e) W is a subspace of V since it's closed under addition and scalar multiplication. Zero vector is $v(x, y) = [0\ 0]^T$, which clearly belongs to W as derivative of 0 is also 0. Proof:

$$v(x, y), u(x, y) \in W \quad v(x, y) + u(x, y) = \begin{bmatrix} v_1(x, y) + u_1(x, y) \\ v_2(x, y) + u_2(x, y) \end{bmatrix}$$

$$\Downarrow$$

$$\nabla \cdot (v + u) = \frac{\delta(v_1 + u_1)}{\delta x} + \frac{\delta(v_2 + u_2)}{\delta y} = \frac{\delta v_1}{\delta x} + \frac{u_1}{\delta x} + \frac{\delta v_2}{\delta y} + \frac{\delta u_2}{\delta y} = (\frac{\delta v_1}{\delta x} + \frac{\delta v_2}{\delta y}) + (\frac{u_1}{\delta x} + \frac{\delta u_2}{\delta y}) = 0 + 0 = 0$$

$$\Downarrow$$

$$v + u \in W$$

$$v(x, y) \in W\ \alpha \in \mathbb{R} \quad \alpha \cdot v(x, y) = \begin{bmatrix} \alpha \cdot v_1(x, y) \\ \alpha \cdot v_2(x, y) \end{bmatrix}$$

$$\Downarrow$$

$$\nabla \cdot (\alpha \cdot u) = \frac{\delta(\alpha \cdot v_1)}{\delta x} + \frac{\delta(\alpha \cdot v_2)}{\delta y} = \frac{\alpha \cdot \delta(v_1)}{\delta x} + \frac{\alpha \cdot \delta(v_2)}{\delta y} = \alpha \cdot (\frac{\delta(v_1)}{\delta x} + \frac{\delta(v_2)}{\delta y}) = \alpha \cdot 0 = 0$$

$$\Downarrow$$

$$\alpha \cdot v \in W$$

Problem 2

a) p_1, p_2, p_3 can be combined into a matrix $[p_1\ p_2\ p_3]$. We can find the row-echelon form of this matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ -3 & 2 & 1 \end{bmatrix} \xrightarrow{r_3+3r_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{r_3+2r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

The rank of the matrix is equal to the number of columns (vectors used to compose it), hence p_1, p_2, p_3 are linearly independent.

b) The dimension of $\mathcal{P}^{(2)}$ is 3 since it depends on 3 variables (coefficients). Since we have exactly 3 vectors p_1, p_2, p_3 and they are all linearly independent, they should span the entire $\mathcal{P}^{(2)}$.

c) By a) and b), p_1, p_2, p_3 are linearly independent and $span(p_1, p_2, p_3) = \mathcal{P}^{(2)}$, hence they form a basis of $\mathcal{P}^{(2)}$ by definition. The coordinates of $q(x) = 1$ in basis p_1, p_2, p_3 are $(-\frac{1}{8}, \frac{1}{4}, \frac{1}{8})$.

Problem 3

a) Note that for any f, x_1 and x_2 are always free. Hence we only need to prove that $x_n = x_{n-1} + x_{n-2}$ holds for addition and multiplication by scalar. Clearly, $\mathcal{F}$ is a vector space under given operations:

$$f_1, f_2 \in \mathcal{F} \qquad f_1 + f_2 = (x_1 + y_1, x_2 + y_2, ...) = (z_1, z_2, ...)$$

$$\begin{aligned} z_n &= z_{n-1} + z_{n-2} \\ &= (x_{n-1} + y_{n-1}) + (x_{n-2} + y_{n-2}) \\ &= (x_{n-1} + x_{n-2}) + (y_{n-1} + y_{n-2}) \\ &= x_n + y_n \quad \Rightarrow \quad f_1 + f_2 \in \mathcal{F} \end{aligned}$$

$$f \in \mathcal{F},\ \alpha \in \mathbb{R} \qquad \alpha \cdot f = (\alpha \cdot x_1, \alpha \cdot x_2, ...) = (z_1, z_2, ...)$$

$$\begin{aligned} z_n &= z_{n-1} + z_{n-2} \\ &= \alpha \cdot x_{n-1} + \alpha \cdot x_{n-2} \\ &= \alpha \cdot (x_{n-1} + x_{n-2}) \\ &= \alpha \cdot x_n \quad \Rightarrow \quad \alpha \cdot f \in \mathcal{F} \end{aligned}$$

$$f_1, f_2 \in \mathcal{F},\ \alpha, \beta \in \mathbb{R} \qquad \alpha \cdot f_1 + \beta \cdot f_2 = (\alpha \cdot x_1 + \beta \cdot y_1, \alpha \cdot x_2 + \alpha \cdot y_2, ...) = (z_1, z_2, ...)$$

$$\begin{aligned} z_n &= z_{n-1} + z_{n-2} \\ &= (\alpha \cdot x_{n-1} + \beta \cdot y_{n-1}) + (\alpha \cdot x_{n-2} + \beta \cdot y_{n-2}) \\ &= (\alpha \cdot x_{n-1} + \alpha \cdot x_{n-2}) + (\beta \cdot y_{n-1} + \beta \cdot y_{n-2}) \\ &= \alpha \cdot (x_{n-1} + x_{n-2}) + \beta \cdot (y_{n-1} + y_{n-2}) \\ &= \alpha \cdot x_n + \beta \cdot y_n \quad \Rightarrow \quad \alpha \cdot f_1 + \beta \cdot f_2 \in \mathcal{F} \end{aligned}$$

b) As mentioned above, each $f \in \mathcal{F}$ has 2 free parameters on which all other parameters depend: x_1 and x_2. Hence $\mathcal{F}$ has two degrees of freedom and its dimension is 2. Two vectors that can form a basis of $\mathcal{F}$ are $f_1 = (1, 0, 1, 1, 2, ...)$ and $f_2 = (0, 1, 1, 2, 3, ...)$.

c) The coordinates of the original f^* in the basis f_1, f_2 defined above are $(1, 1)$.

Problem 4

Denote columns of A as $[c_1\ c_2\ \ldots\ c_n]$. Note that the difference between any two adjacent columns is the vector $(\{1\}^n)^T$. Hence any column c_i in A can be expressed as :

$$c_i = c_1 + k \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{where} \quad k = 0, \ldots, n-1$$

Note that $c_2 - c_1 = (\{1\}^n)^T$, so the same equation can be rewritten as $c_i = c_1 + k \cdot (c_2 - c_1) = (1-k) \cdot c_1 + k \cdot c_2$. Clearly, c_1 and c_2 are linearly independent and $\text{span}(c_1, c_2) = \text{span}(c_1, c_2, \ldots, c_n)$. By definition, $\text{im}A = \text{span}(c_1, c_2)$ hence c_1, c_2 form a basis of the image of A. Note that c_1 and c_2 have no particular significance and any other 2 columns can be chosen.

Similar argument applies for the basis of the coimage of A. Consider columns of A^T, which are essentially transposed rows of A: $\left[r_1^T\ r_2^T\ \ldots\ r_n^T\right]$. This time, the difference between each column is $(\{n\}^n)^T$ and they can be expressed as $r_i^T = (1-k) \cdot r_1^T + k \cdot r_2^T$. Following a similar argument, r_1^T, r_2^T span the entire column space of A^T and since they are linearly independent, r_1^T, r_2^T form a basis of the coimage of A.

To compute a basis for the kernel of A, we can start by finding its row-echelon form. One can begin by starting at the bottom row and subtracting the row above from the current row. When repeated $n-1$ times going up the matrix, this process will leave us with a matrix that has the same r_1 as A but has n everywhere else. Clearly, we can obtain zeroes on all rows below row 2 by subtracting row 2 from them. Finally, we can subtract row 1 fom row 2 n times to end up with the following matrix in row-echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & \ldots & n \\ 0 & -n & -2n & \ldots & n-n^2 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

To obtain the reduced row-echelon form, divide row 2 by $-n$ and subtract row 2 from row 1 twice:

$$\begin{bmatrix} 1 & 0 & -1 & \ldots & 2-n \\ 0 & 1 & 2 & \ldots & n-1 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

We have two basic variables (pivots) and $n-2$ free variables which we can use to express the basic variables. Denote basic variables as x_1, x_2 and free variables as $x_3, \ldots, x_n$. then we can express a general solution for $Ax = 0$ as:

$$x = \begin{bmatrix} \sum_{i=3}^{n}(2-i) * x_i \\ \sum_{i=3}^{n}(i-1) * x_i \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where} \quad x_3, \dots, x_n \in \mathbb{R}$$

Generate $n-2$ vectors (indexed $k = 3, \dots, n$) using the general solution above such that for each vector $x_k = 1$ and all other free variables are 0. These vectors will be linearly independent and they will span the whole kernel of A, hence forming a basis of kernel of A.

Similar argument applies to the basis of the cokernel of A. Repeat the same process with getting the row-echelon form, now using A^T and as a final step, subtract row 1 from row 2 once instead of n times:

$$\begin{bmatrix} 1 & n+1 & 2n+1 & \dots & n^2-n+1 \\ 0 & -n & -2n & \dots & n-n^2 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Divide row 2 by $-n$ and subtract it from row 1 $n+1$ times:

$$\begin{bmatrix} 1 & 0 & -1 & \dots & 2-n \\ 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Note that we end up with exactly the same reduced row-echelon form, hence we can conclude that the basis of kernel of A defined above is also a basis of cokernel of A.

1 Problem 5

See attached Matlab script.