ACM104 Problem Set #4 Solutions

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Problem 1

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 5 \\ 6 \\ 8 \end{bmatrix}$$

$$p(x) = \|Ax - b\|^2 = x^T(A^TA)x - 2\,x^T(A^Tb) + \|b\|^2$$
$$= x^T K x - 2xf + c$$

$$K = A^T \cdot A = \begin{bmatrix} 1 & 0 & 1 & -3 \\ 2 & -2 & 5 & 1 \\ -1 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 & -5 \\ 4 & 34 & -12 \\ -5 & -12 & 12 \end{bmatrix}$$

$$f = A^T b = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} -18 \\ 28 \\ 17 \end{bmatrix}$$

First we need, to check if the matrix K is positive definite. We know that matrix is positive definite if all of its principal minors are positive. Indeed, $\det A_1 = 11 > 0$; $\det A_2 = 11 \cdot 34 - 4^2 = 18 > 0$; $\det A_3 = 11 \cdot 264 - 4 \cdot (-12) - 5 \cdot 122 = 2904 + 48 - 610 = 2342 > 0$; Therefore K is positive definite and there exists a global minimizer $x^* = K^{-1}f$.

$$x^* = K^{-1}f = Kf = \begin{bmatrix} 11 & 4 & -5 \\ 4 & 34 & -12 \\ -5 & -12 & 12 \end{bmatrix}\begin{bmatrix} -18 \\ 28 \\ 17 \end{bmatrix}$$

$$= \frac{1}{2342}\begin{bmatrix} 264 & 12 & 122 \\ 12 & 107 & 112 \\ 122 & 112 & 358 \end{bmatrix}\begin{bmatrix} -18 \\ 28 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \quad \text{(solution)}$$

$$\text{LSE} = \sqrt{\|b\|^2 - b^T A x^*} = \sqrt{\sqrt{125}^2 - 125}$$

$$= 0$$

The least squares error is 0, so in this case x^* is the exact solution.

Problem 2