ACM104 Problem Set #3 Solutions

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October 22, 2017

Problem 1

a) It is possible to express the inner product using the norm:

$$\begin{aligned}
\|u+v\|^2 = \langle u+v, u+v\rangle &= \langle u, u+v\rangle + \langle v, u+v\rangle \\
&= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\
&= \langle u,u\rangle + 2\cdot\langle u,v\rangle + \langle v,v\rangle \\
&= \langle u,u\rangle + 2\cdot\langle u,v\rangle + \langle v,v\rangle \\
&= \|u\|^2 + 2\cdot\langle u,v\rangle + \|v\|^2 \\
&\Downarrow \\
\langle u,v\rangle &= \frac{\|u+v\|^2 - \|u\|^2 - \|v\|^2}{2}
\end{aligned}$$

b) There is only one inner product that can generate a norm. We can prove this by contradiction. Assume there are two distinct inner products $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ that generate the same norm but are not identical. Then, pick two vectors u and v such that $\langle u,v\rangle_1 \neq \langle u,v\rangle_2$. By definition of $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ and norm we have:

$$\begin{gathered}
\sqrt{\langle v,v\rangle_1} = \sqrt{\langle v,v\rangle_2} \qquad \text{for any } \ v \in V \\
\Downarrow \\
\langle u+v, u+v\rangle_1 = \langle u+v, u+v\rangle_2 \qquad \text{for any } \ u, v \in V
\end{gathered}$$

Note that, for any norm $\langle\cdot,\cdot\rangle$, we have:

$$\begin{aligned}
\langle u+v, u+v\rangle &= \langle u, u+v\rangle + \langle v, u+v\rangle \\
&= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\
&= \langle u,u\rangle + 2\cdot\langle u,v\rangle + \langle v,v\rangle
\end{aligned}$$

Apply this to $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$:

$$\langle u+v, u+v \rangle_1 = \langle u, u \rangle_1 + 2 \cdot \langle u, v \rangle_1 + \langle v, v \rangle_1 \tag{1}$$

$$\langle u+v, u+v \rangle_2 = \langle u, u \rangle_2 + 2 \cdot \langle u, v \rangle_2 + \langle v, v \rangle_2 \tag{2}$$

Now subtract (2) from (1) and apply definition of $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ and norm:

$$\begin{aligned}
\langle u+v, u+v \rangle_1 - \langle u+v, u+v \rangle_2 &= \langle u, u \rangle_1 - \langle u, u \rangle_2 + 2 \cdot \langle u, v \rangle_1 - 2 \cdot \langle u, v \rangle_2 + \langle v, v \rangle_1 - \langle v, v \rangle_2 \\
0 &= 0 + 2 \cdot \langle u, v \rangle_1 - 2 \cdot \langle u, v \rangle_2 + 0 \\
2 \cdot \langle u, v \rangle_2 &= 2 \cdot \langle u, v \rangle_1 \\
\langle u, v \rangle_2 &= \langle u, v \rangle_1
\end{aligned}$$

Hence $\langle u, v \rangle_2 = \langle u, v \rangle_1$, what contradicts our assumption and implies that both are the same inner product. Therefore, the inner product generating some norm must be unique.

Problem 2

a) $\langle f, g \rangle_1$ is not an inner product because it is not positive definite. Consider function $f(x) = 1$. Clearly, $f'(x) = 0$, which means $\langle f, f \rangle_1 = \int_0^1 f'(x)f'(x)\ dx = \int_0^1 0\ dx = 0$, but f is not the zero vector.

$\langle f, g \rangle_2$, on the other hand, is an inner product since it satisfies the properties of an inner product: it is bilinear, symmetric and positive-definite.

b) The inner product is:

$$\langle f, g \rangle = \int_0^1 (f(x)g(x) + f'(x)g'(x))\ dx$$

Cauchy-Schwarz inequality:

$$\sqrt{\int_0^1 (f(x)g(x) + f'(x)g'(x))\ dx} \leq \sqrt{\int_0^1 (f(x)^2 + f'(x)^2)\ dx} \cdot \sqrt{\int_0^1 (g(x)^2 + g'(x)^2)\ dx}$$

Triangle inequality:

$$\sqrt{\int_0^1 ((f(x) + g(x))^2 + (f'(x) + g'(x))^2)\ dx} \leq \sqrt{\int_0^1 (f(x)^2 + f'(x)^2)\ dx} + \sqrt{\int_0^1 (g(x)^2 + g'(x)^2)\ dx}$$

c) Starting from $\cos\theta$:

$$\begin{aligned}
\cos\theta &= \frac{\int_0^1 (f(x)g(x) + f'(x)g'(x))\ dx}{\sqrt{\int_0^1 (f(x)^2 + f'(x)^2)\ dx} \cdot \sqrt{\int_0^1 (g(x)^2 + g'(x)^2)\ dx}} \\
&= \frac{\int_0^1 (e^x + 0)\ dx}{\sqrt{\int_0^1 (1^2 + 0^2)\ dx} \cdot \sqrt{\int_0^1 (e^{2x} + e^{2x})\ dx}} \\
&= \frac{[e^x]_0^1}{\sqrt{[x]_0^1} \cdot \sqrt{[e^{2x}]_0^1}} \\
&= \frac{e - 1}{\sqrt{1} \cdot \sqrt{e^2 - 1}} \\
&= \frac{e - 1}{\sqrt{(e-1)(e+1)}} \\
&= \frac{\sqrt{(e-1)}}{\sqrt{(e+1)}} \\
\theta &\approx 0.8233 \text{ rad (4 d.p.)}
\end{aligned}$$

Problem 3

See attached `.png` plots and Matlab files. For part (d), the minimum p value achieved on the last iteration is 1014.6506.

Problem 4

a) Finding the Gram matrix G using $L^2 = \int_0^1 f(x)g(x)\,dx$ inner product:

$$G = \begin{bmatrix} \langle 1,1\rangle & \langle 1,e^x\rangle & \langle 1,e^{2x}\rangle \\ \langle e^x,1\rangle & \langle e^x,e^x\rangle & \langle e^x,e^{2x}\rangle \\ \langle e^{2x},1\rangle & \langle e^{2x},e^x\rangle & \langle e^{2x},e^{2x}\rangle \end{bmatrix}$$
$$= \begin{bmatrix} \int_0^1 1\,dx & \int_0^1 e^x\,dx & \int_0^1 e^{2x}\,dx \\ \int_0^1 e^x\,dx & \int_0^1 e^{2x}\,dx & \int_0^1 e^{3x}\,dx \\ \int_0^1 e^{2x}\,dx & \int_0^1 e^{3x}\,dx & \int_0^1 e^{4x}\,dx \end{bmatrix}$$
$$= \begin{bmatrix} 1 & e-1 & \frac{1}{2}(e^2-1) \\ e-1 & \frac{1}{2}(e^2-1) & \frac{1}{3}(e^3-1) \\ \frac{1}{2}(e^2-1) & \frac{1}{3}(e^3-1) & \frac{1}{4}(e^4-1) \end{bmatrix}$$

b) Note that 1, e^x and e^{2x} are linearly independent since you cannot generate any one function by combining the others using only scalar coefficients. This implies that Gram matrix G must be positive-definite.

c) Using the inner product from Problem 2 results in the matrix G_2 seen below. Note that the matrix itself doesn't matter when it comes to positive-definiteness of G_2: since we're using the same vectors and these vectors are independent, the resultant Gram matrix would always be positive-definite.

$$G_2 = \begin{bmatrix} \langle 1,1\rangle_2 & \langle 1,e^x\rangle_2 & \langle 1,e^{2x}\rangle_2 \\ \langle e^x,1\rangle_2 & \langle e^x,e^x\rangle_2 & \langle e^x,e^{2x}\rangle_2 \\ \langle e^{2x},1\rangle_2 & \langle e^{2x},e^x\rangle_2 & \langle e^{2x},e^{2x}\rangle_2 \end{bmatrix}$$
$$= \begin{bmatrix} \int_0^1 1\,dx & \int_0^1 2\cdot e^x\,dx & \int_0^1 3\cdot e^{2x}\,dx \\ \int_0^1 2\cdot e^x\,dx & \int_0^1 2\cdot e^{2x}\,dx & \int_0^1 3\cdot e^{3x}\,dx \\ \int_0^1 3\cdot e^{2x}\,dx & \int_0^1 3\cdot e^{3x}\,dx & \int_0^1 5\cdot e^{4x}\,dx \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2e-2 & \frac{3}{2}(e^2-1) \\ 2e-2 & e^2-1 & e^3-1 \\ \frac{3}{2}(e^2-1) & e^3-1 & \frac{5}{4}(e^4-1) \end{bmatrix}$$

d) Repeating the point above: Since 1, e^x and e^{2x} are linearly independent, the Gram matrix generated using any inner product for that vector space would be positive-definite. Therefore it's impossible to find $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ such that one generates a positive-definite Gram matrix and the other doesn't.

Problem 5

See attached Matlab script. My calculation revealed that Ted Cruz's Wikipedia page is the most similar to US Constitution.