ACM104 Problem Set #5 Solutions

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November 16, 2017

Problem 1

See attached `ps5problem1Kuzhagaliyev.m` and relevant plot for solutions.

Problem 2

Let $h_{n-1}(x)$ be nth monic Hermite polynomial, then:

$$
\begin{aligned}
h_0(x) &= 1 \\
\\
h_1(x) &= x - \frac{\langle x, 1\rangle}{\|1\|^2} \\
&= x \\
\\
h_2(x) &= x^2 - \frac{\langle x^2, 1\rangle}{\|1\|^2} - \frac{\langle x^2, x\rangle}{\|x\|^2} \cdot x \\
&= x^2 - \frac{\sqrt{2\pi}}{\sqrt{2\pi}} - 0 \\
&= x^2 - 1 \\
\\
h_3(x) &= x^3 - \frac{\langle x^3, 1\rangle}{\|1\|^2} - \frac{\langle x^3, x\rangle}{\|x\|^2} \cdot x - \frac{\langle x^3, x^2-1\rangle}{\|x^2-1\|^2} \cdot (x^2-1) \\
&= x^3 - 0 - \frac{3\sqrt{2\pi}}{\sqrt{2\pi}} \cdot x - 0 \\
&= x^3 - 3x \\
\\
h_4(x) &= x^4 - \frac{\langle x^4, 1\rangle}{\|1\|^2} - \frac{\langle x^4, x\rangle}{\|x\|^2} \cdot x - \frac{\langle x^4, x^2-1\rangle}{\|x^2-1\|^2} \cdot (x^2-1) - \frac{\langle x^4, x^3-3x\rangle}{\|x^3-3x\|^2} \cdot (x^3-3x) \\
&= x^4 - \frac{3\sqrt{2\pi}}{\sqrt{2\pi}} - 0 - \frac{12\sqrt{2\pi}}{2\sqrt{2\pi}} \cdot (x^2-1) - 0 \\
&= x^4 - 3 - 6(x^2-1) \\
&= x^4 - 6x^2 + 3
\end{aligned}
$$

Problem 3

The case with $\langle x, y\rangle_1$ is easy - we just need to find the cross product of v_1 and v_2. The resultant vector will be orthogonal to both of them, and hence orthogonal to any linear combination of the two. Then, the span of the resultant vector will be our $W_1^\perp$.

$$u_1 = v_1 \times v_2 = \begin{bmatrix} 2\cdot 1 - 0 \\ 3\cdot 2 - 1 \\ 0 - 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix} \quad \Rightarrow \quad W_1^\perp = span(u_1) = span(\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix})$$

For $\langle x, y\rangle_2$ we can define two equations using the weighted inner product and the two vectors:

$$\langle u, v_1\rangle = 0 \quad \Rightarrow \quad 1\cdot x + 2\cdot 2\cdot y + 3\cdot 3\cdot z = x + 4y + 9z = 0$$
$$\langle u, v_2\rangle = 0 \quad \Rightarrow \quad 2\cdot x + 0 + 3\cdot 1\cdot z = 2x + 3z = 0$$

We can solve this system of equations to find the general equation for $u \in W_2^\perp$:

$$x = -\frac{3}{2}z \quad \Rightarrow \quad -\frac{3}{2}z + 4y + 9z = 0$$
$$y = -\frac{15}{8}z$$

$$z = 1 \quad \Rightarrow \quad x = -\frac{3}{2},\ y = -\frac{15}{8} \quad \Rightarrow \quad u_2 = \begin{bmatrix} -\frac{3}{2} \\ -\frac{15}{8} \\ 1 \end{bmatrix} \quad \Rightarrow \quad W_2^\perp = span(u_2) = span(\begin{bmatrix} -\frac{3}{2} \\ -\frac{15}{8} \\ 1 \end{bmatrix})$$

Problem 4

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The eigenvalues of this matrix are the solutions to the equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = 0$$

$$\Downarrow$$

$$\begin{aligned} -\lambda(-\lambda)(1-\lambda) - 1(-1+\lambda) &= 0 \\ -\lambda^3 + \lambda^2 - \lambda + 1 &= 0 \end{aligned}$$

By inspection we can conclude that $\lambda = 1$ is a root and hence $(\lambda - 1)$ is a factor, what lets us factorise the equation:

$$\begin{aligned} (\lambda - 1)(-\lambda^2 - 1) &= 0 \\ \lambda = 1 \quad \text{OR} \quad -\lambda^2 - 1 &= 0 \\ \lambda^2 &= -1 \\ \lambda &= \pm i \end{aligned}$$

-1, i and $-i$ are the three distinct eigenvalues of A. We've seen in our equation that all of them had algebraic multiplicity of one. We can find the corresponding eigenvectors:

$$Av = \lambda v \quad \Rightarrow \quad \begin{aligned} -z &= \lambda x \\ y &= \lambda y \\ x &= \lambda z \end{aligned}$$

$$\lambda = 1 \quad \Rightarrow \quad \begin{aligned} -z &= x \\ y &= y \\ x &= z \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 0 \\ y &= \alpha \\ z &= 0 \end{aligned}$$

$$\lambda = i \quad \Rightarrow \quad \begin{aligned} -z &= xi \\ y &= yi \\ x &= zi \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= zi \\ y &= 0 \end{aligned}$$

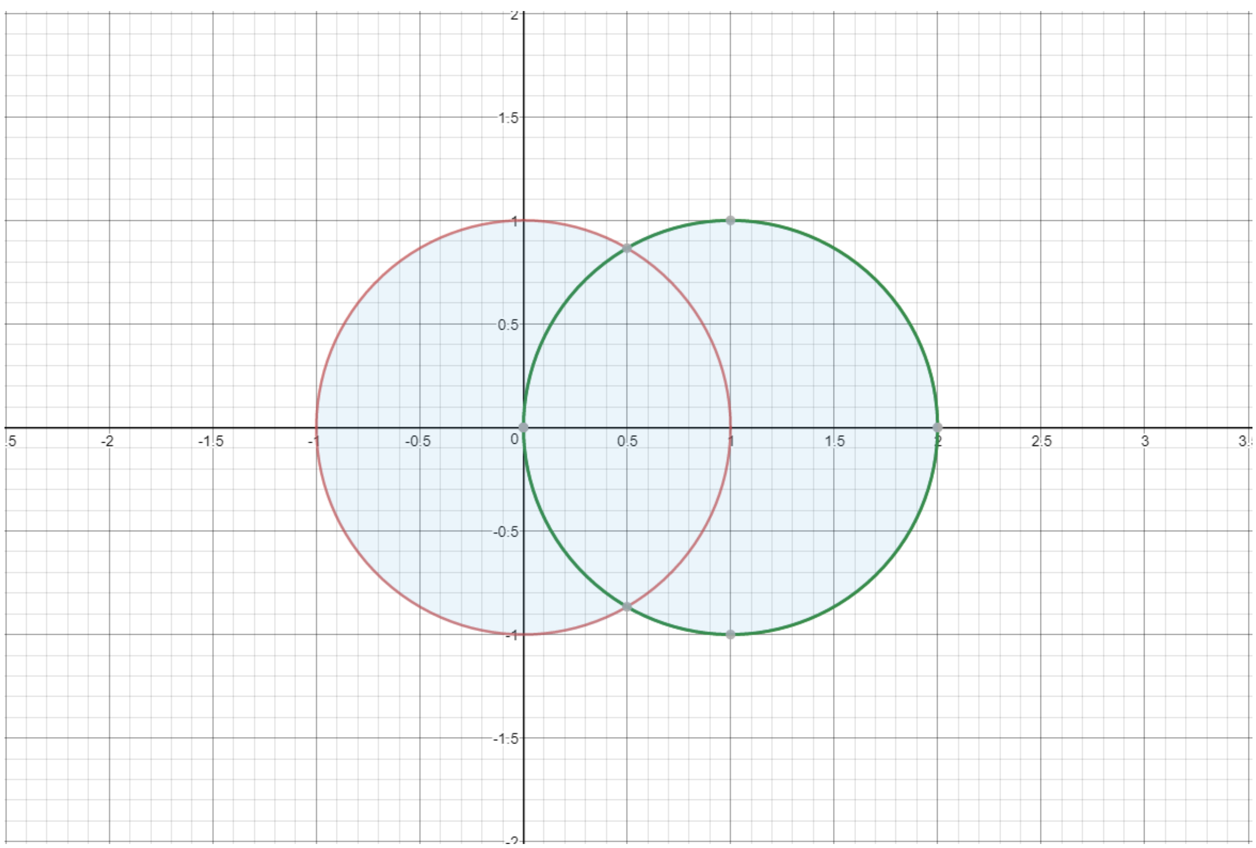
$$\lambda = -i \quad \Rightarrow \quad \begin{aligned} -z &= -xi \\ y &= -yi \\ x &= -zi \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= -zi \\ y &= 0 \end{aligned}$$

From the general equations of the vectors corresponding to the eigenvalues, we can conclude that each eigenvalue has a geometric multiplicity of one, hence all eigenvalues are complete. This implies that A is complete. We can also see that generated eigenvectors span entire $\mathbb{C}^3$.

Problem 5

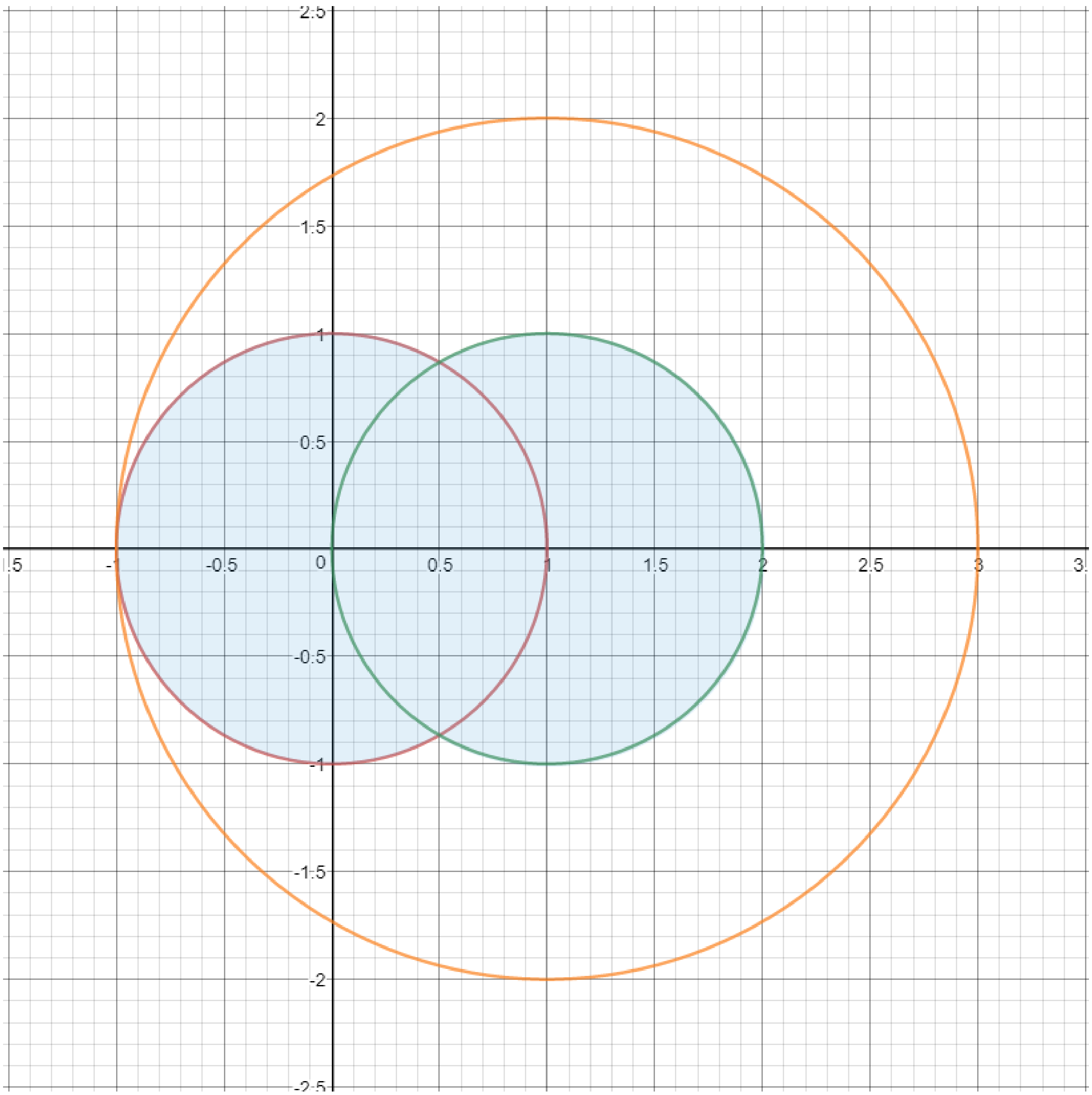
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

a) By a simple calculation we can figure out that the radii of all Gerschgorin disks in this case are 1. The centres for the two possible disks are $(0, 0)$ and $(1, 0)$:



b) By Gerschgorin theorem, we know that $spec(A) \subset D_A$ and $spec(A^T) \subset D_{A^T}$. We also know that eigenvalues of A are the solutions for the equation $det(A - \lambda I) = 0$ and that the eigenvalues of A^T satisfy $det(A^T - \lambda I) = 0$. But $det(A^T - \lambda I) = det((A - \lambda I)^T) = det(A - \lambda I)$ since determinant of a matrix is equal to the determinant of its transpose, so A and A^T have the same eigenvalues. This means $spec(A) = spec(A^T)$, hence both $spec(A) \subset D_A$ and $spec(A) \subset D_{A^T}$ should hold. From this we can conclude that $spec(A) \subset D_A \cap D_{A^T}$.

c) Using the refined Gerschgorin domain, we only have one unit circle centred at $(1, 0)$:



d) The eigenvalues of this matrix are the solutions to the equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{bmatrix} = 0$$

$$\Downarrow$$

$$-\lambda((1-\lambda)^2 + 1) - 0 + 0 = 0$$
$$-\lambda(\lambda^2 - 2\lambda + 1 + 1) = 0$$
$$-\lambda(\lambda^2 - 2\lambda + 2) = 0$$
$$-\lambda(\lambda - 1 + i)(\lambda - 1 - i) = 0$$
$$\lambda = 0 \quad \text{OR} \quad \lambda = 1 - i \quad \text{OR} \quad \lambda = 1 + i$$

The eigenvalues of A are 0, $1 - i$ and $1 + i$, which indeed lie within the refined Gerschgorin domain.