ACM104 Problem Set #4 Solutions

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Problem 1

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 0 \\ 5 \\ 6 \\ 8 \end{bmatrix}$$

$$\begin{aligned} p(x) = \|Ax - b\|^2 &= x^T(A^TA)x - 2\,x^T(A^Tb) + \|b\|^2 \\ &= x^TKx - 2xf + c \end{aligned}$$

$$K = A^T \cdot A = \begin{bmatrix} 1 & 0 & 1 & -3 \\ 2 & -2 & 5 & 1 \\ -1 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 & -5 \\ 4 & 34 & -12 \\ -5 & -12 & 12 \end{bmatrix}$$

$$f = A^Tb = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 1 & 5 & -1 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} -18 \\ 28 \\ 17 \end{bmatrix}$$

First we need to check if the matrix K is positive definite. We know that matrix is positive definite if all of its principal minors are positive. Indeed, $\det A_1 = 11 > 0$; $\det A_2 = 11 \cdot 34 - 4^2 = 18 > 0$; $\det A_3 = 11 \cdot 264 - 4 \cdot (-12) - 5 \cdot 122 = 2904 + 48 - 610 = 2342 > 0$; Therefore K is positive definite and there exists a global minimizer $x^* = K^{-1}f$.

$$
\begin{aligned}
x^* = K^{-1}f = Kf &= \begin{bmatrix} 11 & 4 & -5 \\ 4 & 34 & -12 \\ -5 & -12 & 12 \end{bmatrix} \begin{bmatrix} -18 \\ 28 \\ 17 \end{bmatrix} \\
&= \frac{1}{2342} \begin{bmatrix} 264 & 12 & 122 \\ 12 & 107 & 112 \\ 122 & 112 & 358 \end{bmatrix} \begin{bmatrix} -18 \\ 28 \\ 17 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \quad \text{(solution)} \\
\text{LSE} = \sqrt{\|b\|^2 - b^T A x^*} &= \sqrt{\sqrt{125}^2 - 125} \\
&= 0
\end{aligned}
$$

The least squares error is 0, so in this case x^* is the exact solution.

Problem 2

Note that in both *a)* and *b)* the degree of $p_n(x)$ is 1. We can derive $p_1(x)$ beforehand:

$$
\begin{aligned}
p_1(x) = f(x_o) \cdot L_1(x) + f(x_1) \cdot L_2(x) &= f(x_0) \cdot \frac{x - x_1}{x_0 - x_1} + f(x_1) \cdot \frac{x - x_0}{x_1 - x_0} \\
&= \frac{f(x_0)}{x_0 - x_1} x - \frac{x_1 f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} x - \frac{x_0 f(x_1)}{x_1 - x_0} \\
&= \frac{f(x_0) - f(x_1)}{x_0 - x_1} x + \frac{x_0 f(x_1) - x_1 f(x_0)}{x_0 - x_1}
\end{aligned}
$$

a) Substituting $x_0 = a$ and $x_1 = b$ into $p_1(x)$:

$$
p_1(x) = \frac{f(a) - f(b)}{a - b} x + \frac{a f(b) - b f(a)}{a - b}
$$

$$
\begin{aligned}
\int_a^b f(x)\,dx \approx \int_a^b p_1(x)\,dx &= \int_a^b \left(\frac{f(a)-f(b)}{a-b} x + \frac{af(b)-bf(a)}{a-b} \right) dx \\
&= \left[\frac{f(a)-f(b)}{a-b} \cdot \frac{x^2}{2} + \frac{af(b)-bf(a)}{a-b} \cdot x \right]_a^b \\
&= \frac{f(a)-f(b)}{a-b} \cdot \frac{b^2}{2} + \frac{af(b)-bf(a)}{a-b} \cdot b - \frac{f(a)-f(b)}{a-b} \cdot \frac{a^2}{2} - \frac{af(b)-bf(a)}{a-b} \cdot a \\
&= \frac{f(a)-f(b)}{a-b} \cdot \frac{b^2-a^2}{2} - af(b) + bf(a) \\
&= \frac{1}{2} \cdot (f(b)-f(a))(b+a) - af(b) + bf(a) \\
&= \frac{1}{2} \left[bf(b) + af(b) - bf(a) - af(a) - 2af(b) + 2bf(a) \right] \\
&= \frac{1}{2} \left[bf(b) - af(b) + bf(a) - af(a) \right] \\
&= \frac{1}{2} \left[f(b) + f(a) \right] (b-a) \qquad \text{(trapezoid rule)}
\end{aligned}
$$

b) Substituting $x_0 = \frac{1}{3}(a+b)$ and $x_1 = \frac{2}{3}(a+b)$ into $p_1(x)$:

$$
\begin{aligned}
p_1(x) &= \frac{f(x_0)-f(x_1)}{\frac{1}{3}(a+b) - \frac{2}{3}(a+b)} x + \frac{\frac{1}{3}(a+b)f(x_1) - \frac{2}{3}(a+b)f(x_0)}{\frac{1}{3}(a+b) - \frac{2}{3}(a+b)} \\
&= \left[f(x_1) - f(x_0) \right] \cdot \frac{3x}{a+b} + 2f(x_0) - f(x_1)
\end{aligned}
$$

I didn't replace x_i in $f(x_i)$ to make the notation a bit more compact. I will substitute the actual value in the end.

$$
\begin{aligned}
\int_a^b f(x)\,dx \approx \int_a^b p_1(x)\,dx &= \int_a^b \left(\left[f(x_1) - f(x_0) \right] \cdot \frac{3x}{a+b} + 2f(x_0) - f(x_1) \right) dx \\
&= \left[\frac{3}{2} \left[f(x_1) - f(x_0) \right] \cdot \frac{x^2}{a+b} + 2xf(x_0) - xf(x_1) \right]_a^b \\
&= \frac{3}{2} \left[f(x_1) - f(x_0) \right] \cdot \frac{b^2}{a+b} + 2bf(x_0) - bf(x_1) - \frac{3}{2} \left[f(x_1) - f(x_0) \right] \cdot \frac{a^2}{a+b} - 2af(x_0) + af(x_1) \\
&= \frac{3}{2} \left[f(x_1) - f(x_0) \right] (b-a) + \left[2f(x_0) - f(x_1) \right] (b-a) \\
&= \left[\frac{3}{2} \cdot f(x_1) - \frac{3}{2} \cdot f(x_0) + 2f(x_0) - f(x_1) \right] (b-a) \\
&= \left[\frac{1}{2} f(x_1) - \frac{1}{2} f(x_0) \right] (b-a) \\
&= \frac{1}{2} \left[f(\frac{2}{3}(a+b)) + f(\frac{1}{3}(a+b)) \right] (b-a) \qquad \text{(open rule)}
\end{aligned}
$$

c) Testing the approximations:

$$
\begin{aligned}
\int_0^1 e^x\,dx &= e-1 && = 1.718281\dots \\
&\approx \frac{1}{2}(e+1) && = 1.859140\dots \quad \text{(trapezium rule)} \\
&\approx \frac{1}{2}(e^{\frac{2}{3}}+e^{\frac{1}{3}}) && = 1.671673\dots \quad \text{(open rule)}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\pi sin(x)\,dx &= 1-(-1) && = 2 \\
&\approx \frac{\pi}{2} && = 1.570796\dots \quad \text{(trapezium rule)} \\
&\approx \frac{\pi}{2}(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}) && = 2.720699\dots \quad \text{(open rule)}
\end{aligned}
$$

For e^x, open rule has a smaller error than trapezoid rule, but trapezoid rule performs better for $sin(x)$.

Problem 3

$$y = f(x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 \quad \text{and} \quad \beta^* = (\beta_0^*, \beta_1^*, \beta_2^*, \beta_3^*)^T$$

a) Deriving a system of normal equations on β^*:

$$r_i = y^{(i)} - f(x_1^{(i)}, x_2^{(i)})$$

$$= y^{(i)} - \begin{bmatrix} 1 & x_1^{(i)} & x_2^{(i)} & x_1^{(i)}x_2^{(i)} \end{bmatrix} \begin{bmatrix} \beta_0^* \\ \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ \vdots \\ y^{(m)} \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} & x_1^{(1)}x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & x_1^{(2)}x_2^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_1^{(m)} & x_2^{(m)} & x_1^{(m)}x_2^{(m)} \end{bmatrix} \begin{bmatrix} \beta_0^* \\ \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{bmatrix}$$

$$\Downarrow$$

$$r = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ \vdots \\ y^{(m)} \end{bmatrix} - \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} & x_1^{(1)}x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & x_1^{(2)}x_2^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_1^{(m)} & x_2^{(m)} & x_1^{(m)}x_2^{(m)} \end{bmatrix} \begin{bmatrix} \beta_0^* \\ \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{bmatrix}$$

See attached `ps4problem3Kuzhagaliyev.m` file and plots for other solutions.

Problem 4

See attached `ps4problem4Kuzhagaliyev.m` file and plot for solutions.

Problem 5

See attached `ps4problem5Kuzhagaliyev.m` and plot for solution.