CS 142 Homework Set #1 Timur Kuzhaagaliyev

Notation : true = $\top$, false = $\perp$ SN: 2079376

① a) Using a truth table and definition of $\not\equiv$ twice:

x	y	z	$(x \not\equiv y) \not\equiv z$	$x \not\equiv (y \not\equiv z)$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
$\perp$	$\perp$	$\top$	$\top$	$\top$
$\perp$	$\top$	$\perp$	$\top$	$\top$
$\perp$	$\top$	$\top$	$\perp$	$\perp$
$\top$	$\perp$	$\perp$	$\top$	$\top$
$\top$	$\perp$	$\top$	$\perp$	$\perp$
$\top$	$\top$	$\perp$	$\perp$	$\perp$
$\top$	$\top$	$\top$	$\top$	$\top$

$\Rightarrow \quad (x \not\equiv y) \not\equiv z$

$\equiv$ { By the truth table above }

$x \not\equiv (y \not\equiv z)$

$\Rightarrow \not\equiv$ is associative Q.E.D.

① b) i. $n=1$ $\quad \perp \not\equiv \perp$

$\equiv$ { Def. of $\not\equiv$ }

$\perp$ $\quad \therefore$ statement i is false.

ii. $n=1$ $\quad \top \not\equiv \top$

$\equiv$ { Def. of $\not\equiv$ }

$\perp$

① b) cont.

iii. $n=1$ $\quad T \not\equiv \perp$

$\equiv \{\text{Def. of } \not\equiv\}$

T $\quad \therefore$ statement iii holds

$n=2$ case 1: $T \not\equiv \perp \not\equiv \perp$	case 2: $T \not\equiv T \not\equiv T$
$\equiv \{\text{Def of } \not\equiv \text{ twice}\}$	$\equiv \{\text{Def. of } \not\equiv \text{ twice}\}$
T	T

$\therefore$ statement iii is <u>true</u>

iv. $n=3$ $\quad \perp \not\equiv T \not\equiv T$

$\equiv \{\text{Def. of } \not\equiv \text{ twice}\}$

$\perp$ $\quad \therefore$ statement iv is false

v. Clearly v is false.

① c) Proof by induction: (I) X is true iff the number of predicates $x[i]$ that evaluate to true in the expression is odd.

Base cases: (Let X_n be defined as $x[0] \not\equiv x[1] \not\equiv \ldots \not\equiv x[n]$)

X_0: could be T or $\perp$, so (I) holds (trivial)

X_1:

case 1:	case 2:	case 3:
$T \not\equiv \perp$	$T \not\equiv T$	$\perp \not\equiv \perp$
$\equiv \{\text{Def of } \not\equiv\}$	$\equiv \{\text{Def of } \not\equiv\}$	$\equiv \{\text{Def of } \not\equiv\}$
T	$\perp$	$\perp$

$\therefore$ (I) holds for X_1

Induction hypothesis (I.H.):

Assume that (I) holds for $n=k$

① c) cont.

Inductive step:

Prove that Ⓘ holds for $n = k+1$

Observe that:

X_{k+1}

$\equiv \{ \text{Def of } X_{k+1} \}$

$x[0] \not\equiv x[1] \not\equiv \ldots \not\equiv x[k] \not\equiv x[k+1]$

$\equiv \{ \text{Associativity of } \not\equiv \}$

$(x[0] \not\equiv x[1] \not\equiv \ldots \not\equiv x[k]) \not\equiv x[k+1]$

$\equiv \{ \text{Def. of } X_k \}$

$X_k \not\equiv x[k+1]$

By I.H. we know that Ⓘ holds for X_k, hence there are 4 cases:

Case 1: X_k has even number of true predicates $\iff$ X_k is false

Case 1a: $x[k+1]$ is true	Case 1b: $x[k+1]$ is false
X_{k+1}	X_{k+1}
$\equiv \{ \text{By observation above} \}$	$\equiv \{ \text{By observation above} \}$
$X_k \not\equiv x[k+1]$	$X_k \not\equiv x[k+1]$
$\equiv \{ \text{Substituting values inside} \}$	$\equiv \{ \text{Substituting values inside} \}$
$\bot \not\equiv \top$	$\bot \not\equiv \bot$
$\equiv \{ \text{Def of } \not\equiv \}$	$\equiv \{ \text{Def of } \not\equiv \}$
$\top$	$\bot$
$\therefore X_{k+1}$ has odd number of true predicates and X_{k+1} is true $\Rightarrow$ Ⓘ holds.	X_{k+1} has even number of true predicates and is false $\Rightarrow$ Ⓘ holds.

Case 2: X_k has odd number of true predicates $\Leftrightarrow$ X_k is true

Case 2a: $x[k+1]$ is true

$$
\begin{aligned}
& X_{k+1} \\
\equiv & \{\text{By observation earlier}\} \\
& X_k \not\equiv x[k+1] \\
\equiv & \{\text{Substituting the values in}\} \\
& \top \not\equiv \top \\
\equiv & \{\text{Def. of } \not\equiv\} \\
& \bot
\end{aligned}
$$

X_{k+1} has even number of true predicates and X_{k+1} is false $\Rightarrow$ (I) holds

Case 2b: $x[k+1]$ is false

$$
\begin{aligned}
& X_{k+1} \\
\equiv & \{\text{By observation earlier}\} \\
& X_k \not\equiv x[k+1] \\
\equiv & \{\text{Substituting the values in}\} \\
& \top \not\equiv \bot \\
\equiv & \{\text{Def. of } \not\equiv\} \\
& \top
\end{aligned}
$$

X_{k+1} has odd number of true predicates and X_{k+1} is true $\Rightarrow$ (I) holds

Conclusion:

By base cases and inductive step (I) holds for $\forall n \geq 0$. Q.E.D.

② a) Intuitively, the statement is true. Proof can be found in ② b).

b) Consider De Morgan's law for 2 predicates:

$$\neg(p(x_1)\wedge p(x_2)) \equiv \neg p(x_1) \vee \neg p(x_2)$$

This idea can be extended to any number of predicates n using associativity of $\vee$ and repeatedly applying De Morgan's law for 2 predicates, i.e.:

For $n>2$:

$$\left.\begin{aligned}
&\neg p(x_1) \vee \neg p(x_2) \vee \neg p(x_3) \vee \ldots \vee \neg p(x_n)\\
&\equiv \{\text{Associativity of } \vee\}\\
&(\neg p(x_1) \vee \neg p(x_2)) \vee \neg p(x_3) \vee \ldots \vee \neg p(x_n)\\
&\equiv \{\text{De Morgan's law for 2 predicates}\}\\
&(\neg(p(x_1)\wedge p(x_2))) \vee \neg p(x_3) \vee \ldots \vee \neg p(x_n)\\
&\equiv \{\text{Associativity of } \vee\}\\
&(\neg(p(x_1)\wedge p(x_2)) \vee \neg p(x_3)) \vee \ldots \vee \neg p(x_n)\\
&\equiv \{\text{De Morgan's law for 2 pred.}\}\\
&\neg((p(x_1)\wedge p(x_2))\wedge p(x_3)) \vee \ldots \vee \neg p(x_n)\\
&\equiv \{\text{Assoc. of } \wedge\}\\
&\neg(p(x_1)\wedge p(x_2)\wedge p(x_3)) \vee \ldots \vee \neg p(x_n)
\end{aligned}\right\}\ \alpha$$

$$\equiv \{\text{Repeat } \alpha \ n-1 \text{ times}\}$$

$$\neg(p(x_1)\wedge p(x_2)\wedge p(x_3)\wedge \ldots \wedge \neg p(x_n))$$

$\therefore$ De Morgan's law applies to $n>0$ predicates

Equivalently:

$$\neg(\forall x : r(x))$$

$$\equiv \{\text{Def. of } \forall\}$$

$$\neg(r(x_1)\wedge r(x_2)\wedge \ldots \wedge r(x_n) \wedge \text{true})$$

$$\equiv \{\text{De Morgan's law for } n \text{ predicates}\}$$

$$(\neg r(x_1) \vee \neg r(x_2) \vee \neg r(x_3) \vee \ldots \vee \neg r(x_n) \vee \neg \text{true})$$

$$\equiv \{\text{Def. of } \exists\}$$

$$\exists x : \neg r(x)$$

Using the 2 concepts above we can proceed with the main proof:

$\neg(\forall x : p(x) : q(x))$

$\equiv$ {Def of $\forall$}

$\neg(\forall x : p(x) \Rightarrow q(x))$

$\equiv$ {Definition of implication}

$\neg(\forall x : \neg p(x) \vee q(x))$

$\equiv$ {De Morgan's law for n predicates (proven above)}

$(\exists x : \neg(\neg p(x) \vee q(x)))$

$\equiv$ {De Morgan's laws for 2 predicates}

$(\exists x : \neg\neg(p(x) \wedge \neg q(x)))$

$\equiv$ {$\neg$ is involutive}

$(\exists x : p(x) \wedge \neg q(x))$

$\equiv$ {Def of $\exists$}

$(\exists x : p(x) : \neg q(x))$ Q.E.D

(3) For a state to be in a collection of states corresponding to the fixed point of a program, every action should have no effect when applied to that state.

Clearly, actions in "Unnamed" have no effect when:

$$y = f(7) \wedge (\neg(x=y) \vee x=2)$$

From this we can extract 2 states for the fixed point:

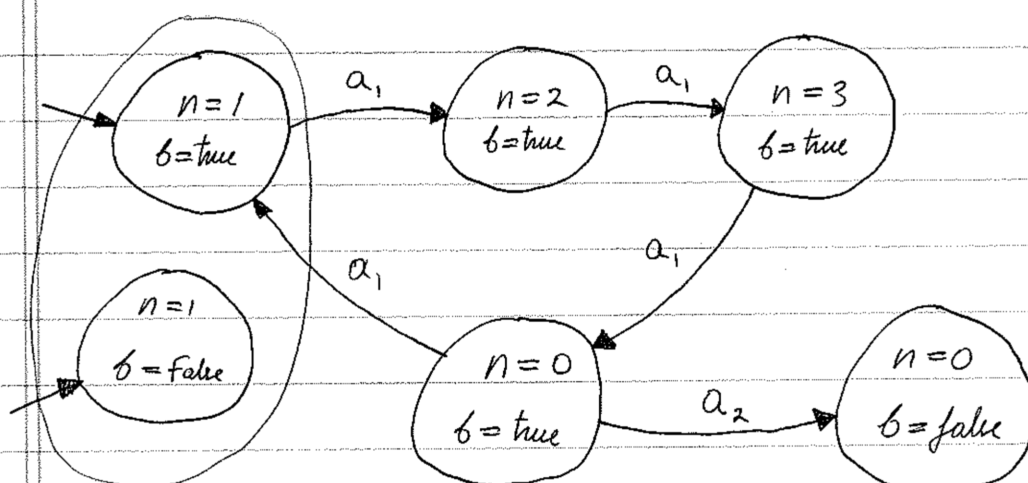
State 1: $y = f(7)$, $x \neq y$ (i.e. $x \in \text{numbers} \setminus \{f(7)\}$)

State 2: $y = f(7)$, $x = 2$

(4) Actions:

a_1) $b \rightarrow n := n +_4 1$

$[]$ a_2) $n=0 \rightarrow b := \text{false}$



* Each state has an implicit "skip" action, which does not change the state. Actions whose guards are false are considered to be "skip"s.