CS 153 Communication Complexity **Timur Kuzhagaliyev**

Problem Set #2 Solutions

Out: May 22 *Due:* **June 1**

1. Solutions:

   (a) We can show that the inequality in the question holds by considering the structure of M_1 and M_2. The breakdown of matrix of M into M_1 and M_2looks like

   $$\begin{pmatrix} \times & X_1 \times T & \times \\ S \times Y_1 & S \times T & S \times Y_2 \\ \times & X_2 \times T & \times \end{pmatrix} \tag{1}$$

   where the highlighted row is matrix M_1, the highlighted column is the matrix M_2 and X_1, S, X_2 with Y_1, T, Y_2 are partitions of X and Y respectively. Note that due to the structure of given matrices, the standard rank of M_1 is bound above by $|S|$ and the standard rank of M_2 is bound above by $|T|$, and the same bounds apply to $triangle - rank$.

   Note also that $S \times T$ is a zero matrix - this means that regardless of how rows or columns from $S \times T$ are permuted within M_1 or M_2, they will not change the $triangle - rank$. As such, there is no reason to permute rows indexed by S in M_2 and no reason to permute columns indexed by T in M_1. This creates a convenient "mask" that lets us permute rows and columns of M and build desired square submatrices inside M_1 and M_2 without undoing any progress.

   Now we can put the two points from above together to define a procedure that proves the desired inequality. We start off with M. Next we permute rows and columns of M to build the largest square submatrix with ones in diagonals on zeros below the diagonal in both M_1 and M_2 (showing $triangle - rank$ of these matrices). As discussed previously, we can do that without any clashes.

   Denote such square submatrices S_1 in M_1 and S_2 in M_2. Finally, we permute columns of resultant M to put S_1 into the top left corner of block $S \times Y_2$, and then permute rows of resultant M to put S_2 into the bottom right corner of block $X_1 \times T$. As a result of this operation, we have transformed M in such a way that it now contains a square matrix that is a tensor product of S_1 and S_2 (with the exception of the top right corner which contains arbitrary values and doesn't affect $triangle - rank$). Hence we have shown that if matrices M_1 and M_2 have $triangle - rank$ of m and n respectively, we can permute rows and columns of M to build a square submatrix of side length $m + n$ with ones on the diagonal and zeroes below the diagonal, showing that the $triangle - rank$ of matrix M must be at least $m + n$.

2. We can assume $X \times Y$ is a square matrix - if it isn't, we can just pad out one of the sides to make it square.

   TODO: *Can show this by proving that there must a big monochromatic rectangle of size $\geq 2^{-c(r)}|X||Y|$, then showing that we can recursively say yes or no to recurse into one of the smaller rectangles, at every step either halving the rank of making the matrix much smaller.*

3.