0.1 Graph Aware Determinism

When treating the "Network" as an *opaque cloud*, it's easy to underestimate how varied network partitions when link failures are asymmetrical: A can see B, but B can't see A. In a 4 node setup, there are over 1295 potential partitions, and a flaky network can reproduce them all. From a distributed systems (event ordering in a cluster) as an availability equation, we can easily overestimate how reliable they are, by 3 orders of magnitude.

Link failures are invisible (hidden) in a Clos. They are 100% Visible to us in a local graph of *triangular* relationships.

And that's only the clean (binary) binary failures. Real system *flakey* connections are much worse.

0.1.1 Transactions need a coordinator?

The Æthernet protocol is designed to be exquisitely sensitive to packet loss and corruption We monitor, detect, diagnose link failures, and recover reversibly and automatically.

0.1.2 A Resilience Metric for Mesh Networks

0.1.3 Graph Laplacian and Algebraic Connectivity

The Graph Laplacian. For a simple, undirected graph $G = (V, E)$ with $n = |V|$ vertices, the *combinatorial Laplacian* matrix L is defined as

$$L = D - A,$$

where

- A is the $n \times n$ adjacency matrix, with $A_{ij} = 1$ if there is an edge between i and j, and 0 otherwise,
- D is the $n \times n$ diagonal *degree matrix*, whose diagonal entries are $D_{ii} = \deg(i)$.

The Laplacian L is central in spectral graph theory, encoding many connectivity properties of G.

Algebraic Connectivity (λ_2). Let the eigenvalues of L be ordered as

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

The second-smallest eigenvalue, λ_2, is the *algebraic connectivity* (or Fiedler value). It satisfies

- $\lambda_2 > 0$ if and only if G is connected,
- A larger λ_2 generally indicates stronger connectivity and a larger cut is required to disconnect G.

Thus, λ_2 is often seen as a "spectral" measure of how robustly G remains connected under certain disruptions.

0.1.4 Classical Connectivity Measures

Beyond λ_2, there are other classical measures:

1. **Edge Connectivity** $\lambda(G)$: The minimum number of edges whose removal disconnects G.
2. **Vertex Connectivity** $\kappa(G)$: The minimum number of vertices whose removal disconnects G.
3. **Expansion or Isoperimetric Constants**: Relate cut sizes to the cardinalities of sets being separated.

These capture *global* connectivity but may not reflect the incremental or adversarial removal of edges in a constrained-valency network.

0.1.5 Incremental Link Failures in Constrained-Valency Networks

In HPC or data-center systems (e.g. with IPUs or smartNICs), each node has limited valency (e.g. 8 ports), and edges can fail one by one. A single λ_2 value may not capture how partial or progressive failures degrade connectivity.

Why a single λ_2 may not suffice.

- λ_2 is a *one-shot* global measure. It does not directly model how connectivity degrades as edges fail in sequence.
- Some topologies might remain connected but experience severe bottlenecks after a few critical edges fail, which does not show up immediately in a single baseline λ_2.

0.1.6 Potential Approaches for a "Resilience Metric"

0.1.7 Spectral-Based Extensions

(a) Expected λ_2 under random failures. If edges fail independently with probability p, form a random subgraph G_p. One could define:

$$\mathbb{E}\left[\lambda_2(G_p)\right]$$

as a measure of *average* resilience. Larger expected algebraic connectivity implies better tolerance to random edge losses.

Worst-case sequence of λ_2 values. Define:

$$R(k) \;=\; \min_{\substack{F \subseteq E \\ |F|=k}} \lambda_2\big(G - F\big),$$

where $G - F$ is the graph with edges F removed. $R(k)$ measures the smallest λ_2 achievable *after k edge removals*. A graph is more resilient if $R(k)$ remains high for larger k. If $R(k)$ drops to 0, it indicates that with k removed edges, G can be disconnected.

0.1.8 Connectivity-Based Ideas

(a) k-Edge Connectivity Functions. Beyond the single value of $\lambda(G)$ (the edge connectivity), define

$$\phi(k) \;=\; \min_{\substack{F \subseteq E \\ |F|=k}} \Big(\text{size of the largest connected component of } G - F\Big).$$

If $\phi(k)$ remains large, it means removing any k edges fails to isolate more than a small fraction of nodes. This complements λ_2 by focusing on *component sizes*.

(b) Edge-disjoint path counts. Using Menger's Theorem, one can track the number of edge-disjoint paths between certain pairs of nodes. Higher numbers of disjoint paths generally imply more resilient connectivity.

0.1.9 Weighted or Dynamic Laplacian

A *dynamic* Laplacian $L(\mathbf{w})$ might assign weights w_e to edges. If an edge is fully failed, $w_e = 0$. Then one can track how $\lambda_2\big(L(\mathbf{w})\big)$ evolves as edges degrade from weight 1 to weight 0, either in random or adversarial patterns.

0.1.10 A Concrete Proposal

A practical "resilience function" might be:

$$R(k) \;=\; \min_{\substack{F \subseteq E \\ |F|=k}} \lambda_2\big(G - F\big),$$

where the minimum is taken over all subsets F of k edges. Then:

- $R(0) = \lambda_2(G)$ is the baseline algebraic connectivity.
- If $R(k) > 0$, the graph *cannot be disconnected* by removing any k edges.
- The rate at which $R(k)$ decreases with k reflects how fast the network's connectivity deteriorates under incremental failures.

0.1.11 Computational Observations

Exact computation of $R(k)$ can be expensive for large graphs because there are $\binom{|E|}{k}$ subsets. One may:

- Use *heuristics* or *approximation algorithms* to identify critical edges,
- Leverage *min-cut* or *max-flow* bounds to quickly estimate how easy it is to disconnect the graph,
- Perform *sampling* over subsets F if a random measure of resilience suffices.
- The *graph Laplacian* (and in particular λ_2) is a powerful spectral tool. It already gives a measure of connectivity robustness.

- For *incremental* or *adversarial* link failures, a single λ_2 value may not capture the full picture. A *function* $R(k)$ over subsets of size k can indicate how robustly the graph handles multiple simultaneous failures.
- In *constrained-valency* networks, certain edges are more critical, because each node has fewer possible alternate paths. Thus, a spectral-based metric that accounts for edge removals (like $R(k)$) can better reflect real-world vulnerability.
- Combined with classical connectivity measures (e.g. $\lambda(G)$, $\kappa(G)$), a Laplacian-based incremental approach provides a practical, mathematically grounded way to define and quantify *resilience* of a network topology.