Assignment 4

Due: Friday, July 22 at 4:00pm

1. [4 points] Let $\Sigma = \{0, 1\}$, and assume A and B are Turing-recognizable languages such that $A \cup B = \Sigma^*$. Prove that there exists a decidable language $C \subseteq \Sigma^*$ such that

$$A \cap \overline{B} \subseteq C \qquad \text{and} \qquad \overline{A} \cap B \subseteq \overline{C}.$$

2. [4 points] Let Σ be an alphabet, and assume that we have fixed a scheme for encoding every possible DTM M as a string $\langle M \rangle \in \Sigma^*$ in the usual way, and define a language $A \subseteq \Sigma^*$ as

$$A = \big\{ \langle M \rangle \, : \, M \text{ is a DTM that halts on at least one input string} \big\}.$$

   Prove that A is not decidable.

3. [6 points] Define two languages as follows:

$$\begin{aligned} \mathrm{E}_{\mathrm{DTM}} &= \{ \langle M \rangle \, : \, M \text{ is a DTM with } \mathrm{L}(M) = \varnothing \} \\ \mathrm{DISJ}_{\mathrm{DTM}} &= \{ \langle M_1, M_2 \rangle \, : \, M_1 \text{ and } M_2 \text{ are DTMs with } \mathrm{L}(M_1) \cap \mathrm{L}(M_2) = \varnothing \} \, . \end{aligned}$$

   We already discussed $\mathrm{E}_{\mathrm{DTM}}$ in lecture. The language $\mathrm{DISJ}_{\mathrm{DTM}}$ contains all encodings $\langle M_1, M_2 \rangle$ of pairs of DTMs whose corresponding languages are *disjoint*.

   (a) Prove that $\mathrm{E}_{\mathrm{DTM}} \leq_m \mathrm{DISJ}_{\mathrm{DTM}}$.
   (b) Prove that $\mathrm{DISJ}_{\mathrm{DTM}} \leq_m \mathrm{E}_{\mathrm{DTM}}$.

   (When answering this question you should assume that $\mathrm{E}_{\mathrm{DTM}}$ and $\mathrm{DISJ}_{\mathrm{DTM}}$ are languages over the same alphabet Σ.)

4. [4 points] Prove that there does not exist a DTM M that simultaneously satisfies both of the following two properties:

   (i) If K is a DTM that halts on all input strings over its alphabet and $\mathrm{L}(K)$ is infinite, then M accepts $\langle K \rangle$.
   (ii) If K is a DTM that halts on all input strings over its alphabet and $\mathrm{L}(K)$ is finite, then M does not accept $\langle K \rangle$.

5. [6 points] Let $\Sigma = \{0, 1\}$ and let $A, B \subseteq \Sigma^*$ be languages.

   (a) Prove that if A and B are both in NP, then the union $A \cup B$ is also in NP.
   (b) Prove that if A is NP-complete, B is in P, $A \cap B = \varnothing$, and $A \cup B \neq \Sigma^*$, then $A \cup B$ is NP-complete.

6. [1 point] For each of the questions above, list the full name of each of your 360 classmates with whom you worked on that question. (If you didn't work with anyone, that is fine: just indicate that you worked alone.)