Proofs. Induction

1. Prove that $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is an irrational number.

2. Given 50 distinct positive integers strictly less than 99, prove that some two of them sum to 99.

3. Show that any odd integer x in the range $10^9 < x < 2 \cdot 10^9$ containing all ten digits $0, 1, \ldots, 9$, must have consecutive even digits.

4. Prove that every integer (positive, negative, or zero) can be written in the form $\sum_i \pm 3^i$, where the exponents i are distinct non-negative integers. For example

$$\begin{aligned} 42 &= 3^4 - 3^3 - 3^2 - 3^1, \\ 25 &= 3^3 - 3^1 + 3^0, \\ 17 &= 3^3 - 3^2 - 3^0. \end{aligned}$$

5. Prove that given an unlimited supply of 6-cent coins, 10-cent coins, and 15-cent coins, one can make any amount of change larger than 29 cents.

6. Good coins weigh 10g, bad coins weigh 9g. Given four coins and an accurate scale, determine which coins are good and which are bad with at most three weighings.

7. A *bitstring* is a finite non-empty sequence of 0s and 1s. Let B^n be the set of bitstrings of lenth n and let $B^{<n}$ denote the set of bistrings of length less than n. Prove that it does not exists a one-to-one function $f : B^n \to B^{<n}$. Deduce that it is impossible to come up with a *lossless* compression algorithm that always makes strings shorter. Discuss.

8. Prove that $3^n \geq n^3$ for all positive integers n.

***9.** Show that for each positive integer n we can find an n-digit number with all its digits odd which is divisible by 5^n.