Properties of CFGs

1. Let $L \subseteq \Sigma^*$ be a regular language, where $\Sigma = \{0, 1\}$. Prove that the languages

$$L_1 = \{ww^R | w \in L\}$$

and

$$L_2 = \{uv^R | u, v \in L \text{ and } |u| = |v|\}$$

are context-free.

2. Let $L \subseteq \Sigma^*$ be a context-free language over some alphabet Σ. Prove that the language L^R is context-free.

3. Show that the grammar $S \to aS|aSbS|\epsilon$ is ambiguous and find an unambiguous grammar that generates the same language.

4. Let G be a context-free grammar in Chomsky normal form. Prove that all parse trees for strings of length n have $2n - 1$ internal nodes (that is, nodes labeled by non-terminals).

5. Decide whether each the following statements is true or false. Give a short proof for each statement you believe is true, and give a counter-example to each statement you believe is false.

a) For any context-free languages L_1 and L_2 over the alphabet $\{0, 1\}$, the language $L_1 - L_2$ is context-free.

b) If L is not context-free and F is finite, then $L - F$ is not context-free.

c) The string $aabbabba$ is in the language generated by the following grammar.

$$S \to aaB$$
$$A \to bBb|\epsilon$$
$$B \to Aa$$

6. Consider the following context-free grammar G over the alphabet $\{a, b\}$.

$$S \to aSb|bY|Ya$$
$$Y \to bY|aY|\epsilon$$

Give a simple description of $L(G)$ in English. Use that description to give a CFG for the complement of $L(G)$.

7. Let $\Sigma = \{0, 1\}$ and let $L \subseteq \Sigma$ be a context-free language. Prove that the following languages are also context-free:

a) $\text{Prefix}(L) = \{x \in \Sigma^* |$ there exists $v \in \Sigma^*$ such that $xv \in L\}$.

b) $\text{Suffix}(L) = \{x \in \Sigma^* |$ there exists $u \in \Sigma^*$ such that $ux \in L\}$.

c) $\text{EraseOne}(L) = \{uv | \ u, v \in \Sigma^*$ and there exists $\sigma \in \Sigma$ such that $u\sigma v \in L\}$.