CS 360 Introduction to the Theory of Computing

Exam 2 Practice Problems

Exam 2 will cover material up to and including Lecture 14. The sample problems below are representative of some of the sorts of problems you might see on the exam.

1. This problem is a variation on one concerning balanced parentheses from Assignment 2, but this time there are two types of parentheses.

   Let $\Sigma = \{\, (\,,\,)\,,\,[\,,\,]\,\}$. In words, the symbols in this alphabet correspond to *left parenthesis*, *right parenthesis*, *left square-bracket*, and *right square-bracket*. To say that a string w over the alphabet Σ is *properly balanced* means that by repeatedly removing either of the substrings ( ) or [ ] from w, you can eventually reach ε.

   These are examples of properly balanced strings:

   $$[\,(\,)\,(\,)\,]\,[\,]\,,\quad (\,(\,[\,]\,(\,)\,)\,)\,,\quad [\,[\,]\,]\,,\quad \text{and}\quad \varepsilon.$$

   These are examples of strings that are not properly balanced:

   $$[\,(\,]\,)\,,\quad [\,[\,(\,)\,]\,[\,]\,,\quad \text{and}\quad [\,[\,]\,]\,)\,[\,].$$

   (a) Prove that the language consisting of all properly balanced strings over the alphabet Σ is context-free.

   (b) Prove that the language consisting of all strings over the alphabet Σ that are not properly balanced is context-free.

2. Give context-free grammars for each of the following languages:

   $$\begin{aligned} A &= \{x \in \{0,1\}^* \,:\, x = x^R \text{ and } |x|_1 \text{ is divisible by } 3\},\\ B &= \{0^n 1^m 0^k \,:\, n,m,k \in \mathbb{N},\ m = n+k \text{ or } n = m+k\},\\ C &= \{0^n 1^m \,:\, m \leq n \leq 2m\},\\ D &= \{0^n 1^m \,:\, n \notin \{m, 2m\}\}. \end{aligned}$$

   You do not need to prove that your grammars are correct.

3. Let $\Sigma = \{0,1\}$, and suppose $A \subseteq \Sigma^*$ is a regular language. Define a new language $B \subseteq \Sigma^*$ as

   $$B = \{xy \in \Sigma^* \,:\, |x| = |y| \text{ and there exists a symbol } \sigma \in \Sigma \text{ such that } x\sigma y \in A\}.$$

   In words, B contains those strings that can be obtained by choosing a string of odd length from A and removing the middle symbol. Prove that B is context-free.

   Note that this problem is similar to one from Assignment 2, but it is different: here we are removing the middle symbol rather than adding it.

4. Let $\Sigma = \{0,1\}$. For any two strings $x, y \in \Sigma^*$ with $|x| = |y|$, define $x \oplus y \in \Sigma^*$ to be the bitwise XOR of x and y. For instance, $00101 \oplus 10110 = 10011$. As you might expect, we define $\varepsilon \oplus \varepsilon = \varepsilon$.

   Suppose $A \subseteq \Sigma^*$ is a regular language. Define $B \subseteq \Sigma^*$ as

$$B = \{xy^R \;:\; x, y \in \Sigma^*,\ |x| = |y|, \text{ and } x \oplus y \in A\}.$$

   Prove that B is context-free.

5. Define a language $A \subseteq \{0,1\}^*$ as follows:

$$A = \{0^n 1^{n \cdot m} \;:\; n, m \in \mathbb{N}\}.$$

   Prove that A is not context-free.

   If you choose, feel free to make use of the following fact without proving it:

   > If $n, j, k \in \mathbb{N}$ are integers satisfying $n \geq 2$ and $1 \leq j + k \leq n$, then $n^2 + j$ does not evenly divide $n^3 + k$.

6. Let Σ be an alphabet. For any two strings $x, y \in \Sigma^*$ having equal length n, define

$$\text{shuffle}(x, y) \in \Sigma^*$$

   to be the string having length $2n$ that is obtained by alternating back and forth between the symbols in x and y, starting with the first symbol of x. For example, if $\Sigma = \{0,1\}$, then

$$\text{shuffle}(110, 010) = 101100.$$

   One may also define this operation recursively as follows:

   (i) $\text{shuffle}(\varepsilon, \varepsilon) = \varepsilon$, and
   (ii) $\text{shuffle}(\sigma x, \tau y) = \sigma\tau\,\text{shuffle}(x, y)$ for all $\sigma, \tau \in \Sigma$ and $x, y \in \Sigma^*$ with $|x| = |y|$.

   Give an example of an alphabet Σ and two context-free languages $A, B \subseteq \Sigma^*$ for which the language

$$C = \{\text{shuffle}(x, y) \;:\; x \in A,\ y \in B, \text{ and } |x| = |y|\}$$

   is not context-free. You do not need to prove that C is not context-free—just give the languages A and B.

7. Prove that the following languages are all decidable:

$$A = \left\{ \langle D, k \rangle \;:\; \begin{array}{l} D \text{ is a DFA, } k \in \mathbb{N}, \text{ and } \mathrm{L}(D) \text{ does not contain} \\ \text{any strings of length } k \end{array} \right\},$$

$$B = \left\{ \langle G, k \rangle \;:\; \begin{array}{l} G \text{ is a CFG, } k \in \mathbb{N}, \text{ and } \mathrm{L}(G) \text{ does not contain} \\ \text{any strings of length } k \end{array} \right\},$$

$$C = \{\langle D \rangle \;:\; D \text{ is a minimal DFA}\}.$$

   (A minimal DFA is one that has the smallest possible number of states needed to recognize its language.)

8. Let Σ be an alphabet and let $w \in \Sigma^*$ be a string. Let us define $\text{rotate}(x) \in \Sigma^*$ to be the string that you would obtain by removing the rightmost symbol of w and then adding that symbol to the left-hand side of w. A recursive definition of this operation is as follows:

   (i) $\text{rotate}(\varepsilon) = \varepsilon$, and

   (ii) $\text{rotate}(w\sigma) = \sigma w$ for all $\sigma \in \Sigma$ and $w \in \Sigma^*$.

   Now suppose that $\Sigma = \{0, 1\}$ and $A \subseteq \Sigma^*$ is a context-free language. Prove that the language

$$B = \big\{\text{rotate}(w) \,:\, w \in A\big\}$$

   is also context-free.