Question 1. Prove that given an unlimited supply of 6-cent coins, 10-cent coins, and 15-cent coins, one can make any amount of change larger than 29 cents.

Answer:
More formally, we are trying to prove that given any $n > 29$ we can create n using $6, 10$, and 15 cent coins. It is not naturally intuitive to see how to use induction here. So think about what you would do in real life, you would grab one of the coins, say 6-cent, then try to make $n - 6$ using 6,10 and 15 cent coins. Keep doing this until it hopefully it works out.

Now the best way to explain the strategy on how to attack this type of induction is the following:

- Pick the smallest coin. (6-cent)
- Notice that if we can create 30, 31, 32, 33, 34, and 35 explicitly, then adding a 6-cent coin in any of those cases would get you 36, 37, 38, 39, 40 and 41. You could keep doing this until you get to any n. So we can reduce this problem to a problem of adding a 6-cent coin.
- Explicitly create 30, 31, 32, 33, 34 and 35 as the base cases.
- Assuming that $k - 6$ for $k > 35$ can be create using 6, 10, 15 cent coins, add 6-cent coin to make k.
- Conclude that by mathematical induction, we have shown that for all $n > 29$, n can be made with any combination of 6, 10, 15 cent coins.

Proof:
<u>Base Cases:</u>

$n = 30 \ = 15 + 15$

$n = 31 \ = 15 + 10 + 6$

$n = 32 \ = 10 + 10 + 6 + 6$

$n = 33 \ = 15 + 6 + 6 + 6$

$n = 34 \ = 10 + 6 + 6 + 6 + 6$

$n = 35 \ = 15 + 10 + 10$

<u>Inductive Case:</u>
Assume $n - 6$ for all $n > 35$ can be created using 6, 10 and 15 cent coins. We want to show that n can be created using 6, 10, and 15 cent coins. We know that $n - 6$ can be created with 6, 10 and 15 cent coins by the inductive hypothesis. Add a 6-cent coin to $n - 6$ change, this is allowed since we can use 6-cent coins, and now $(n - 6) + 6 = n$ is made with $6, 10$ and 15 cent coins.

By mathematical induction we have shown that for all $n > 29$, n can be made with any combination of 6, 10, 15 cent coins.

Question 2. Prove that $2^n \geq n^2$ for all $n \geq 4$. You can assume $k^2 > 4k + 2$ for $k \geq 5$.

Answer:

Base Case: $n = 4$

$2^n = 2^4 = 16 \geq 16 = 4^2 = 4^n$

Inductive Case:
For arbitrary k such that $k \geq 5$, assume $2^{(k-1)} > (k-1)^2$. We want to show that $2^k > k^2$.

$$\begin{aligned}
2^k &= 2(2^{k-1}) \\
&\geq 2(k-1)^2 && \text{(By I.H.)} \\
&= 2(k^2 - 2k - 1) \\
&= 2k^2 - 4k - 2 \\
&= k^2 + (k^2 - 4k - 2) \\
&\geq k^2 && \text{Since } k^2 > 4k + 2 \text{ for } k \geq 5
\end{aligned}$$

Therefore, by mathematical induction we have shown that for all $2^n \geq n^2$ for all $n \geq 4$.

Question 3. Given 50 distinct positive integers strictly less than 99, prove that at least one pair of those integers sum to 99. (Note that this version of the question addresses an edge case problem with the original question, there is no way to sum to 99 if all the numbers (50-99) are chosen, so we restrict to choosing up to 98.).

Answer:
By the way that the question is worded, we can pick 50 numbers between 1 and 98 inclusive. Intuitively summing to 99 could be an issue if we chose all small numbers. Lets check this out!

Pick 1-50, the smallest values possible, we can see that 49 + 50 =99.

Whats wrong with this solution? Although it proves the edge case, and its pretty intuitive that any other group of 50 integers would also work, we didn't prove it in general. What we need is the pigeonhole principle!

The pigeonhole principle states that given n pigeons, and $m < n$ holes, there will be atleast 2 pigeons in one of the holes.

What we need to do is figure out what the pigeons and holes are in our problem. Let the pigeons be the positive integers, and let the holes be all possible pairs of integers between 1 and 99 inclusive that add up to 99:

$$(1, 98)(2, 97)(3, 96) \ldots (47, 52)(48, 51)(49, 50)$$

we can see that there are 49 holes.

Since we have 50 integers, and 49 pairs, there must be at least one pair that has two integers in it. Therefore atleast one pair of number from the 50 chosen integers add up to 99.