From Kleinberg Notes.

Example 3. We want to construct a total TM that accepts its input string if the length of the string is prime. We will give a TM implementation of the *sieve of Eratosthenes*, which can be described informally as follows. Say we want to check whether n is prime. We write down all the numbers from 2 to n in order, then repeat the following: find the smallest number in the list, declare it prime, then cross off all multiples of that number. Repeat until each number in the list has been either declared prime or crossed off as a multiple of a smaller prime.

For example, to check whether 23 is prime, we would start with all the numbers from 2 to 23:

2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23

In the first pass, we cross off multiples of 2:

$\cancel{2}$ 3 $\cancel{4}$ 5 $\cancel{6}$ 7 $\cancel{8}$ 9 $\cancel{10}$ 11 $\cancel{12}$ 13 $\cancel{14}$ 15 $\cancel{16}$ 17 $\cancel{18}$ 19 $\cancel{20}$ 21 $\cancel{22}$ 23

The smallest number remaining is 3, and this is prime. In the second pass we cross off multiples of 3:

$\cancel{2}$ $\cancel{3}$ $\cancel{4}$ 5 $\cancel{6}$ 7 $\cancel{8}$ $\cancel{9}$ $\cancel{10}$ 11 $\cancel{12}$ 13 $\cancel{14}$ $\cancel{15}$ $\cancel{16}$ 17 $\cancel{18}$ 19 $\cancel{20}$ $\cancel{21}$ $\cancel{22}$ 23

Then 5 is the next prime, so we cross off multiples of 5; and so forth. Since 23 is prime, it will never be crossed off as a multiple of anything smaller, and eventually we will discover that fact when everything smaller has been crossed off.

Now we show how to implement this on a TM. Suppose we have a^p written on the tape. We illustrate the algorithm with $p = 23$.

$$\vdash a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a \sqcup\sqcup\sqcup\cdots$$

If $p = 0$ or $p = 1$, reject. We can determine this by looking at the first three cells of the tape. Otherwise, there are at least two a's. Erase the first a, scan right to the end of the input, and replace the last a in the input string with the symbol \$. We now have an a in positions $2, 3, 4, \ldots, p-1$ and \$ at position p.

$$\vdash \sqcup\, a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\, \$ \sqcup\sqcup\sqcup\cdots$$

Now we repeat the following loop. Starting from the left endmarker $\vdash$, scan right and find the first nonblank symbol, say occurring at position m. Then m is prime (this is an invariant of the loop). If this symbol is the \$, we are done: $p = m$ is prime, so we halt and accept. Otherwise, the symbol is an a. Mark it with a $\widehat{\ }$ and everything between there and the left endmarker with $\acute{\ }$.

$$\vdash \acute{\sqcup}\,\widehat{a}\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\, \$ \sqcup\sqcup\sqcup\cdots$$

We will now enter an inner loop to erase all the symbols occurring at positions that are multiples of m. First, erase the a under the $\widehat{\ }$. (Formally, just write the symbol $\widehat{\sqcup}$.)

$$\vdash \acute{\sqcup}\,\widehat{\sqcup}\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\, \$ \sqcup\sqcup\sqcup\cdots$$

Shift the marks to the right one at a time a distance equal to the number of marks. This can be done by shuttling back and forth, erasing marks on the left and writing them on the right. We know when we are done because the $\widehat{\ }$ is the last mark moved.

$$\vdash \sqcup\,\sqcup\,\acute{a}\,\widehat{a}\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\,a\, \$ \sqcup\sqcup\sqcup\cdots$$

When this is done, erase the symbol under the $\hat{}$. This is the symbol occurring at position $2m$.

$$\vdash \sqcup \sqcup \acute{a} \hat{\sqcup} a a a a a a a a a a a a a a a a a a \$ \sqcup \sqcup \sqcup \cdots$$

Keep shifting the marks and erasing the symbol under the $\hat{}$ in this fashion until we reach the end.

$$\vdash \sqcup \sqcup a \sqcup a \sqcup a \sqcup a \sqcup a \sqcup a \sqcup a \sqcup a \sqcup a \sqcup a \sqcup \acute{\$} \hat{\sqcup} \sqcup \sqcup \cdots$$

If we find ourselves at the end of the string wanting to erase the \$, reject—p is a multiple of m but not equal to m. Otherwise, go back to the left and repeat. Find the first nonblank symbol and mark it and everything to its left.

$$\vdash \acute{\sqcup} \acute{\sqcup} \hat{a} \sqcup a \sqcup a \sqcup a \sqcup a \sqcup a \sqcup a \sqcup a \sqcup a \sqcup a \sqcup \$ \sqcup \sqcup \sqcup \cdots$$

Alternately erase the symbol under the $\hat{}$ and shift the marks until we reach the end of the string.

$$\vdash \sqcup \sqcup \sqcup \sqcup a \sqcup a \sqcup \sqcup \sqcup a \sqcup a \sqcup \sqcup \sqcup a \sqcup a \sqcup \sqcup \acute{\sqcup} \acute{\$} \hat{\sqcup} \sqcup \sqcup \cdots$$

Go back to the left and repeat.

$$\vdash \acute{\sqcup} \acute{\sqcup} \acute{\sqcup} \acute{\sqcup} \hat{a} \sqcup a \sqcup \sqcup \sqcup a \sqcup a \sqcup \sqcup \sqcup a \sqcup a \sqcup \sqcup \sqcup \$ \sqcup \sqcup \sqcup \cdots$$

If we ever try to erase the \$, reject—p is not prime. If we manage to erase all the a's, accept.