## CS 360 Introduction to the Theory of Computing

# Exam 1 Practice Problems

Here are some practice problems that may help you to prepare for your first exam. Most of these problems come from old exams or homework assignments—and for a couple of them you very well may find solutions in the notes.

1. The following problems each have a short answer, perhaps just a few sentences and maybe an equation or two. Try to make your answers clear and to the point—and choose the simplest answer whenever possible.

   (a) Suppose that $A$ and $B$ are nonempty sets for which $B \subseteq A$, and assume that there exists an onto function $f$ of the form $f : \mathbb{N} \to A$. Prove that there exists an onto function of the form $g : \mathbb{N} \to B$. What do you conclude from this fact about any subset of a countable set?

   (b) Suppose that $A, B \subseteq \{0,1\}^*$ are languages for which it holds that $A \subseteq B$ and $B$ is regular. Is it necessarily the case that $A$ is regular? Give a short argument in support of your answer.

2. Decide whether each the following statements is true or false, and defend your answer: give a short proof for each statement you believe is true, and give a counter-example to each statement you believe is false. Assume that $\Sigma = \{0,1\}$, and that $A$ and $B$ are languages over $\Sigma$ for each of the statements.

   (a) If $A^*$ is regular then $A$ is also regular.

   (b) If $A$ and $B$ are nonregular, then $AB$ is also nonregular.

   (c) If $A$ is nonregular, then there exists a nonregular language $C \subseteq \Sigma^*$ such that $A \cap C$ is finite.

   (d) If $A$ and $B$ are nonregular, then $A \cup B$ is not finite.

   (e) If $A$ is nonregular, $B$ is regular, and $A \cap B = \varnothing$, then $A \cup B$ is nonregular.

3. Suppose that $\Sigma$ is an alphabet and $A \subseteq \Sigma^*$ is a regular language. Define a new language $B \subseteq \Sigma^*$ as follows:
$$B = \{x\sigma y \,:\, x, y \in \Sigma^*,\ \sigma \in \Sigma \text{ and } xy \in A\}.$$
   In words, $B$ consists of all strings that can be obtained by inserting exactly one symbol $\sigma \in \Sigma$ somewhere into any string in $A$. Prove that $B$ is regular.

4. (This one is a variation on the previous problem.) Let $\Sigma$ be an alphabet, and let $A$ and $B$ be regular languages over the alphabet $\Sigma$. Prove that the following language is regular:
$$C = \{uvw \,:\, u, v, w \in \Sigma^*,\ uw \in A \text{ and } v \in B\}.$$

5. Let $\Sigma = \{0,1\}$ and let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA, where $Q = \{q_0, \ldots, q_{n-1}\}$ for some positive integer $n$. Suppose further that an NFA $N$ is formed by adding a (non-accepting) state $q_n$ to $M$, along with any number of transitions to or from $q_n$.

   In more formal terms, you are to assume that $N = (Q \cup \{q_n\}, \Sigma, \mu, q_0, F)$ for some choice of a transition function

$$\mu : (Q \cup \{q_n\}) \times (\Sigma \cup \{\varepsilon\}) \to \mathcal{P}(Q \cup \{q_n\})$$

   that satisfies these properties:

   - $\mu(q_j, \sigma) = \{\delta(q_j, \sigma)\}$ or $\mu(q_j, \sigma) = \{\delta(q_j, \sigma),\ q_n\}$ for each $j \in \{0, \ldots, n-1\}$ and $\sigma \in \Sigma$,
   - $\mu(q_j, \varepsilon) = \varnothing$ or $\mu(q_j, \varepsilon) = \{q_n\}$ for each $j \in \{0, \ldots, n-1\}$, and
   - $\mu(q_n, \sigma)$ is an arbitrary subset of $Q \cup \{q_n\}$ for each $\sigma \in \Sigma \cup \{\varepsilon\}$.

   Decide whether each of the following statements is true or false, and defend each answer with a short, high-level proof sketch or a counter-example.

   (a) $\mathrm{L}(M) \subseteq \mathrm{L}(N)$.

   (b) The language of all strings that are accepted by $N$, but not accepted by $M$, is regular.

   (c) The language of all strings $x$ that are accepted by a computation of $N$ that visits the state $q_n$ at least once (i.e., for which there exists a sequence of states that includes $q_n$ and satisfies the definition for acceptance of $x$ by $N$) is regular.

   (d) The minimal DFA for $\mathrm{L}(N)$ is at least as large as the minimal DFA for $\mathrm{L}(M)$.

   (The minimal DFA for any regular language is the unique DFA that has the smallest possible number of states among all the DFAs for that language.)

6. Let $\Sigma$ be an alphabet. For any two strings $x, y \in \Sigma^*$ having equal length, define

$$\text{shuffle}(x, y) \in \Sigma^*$$

   to be the string that is obtained by alternating back and forth between the symbols in $x$ and $y$, starting with the first symbol of $x$. For example, if $\Sigma = \{0,1\}$, then

$$\text{shuffle}(110, 010) = 101100.$$

   One may also define this operation recursively as follows:

   (i) $\text{shuffle}(\varepsilon, \varepsilon) = \varepsilon$, and

   (ii) $\text{shuffle}(\sigma x, \tau y) = \sigma\tau\,\text{shuffle}(x, y)$ for all $\sigma, \tau \in \Sigma$ and $x, y \in \Sigma^*$ with $|x| = |y|$.

   Prove that, for every choice of an alphabet $\Sigma$ and regular languages $A, B \subseteq \Sigma^*$, it holds that the language

$$C = \big\{\text{shuffle}(x, y) \,:\, x \in A,\ y \in B,\ \text{and } |x| = |y|\big\}$$

   is also regular.

7. Prove that the following two languages are both nonregular:

$$A = \{0^n 1^m \,:\, n, m \in \mathbb{N},\, 99n \geq 100m\}$$
$$B = \{w \in \{0,1,2\}^* \,:\, 99\,|w|_0 \geq 100\,|w|_1\}.$$

8. Let $A = \{0^n 1^n \,:\, n \in \mathbb{N}\}$ and suppose that $B \subseteq \{0,1\}^*$ is a language for which $B \subseteq A$. Prove that $B$ is regular if and only if $B$ is finite.