CS 360 Introduction to the Theory of Computing

Final Exam Practice Problems

The final exam for CS 360 will be comprehensive: you should be prepared for problems on all of the material covered in the course. Some practice problems follow—but be sure to also look back at all of the homework assignments, exams, solutions, and practice problems from earlier in the course.

1. Let $\Sigma = \{0,1\}$. For every language $A \subseteq \Sigma^*$, define a new language $\operatorname{half}(A) \subseteq \Sigma^*$ as follows:

$$\operatorname{half}(A) = \big\{x \in \Sigma^* \,:\, \text{there exists } y \in \Sigma^* \text{ such that } |y| = |x| \text{ and } xy \in A\big\}.$$

   In words, a string is in $\operatorname{half}(A)$ if and only if it is the first half of any even-length string in A.

   (a) Prove that if A is Turing-recognizable, then $\operatorname{half}(A)$ is Turing-recognizable.

   (b) Prove that if A is regular, then $\operatorname{half}(A)$ is regular. (This one is probably too hard for an exam, but it is good practice anyway.)

2. Determine whether each of the following statements is true or false.

   (a) If a language A can be decided by a Turing machine having two tapes, then A is also decidable by a Turing machine having just one tape.

   (b) If A is a decidable language, then either A is infinite or A is regular (or both).

   (c) If M_1 and M_2 are DTMs having the same input alphabet, then the union $\mathrm{L}(M_1) \cup \mathrm{L}(M_2)$ is a Turing-recognizable language.

   (d) Every Turing-recognizable language $A \subseteq \{0\}^*$ over the single-symbol alphabet $\{0\}$ is also decidable.

   (e) If A is a Turing-recognizable language and B is a context-free language, then the intersection $A \cap B$ is a decidable language.

   (f) If A is a decidable language and B is a context-free language, then the intersection $A \cap \overline{B}$ is a decidable language.

   (g) If $A \subseteq \{0,1\}^*$ is a Turing-recognizable language, then there exists a regular language $B \subseteq \{0,1\}^*$ and a decidable language $C \subseteq \{0,1\}^*$ such that

$$A = \{w \in \{0,1\}^* \,:\, \text{there exists } x \in B \text{ such that } wx \in C\}.$$

   (h) If $B \subseteq \{0,1\}^*$ is a decidable language, then the language

$$A = \{w \in \{0,1\}^* \,:\, \{w\}^* \cap B \neq \varnothing\}$$

   is also decidable.

3. Prove that the following language is decidable:

$$\text{ONE}_{\text{CFG}} = \{\langle G\rangle \,:\, G \text{ is a CFG such that } \mathrm{L}(G) \text{ contains exactly one string}\}.$$

4. Prove that the following language is not Turing recognizable:

$$\text{ONE}_{\text{DTM}} = \{\langle M \rangle \; : \; M \text{ is a DTM such that } \mathrm{L}(M) \text{ contains exactly one string}\}.$$

(If you are not able to prove that this language is non-Turing-recognizable, try proving it is undecidable instead.)

5. Let $\Sigma = \{0,1\}$ and suppose that $A \subseteq \Sigma^*$ is a Turing-recognizable language. Prove that there exists a decidable language $B \subseteq \Sigma^*$ such that

$$x \in A \Leftrightarrow \left(\exists y \in \Sigma^*\right)\left[\langle x, y\rangle \in B\right]$$

for every $x \in \Sigma^*$.

6. Define a language B as follows:

$$B = \left\{ \langle M, x\rangle \; : \; \begin{array}{l} M \text{ is a DTM, } x \text{ is a string over the input alphabet of } M, \\ \text{and } M \text{ accepts at least one string } y \text{ with } y \leq x \end{array} \right\}$$

The notation $u \leq v$ means that either $u = v$ or u comes before v with respect to the lexicographic ordering (of strings over the input alphabet of M).

(a) Prove that B is Turing recognizable.
(b) Prove that B is undecidable.

7. Let $\Sigma = \{0,1\}$ and suppose that $A \subseteq \Sigma^*$ is context-free. Prove that the language

$$B = \{u1v1w \; : \; u, v, w \in \Sigma^* \text{ and } uvw \in A\}$$

is also context-free. In words, B is the language consisting of all strings that can be obtained by choosing a string from A and inserting exactly two occurrences of the symbol 1 into that string.

8. Let $\Sigma = \{0,1\}$ and suppose that $A \subseteq \Sigma^*$ is context-free. Prove that the language

$$B = \{uvw \; : \; u, v, w \in \Sigma^* \text{ and } u1v1w \in A\}$$

is also context-free. In words, B is the language consisting of all strings that can be obtained by choosing a string from A and removing exactly two occurrences of the symbol 1 from that string.

9. This problem and the next are variants of the previous two for regular languages in place of context-free languages. Let $\Sigma = \{0,1\}$ and suppose that $A \subseteq \Sigma^*$ is regular. Prove that the language

$$B = \{u1v1w \; : \; u, v, w \in \Sigma^* \text{ and } uvw \in A\}$$

is also regular.

10. Let $\Sigma = \{0,1\}$ and suppose that $A \subseteq \Sigma^*$ is regular. Prove that the language

$$B = \{uvw \,:\, u,v,w \in \Sigma^* \text{ and } u1v1w \in A\}$$

is also regular.

11. For any nonempty string $x = \sigma_{n-1} \cdots \sigma_0$ over the alphabet $\{0,1\}$, define

$$b(x) = \sum_{k=0}^{n-1} \sigma_k \cdot 2^k,$$

and define $b(\varepsilon) = 0$. For a nonempty string x, we have that $b(x)$ is the number represented by x in binary notation, allowing any number of leading zeros.

Define a language $A \subseteq \{0,1,\#\}^*$ as follows:

$$A = \big\{x\#y^R \,:\, x,y \in \{0,1\}^*,\ |x| = |y|, \text{ and } b(x) + b(y) \text{ is divisible by } 3\big\}.$$

Prove that A is context-free.

12. Let Σ be an alphabet and let $w \in \Sigma^*$ be a string. Let us define $\text{rotate}(w) \in \Sigma^*$ to be the string that you would obtain by removing the rightmost symbol of w and then adding that symbol to the left-hand side of w. A recursive definition of this operation is as follows:

    (i) $\text{rotate}(\varepsilon) = \varepsilon$, and
    (ii) $\text{rotate}(w\sigma) = \sigma w$ for all $\sigma \in \Sigma$ and $w \in \Sigma^*$.

    Now suppose that $\Sigma = \{0,1\}$ and $A \subseteq \Sigma^*$ is a regular language. Prove that the language

$$B = \big\{\text{rotate}(w) \,:\, w \in A\big\}$$

is also regular.

13. Prove that the language $A = \big\{0^n 1^m \,:\, n,m \in \mathbb{N},\ n = 2^m\big\}$ is not context-free.

14. Let Σ be an alphabet and suppose that $f : \Sigma^* \to \Sigma^*$ is a computable function with the property that

$$x \leq y \;\Rightarrow\; f(x) \leq f(y)$$

for all strings $x,y \in \Sigma^*$. (For two strings $u,v \in \Sigma^*$ we interpret $u \leq v$ to mean that either $u = v$ or u comes before v in the lexicographic ordering of Σ^*.)

Prove that $\text{range}(f) = \{f(x) \,:\, x \in \Sigma^*\}$ is a decidable language.

15. Define a language

$$A = \big\{\langle M\rangle \,:\, M \text{ is a DTM that accepts at least one string of even length}\big\}.$$

    (a) Prove that A is Turing recognizable.
    (b) Prove that $\text{HALT} \leq_m A$.