Computer Science 331

Graph Search: Breadth-First Search

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Lecture #33

Outline

Breadth-First Search

Another way to search a connected component of a graph

Given a graph $G = (V, E)$ and *source vertex* s, the algorithm finds a *breadth-first* tree with root s, that is, a subgraph $\widehat{G} = (\widehat{V}, \widehat{E})$ such that

- $\widehat{G}$ is a tree
- for every vertex $v \in V$, $v \in \widehat{V}$ if and only if c is reachable from s (that is, there is a path from s to v in G — so $\widehat{G}$ is a spanning tree for a connected component of G)
- for each vertex $v \in G$, the simple path from v up to s in $\widehat{G}$ (in which each edge is an edge from a node to its parent in the tree) is a *shortest* path from v to s in G.

The version of the algorithm presented here also returns the *distance* from v to s in G (that is, the *length* of a shortest path from v to s).

Idea

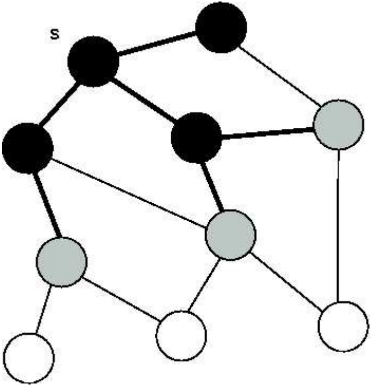
Begin with s; expand the boundary between "discovered" and "undiscovered" vertices uniformly across the breadth of the boundary

As in DFS, Vertices are coloured during the search

- All vertices are initially **white**, s is almost immediately coloured **grey**.
- All white vertices are "undiscovered."
- "Discovered" vertices are either grey or black. Vertices on the boundary between discovered and undiscovered vertices are **grey**. Other discovered vertices are **black**.

Unlike DFS, when a grey vertex t is processed, all white neighbours are recoloured grey; t is then coloured black.

Typical Search Pattern



Specification of Requirements

Precondition: $G = (V, E)$ is a graph and $s \in V$

Postcondition:

- One value returned is a function $\pi : V \to V \cup \{NIL\}$ defining (with s) a *predecessor subgraph*, that is a spanning tree for the connected component of G containing s
- For each vertex v in the above spanning tree, the simple path from v to s in this tree is a *shortest path* from v to s in the graph G
- Another value returned is a function $d : V \to \mathbb{N} \cup \{+\infty\}$; for each vertex $v \in V$, $d[v]$ is the distance from v to s (so that $d[v] = +\infty$ if and only if there is no path from v to s in G).
- The graph G has not been changed.

Data and Data Structures

The following information is maintained for each $u \in V$:

- $colour[u]$: Colour of u
- $d[u]$: Distance of u from s
- $\pi[u]$: Parent of u in tree being constructed

In order to ensure that the search is performed in a “breadth-first” way, a **queue** is used to store grey nodes

Pseudocode

BFS(G, s)

{Initialization}

for each vertex $u \in V$ **do**

$colour[u] = \text{white}$

$d[u] = +\infty$

$\pi[u] = \text{NIL}$

end for

$colour[s] = \text{grey}$

$d[s] = 0$

$\pi[s] = \text{NIL}$

Initialize queue Q to be empty

$Q.\text{add}(s)$

Pseudocode

BFS(G, s)
  {Initialization}
  for each vertex $u \in V$ **do**
    $colour[u] = \text{white}$ {mark all vertices as undiscovered}
    $d[u] = +\infty$
    $\pi[u] = \text{NIL}$
  end for
  $colour[s] = \text{grey}$
  $d[s] = 0$
  $\pi[s] = \text{NIL}$
  Initialize queue Q to be empty
  $Q.\text{add}(s)$

Pseudocode

BFS(G, s)
  {Initialization}
  for each vertex $u \in V$ **do**
    $colour[u] =$ white   {mark all vertices as undiscovered}
    $d[u] = +\infty$
    $\pi[u] =$ NIL
  end for
  $colour[s] =$ grey   {start with source vertex s}
  $d[s] = 0$
  $\pi[s] =$ NIL
  Initialize queue Q to be empty
  Q.add(s)

Pseudocode

BFS(G, s)
{Initialization}
for each vertex $u \in V$ **do**
    $colour[u] = \text{white}$ {mark all vertices as undiscovered}
    $d[u] = +\infty$
    $\pi[u] = \text{NIL}$
end for
$colour[s] = \text{grey}$ {start with source vertex s}
$d[s] = 0$ {path from s to itself has distance 0}
$\pi[s] = \text{NIL}$
Initialize queue Q to be empty
$Q.\text{add}(s)$

Pseudocode

BFS(G, s)

{Initialization}

for each vertex $u \in V$ **do**

$colour[u] =$ white {mark all vertices as undiscovered}

$d[u] = +\infty$

$\pi[u] =$ NIL

end for

$colour[s] =$ grey {start with source vertex s}

$d[s] = 0$ {path from s to itself has distance 0}

$\pi[s] =$ NIL {s is the root of the BFS tree (no parent)}

Initialize queue Q to be empty

Q.add(s)

Pseudocode

BFS(G, s)

{Initialization}

for each vertex $u \in V$ **do**

$colour[u] = \text{white}$ {mark all vertices as undiscovered}

$d[u] = +\infty$

$\pi[u] = \text{NIL}$

end for

$colour[s] = \text{grey}$ {start with source vertex s}

$d[s] = 0$ {path from s to itself has distance 0}

$\pi[s] = \text{NIL}$ {s is the root of the BFS tree (no parent)}

Initialize queue Q to be empty

$Q.\text{add}(s)$ {add first grey node s to the queue}

Pseudocode, Continued

while (Q is not empty) **do**
  $u = Q$.remove()
  for each $v \in Adj[u]$ **do**
    {examine neighbours of u}
    if $colour[v] ==$ white **then**
      $colour[v] =$ grey
      $d[v] = d[u] + 1$
      $\pi[v] = u$
      Q.add(v)
    end if
  end for
  $colour[u] =$ black
end while
return π, d

Pseudocode, Continued

while (Q is not empty) **do**
  $u = Q$.remove()
  for each $v \in Adj[u]$ **do**
    {examine neighbours of u}
    if $colour[v] ==$ white **then**
      $colour[v] =$ grey {discover each undiscovered neighbour}
      $d[v] = d[u] + 1$
      $\pi[v] = u$
      Q.add(v)
    end if
  end for
  $colour[u] =$ black
end while
return π, d

Pseudocode, Continued

while (Q is not empty) **do**
  $u = Q$.remove()
  for each $v \in Adj[u]$ **do**
    {examine neighbours of u}
    if $colour[v] ==$ white **then**
      $colour[v] =$ grey {discover each undiscovered neighbour}
      $d[v] = d[u] + 1$ {shortest path: s to u followed by (u, v)}
      $\pi[v] = u$
      Q.add(v)
    end if
  end for
  $colour[u] =$ black
end while
return π, d

Pseudocode, Continued

while (Q is not empty) **do**
  $u = Q$.remove()
  for each $v \in Adj[u]$ **do**
    {examine neighbours of u}
    if $colour[v] ==$ white **then**
      $colour[v] =$ grey {discover each undiscovered neighbour}
      $d[v] = d[u] + 1$ {shortest path: s to u followed by (u, v)}
      $\pi[v] = u$ {u is the predecessor on the shortest path}
      Q.add(v)
    end if
  end for
  $colour[u] =$ black
end while
return π, d

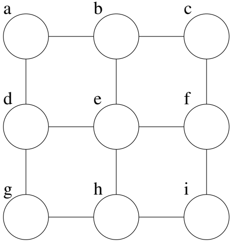
Pseudocode, Continued

```
while (Q is not empty) do
   u = Q.remove()
   for each v ∈ Adj[u] do
      {examine neighbours of u}
      if colour[v] == white then
         colour[v] = grey    {discover each undiscovered neighbour}
         d[v] = d[u] + 1    {shortest path: s to u followed by (u, v)}
         π[v] = u    {u is the predecessor on the shortest path}
         Q.add(v)    {examine neighbours of v}
      end if
   end for
   colour[u] = black
end while
return π, d
```

Pseudocode, Continued

while (Q is not empty) **do**
  $u = Q$.remove()
  for each $v \in Adj[u]$ **do**
    {examine neighbours of u}
    if $colour[v] ==$ white **then**
      $colour[v] =$ grey {discover each undiscovered neighbour}
      $d[v] = d[u] + 1$ {shortest path: s to u followed by (u, v)}
      $\pi[v] = u$ {u is the predecessor on the shortest path}
      Q.add(v) {examine neighbours of v}
    end if
  end for
  $colour[u] =$ black {all neighbours of u have been discovered}
end while
return π, d

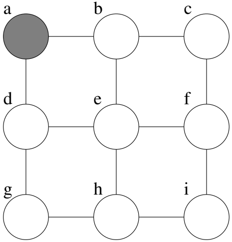
Example



Q

	a	b	c	d	e	f	g	h	i
d									
π									

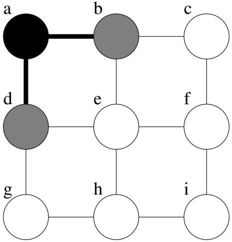
Example



Q a

	a	b	c	d	e	f	g	h	i
d	0	∞	∞	∞	∞	∞	∞	∞	∞
π	NIL	NIL	NIL	NIL	NIL	NIL	NIL	NIL	NIL

Example

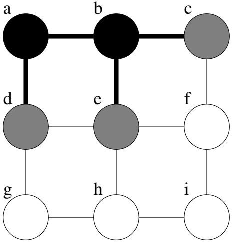


Q

b	d

	a	b	c	d	e	f	g	h	i
d	0	1	∞	1	∞	∞	∞	∞	∞
π	NIL	a	NIL	a	NIL	NIL	NIL	NIL	NIL

Example

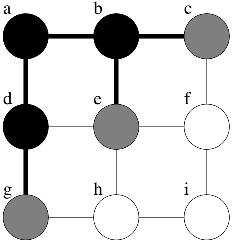


Q

d	c	e

	a	b	c	d	e	f	g	h	i
d	0	1	2	1	2	∞	∞	∞	∞
π	NIL	a	b	a	b	NIL	NIL	NIL	NIL

Example

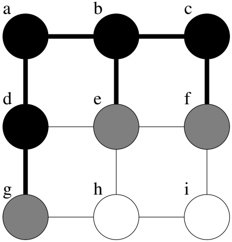


Q

c	e	g

	a	b	c	d	e	f	g	h	i
d	0	1	2	1	2	∞	2	∞	∞
π	NIL	a	b	a	b	NIL	d	NIL	NIL

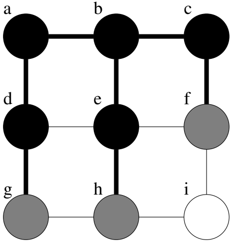
Example



Q	e	g	f

	a	b	c	d	e	f	g	h	i
d	0	1	2	1	2	3	2	∞	∞
π	NIL	a	b	a	b	c	d	NIL	NIL

Example

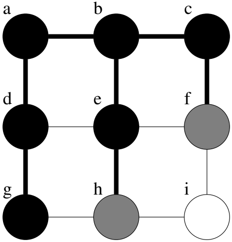


Q

g	f	h

	a	b	c	d	e	f	g	h	i
d	0	1	2	1	2	3	2	3	∞
π	NIL	a	b	a	b	c	d	e	NIL

Example

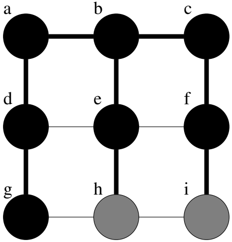


Q

f	h

	a	b	c	d	e	f	g	h	i
d	0	1	2	1	2	3	2	3	∞
π	NIL	a	b	a	b	c	d	e	NIL

Example

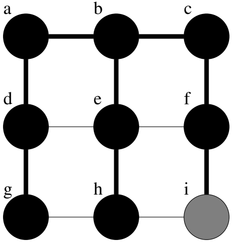


Q

h	i

	a	b	c	d	e	f	g	h	i
d	0	1	2	1	2	3	2	3	4
π	NIL	a	b	a	b	c	d	e	f

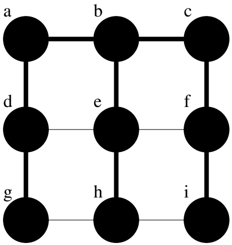
Example



Q $\boxed{\text{i}}$

	a	b	c	d	e	f	g	h	i
d	0	1	2	1	2	3	2	3	4
π	NIL	a	b	a	b	c	d	e	f

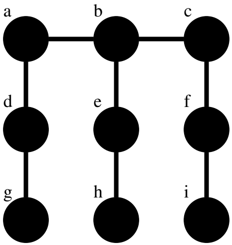
Example



Q

	a	b	c	d	e	f	g	h	i
d	0	1	2	1	2	3	2	3	4
π	NIL	a	b	a	b	c	d	e	f

Example



Q $\square$

	a	b	c	d	e	f	g	h	i
d	0	1	2	1	2	3	2	3	4
π	NIL	a	b	a	b	c	d	e	f

Useful Properties Concerning Colours

Each of the following properties hold immediately before each execution of the outer `while` loop and before each execution of the inner `for` loop.

- All nodes that have never been added to the queue are `white`.
- All nodes that are currently of the queue are `grey`.
- All nodes that have been of the queue but later removed from it are `black`.
- All nodes that have been included in the predecessor subgraph (for π) are either `grey` or `black`. All other nodes are `white`.

It is also clear — by inspection of the code —that the colour of a node is never changed again once it becomes `black`.

Useful Property of Distances

The *shortest-path distance* $\delta(s, v)$ from s to v is the minimum number of edges on a path from s to v.

Lemma 1

Let $G = (V, E)$ be an undirected graph, and let $s \in V$ be an arbitrary vertex. Then, for every edge $(u, v) \in E$, $\delta(s, v) \leq \delta(s, u) + 1$.

Proof.

If u is reachable from s :

□

Useful Property of Distances

The *shortest-path distance* $\delta(s, v)$ from s to v is the minimum number of edges on a path from s to v.

Lemma 1

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Proof.

If u is reachable from s :

- one path from s to v : shortest path to u followed by edge (u, v)

$\square$

Useful Property of Distances

The *shortest-path distance* $\delta(s, v)$ from s to v is the minimum number of edges on a path from s to v.

Lemma 1

Let $G = (V, E)$ be an undirected graph, and let $s \in V$ be an arbitrary vertex. Then, for every edge $(u, v) \in E$, $\delta(s, v) \leq \delta(s, u) + 1$.

Proof.

If u is reachable from s :

- one path from s to v : shortest path to u followed by edge (u, v)
- shortest path to v is at most as long as this path ($\delta(s, u) + 1$)

□

Useful Property of Distances

The *shortest-path distance* $\delta(s, v)$ from s to v is the minimum number of edges on a path from s to v.

Lemma 1

Let $G = (V, E)$ be an undirected graph, and let $s \in V$ be an arbitrary vertex. Then, for every edge $(u, v) \in E$, $\delta(s, v) \leq \delta(s, u) + 1$.

Proof.

If u is reachable from s :

- one path from s to v : shortest path to u followed by edge (u, v)
- shortest path to v is at most as long as this path $(\delta(s, u) + 1)$

Otherwise, $\delta(s, u) = \infty$ and the inequality holds. □

Lemma: Distance Inequality

Lemma 2

Let $G = (V, E)$ be an undirected graph, suppose BFS is run on G from a given source vertex $s \in V$, and suppose that this algorithm terminates. Then (on termination of the algorithm), for each vertex $v \in V$ if $d[v]$ is a nonnegative integer then the sequence of edges

$$(v, \pi[v]), (\pi[v], \pi[\pi[v]]), \ldots$$

starting from v, and following edges from each vertex to its parent, forms a path of length $d[v]$ from v to s in G.

Method of Proof.

Prove that this property is satisfied at the both the beginning and end of each execution of the body of the `while` loop, using induction on the number of executions. □

Lemma: Enqueued Vertices

Lemma 3

Suppose that, at the beginning of an execution of the body of the `while` *loop, the queue Q contains vertices $\langle v_1, v_2, \ldots, v_r \rangle$, where v_1 is the head of Q and v_r is the tail of Q. Then $d[v_r] \leq d[v_1] + 1$ and $d[v_i] \leq d[v_{i+1}]$ for $1 \leq i \leq r-1$.*

Method of Proof.

Use induction on the number of nodes that have already been *removed* from the queue at this point. □

Note: One can show (indeed, one likely *does* show as part of the above proof) that this property is satisfied at the *end* of each execution of the body of the loop when the queue is nonempty, as well.

Lemma: Distance and Queue Order

Lemma 4

Suppose that vertices v_i and v_j are added to the queue during the execution of BFS, and that v_i is added before v_j. Then $d[v_i] \leq d[v_j]$ at the time v_j is added to the queue.

Method of Proof.

Use induction on the number of vertices that were added to the queue *between* v_i and v_j. Follows from Lemma 3, and the fact that each vertex only receives a finite d value once. $\square$

Lemma: Correctness of Distance

Lemma 5

If a vertex v is added to the queue at any point during the execution of the BFS algorithm, then v is reachable from s. Furthermore, the value $d[v]$ that is set immediately before v is added to the queue is equal to $\delta(s, v)$.

See lecture supplement for complete proof.

Lemma: Completeness of Predecessor Subgraph

Lemma 6

Suppose the BFS algorithm is run with an undirected graph $G = (V, E)$ and vertex $s \in V$ as input. If the algorithm terminates then, on termination, the predecessor subgraph for the function π and vertex s is a spanning tree for the connected component of G that includes s.

Method of Proof.

Lemma 2 implies that every vertex in the predecessor subgraph (defined by s and π) is reachable from s.

The fact that "if v is reachable from s then v is included in the predecessor subgraph" — so that the predecessor subgraph is a spanning tree for the (entire) connected component containing s — can be proved by induction on the distance from v to s. $\square$

Partial Correctness of Breadth-First Search

Theorem 7

Let $G = (V, E)$ be a directed or undirected graph, and suppose BFS is run on G from a given source vertex $s \in V$. Then each of the following properties is satisfied on termination of the algorithm (if it terminates):

- *The predecessor subgraph $G_p = (V_p, E_p)$ for the function π and vertex s is a spanning tree for the connected component of G that contains s.*
- *For all $v \in V$, $d[v]$ is the length of a shortest path from v to s in G, and $d[v] = +\infty$ if and only if v is not reachable from s.*
- *For every $v \in V$ that is reachable from s, the path from s to v in G_p is also a shortest path from s to v in G.*

Proof.

Consequence of the previous lemmas. $\square$

Termination and Efficiency

Theorem 8

Let $G = (V, E)$ be a directed or undirected graph, and suppose BFS is run on G from a given source vertex $s \in V$. Then the algorithm terminates after performing $\Theta(|V| + |E|)$ operations.

Proof.

Exercise (can be completed by modifying the analysis of the DFS algorithm). $\square$

References

Further Reading and Java Code:

- **Introduction to Algorithms**, Chapter 22.2
- **Data Structures: Abstraction and Design Using Java**, Chapter 10.4