Computer Science 331
Binary Search Trees

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Lectures #18–19

Outline

The Dictionary ADT

A *dictionary* is a finite set (no duplicates) of elements.

Each element is assumed to include

- A **key**, used for searches.
  - Keys are required to belong to some ordered set.
  - The keys of the elements of a dictionary are required to be distinct.
- Additional **data**, used for other processing.

Permits the following operations:

- search by key
- insert (key/data pair)
- delete an element with specified key

Similar to Java's `Map` (unordered) and `SortedMap` (ordered) interfaces.

Binary Tree

A **binary tree** T is a hierarchical, recursively defined data structure, consisting of a set of **vertices** or **nodes**.

A binary tree T is **either**

- an "empty tree,"

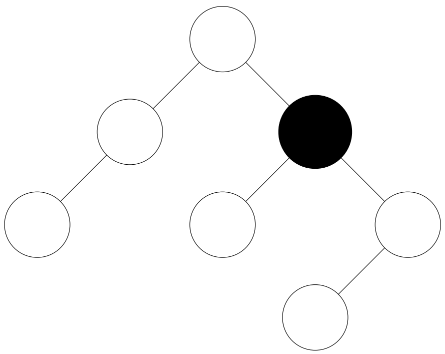
or

- a structure that includes
  - the **root** of T (the node at the top)
  - the **left subtree** T_L of T ...
  - the **right subtree** T_R of T ...

... where both T_L and T_R are also binary trees.

Example and Implementation Details

Example:



Each node has a:

- parent:
- left child:
- right child:

Ie, parent of greatest ancestor is null, children of youngest children are null

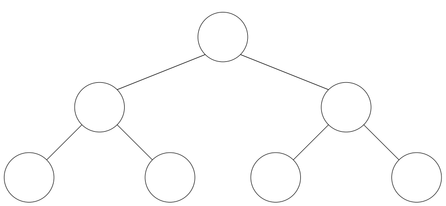
Additional Terminology

Additional terms related to binary trees:

- **siblings**: two nodes with same parent
- **descendant (of N)**: any node occuring in tree with root N
- **ancestor (of N)**: root of any tree containing node N
- **leaf**: node with no children
- **size**: number of nodes in the tree
- **depth (of N)**: length (# of edges) of path from root to N
- **height**: length of longest path from root to a leaf (height(emptyTree) = -1)

Note: depth and height are sometimes (as in the text) defined in terms of number of nodes as opposed to number of edges.

Size vs. Height: One Extreme



- Size: 7
- Height: 2
- Relationship: Sum from i = 0 to h of 2^i

Also equal to 2^(h+1)-1

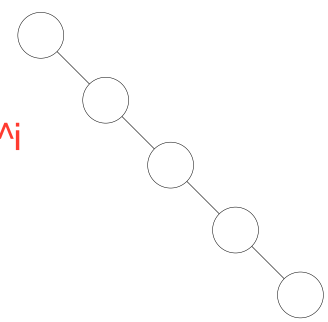
So, h = log_2(n+1)-1

This binary tree is said to be *full*:

- all leaves have the same depth
- all non-leaf nodes have exactly two children

Upper bound: a binary tree of height h has size *at most*

Size vs. Height: Another Extreme



- Size: 5
- Height: 4
- Relationship: n=h-1

Essentially a linked list!

Lower bound: a binary tree with height h has size *at least*

Binary Search Tree

A **binary search tree** T is a data structure that can be used to store and manipulate a finite ordered set or mapping.

- T is a binary tree
- Each element of the dictionary is stored at a node of T, so

$$\text{set size} = \text{size of } T$$

- In order to support efficient searching, elements are arranged to satisfy the **Binary Search Tree Property** ...

Binary Search Tree Property

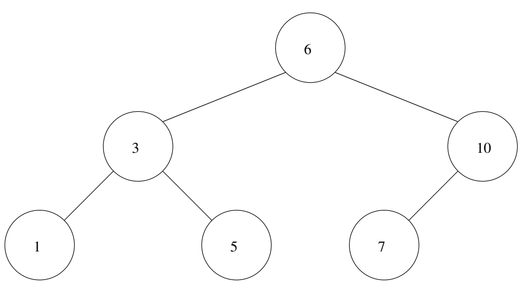
Binary Search Tree Property: If T is nonempty, then

- The left subtree T_L is a binary search tree including all dictionary elements whose keys are *less than* the key of the element at the root
- The right subtree T_R is a binary search tree including all dictionary elements whose keys are *greater than* the key of the element at the root

Example

One binary search tree for a dictionary including elements with keys

$$\{1, 3, 5, 6, 7, 10\}$$



Binary Search Tree Data Structure

```
public class BST<E extends Comparable<E>,V> {
  protected bstNode<E,V> root;
  ...

  protected class bstNode<E,V> {
    E key;
    V value;
    bstNode<E,V> left;
    bstNode<E,V> right;
    ...
  }
}
```

`bstNode` can also include a reference to its parent

Specification of "Search" Problem:

Precondition 1:

a) T is a BST storing values of some type V along with keys of type E

b) key is an element of type E stored with a value of type V in T

Postcondition 1:

a) Value returned is (a reference to) the value in T with key key

b) T and key are not changed

Precondition 2: same, but key is not in T

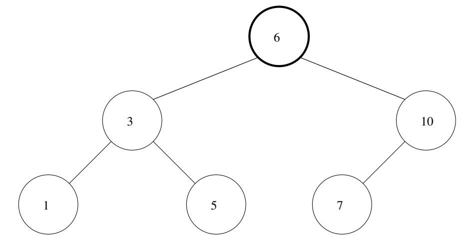
Postcondition 2:

a) A notFoundException is thrown

b) T and key are not changed

Searching: An Example

Searching for 5:



Nodes Visited:

- Start at 6 : Since 5<6, search left subtree
- Next node Since 5>3, search right subtree
- Next node equal to key,so we are finished

A Recursive Search Algorithm

```
public V search(bstNode<E,V> T, E key)
    throws notFoundException {
  if (T == null)

  else if (key.compareTo(T.key) == 0)

  else if (key.compareTo(T.key) < 0)

  else

}
```


Partial Correctness

Proved by induction on the height of T:

Termination and Running Time

Let Steps(T) be the number of steps used to search in a BST T in the worst case. Then there are positive constants c_1, c_2 and c_3 such that

$$\texttt{Steps(T)} \leq \begin{cases} c_1 & \text{if height}(\texttt{T}) = -1, \\ c_2 & \text{if height}(\texttt{T}) = 0, \\ c_3 + \max(\text{Steps}(\texttt{T.left}), \text{Steps}(\texttt{T.right})) & \\ & \text{if height}(\texttt{T}) > 0. \end{cases}$$

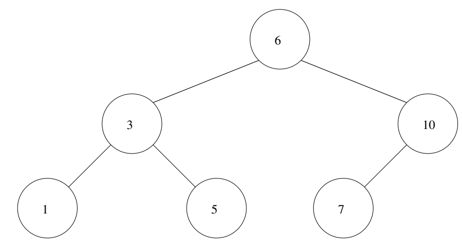
Exercise: Use this to prove that

$$\text{Steps}(\texttt{T}) \leq c_3 \times \text{height}(\texttt{T}) + \max(c_1, c_2)$$

Exercise: Prove that $\text{Steps}(\texttt{T}) \geq \text{height}(\texttt{T})$ as well.

$\Longrightarrow$ The worst-case cost to search in T is in $\Theta(\text{height}(\texttt{T}))$.

Minimum Finding: The Idea



Idea:

-
-

Example:

A Recursive Minimum-Finding Algorithm

```
// Precondition:  T is non-null
// Postcondition:  returns node with minimal key,
//   null if T is empty

public bstNode<E,V> findMin(bstNode<E,V> T) {
  if (T == null)

  else if (T.left == null)

  else

}
```


Analysis: Correctness and Running Time

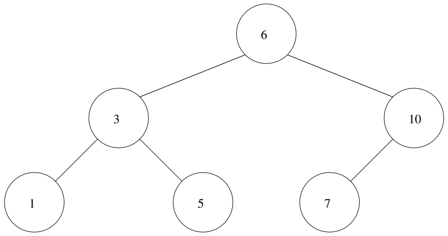
Partial Correctness (tree of height h):

- Exercise (similar to proof for `Search`)

Termination and Bound on Running Time (tree of height h):

-
-

Insertion: An Example



Idea:

Nodes Visited (inserting 9):

- Start at 6 :
- Next node
- Next node
- Next node

A Recursive Insertion Algorithm

```
// Non-recursive public function calls recursive worker function
public void insert(E key, V value)
  { root = insert(root, key, Value); }

protected
bstNode<E,V> insert(bstNode<E,V> T, E newKey, V newValue) {
  if (T == null)

  else if (newKey.compareTo(T.key) < 0)

  else if (newKey.compareTo(T.key) > 0)

  else

  return T;
}
```


Analysis: Correctness and Running Time

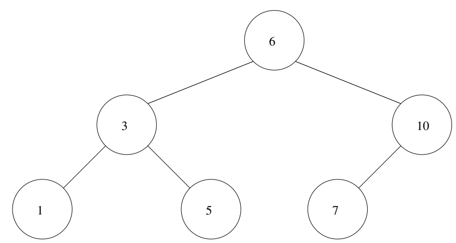
Partial Correctness (tree of height h):

- Exercise (similar to proof for `Search`)

Termination and Bound on Running Time (tree of height h):

- worst case running time is $\Theta(h)$ (and hence $\Theta(n)$)
- Proof: exercise

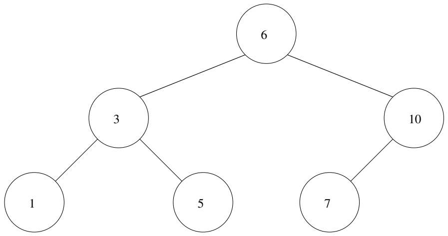
Deletion: Four Important Cases



Key is/has ...

1. Not Found (Eg: Delete 8)
2. At a Leaf (Eg: Delete 7)
3. One Child (Eg: Delete 10)
4. Two Children (Eg: Delete 6)

First Case: Key Not Found



Idea:

Nodes Visited (delete 8):

- Start at 6 :
- Next node
- Next node
- Next node

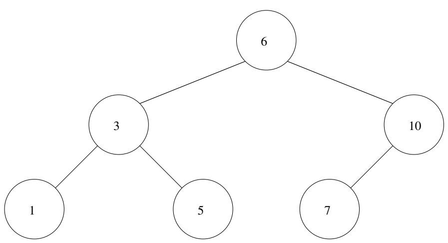
Algorithm and Analysis

```
protected bstNode<E,V> delete(bstNode<E,V> T, E key) {
  if (T != null) {
    if (key.compareTo(T.key) < 0)
      T.left =  delete(T.left, key);
    else if (key.compareTo(T..key) > 0)
      T.right = delete(T.right,key);
    else if ...
      // found node with given key
  }
  else
    throw new notFoundException();
  return T;
}
```

Correctness and Efficiency For This Case:

- tree is not modified if key is not found (base case will be reached)
- worst-case cost $\Theta(h)$ (same as search)

Second Case: Key is at a Leaf



Idea:

Nodes Visited (delete 7):

- Start at 6 :
- Next node
- Next node

Algorithm and Analysis

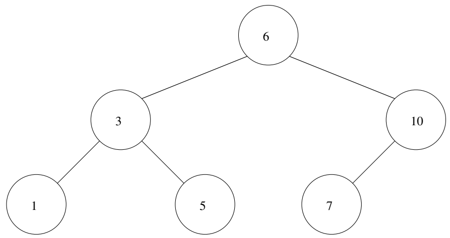
Extension of Algorithm:

```
else if () T.left == null && T.right == null
             T = null;
```

Correctness and Efficiency For This Case:

- Tests whether the node is a leaf
- Replacing T with null deletes the leaf at T
-
-

Third Case: Key is at a Node with One Child



Idea:

Nodes Visited (delete 10):

- Start at 6 : Delete 10 from right subtree since 10 > 6
- Next node Set pointer to right child of parent to child to 10

Algorithm and Analysis

Extension of Algorithm:

```
else if (T.left == null)
```

T = T.right

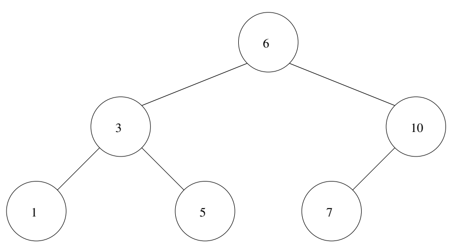
```
else if (T.right == null)
```

T = T.left

Correctness and Efficiency For This Case:

- T is replaced with one non-empty subtree
  - Node originally at T is deleted
  - BST property still holds (new subtree at T still contains keys that were in the old subtree.
- Worst case is theta (h) (theta (h) to locate, theta (1) to delete.

Fourth Case: Key is at a Node with Two Children



Idea:

Nodes Visited (delete 6):

- Start at 6 : Found node to delete
- replace data at node with data from node of minimum key in right subtree
- delete node with minimal key from right subtree

Algorithm and Analysis

Extension of Algorithm:

```
else {
```

bstNode<E,V> min = findMin(T.right);
T.key = min.key;
T.value = min.value;
T.right = delete(T.right, T.key)

```
}
```

Correctness and Efficiency For This Case:

- BST property holds: all entries in new right subtree have keys > the smallest key from the original right subtree
- worst case cost is theta (h)
  - findMin costs theta h
  - recursive call deletes a node with at most one child from a tree of height<height (cost is theta h)

More on Worst Case

All primitive operations (`search`, `insert`, `delete`) have worst-case complexity $\Theta(n)$

- all nodes have exactly one child (i.e., tree only has one leaf)
- Eg. will occur if elements are inserted into the tree in ascending (or descending) order

On average, the complexity is $\Theta(\log n)$

- Eg. if the tree is full, the height of the tree is $h = \log_2(n+1) - 1$
- the height of a randomly constructed tree (inserting n elements uniformly randomly) is $3\log_2 n$ for sufficiently large n (see lecture supplement)

Need techniques to ensure that all trees are close to full

- want $h \in \Theta(\log n)$ in the worst case
- one possibility: red-black trees (next three lectures)

References

Introduction to Algorithms, Chapter 12

and,

Data Structures: Abstraction and Design Using Java, Chapter 6.1-6.4