1. A. We note the midpoints of the sides of the square form the corners of the diamonds. The side length of the diamond is therefore the hypotenuse of the triangle formed with the legs extending from the corner of the square to the midpoints.

Since the side lengths of the square are 1.0, half of that is 0.50. So, the length of the diagonal is

$$s = \sqrt{(0.50)^2 + (0.50)^2} \quad \text{(By applying Pythagorean Theorem)}$$
$$= \sqrt{2}/2$$
$$\approx 0.71$$

The side length of the diamond nested immediately inside is $\sqrt{2}/2$, or approximately 0.71.

B. To find side length of square nested immediately inside of above diamond, we again note that it is the hypotenuse of triangle formed with legs of length half of sides of diamond. Applying Pythagorean Theorem again, we get

$$s = \sqrt{(\sqrt{2}/4)^2 + (\sqrt{2}/4)^2}$$
$$= 0.50$$

The side length of the square nested immediately inside diamond of 1A has side length of 0.50.

C. We note that the side length of each successive diamond/square is $\sqrt{2}/2$ times the side length of the previous square/diamond. This comes from observation that for side length of a, the side length of next shape would be

$$s = \sqrt{(a/2)^2 + (a/2)^2} = \sqrt{\frac{a^2}{4} + \frac{a^2}{4}} = \sqrt{\frac{a^2}{2}}$$
$$= a/\sqrt{2} = \left(\frac{\sqrt{2}}{2}\right)a$$

Thus, since each iteration is comprised of a square and a diamond, this means the side lengths of each iteration is $1/2$ as much as the previous iteration. Thus, if $\alpha = 1000$ to start with, we simply repeatedly divide by 2 to get the side length of the next square. The results are below

α	$\alpha/2$	level
1000	500	1
500	250	2
250	125	3
125	62.5	4
62.5	31.25	5
31.25	15.625	6
15.625	7.8125	7
7.8125	3.90625	8
3.90625	1.953125	9
1.953125	0.9765625	10

You can therefore draw 10 levels in this pattern before the smallest square is less than 1 pixel wide, since on level 11, the width of square would be less than 1 (it would be 0.9765625)

We note that when the smallest square is 1.953125, the diamond nested inside is 1.381... wide, which is also greater than 1 pixel

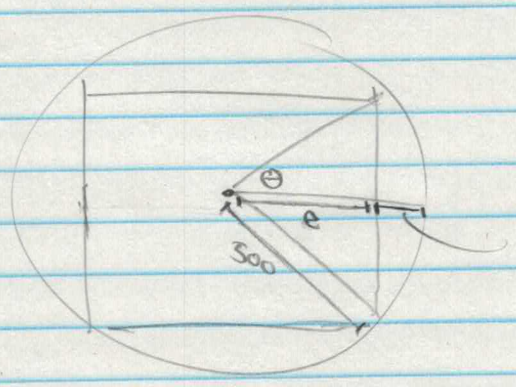
* Note we use constant factor $\frac{1000}{11\pi}$ to ensure that spiral fills 1000x1000 pixel window of 2c.

2\. A. An example of a parametric equation for a spiral curve that starts at the origin and makes 3 complete turns as it moves radially outwards is the Archimedes Spiral, given by the equation

$$x(u) = \frac{1000}{11\pi} u\cos(u) \qquad \text{for } u = 0 \text{ to } 6\pi$$
$$y(u) = \frac{1000}{11\pi} u\sin(u)$$

B. To approximate a 1000 pixel diameter circle using equal line segments, we note that to ensure that we are no more than 1 pixel off from true circle, the line segment formed by the midpoint of the lines used to approximate the circle and the actual circle has to be less than 1 pixel.

For instance, if we use 4 lines, then



This has to be less than 1 pixel

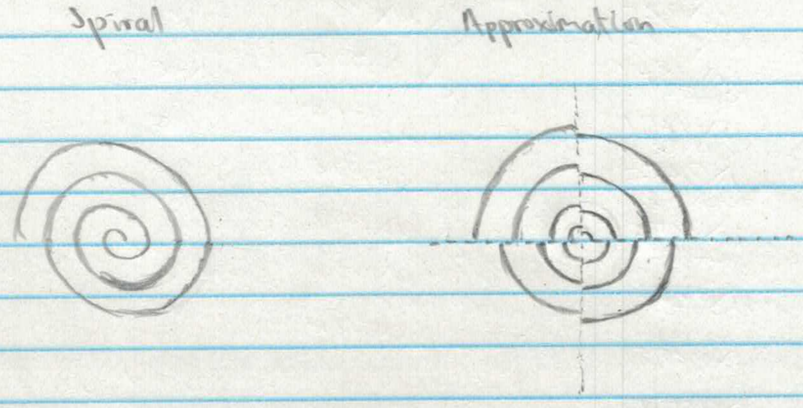
To find the number of segments needed, which we call n, we note that n has to satisfy

$500 - 500\cos\left(\frac{180^\circ}{n}\right) \leq 1$, where 500 is the radius, $\frac{180^\circ}{n}$ is θ, and $500\cos\left(\frac{180^\circ}{n}\right)$ is the length e.

We let $n = 3$, and increment by 1 until the result is ≤ 1. Doing this, we find that $n = 50$. Therefore, we need 50 line segments drawn to ensure we are no more than 1 pixel off true circle.

c. Since it is difficult to reach an exact answer, we will approximate by applying the method used in 2B.
We will split the spiral into 1/4's quadrants, and suppose that the spiral quadrant requires the same amount of lines as a circle of that size:

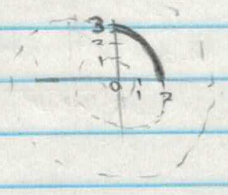
For instance, consider the spiral and our approximation



We note that this approximation gets more accurate as we take more divisions.
However, because that is impractical, we will simply use 1/4 for these calculations.

For our calculation, we will divide the spiral into 4 quadrants. Then, take the average "radius" by taking 1/2 of the radius as the spiral enters quadrant, and when it leaves.

ie



For this segment, we average, so radius of 2.5.

This number will replace 500 from 2B as our radius. Since these quadrants correspond to 1/4 of a circle, after finding n, we divide n by 4 to account for this.

- Recall $x(u) = \frac{1000}{11\pi} u\cos(u)$, $y(u) = \frac{1000}{11\pi} u\sin(u)$ for $u = 0$ to 6π

Furthermore, we note for our spiral that dividing into 4 quadrants means finding coordinates corresponding to $u = \frac{\pi}{2} n$ for $n = 0$ to 12.

→ (how far point on spiral is from origin)

Thus, we find our "radius" points are:

n	u	radius	average radius
0	0	0	
1	$\pi/2$	$500/11$	$250/11$
2	π	$1000/11$	$750/11$
3	$3\pi/2$	$1500/11$	$1250/11$
4	2π	$2000/11$	$1750/11$
5	$5\pi/2$	$2500/11$	$2250/11$
6	3π	$3000/11$	$2750/11$
7	$7\pi/2$	$3500/11$	$3250/11$
8	4π	$4000/11$	$3750/11$
9	$9\pi/2$	$4500/11$	$4250/11$
10	5π	$5000/11$	$4750/11$
11	$11\pi/2$	$5500/11$	$5250/11$
12	6π	$6000/11$	$5750/11$

Using the 12 average radiuses, call them r, we find the n for each r such that $r - r\cos(180°/n) \leq 1$. The n values are below:

11, 19, 24, 28, 32, 36, 39, 42, 44, 47, 49, 51

Adding these number of line segments and dividing by 4, we get $423/4 = 105.75 \rightarrow 106$.

Therefore, we need around 106 line segments to approximate the spiral to an accuracy of 1 pixel

Note that the constant $1000/11\pi$ was chosen so that the spiral would take up 1000 pixels horizontally.

That is, when $u = 5\pi$, we get the leftmost pixel, and when $u = 6\pi$, we get the rightmost. Thus,

$$x(5\pi) = \frac{1000}{11\pi} 5\pi(-1) = -5000/11$$
$$x(6\pi) = \frac{1000}{11\pi} 6\pi(1) = 6000/11$$

we note that these points are $6000/11 - (-5000/11)$ $= 11000/11 = 1000$ apart.

3. A. For the point at location ①, we note that the x-coordinate is 0.5. We apply Pythagorean Theorem by taking right triangle formed by splitting the largest equilateral triangle in half vertically, where one of the legs has side length 0.5, and hypotenuse is length 1.

$$
\begin{aligned}
y &= \sqrt{(hyp)^2 - (leg)^2} \\
&= \sqrt{1^2 - (0.5)^2} \\
&= \sqrt{1 - 0.25} \\
&= \sqrt{.75} \\
&= \sqrt{3}/2 \approx .87
\end{aligned}
$$

① has xy coordinates of (0.50, 0.87)

For point at location ②, y-coordinate is half has high as that of ①, so y-coordinate is $\sqrt{3}/4$, or approximately 0.43. The x-coordinate is 3/4 the length of one side, so it is 0.75.

② has xy coordinates of (0.75, 0.43)

For point at location ③, x-coordinate is 3/4 of 0.5 (half length of one side), so it is 0.375. The y-coordinate is 1/2 as high as ②, so it is $\sqrt{3}/8$, or approximately 0.22.

③ has xy coordinates of (0.375, 0.22)

B. We note that at each iteration, we have 8 times more squares than before. Thus, if we consider the first iteration to be the one with 8 squares, we solve

$8^n = 1000000$ and round down to get the answer.
So, $n = \log_8 1000000$, so $n = 6.64... \rightarrow 6$.

We can generate up to the 6th iteration of 2D Menger Sponge before we exceed our maximum capacity