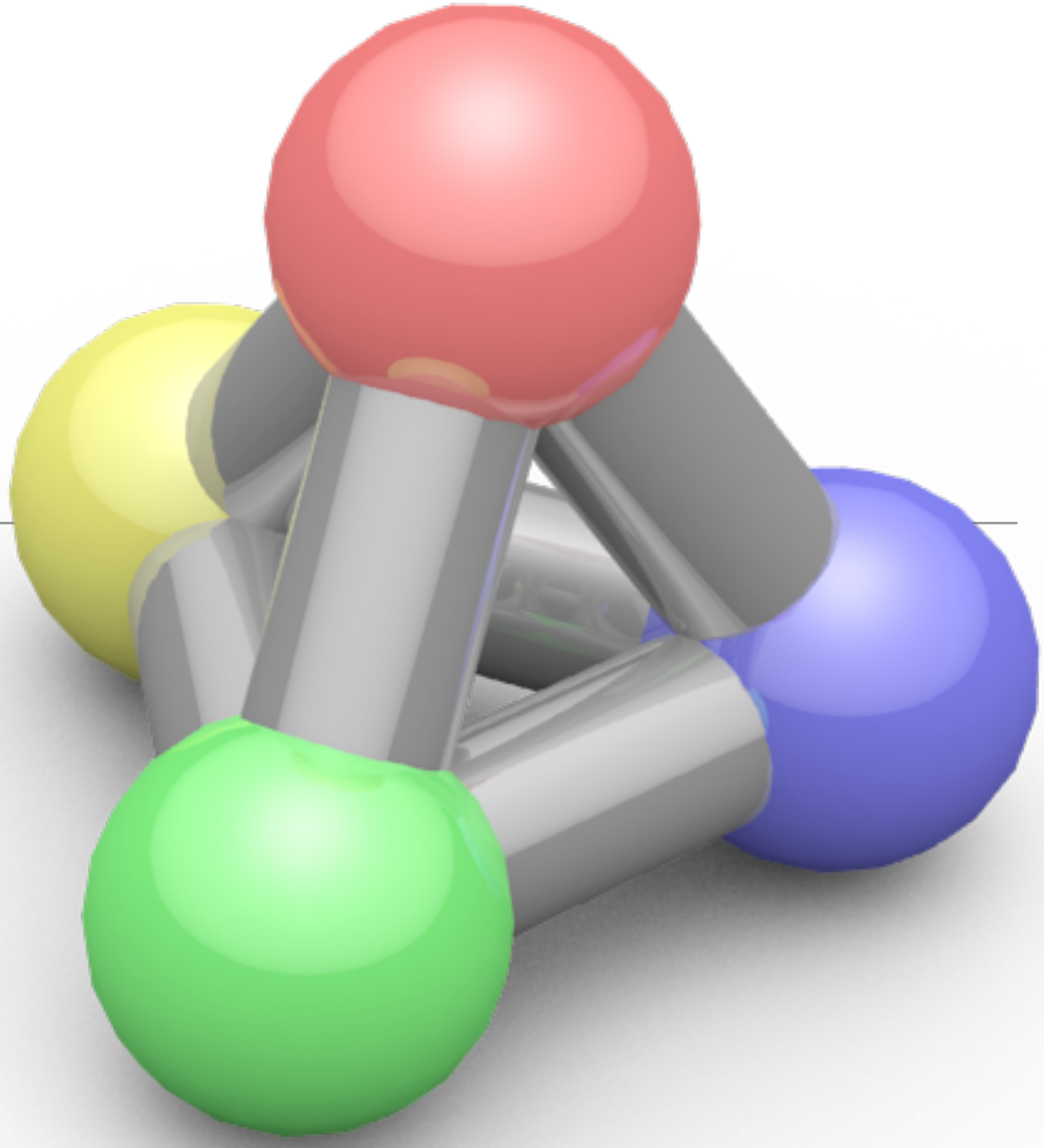
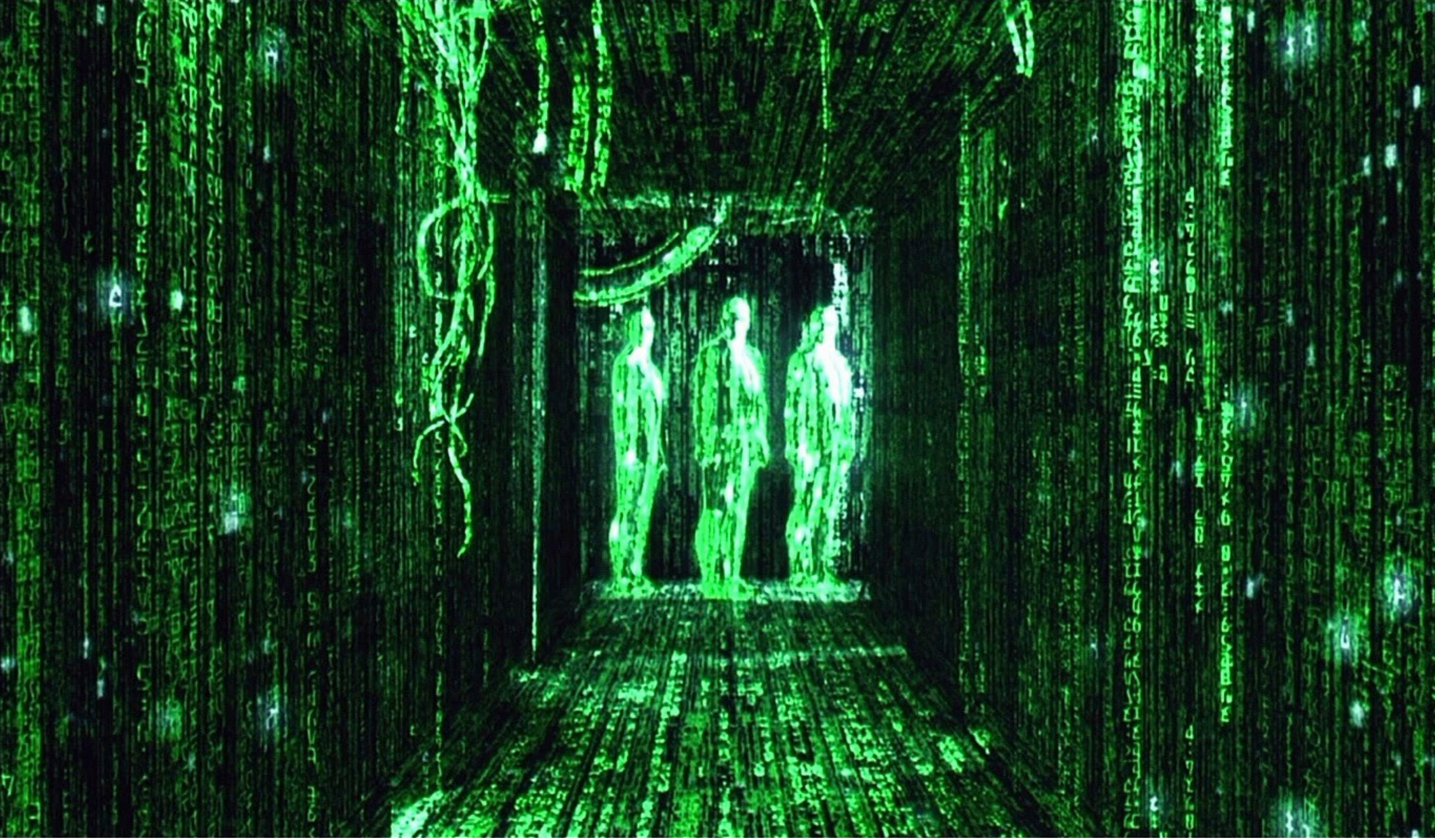
Transformation Matrices



CPSC 453 – Fall 2016
Sonny Chan

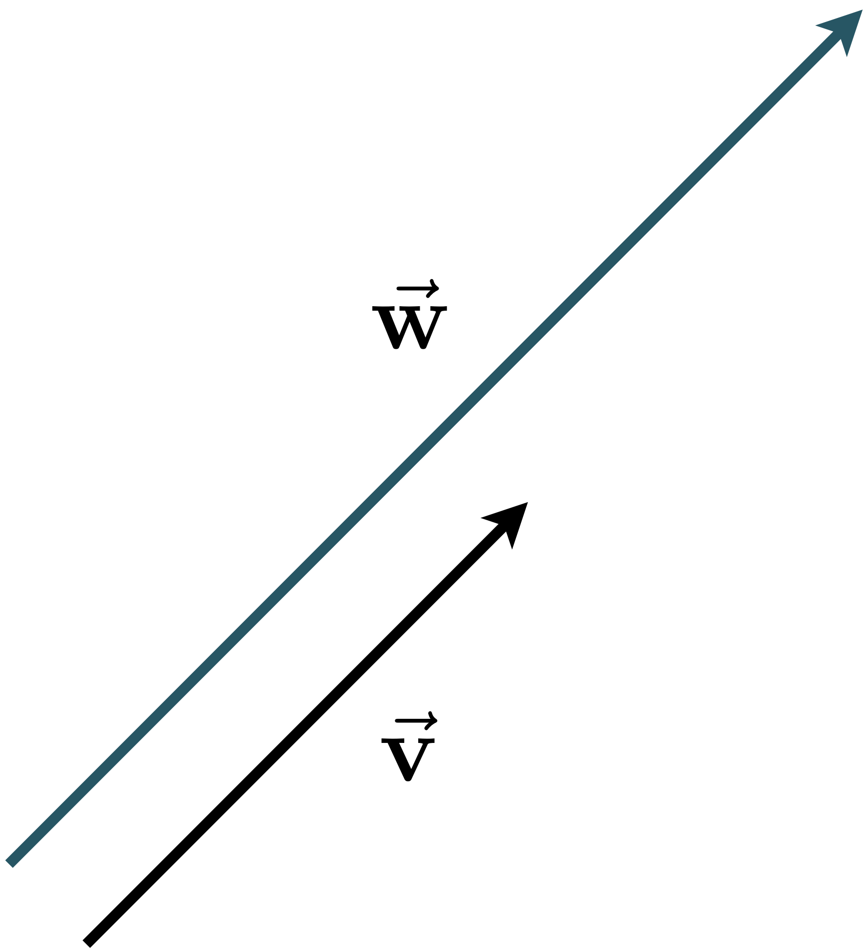
Today’s Outline

- Transformations in matrix form
  - scalar multiplication
  - rotation / change of basis
  - translation / displacement
  - other types of transformations



Matrix Forms | of vector operations

Scalar Multiplication



- in geometric form:

$$\vec{\mathbf{w}} = s\vec{\mathbf{v}}, \ s \in \mathbb{R}$$

- in matrix form:

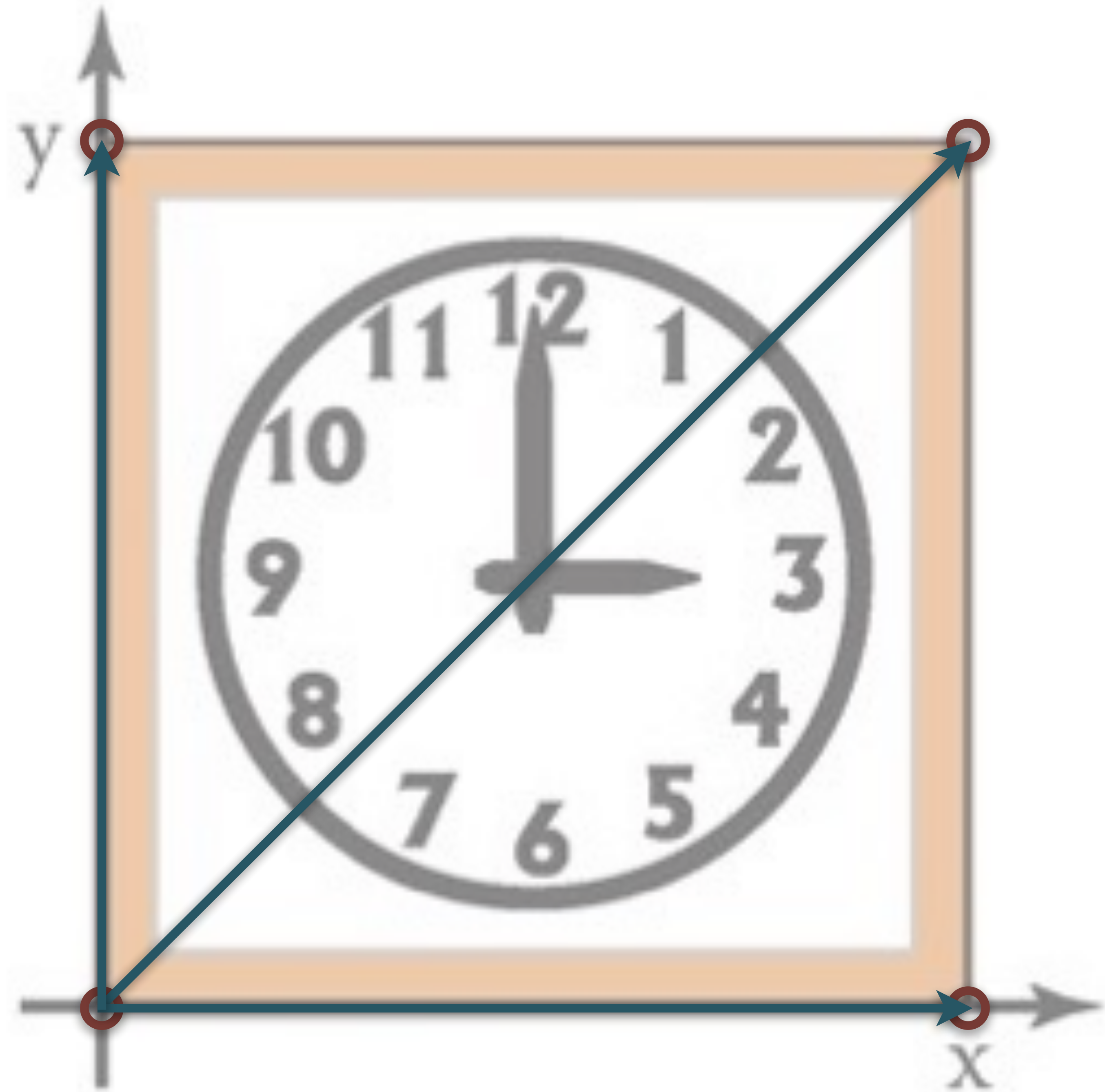
$$[\vec{\mathbf{w}}] = \mathbf{S}\,[\vec{\mathbf{v}}]\,, \ \mathbf{S} \in \mathbb{R}^{2\times 2}$$

- what is **S**?

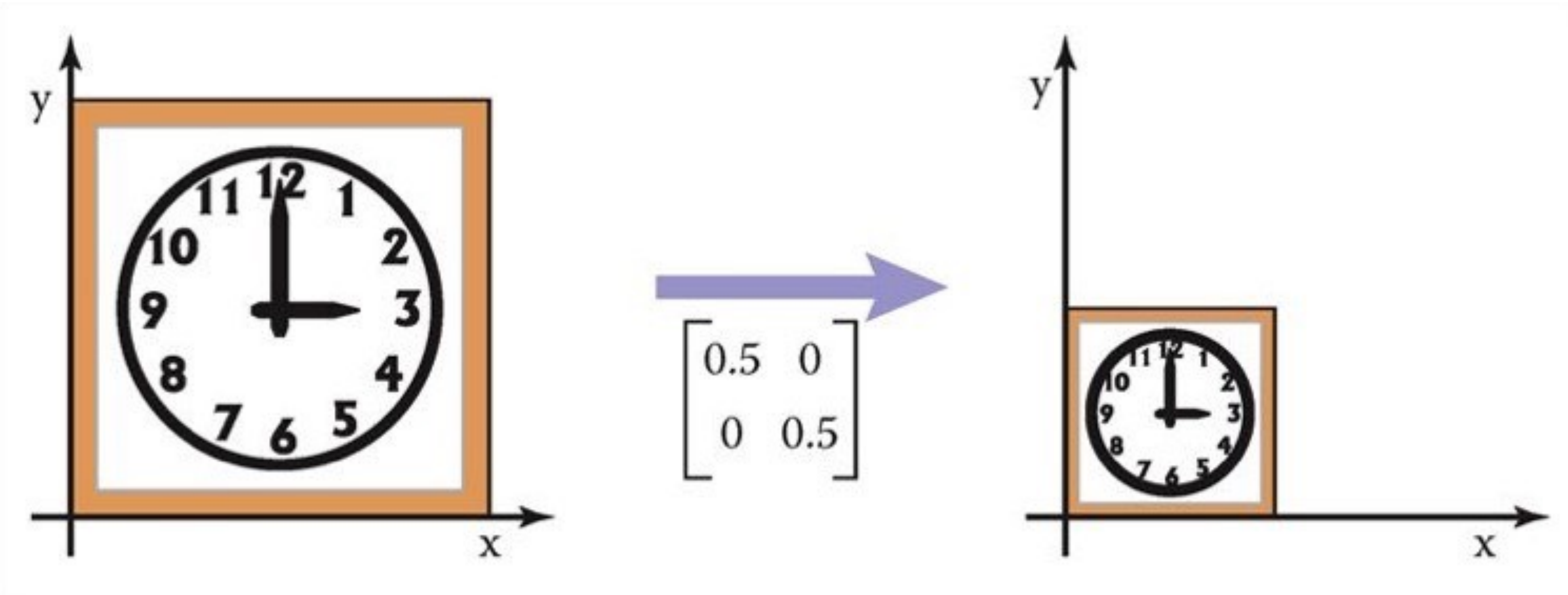
$$\mathbf{S} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

Our Example Clock

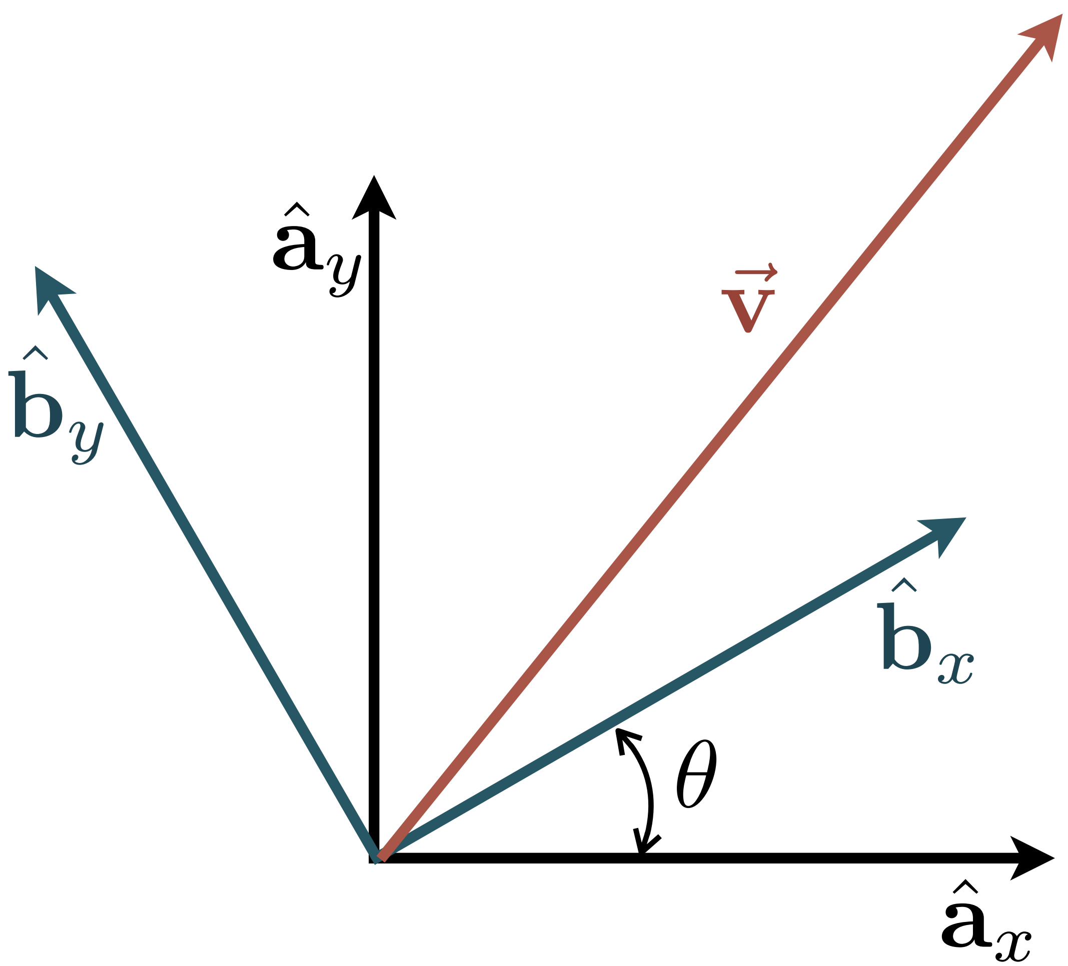
(as borrowed from your textbook)



Uniform Scaling



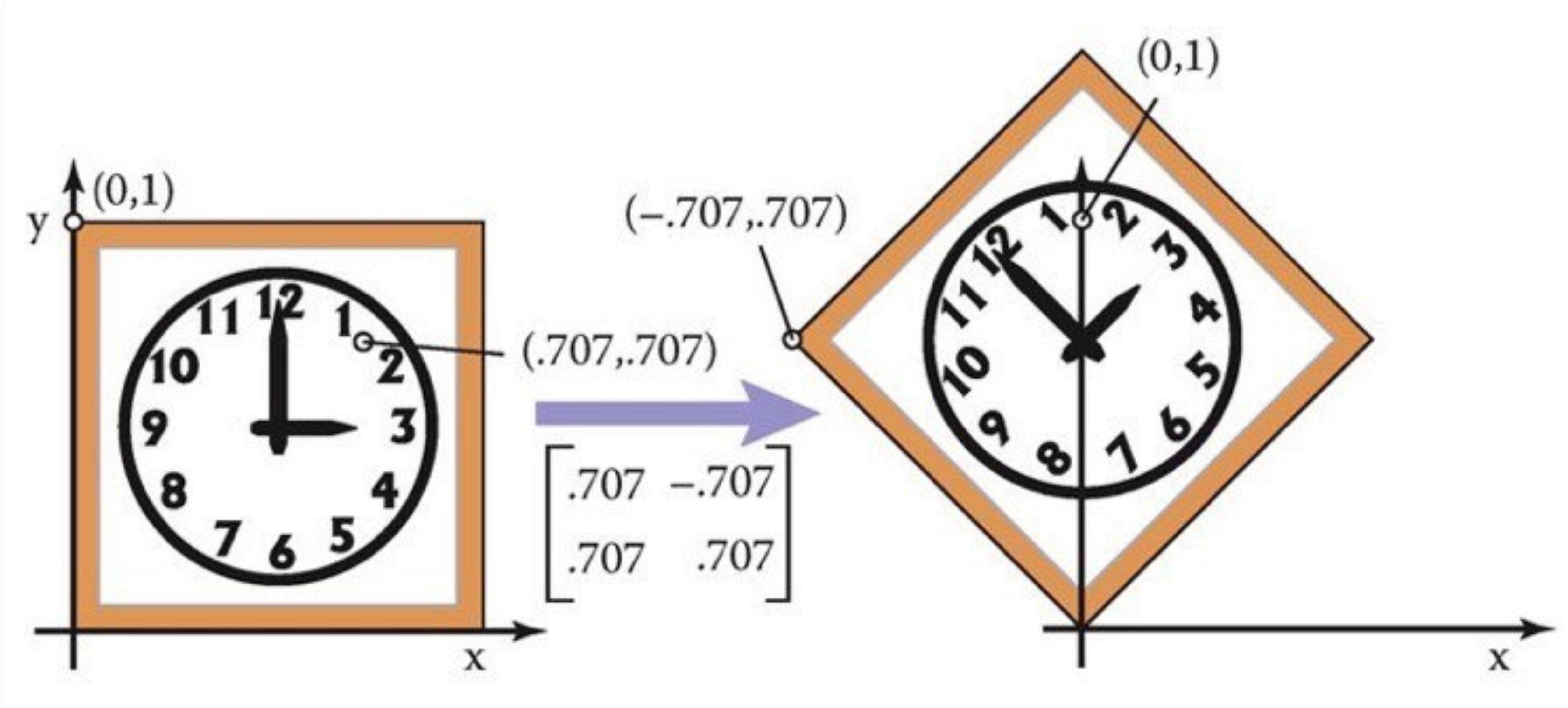
Rotation (Change of Basis)



$$[\vec{\mathbf{v}}]_A = {}^A\mathbf{R}^B \, [\vec{\mathbf{v}}]_B \,,$$

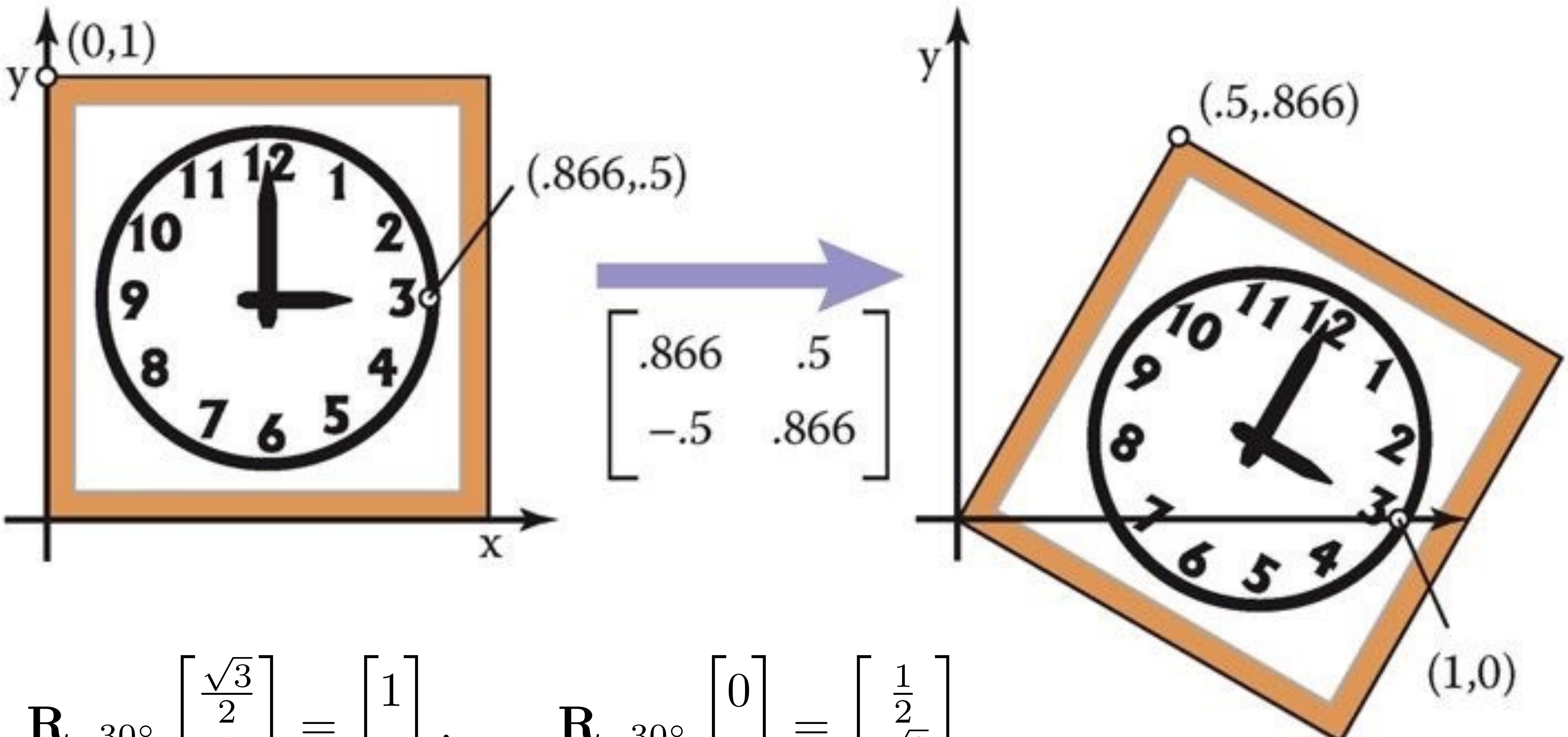
$${}^A\mathbf{R}^B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation by 45°



$$\mathbf{R}_{45^\circ}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}, \qquad \mathbf{R}_{45^\circ}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

Rotation by -30°



$$\begin{bmatrix} .866 & .5 \\ -.5 & .866 \end{bmatrix}$$

$$\mathbf{R}_{-30^\circ} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{R}_{-30^\circ} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

General Transforms

What happens if we put in generic values for the matrix **M**?

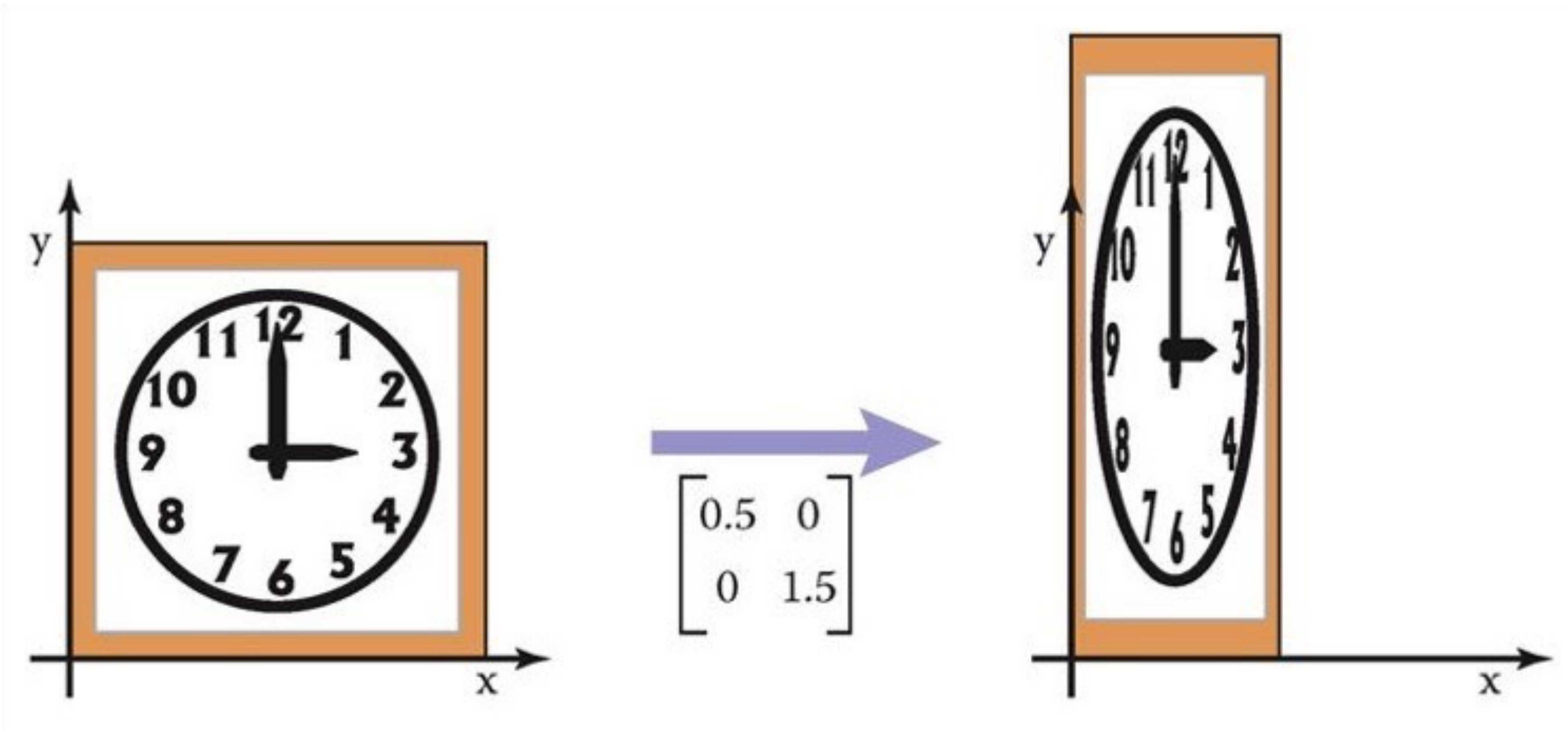
$$[\vec{\mathbf{w}}] = \mathbf{M}\,[\vec{\mathbf{v}}]\,,$$

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

What does this matrix do?

$$\mathbf{M} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

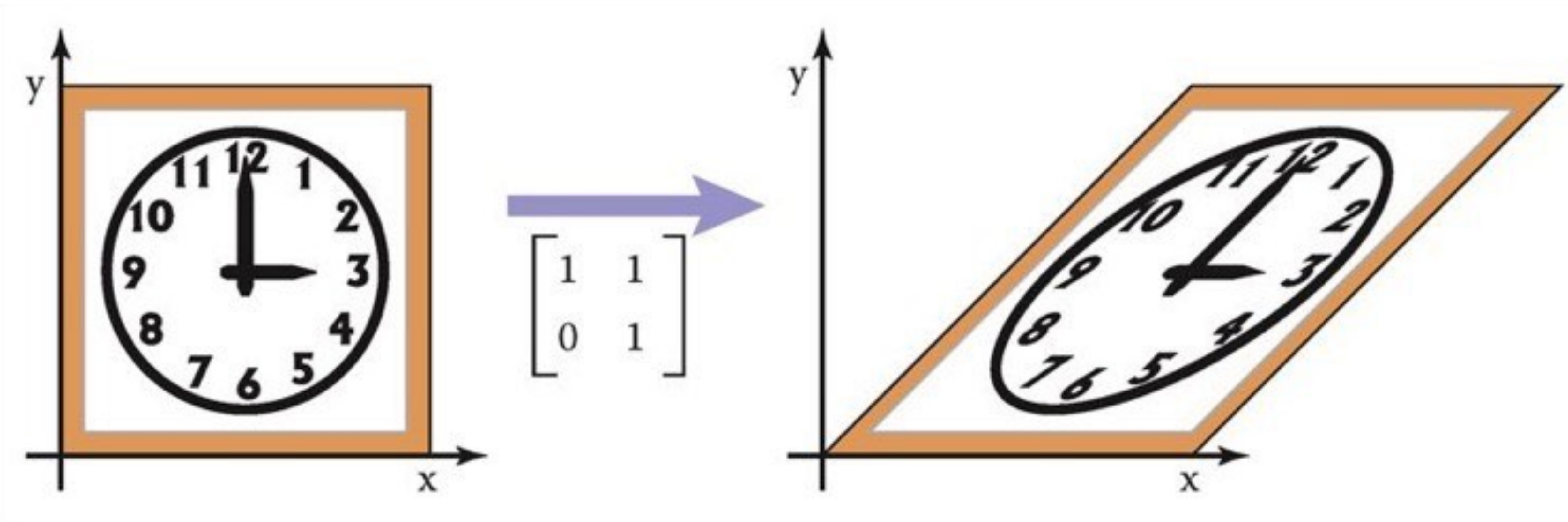
Non-Uniform Scaling



What does this matrix do?

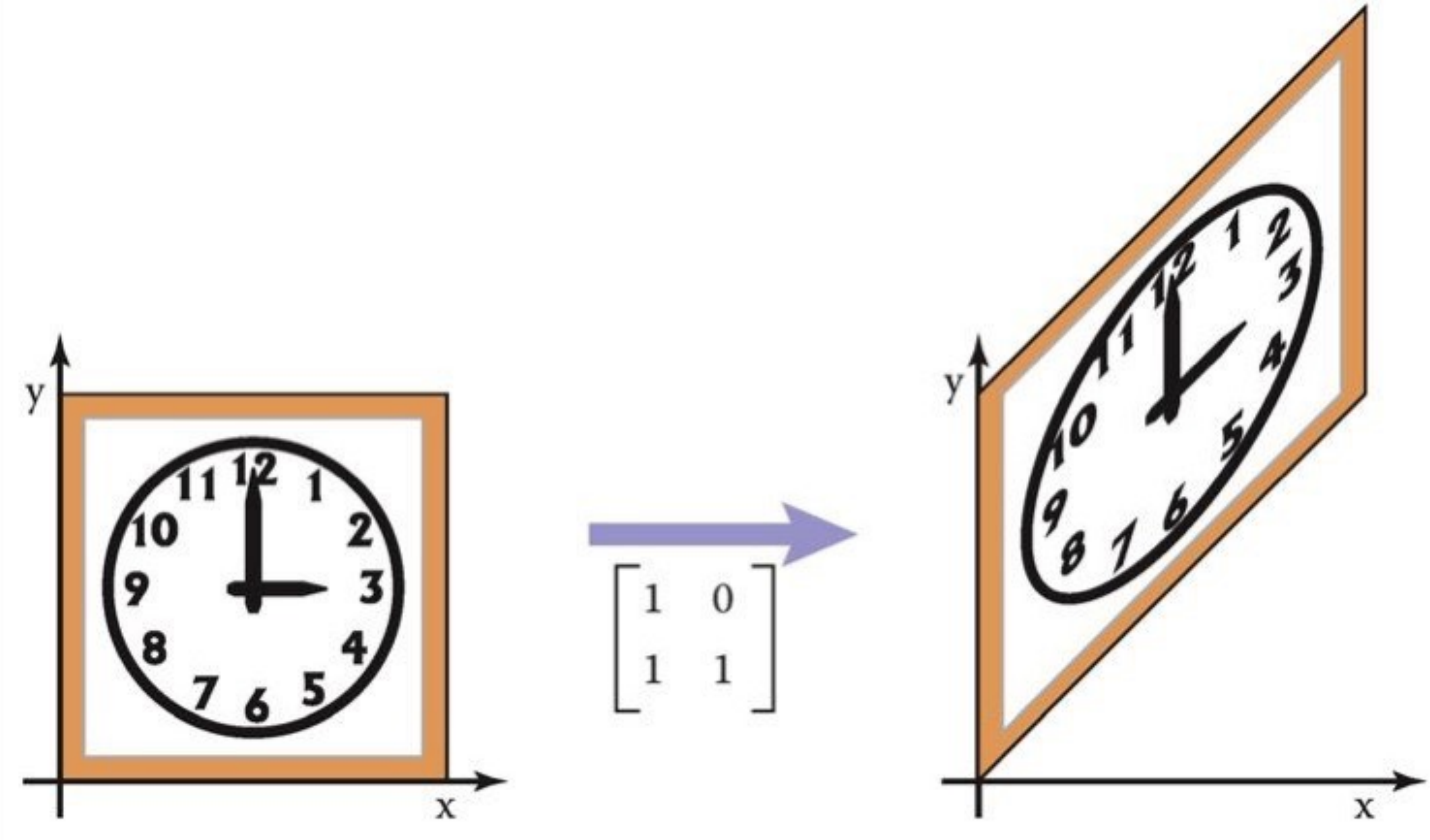
$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Shearing



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

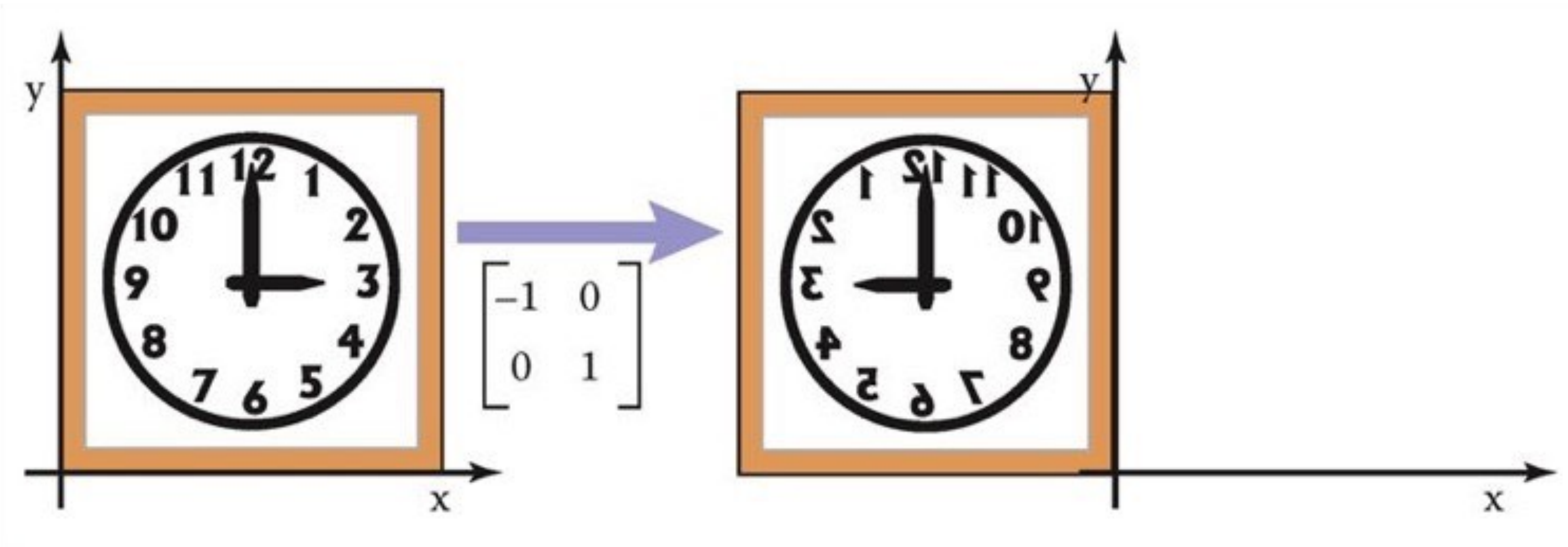
Shearing



What does this matrix do?

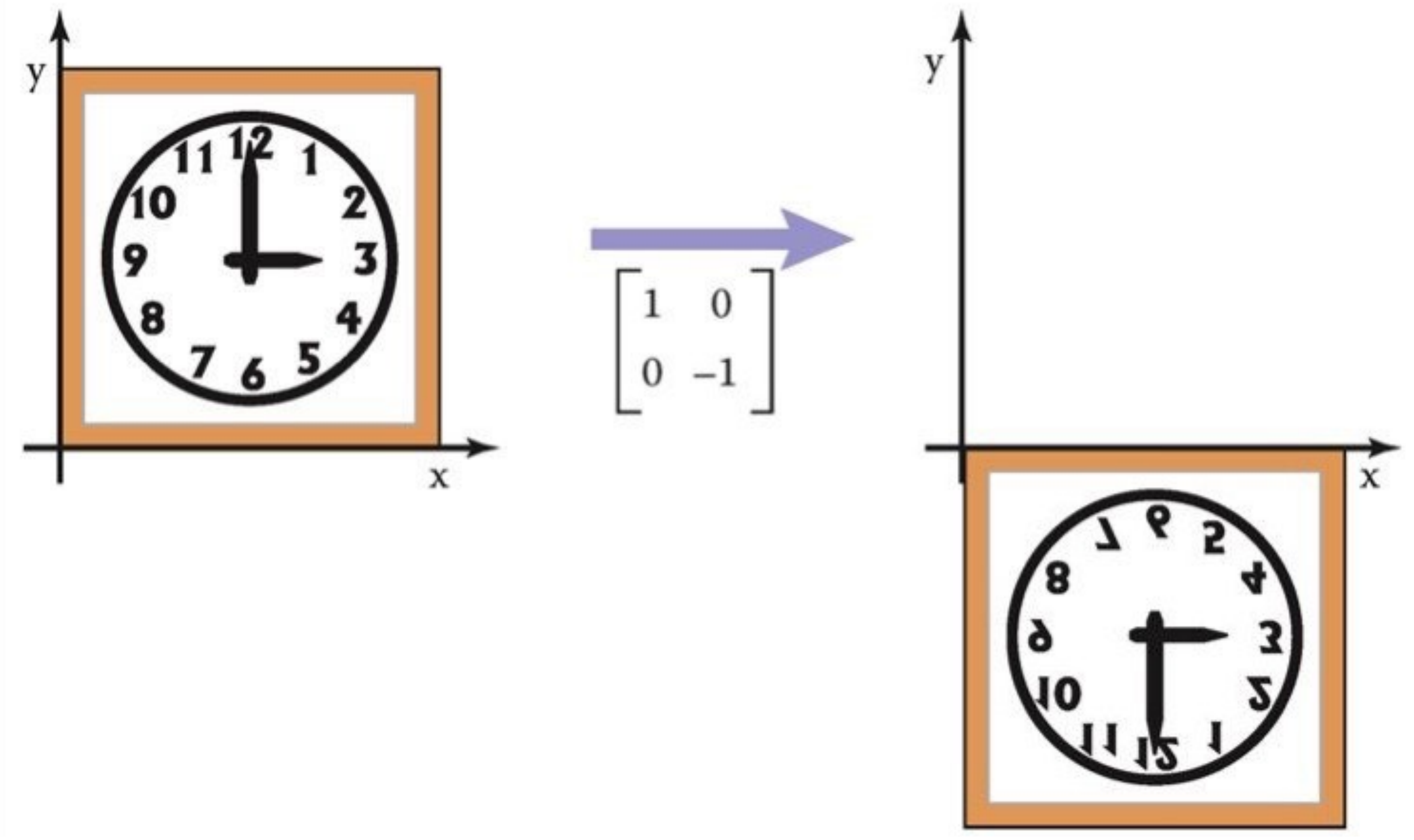
$$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Flipping



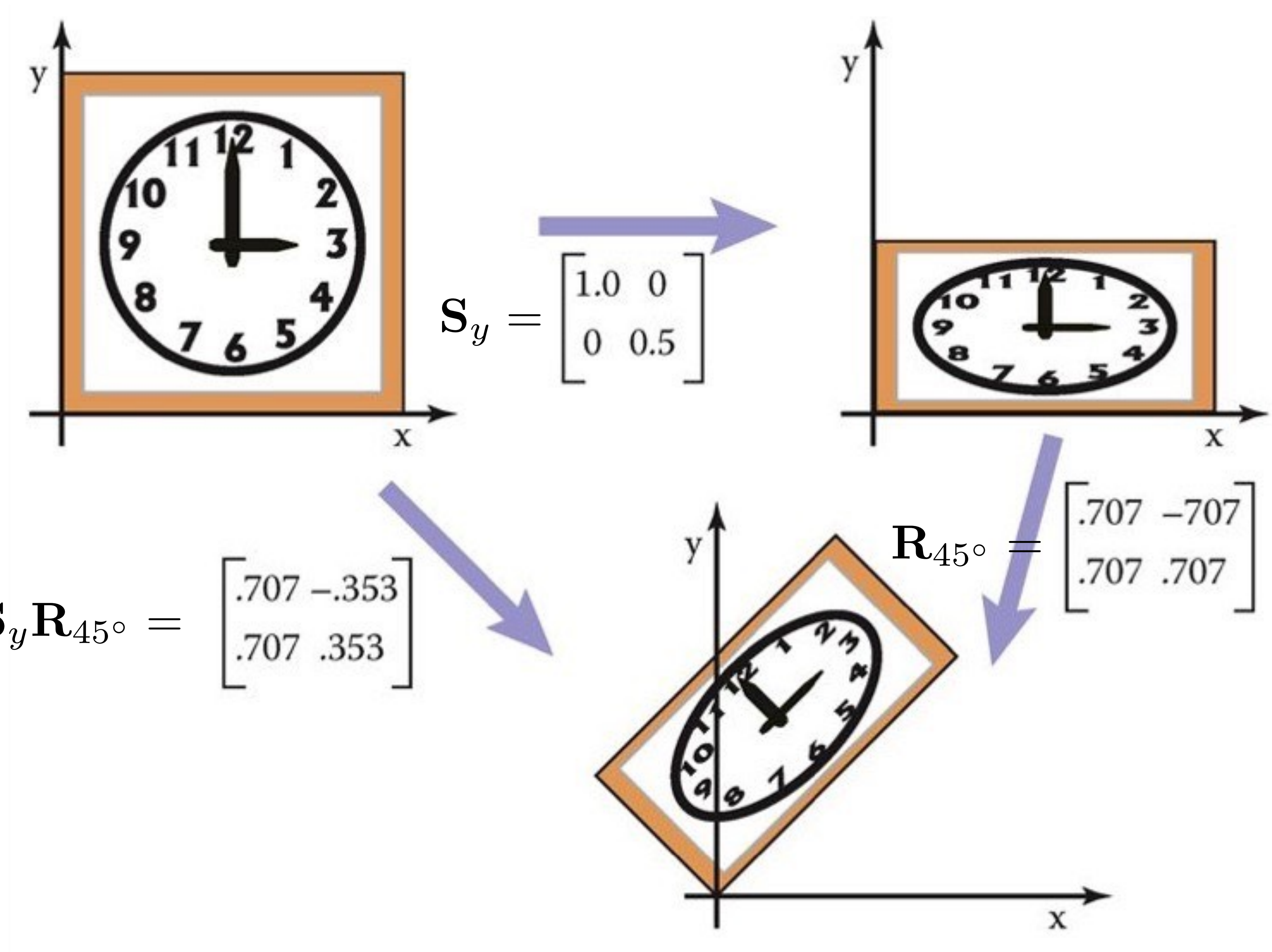
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

Flipping



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Composing Transformations



$$\mathbf{S}_y = \begin{bmatrix} 1.0 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\mathbf{R}_{45^\circ} = \begin{bmatrix} .707 & -707 \\ .707 & .707 \end{bmatrix}$$

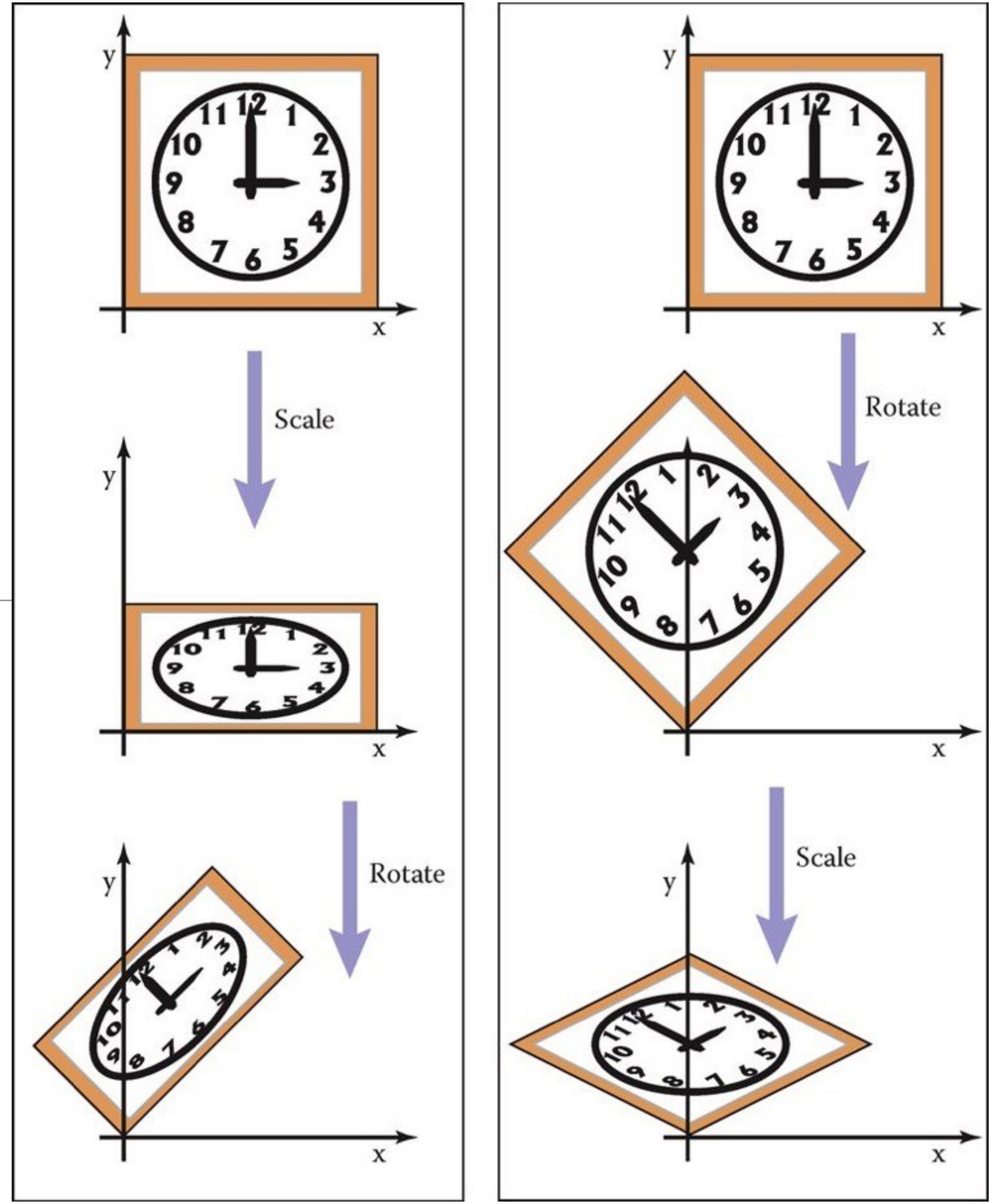
$$\mathbf{S}_y\mathbf{R}_{45^\circ} = \begin{bmatrix} .707 & -.353 \\ .707 & .353 \end{bmatrix}$$

When composing transformations,

does order matter?

Order Matters!

$\mathbf{S}_y \mathbf{R}_{45^\circ} \neq \mathbf{R}_{45^\circ} \mathbf{S}_y$



What about displacements or

translations?

Affine Transforms

- Matrix multiplication gives as a linear transform:

$$x' = ax + by \qquad\qquad \text{for} \quad \mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$y' = cx + dy$$

- For translation, we need an affine transform:

$$x' = x + u \qquad\qquad \text{for} \quad \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{R}^2$$
$$y' = y + v$$

Homogeneous Coordinates

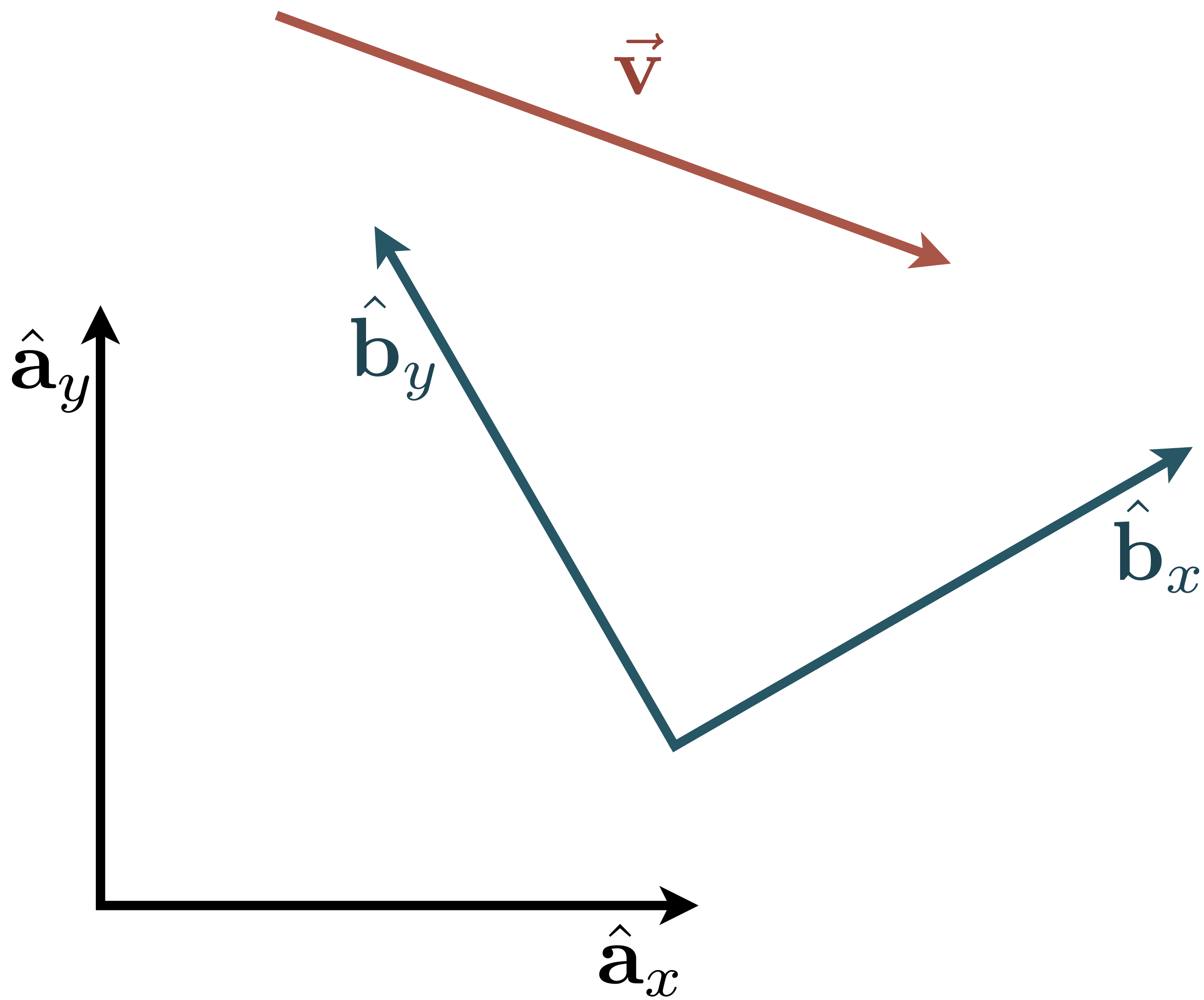
- A brilliantly convenient “trick” is to add an extra coordinate to our vectors and matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- The result multiplication gives us an affine transform:

$$x' = ax + by + u$$
$$y' = cx + dy + v$$

Hold on a second!



$$\vec{\mathbf{v}} = v_1 \hat{\mathbf{a}}_x + v_2 \hat{\mathbf{a}}_y = \begin{bmatrix} v_1 \\ v_2 \\ v_h \end{bmatrix}$$

If we want $[\vec{\mathbf{v}}]_B \Rightarrow [\vec{\mathbf{v}}]_A$,
then what is v_h?

free vectors

versus

bound vectors

Reference Frame Transformations

- Our final homogeneous transformation matrix to go between references frames A and B looks like this:

$$ {}^{A}\mathbf{T}^{B} = \begin{bmatrix} {}^{A}\mathbf{R}^{B} & \vec{\mathbf{r}}^{B/A} \\ 0 & 1 \end{bmatrix} $$

- Where free vectors and bound (position) vectors are encoded differently:

$$ [\vec{\mathbf{v}}] = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \qquad \left[\vec{\mathbf{r}}^{P/B}\right] = \begin{bmatrix} r_1 \\ r_2 \\ 1 \end{bmatrix} $$

Things to Remember

- Operations on vectors can be encoded as matrices
- Transforms can be composed by matrix multiplication
    - order matters!
- Homogeneous coordinates allow translations to be encoded in matrix form as well
- Free vectors and bound (position) vectors must be encoded differently