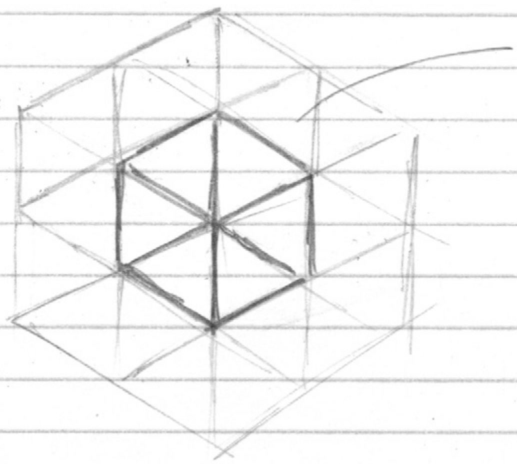
1. A. We first note that $360° / 30° = 12$. That is, if we evaluate azimuth and altitude parameters at $30°$ intervals, it would take 12 readings to obtain a complete circle around the sphere. Since this is symmetrical, to answer the question, we divide the sphere into 8 parts in 3-Dimensional space, so we simply evaluate number of triangles in $1/8$ of sphere, then multiply by 8 to get number of triangles in sphere.



Evaluate $1/8$ corner of sphere that would fit in this cube.

That is, we need to consider 3 altitude and 3 azimuth readings, not counting starting position. There are 4 azimuth readings for every altitude reading not at the north / south pole. This means we have $3 \times 2 = 6$ quadrilaterals $\rightarrow$ 12 triangles. We also have 3 more triangles extending from the pole. That is, we have 15 triangles in $1/8$ of sphere, so we have $15 \times 8 = 120$ triangles in the sphere.

b. The smallest triangle would be the one formed near the poles, whereas the largest triangle would be near the "equator". We use a unit vector to compare ratios.

We now determine position vector of any arbitrary small triangle and large triangle using $r=1$ and changing θ and ϕ

Small triangle has first point at pole $(0,0,1)$. Second point is changing altitude θ to $30°$, gives $(\sin(30)\cos(0), \sin(30)\sin(0), \cos(30)) = (1/2, 0, \sqrt{3}/2)$. Third point is at azimuth of $30°$ from second, so $(\sin(30)\cos(30), \sin(30)\sin(30), \cos(30)) = (\sqrt{3}/4, 1/4, \sqrt{3}/2)$.

Large triangle has first point at $60°$ altitude, which gives $(\sin(60)\cos(0), \sin(60)\sin(0), \cos(60)) = (\sqrt{3}/2, 0, 1/2)$. Second point at $30°$ azimuth from first is $(\sin(60)\cos(30), \sin(60)\sin(30), \cos(60)) = (3/4, \sqrt{3}/4, 1/2)$. Third point is $30°$ altitude from second, so $(\sin(90)\cos(30), \sin(90)\sin(30), \cos(90)) = (\sqrt{3}/2, 1/2, 0)$.

To summarize, small triangle coordinates at $(0,0,1)$, $(1/2, 0, \sqrt{3}/2)$ and $(\sqrt{3}/4, 1/4, \sqrt{3}/2)$. Large triangle coordinates at $(\sqrt{3}/2, 0, 1/2)$, $(3/4, \sqrt{3}/4, 1/2)$ and $(\sqrt{3}/2, 1/2, 0)$.

We now take $1/2$ of magnitude of cross product to get area.

$$\text{Small: } \|AB \times AC\|/2 = \|(1/2, 0, \sqrt{3}/2 - 1) \times (\sqrt{3}/4, 1/4, \sqrt{3}/2 - 1)\|/2$$
$$= \|((2-\sqrt{3})/8, 7/8 - \sqrt{3}/2, 1/8)\|/2$$
$$= \sqrt{15(7-4\sqrt{3})}/16 \approx 0.0648602$$

$$\text{Large} = \|AB \times AC\|/2 = \|(3/4 - \sqrt{3}/2, \sqrt{3}/4, 0) \times (0, 1/2, -1/2)\|/2$$
$$= \|(-\sqrt{3}/8, 3/8 - \sqrt{3}/4, 3/8 - \sqrt{3}/4)\|/2$$
$$= \sqrt{45 - 24\sqrt{3}}/16 \approx .115765$$

A, B, C denote new points on large triangle

$$\text{Ratio of smallest to largest} = .0648602/.115765 = 0.560275$$

The ratio between the smallest triangle to the largest is approximately 0.560:1.

Hilroy

2. A. We note that the cartesian coordinates x, y, z are given by
$z = r\sin(\theta)\cos(\phi)$, $x = r\sin(\theta)\sin(\phi)$, $y = r\cos(\theta)$,
Thus, we have

$$
\begin{aligned}
z &= 4\sin(60^\circ)\cos(135^\circ) &= -\sqrt{6} \\
x &= 4\sin(60^\circ)\sin(135^\circ) &= \sqrt{6} \\
y &= 4\cos(60^\circ) &= 2
\end{aligned}
$$

The camera's position in (x, y, z) coordinates is therefore
$(\sqrt{6}, 2, -\sqrt{6})$

B. We note that $\hat{c}_z$ is simply the normalized unit vector of the position vector. That is, it becomes $\left(\frac{\sqrt{3/2}}{2}, 1/2, \frac{-\sqrt{3/2}}{2}\right)$.

The up vector is $(0,1,0)$. We can use this to find c_x, which is the cross product of $\vec{up} \times c_z =$
$(0,1,0) \times \left(\frac{\sqrt{3/2}}{2}, 1/2, \frac{-\sqrt{3/2}}{2}\right) = \left(\frac{-\sqrt{3/2}}{2}, 0, \frac{-\sqrt{3/2}}{2}\right)$.
Normalized, $c_x = (-1/\sqrt{2}, 0, -1/\sqrt{2})$.

We can find c_y, which is the cross product of $c_z \times c_x$
$= \left(\frac{\sqrt{3/2}}{2}, \frac{1}{2}, \frac{-\sqrt{3/2}}{2}\right) \times (-1/\sqrt{2}, 0, -1/\sqrt{2}) = (-1/2\sqrt{2}, \sqrt{3}/2, 1/2\sqrt{2})$

Therefore the three camera frame basis vectors are
$c_x = (-\sqrt{2}/2, 0, -\sqrt{2}/2)$, $c_y = (-1/2\sqrt{2}, \sqrt{3}/2, 1/2\sqrt{2})$ and
$c_z = \left(\frac{\sqrt{3/2}}{2}, 1/2, \frac{-\sqrt{3/2}}{2}\right)$ when expressed as measures in the world frame basis

3. A. To determine ${}^cM^w$, we compose the rotation and translation to obtain ${}^cM^w =$

negative of position camera.

$$
\begin{array}{c}
c_x \rightarrow \\
c_y \rightarrow \\
c_z \rightarrow \\
\\
\end{array}
\begin{bmatrix}
-\sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\
-\frac{1}{2\sqrt{2}} & \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & 0 \\
\frac{\sqrt{3/2}}{2} & 1/2 & \frac{-\sqrt{3/2}}{2} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & -\sqrt{6} \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & \sqrt{6} \\
0 & 0 & 0 & 1
\end{bmatrix}
=
\begin{bmatrix}
-1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\
-\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{2}} & 0 \\
\frac{\sqrt{3/2}}{2} & 1/2 & \frac{-\sqrt{3/2}}{2} & -4 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

3. B. Since it is rotated 45° counterclockwise, around z-axis, this means the rotation matrix is

$$\begin{bmatrix} \cos(45) & -\sin(45) & 0 & 0 \\ \sin(45) & \cos(45) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since it is then displaced by 1 unit along the x-axis, this is a translation of

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since we want ${}^{W}M^{B}$ as opposed to ${}^{B}M^{W}$, we compose to get ${}^{W}M^{B} =$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(45) & -\sin(45) & 0 & 0 \\ \sin(45) & \cos(45) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(45) & -\sin(45) & 0 & 1 \\ \sin(45) & \cos(45) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. A. We first apply ${}^{W}M^{B}$ to $[r^{P/B}]_B$ to convert from B basis to world basis. Doing this, we get

$$\begin{bmatrix} \cos 45 & -\sin 45 & 0 & 1 \\ \sin 45 & \cos 45 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Now, since this is the point P on surface expressed in world frame, the light direction l comes from finding the vector from point P to light source, which is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

This is the light direction vector expressed in world frame. Converting to camera frame coordinates, we apply ${}^{C}M^{W}$ from part 3A to get

$$\begin{bmatrix} -1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & 0 \\ \frac{\sqrt{3}/2}{2} & 1/2 & -\frac{\sqrt{3}/2}{2} & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{3}/2 \\ -1/2 \\ 0 \end{bmatrix}$$

← This is the light direction vector l expressed in camera frame.

(remove the 0)

b. The view direction vector starts at point P and points in direction e. This is in the direction of the camera, which we found cartesian coordinates for in 2A. So, the view direction vector becomes

$$\begin{bmatrix} \sqrt{6} \\ 2 \\ -\sqrt{6} \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{6} \\ 1 \\ -\sqrt{6} \\ 0 \end{bmatrix}$$

This is the view direction vector e expressed in world frame. Converting to camera frame coordinates, we apply ${}^{C}M^{W}$ to get

$$\begin{bmatrix} -1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & 0 \\ \frac{\sqrt{3}/2}{2} & 1/2 & -\frac{\sqrt{3}/2}{2} & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{6} \\ 1 \\ -\sqrt{6} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{3}/2 \\ 7/2 \\ 0 \end{bmatrix}$$

This is the view direction vector e expressed in camera frame.

(remove 0)

Hilroy

5. A. We have 27.3 days required for a full $360°$ rotation around the Earth by the moon. Since 1 second of animation corresponds with 1 day of real time, this means the animated moon travels full $360°$ in 27.3 seconds. Since we have 60 frames per second, we need a total of $60 \times 27.3 = 1638$ frames. Thus, we need to rotate $360/1638$ degrees per frame, which is around 0.2198 degrees per frame.

B. The perceived orbital period would result from adding the contributions in rotation from both the sun and the Earth. Since the Earth has an orbital period of 365 days, this means it makes a $360/365$ degree rotation each day. Similarly, the sun makes a $360/25.4$ degree rotation each day. Adding this means a total of $360/365 + 360/25.4 = 140544/9271 \approx 15.16°$ rotation each day. $360/15.16 = 23.75$. Thus, the perceived orbital period of the Earth would be approximately 23.75 days. In our program, the orbital period would therefore be around 23.75 seconds.

Hilroy