10.2 Elastic Deformation in Solids

Hooke's Law -- Linear elasticity of materials

Stress/strain = constant. (up to about 1% elongation in metals)

10.2.1 Young's Modulus for Linear Deformation

For one-directional normal stress

$$s_y = Ee_y$$

E is called Young's modulus or the modulus of elasticity. Its units are Pa.

Table 10-1 lists values of E for several materials. E is generally higher for higher melting point materials.

10.2.2 Poisson's Ratio

When we apply stress in one direction, it causes strain not only in the direction of the stress but also in other directions. The extent of deformation in the other directions is proportional to the deformation in the direction of applied stress. The constant of proportionality is an important property of the material involved. It is called Poisson's ratio.

Poisson's Ratio (denoted by ν)

$$n = \frac{-e_x}{e_y} = \frac{-e_x E}{s_y}$$

$$or$$

$$e_x = \frac{-ns_y}{E}$$

For a three-dimensional body, stressed in all three directions,

$$e_x = \frac{s_x}{E} - \frac{n}{E}\left(s_y + s_z\right)$$

$$e_y = \frac{s_y}{E} - \frac{n}{E}\left(s_x + s_z\right)$$

$$e_z = \frac{s_z}{E} - \frac{n}{E}\left(s_y + s_x\right)$$

Where, σ_x, σ_y, σ_z are stress components, and ε_x, ε_y, ε_z are strain components.

10.2.3 The bulk Modulus for Volume Change

Consider a rectangular body, x by y by z, under stress. The new dimensions would be:
$(1+\varepsilon_x)x$, $(1+\varepsilon_y)y$, $(1+\varepsilon_z)z$

New volume would be
$(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)xyz$

Original volume was xyz

Fractional increase in volume = $\Delta V/V = [(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - 1]$
When strains are small, their products are much smaller and can be neglected
Therefore, $\Delta V/V = \varepsilon_x + \varepsilon_y + \varepsilon_z$

We can now substitute for each of ε terms from previously obtained equations for stresses.

$$\frac{\Delta V}{V} = \frac{1-2n}{E}(s_x + s_y + s_z)$$

This provides an equation to calculate the change in volume when all stress components are present.

Let us look at compression of solids under pressure, for example solids in deep-sea environment.

Compression is negative, therefore $\sigma_x = \sigma_y = \sigma_z = -P$

Then

$$\frac{\Delta V}{V} = \frac{1-2n}{E}(-P-P-P) = \frac{-3(1-2n)P}{E} \equiv -\frac{P}{K}$$

Where, $K = \dfrac{E}{3(1-2n)}$ is called the bulk modulus of elasticity. It is a function of E and ν.

Since,

$$\frac{\Delta V}{V} = -\frac{P}{K}$$

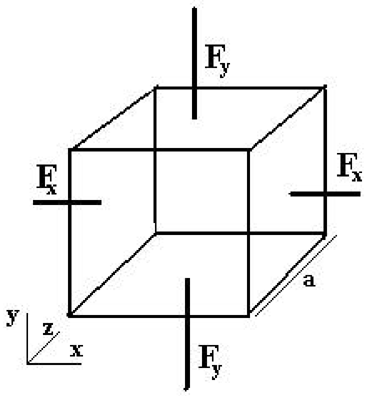
$$\frac{1}{K} = -\frac{1}{P}\frac{\Delta V}{V} = -\frac{1}{V}\frac{\Delta V}{P}$$

1/K is the fractional change in volume with pressure. It is the isothermal compressibility of solids. This is similar to the isothermal compressibility of liquids, in Chapter 7.

$$\beta_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$$

Example Problem

Consider a metal that has a cubic shape and which is subject to forces in the x and y directions as shown in the figure. Note that there are no forces applied in the z-direction.



(a) Show that if the tensile force in the x-direction and the compressive force in the y-direction have the same magnitude, then there is no change in the volume of the cube, i.e. ΔV=0. (**6 marks**)

For the system of forces shown in the figure we have:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E},$$
$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E}, \qquad \Rightarrow \frac{\Delta V}{V} \approx \varepsilon_x + \varepsilon_y + \varepsilon_z = (1-2\nu)\frac{\sigma_x + \sigma_y}{E}$$
$$\varepsilon_z = -\nu\frac{\sigma_x + \sigma_y}{E},$$

But $\sigma_x = F_x/A$ and $\sigma_y = -F_y/A$. Therefore if $|F_x| = |F_y|$, then $\sigma_x + \sigma_y = 0$ or Δ**V=0**.

(b) It is found that if the magnitudes of the forces in the x and y directions are equal to 200N and 100N respectively, the relative change in the volume of the cube $\Delta V/V$, is equal to 10^{-5}. We give a=1cm, and the Poisson ratio ν=0.4. Determine:

(i) The value of the Young modulus of the metal. (**3 marks**)

$\sigma_x = F_x/A = F_x/a^2 = 200/(0.01)^2 = 2*10^6$ Pa.
$\sigma_y = -F_y/A = -F_y/a^2 = -100/(0.01)^2 = -10^6$ Pa.
$\sigma_z = 0$
Therefore from Eq. 10.16

$$E = (1-2\nu)\frac{\sigma_x + \sigma_y + \sigma_z}{\left(\frac{\Delta V}{V}\right)} = (1-2\times 0.4)\frac{10^6}{10^{-5}} = 20\,GPa.$$

(ii) The value of the Shear modulus of the metal. (**3 marks**)

$$G = \frac{E}{2(1+\nu)} = \frac{20GPa}{2(1+0.4)} = 7.14GPa.$$

(iii) The value of the strain in the x-direction, ε_x. (**3 marks**)

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = \frac{2.10^6}{2.10^{10}} - 0.4\frac{(-10^6)}{2.10^{10}} = 1.2\times 10^{-4}$$