9.2.4 The Rate of Heat Conduction

The rate of heat flow in solids depends on:

- Geometry of the solid (plate, cylinder, sphere)
- Temperature distribution
- Thermal conductivity, κ

We will look at the simpler case of steady-state heat flow only. Steady-state means the temperature distribution is not changing with time. The simplest geometry is that of a flat plate in which the cross-sectional area for heat flow remains constant.

$$Q = -kA\frac{dT}{dx} = -kA\frac{\Delta T}{\Delta x}$$

If you know κ, A and the temperature gradient, the amount of heat energy flowing through the surface can be calculated.

Composite (Layered) Walls

- Good heat insulators often do not have good mechanical properties.
- Composite walls are used where good heat insulation needs to be combined with good mechanical properties.
- Often several layers may be involved.

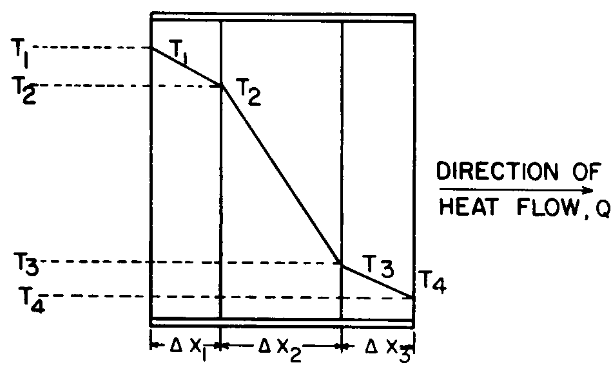

Figure 9-6 Temperature Profile for a Composite Wall

Analysis of heat flow through a composite wall

- ***At steady-state the amount of heat flow is same in every layer, while thermal conductivity and temperature gradients are different, i.e.***

$$\boldsymbol{Q_1 = Q_2 = Q_3 = Q}$$

$$Q_1 = -k_1 A \frac{T_2 - T_1}{\Delta x_1} = k_1 A \frac{T_1 - T_2}{\Delta x_1}$$ For layer # 1. Similarly for layer #2,

$$Q_2 = -k_2 A \frac{T_3 - T_2}{\Delta x_2} = k_2 A \frac{T_2 - T_3}{\Delta x_2}$$ and for layer #3,

$$Q_3 = -k_3 A \frac{T_4 - T_3}{\Delta x_3} = k_3 A \frac{T_3 - T_4}{\Delta x_3}$$

These equations can be rearranged to make them explicit for the temperature difference. Also, Q_1, Q_2 and Q_3 can be replaced by Q.

$$T_1 - T_2 = \frac{Q(\Delta x_1)}{k_1 A}$$ For layer #1

$$T_2 - T_3 = \frac{Q(\Delta x_2)}{k_2 A}$$ For layer #2

$$T_3 - T_4 = \frac{Q(\Delta x_3)}{k_3 A}$$ For layer #3

Taking the sum of these equations,

$$T_1 - T_4 = \frac{Q}{A}\left[\frac{\Delta x_1}{k_1} + \frac{\Delta x_2}{k_2} + \frac{\Delta x_3}{k_3}\right]$$

This can be rearranged to get an expression for Q.

$$Q = \frac{A(T_1 - T_4)}{\left[\frac{\Delta x_1}{k_1} + \frac{\Delta x_2}{k_2} + \frac{\Delta x_3}{k_3}\right]}$$

Resistance to heat flow (insulation value) is defined as $R = \frac{\Delta x}{k}$.

For the three layers, we can write

$$R_1 = \frac{\Delta x_1}{k_1}$$

$$R_2 = \frac{\Delta x_2}{k_2}$$

$$R_3 = \frac{\Delta x_3}{k_3}$$

The heat flow in terms of the resistances of the layers is

$$Q = \frac{(T_1 - T_4)A}{R_1 + R_2 + R_3} = \frac{A(T_1 - T_4)}{R}$$

$$or, \ \frac{Q}{A} = -\frac{\Delta T}{R}$$

Heat flux is proportional to ΔT and inversely proportional to R.
Analogous to electric current ($I = \Delta E/R$)

Heat flow through cylindrical elements (pipes)

In this case, heat flows in the radial direction and is given by

$$Q = -kA\frac{dT}{dr}$$

The area for heat flow for a given length L of the pipe is, $A = 2\pi rL$. Note that the area is not constant, it increases with r.

$$Q = -2pkrL\frac{dT}{dr}$$

$$or, \ Q\frac{dr}{r} = -2pkLdT$$

Integrating both sides between r_1 and r_2 gives,

$$Q\int_{r_1}^{r_2} \frac{dr}{r} = -2pkL\int_{T_1}^{T_2} dT$$

Or

$$Q\ln\left(\frac{r_2}{r_1}\right) = 2pkL(T_1 - T_2)$$

or,

$$Q = \frac{2pkL(T_1 - T_2)}{\ln\left(\frac{r_2}{r_1}\right)}$$

If a composite cylindrical wall is involved (e.g. metal pipe clad with layers of insulation)

$$Q = \frac{-2pL\Delta T}{\dfrac{\ln(r_2/r_1)}{k_1} + \dfrac{\ln(r_3/r_2)}{k_2} + \dfrac{\ln(r_4/r_3)}{k_3} + ...}$$

Similar equations can be developed for spherical surfaces.

Example Problem:
A cylindrical pipe with a 5 cm inner diameter is used to carry a hot fluid. The pipe's wall is 5 mm thick and has a thermal conductivity of 25 W/(m.K). Determine the rate of heat loss per unit length (in units of kW/m) if the temperature of the hot fluid is 300 ^{o}C, and the outer surface of the cylinder remains at the ambient temperature of 25 ^{o}C.

R_1 = 2.5cm
R_2 = 2.5 + 0.5 = 3cm

$$Q = \frac{2\pi \kappa L(T_1 - T_2)}{\ln\left(\dfrac{r_2}{r_1}\right)}$$

$$\frac{Q}{L} = \frac{2\pi\kappa(T_1 - T_2)}{\ln\left(\dfrac{r_2}{r_1}\right)} = \frac{2\pi \times 25 \times (300 - 25)}{\ln(3/2.5)} = 236.93\,\mathrm{kW/m}$$