8.3 Flow of Viscous (Real) Fluids

Real fluids experience a certain resistance to flow. Overcoming this resistance requires work. It means that a portion of the total mechanical energy (pressure + gravitational potential + kinetic energy) is dissipated (converted to heat) during the flow.

For low flow rates, the heat generated is too small to make any significant difference in temperature. At high flow velocities in well-insulated systems, the temperature rise due to energy dissipation can be quite large.

When the velocity is low, fluid particles move in smooth streamlines.

When the flow velocity is high, eddies or circulatory motion of fluids takes place. This increases the rate of energy dissipation.

Thus there are two distinct patterns of flow: Laminar (no eddies) and Turbulent (eddies)

The transition from laminar flow turbulent flow occurs at threshold set of conditions that are predictable.

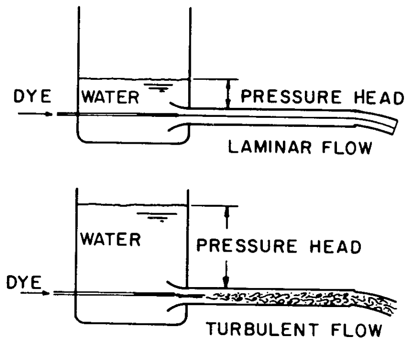


Figure 8-4 An Apparatus for Reynolds' Experiment

Reynolds Number

A dimensionless ratio of parameters

$R_e = \dfrac{D\bar{u}\ r}{m}$, Where,

D = pipe diameter, m

$\bar{u}$ = average velocity, m/s

ρ= density, kg/m^3

μ= viscosity, Pa.s or kg/m.s

Laminar flow in pipes when Re < 2,100
Turbulent flow in pipes when Re > 4,000

For 2100 < Re < 4000, Intermediate or transition region. It means some turbulence has started but it is not fully developed.

Read Section 8.3.1 "The Boundary Layer" yourself.

8.4 Laminar Flow in Pipes; The Hagen Poiseulle Eqn.

Consider:

- Steady flow, (no time dependent changes in velocity, i.e. no acceleration)
- Horizontal pipe
- Real liquid at constant temperature (constant density)
- Newtonian behaviour
- Laminar flow

Consider the force balance on the fluid element represented by the disk.

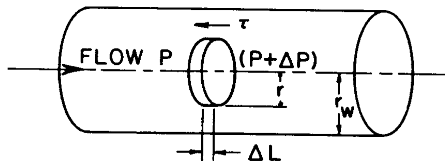


Figure 8-6 Flow Through a Pipe

Pressure force on the left side = $\pi R^2 P$

Pressure force on the right side = $\pi R^2 (P+\Delta P)$

Shear force (friction) on the edge = $-2\pi r \Delta L \tau$

Force balance (net force = zero, since no acceleration)

$$pr^2 P - pr^2 (P - \Delta P) - 2pr\Delta Lt = F_n = 0$$

Divide by $\pi r^2 \Delta L$ and rearrange to get

$$\frac{\Delta P}{\Delta L} + \frac{2t}{r} = 0$$

The expression: $\Delta P/\Delta L + 2\tau/r = 0$ is true for all radii of the disk.

When $r = r_w$,

$$\frac{\Delta P}{\Delta L} + \frac{2t_w}{r_w} = 0$$

Since the value of $\Delta P/\Delta L$ is same for all r,

$$\frac{t_w}{r_w} = \frac{t}{r} = \text{constant, therefore}$$

$$t = C\,r$$

8.4.1 Velocity distribution in a Pipe

For a Newtonian Fluid

$$t = -m \frac{du}{dy}$$

Substitute this into $\dfrac{t_w}{r_w} = \dfrac{t}{r}$

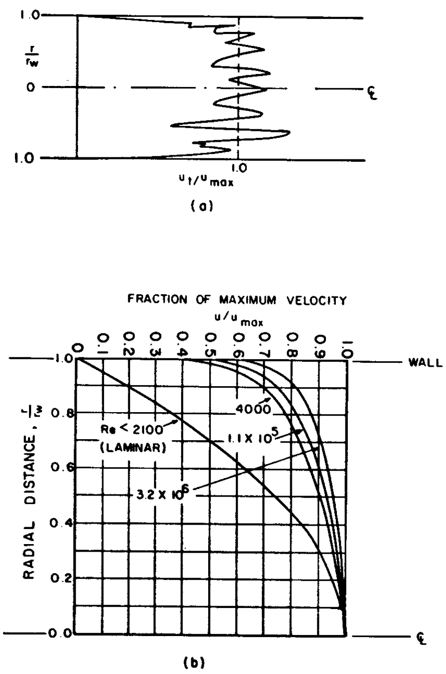


Figure 8-8 Velocity Profiles in Pipes
(a) Local Random Velocities in Turbulent Flow.
(b) Average Local Velocities in Laminar and Turbulent Flow.

$$-m\frac{du}{dy}=\frac{t_w}{r_w}r,\ or$$

$$du=-\frac{t_w rdr}{r_w m}$$

By integrating this differential equation

$$\int_0^u du=-\int_{r_w}^{r}\frac{t_w}{r_w m}dr$$. Limits are: at r = r_w, u = 0 (the no slip condition)

Therefore,

$$u=\frac{t_w}{r_w m}\left.\frac{r^2}{2}\right|_{r_w}^{r}=\frac{t_w}{2r_w m}\left[r_w^2-r^2\right]$$

or

$$u=\frac{t_w r_w}{2m}\left[1-\frac{r^2}{r_w^2}\right]$$

The velocity u at any radius r is given by the above equation.

If you plot u vs. r/r_w, the result is a parabola. The velocity profile for laminar flow in circular pipes is always parabolic.

At r = r_w, u = 0
At r = 0 , u = u_{max} =

Therefore in terms of u_{max},

$$u=u_{\max}\left[1-\left(\frac{r}{r_w}\right)^2\right]$$

$$or,\quad \frac{u}{u_{\max}}=\left[1-\left(\frac{r}{r_w}\right)^2\right]$$

8.4.2 Average Velocity and Volumetric Flow Rate

We have seen that u varies from zero at the walls to a maximum value at the pipe axis. For any given volumetric flow rate, the average velocity is simply the ratio of volumetric flow rate and cross-sectional area.

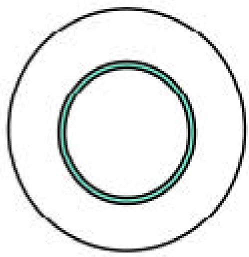
$$\bar{u} = \frac{Q}{A} = \frac{Q}{\frac{1}{4}\pi D^2}$$

What is the relationship between average velocity and the maximum velocity?

This relationship can be determined by evaluating the total flow rate as a summation (or integration) of flow rates representing small concentric rings in the cross-sectional area.

The flow rate through a concentric ring between r and r+dr is given by velocity X cross-sectional area of the ring.

$dQ = (2\pi\ r\ dr)u = (2\pi\ r\ dr)u_{max}\ [1- (r/r_w)^2]$



Integrating from r= 0 to r = r_w,

$$Q = \int_0^Q dQ = 2\pi\ u_{max} \int_0^{r_w} \left[1 - \left(\frac{r}{r_w}\right)^2\right] r dr = \frac{u_{max}}{2}(\pi\ r_w^2)$$

Now we have two expressions for Q

By definition: $Q = (\pi\ r_w^2)\bar{u}$ and

$$Q = \frac{u_{max}}{2}(\pi\ r_w^2)$$

Equating the two expressions we get, $\bar{u} = \dfrac{u_{max}}{2}$

Therefore for laminar flow of a newtonian fluid in a circular pipe the maximum velocity is equal to twice the average velocity.

8.4.3 Working Equation for Laminar Flow

Before we derive the important "working" equation for laminar flow, one other item needs to be looked at.

The equation $\dfrac{\Delta P}{\Delta L} + \dfrac{2\tau_w}{r_w} = \dfrac{\Delta P}{\Delta L} + \dfrac{2\tau}{r} = 0$ was derived for a horizontal pipe. What if the pipe is not horizontal?

Let us look at an inclined pipe. Now the influence of gravity can not be neglected. Now the flow is by gravity as well as by the pressure difference. The force balance for this case must include the gravitational force as well. We get

$$\frac{\Delta P}{L} + rg\frac{h_2 - h_1}{L} + \frac{2t}{r} = 0$$

At the wall,

$$\frac{\Delta P}{L} + rg\frac{h_2 - h_1}{L} + \frac{2t_w}{r_w} = 0$$

Now we substitute the expression for τ_w in terms of u

$$t_w = \frac{4m\bar{u}}{r_w}$$

Therefore,

$$\frac{\Delta P}{L} + rg\frac{h_2 - h_1}{L} + \frac{8m\bar{u}}{r_w^2} = 0$$

or

$$-\left[\frac{\Delta P}{L} + rg\frac{h_2 - h_1}{L}\right] = \frac{8m\bar{u}}{r_w^2} = \frac{32m\bar{u}}{D^2}$$

This equation is known as Hagen-Poiseulle equation.
In terms of the volumetric flow rate:

$$Q = \frac{p}{4}D^2\bar{u} \quad \text{or} \quad \bar{u} = \frac{4Q}{pD^2}$$

Substituting for $\bar{u}$ in the flow equation, we get

$$-\left[\frac{\Delta P}{L} + rg\frac{h_2 - h_1}{L}\right] = \frac{8mQ}{pr_w^4} = \frac{128mQ}{pD^4}$$

This equation is the working equation for laminar flow in pipes.