1. Hydrostatic forces

a. Solids, liquids and gases

It may be worth our while to review briefly some elementary ideas about the three classical states of matter, as summarized in the following table.

State	Microscopic aspects	Macroscopic aspects
GAS	Very weak intermolecular forces. Particles move freely at high speed.	Low density Does not resist shear
LIQUID	Intermolecular forces strong enough to keep particles close, but not enough to prevent them from sliding past each other.	High density Doesn't bear shear if at rest
SOLID	Very strong intermolecular forces. Particles oscillate around fixed positions.	High density Resists shear

As a consequence, gases are highly compressible and, in their random motion, the molecules fill the recipient in which they are placed. Liquids are quite incompressible. When placed in the gravitational field of the Earth, they flow to the lower parts of a recipient and exhibit a perfectly horizontal free surface at the top (except for the formation of menisci near the boundary). As a consequence of their inability to support any shearing forces, gases and liquids (at rest) experience, at a given point, the same *internal pressure* in all directions. This fact is known as *Pascal's principle*. More generally, Blaise Pascal (1623-1662)[1] affirmed that if an increase of pressure is applied at any point in a fluid, it is transmitted equally to all parts of the fluid in all directions. To understand why it is that the absence of internal shear forces implies the equality of pressure (normal forces per unit area) in all directions, it is perhaps helpful to think of a vertical column of water subjected to its own weight. If you now consider a section of this column at, say, 45 degrees from the horizontal, and look at the FBD above the cut, how can you balance a vertical force if not with an equal and opposite (internal) vertical force? But, since the section is inclined, this vertical internal force will have both an axial and a shear component, which the liquid cannot bear. It, therefore, would collapse, unless lateral pressures are applied (by the walls of the tubular container).

b. Hydrostatic pressure in the gravitational field

Having established that the pressure in a liquid at rest is independent of direction, we want to quantify its magnitude (that is, its variation from point to point) when the liquid is at rest under the action of a constant gravitational field g. By again considering a column of liquid, it is not difficult to conclude that the pressure p at a point in the liquid (assuming constant density ρ) is given by

$$p = \rho g h.$$

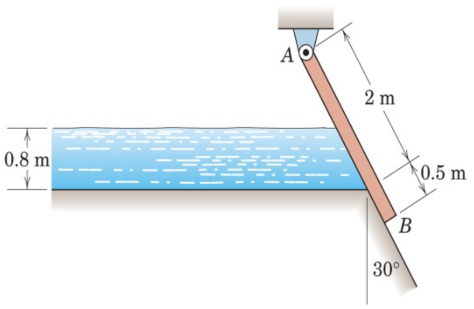
[1] Pascal was a remarkable genius who influenced pure and applied mathematics, physics and philosophical and religious thought. At the age of 18 he invented and built a mechanical calculator, but this was not his first piece of mathematical work, having already published an original Essay on Conic Sections two years earlier. His religious philosophy, implying that rational thought is not the only access to truth, is sometimes summarized in his best-known adage: "Le coeur a ses raisons que la raison ne connaît point".

In this formula, h represents the depth of the point measured vertically from the horizontal free surface of the liquid. This formula provides us with the pressure due to the liquid alone. It does not include the atmospheric pressure exerted by the air on the surface of the liquid. According to Pascal's principle, this atmospheric pressure should be added uniformly to every point of the liquid. In many practical applications, however, the effect of the atmospheric pressure can be considered as cancelling out, since air permeates all the available space (unless, for instance, a vacuum is maintained artificially on one side of the structure being analyzed).

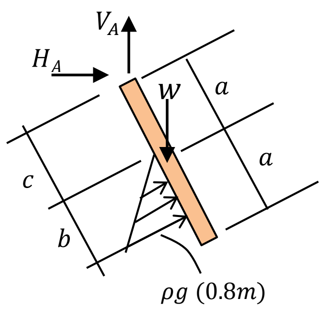
The density of fresh water under usual ranges of temperature and pressure is approximately $\rho = 1000\ kg/m^3$, which leads to a specific weight of $\gamma = \rho g = 9807\ N/m^3$ on the surface of the earth. In US standard units, the result is $\gamma = 62.43\ lb/ft^3$.

2. <u>Examples</u>

a. <u>Opening a gate</u>: (Problem 5.191) Fresh water in a channel is contained by the uniform 2.5 *m* plate freely hinged at *A*. If the gate is designed to open when the depth of the water reaches 0.8 *m*, as shown in the figure, what must the weight *w* (in *N/m* of horizontal length into the paper) of the gate be? Density of fresh water = 1000 kg/m^3.



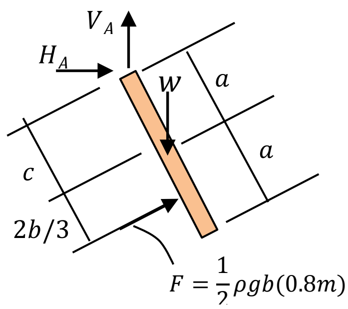
<u>Solution</u>: The gate will be about to open when the reaction at B vanishes. Indeed, adding any water will raise the level and will require a negative reaction, which the support at B is unable to provide. The free-body diagram corresponding to this state of impending opening is, therefore, as depicted below. The equation of moments with respect to point A delivers



$$a = \frac{2.5m}{2} = 1.25m$$

$$b = \frac{0.8m}{\cos 30°} = 0.924m$$

$$c = 2m - b = 1.076m$$

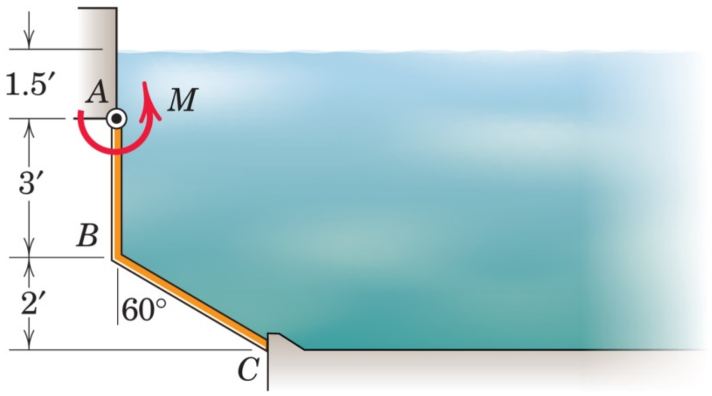


$$\Sigma M_A = -wa(\sin 30°) + F\left(c + \frac{2b}{3}\right) = 0$$

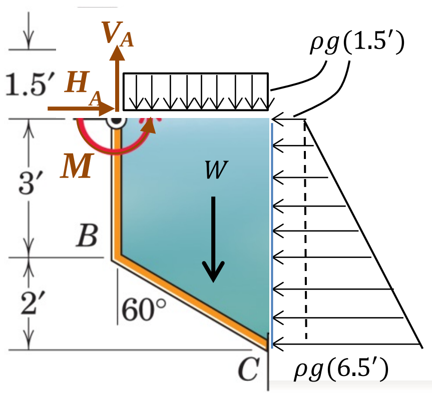
From here we obtain

$$w = \frac{F\left(c+\frac{2b}{3}\right)}{a(\sin 30°)} = 9810\ N/m$$

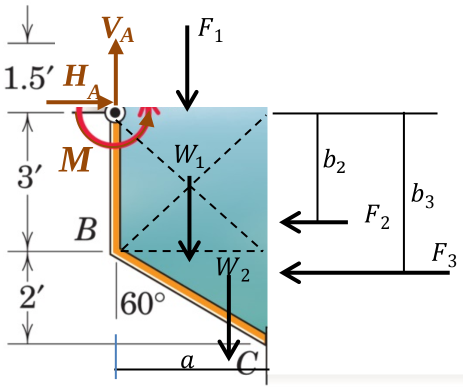
b. Keeping a gate in place: (Problem 5.201) A gate is used to hold fresh water in storage. Determine the required moment M to just hold the gate closed against the lip of the container at C if the width of the gate is 5 ft. Neglect the weight of the gate. The specific weight of fresh water is $\rho g = 62.43\ lb/ft^3$.



Solution: We can solve this problem following exactly the same technique as in the previous problem, namely, drawing a correct FBD of the gate itself subjected to the water pressure and the applied couple. An alternative, often more elegant and efficient, procedure consists of including part of the water in the FBD. Why? If the included part of the water is bounded by vertical and horizontal planes, it is often easier to avoid complicated shapes of the solid surface (particularly curved shapes, as we shall see in the next problem). There is always a price to pay for any such benefit. In this case, the price is that we must also include the weight of the portion of water included in the FBD. In this particular example, both procedures (including or not including a portion of the water) are roughly equivalent in terms of amount of computations. For this reason, it is strongly recommended that you solve it also by the direct method used in the previous problem. You may learn more from this comparison than from solving more and more examples.



Just as in the previous example, the reaction at C is set to zero, since the problem specifies the minimum value of M to achieve closure (without pushing against the lip). We now redraw the FBD replacing the distributed loads with their equivalent resultants.



$$a = 2' \tan 60° = 3.464'$$

$$b_2 = 0.5(3' + 2') = 2.5'$$

$$b_3 = (2/3)(3' + 2') = 3.33'$$

$$W_1 = \rho g(3')a(5') = \rho g\ 51.96 ft^3$$

$$W_2 = 0.5\ \rho g\ (2')a(5') = \rho g\ 17.32\ ft^3$$

$$F_1 = \rho g\ (1.5')a(5') = \rho g\ 25.98\ ft^3$$

$$F_2 = \rho g\ (1.5')(5')(5') = \rho g\ 37.5\ ft^3$$

$$F_3 = 0.5\ \rho g\ (6.5' - 1.5')(5')(5') = \rho g\ 62.5\ ft^3$$

Notice that the weight of the water has been decomposed into two conveniently chosen components. The equation of moments about the hinge at A yields

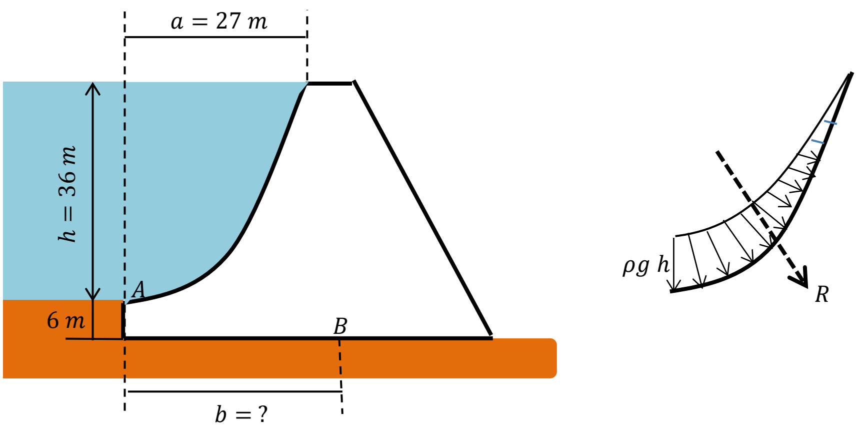
$$\Sigma M_A = M - \frac{W_1 a}{2} - W_2 \frac{2a}{3} - \frac{F_1 a}{2} - F_2 b_2 - F_3 b_3 = 0$$

Using the values indicated in the diagram, we obtain

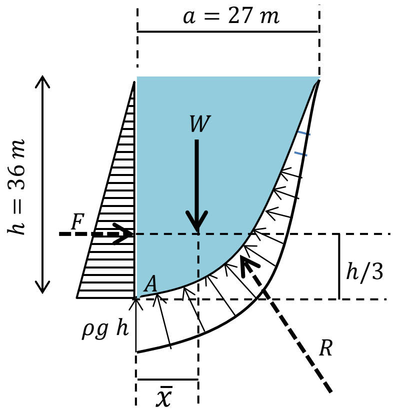
$$M = 29{,}771 \; lb \cdot ft$$

c. A water dam: One of the most important engineering applications of hydrostatic forces (beyond water tanks and chemical reactors) is the design and analysis of water dams. These are massive projects that involve environmental, electrical, mechanical and civil engineers as well as politicians, social scientists, economists and advocates. From the point of view of Statics, beyond considerations of equilibrium, there are problems of *stability*. The dam is held in place, among other factors, by the weight of the solid structure and the resistance of the ground to normal pressure and to horizontal sliding. Forces of *friction* play an important role. Moreover, the designers have to guarantee that the dam is stable against tipping. The problem below, without entering into such details, is somewhat illustrative of these considerations. At the very least, we want the resultant of the water forces to intersect the base of the dam. This explains in part the curved (rather than straight and vertical) shape of the dam wall in contact with the water.

Problem 5.211: The fresh-water side of the concrete dam shown has the shape of a vertical parabola with vertex at A. Determine the position b of the base point B through which the resultant force of the water acts against the dam face.



Solution: We could reason (correctly) as follows: Since we are asked to find the point of intersection of the resultant force of the water, we should, in principle, be able to determine this resultant R (shown schematically on the right side of the figure above) by integration of the pressure over the length of the parabolic curve. This is a mathematical exercise of some difficulty, but certainly well within your capabilities. On the other hand, as we saw in the previous problem, we could opt for the alternative procedure of including a portion of the water in the free-body diagram and thus achieve a certain simplification. With this option in mind, we draw below the corresponding FBD. From this picture we can establish that the forces F (resultant of water pressure on the vertical wall), W (weight of the parabolic segment of water) and R (reaction of the curved wall upon the water) must be in equilibrium. In other words, we can obtain the force R exerted by the water on the parabolic wall as the resultant of the forces F and W.



But these two forces are relatively easy to obtain. Indeed, we have

$$F = 0.5\,(\rho g h)h$$

The weight W (per unit length in the direction perpendicular to the drawing) is obtained by multiplying the specific weight of water times the area within the parabolic segment. Either by direct integration or by checking the value of this area in the table at the end of the book we obtain

$$W = \rho g\,\frac{2}{3}\,a\,h$$

Again, whether by integration or from the table, we obtain that the location of the centroid of this parabolic segment is given by

$$\bar{x} = \frac{3}{8}\,a$$

Now, if the resultant of these two forces passes through point B, then certainly the moment with respect to B must vanish (just as was the case in a question of the first midterm)! From this condition we obtain

$$\Sigma\, M_B = -F\left(\frac{h}{3} + 6m\right) + W(b - \bar{x}) = 0$$

Solving for b we obtain

$$b = \bar{x} + \frac{F\left(\frac{h}{3} + 6m\right)}{W} = \frac{3}{8}a + \frac{0.5(\rho g\,h)h\left(\frac{h}{3} + 6m\right)}{\rho g\,\frac{2}{3}\,a\,h} = 28.125\,m$$