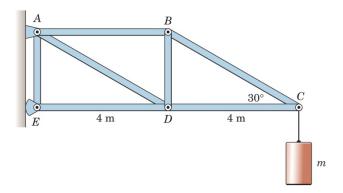
1. Analysis vs. design

Among the important activities that engineers engage in during the exercise of their profession, we can mention analysis and design. Although the boundary between these two tasks is fuzzy at best, it is always helpful to have at least a general idea of what they entail. *Analysis* (and, in particular, *structural analysis*) is entirely governed by the left side of the brain. Given a structure with all its details (dimensions, materials, supports and loads) the structural analyst enforces the laws of equilibrium (and perhaps other laws) to calculate the internal forces in the structure. In this way, the engineer can find out whether or not the given structure can safely bear the applied loads. The result of this analysis can sometimes be encapsulated in a single number called the *safety factor* of the structure.

Design, on the other hand, is a creative activity that involves both hemispheres of the brain. It usually involves also other important engineering activities such as interacting with architects, clients, governments and other stakeholders. The point of departure of a design is a general statement of the function that a structure must perform. It may be that we want to build a bridge over a river to allow for the traffic of cars and trucks. Clearly, there may be many solutions to this problem. Once a solution has been chosen in terms of the location, the kind of bridge (truss, suspension, cable-stayed, pontoon), the number of lanes and other such factors, the problem is narrowed down to the choice of specific sizes and materials to be used. But even at this stage there is still a considerable amount of latitude left for the engineering team to make decisions that affect the cost, the time limits, the functionality and the aesthetics of the structure. At each step of the way, the engineers must also perform an analysis of each of the proposed alternatives to ensure that the load can be carried safely. It follows, therefore, that design is a much more complex activity than analysis.

In this course, we naturally focus our attention on the analysis of some rather simple structures and do not engage in design in any serious way. Nevertheless, some problems of analysis involve considerations akin to those of design. For example, instead of specifying the load acting on a given structure, we may specify the *limiting strength* of each member of the structure and then ask ourselves what is the maximum load that can be applied to the structure without exceeding the strength of any of the members. By limiting strength of a member (in the case of a truss) we mean the maximum internal force (tensile or compressive) that the member can safely carry (before it breaks or before it becomes unsafe). This kind of incipient design problem is exemplified in the following exercise.

2. Example (P 4.11) If the maximum tensile force in any of the truss members must be limited to 24 kN, and the maximum compressive force must be limited to 35 kN, determine the largest permissible mass m that may be supported by the truss.



The solution strategy for a question of this kind is the following:

- (a) Leave the value of the external load as a variable W.
- (b) Solve for the internal forces in all the bars *in terms of W*. Namely, each bar will carry an internal force of the form $N_i = \alpha_i W$. In this expression, N_i is the internal force in bar number i, and α_i is a numerical coefficient that may be positive (tension) or negative (compression). The zero-force members are clearly irrelevant.
- (c) If $\alpha_i > 0$, in our specific example, we will stipulate the safety condition

$$N_i = \alpha_i W \le 24 \ kN$$

Hence we obtain the condition

$$W \leq \frac{24 \; kN}{\alpha_i}$$

(d) Similarly, if $\alpha_i < 0$ we stipulate

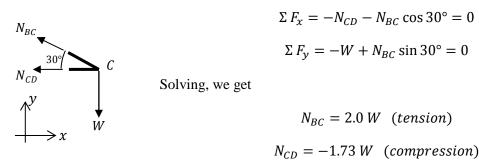
$$|N_i| = |\alpha_i|W \le 35 \ kN$$

We obtain the condition

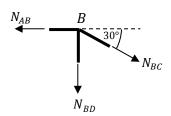
$$W \le \frac{35 \, kN}{|\alpha_i|}$$

(e) In this way, we obtain a total of *n* inequalities, where *n* is the number of bars (excluding zero-force members). Since all these inequalities must be satisfied (otherwise one or more members would violate the safety condition) we choose the value of *W* corresponding to the *smallest* right-hand side. Thus, we obtain a structure for which the weakest link governs the design. All bars amply satisfy their stipulated safety conditions, except that one for which the safety condition is just barely satisfied.

We start from node C (since it has just two unknown internal forces) and obtain



Moving to node B yields



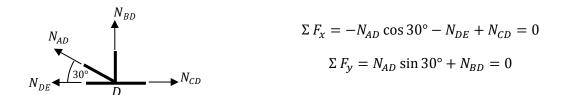
$$\Sigma F_x = -N_{AB} + N_{BC} \cos 30^\circ = 0$$

$$\Sigma F_y = -N_{BD} - N_{BC} \sin 30^\circ = 0$$

Using the value of N_{BC} obtained above, we get

$$N_{BD} = -1.0 W$$
 (compression)
 $N_{AB} = 1.73 W$ (tension)

From the FBD of node *D* we conclude that



The forces in the two new bars are, therefore,

$$N_{AD} = 2.0 W$$
 (tension)
 $N_{DE} = -3.46 W$ (compression)

Finally, at node *E* we see, by inspection, that AE is a zero-force member.

Having obtained all the internal forces in terms of the external load W, we proceed to distinguish between tensile and compressive members and to write the corresponding safety inequalities. The result is

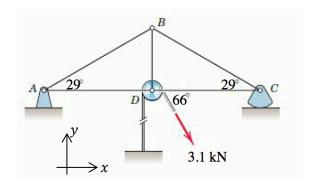
Bar BC:
$$W \le \frac{24 \ kN}{2.0} = 12 \ kN$$

Bar AB: $W \le \frac{24 \ kN}{1.73} = 13.9 \ kN$
Bar AD: $W \le \frac{24 \ kN}{2.0} = 12 \ kN$
Bar CD: $W \le \frac{35 \ kN}{1.73} = 20.2 \ kN$
Bar BD: $W \le \frac{35 \ kN}{1.0} = 35 \ kN$
Bar DE: $W \le \frac{35 \ kN}{3.46} = 10.1 \ kN$

On the basis of these results we conclude that the largest value of the weight W that can be applied without violating these inequalities is $10.1 \, kN$. The weakest link in this structure is bar DE. The largest mass that can be safely suspended from point C is, accordingly

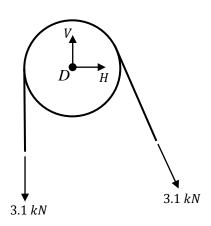
$$m = \frac{W}{g} = \frac{10.1 \, kN}{9.8 \, m \, s^{-2}} = 1030 \, kg$$

3. One more example (involving a pulley)



Determine the force in each member of the truss.

<u>Solution</u>: We start by drawing and analyzing the FBD of the pulley. Our intention in doing this is to calculate the force exerted by the pulley (via its axle) onto node *D* of the truss.



The equilibrium equations are

$$\Sigma F_x = (3.1 \, kN) \cos 66^\circ + H = 0$$

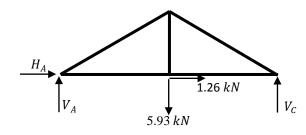
$$\Sigma F_y = -3.1 \, kN - (3.1 \, kN) \sin 66^\circ + V = 0$$

<u>Note</u>: The equation of moments does not need to be imposed, since it has been already used to establish that the tension is the same on both sides of the rope.

Thus, we conclude that

$$H = -1.26 \, kN$$
 (opposite to shown)
 $V = 5.93 \, kN$ (as shown)

The forces applied by the pulley on the truss are, of course, opposite to these ones. The FBD of the truss is, therefore, as depicted below.



The support reactions are obtained from the equilibrium equations

$$\Sigma F_x = H_A + 1.26 \, kN = 0$$

$$\Sigma F_y = V_A - 5.93 \, kN + V_C = 0$$

$$\Sigma M_A = -(5.93 \, kN) \, \overline{AD} + V_C(2 \, \overline{AD}) = 0$$

Solving these equations, we obtain

$$H_A = -1.26 \, kN$$
 (opposite to shown)
 $V_C = 2.965 \, kN$ (as shown)
 $V_A = 2.965 \, kN$ (as shown)

Remark: these reactions could have been obtained by inspection on considerations of symmetry.

The FBD of joint C and the corresponding equations of equilibrium are

$$\Sigma F_{x} = -N_{CD} - N_{BC} \cos 29^{\circ} = 0$$

$$\Sigma F_{y} = N_{BC} \sin 29^{\circ} + 2.965 \ kN = 0$$

$$N_{CD}$$
Thus we obtain the internal forces
$$N_{BC} = -6.12 \ kN \quad (compression)$$

$$N_{CD} = 5.35 \ kN \quad (tension)$$

We could obtain the forces in the remaining bars by considerations of symmetry with respect to the vertical loads, but we might as well continue with the standard procedure. At node *B*, the sum of forces in the horizontal direction immediately renders

$$N_{AB} = N_{BC} = -6.12 \, kN \quad (compression)$$

Moreover, in the vertical direction, we obtain

$$N_{BD} = 2(6.12 \, kN) \sin 29^{\circ} = 5.93 \, kN \quad (tension)$$

This result should not be surprising, since it would arise trivially from the equation of equilibrium in the vertical direction at node D. The equation of equilibrium in this node in the horizontal direction yields

$$N_{AD} = 1.26 \, kN + N_{CD} = 6.61 \, kN \, (tension)$$

4. The 'method' of sections

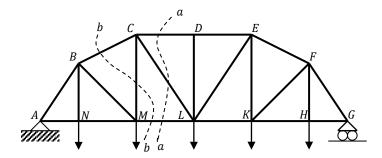
Whereas the method of joints is a legitimate general method for the solution of any statically determinate truss, the so-called *method of sections* only provides the solution in certain cases. In this sense, it is not a general method. On the other hand, when it is applicable, it has the advantage over the method of joints that *it delivers the force in a single bar*, without the need of solving for the whole truss.

The condition under which the method of sections can be used to determine the internal force in specific bar of a truss is the following:

There must exist a section (or cut) splitting the truss into two separate parts and passing exactly through 3 non-concurrent bars (including the desired one).

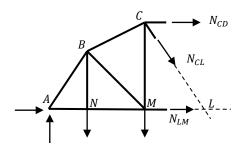
As an example, consider a railway bridge as shown and assume that we want to determine the internal force in a few individual bars, such as CD, CL and CM. We immediately see that, as far as bar CD is concerned there is just one cut that will satisfy the condition. We have indicated it with the letter a. This cut turns out to be also good for bar CL. For bar CM, we need a different cut, that we have represented with the letter b. Sometimes, it is possible to find more than one cut that will serve the purpose. For example, if we want to determine the internal force in bar ML, we could use either cut a or cut b.

The procedure is now as follows:



- (i) Find the reactions at the supports. This step is sometimes not required, as discussed in the description of step (iii).
- (ii) Find, if possible, a section (or cut) that affects the desired bar, cuts through exactly 3 non-concurrent bars and divides the truss into 2 separate parts.
- (iii) Draw the free-body diagram of one of the two parts into which the truss is divided by the cut. In general, it is wiser to choose the smaller part, or that which has the smaller number of external forces or, if available, a part that does not include any support. If the latter is the case, then the calculation of the reactions becomes unnecessary.
- (iv) The 3 equilibrium equations corresponding to the chosen FBD permit the calculation of the internal forces in the 3 bars severed by the cut. This step can be further simplified by decoupling these 3 equations completely. This can always be achieved by taking moments with respect to the point of intersection of 2 of the bars. If these 2 bars happen to be parallel, then (instead of the equation of moments) the equation of sum of force components in the direction perpendicular to these bars will do the job.

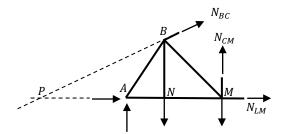
Consider cut *a* in our example. The FBD of the part of the truss to the left of the cut is shown below. We are not interested at this moment in numerical values (of the reactions or the external forces), since we only want to show the essence of the method, but these values are known, since we know the given external forces and we have calculated the reactions (using the FBD of the whole structure).



Clearly, if we take an equation of sum of moments with respect to point L (namely, the point of intersection of the other 2 bars affected by the cut) the force N_{CD} will emerge as the only unknown quantity in this equation. In the same FBD, if we want to obtain N_{CL} , the remaining 2 bars happen to be parallel (and horizontal). Writing the equation of sum sum of forces in the vertical direction yields immediately the desired result, since the horizontal bars make no contribution to this equation.

To obtain the internal force in bar CM, we need the cut b. The corresponding FBD to the left of the cut is shown below. The internal force N_{CM} is obtained by finding the point P of intersection of bars BC and LM. If point P is not easily located, we can always resort to writing the 3 standard equilibrium

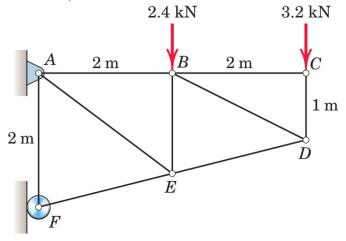
equations and solving the system. In this way, we obtain also the forces in the remaining 2 bars (which may not have been required).



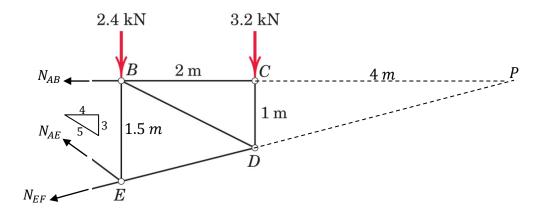
<u>Important remark</u>: For the central member *DL* there is no section that would cut through it and another two non-concurrent members. As a consequence, *the method of sections cannot be used for this member* of the truss. When this happens, one may still profit from the method of sections in combination with the method of joints. In this case, it so happens that *DL* is a zero-force member. The reasoning leading to this conclusion is, of course, based on the method of joints.

5. Example

Problem 4,32: Determine the force in member AE of the loaded truss.



Solution: Cutting through bars AB, AE and EF, and choosing the part of the structure to the right of this cut (so as to avoid the calculation of reactions!) we obtain the FBD shown below. The point P where bars AB and EF intersect is easily located in this case by a simple



proportion (i.e., since the height went from 2m to 1m when advancing 4m horizontally, it will take exactly another 4m for the height to go down to zero). Taking moments with respect to P and using Varignon's lemma, we write

$$\Sigma M_P = (2.4 \, kN)(6m) + (3.2 \, kN)(4m) - 0.8N_{AE}(1.5 \, m) - 0.6N_{AE}(6m) = 0$$

Solving this equation we obtain the result as

$$N_{AE} = 5.67 \, kN \quad (tension)$$

6. Frames

Just as trusses, *frames* are structures made up of slender members. In the case of trusses, however, severe limitations were imposed upon the geometry of the members (they had to be straight), their interconnections (they had to be pin-jointed), the loads (they had to be forces applied at the nodes) and the supports (they had to be hinges or rollers at the nodes). None of these restrictions apply to frames. The members may be curved and may be connected arbitrarily, the loads may be arbitrary and may include couples, the supports may be applied anywhere and may include fully fixed supports.

Frames constitute the skeleton of many structures in wide use. A skyscraper under construction, whether made of steel or reinforced concrete, clearly reveals its underlying framed structure. Frames are also involved in the chassis of automobiles, aircraft structures, bicycles, antenna towers, bridges, industrial buildings, skeletons of vertebrates, trees and innumerable other instances. Not all structures, however, are frames. Whereas a frame is generally understood as an assembly of slender components in which one dimension predominates over the other two, *plates* and *shells* are structural elements in which two dimensions prevail over the third, such as is the case in gas storage tanks and the hull of ships. There are also genuinely three-dimensional structures in which all three dimensions are of comparable extent, such as in the case of water dams and certain machine components.



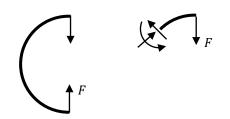
... but not everything is a frame.



7. Internal forces in plane frames

The peculiar property of trusses (as a particular case of frames) is that *in a truss every member is a two-force member*. Moreover, since the members of a truss are straight, *the internal forces in a truss boil down to either tension or compression* acting on the normal cross sections of each member.

This is no longer the case in general frames. Even when a member of a frame is a two-force member, if the member is not straight we can no longer describe the internal forces in the member by a single number. Just as we did in trusses, we will restrict our study to the case of *plane* frames, when the axes of all the members as well as the external forces lie in one and the same plane. Consider, as an example, a plane curved member (such as a bow) in equilibrium under the action of two forces, as shown below. We ask ourselves: what are the internal forces keeping this member as an integral unit? To answer this question, we know exactly what we have to do. We need to consider a subsystem, as



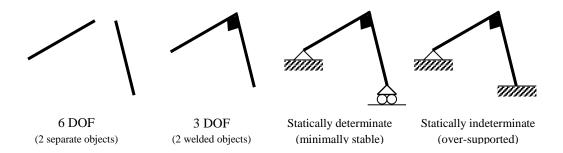
we have done before. Let us, accordingly, consider one quarter of the member and draw its free-body diagram. To maintain it in equilibrium, the contact (gluing) forces between this part of the member and the remainder must amount to a force that balances \boldsymbol{F} . But since contact forces act within the cross section, to produce a system statically equivalent to $-\boldsymbol{F}$ we need a force and a couple. Moreover, the force is not

necessarily perpendicular to the cross section, so we can resolve into a normal and a tangential component. So, in contradistinction with the case of a truss member, we see that in a frame member the internal forces will in general consist of a *normal force* (or *axial force*) (as before, a tension or a compression), a *tangential* (or *shear*) force and a *couple* (or *bending moment*). Since we have at our disposal three equilibrium equations, we conclude that the internal forces (normal force, shear force and bending moment) can be determined at any given cross section. Even when the members are not curved but straight, if we are not dealing with a two-force member we will always possibly have these three internal forces. This will be an important part of our study of frames later. But for now we want to solve the problem of the determination of the support reactions in a frame.

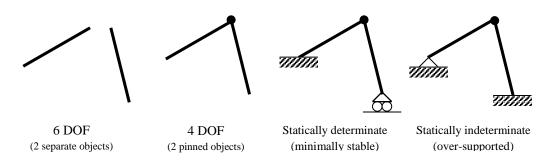
8. Statically determinate and indeterminate frames

As we have already learned, a rigid object in the plane has exactly 3 degrees of freedom (DOF). In other words, to specify the location of a rigid object in the plane we need to provide 3 parameters. They can be the 2 coordinates of one point of the object and the angle formed by some line attached to the object with, say, the x axis. Consider 2 members of a frame, as shown in the figure below. To specify the location of these two members we need 6 parameters (3 for each member). But if these two members are connected, it should be clear that the number of degrees of freedom is no longer 6, but a smaller number. Suppose that these two members have been welded together at one end, to form a bracket. By a weld we mean that the cross sections joined must experience always the same displacements (in the x and the y directions, say) and the same rotation. As a consequence of this weld, therefore, the bracket functions as a single rigid object. We conclude that a weld reduces the degrees of freedom (of the system of the two members it joins) by 3. To support the bracket (so that it will be in equilibrium under the action of any system of external forces) we need to supply exactly 3 conditions of support, such as those provided by a hinge (2 conditions) and a roller (1 condition). Alternatively, we can fix the bracket by a single fixed support (3 conditions). If we detach the bracket from its supports, we have at our disposal exactly 3 independent equations of statics. The system is, therefore, externally statically determinate. Any more supports, and we no longer have enough

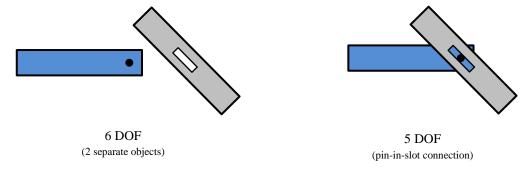
equations of statics to solve it. The system becomes *statically indeterminate*.



Two members can also be connected by a frictionless pin, just as in a truss. In this case the pin prevents the relative displacements between the sections joined, but does not impose any restriction on their relative rotation. In other words, the pin reduces the number of degrees of freedom by 2. To render the resulting combination stable under the action of any external forces we need to supply 4 support conditions, such as those provided by 2 hinges or, alternatively, one fixed support and a roller.



There is still another way to connect two members of a frame. This connection consists of a pin running within a slot. This kind of connection only precludes one component of the relative displacement, namely, the one perpendicular to the slot. Accordingly, it reduces the number of degrees of freedom by just 1.



In this course we will be only dealing with statically determinate systems, so that we will always have enough equations to solve for the external reactions.

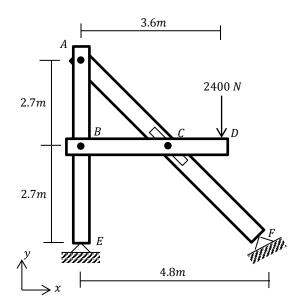
9. General method of solution

If the frame consists of a rigid structure (such as a truss triangle or a bracket) then there will be exactly 3 supports and we can find the reactions by drawing the FBD of the whole structure detached from its supports and formulating 3 independent equations of equilibrium (such as the sum of force components in 2 directions and the sum of moments with respect to a point in the plane).

On the other hand, if the frame, when detached from its supports, is not a rigid object we will have (as discussed above) more than 3 conditions of support and the 3 equations arising from the FBD of the structure as a whole will be insufficient to find the reactions. The keyword for the solution of these frames is: *dismemberment*. If two members are joined by means of a pin or a pin-in-slot connection, we split them apart and draw the corresponding free-body diagrams. If two members are welded together (forming, say, a bracket) they are considered as one. Once we have dismembered the structure in this way, we write 3 equations of equilibrium for each FBD and we obtain exactly the number of equations needed. We can always substitute the FBD of the whole structure for the FBD of one of the members.

Let us illustrate this technique by means of a numerical example.

Example (Beer and Johnston, 3rd SI edition)



Determine the components of all the forces acting on member *ABE*.

Solution: For the purpose of deeper understanding, we will follow the procedure of dismemberment to the letter, without taking advantage of possible simplifications when we are only looking for the forces in one of the members, as prescribed by the problem. Moreover, we will not use initially the FBD of the body as a whole, which we could always do to replace the FBD of one of the members. Notice that the supports at *E* and *F* are hinges fixed to the ground and that the connecting pin at *C* runs in a slot. In the figure below, we draw the free-body diagrams of each one of the 3 members. A very important detail is that the

forces at the points of interconnection are internal forces as far as the structure is concerned. Consequently, they abide by the principle of action and reaction and have to be consistently indicated in the diagrams of the connected members.

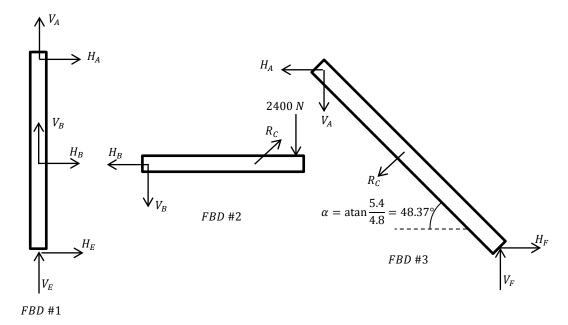
The equilibrium equations for FBD #1 are

$$\Sigma F_{x} = H_A + H_B + H_E = 0 \tag{1}$$

$$\Sigma F_{y} = V_A + V_B + V_E = 0 \tag{2}$$

$$\Sigma M_A = H_E(5.4m) + H_B(2.7m) = 0 \tag{3}$$

For FBD #2 we obtain



$$\Sigma F_{\chi} = -H_B + R_C \sin \alpha = 0 \tag{4}$$

$$\Sigma F_y = -V_B + R_C \cos \alpha - 2400N = 0 \tag{5}$$

$$\Sigma M_B = (R_C \cos \alpha)(2.4m) - 2400 N (3.6m) = 0$$
 (6)

Finally, FBD #3 yields the equilibrium equations

$$\Sigma F_{x} = -H_{A} - R_{C} \sin \alpha + H_{F} = 0 \tag{7}$$

$$\Sigma F_{v} = -V_A - R_C \cos \alpha + V_F = 0 \tag{8}$$

$$\Sigma M_F = (R_C \sin \alpha)(2.7m) + R_C \cos \alpha (2.4m) + H_A(5.4m) + V_A(4.8m) = 0$$
 (9)

Starting from Equation (6), we obtain

$$R_C = 5419 N$$
 (as shown)

From Equation (4)

$$H_B = 4051 N$$
 (as shown)

Equation (5) yields

$$V_B = 1200 N$$
 (as shown)

We move now to Equation (3) and get

$$H_E = -2025.5 N$$
 (opposite to shown)

Equation (1) yields

$$H_A = -2025.5 N$$
 (opposite to shown)

Equation (9) gives us

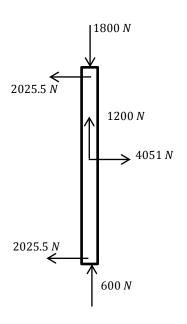
$$V_A = -1800 \, N \quad (opposite \ to \ shown)$$

Finally, from Equation (2) we get

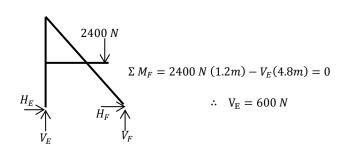
$$V_E = 600 n$$
 (as shown)

The forces acting on member *ABE* are shown in the diagram below. In particular, this is not a two-force member but rather a three-force member. Its internal forces at any given cross section involve axial force, shear force and bending moment.

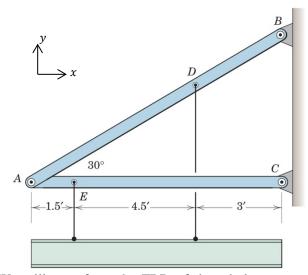
This completes the solution.



<u>Remark</u>: As pointed out, we could have used the FBD of the whole structure to replace the FBD of one of the members. This idea would have somewhat accelerated the solution. In particular, taking moments with respect to point F would have immediately delivered the vertical reaction V_E . Let us check this for the sake of verification.



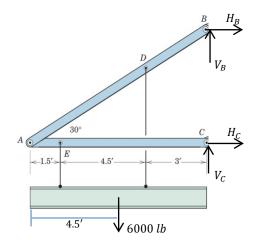
10. An example from our textbook



We will start from the FBD of the whole structure. This FBD involves only the reactions at the supports, as shown. We will use it a substitute for the FBD of

Problem 4.78: Determine the magnitude of the pin reactions at A, B and C, due to the 6000 lb beam.

<u>Solution</u>: Notice that the hanging beam is not completely supported, since it can still rock sideways. This is not a problem as long as horizontal forces are not applied to the beam.



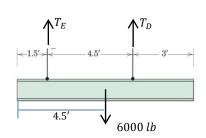
one of the members. We obtain the following three equilibrium equations

$$\Sigma F_x = H_B + H_C = 0$$
 (1)

$$\Sigma F_y = V_B + V_C - 6000 \ lb = 0$$
 (2)

$$\Sigma M_B = H_C (9' \tan 30^\circ) + (6000 \ lb)(4.5') = 0$$
 (3)

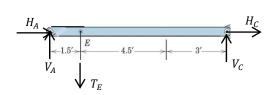
Our next FBD will be that of the hanging beam. Recall that in this diagram the sum of the horizontal components is trivially zero, since the beam is not supported horizontally. The remaining two equilibrium equations are



$$\Sigma F_y = T_B + T_E - 6000 \ lb = 0$$

$$\Sigma M_D = -T_E(4.5') + 6000 \ lb \ (1.5') = 0$$
(5)

We need to consider just one more FBD. We will choose the horizontal member AEC. We obtain the equilibrium equations:



$$\Sigma F_x = H_A + H_C = 0 (6)$$

$$\Sigma F_y = V_A + V_C - T_E = 0 (7)$$

$$\Sigma M_A = V_C(9') - T_E(1.5') = 0 (8)$$

We have 8 linear equations for the 8 unknown quantities H_A , V_A , H_B , V_B , H_C , V_C , T_D , T_E . From Equation (5) we obtain

$$T_E = 2000lb$$
 (as shown, i. e. tension in wire)

Using this result in Equation (8) we get

$$V_C = 333 lb$$
 (as shown)

Combining with Equation (7) yields

$$V_A = 1666 lb (as shown)$$

Equation (3) on its own delivers

$$H_C = -5196 lb$$
 (opposite to shown)

Substituting into Equation (1) we obtain

$$H_B = 5196 lb$$
 (as shown)

From Equation (6) we obtain

$$H_A = 5196 lb$$
 (as shown)

Finally, Equation (2) delivers

$$V_B = 5667 \ lb$$

The magnitude of the reactions at B and C are, respectively,

$$R_B = \sqrt{H_B^2 + V_B^2} = 7689 \ lb$$
 $R_C = \sqrt{H_C^2 + V_A^2} = 5207 \ lb$

The magnitude of the force transmitted by the pin at A is

$$R_A = \sqrt{H_A^2 + V_A^2} = 5457 \ lb$$

<u>Suggested exercise</u>: In the problem above, replace the FBD of the total structure by the FBD of the inclined bar and verify that you obtain the same final results.