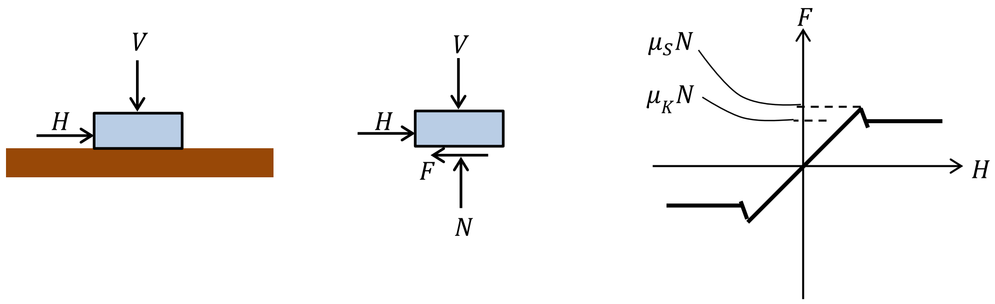
1. Friction

Tribology is the discipline that studies the relative motion of two surfaces in contact. This is an area of active research as scientists and engineers, with the aid of devices such as the atomic-force microscope, learn more about the origin of the *frictional forces* between the surfaces at play. Phenomenological models of friction have existed now for several centuries. In this course, we will deal only with *dry friction*, namely, with the forces that try to resist the relative motion of two solid surfaces in contact. When fluids are also involved, we talk about *lubrication*. *Fluid friction* studies the viscous forces within a liquid.

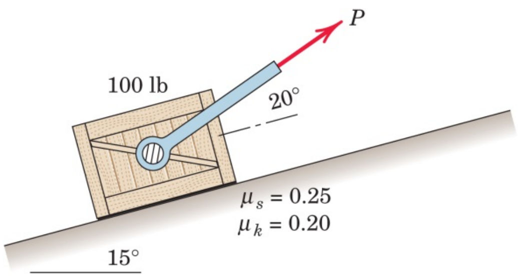
Based on empirical evidence, as far back as the late 17th century, Guillaume Amontons (1663-1705) formulated the two laws of dry friction that bear his name. These laws were later enhanced by the work of Charles-Augustin de Coulomb (1736-1806) on kinetic friction. Instead of introducing these laws formally, let us resort to the description of the result of a possible experiment of two solid surfaces pressed against each other in the normal direction to the surface of contact and subjected to a force parallel to it, as shown in the figure below. For descriptive purposes, let us think of a solid object made of some material lying on a horizontal floor made of another material. The object is pressed against the floor by means of a downward vertical force of magnitude V (that includes the weight) and subjected to a horizontal force of magnitude H applied very close to the floor. There are two possibilities: (a) the object remains at rest; (b) the object moves horizontally. For case (a), a free-body diagram of the object reveals that the reaction of the floor has two components, N and F, respectively balancing V and H. The components N and F are known, respectively, as the *normal force* and the *friction force*. For case (b), since motion can develop only in the direction of the surface of contact, the equilibrium equation $N = V$, remains valid. We only need to determine under which condition the transition from rest to motion happens. This is shown in the graph of the friction force F versus H. For most materials (within a reasonable range of the normal force N that does not crush the object) it is observed that the transition from rest to motion happens when the threshold value $\mu_S N$ is reached, where μ_S is the *coefficient of static friction*. This coefficient is remarkably constant and is a characteristic of the two materials in contact. As an example, for contact between two metals μ_S is in the range of 0.5 to 0.8. For wood on metal, the range is between 0.2 and 0.6. For Teflon on metal, the coefficient is of the order of 0.04. For synovial joints (such as the knee) in humans, a typical value is 0.01. These are only typical values. Every particular case has to be studied in more detail. Clearly, the friction coefficient depends also on the polish of the surfaces, the presence of impurities (such as dust), the temperature and the humidity.

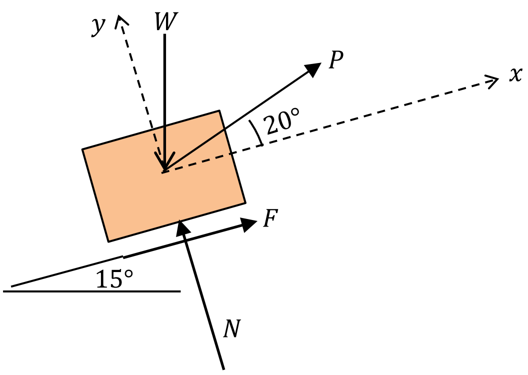


Once the threshold is reached and motion ensues, it is (somewhat surprisingly) observed that the frictional force is usually reduced and remains almost constant at a value equal to $\mu_K N$, where $\mu_K \leq \mu_S$ is the *coefficient of kinetic friction.* Clearly, once motion starts, the equation of equilibrium in the direction of the surface is no longer valid (unless the body moves at a constant speed). As we know from everyday experience, it is easier to push an object that is already in motion than to initiate its motion in the first place. Notice that the frictional force always opposes the motion. This is in accordance with the second law of thermodynamics. Otherwise, we would be able to create (rather than dissipate as heat) mechanical energy.

Solving problems involving frictional forces requires the consideration of various possibilities. For this reason, they can be somewhat more demanding than usual questions of finding reactions and internal forces. We will try to illustrate a few typical instances by means of numerical examples.

2. A ramp (Problem 6.3): The force P is applied to the 100 lb block when it is at rest. Determine the magnitude and direction of the friction force exerted by the surface on the block if (a) $P = 0$, (b) $P = 40\ lb$, (c) $P = 60\ lb$. (d) What is the value of P required to initiate motion of the crate up the incline?





Solution: We draw a free-body diagram in which the unknown frictional force on the crate has been provisionally assumed to point up the ramp (as if the tendency of motion were downwards). For each case, we will determine whether or not this assumption has been correct. For convenience, we adopt a rotated x, y coordinate system. We assume the dimensions of the crate to be small enough so that we don't worry about the problem of tipping and the associated equation of sum of moments. The equilibrium equations (assuming no motion occurs) are:

$$\Sigma F_x = F - W \sin 15° + P \cos 20° = 0 \qquad (1)$$

$$\Sigma F_y = N - W \cos 15° + P \sin 20° = 0 \qquad (2)$$

An important observation is that, whether or not there is motion, the equation of equilibrium in the direction perpendicular to the surface of contact is applicable. Why? The putative motion is rectilinear (along the incline) and, therefore, its acceleration vector is along the incline. It follows that the total force perpendicular to the incline (according to Newton's second law) must vanish. In other words, regardless of the state of motion, we can read off

$$N = W \cos 15° - P \sin 20°$$

(a) For $P = 0$ we obtain

$$N = W \cos 15°$$

If we assume no motion, Equation (1) yields

$$F = W \sin 15°$$

To decide whether or not motion occurs, we compare with the threshold value, namely, we check whether the ratio of the (absolute value) of F and N is larger or smaller than the coefficient of static friction. We obtain

$$\frac{|F|}{N} = \frac{\sin 15°}{\cos 15°} = \tan 15° = 0.268 > 0.25$$

We conclude that the crate cannot be kept at rest, since the maximum available friction force is insufficient to satisfy the equilibrium condition. Thus, the crate is moving downwards. According to the laws of kinetic friction, the friction force during motion is independent of the speed, as long as the speed doesn't vanish. This force is given by the product of the coefficient of kinetic friction and the compressive normal force between the surfaces. Thus, we obtain the final result

$$F = \mu_K N = (0.20)(100\ lb) \cos 15° = 19.32\ lb$$

The motion of the crate is downwards and this friction force opposes it.

Measuring coefficients of friction: The solution of part (a) of this problems shows, incidentally, how a coefficient of friction between two materials, *A* and *B*, may be determined experimentally. A block of material *A* is placed over a hinged ramp of material *B*, whose angle θ of inclination can be varied. Starting from the horizontal configuration ($\theta = 0$), the ramp angle is increased until motion starts. The trigonometric tangent of this angle equals the coefficient of static friction. Once motion starts the angle is gradually decreased until motion ceases. At this moment, the tangent of the angle θ equals the coefficient of kinetic friction.

(b) For $P = 40\ lb$ we first enforce both equilibrium equations, (1) and (2), assuming that there is no motion. The result is

$$F = -11.71\ lb$$

$$N = 82.91\ lb$$

It is important to verify that the normal force has the correct sign, namely, that the body is actually pressed against the ramp. We now check the friction force to verify that the threshold has not been exceed, that is,

$$\frac{|F|}{N} = \frac{11.71}{82.91} = 0.141 < 0.25$$

The physical interpretation of this inequality is that the friction between the two objects is able to with stand the tangential component of the reaction and, therefore, the object is at rest. Moreover, the negative sign indicates that the force acts downwards and that, accordingly, an increase in the magnitude of P will end up causing an upward motion.

(c) For $P = 60\ lb$, the solution of the purported equilibrium equations is

$$F = -30.50\ lb$$

$$N = 76.07\ lb$$

Checking for the static friction threshold, we obtain

$$\frac{|F|}{N} = \frac{30.50}{76.06} = 0.401 > 0.25$$

We conclude that motion ensues. Moreover, from the sign of the result, we infer that motion takes place upwards (recall that the frictional forces oppose the motion). The force of friction is obtained from the coefficient of kinetic friction as

$$|F| = \mu_K N = (0.20)(76.06) = 15.21\ lb$$

(d) The force necessary to initiate motion up the incline should be an intermediate value between 40 lb and 60 lb. To determine its exact value, we set

$$F = -\mu_S\, N$$

The negative sign indicates that, in the FBD we drew, the force will be directed downwards, as behooves an upward motion. Equations (1) and (2) become

$$\Sigma\, F_x = -\mu_S N - W \sin 15° + P \cos 20° = 0 \qquad (3)$$

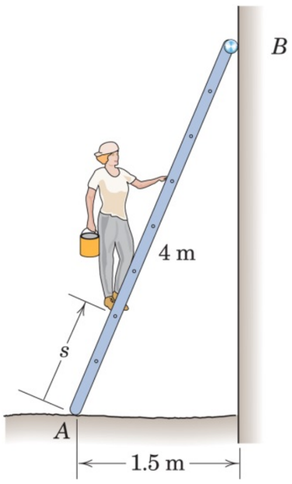
$$\Sigma\, F_y = N - W \cos 15° + P \sin 20° = 0 \qquad (4)$$

These equations can now be solved for N and P. In particular, we obtain

$$P = \frac{\sin 15° + \mu_S \cos 15°}{\cos 20° + \mu_S \sin 20°} W = 48.80\ lb$$

[Additional challenge: Determine the smallest value of P necessary to prevent downward motion.]

3. A ladder: The ramp and the ladder are among the most common paradigms for friction problems. We will discuss a few more paradigms later.

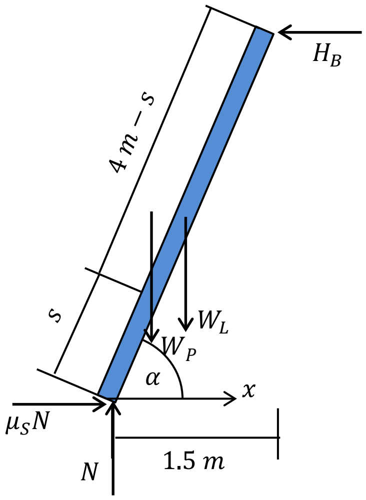


Problem 6.33: Determine the distance s to which the 90 kg painter can climb without causing the 4 m ladder to slip at its lower end. The top of the 15 kg ladder has a small roller and at the ground the coefficient of friction is 0.25. The mass centre of the painter is directly above her feet.

Solution: The angle α between the ladder and the ground is

$$\alpha = \cos^{-1}\frac{1.5}{4} = 67.98°$$

The FBD of the ladder is shown below. Notice that since we are asked for the maximum distance that can be sustained without motion, we are clearly under conditions of *impending motion*, implying that the frictional force is equal to the threshold value $\mu_S N$. Moreover, the frictional force on the ladder acts towards the right, since motion is impending towards the left.



Since we are not interested in the reaction at B, we'll consider two equilibrium equations that exclude it, that is,

$$\Sigma F_y = N - W_P - W_L = 0$$

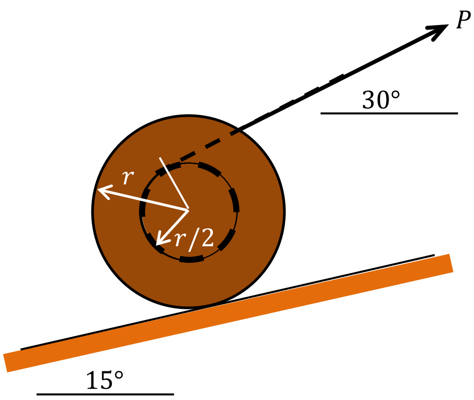
$$\Sigma M_B = -N(1.5m) + \mu_S N(4m)\sin\alpha + W_P(4m - s)\cos\alpha + W_L(0.75m) = 0$$

Solving for s we obtain

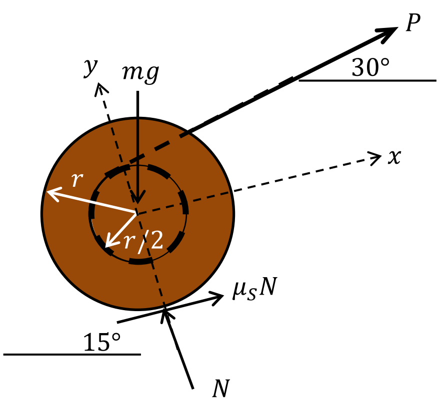
$$s = \frac{(W_P + W_L)(-1.5m + \mu_S(4m)\sin\alpha) + W_P(4m)\cos\alpha + W_L(0.75m)}{W_P\cos\alpha} = 2.55\,m$$

Notice, incidentally, that this result is independent of the value of the gravity constant (as long as it doesn't vanish). In other words, if you were using the ladder to go back up to a lunar module that has landed on the moon, the result would still be 2.55 m. Naturally, the reactions themselves would depend on the value of gravity.

4. A spool: A somewhat more sophisticated paradigm is offered by the unwinding of a cable from a spool (or, if you prefer, a yo-yo).



Problem 6.6: Determine the minimum coefficient of static friction μ_S which will allow the drum of mass m with fixed inner hub to be rolled up the incline at a steady speed without slipping. What are the corresponding values of the force P and the friction force F?



Solution: For a wheel rolling without sliding on a line at a constant speed the laws of equilibrium apply as if it were at rest. Moreover, since we are looking for the minimum value of the coefficient of static friction to maintain this equilibrium, we set the friction force to its threshold. Any less friction and we wouldn't be able to prevent the drum from sliding downwards. The FBD is shown. We have at our disposal three equilibrium equations. We start by formulating the equation of moments about the centre C of the drum, namely,

$$\Sigma M_C = \mu_S N\, r - P\,\frac{r}{2} = 0$$

Notice that the moment arm for the force P is exactly $r/2$, since, while unwinding, the cable is always tangential to the inner hub! The equations of force yield

$$\Sigma F_x = \mu_S N + P\cos(30° - 15°) - mg\sin 15° = 0$$

$$\Sigma F_y = N + P\sin(30° - 15°) - mg\cos 15° = 0$$

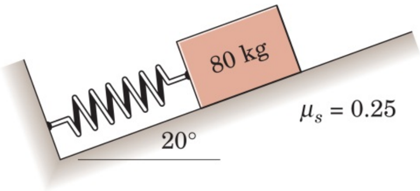
We have obtained a system of 3 equations, which we can solve for μ_S, P and N. The result is

$$\mu_s = \frac{\sin 15°}{\cos 15°(1 + 2\cos 15°) - 2\sin^2 15°} = 0.0959$$

$$N = 0.9202\, mg \qquad \therefore \qquad F = \mu_S N = 0.0883\, mg$$

$$P = 0.1765\, mg$$

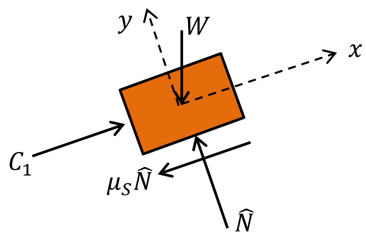
5. A block and spring: This is a variant of the ramp problem, except that the applied force arises from the elastic stretching or contraction of a linear spring.



Problem 6.121: The 80-*kg* block is placed on the 20° incline against the spring and released from rest. The coefficient of static friction between the block and the incline is 0.25. (*a*) Determine the maximum and minimum values of the initial compression force C in the spring for which the block will not slip upon release. (*b*) Calculate the magnitude and direction of the friction force acting on the block if the spring compression force is $C = 200\ N$.

Solution: If the spring is compressed, as prescribed by the problem, it will exert an upward force on the block. On the other hand, the weight of the block will have a downward component along the ramp. As a consequence of these two competing tangential forces, we can have a tendency to move up or down the ramp.

(a) We are asked to obtain the limiting values of the spring compression for impending upward and impending downward motion. For the first case (upward motion) the FBD is shown below. The equations of equilibrium are



Impending upward motion

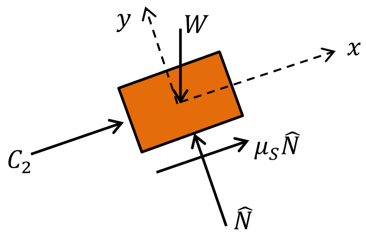
$$\Sigma F_x = C_1 - \mu_S \hat{N} - W\sin 20° = 0$$

$$\Sigma F_y = \hat{N} - W\cos 20° = 0$$

Solving for $\hat{N}$ and C_1, we obtain

$$\hat{N} = W\cos 20° = 737\ N$$

$$C_1 = \mu_S N + W\sin 20° = 453\ N$$



Impending downward motion

Similarly, for impending downward motion, we obtain

$$\Sigma\, F_x = C_2 + \mu_S \hat{N} - W \sin 20° = 0$$

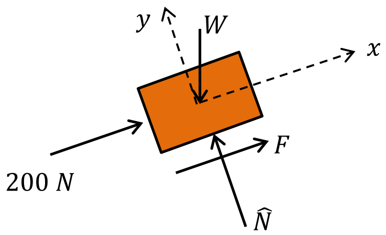
$$\Sigma\, F_y = \hat{N} - W \cos 20° = 0$$

Solving for $\hat{N}$ and C_2, we obtain

$$\hat{N} = W \cos 20° = 737\ N$$

$$C_2 = -\mu_S N + W \sin 20° = 84\ N$$

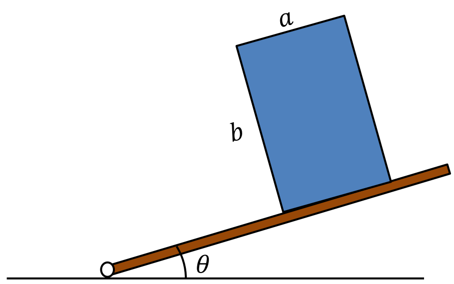
(b) If the force in the spring is of 200 N, it being within the interval determined by the two values obtained above, it is clear that the body remains at rest. The friction force is obtained from a standard free-body diagram, as shown. Notice that the normal force $\hat{N}$ remains as before at the value of 737 N. From equilibrium in the x direction, we obtain



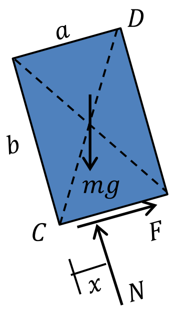
$$\Sigma\, F_x = 200N + F - W \sin 20° = 0$$

which yields $F = 68.4\ N$ (upwards, as shown).

6. A fridge on a ramp: In all the problems shown so far, the implicit assumption has been that the objects involved can be considered as very flat, so that the problem of *tipping* is ignored. We know, however, from everyday experience, that if we try to push a fridge by applying a horizontal force near the top, the fridge is more likely to tip over rather than to slide. If the force is applied at a point closer to the base, sliding is likely to occur first. In other words, an equation of sum of moments may be involved in the situation. A simple instance is provided by the following example.



Problem 6.16: The homogeneous rectangular block of mass m rests of the inclined plane which is hinged about a horizontal axis at O. If the coefficient of static friction between the block and the plane is μ, specify the conditions which determine whether the block tips before it slips or slips before it tips as the angle θ is gradually increased.



Solution: The FBD of the block is shown. An important detail is that the reaction of the incline is not necessarily uniformly distributed over the base of the block. If you think, for example, of the legs of the fridge, the legs on the lower side will be more loaded than those on the upper side. Consequently, the resultant normal reaction will act at an unknown distance x from the lower corner C. When will tipping be impending (about to happen)? Clearly, when $x = 0$. In other words, when the resultant N is about to exit from the picture. Since the sum of moments with respect to C must vanish, we conclude that the angle θ_{tip} corresponding to this situation occurs when the line of action of the weight passes through C. Put differently, when the diagonal CD is vertical. We conclude that

$$\tan\theta_{tip} = \frac{a}{b}$$

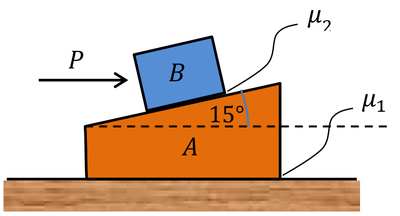
As far as sliding is concerned, we know from the ramp problem (or by rewriting the equations of force equilibrium along the ramp and its normal) that impending sliding occurs when

$$\tan\theta_{slip} = \mu$$

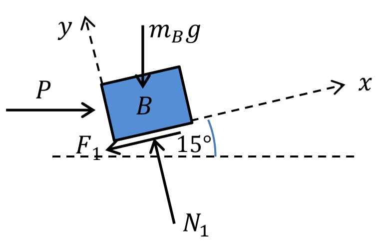
We conclude that for slip to happen before tipping, we must have

$$a > \mu\, b$$

7. A stack: When we have a stack of blocks, such as a pile of books, and we apply a horizontal force, say, at the top member of the stack, we could have several possibilities of relative motion between these members. If, for example, the coefficient of friction between the elements is very high while the coefficient of friction with the floor is very low, it is logical to expect that the whole stack will move as a single object. It may also happen, depending on the values of the masses and the coefficients of friction involved, that several relative motions will ensue. The following example illustrates a stack made of a wedge and a block riding on it.



Problem 6.30: The horizontal force $P = 50N$ is applied to the upper block with the system initially stationary. The block masses are $m_A = 10\ kg$ and $m_B = 5\ kg$. Determine if and where the slippage occurs for the following conditions on the coefficients of static friction; (a) $\mu_1 = 0.40, \mu_2 = 0.50$ and (b) $\mu_1 = 0.30$, $\mu_2 = 0.60$. Assume that the coefficients of kinetic friction are 75 percent of the static values.



Solution: (a) We start by checking whether or not block B is in motion with respect to block A. As always, we draw the corresponding FBD to determine the friction force required to maintain equilibrium and compare it with the threshold value. The equilibrium equations in the x, y directions yield

$$\Sigma\, F_x = -F_1 + P\cos 15° - m_B g \sin 15° = 0$$

$$\Sigma\, F_y = N_1 - P\sin 15° - m_B g \cos 15° = 0$$

Solving this simple system, we obtain

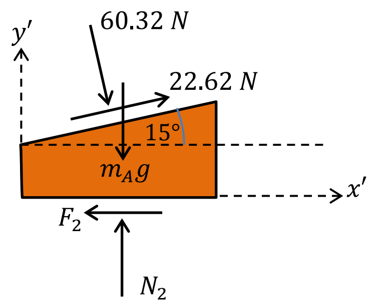
$$N_1 = 60.32\ N \qquad F_1 = 35.60\ N$$

We check

$$\frac{|F_1|}{N_1} = \frac{35.60}{60.32} = 0.59 > 0.5$$

We conclude that body B is in motion relative to A. We still need to determine whether body A is at rest or in motion. For this purpose, we draw a free-body diagram of A. There is, however, a subtlety to be observed. Since B is moving, the frictional force F_1' of interaction between the two bodies is governed by the coefficient of kinetic friction. Its value is

$$F_1' = (0.75)(0.5)N_1 = 22.62\ N$$



Notice that the value of N_1 is not affected, since there is no motion in the normal direction and, consequently, the equilibrium equation in the y direction is still valid. The forces acting on A at the surface of contact between the two bodies are the reactions to the forces acting on B.

The equilibrium equations are

$$\Sigma F_x' = -F_2 + (22.62\ N)\cos 15° + (60.32\ N)\sin 15° = 0$$

$$\Sigma F_y' = N_2 - m_A g + (22.62\ N)\sin 15° - (60.32\ N)\cos 15° = 0$$

Solving for N_2 and F_2, we obtain

$$N_2 = 150.51\ N \qquad F_2 = 37.46\ N$$

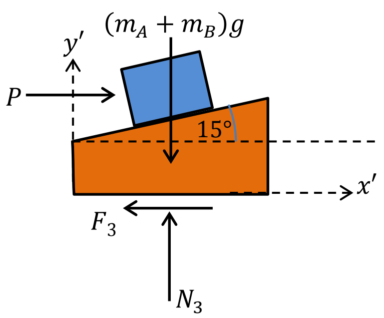
Finally, we check that

$$\frac{|F_2|}{N_2} = \frac{37.46}{150.51} = 0.25 < 0.40$$

We conclude that block A does not move.

(b) The only difference between parts (a) and (b) resides in the different values of the friction coefficients. When analyzing block B, we obtained values for the forces at the interface under the assumption of equilibrium. We now check that

$$\frac{|F_1|}{N_1} = \frac{35.60}{60.32} = 0.59 < 0.60$$



We conclude that B does not move with respect to A, since it just squeezes under the threshold. The two blocks, therefore, function as a unit and we can analyze the FBD of the combined group, as shown in the figure.

The equilibrium equations are

$$\Sigma F'_x = -F_3 + P = 0$$

$$\Sigma F'_y = N_3 - (m_A + m_B)g = 0$$

We obtain

$$N_3 = 147.15\ N \qquad F_3 = 50\ N$$

Finally, we check

$$\frac{|F_3|}{N_3} = \frac{50}{147.15} = 0.34 > 0.3$$

We conclude that the two-block complex moves as a unit under the action of the applied force.