Force and couple systems in three dimensions

There are no substantial differences in the treatment of three-dimensional problems as compared with their two-dimensional counterparts. Given a system of forces, it can always be reduced to a point to obtain a force-couple system. Equilibrium is tantamount to the vanishing of this system (for one, and therefore for every, point. The difficulties in three dimensions, if any, may stem only from the necessity to interpret the data with care. In two dimensions we can sometimes get away with computations based on simple trigonometric or geometric considerations on the basis of a representative drawing of the situation at hand. In three dimensions, on the other hand, the best policy is to use a formulation based on components in a Cartesian frame. A point with coordinates x, y, z can be associated with its *position vector* $\boldsymbol{r}$ defined as

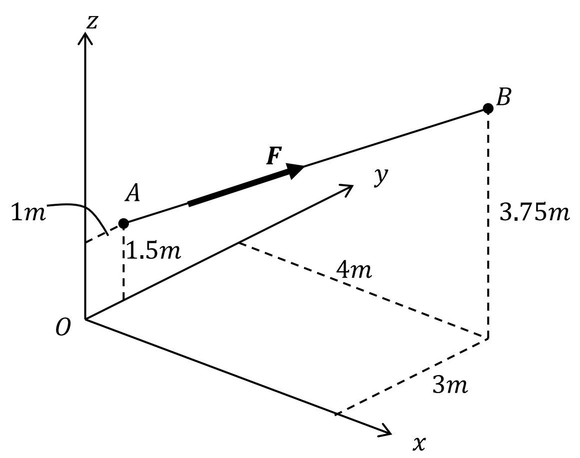
$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

In this expression, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vectors along the respective coordinate axes. Any vector $\boldsymbol{v}$ (force or couple) can be expressed in components as

$$\boldsymbol{v} = v_x\,\boldsymbol{i} + v_y\,\boldsymbol{j} + v_z\,\boldsymbol{k}$$

Consider a few examples of translation of the data into this vector-component format.

a) Based on Problem 2.102 of the text, a force $\boldsymbol{F}$ of magnitude $F = 6kN$ is given along a line AB, as shown. We are asked to express it in terms of components and to find the angle it forms with the x axis.
Solution: We start by noticing that any vector $\boldsymbol{F}$ can be written in terms of its magnitude F and a unit vector $\boldsymbol{n}$ along its line of action as



$$\boldsymbol{F} = F\boldsymbol{n}$$

The vector $\boldsymbol{n}$ is a purely geometric entity. Given two points A, B with respective position vectors $\boldsymbol{r}_A, \boldsymbol{r}_B$, it is clear that the vector $\boldsymbol{r}_{AB}$ from A to B is given by

$$\boldsymbol{r}_{AB} = \boldsymbol{r}_B - \boldsymbol{r}_A$$

The corresponding unit vector $\boldsymbol{n}$, therefore, is obtained by dividing $\boldsymbol{r}_{AB}$ by its magnitude, namely,

$$\boldsymbol{n} = \frac{\boldsymbol{r}_{AB}}{AB}$$

In our particular example, we have

$$\boldsymbol{r}_A = (1m)\boldsymbol{j} + (1.5m)\boldsymbol{k} \qquad \boldsymbol{r}_B = (4m)\boldsymbol{i} + (3m)\boldsymbol{j} + (3.75m)\boldsymbol{k}$$

whence

$$\boldsymbol{r}_{AB} = \boldsymbol{r}_B - \boldsymbol{r}_A = (4m)\boldsymbol{i} + (2m)\boldsymbol{j} + (2.25m)\boldsymbol{k}$$

Finally,

$$\boldsymbol{n} = \frac{\boldsymbol{r}_{AB}}{AB} = \frac{4\boldsymbol{i} + 2\boldsymbol{j} + 2.25\boldsymbol{k}}{\sqrt{4^2 + 2^2 + 2.25^2}} = 0.799\boldsymbol{i} + 0.399\boldsymbol{j} + 0.449\boldsymbol{k}$$

If we take the dot product of this (unit) vector with the (unit) vector $\boldsymbol{i}$, we obtain the cosine of the angle θ_x between them. Thus, $\cos\theta_x = 0.799$, or

$$\theta_x = 36.965°$$

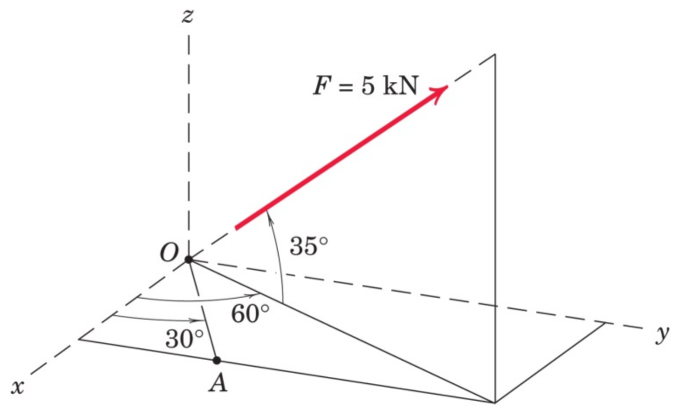
Note that for the angles between a vector and the coordinate axes we will always, by convention, choose a value between 0 and 180°. Thus, the value $-36.965°$ is discarded (although it has the same cosine as $+36.965°$). We have obtained the answer to our second question before solving the first one. The force vector is given by

$$\boldsymbol{F} = (6kN)\boldsymbol{n} = (4.794\boldsymbol{i} + 2.394\boldsymbol{j} + 2.694\boldsymbol{k})\ kN$$

b) Problem 2.103: Express the 5*kN* force **F** as a vector in terms of **i**, **j**, **k**. Determine the projection of **F** onto the *x*-axis and onto the line OA.

<u>Solution</u>: From the figure, the projection of the force $\boldsymbol{F}$ onto the z axis is

$$F_z = (5kN)\sin 35° = 2.868\ kN$$



The projection of $\boldsymbol{F}$ onto the plane xy has a magnitude of $(5kN)(\cos 35°) = 4.096kN$. This segment can be further projected onto the x and y axes to obtain

$$F_x = (4.096kN)(\cos 60°) = 2.048kN$$

and

$$F_y = (4.096kN)(\sin 60°) = 3.547kN$$

Finally,

$$\boldsymbol{F} = (2.048\boldsymbol{i} + 3.547\boldsymbol{j} + 2.868\boldsymbol{k})kN$$

The projection of a vector $\boldsymbol{F}$ on a direction specified by the unit vector $\boldsymbol{m}$ is defined as the dot product $\boldsymbol{F}\cdot\boldsymbol{m}$. Thus, we obtain

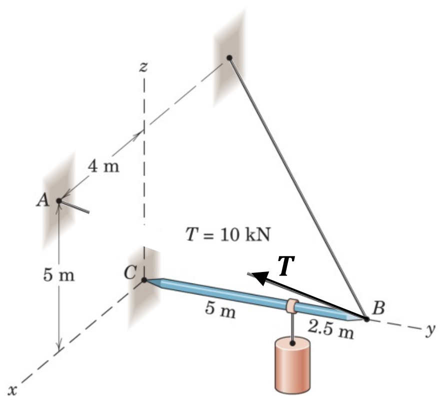
$$F_x = \boldsymbol{F}\cdot\boldsymbol{i} = 2.048kN$$

Analogously,

$$F_{OA} = \boldsymbol{F}\cdot\boldsymbol{m}_{OA} = \boldsymbol{F}\cdot(\cos 30°\,\boldsymbol{i} + \sin 30°\,\boldsymbol{j})$$

$$= (2.048kN)(0.866) + (3.547kN)(0.5) = 3.547kN$$

c) Problem 2.110: The tension in the supporting cable AB is 10 *kN*. Write the force that the cable exerts on the boom BC as a vector $\boldsymbol{T}$. Determine the angles that the line of action of $\boldsymbol{T}$ forms with *x*, *y*, and z.



<u>Solution</u>: At the outset, let us clarify that *we have not yet explained what is meant by a tension in a cable*. So, we are using this notion

cavalierly. At any rate, if we were to cut the cable AB, we would have to apply a force at B pointing from B towards A. This problem is very similar to problem (a) above. We have

$$\boldsymbol{T} = (10kN)\boldsymbol{n}_{BA}$$

We calculate

$$\boldsymbol{n}_{BA} = \frac{\boldsymbol{r}_{BA}}{BA} = \frac{\boldsymbol{r}_A - \boldsymbol{r}_B}{BA}$$

But

$$\boldsymbol{r}_A = (4m)\boldsymbol{i} + (5m)\boldsymbol{k} \qquad \boldsymbol{r}_B = (7.5m)\boldsymbol{j}$$

Thus,

$$\boldsymbol{n}_{BA} = \frac{(4m)\boldsymbol{i} - (7.5m)\boldsymbol{j} + (5m)\boldsymbol{k}}{\sqrt{4^2 + 7.5^2 + 5^2}\, m} = 0.406\boldsymbol{i} - 0.760\boldsymbol{j} + 0.507\boldsymbol{k}$$

Therefore,

$$\boldsymbol{T} = (10kN)\boldsymbol{n}_{BA} = (4.06\boldsymbol{i} - 7.60\boldsymbol{j} + 5.07\boldsymbol{k})\ kN$$

The angles $\theta_x, \theta_y, \theta_z$ are obtained by recognizing that the components of $\boldsymbol{n}_{BA}$ are precisely their respective cosines. Thus

$$\theta_x = \cos^{-1} 0.406 = 66.05°$$
$$\theta_y = \cos^{-1}(-0.760) = 139.46°$$
$$\theta_z = \cos^{-1} 0.507 = 59.54°$$

d) Our examples so far have dealt with forces alone, but we need to show a few examples to review the calculation of the moment of a force about a point in space, as we have done in previous weeks. The moment of a force $\boldsymbol{F}$ with respect to a point P, is defined (as we have done in previous lectures) as the cross product

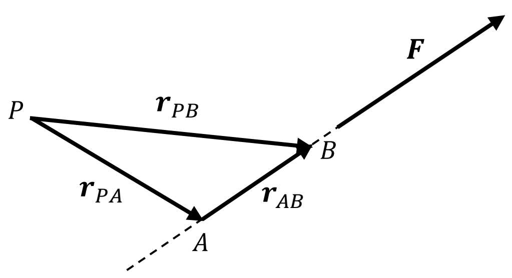
$$\boldsymbol{M} = \boldsymbol{r}_{PA} \times \boldsymbol{F}$$

In this formula, A is *any* point along the line of action of the force. You may ask, how can we be sure that the choice of this point A is arbitrary? Let B be another point along the same line of action, as shown in the figure. Then we can write

$$\boldsymbol{r}_{PB} = \boldsymbol{r}_{PA} + \boldsymbol{r}_{AB}$$

Invoking the *distributive property* of the cross product, we can write

$$\boldsymbol{r}_{PB} \times \boldsymbol{F} = (\boldsymbol{r}_{PA} + \boldsymbol{r}_{AB}) \times \boldsymbol{F} = \boldsymbol{r}_{PA} \times \boldsymbol{F} + \boldsymbol{r}_{AB} \times \boldsymbol{F}$$



But, on the other hand, we know that

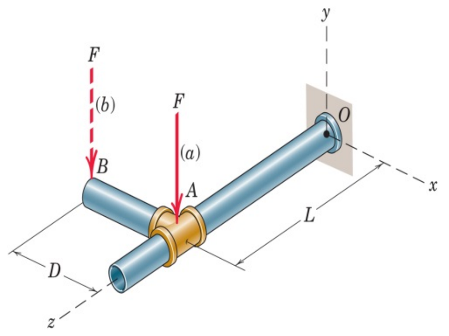
$$\boldsymbol{r}_{AB} \times \boldsymbol{F} = \boldsymbol{0}$$

Indeed, the vectors $\boldsymbol{r}_{AB}$ and $\boldsymbol{F}$ are collinear! Thus we obtain

$$\boldsymbol{r}_{PB} \times \boldsymbol{F} = \boldsymbol{r}_{PA} \times \boldsymbol{F} = \boldsymbol{M}$$

We conclude, in fact, that the calculation of the moment of a force with respect to a point is *consistent with the principle of transmissibility*. In other words, the moment of a force with respect to a point is not affected by translating the force along its line of action. On the other hand, as we already know, to transfer a force to a different parallel line of action entails a price. The price to be paid for this operation is precisely equal to the moment of the original force with respect to any point along the new line of action.

Let us look at Problem 2.123 from the text. It asks us to determine the moment about *O* of the force of magnitude *F* for the case (a) when the force $\boldsymbol{F}$ is applied at *A* and for the case (b) when the force is applied at *B*.



Solution: We will proceed systematically, although in this relatively simple case one should be able to obtain the answers by inspection. The force is given by

$$\boldsymbol{F} = -F\boldsymbol{j}$$

For part (a) we need

$$\boldsymbol{r}_{OA} = L\,\boldsymbol{k}$$

We obtain

$$\boldsymbol{M} = \boldsymbol{r}_{OA} \times \boldsymbol{F} = (L\boldsymbol{k}) \times (-F\boldsymbol{j}) = FL\,\boldsymbol{i}$$

We have exploited the fact that, according to the right-hand rule, we have

$$\boldsymbol{i} \times \boldsymbol{j} = -(\boldsymbol{j} \times \boldsymbol{i}) = \boldsymbol{k} \qquad \boldsymbol{j} \times \boldsymbol{k} = -(\boldsymbol{k} \times \boldsymbol{j}) = \boldsymbol{i} \qquad \boldsymbol{k} \times \boldsymbol{i} = -(\boldsymbol{i} \times \boldsymbol{k}) = \boldsymbol{j}$$

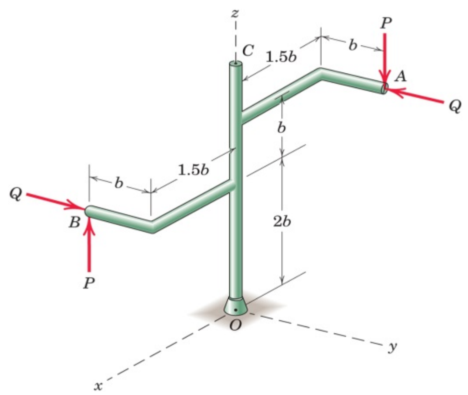
For part (b) we need

$$\boldsymbol{r}_{OB} = -D\,\boldsymbol{i} + L\,\boldsymbol{k}$$

The moment of the force applied at *B* is, therefore,

$$\boldsymbol{M}' = \boldsymbol{r}_{OB} \times \boldsymbol{F} = (-D\,\boldsymbol{i} + L\,\boldsymbol{k}) \times (-F\,\boldsymbol{j}) = FD\,\boldsymbol{k} + FL\,\boldsymbol{i}$$

e) Problem 2.131: Determine the combined moment of the two pairs of forces about point *O* and about point *C*, knowing that *P*=4*kN*, *Q*=7.5*kN*, *b*=3*m*.



Solution: Each of the given pair of forces clearly forms a couple. Recall that a couple (defined as two parallel forces of equal magnitude but opposite directions) has a moment that is *independent of the point with respect to which moments are taken*. This is the very essence of the concept of couple. [For a review, see our notes from two weeks ago]. Recall that this concept is so crucial that we have given to it a card of citizenship in the country of Statics equal to that of a force.

Having recognized that our forces are, in fact, equivalent to two couples, the answer to both questions is identical. Let us, therefore, calculate the moment of the forces with respect to, say, point *B*. The forces at *B* *contribute* a zero moment while the forces at *A* produce the moment

$$\boldsymbol{M} = \boldsymbol{r}_{BA} \times (-P\,\boldsymbol{k} - Q\,\boldsymbol{j})$$

We have

$$\boldsymbol{r}_A = -1.5\,b\,\boldsymbol{i} + b\,\boldsymbol{j} + 3b\,\boldsymbol{k} \qquad \boldsymbol{r}_B = 1.5\,b\,\boldsymbol{i} - b\,\boldsymbol{j} + 2b\,\boldsymbol{k}$$

Therefore,

$$\boldsymbol{r}_{BA} = \boldsymbol{r}_A - \boldsymbol{r}_B = -3b\,\boldsymbol{i} + 2b\,\boldsymbol{j} + b\,\boldsymbol{k}$$

Finally,

$$\boldsymbol{M} = (-3b\,\boldsymbol{i} + 2b\,\boldsymbol{j} + b\,\boldsymbol{k}) \times (-P\,\boldsymbol{k} - Q\,\boldsymbol{j})$$

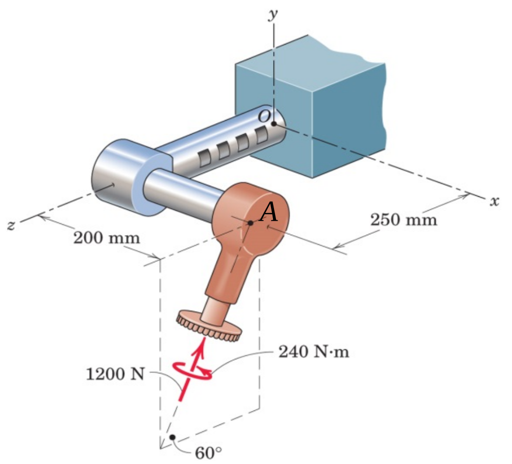
For the sake of variety, let us calculate the cross product on the basis of the determinant formula (noticing, however, that we could also have done this by using the distributive property and the known values of the cross products between the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$). We write

$$\boldsymbol{M} = \begin{vmatrix} \boldsymbol{i} & j & k \\ -3b & 2b & b \\ 0 & -Q & -P \end{vmatrix} = (-2P + Q)b\,\boldsymbol{i} - 3Pb\,\boldsymbol{j} + 3Qb\,\boldsymbol{k}$$

Using the numerical values provided, we obtain

$$\boldsymbol{M} = (-1.5\,\boldsymbol{i} - 36\,\boldsymbol{j} + 67.5\,\boldsymbol{k})\;kN\,m$$

f) Problem 2.146: The special purpose milling cutter is subjected to the force of 1200 *N* and a couple of 240 *Nm* as shown. Determine the moment of this system about point *O*.



Solution: Note that by the principle of transmissibility we may consider the force applied at the point *A*, which was not labeled in the original figure in the text. The couple, as you know, is a free vector and can be applied anywhere. Note, by the way, that a couple and a force along parallel lines of action constitute what is called a *wrench*. This is the kind of action that we exert on a screwdriver, while pushing along and turning around the same axis. The position vector of point *A* is seen to be given by

$$\boldsymbol{r}_A = \boldsymbol{r}_{OA} = (200\,\boldsymbol{i} + 250\,\boldsymbol{k})\;mm$$

The unit vector associated with the force and the couple is obtained (from the figure) as

$$\boldsymbol{n} = (\sin 60°)\boldsymbol{j} - (\cos 60°)\boldsymbol{k}$$

The force is, therefore,

$$\boldsymbol{F} = (1200\;kN)\boldsymbol{n} = (1039\,\boldsymbol{j} - 600\,\boldsymbol{k})\;N$$

Similarly, the couple is

$$\boldsymbol{C} = (240\;kN\,m)\boldsymbol{n} = (207.8\,\boldsymbol{j} - 120\,\boldsymbol{k})\;Nm$$

Notice the important detail that we have used the right-hand rule by placing our curved fingers in accordance with the curved arrow and letting the thumb show the way. The total moment of the system about point O is

$$\boldsymbol{M}_O = \boldsymbol{r}_A \times \boldsymbol{F} + \boldsymbol{C} = \begin{vmatrix} \boldsymbol{i} & j & k \\ 0.20m & 0 & 0.25m \\ 0 & 1039N & -600N \end{vmatrix} + \boldsymbol{C}$$

We are invoking the property of couples in the sense that they have a constant moment (independent of the point with respect to which moments are taken). Performing the operations indicated, we obtain the final result as

$$\boldsymbol{M}_O = (-259.75\,\boldsymbol{i} + 327.80\,\boldsymbol{j} + 87.80\,\boldsymbol{k})\; N\,m$$

The moment of a force about an axis

(i) Intuitive considerations

Roughly speaking, the moment of a force about a point provides us with an idea of the ability of a force to 'produce a tendency to rotate' about a point. This description should not be taken literally, but it is enough to elicit a mental image. For example, if we think of a ball suspended from the roof with a rope (like a pendulum), then hitting the ball with a force will cause the pendulum to swing in the plane formed by the force and the point of suspension. The moment of the applied force with respect to the point of suspension will vanish only if the force happens to pass through that point.

In the same intuitive vein, let us consider now a rigid object supported (partially) by means of a line of hinges, such as a door or the cover of a grand piano. If we think of the door with its hinges along a vertical axis, it should be clear that any vertical will not make the door rotate at all about this axis! Yet, the moment of a vertical force applied, say, at the door handle with respect to any point in the line of hinges does not vanish. In fact, any force in the plane of the door is completely ineffective in making the door rotate about the hinge line. We want, therefore, to introduce a concept that would represent the relevance (i.e. the "moment")[1] of a force inasmuch as its tendency to make an object rotate about an axis.

(ii) Formal definition

Let $\boldsymbol{F}$ be a force in space and let n be an oriented line. Let P be any point on n and let $\boldsymbol{n}$ be the unit vector in the direction of n. The *moment of* $\boldsymbol{F}$ with respect to n is defined as the *projection* onto n of the moment of $\boldsymbol{F}$ with respect to P.

Recalling that the projection of a vector $\boldsymbol{v}$ onto a direction with unit vector $\boldsymbol{m}$ is the dot product $\boldsymbol{v} \cdot \boldsymbol{m}$, we conclude that the moment of $\boldsymbol{F}$ about n is obtained as

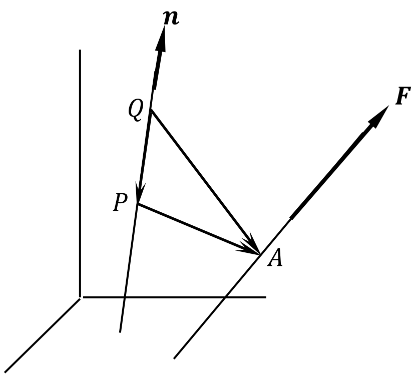
$$M_n^F = \boldsymbol{M}_{\boldsymbol{P}}^{\boldsymbol{F}} \cdot \boldsymbol{n} = (\boldsymbol{r}_{\boldsymbol{PA}} \times \boldsymbol{F}) \cdot \boldsymbol{n}$$

In this equation, A is any point along the line of action of $\boldsymbol{F}$. Several remarks are in order:

[1] Thus, we may say: "The government has adopted a decision of great moment".

- The moment of a force with respect to a line *is a scalar quantity*. It may be, of course, positive or negative. Reversing the orientation of the line n changes the sign of the result. If desired, one can also define the *vector moment with respect to an axis* by multiplying the scalar value M_n^F by the unit vector $\boldsymbol{n}$, that is, $\boldsymbol{M}_n^F = M_n^F\,\boldsymbol{n}$.

- The point P along the line n is arbitrary, just as the point A along the line of action of $\boldsymbol{F}$ is arbitrary. Let us prove that this is the case. Let Q be a different point along n. Then we can write $\boldsymbol{r}_{QA} = \boldsymbol{r}_{QP} + \boldsymbol{r}_{PA}$, as shown in the figure. We calculate



$$\boldsymbol{r_{QA}} \times \boldsymbol{F} = \left(\boldsymbol{r_{QP}} + \boldsymbol{r_{PA}}\right) \times \boldsymbol{F} = \boldsymbol{r_{QP}} \times \boldsymbol{F} + \boldsymbol{r_{PA}} \times \boldsymbol{F}$$

Dot-multiplying by $\boldsymbol{n}$ yields

$$\left(\boldsymbol{r_{QA}} \times \boldsymbol{F}\right) \cdot \boldsymbol{n} = \left(\boldsymbol{r_{QP}} \times \boldsymbol{F}\right) \cdot \boldsymbol{n} + \left(\boldsymbol{r_{PA}} \times \boldsymbol{F}\right) \cdot \boldsymbol{n}$$

But, by definition of cross product, the vector $\boldsymbol{r_{QP}} \times \boldsymbol{F}$ is perpendicular to $\boldsymbol{r}_{QP}$ and, therefore, its dot product with $\boldsymbol{n}$ (which lies along the line PQ) vanishes. Thus we obtain

$$\left(\boldsymbol{r_{QA}} \times \boldsymbol{F}\right) \cdot \boldsymbol{n} = \left(\boldsymbol{r_{PA}} \times \boldsymbol{F}\right) \cdot \boldsymbol{n}$$

This shows that the choice of the point P is, indeed, arbitrary.

- If the force $\boldsymbol{F}$ and the line n happen to be in the same plane, the moment of $\boldsymbol{F}$ with respect to n necessarily vanishes. The proof is straightforward. If $\boldsymbol{F}$ and n share a plane, then the moment of $\boldsymbol{F}$ with respect to any point P in n is perpendicular to that plane and, therefore, perpendicular to $\boldsymbol{n}$.

- <u>Expression in terms of a determinant</u>: As you know, the cross product of two vectors $\boldsymbol{u}, \boldsymbol{v}$ can be expressed in terms of their components in a right-handed Cartesian coordinate system as the determinant

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \left(u_y v_z - u_z v_y\right)\boldsymbol{i} + \left(u_z v_x - u_x v_z\right)\boldsymbol{j} + \left(u_x v_y - u_y v_x\right)\boldsymbol{k}$$

On the other hand, the dot product of two vectors $\boldsymbol{w}, \boldsymbol{n}$ in terms of their components can be written as

$$\boldsymbol{w} \cdot \boldsymbol{n} = w_x n_x + w_y n_y + w_z n_z$$

It follows that the *double product* $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{n}$ can be represented by the determinant

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{n} = \begin{vmatrix} n_x & n_y & n_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

- Consider, for example, the moment of a force $\boldsymbol{F}$ applied at a point A with coordinates x, y, z, about the origin of the coordinate system. We obtain

$$\boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ F_x & F & F_z \end{vmatrix} = \left(yF_z - zF_y\right)\boldsymbol{i} + \left(zF_x - xF_z\right)\boldsymbol{j} + \left(xF_y - yF_x\right)\boldsymbol{k}$$

To compute the moment with respect to the x axis, we have to dot-multiply this expression by the unit vector $\boldsymbol{i}$, and similarly for the moments about the other two axes. Thus we obtain the result that the moments of a force with respect to the coordinate axes are nothing but the (scalar) components of the (vector) moment with respect to the origin. That is,

$$M_x = yF_z - zF_y \qquad M_y = zF_x - xF_z \qquad M_z = xF_y - yF_x$$

(iii) Example: In Problem 2.146 solved above, we are now asked to compute the moment of the system about the line n passing through the origin O and bisecting the *y* and *z* axes. The unit vector along this line is given by $\boldsymbol{n} = \frac{\sqrt{2}}{2}(\boldsymbol{j} + \boldsymbol{k})$.

Solution: We have obtained

$$\boldsymbol{M}_O = (-259.75\,\boldsymbol{i} + 327.80\,\boldsymbol{j} + 87.80\,\boldsymbol{k})\ N\,m$$

The moment about the line n is, therefore,

$$M_n = \boldsymbol{M}_O \cdot \boldsymbol{n} = \frac{\sqrt{2}}{2}(327.80 + 87.80) = 293.87\ N\,m$$