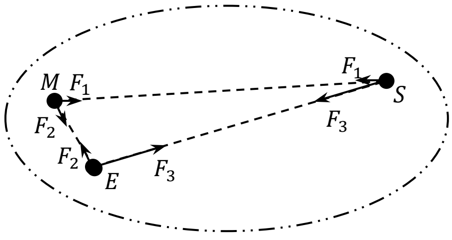
1. <u>Structures</u>

   a. <u>Definition</u>: A *structure* is a collection of (rigid) objects, or *members*, assembled together by means of *connections* (such as frictionless pins). We have already encountered some examples of structures, but we want to undertake a more systematic study of some common classes of structures such as trusses and frames.

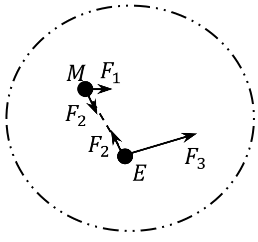
   b. <u>External and internal forces</u>: In studying the conditions of equilibrium of a rigid object, we have focused our attention mainly on the determination of the support reactions that arise from the application of external forces. On the other hand, when, as engineers, we want to investigate the strength of a structure or of a member thereof, we realize that these external forces and reactions don't tell the whole story. If we think for a moment on an intuitive basis, the failure of a structural member occurs when the atomic bonds holding the piece together are broken. If we think of a bar, such as a piece of chalk, the consequence of overcoming these bonds is a separation of the piece at a cross section. The two separate pieces of the broken bar expose the two sides of the cross section. These two sides were held together by the resultant of all the atomic bonding forces abiding by the principle of action and reaction. These *internal forces*, however, do not make an appearance, nor otherwise manifest themselves, in the free-body diagram used to calculate the support reactions. We would like, therefore, to investigate whether Statics is capable to give us some information about these internal forces.

   c. <u>A planetary analogy</u>: When studying a phenomenon from a physical point of view, we start by defining the part of the universe under consideration. This idea leads to the concept of a *physical system*. In our case, we will be speaking of purely *mechanical systems* (as opposed, say, to *thermo-dynamical systems*, where exchanges of thermal and chemical energy take place). A mechanical system consists of a (continuous or discrete) collection of material particles that, for the purposes of analysis, has been detached from the rest of the universe by means of a definite boundary. In other words, given a particle in the universe, it either belongs to the system or is external to the system. Particles in general interact with each other by means of mechanical interactions that we call *forces*. Let us adopt, accordingly, the following definition: A force is said to be *external* if it represents an interaction between a part of the system and the rest of the universe. A force is said to be *internal*, if it represents an interaction between two parts of the system.
   As an example, consider a planetary system. If it is far enough from other such systems, we can assume that there are no external forces. The internal forces are the result of the mutual gravitational attractions. These forces keep the planets orbiting (rather than flying apart) and are also responsible for phenomena such as the tides. Can we make these important forces manifest? Fortunately, yes. But to do so *we need to consider a subsystem*. Take as an example the system comprised by the sun, the earth and the moon.



System

External forces: None
Internal forces: F_1, F_2, F_3

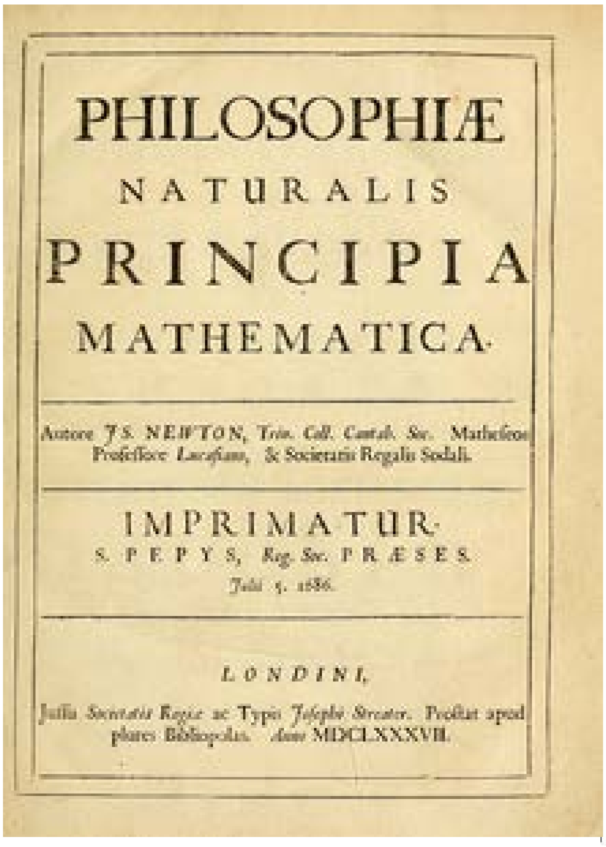


Subsystem

External forces: F_1, F_3
Internal forces: F_2

Consider now the subsystem made up by the moon and the earth alone. The gravitational interactions between the earth and the sun and between the moon and the sun (which were before part of the internal forces) are now external forces, since the sun has been excluded from the subsystem! In other words, *to expose an internal force all we have to do is to consider a subsystem whose boundary interferes with the transmission of that force*.

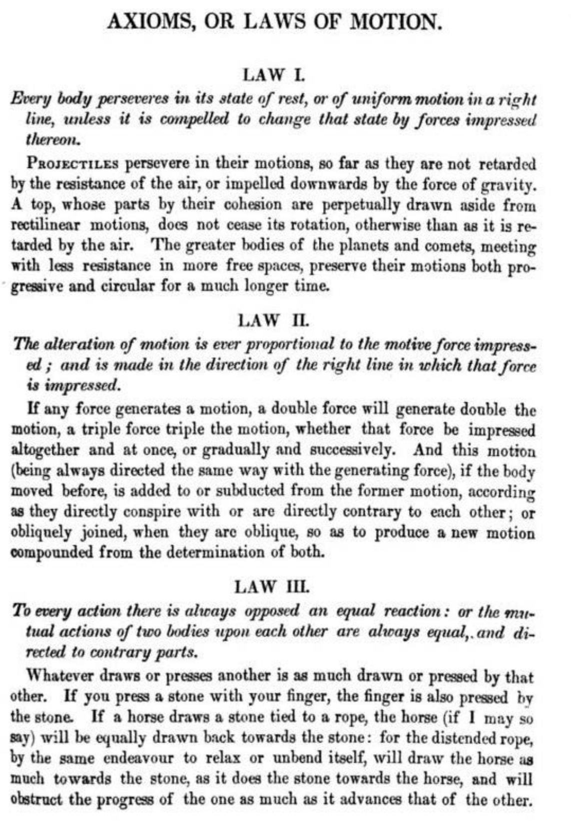
d. The simplest example: The planetary system is made up of discrete points, but we are interested in continuous systems, such as bars, beams and plates. In the case of discrete systems, the internal forces are forces acting at a distance. In continuous systems, on the other hand, the internal forces (such as those resulting from the interactions of nearby molecules) are *contact forces*, which can be more elaborate than distant forces. The simplest example of internal contact forces is the case of a straight two-force member, such as a rope pulled at its ends or a column sustaining a heavy water tank.
Newton himself used such an example when explaining his third axiom. He says: "If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone: for the distended rope, by the same endeavour to relax or unbend itself, will draw the horse as much towards the stone, as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other."


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore JS. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

LONDINI,
Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.


AXIOMS, OR LAWS OF MOTION.

LAW I.

Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.

Projectiles persevere in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are perpetually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in more free spaces, preserve their motions both progressive and circular for a much longer time.

LAW II.

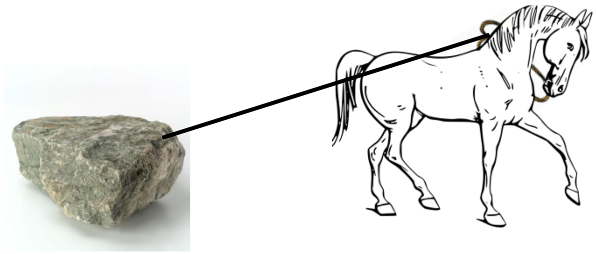
The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subducted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

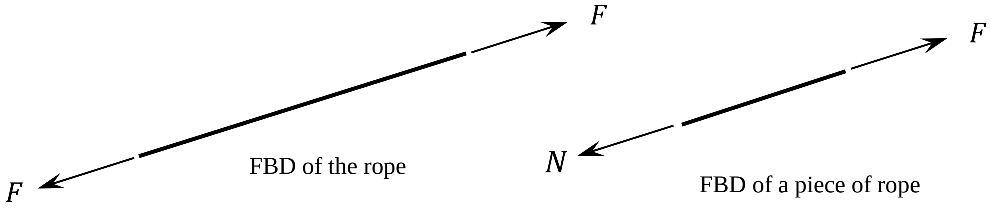
LAW III.

To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone: for the distended rope, by the same endeavour to relax or unbend itself, will draw the horse as much towards the stone, as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other.



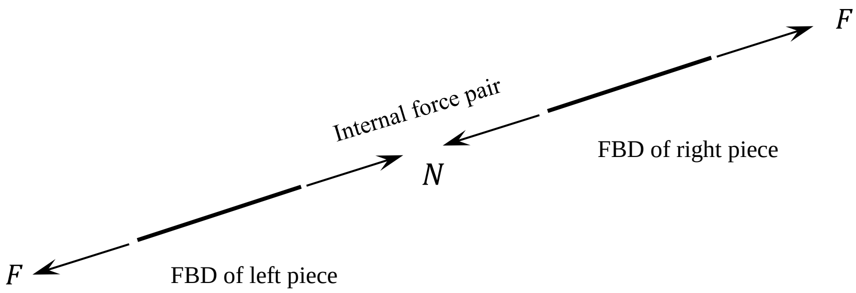
Newton’s horse and stone



By choosing as our subsystem a piece of the rope (say from the midpoint up to the horse attachment) we expose the internal force N. From this simple free-body diagram we conclude that

$$N = F$$

Although (in this case) numerically equal to the external force, the internal force N conveys a different concept. It represents the resultant of the molecular interactions at the cross section. The force N in the FBD on the right side, represents the forces exerted by the part to the left of the cut on the chosen subsystem (to the right of the cut). The force exerted by the right on the left is clearly equal in magnitude and opposite in direction, as shown in the figure below. These forces are responsible for preserving the integrity of the structure. When we speak about the internal force we refer to the action-reaction pair. We say that the internal force N is *tensile* if the forces are directed toward each other, like in the case of the rope. The opposite case, when the forces point away from each other is called *compression*. By convention, we will assign a positive sign to tension and a negative sign to compression.



2. Trusses

A *truss* is a structure made entirely from prismatic (straight) two-force members. How can we make sure that all the members of the truss are actually two-force members? The answer is simple. We connect the members (at their ends) by means of frictionless pins. The points of connection between members are called *nodes* or *joints*. Moreover, we will consider that the external forces are applied exclusively at these nodes. No couples are permitted as external loads. Finally, the supports, applied exclusively at the nodes, are hinges (either fixed or mounted on rollers). Under these conditions, each member is a two-force member. In practice, some or all of these conditions are violated, but it can be shown that if the members are slender enough the error made by considering them as two-force members is very small. Hence follows the enormous success of these ubiquitous structures. They are relatively light-weight and inexpensive, while providing great strength. Historically, they made possible the widespread extension of railway lines by enabling the construction of bridges of large spans over rivers and mountainous areas.

A truss is said to be *plane* or two-dimensional if the axes of all the (prismatic) members lie in one and the same plane and so do all the external forces (applied or support reactions). The basic plane truss is just a triangle. It is not difficult to convince ourselves that a triangle (three bars articulated at their ends) constitutes a rigid object. Each pin reduces the 9 original degrees of freedom (3×3) by 2 (the relative displacements at the joined ends of the two bars joined). Since we have 3 pins, we obtain $9 - 3 \times 2 = 3$. So, the resulting triangle has exactly 3 degrees of freedom, as expected for a rigid object in the plane. How can we build up more complicated trusses? One way to do so is the following. Starting from the triangle, we add two more bars (starting at two of the existing nodes) and connect them at a new node. Now we have 5 bars and 4 nodes. We can continue this process as many times as we want: two new bars for every new node. If we denote by n the number of nodes and by b the number of bars, we obtain the equation

$$b - 3 = 2(n - 3)$$

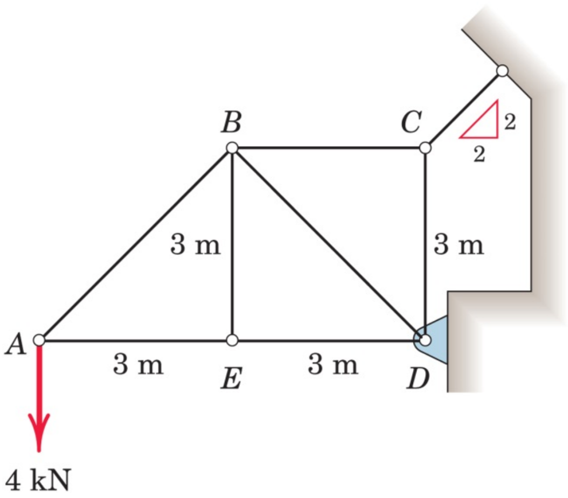
In other words,

$$b = 2n - 3$$

Not all trusses are generated in this way. Those that are, are called *simple trusses*. Notice that by construction, a simple truss has exactly the number of bars to obtain rigidity. If one bar is eliminated, the structure acquires a degree of freedom and becomes a mechanism. If, on the contrary, more bars are added, we obtain a rigid object, but we will find out that the equations of statics are not sufficient to determine all the internal forces. We say that such a truss is *internally statically indeterminate*, and we will not be able to study it in this course. Notice also that a simple truss (as a rigid object) will need exactly 3 conditions of support (as provided by a hinge and a roller or, alternatively, by 3 roller supports).

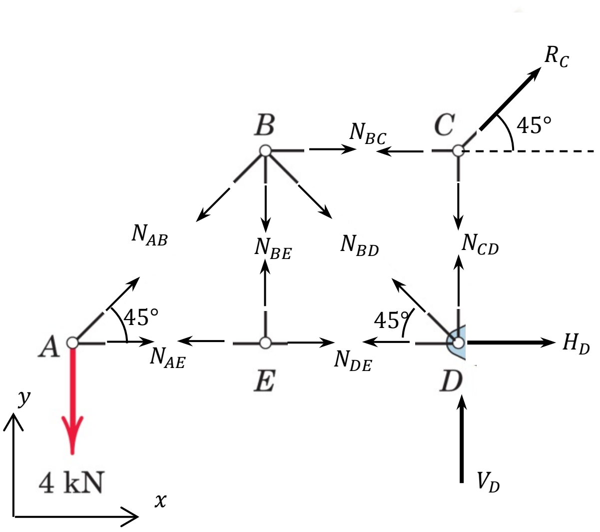
3. The method of joints

The most basic and general method for finding the internal forces in all the members of a truss, known as *the method of joints*, is based on the fact that we can establish the equilibrium equations of each and every node of the truss and obtain 2 independent equations. More precisely, if we cut a small vicinity of a node, as it were, with a cookie cutter, we obtain a concurrent system of forces acting on the free-body diagram of that small object. We have at our disposal the two equations of equilibrium of forces. [Since the system is concurrent, the equation of moments is satisfied automatically]. How many equations we have in total? Clearly, we have $2n$ equations. How many unknown quantities? We have one internal force per bar plus 3 reactions, that is, $b + 3$. But, for a simple truss, as we just saw, $b + 3 = 2n$. In other words, we have precisely the same number of equations as of unknown static quantities! The equations of equilibrium at the nodes are mutually independent. We may choose to replace 3 of these equations by equations of equilibrium (of forces and moments) of the whole structure, thus obtaining the support reactions first. Let us put all these facts together in an example.



Example: Problem 4.5 in our text. We want to solve for all the forces and support reactions in the truss shown.

Solution: Although there are some simplifications that can be implemented (as we shall discuss later), let us apply the method in its full generality (before engaging into any clever 'tricks'). What matters, after all, is the general philosophy rather than the particular instances. In our day, with the use of automated programmable devices, the methods are applied in their full generality, since there is little to be gained by implementing costly artificial intelligence components that result in the reduction of the computing time by a couple of milliseconds. So, let us draw the free-body diagrams of each and every joint and see what happens. It is a good policy to draw (à la cookie-cutter) the nodes with a little piece of each concurrent bar. In this way, we can intuit tension as a pulling force, and vice versa. For generality, let us assume that all the internal forces are tensile. A negative numerical answer, therefore, will mean compression. We have here a total of 5 independent FBDs. The equations of sums of forces in two directions yield, therefore, the following 10 equations of equilibrium:



Node	$\Sigma F_x = 0$	$\Sigma F_y = 0$
A	$N_{AE} + N_{AB}\cos 45° = 0$	$-4\,kN + N_{AB}\sin 45° = 0$
B	$-N_{AB}\cos 45° + N_{BD}\cos 45° + N_{BC} = 0$	$-N_{AB}\sin 45° - N_{BE} - N_{BD}\sin 45° = 0$
C	$-N_{BC} + R_C\cos 45° = 0$	$-N_{CD} + R_C\sin 45° = 0$
D	$-N_{DE} - N_{BD}\cos 45° + H_D = 0$	$V_D + N_{BD}\sin 45° + N_{CD} = 0$
E	$-N_{AE} + N_{DE} = 0$	$N_{BE} = 0$

Starting at node A, 2nd equation, we obtain

$$N_{AB} = 5.66\,kN \;\; (tension)$$

From the same node, 1st equation, we obtain

$$N_{AE} = -4\ kN \quad (compression)$$

Moving to node *E*, we get

$$N_{DE} = -4kN \quad (compression)$$

and

$$N_{BE} = 0 \quad (zero - force\ member)$$

We proceed to node *B* and obtain

$$N_{BD} = -5.66\ kN \quad (compression)$$

$$N_{BC} = 8\ kN \quad (tension)$$

Node *C* yields

$$R_C = 11.32\ kN \quad (as\ shown)$$

$$N_{CD} = 8\ kN \quad (tension)$$

Finally, from node *D*,

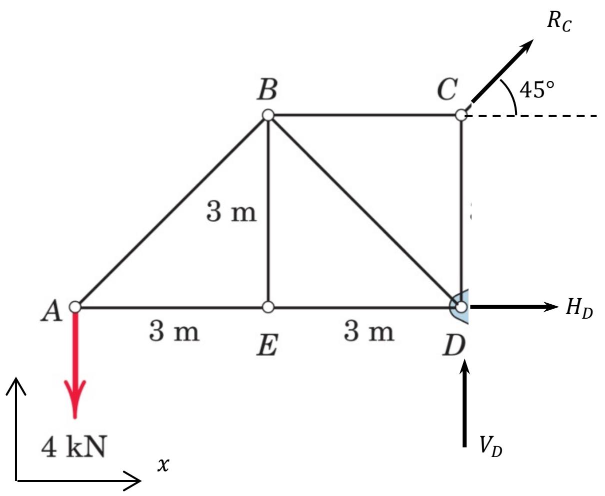
$$H_D = -8\ kN \quad (opposite\ to\ shown)$$

$$V_D = -4\ kN \quad (opposite\ to\ shown)$$

<u>Alternative procedure</u>: The method of joints in its raw version, which we have displayed above, is applicable to any statically determinate truss. It always provides the necessary number of equations to solve for the internal forces and the external support reactions. In some cases, however, we may end up with systems of equations that cannot be solved in the domino-like fashion that we encountered in the previous example. For the case of simple trusses (namely, those created by the process of adding two bars and one node, as described above) the following procedure guarantees that the truss can always be solved systematically like the example above. The procedure runs as follows:

a) Draw the FBD of the truss as a whole and determine the support reactions

b) Find a node at which there are only two unknown internal forces. For a simple truss, such a node will always exist.
c) Draw the FBD of this node and, by implementing the two equilibrium equations, find the two unknown internal forces thereat.
d) Repeat steps b and c as many times as needed.



Let us apply this (equivalent) version of the method of joints to our example. We start by drawing the FBD of the whole truss and determining the support reactions, namely,

$$\sum F_x = H_D + R_C \cos 45° = 0$$

$$\sum F_y = -4\ kN + V_D + R_C \sin 45° = 0$$

$$\sum M_C = (4kN)(6m) + H_D(3m) = 0$$

The solution of this system is easily found as

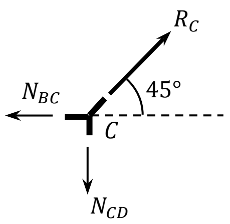
$$H_D = -8\ kN \quad (opposite\ to\ shown)$$

$$R_C = 11.32\ kN \quad (as\ shown)$$

$$V_D = -4\ kN \quad (opposite\ to\ shown)$$

Note that these results coincide with the ones obtained before (and that, even without implementing the new procedure, these reactions can be used to double check the numerical correctness of the previous results).

For step b, we have two choices: node *A* and node *C*. Let us start from node *C*, as shown in the following FBD and associated equilibrium equations:



$$\Sigma F_x = -N_{BC} + (11.32\ kN) \cos 45° = 0$$

$$\Sigma F_y = -N_{CD} + (11.32\ kN) \sin 45° = 0$$

We can immediately solve for these two unknown internal forces as

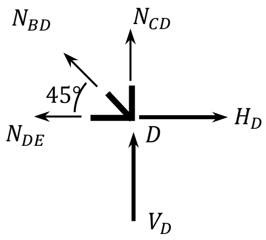
$$N_{BC} = 8\ kN \quad (tension)$$

$$N_{CD} = 8\ kN \quad (tension)$$

The next node cannot be *B* because we still have 3 unknown internal forces converging thereto. Instead, we can move to node *D*, since only the forces in bars BD and DE are unknown. The corresponding FBD and the associated equilibrium equations are

$$\Sigma F_x = -N_{DE} - N_{BD}\cos 45° + (-8\ kN) = 0$$

$$\Sigma\ F_y = (-4\ kN) + 8\ kN + N_{BD}\sin 45° = 0$$

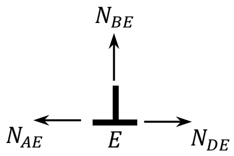


The solution is immediately obtained as

$$N_{BD} = -5.66\ kN\quad (compression)$$

$$N_{DE} = -4\ kN\quad (compression)$$

The next available node is *E*.

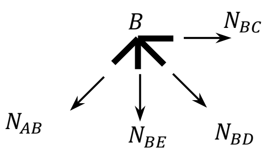


$$\Sigma\ F_x = \ -N_{AE} + (-4\ kN) = 0$$

$$\Sigma F_y = N_{BE} = 0$$

Thus, $N_{BE} = 0$ and $N_{AE} = -4kN\quad (compression)$.

Now we can move to *A* or to *B*. In both cases, we have a superabundance of equations due to the fact that we already calculated the reactions and thus exploited 3 equilibrium equations. Let us move to *B*, to make matters more interesting.



$$\Sigma F_x = -N_{AB}\cos 45° + (-5.66\ kN)\cos 45° + 8\ kN = 0$$

Solving, we obtain

$$N_{AB} = 5.66\ kN\quad (tension)$$

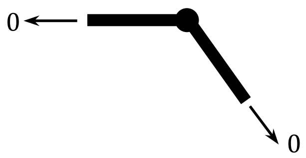
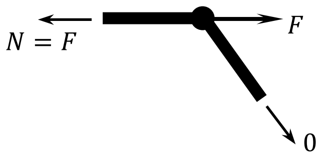
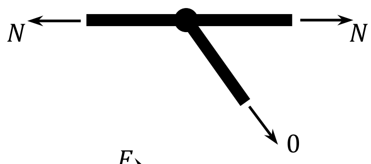
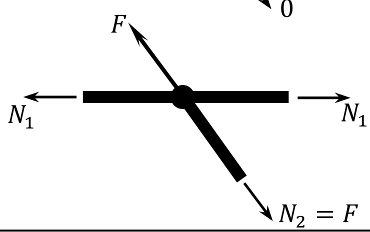
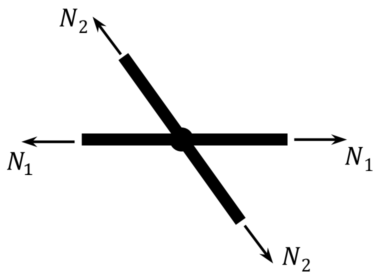
We have 3 equations left (sum of forces in the *y* direction at *B* and two equations at *A*). If there are no mistakes, these equations should be satisfied identically.

4. <u>Zero-force members</u>

Let us have a fresh look at the problem we have just solved and place ourselves at the beginning of the process, before any calculations have taken place. Consider node *E*, at which 3 bars converge, two of which are collinear. Notice, moreover, that this particular node is unloaded. Clearly, two collinear forces cannot balance a force in a different direction! In other words, if we imagine the FBD of this node and we mentally sum forces in a direction perpendicular to the collinear bars, we must conclude that the force in the third bar must vanish. Thus, without any calculation, we conclude that the third bar is a *zero-force member*. Moreover, the two collinear bars must have identical internal forces. We conclude, therefore, that a good (but by no means strictly necessary) policy when solving a truss is to identify as many zero-force members as possible before undertaking the solution proper.

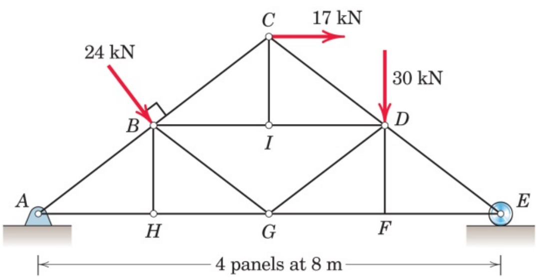
Note that the designation “zero-force member” is somewhat misleading, since this is not a property of the member itself, but rather of the loading. Had we applied a force at node E in an arbitrary direction, we would not have obtained a zero-force member at that node. On the other hand, if the external force had coincided in direction with the third bar (BE), we would still have concluded that the internal forces in the collinear bars are equal to each other, and that the external force is balanced by the internal force in the third bar.

<u>Warning</u>: On the basis of these observations, let us try to give a more or less complete list of occurrences of this kind. This list may allow us to save some work. On the other hand, there is a certain danger in using it carelessly. After all, the zero-force members will arise automatically in the process of drawing the free-body diagrams of the nodes, so there is not much computational work saved by identifying the zero-force members a priori.

<table>
<tr><td rowspan="3">2 members</td><td></td><td>Unstable</td></tr>
<tr><td>
</td><td>No load</td></tr>
<tr><td>
</td><td>Load aligned
with one bar</td></tr>
<tr><td rowspan="2">3 members</td><td>
</td><td>No load</td></tr>
<tr><td>
</td><td>Load aligned with
non-collinear bar</td></tr>
<tr><td>4 members</td><td>
</td><td>No load</td></tr>
</table>

<u>Important exercise</u>: Verify, by reasoning on free-body diagrams, that each of the claims made in the table above is correct.

Example: (P4.16) Determine the force in each member of the loaded truss. All triangles are 3-4-5. Identify in advance all the zero-force members



Solution:

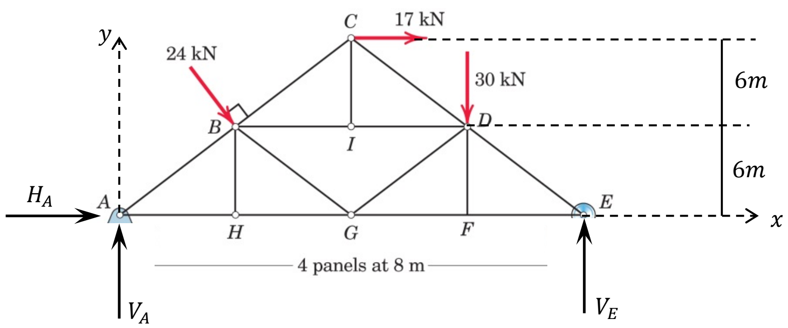
(a) Zero-force and special members: By inspection, members *BH*,*DF* and *CI* are zero-force members, that is

$$N_{BH} = N_{DF} = N_{CI} = 0$$

Also, we have

$$N_{AH} = N_{GH} \qquad N_{FG} = N_{EF} \qquad N_{BI} = N_{DI}$$

(b) Support reactions: We start by drawing a free-body diagram of the structure as a whole. Incidentally, we notice that this is not a simple truss. Indeed, it cannot be generated from an initial triangle by a process of systematic addition of two new bars and one new node. On the other hand, the truss satisfies the condition $b = 2n - 3$. Whether or not the truss is simple, it is not difficult to show that this condition is a *necessary condition for rigidity* of a truss.



FBD

$$\Sigma F_x = H_A + (24\ kN)(0.6) + 17\ kN = 0$$

$$\Sigma F_y = V_A - (24\ kN)(0.8) - 30\ kN + V_E = 0$$

$$\Sigma M_E = -V_A(32m) - (24kN)(0.6)(6m) + (24kN)(0.8)(24m) - (17kN)(12m) + (30\ kN)(8m) = 0$$

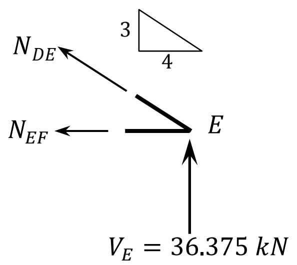
We obtain

$$H_A = -31.4\ kN \quad \leftarrow$$

$$V_A = 12.825\ kN \quad \uparrow$$

$$V_E = 36.375\ kN \quad \uparrow$$

(c) Internal forces in bars: Since this is not a simple truss, there exists the possibility that we may not be able to apply our “domino effect” and, therefore, we may be forced to solve simultaneous equations. On the other hand, having identified a few zero-force members, in this case we will manage to move from node to node in such a way that we never encounter a node with more than two unknown converging internal forces. Let us start from node *E*. Since we have already determined the reaction of the support, we have a case of just two unknown quantities for which we can exploit the sums of forces in two directions.



$$\Sigma F_x = -N_{EF} - 0.8\ N_{DE} = 0$$

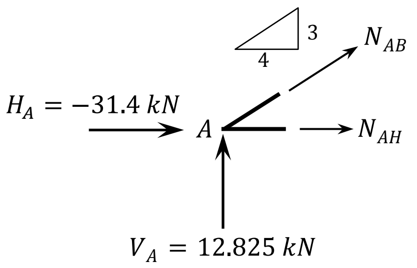
$$\Sigma F_Y = 36.375\ kN + 0.6\ N_{DE} = 0$$

We obtain

$$N_{DE} = -60.625\ kN \quad (compr.)$$

$$N_{EF} = 48.5\ kN \quad (tension)$$

We now move to node *A* and obtain the following results



$$\Sigma F_x = -31.4\ kN + N_{AH} + 0.8\ N_{AB} = 0$$

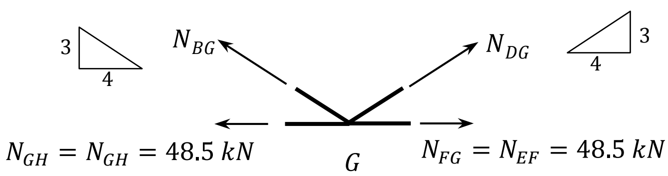
$$\Sigma F_y = 12.825\ kN + 0.6\ N_{AB} = 0$$

These equations yield

$$N_{AB} = -21.375\ kN \quad (compr.)$$

$$N_{AH} = 48.5\ kN \quad (tension)$$

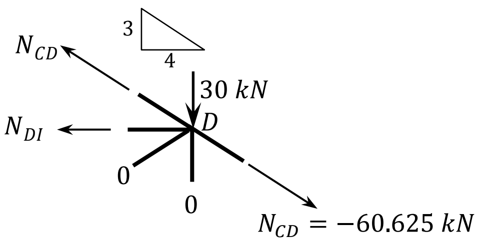
Nodes *H* and *F* have already been taken into consideration when we identified the zero-force and special members. Let us point out, however, that this is not an impressive achievement. If we hadn't noticed that these nodes are special, we would have moved to them, drawn a simple FBD and immediately concluded that the forces in the vertical members is zero and the horizontal members at each node carry the same internal force. At any rate, we can now move to node *G*. Our FBD looks as shown below.



Notice that by chance the forces in members GH and FG happen to be equal. We immediately conclude that

$$N_{BG} = N_{DG} = 0$$

For node *D* we obtain



$$\Sigma\, F_x = -0.8N_{CD} - N_{DI} + 0.8(-60.625kN) = 0$$

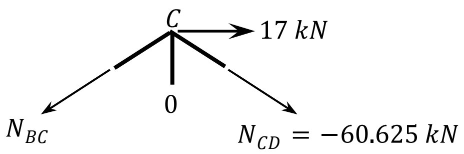
$$\Sigma\, F_y = -30kN + 0.6N_{CD} - 0.6(-60.625kN) = 0$$

The solution of these equations is

$$N_{CD} = -10.625\ kN \quad (compr.)$$

$$N_{DI} = -40.0\ kN \quad (compr.)$$

Finally, for node C,

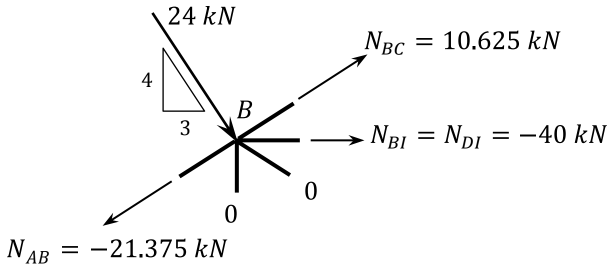


By sum of forces in the vertical direction, we obtain immediately

$$N_{BC} = -N_{CD} = 10.625\ kN \quad (tension)$$

Let us check the consistency of the solution by verifying that the sum of forces in the horizontal direction vanishes. Indeed,

$$\Sigma\, F_x = -0.8(10.625\ kN) + 0.8(-10.625\ kN) + 17\ kN = 0 \quad \surd$$



Node B (which has not been used) can now be exploited as a final check. As you may recall, the reason why we are left with 3 extra equations at the nodes is that we have used the three equations of global equilibrium when finding the reactions.

$$\Sigma\, F_x = -0.8(-21.375\ kN) + 0.8(10.625\ kN) + (-40\ kN) + 0.6(24\ kN) = 0 \quad \surd$$

$$\Sigma\, F_y = -0.6(-21.375\ kN) + 0.6(10.625\ kN) - 0.8(24\ kN) = 0 \quad \surd$$

Final remark: We emphasize again that finding zero-force and other special members *is not a necessary part of the theory or of the solution procedure*. The important concept is that each node of the truss provides us with two equilibrium equations. If the truss is statically determinate (as it will be the case always in this course) these equations are exactly all you need in order to solve for the internal forces and the support reactions. All the rest (such as finding the reactions first, identifying two-force members, and so on) is just embellishment. If you were to program the procedure for a computer, and if the number of nodes is n, you would just have the program formulate the $2n$ nodal equations and solve the system of $2n$ equations with $2n$ unknowns. There is really nothing else to the theory of statically determinate plane trusses.