1. Introduction

Let us recapitulate and see where we stand so far. We have assumed the following:

(a) Forces can be represented as vectors.
(b) The combined effect of several forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \dots, \boldsymbol{F}_n$ whose lines of action intersect at one and the same point is equal to the effect of the single force $\boldsymbol{R}$ obtained as their vector sum, that is,
$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots + \boldsymbol{F}_n.$$
This total force is called the *resultant* of the *concurrent* forces.
(c) As far as the main topic of this course is concerned (namely, equilibrium of isolated systems), the effect of a force is not affected by translating it over its line of action. In other words, the point of application of a force does not play any role as long as the line of action is preserved and as long as the force is assumed to be acting on the same rigid body. This idea is sometimes called the *principle of transmissibility*.[1]

Digression No. 1: Another way to formulate the principle of transmissibility is by declaring that forces are *sliding vectors*. A sliding vector has magnitude, direction and line of action, but no definite point of application. For the mathematically inclined, a sliding vector is an *equivalence class* of vectors.

Apart from these assumptions, we have also *defined* the concept of the moment $\boldsymbol{M}_P$ of a force $\boldsymbol{F}$ with respect to a point P as the cross product $\boldsymbol{M}_P = \boldsymbol{r}_{PQ} \times \boldsymbol{F}$. In this formula, $\boldsymbol{r}_{PQ}$ is the vector $\overrightarrow{PQ}$, where Q is *any* point in the line of action of $\boldsymbol{F}$. We have solved many examples showing the calculations leading to the moment of a force with respect to a point, but so far we have not done anything or made any statement of physical significance with this important concept. If you now look back at point (b) above, you see that we have still an open question, namely: What is the combined effect of several forces if they are *not concurrent*, that is, if they don't all meet at one point? In some cases, the answer can be attained by judiciously invoking the principle of transmissibility as many times as needed. But this technique does not work for the general case. For example, consider two forces of equal magnitude along parallel lines acting in opposite directions. They seem to add up to zero! Such a system is known as a force *couple*. Intuitively, as when we turn the steering wheel of a car, we understand that this may have something to do with the tendency of a body to rotate under the action of such a couple. Since this is a course on *Statics* (as opposed to *Dynamics*), we will try to find conditions to prevent such rotations from happening. So, you see that these couples will have to play some role. Apart from this simple example of a couple, we can also think about forces in space. In this case, as you know, two lines do not need to be parallel to not have any points in common. What this means is that in 3 dimensions the non-concurrent systems of forces constitute the rule rather than the exception. For all these reasons, it will turn out that the intellectual effort we invested in the concept of the moment of a force about a point has not been wasted, but will bear fruit in the study of general systems of forces.

2. Couples

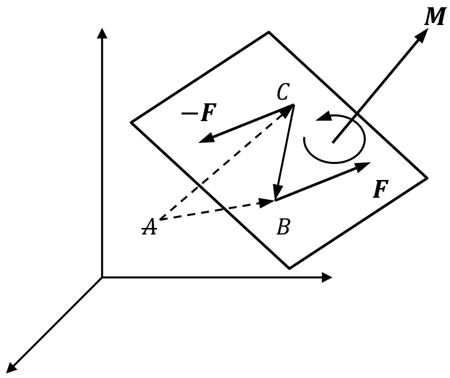
Consider two forces, $\boldsymbol{F}$ and $-\boldsymbol{F}$, acting along parallel lines, as shown in the figure below. Consider, moreover, an arbitrary point A and let us calculate the sum of the moments of the two forces with respect to this point. To do this, as we know from the definition of moment of a force, we need to

[1] As principles go, transmissibility has to be taken with a grain of salt. [Think of a bar in tension and the same bar in compression by means of two collinear and opposite forces. They are certainly not the same!]

choose two points, B and C, say, one on each of the lines of action. By definition of moment, and using the distributive property of the cross product with respect to the sum of vectors, we write

$$\boldsymbol{M}_A = \boldsymbol{r}_{AB} \times \boldsymbol{F} + \boldsymbol{r}_{AC} \times (-\boldsymbol{F}) = (\boldsymbol{r}_{AB} - \boldsymbol{r}_{AC}) \times \boldsymbol{F}$$

But the vector difference between two vectors issuing from the same point (A) is given by the vector joining the tips of these two vectors, namely, the vector $\overrightarrow{CB}$. We conclude that



$$\boldsymbol{M}_A = \boldsymbol{r}_{CB} \times \boldsymbol{F}$$

We infer, somewhat unexpectedly, that the moment of this system of forces is actually *independent of the point with respect to which moments are taken*. If we call $\boldsymbol{r}$ a vector from any point of the line of action of the second force to any point on the line of action of the first, the moment of the system is given by $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$, without any need to specify the point A. The magnitude of $\boldsymbol{M}$ is equal to Fd, where d is the distance between the two lines of action.

But there is more. Let us assume that we have a different pair of equal-magnitude and opposite forces acting along parallel lines and that, by chance, they happen to have the same moment $\boldsymbol{M}$ as the previous system. *Two such systems cannot be distinguished*, at least as far as the topic of this course is concerned. Just as, according to the principle of transmissibility, we cannot distinguish between two forces that differ only by their point of application on the same line of action, we also cannot distinguish between two systems of forces each consisting of a pair of equal-magnitude and opposite forces acting along parallel lines of action if they give rise to the same total moment $\boldsymbol{M}$.

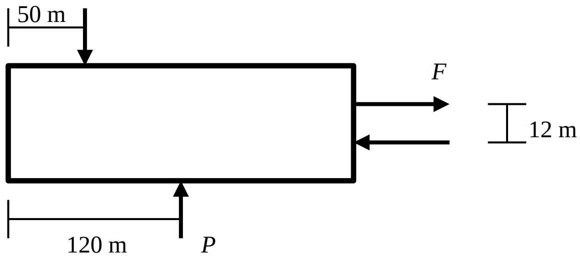
This observation gives rise to the definition of a *couple* as an entity with a given moment $\boldsymbol{M}$. This entity can be realized by many (infinite) systems of two forces, but is not necessarily identifiable with any one of them in particular. You see that this is similar to saying that a force is not necessarily identified with any one of the vectors along a fixed line of action. In the case of a couple, the vector $\boldsymbol{M}$ not only does not have a fixed point of application, but it does not even have a particular line of action. You may place it anywhere, as long as it has the same components in a given coordinate system.

Digression No. 2: Forces, as we have seen, are sliding vectors. Couples are *free vectors*. By this we mean that two couples are considered to be equivalent if they act along parallel lines (that is, if they are perpendicular to the same plane) and have the same magnitude and direction.

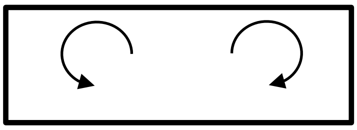
Notice that for two force pairs to represent the same couple, they must necessarily live in the same plane or in parallel planes. An alternative way to represent a couple is to draw a circular arrow in a plane perpendicular to $\boldsymbol{M}$, always abiding by the right-hand rule, as shown in the figure.

Given two couples, they can be added as vectors to produce a resultant couple. We postulate that the effect of two couples acting on a rigid body is the same as the effect of their vector sum. From now on, we will attach equal importance to couples as to forces. They both will have the same legitimate citizenship in the world of Statics

3. <u>Example</u>

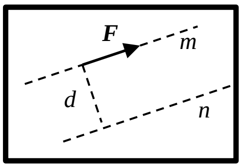


Let us solve Problem 2.65 from the text. We have a ship trying to turn by applying equal thrusts of magnitude F=300 kN each in opposite directions by means of a pair of propellers. The action of the propellers, however, is counteracted by a pair of tugboats, as shown. We are asked what forces *P* must the tugs apply. In other words, we have to find another couple that has the same magnitude but opposite direction to the one produced by the propellers. The couple produce by the propellers is $M = -(300kN)(12m) = -3600kN\ m$. The negative sign indicates that the couple is acting clockwise. The tugboats, therefore, must apply a positive couple of the same magnitude, that is, $P(70m) = +3600kN\ m$. We thus obtain the result as $P = 51.43\ kN$. Here we have an example that shows that the same couple can be produced by any system of parallel forces in a given plane and that the magnitude of the forces can be made as small as desired by increasing the distance as much as desired (or possible). For the sake of clarity, we can also draw the free-body diagram in terms of curved arrows, as shown. Note that the curved arrows (representing free vectors) can be placed anywhere.

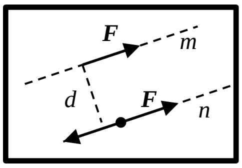


4. <u>Force-couple systems</u>

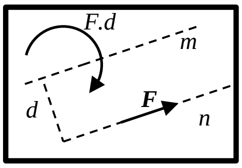
According to the principle of transmissibility we are allowed to move a force along its line of action without altering its effect on a rigid body. Can we move the force to a parallel line? The answer is: yes, but at a price. This price is given by a couple. To understand why this is so, consider (for simplicity) a force ***F*** acting in the *x,y* plane. We want to move this force from its current line of action *m* to a new parallel line of action *n* in the plane. Since two forces of equal magnitude and opposite directions along the same line of action amount to a zero effect, we will add such a system along the desired line of action *n*. The magnitude of the new forces can be made equal to the magnitude of ***F.*** We have now three forces of the same magnitude. Two of them form a couple and the third, is equal to ***F*** in both magnitude and direction but acts on the new line of action *n*, as desired. The magnitude of the couple (the price to be payed) is equal to the magnitude of the force times the distance *d* between the lines. This couple is equal (as a vector) to the moment of the original force with respect to any point on *n*.



Original



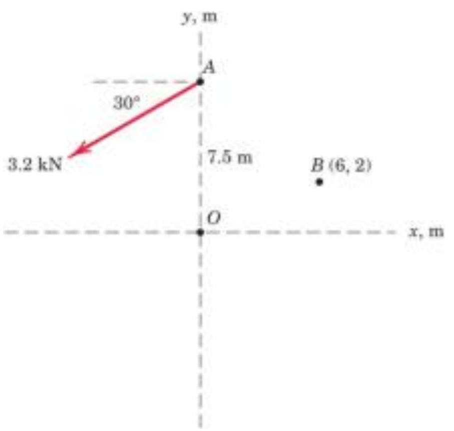
Intermediate



Final

We conclude that a single force can be replaced by a force-couple system, whereby the force has been transferred to a parallel line of action and a couple of appropriate magnitude and sense has been added.

5. Example



In Problem 2.62 we are asked to replace the given force by a force couple system (a) at point O, and (b) at point B.

(a) The couple to be introduced, according to our reasoning above, is equal to the moment of the force with respect to point O. We may use Varignon's lemma, which yields $M_O = (F\cos 30°)(7.5m) = 20.78\ kN\ m$. Note: We have adopted the convention that CCW is positive; moreover, the y component at A passes through O and, therefore, provides a zero contribution to the moment. In short, the answer is that the force $\boldsymbol{F}$ at A can be replaced by a force equal to $\boldsymbol{F}$ at O plus a couple given by the vector $\boldsymbol{M}_O = (20.78\ kN\ m)\boldsymbol{k}$.

(b) Following the same idea, we now calculate the moment of $\boldsymbol{F}$ about point B, using Varignon's lemma. We obtain $M_B = (F\cos 30°)(7.5m - 2m) + (F\sin 30°)(6m) = 24.84\ kN\ m$. Note: We have determined the sense of rotation of both components to be CCW and then computed the distances as positive numbers. Alternatively, if you want to be more systematic, you could use the cross product and let the formulas do the job. The answer is: The force $\boldsymbol{F}$ at A can be replaced by a force equal to $\boldsymbol{F}$ at B plus a couple given by the vector $\boldsymbol{M}_B = (24.84\ kN\ m)\boldsymbol{k}$.

6. Reduction of a system of forces to a point

After having assigned equal status to couples as to forces, we have opened the door to formulate one of the most important statements in this course:

Every system of forces (and possibly couples) can be reduced to an equivalent force-couple system, namely, to a single force $\boldsymbol{R}$ *(passing through any pre-assigned point P) and a single couple* $\boldsymbol{C}$.

To convince ourselves that this is the case, we need only choose any point P and repeat the process above for each of the n forces $\boldsymbol{F}_i \quad (i = 1,2,\dots,n)$. We thus obtain a concurrent system of forces at P, which can replace by their vector sum $\boldsymbol{R} = \boldsymbol{F}_1 + \cdots + \boldsymbol{F}_n$, and a collection of couples, which we can also add, to obtain a couple $\boldsymbol{C}$. We say that the system has been reduced to the point P. Notice that the resultant force $\boldsymbol{R}$ is *independent of the point P chosen*. The couple $\boldsymbol{C}$, on the other hand, depends in general of this choice.

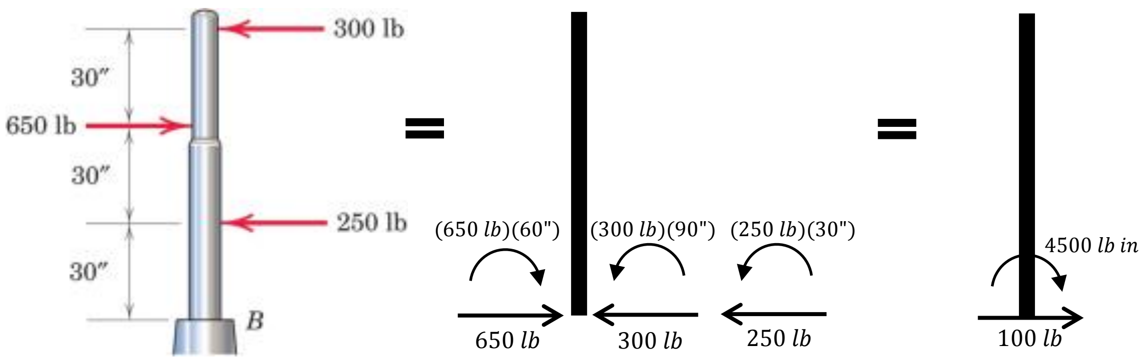
7. Resultant of a system of a system of forces in a plane

In the particular case of a system of forces acting *in a common plane* it is possible to go one step further. Indeed, by reversing the procedure we used to move the line of action of a force at the price of introducing a couple, we can start from a force-couple system and move the force to a new line of action that exactly counteracts the given couple. All we have to do, given a force and a couple, is to read the three parts of the figure ("original", "intermediate", "final") backwards and calculate the distance *d* by dividing the magnitude of the couple by the magnitude of the force. When this is done, the force ***R*** is called *the* resultant of the system. Note that if the vector sum of the forces happens to vanish, the system reduces to a pure couple.

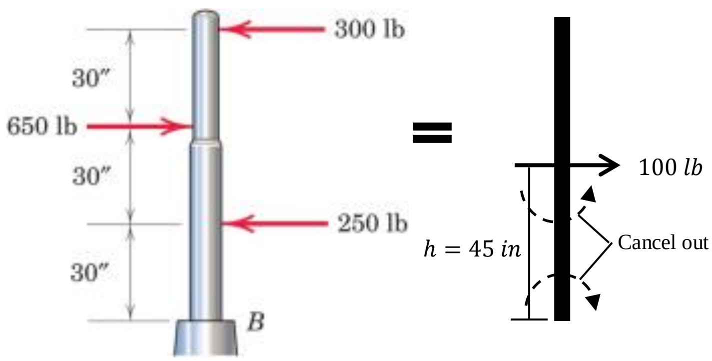
Digression No. 3: For a general (spatial) system of forces and couples, the reduction to a single force (i.e., the elimination of the couple) is not possible. The best that can be done is to replace the system by a so-called *wrench* (or *screw*), consisting of a force ***R*** and a couple ***C*** parallel to the force.

8. Example

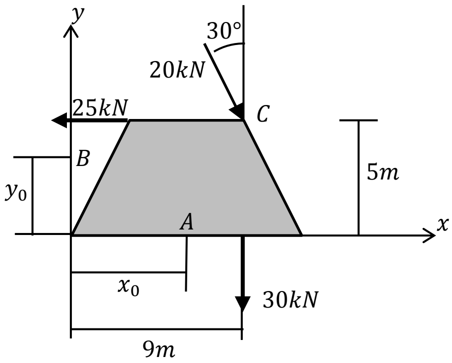
We are asked to determine the height *h* above the base *B* at which the resultant of the three forces acts.
We will solve this problem in two stages as follows: (a) We will reduce the system of three forces to a force-couple system at any point, such as *B*. (b) We will `eliminate' the resulting couple by moving the force to a judiciously chosen new line of action.
(a) The calculations in this case are straight forward, since the distances are explicitly given. All we have to worry about is the correct sense (CW or CCW) of the couples.



(b) Now, starting from the reduced force couple system, we move the force so as to produce a couple equal and opposite to the 4500 lb in. Thus, we have to move the force upwards by an amount of $h = \frac{4500\ lb\ in}{100\ lb} = 45\ in.$



9. <u>Example</u>



Based on Problem 2.96 of the text, we have a rigid body of a trapezoidal shape and dimensions shown on which three forces act. We are asked to find the points on the *x*- and *y*-axes through which the resultant of these forces must pass.

This problem can be solved in different ways, such as actually finding the resultant and determining where its line of action intercepts the axes. For the sake of variety, however, we are going to proceed somewhat differently.

Clearly, the sum of the moments of the three given forces with respect to any point lying on the line of action of the (unknown) resultant must vanish. [Question: Why?] This being the case, let us consider a point *A* on the *x*- axis and let us denote its *x*- coordinate by x_0. We take moments with respect to this putative *x*-intercept $(x_0, 0)$ and equate the result to zero, namely,

$$(25kN)(5m) - (30kN)(9m - x_0) - (20\ kN)(\cos 30°)(9m - x_0) \\ - (20\ kN)(\sin 30°)(5m) = 0$$

Solving, we obtain

$$x_0 = \frac{-(25)(5) + (30)(9) + (20\cos 30°)(9) + (20)(\sin 30°)(5)}{30 + 20\cos 30°}\ m = 7.42m$$

Analogously, if we take a point *B* on the *y*-axis with an ordinate y_0 and if we equate to zero the moment of the three forces with respect to this point $(0, y_0)$, we obtain

$$(25kN)(5m - y_0) - (30kN)(9m) - (20\ kN)(\cos 30°)(9m) \\ - (20\ kN)(\sin 30°)(5m - y_0) = 0$$

Solving,

$$y_0 = \frac{-(25)(5) + (30)(9) + (20\cos 30°)(9) + (20)(\sin 30°)(5m)}{-25 + 20\sin 30°}\ m = -23.39m$$

We conclude that the line of action of the resultant is determined by these two intercepts.

Digression No. 4: Although we have used a justifiably general procedure, we could have started in this particular case by noticing that the given system of forces is actually concurrent! Indeed, all three forces concur at the vertex *C*. We use this fact to verify our solution. Thus, we must have equal slopes

$$\frac{y_C - y_A}{x_C - x_A} = \frac{y_A - y_B}{x_A - x_B}$$

$$\frac{5 - 0}{9 - 7.42} = \frac{0 - (-23.39)}{7.42 - 0} = 3.15 \qquad \surd$$

Further confirmation is obtained by calculating the resultant vector as

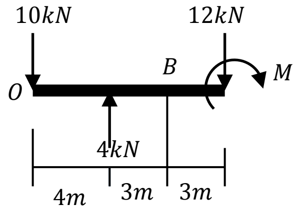
$$\boldsymbol{R} = \left((-25 + 20\sin 30°)\mathbf{i} + (-20\cos 30° - 30)\boldsymbol{j}\right)\ kN$$

The slope is

$$\frac{-20\cos 30° - 30}{-25 + 20\sin 30°} = 3.15 \qquad \surd$$

10. <u>Example</u>

Just as forces, couples too can be applied directly to solid objects, at least as an ideal approximation to reality. The following example is based on Problem 2.86 of the text. In the beam shown it is required to adjust the value of the applied couple M so that the resultant of the system passes through point B. Moreover, we are asked to find the equivalent force-couple system at O.



<u>Solution</u>: We reason as follows. Since the resultant must pass through B, it should be clear that the total moment (of the forces and the applied couple) must vanish. [Recall that we define 'the' resultant as a single force (without a couple) that is equivalent to the system]. Thus we obtain the condition

$$\Sigma M_B = (10kN)(7m) - (4kN)(3m) - (12kN)(3m) - M = 0$$

We obtain

$$M = 22kNm$$

Notice that a positive M means "as shown" in the figure, since we already assumed it to have a definite sense (CW). Had we pictured M as CCW, the result would have been negative, meaning "opposite to shown in figure". The resultant is obtained by adding the forces, which in this case is an elementary task. Finally, to find an equivalent force-couple system at O we introduce the corresponding couple of $(18kN)(7m) = 126kNm$, as shown below.