1. <u>Reduction of a 3D system of forces and couples to a point</u>

We have already established (in the third installment of these notes) the following general principle:

Every system of forces (and couples) can be reduced to an equivalent force-couple system, namely, to a single force $\boldsymbol{R}$ (passing through any pre-assigned point P) and a single couple $\boldsymbol{C}$.

This statement is one of the cornerstones of Statics. Recall that two systems of forces and couples are said to be *statically equivalent* if: (i) the vector sum of the forces is the same for both systems; (ii) the sum of the moments of all forces and couples with respect to any point P is the same for both systems. It is not difficult to prove that if the first condition is satisfied, then, if the second condition holds true for one point, it also holds true for any other point.

According to the theory that we developed in previous weeks, the procedure to reduce any system of forces and couples to an equivalent system consisting of a single force $\boldsymbol{R}$ passing through a point P and a couple $\boldsymbol{C}$ is the following

a) Let the forces of the system be labeled as $\boldsymbol{F}_1, \dots, \boldsymbol{F}_n$. Then the force $\boldsymbol{R}$ is obtained as the vector sum $\boldsymbol{F} = \boldsymbol{F}_1 + \cdots + \boldsymbol{F}_n$.
b) Let the couples of the system (if any) be labeled as $\boldsymbol{C}_1, \dots, \boldsymbol{C}_m$. Then the couple $\boldsymbol{C}$ is the vector sum of these m couples plus the vector sum of the moments of the n forces with respect to point P, namely,

$$\boldsymbol{C} = \boldsymbol{C}_1 + \cdots + \boldsymbol{C}_m + \boldsymbol{M}_P^{\boldsymbol{F}1} + \cdots + \boldsymbol{M}_P^{\boldsymbol{F}n}$$

Stated more explicitly, let $A_1, \dots, A_n$ be, respectively, points along the lines of action of the forces $\boldsymbol{F}_1, \dots, \boldsymbol{F}_n$. Then the couple $\boldsymbol{C}$ is given by

$$\boldsymbol{C} = \boldsymbol{C}_1 + \cdots + \boldsymbol{C}_m + \boldsymbol{r}_{PA_1} \times \boldsymbol{F}_1 + \cdots + \boldsymbol{r}_{PA_m} \times \boldsymbol{F}_m$$

or more compactly as

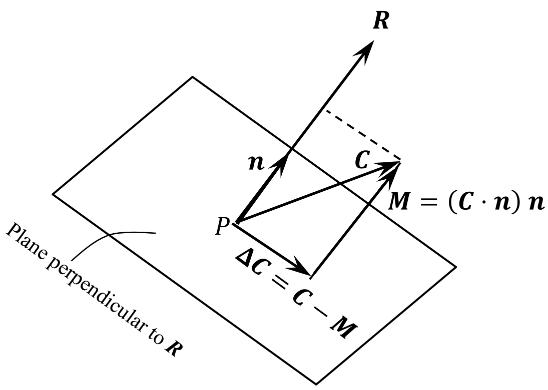
$$\boldsymbol{C} = \sum_{i=1}^{\boldsymbol{m}} \boldsymbol{C}_{\boldsymbol{i}} + \sum_{j=1}^{\boldsymbol{n}} \boldsymbol{r}_{PA_j} \times \boldsymbol{F}_j$$

Notice that the resultant force $\boldsymbol{R}$ is *independent of the point P chosen*. The couple $\boldsymbol{C}$, on the other hand, depends in general of this choice.

2. <u>Main difference with the 2D case</u>

The principle established in the previous paragraph applies equally well to systems in 2 or 3 dimensions. In the 2D case, however, we were able to effect a further simplification. Indeed, by composing the force $\boldsymbol{R}$ with the couple $\boldsymbol{C}$, it was possible to eliminate the couple altogether and obtain an equivalent system consisting of just the resultant force $\boldsymbol{R}$ acting along a new (parallel) line of action. This further reduction is in general not affordable in three dimensions. To understand why this is the case, consider a force $\boldsymbol{F}$ in space. We know full well that if we want to move this force to a new parallel line of action, we must pay a "price" consisting of the moment of $\boldsymbol{R}$ with respect to a point in the new line of action. But, by the properties of the vector product, this moment is a vector *necessarily perpendicular to* $\boldsymbol{R}$! In other words, if the couple $\boldsymbol{C}$ has a non-vanishing projection in the line of action of $\boldsymbol{R}$, there is no way to eliminate this component of $\boldsymbol{C}$. It follows from this brief reasoning that the most that we can hope for is the elimination of the components of $\boldsymbol{C}$ on a plane perpendicular to $\boldsymbol{R}$. This elimination is always possible by choosing the new line of action judiciously. We conclude therefore that:

In contradistinction with the 2D case, a 3D system of forces and couples cannot in general be reduced to a single force, but only to a so-called "wrench", consisting of a force $\boldsymbol{R}$ and a couple $\boldsymbol{M}$ along the direction of the line of action of $\boldsymbol{R}$. This is the kind of force-couple system that we exert on a screwdriver or on a tunnel digging device.

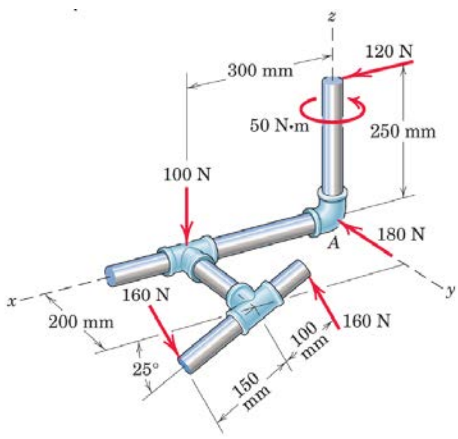


In the figure, we show the vectors $\boldsymbol{R}$ and $\boldsymbol{C}$ obtained after reducing the system to a point P. The couple $\boldsymbol{C}$ can be resolved into the sum of a vector $\boldsymbol{M}$ in the direction of $\boldsymbol{R}$ and a vector lying in the plane perpendicular to $\boldsymbol{R}$. To obtain $\boldsymbol{M}$, we first obtain the unit normal along $\boldsymbol{R}$ and then evaluate the projection as $M = \boldsymbol{C} \cdot \boldsymbol{n}$. This is a scalar quantity. The corresponding vector $\boldsymbol{M}$ is obtained immediately as $\boldsymbol{M} = M\,\boldsymbol{n}$. The difference $\Delta\boldsymbol{C} = \boldsymbol{C} - \boldsymbol{M}$ is, therefore, the component perpendicular to $\boldsymbol{R}$. This should be clear by construction. We can verify this perpendicularity by the following simple calculation

$$\Delta\boldsymbol{C} \cdot \boldsymbol{n} = (\boldsymbol{C} - \boldsymbol{M}) \cdot \boldsymbol{n} = \boldsymbol{C} \cdot \boldsymbol{n} - M\,\boldsymbol{n} \cdot \boldsymbol{n} = M - M = 0$$

By moving the force $\boldsymbol{R}$ to a parallel line of action we can take care of the component $\Delta\boldsymbol{C}$, but there is no way to eliminate the component $\boldsymbol{M}$. In other words, the system can be further reduced to the wrench formed by a force equal to $\boldsymbol{R}$ at some point Q and the couple $\boldsymbol{M}$.

3. <u>Example</u>: This is taken from Problem 2.155 in the text. We are asked to find an equivalent force-couple system at A.



<u>Solution</u>: We could organize the calculation in tabular form and proceed more systematically, but we choose to scan the figure intuitively and indicate the corresponding operations. Adding the forces, we obtain

$$\boldsymbol{R} = (120N)\boldsymbol{i} - (180N)\boldsymbol{j} - (100N)\boldsymbol{k}$$

Needless to say, the 50 Nm couple and the couple formed by the two 160N forces do not contribute to the equation of sum of forces, for obvious reasons.

Notice that this last couple is represented by a vector along the *y* axis. We obtain, therefore,

$$\boldsymbol{C} = (50Nm)\boldsymbol{k} + (0.1m + 0.15m)(160N)\boldsymbol{j} + (0.25m\ \boldsymbol{k}) \times (120N\ \boldsymbol{i}) + (0.3m\ \boldsymbol{i}) \times (-100N\ \boldsymbol{k})$$

Performing the operation indicated, we finally get

$$\boldsymbol{C} = (100\ \boldsymbol{j} + 50\ \boldsymbol{k})\ Nm$$

Although not specified in the problem, let us try to reduce this result (that is, the pair $\boldsymbol{R}, \boldsymbol{C}$) to a wrench (that is, to a pair $\boldsymbol{R}, \boldsymbol{M}$), where $\boldsymbol{M}$ acts along the line of action of $\boldsymbol{R}$. The unit vector $\boldsymbol{n}$ parallel to $\boldsymbol{R}$ and oriented accordingly is given by

$$\boldsymbol{n} = \frac{\boldsymbol{R}}{R} = \frac{120\boldsymbol{i} - 180\boldsymbol{j} - 100\boldsymbol{k}}{\sqrt{120^2 + 180^2 + 100^2}} = 0.504\ \boldsymbol{i} - 0.755\ \boldsymbol{j} - 0.420\ \boldsymbol{k}$$

The projection $\boldsymbol{C}$ onto $\boldsymbol{n}$ is $M = \boldsymbol{C} \cdot \boldsymbol{n} = (-75.5 - 21.0)Nm = -96.5\ Nm$. The moment $\boldsymbol{M}$ in the wrench is, therefore,

$$\boldsymbol{M} = M\ \boldsymbol{n} = (-48.64\ \boldsymbol{i} + 72.86\ \boldsymbol{j} + 40.53\ \boldsymbol{k})\ Nm$$

The remainder of the couple is the difference

$$\boldsymbol{\Delta C} = \boldsymbol{C} - \boldsymbol{M} = (48.64\ \boldsymbol{i} + 27.14\ \boldsymbol{j} + 9.47\ \boldsymbol{k})\ Nm$$

If everything has been done correctly, this vector $\Delta\boldsymbol{C}$ should be perpendicular to $\boldsymbol{n}$. Let us verify that this is indeed the case. We calculate

$$\Delta\boldsymbol{C} \cdot \boldsymbol{n} = (48.64)(0.504\ ) + (27.14)(-0.755) + \ (9.47)(-0.420\ ) = 0.05 \cong 0$$

Let us find the point *B* at which the new line of action intersects the plane *xy*. Clearly, in order to eliminate $\Delta\boldsymbol{C}$, the moment of $\boldsymbol{R}$ (acting at *A*) with respect to this point *B* must be equal to $-\Delta\boldsymbol{C}$. Denoting the coordinates of *B* by $(x_B, y_B, 0)$, we obtain

$$(x_B\boldsymbol{i} + y_B\boldsymbol{j}) \times (120\boldsymbol{i} - 180\boldsymbol{j} - 100\boldsymbol{k})N = -(48.64\ \boldsymbol{i} + 27.14\ \boldsymbol{j} + 9.47\ \boldsymbol{k})\ Nm$$

We obtain, therefore, the three scalar equations

$$-100y_B = -48.64m$$
$$100x_B = -27.14m$$
$$180x_B + 120y_B = 9.47m$$

The determinant of the coefficient matrix of this system is zero, indicating (as it should) that only two of the three equations are linearly independent. Solving the first two, we obtain

$$x_B = -0271\ m \qquad y_B = 0.486\ m$$

It can be checked that this solution also satisfies the third equation (within the round-off error of our calculations). The physical reason for the linear dependence of the three equations is that we know a priori that the moment will necessarily be contained in a plane perpendicular to $\boldsymbol{R}$.

4. Equilibrium

We have already introduced and discussed the concept of equilibrium in a two-dimensional context. You can refer back to the 4th instalment of these notes for general considerations. As we did in the 2D case, it is possible to *define* equilibrium by stating that *a system of forces is in equilibrium whenever the equivalent force-couple system, obtained by reducing the system to a point, vanishes*. In other words, equilibrium implies that *the vector sum of all the forces of the system as well as the vector sum of the moments of all forces (and couples) with respect to any point vanish*. We obtain, accordingly, the two vector equations

$$\boldsymbol{R} = \boldsymbol{F}_1 + \cdots + \boldsymbol{F}_n = \sum_{j=1}^{\boldsymbol{n}} \boldsymbol{F}_j = \boldsymbol{0}$$

$$\boldsymbol{C} = \sum_{i=1}^{\boldsymbol{m}} \boldsymbol{C}_{\boldsymbol{i}} + \sum_{j=1}^{n} r_{PA_j} \times F_j = \boldsymbol{0}$$

The first equation stipulates that the sum of all forces vanishes and the second equation stipulates that the sum of all moments with respect to some point P vanishes as well. Recall that the moment of a couple is a constant, independent of P. If there are couples applied, these couples do not interfere with the equation of sum of forces. Thus, the number of summands in the first equation is, in general, larger than in the second. If we omit this important detail, it is possible to express the two equations above more compactly as

$$\sum \boldsymbol{F} = \boldsymbol{0}$$

$$\sum \boldsymbol{M}_P = \boldsymbol{0}$$

In terms of components in a Cartesian coordinate system, we obtain 6 scalar equations, namely,

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$

$$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0$$

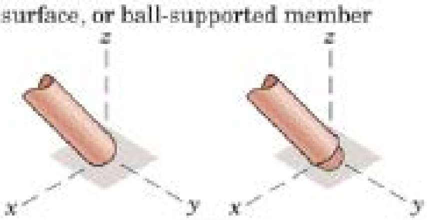
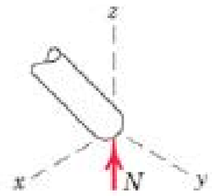
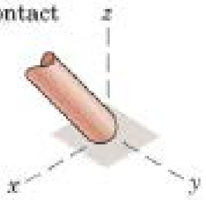
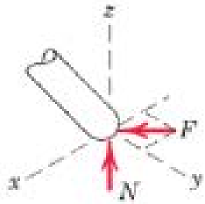
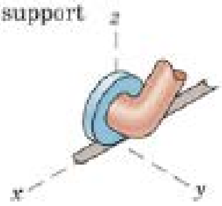
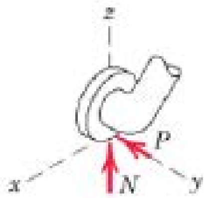
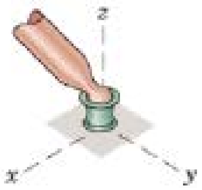
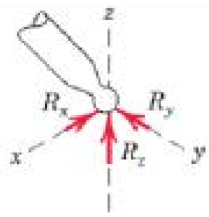
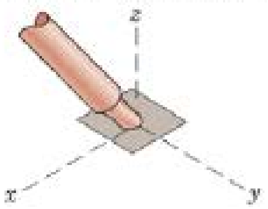
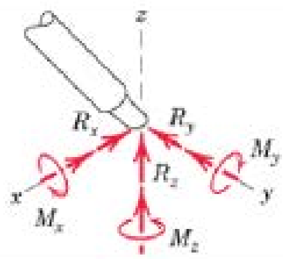
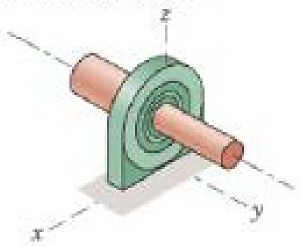
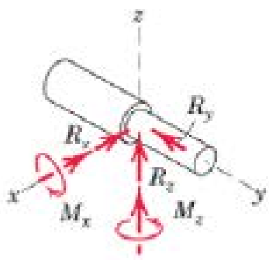
In these equations F_x, F_y, F_z represent the components of the forces and M_x, M_y, M_z represent the components of the moments of the couples and of the forces with respect to a fixed point (the origin, say). Recall that, physically speaking, these components describe the moments with respect to the axes x, y, z, respectively.

The same caveats apply in 3D as in 2D. The first warning has to do with the fact that it is implicitly assumed that these equations are formulated with respect to an *inertial observer*. This is a central concept in classical physics, but we are not going to discuss it any further here. Suffice it to say that the celebrated law of Newton ($\boldsymbol{F} = m\boldsymbol{a}$) is not valid if you are measuring the acceleration with respect to an observer in a merry go round. The second point to be made is that, even if the observer is inertial, the equations of equilibrium are only necessary for the absence of motion. If we want to make sure that a rigid object does not move at all with respect to an inertial observer, we need to provide *supports* that suppress all degrees of freedom of the object regardless of the forces and couples applied. These supports achieve this objective by applying to the object the so-called *support reactions*. These reactions can be forces in the direction of a suppressed displacement or couples along an axis about which a rotation is suppressed. The equations of equilibrium, accordingly, will be satisfied by the system composed of the applied external forces and the support reactions. This total system can be drawn in a free-body diagram.

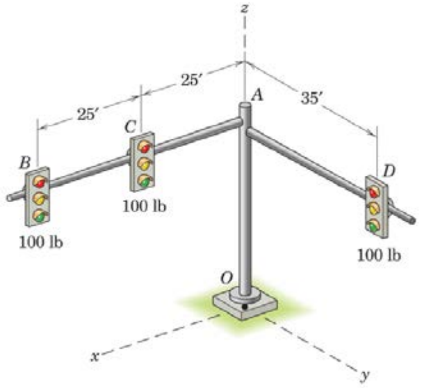
How many degrees of freedom in space does a rigid object have? It can be shown that the most general motion of a rigid object consist of a translation followed by a rotation about a point. A translation can be pinned down in terms of the coordinates of a point P in the body, while a rotation about this point can be described in terms of 3 successive rotations about three distinct axes (such as x, y, z) at P. In short, the most general rigid body motion can be described by means of 6 independent parameters. It follows that, to obtain a minimally supported structure that will be in equilibrium for all applied external forces, we need to provide exactly 6 conditions of support. Any less, and the structure is a mechanism. Any more, and the structure becomes *statically indeterminate*. Thus we see that the six conditions of equilibrium exactly match the number of supports needed to suppress the 6 degrees of freedom. A minimally supported stable structure is, thus, *statically determinate*, since the equations of Statics alone are sufficient to uniquely determine the support reactions. These are the only structures that we can treat in this course.

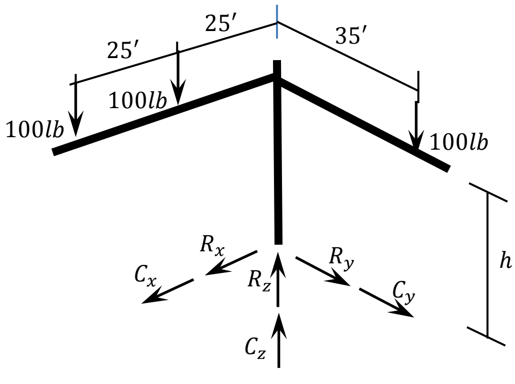
Our textbook provides a useful description of some of the most common types of support, including the reaction (force and/or couple) components that the support can apply. We reproduce this table below, adding the number of degrees of freedom (and, therefore, of reaction components) provided by each support. Given a particular example, it is imperative to draw a free-body diagram (FBD) of the structure *detached from the rest of the universe*, and clearly indicate the support reactions applied by the severed supports. Moreover, it is important to check that the number of support reaction components is exactly six.

Note: If the physical dimensions of the object are very small so that the body can be considered as a point, we do not need to enforce the equations of moments. Correspondingly, the number of degrees of freedom is exactly 3, since in classical physics the notion of rotation of a point is meaningless. Moreover, all the applied forces must concur at the point and no couples can be applied.

Type of contact	Drawing	Forces on member	
1. Member in contact with smooth surface, or ball-supported member			1
2. Member in contact with rough surface			3
3. Roller or wheel support with lateral constraint			2
4. Ball-and-socket joint			3
5. Fixed connection (embedded or welded)			6
6. Thrust-bearing support			5

5. Example: (Problem 3.65) The vertical and horizontal poles at the traffic light assembly are erected first. Determine the additional force and moment reaction s at the base O caused by the addition of the three 100 lb traffic signals B, C, D.
Solution: As in all equilibrium problems, we start by drawing a free-body diagram, clearly indicating the applied forces and couples and the corresponding reaction, as shown in the figure. The support at O is a fixed connection. It restricts all displacements and rotations. Correspondingly, we apply (in the FBD) the 3 components of the force reaction and the 3 components of reactive couple. We assume these components to be positive and let the algebra determine their final signs. Notice that the dimension h (not given) will play no role in the final answer.





We will write the equations of statics in terms of components in the given x, y, z system and by taking moments about the origin O. The position vectors of points B, C, D are, respectively,

$$\boldsymbol{r}_{OB} = (50\boldsymbol{i} + h\boldsymbol{k})ft, \;\; \boldsymbol{r}_{OC} = (25\boldsymbol{i} + h\boldsymbol{k})ft\,, \;\; \boldsymbol{r}_{OD} = (35\boldsymbol{j} + h\boldsymbol{k})ft$$

We obtain the 2 vector equations

$$-(300\ lb)\boldsymbol{k} + R_x\boldsymbol{i} + R_y\boldsymbol{j} + R_z\boldsymbol{k} = \boldsymbol{0}$$

and

$$\boldsymbol{r}_{OB} \times (-100lb\ \boldsymbol{k}) + \boldsymbol{r}_{OC} \times (-100lb\ \boldsymbol{k}) + \boldsymbol{r}_{OD} \times (-100lb\ \boldsymbol{k}) + M_x\boldsymbol{i} + M_y\boldsymbol{j} + M_z\boldsymbol{k} = \boldsymbol{0}$$

or, performing the operations indicated,

$$(7500\,\boldsymbol{j} - 3500\,\boldsymbol{i})\ lb\ ft + C_x\boldsymbol{i} + C_y\boldsymbol{j} + C_z\boldsymbol{k} = \boldsymbol{0}$$

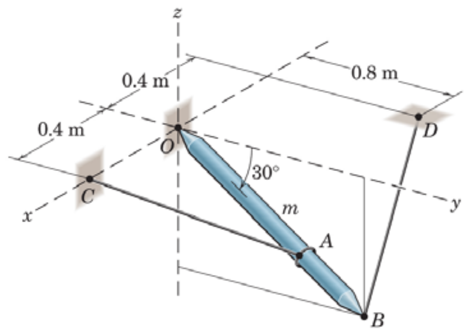
Solving this system component by component we obtain the result as

$$R_x = 0 \quad R_y = 0 \quad R_z = 300\ lb \quad C_x = 3500\ lb\ ft \quad C_y = -7500\ lb\ ft \quad C_z = 0$$

Notice that the reactive couple $\boldsymbol{C} = (3500\ \boldsymbol{i} - 7500\ \mathbf{j})\ lb\ ft$ satisfies our intuition. It corresponds to the attempt by the traffic signals to bend the vertical pole in a vertical plane but not to cause any twisting of this pole about its own axis, as would be the case if horizontal (wind, say) forces were acting on the traffic signals. Put differently, there is no wrench effect.

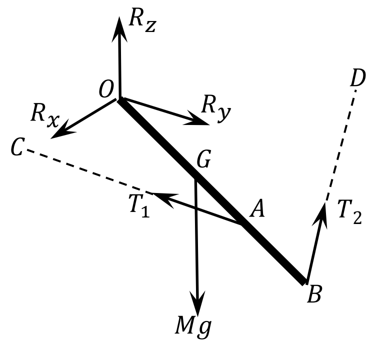
6. Example: (Problem 3.74) The uniform slender rod of mass M is suspended by a ball-and-socket joint at O and two cables. Determine the force reactions at O and the tension in each cable.

Solution: At the outset we must warn that this system is under-supported, since only five conditions of support have been imposed, rather than the necessary six. A cursory examination of the system reveals, indeed, that the bar OB is free to rotate around its own axis. To justify this inconsistency, however, we rely on the fact that the slenderness of the bar is such that the diameter of its cross section can be considered as zero. In this case, just like we claim that a particle cannot sustain moments or rotations, we can claim that this bar cannot sustain moments around its own axis. We will impose only two equations of moments around any two axes that do not coincide with the axis of the bar. If we were to check a third equation of moments, it would result in a linear combination of the other two.



$\overline{OB} = 1.2$ m, $\overline{OA} = 0.9$ m

Since the bar is uniform, the weight can be assumed to be concentrated at its mid-point G. It is convenient to tabulate the coordinates of all the important points.



FBD

	x (m)	y (m)	z (m)
O	0	0	0
A	0	$0.9\cos 30°$	$-0.9\sin 30°$
B	0	$1.2\cos 30°$	$-1.2\sin 30°$
C	0.4	0	0
D	-0.4	0.8	0
G	0	$0.6\cos 30°$	$-0.6\sin 30°$

On the basis of these data, we calculate the following unit vectors

$$\boldsymbol{n}_{AC} = \frac{\boldsymbol{r}_C - \boldsymbol{r}_A}{AC} = 0.406\,\boldsymbol{i} - 0.791\,\boldsymbol{j} + 0.457\,\boldsymbol{k}$$

$$\boldsymbol{n}_{BD} = \frac{\boldsymbol{r}_D - \boldsymbol{r}_B}{BD} = -0.527\,\boldsymbol{i} - 0.315\,\boldsymbol{j} + 0791\,\boldsymbol{k}$$

$$\boldsymbol{n}_{OB} = \frac{\boldsymbol{r}_B - \boldsymbol{r}_O}{OB} = 0.866\,\boldsymbol{j} - 0.500\,\boldsymbol{k}$$

We are in a position to write the force equilibrium equations as

$$R_x\boldsymbol{i} + R_y\boldsymbol{j} + R_z\boldsymbol{k} - Mg\boldsymbol{k} + T_1\boldsymbol{n}_{AC} + T_2\boldsymbol{n}_{BD} = \boldsymbol{0}$$

In components, we obtain

$$R_x + 0.406\,T_1 - 0.527\,T_2 = 0$$
$$R_y - 0.791\,T_1 - 0.315\,T_2 = 0$$
$$R_z - Mg + 0.457\,T_1 + 0.791\,T_2 = 0$$

Taking moments about O yields

$$\underbrace{(0.6m)\boldsymbol{n}_{OB}}_{\boldsymbol{r}_{OG}} \times (-Mg\boldsymbol{k}) + \underbrace{(0.9m)\boldsymbol{n}_{OB}}_{\boldsymbol{r}_{OA}} \times T_1\boldsymbol{n}_{AC} + \underbrace{(1.2m)\boldsymbol{n}_{OB}}_{\boldsymbol{r}_{OB}} \times T_2\boldsymbol{n}_{BD} = \boldsymbol{0}$$

In components, carefully performing the operations indicated, we obtain the three scalar equations

$$-0.52Mg + 0.633\,T_2 = 0$$
$$-0.183\,T_1 + 0.316\,T_2 = 0$$
$$-0.316\,T_1 + 0.547\,T_2 = 0$$

Note that, as anticipated, the third equation is equivalent to the second and can be discarded.

From the first of these last three equations we obtain

$$T_2 = 0.821\,Mg$$

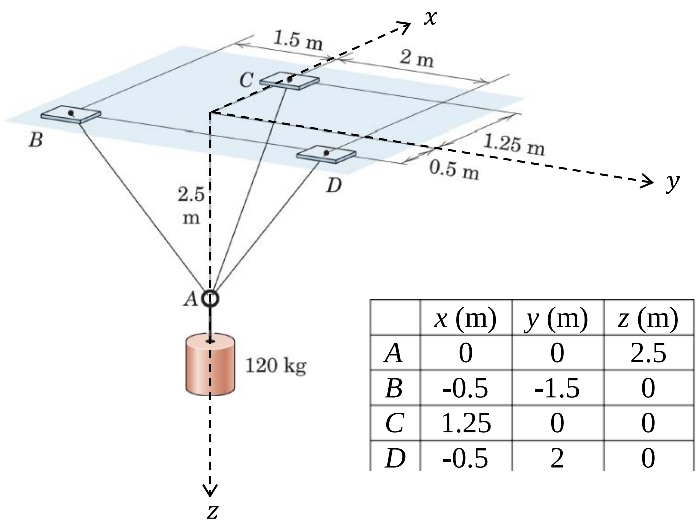
From here (using any of the last two equations) it follows that

$$T_1 = 1.418\,Mg$$

Plugging these values in the equations of force, we complete the solution as

$$R_x = -0.143\,Mg$$
$$R_y = 1.380\,Mg$$
$$R_z = -0.297\,Mg$$

7. Example: (Problem 3.63) Determine the tension in each of the cables.



	x (m)	y (m)	z (m)
A	0	0	2.5
B	-0.5	-1.5	0
C	1.25	0	0
D	-0.5	2	0

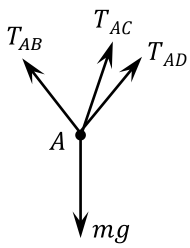
Solution: This a system of concurrent forces and, accordingly, we need only consider the equation of summation of forces (since the moments with respect to the point of concurrence vanish automatically). We start by tabulating the coordinates of the relevant points. Since the coordinate system has not been specified, we adopt the one indicated in the figure.

The unit vectors along the cables are obtained as

$$\boldsymbol{n}_{AB} = \frac{\boldsymbol{r}_B - \boldsymbol{r}_A}{AB} = -0.169\,\boldsymbol{i} - 0.507\,\boldsymbol{j} - 0.845\,\boldsymbol{k}$$

$$\boldsymbol{n}_{AC} = \frac{\boldsymbol{r}_C - \boldsymbol{r}_A}{AC} = 0.447\,\boldsymbol{i} - 0.894\,\boldsymbol{k}$$

$$\boldsymbol{n}_{AB} = \frac{\boldsymbol{r}_D - \boldsymbol{r}_A}{AD} = -0.154\,\boldsymbol{i} + 0.617\,\boldsymbol{j} - 0.772\,\boldsymbol{k}$$



From the free-body diagram of point *A*, we can write the vector equilibrium equation as

$$T_{AB}\,\boldsymbol{n}_{AB} + T_{AC}\,\boldsymbol{n}_{AC} + T_{AD}\,\boldsymbol{n}_{AD} + mg\,\boldsymbol{k} = \boldsymbol{0}$$

This vector equation yields the three scalar equations

$$-0.169\,T_{AB} + 0.447\,T_{AC} - 0.154\,T_{AD} = 0$$
$$-0.507 T_{AB} + 0.617\,T_{AD} = 0$$
$$-0.845\,T_{AB} - 0.894\,T_{AC} - 0.772\,T_{AD} + mg = 0$$

Solving this linear system, we obtain

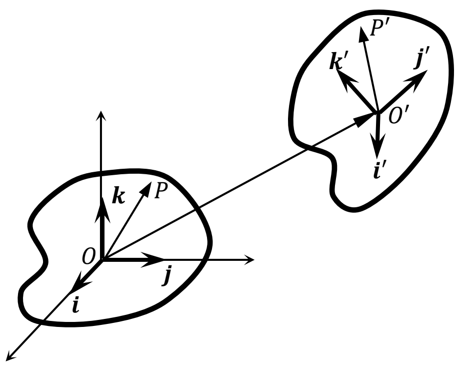
$$T_{AB} = 0.483\,mg \qquad T_{AC} = 0.319\,mg \qquad T_{AD} = 0.397\,mg$$

With $m = 120kg$ and $g = 9.81\,ms^{-2}$,

$$T_{AB} = 569\,N \qquad T_{AC} = 376\,N \qquad T_{AD} = 467\,N$$

8. **<u>Algebraic digression</u>**: <u>Rigid-body motion and orthogonal matrices</u>

We have already intimated in previous weeks that there is a very close connection between Linear Algebra and Statics. When describing the degrees of freedom of a rigid object in space we mentioned that they are 6 in number and that they can be viewed as three components of the displacements of a point followed by three rotations about the coordinate axes. We want to investigate how these rotations can be represented by means of a matrix and what kind of matrix this is. Consider a rigid object and imagine that the unit triad $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ is embedded in it. As the object moves, it will carry the origin O to a point O' and the vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ to new vectors $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$. Since the body is rigid, these new vectors are of unit length and mutually perpendicular (orthogonal). But these vectors (like any other vector) can be expressed in terms of the basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. Thus, for the vector $\boldsymbol{i}'$ we can write



$$\boldsymbol{i}' = (\boldsymbol{i}' \cdot \boldsymbol{i})\,\boldsymbol{i} + (\boldsymbol{i}' \cdot \boldsymbol{j})\,\boldsymbol{j} + (\boldsymbol{i}' \cdot \boldsymbol{k})\,\boldsymbol{k} = i'_x\,\boldsymbol{i} + i'_y\,\boldsymbol{j} + i'_z\,\boldsymbol{k}$$

Let us form the matrix

$$[R] = \begin{bmatrix} i'_x & j'_x & k'_x \\ i'_y & j'_y & k'_y \\ i'_z & j'_z & k'_z \end{bmatrix}$$

It is not difficult to verify that $[R]^T[R] = [I]$. This result follows directly from the fact that the vectors $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ are of unit length and perpendicular. Another way to state this result is to say that $[R]^T = [R]^{-1}$, namely, the transpose of $[R]$ is equal to its inverse! A matrix with this property is said to be an *orthogonal matrix*.

Let P be a point in the body with coordinates x_P, y_P, z_p and let P' denote the position it occupies after the rigid motion takes place. Because of rigidity, it should be clear that

$$\boldsymbol{r}_{O'P'} = x_P\ \boldsymbol{i}' + y_P\ \boldsymbol{j}' + z_P\boldsymbol{k}'$$

Notice that the components are the same x_P, y_P, z_p, due to the assumed rigidity. Only the basis has changed. Now,

$$\boldsymbol{r}_{OP'} = \boldsymbol{r}_{OO'} + \boldsymbol{r}_{O'P'}$$

If we want to obtain the x coordinate of the point P', all we need to do is to take the dot product of this expression with $\boldsymbol{i}$, just as we would do for any other position vector. In other words, we obtain

$$x_{P'} = x_{O'} + x_P\ i'_x + y_P\ j'_x + z_P\ k'_x$$

Proceeding in a similar way for the other two coordinates of P', we can collect all these results into the matrix equation

$$\begin{Bmatrix} x_{P'} \\ y_{P'} \\ z_{P'} \end{Bmatrix} = \begin{Bmatrix} x_{O'} \\ y_{O'} \\ z_{O'} \end{Bmatrix} + [R] \begin{Bmatrix} x_P \\ y_P \\ z_P \end{Bmatrix}$$

This result can be expressed as the fact that every rigid-body motion can be completely encapsulated in a vector representing the displacement of the point initially at the origin of the coordinates and an orthogonal matrix $[R]$. A square matrix in 3 dimensions has, in principle, 9 independent entries. But since $[R]$ is an orthogonal matrix, it satisfies 6 independent identities (the columns are of unit length and mutually perpendicular), so that we are left with just 3 independent entries.

Note: Not all orthogonal matrices represent rotations. It can be shown that the most general orthogonal matrix can represent a rotation followed by a reflection about a plane (like with a mirror). It can also be shown that an orthogonal matrix represents a pure rotation if, and only if, its determinant is equal to $+1$.

Exercise: Prove that the determinant of an orthogonal matrix is either $+1$ or -1.

Exercise: Prove that in dimension 2 the most general orthogonal matrix representing a rotation is necessarily of the form

$$[R] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

In this equation, θ represents the angle of rotation. [In dimension 3, the most general rotation matrix can be represented by means of 3 angles, such as the so-called *Euler angles*, used in the description of gyroscopic motion].