University of Calgary

Schulich School of Engineering

Engineering Statics

ENGG 202

Winter 2016

Instructors:

Ahmad Ghasemloonia

Marcelo Epstein

Robyn Paul

These notes belong to: ___________________________________

These lecture notes will be supplemented with additional information and explanation by the instructor during lectures periods.

Preface

These lecture notes were initially developed by Ahmad Ghasemloonia for teaching Engineering Statics at the University of Calgary in the Spring semester 2015. They were adapted with permission from Engineering Statics (Hibbeler, 13th edition), lecture notes of ENGG 202 (The University of Calgary, Winter semester 2015) and lecture notes of Dr. Neil Hookey (Memorial University, 2015). The structure of the materials and examples are inspired by my experience while conducting tutorials at Cahill Engineering One Help Centre and substitute lecturing duties at Memorial University. A special thank you to Dr. Geoff Rideout and Dr. Neil Hookey at Memorial University for their expert discussions on teaching philosophies, preparation of course notes and lecturing of engineering course. Robyn Paul at the University of Calgary provided input and edits for the improved notes for the Winter 2016 version.

Ahmad Ghasemloonia, Ph.D., P.Eng.

Winter 2016

Table of Contents

Chapter 1

Introduction and General Principles

Chapter 1: Introduction and General Principles

1.1: Mechanics

Mechanics is a branch of the physical sciences that is concerned with the state of rest or motion of bodies that are subjected to the action of forces. In general, this subject can be subdivided into three branches:

- Rigid-body mechanics
- Deformable-body mechanics
- Fluid mechanics

In this course we will study rigid-body mechanics (bodies do not deform from applied forces). Rigid body mechanics is divided into two areas:

- Statics – equilibrium of bodies (*i.e.* those at rest or moving with a constant velocity)
- Dynamics – the accelerated motion of bodies

Galileo Galilei (1564–1642) was one of the first major contributors to this field. His work consisted of experiments using pendulums and falling bodies. The most significant contributions in dynamics, however, were made by Isaac Newton (1642–1727), who is noted for his formulation of the three fundamental laws of motion and the law of universal gravitational attraction. Shortly after these laws were postulated, important techniques for their application were developed by such notables as Euler, D'Alembert, Lagrange, and others.

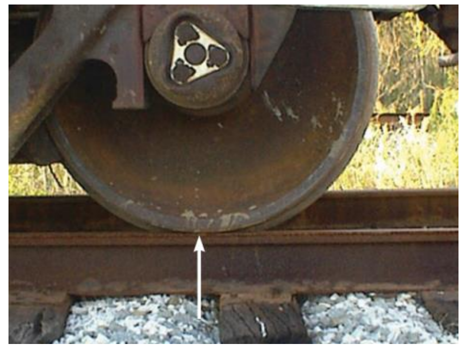
1.2: Definitions

Particle: A particle has a mass, but a size that can be neglected. When a body is idealized as a particle, the principles of mechanics are simplified since the **geometry** of the body will **NOT** be involved in the analysis of the problem.

Rigid body: A rigid body can be considered as a large number of particles where all particles remain at a fixed distance from one another. This distance remains fixed both before and after applying a load.

- The rigid body assumption is suitable for analysis because typically actual deformations occurring in structures, machines, mechanisms, and the like are relatively small.
- In rigid bodies **geometry** of the body will be **considered** in the analysis of the problem.
- The body's shape does not change when applying loads, thus we do not consider the material type.

Concentrated force: A concentrated force represents the effect of a loading which is assumed to act at one point on a body. A load can be represented by a concentrated force when the area over which the load is applied is very small relative to the overall size of the body. An example would be the contact force between a wheel and the ground.



1.3: Newton's Three Laws of Motion

These rules are defined with respect to a non-accelerating frame of reference (*i.e.* fixed, or moving in a straight line at constant speed).

1st Law: Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it (concept of inertia). Uniform motion includes:

- Particles at rest
- Particles moving in a straight line with constant speed

2nd Law: A particle acted upon by an external force will experience acceleration proportional to the force. If the particle has mass, m:

$$\sum \vec{F} = m\vec{a}$$

- For Statics: $\sum \vec{F} = 0$.

3rd Law: For every action there is an equal, opposite and collinear reaction.

1.4: Units of Measurement

Two measurement units are commonly used in engineering applications:

- SI Units: The International System of units (SI)
- U.S. Customary system of units (FPS)

Measurement unit	**SI Units**	**FPS Units**
Length	meters (m)	feet (ft)
Time	seconds (s)	seconds (s)
Mass	kilograms (kg)	slug
Force	newton (N)	pounds (lb)
Gravity Acceleration	$g = 9.8\ \frac{m}{s^2}$	$g = 32.2\ \frac{ft}{s^2}$

Table below provides a set of direct conversion factors between FPS and SI units for basic quantities.

Conversion Factors			
Quantity	Unit of Measurement (FPS)	Equals	Unit of Measurement (SI)
Force	lb		4.448 N
Mass	slug		14.59 kg
Length	ft		0.3048 m

<u>Note</u>: Although discussed in Chapter 1 of the textbook, in ENGG 202 students need not concern themselves with significant digits. Rather, it is just important to carry enough figures and not round too much in intermediate steps. This will ensure the resulting answer is not significantly affected. At the same time, showing too many digits after the decimal place is also not necessary and only makes it more difficult to present clear solutions for grading on examinations.

1.5: General Procedure for Analysis

The most effective way of learning the principles of engineering mechanics is to **solve problems**. To be successful, it is important to always present your work in a logical and orderly manner, as suggested:

- Read the problem carefully
- **Draw** any necessary diagrams with a **large scale**
- **Summarize** the problem data
- Apply the relevant principles, generally in mathematical form
- Check the equations are **dimensionally homogeneous**
- Solve the necessary equations
- **Clearly indicate** the final answer with no more than two significant figures
- Review the answer with **technical judgment & common sense** to determine if it seems reasonable

Example 1-1: If an object has a mass of 40 slugs, determine its mass in kilograms.

Example 1-2: If a man weighs 155 lb on earth, specify (a) his mass in slugs, (b) his mass in kilograms, and (c) his weight in newtons.

Chapter 2

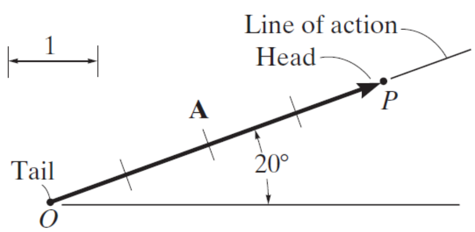
Force Vectors

Chapter 2: Force Vectors

2.1: Scalars and Vectors

All physical quantities in engineering mechanics are measured using either scalars or vectors.

- **Scalar**: A scalar is any positive or negative physical quantity that can be completely specified by its magnitude. Examples of scalar quantities include length, mass and time.
- **Vector**: A quantity that has a magnitude and direction. Examples of vectors encountered in statics are force, position and moment.



A car on the highway is going:

- a **speed** of $100\ km/h$ – this is a scalar (there is no information about direction)
- a **velocity** of the car would be represented as, for example, $100\ km/h$, North – this is a vector (magnitude and direction)

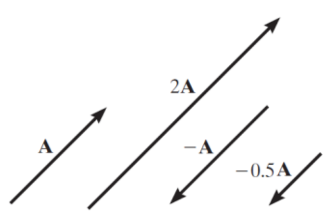
Vector quantities are represented by boldface letters such as **A**, or by simply drawing an arrow above it, $\vec{A}$.

2.2: Force Vector Operations

A force is a vector quantity since it has a specified magnitude, direction, and sense.

2.2.1: Multiplication and Division of a Force Vector by a Scalar

If a force vector is multiplied by a positive scalar, its magnitude is increased by that amount. Multiplying by a negative scalar will also change the directional sense of the force vector.



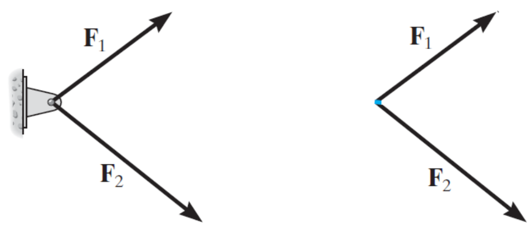
2.2.2: Resultant Force

Force vectors can be added together with:

- the parallelogram law of addition
- the triangle rule
- the algebraic method (will be discussed in Section 2.3)

Parallelogram law of force vector addition: The two component forces, $\vec{F}_1$ and $\vec{F}_2$, acting on a pin can be added together to form the resultant force, $\vec{F}_R = \vec{F}_1 + \vec{F}_2$, with the parallelogram law of addition using the following procedure:

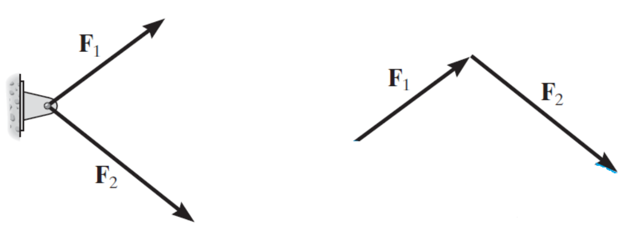
- Join the tails of the components at a point to make them concurrent.
- From the head of $\vec{F}_1$, draw a line parallel to $\vec{F}_2$. Draw another line from the head of $\vec{F}_2$ parallel to $\vec{F}_1$. These two lines intersect at point *P* to form the adjacent sides of a parallelogram.
- The diagonal of this parallelogram that extends to P forms $\vec{F}_R$, representing the resultant vector.



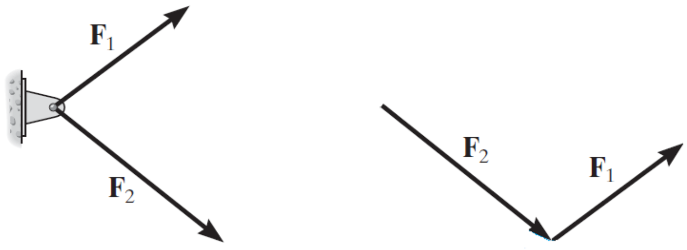
NOTE: Force vectors in this method must be drawn **to scale**!!!

Triangle rule of force vector addition: We can also add $\vec{F}_2$ to $\vec{F}_1$ using the triangle rule, which is a special case of the parallelogram law. Force vector $\vec{F}_1$ is added to force vector $\vec{F}_2$ in a "head-to-tail" fashion.

- Connect the head of $\vec{F}_1$ to the tail of $\vec{F}_2$
- The resultant $\vec{F}_R$ extends from the tail of $\vec{F}_1$ to the head of $\vec{F}_2$.



In a similar manner, $\vec{F}_R$ can also be obtained by adding $\vec{F}_1$ to $\vec{F}_2$.

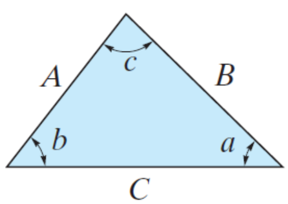


By comparison, it is evident that vector addition is commutative; in other words, the vectors can be added in either order, *i.e.*, $\vec{F}_R = \vec{F}_1 + \vec{F}_2 = \vec{F}_2 + \vec{F}_1$

Trigonometry rule: Starting with the triangular head-to-tail addition of components:

- the magnitude of the resultant force is determined using the law of cosines
- the direction of the resultant force is determined from the law of sines
- the magnitudes of two force components are determined from the law of sines

NOTE: This method can be used to add 2 vectors only and scale is not important.

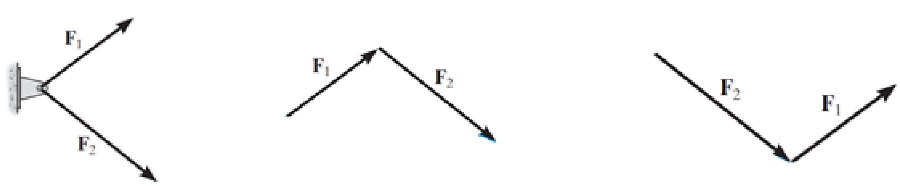


Cosine law:

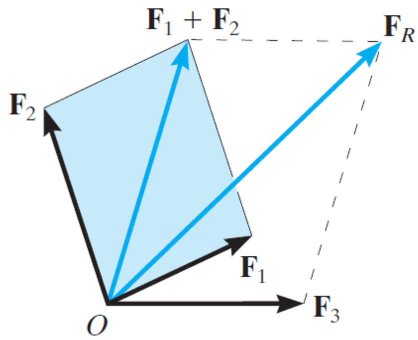
$$C = \sqrt{A^2 + B^2 - 2AB \cos c}$$

Sine law:

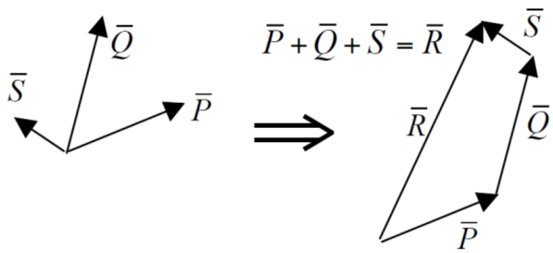
$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$$



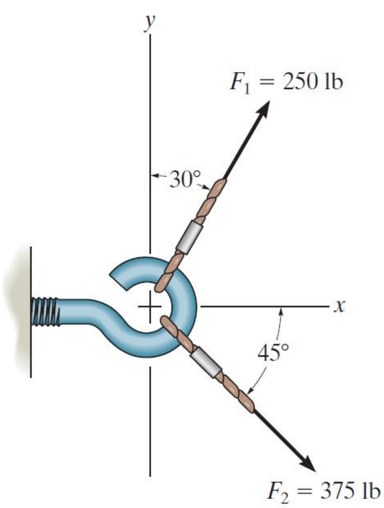
Addition of several force vectors: If more than two forces are to be added, successive applications of the parallelogram law can be carried out to obtain the resultant force. For example, if three forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ act at a point *O*, the resultant of any two of the forces is found, say, $\vec{F}_1 + \vec{F}_2$, and then this resultant is added to the third force, yielding the resultant of all three forces; *i.e.*, $\vec{F}_R = (\vec{F}_1 + \vec{F}_2) + \vec{F}_3$.



Also, the triangle rule can also be implemented to add several force vectors.



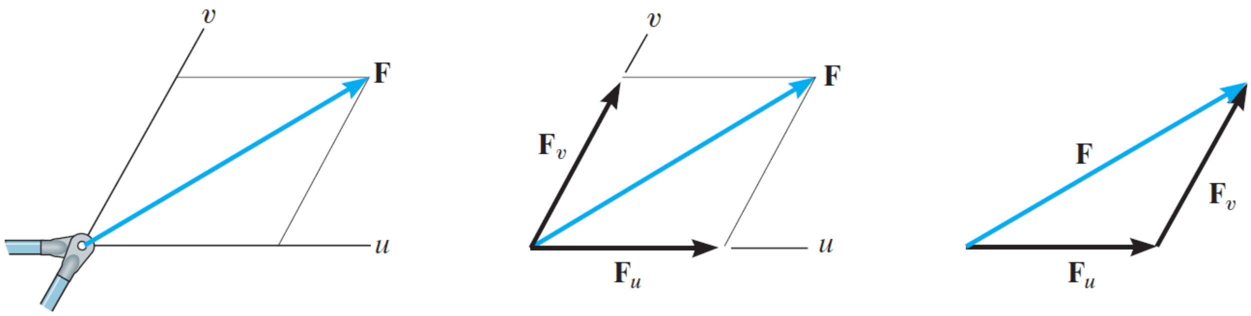
Example 2.1: Determine the magnitude of the resultant force $\vec{F}_R = \vec{F}_1 + \vec{F}_2$ and its direction, measured counterclockwise from the positive *x*-axis. Solve the problem with both the parallelogram and the triangle rules of vector addition.



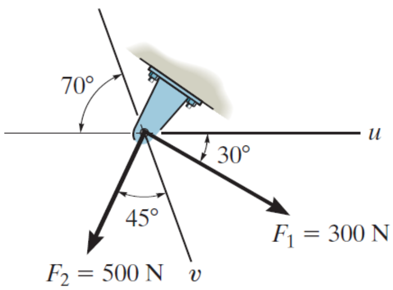
2.2.3: Finding the Components of a Force

Sometimes a force must be resolved into two components to study the pulling or pushing effect in two specific directions. For example, in the figure below, $\vec{F}$ is to be resolved into two components along the two members, defined by the *u* and *v* axes. To determine the magnitude of each component:

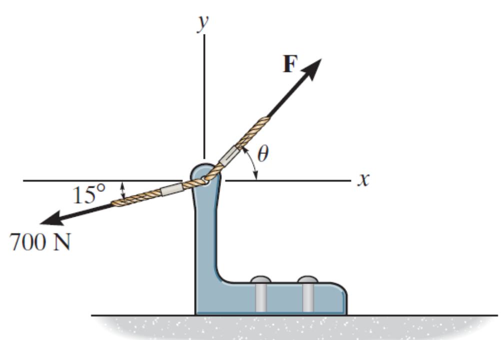
- A parallelogram is constructed by drawing lines starting from the head of $\vec{F}$
  - One line is parallel to *u*, and the other line is parallel to *v*
  - These lines then intersect with the *v* and *u* axes, forming a parallelogram
- To find force components $\vec{F}_u$ and $\vec{F}_v$, the tail of $\vec{F}$ is joined to the intersection points on the axes.
- This parallelogram can then be reduced to a triangle, which represents the triangle rule.
- From this, the law of sines can be used to determine the unknown magnitudes of the components.



Example 2.2: Resolve force $\vec{F}_2$ into components acting along the *u* and *v* axes and determine the magnitudes of the components.



Example 2.3: If the magnitude of the resultant force is to be 500 N, directed along the positive *y* axis, determine the magnitude of force $\vec{F}$ and its direction θ.



2.3: Addition of a System of Coplanar Forces

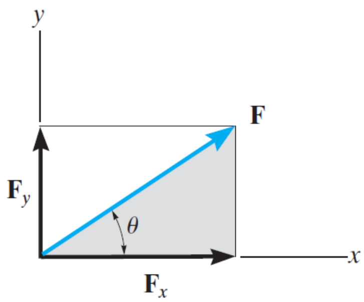
The graphical methods discussed are good to show physically what is happening. However, they are limited to two-dimensions and are inconvenient in real problems or for more than two coplanar forces.

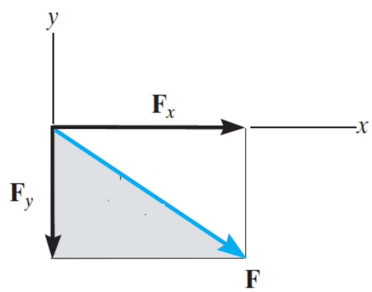
We need a mathematical means for adding vectors with components along a standard coordinate system.

- For coplanar forces we will use an x, y (Cartesian) coordinate system (x and y are perpendicular)
- For analytical work we can represent these components in one of two ways
  - Scalar notation
  - Cartesian vector notation

2.3.1: Scalar Notation

The rectangular components of force $\vec{F}$ in the figure below are found using the parallelogram law, so that $\vec{F} = \vec{F}_x + \vec{F}_y$. Because these components form a right triangle, they can be resolved as:

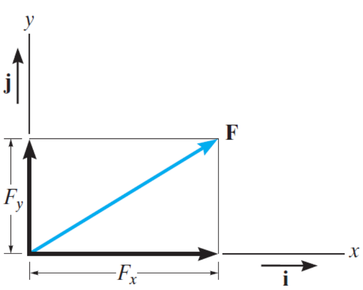




NOTE: This positive and negative scalar notation is only for computational purposes, NOT for graphical representations in figures.

2.3.2: Cartesian Vector Notation

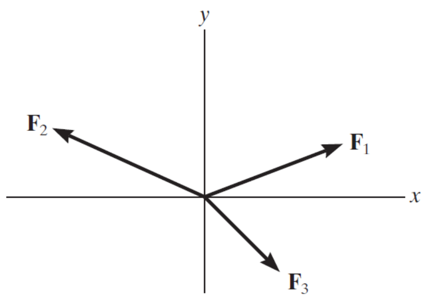
The *x* and *y* force components can also be represented in terms of the Cartesian unit vectors $\hat{\imath}$ and $\hat{\jmath}$. They are **unit vectors** because they have a dimensionless magnitude of 1, and they designate the directions of the *x* and *y* axes, respectively. Since the magnitude of each component of $\vec{F}$ is always a positive quantity (as represented by the positive scalars F_x and F_y), we can express $\vec{F}$ as a Cartesian vector:



2.3.1: Coplanar Force Resultants

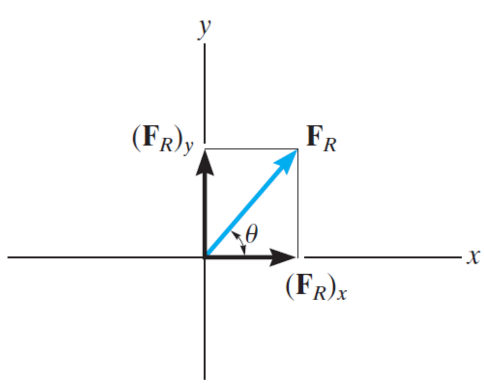
We can use either of the two methods described above to determine the resultant of many coplanar forces:

- Each force is first resolved into its *x* and *y* components.
- The respective components are added using scalar algebra since they are collinear.
- The resultant force is formed by adding the resultant components using the parallelogram law.

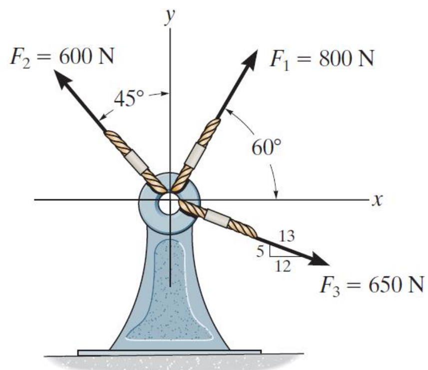


We can represent the components of the resultant force of any number of coplanar forces symbolically by the algebraic sum of the *x* and *y* components of all the forces:

Once these components are determined, they can be sketched along the *x* and *y* axes with their proper direction. The resultant force is determined from vector addition:

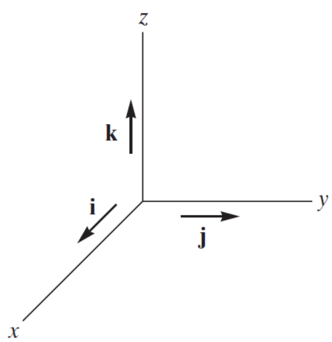


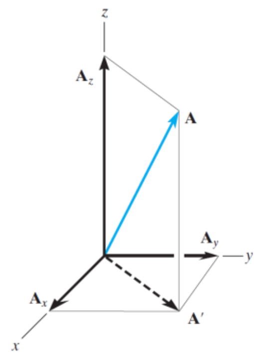
Example 2.4: Determine the magnitude of the resultant force and its direction.



2.4: Cartesian Vectors- 3D

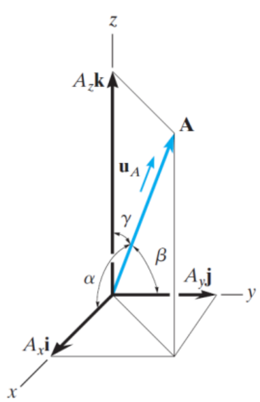
In three dimensions, the set of **Cartesian unit vectors**, $\hat{\imath}$, $\hat{\jmath}$, k, designates the directions of the *x*, *y*, *z* axes, respectively. A vector may have one, two, or three rectangular components along the *x*, *y*, *z* coordinate axes, depending on how the vector is oriented relative to the axes.





It is always possible to obtain the magnitude of $\vec{A}$ provided it is expressed in Cartesian vector form.

We define the direction of $\vec{A}$ by the coordinate direction angles α, β, and γ. These angles are measured between the tail of $\vec{A}$ and the positive *x*, *y*, *z* axes. These angles are the **direction cosines** of $\vec{A}$. To determine direction cosines, consider the projection of $\vec{A}$ onto the *x*, *y*, *z* axes:



These direction cosines are easily obtained by forming a **unit vector** $\vec{u}_A$ in the direction of $\vec{A}$. If $\vec{A}$ is expressed in Cartesian vector form, $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$, then $\vec{u}_A$ will have a magnitude of **one** and be dimensionless provided $\vec{A}$ is divided by its magnitude:

NOTE: To add 3D vectors is similar to adding vectors in 2D (in a plane):

Example 2.5: For force $\vec{F} = 60\hat{\imath} - 50\hat{\jmath} + 40\hat{k}\ N$:

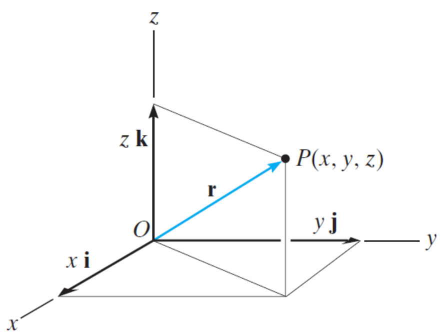
a) Determine the magnitude and coordinate direction angles of $\vec{F}$.

b) Find the unit vector of $\vec{F}$.

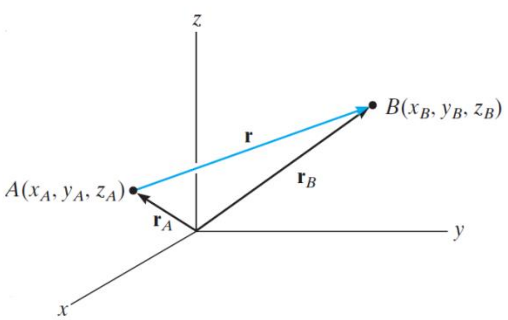
c) Find the resultant force of a system including forces $\vec{F}$ and $\vec{A} = -60\hat{\imath} + 20\hat{\jmath} - 30\hat{k}\ N$.

2.5: Position and Force Vectors

A position vector $\vec{r}$ is defined as a fixed vector which locates a point in space relative to another point. For example, if $\vec{r}$ extends from the origin (O), to point $P(x, y, z)$, then $\vec{r}$ in Cartesian vector form is:

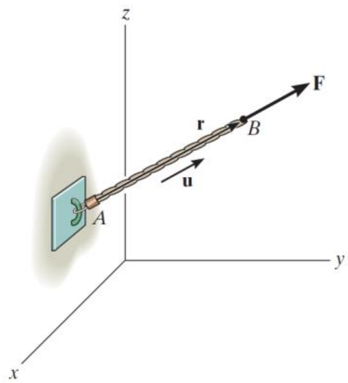


Generally, the position vector may be directed from point A to B in space. Then, the position vector $\vec{r}$ is defined as:

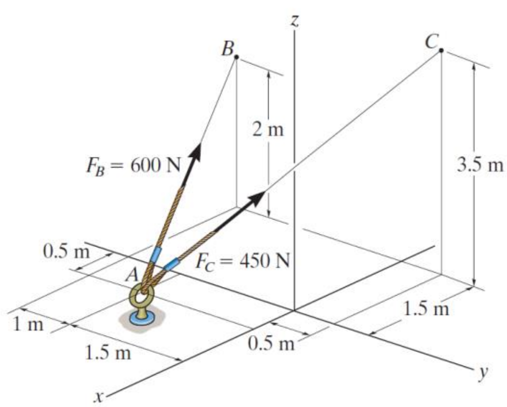


Often in 3D problems, the force direction is specified by two points that pass through its line of action. In figure below the force $\vec{\mathrm{F}}$ is directed along the cord AB and can be formulated as a Cartesian vector:

- It has the same direction and sense as position vector $\vec{r}$ from point A to B along the cord.
- This direction is specified by the unit vector $\vec{u}_r$ of position vector $\vec{r}$.



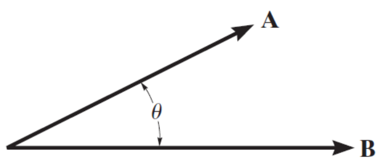
Example 2.6: Express $\vec{F}_B$ in Cartesian vector form.



2.6: Dot Product

Occasionally in statics, we must find the angle between two lines or the components of a force parallel and perpendicular to a line. In 2D, these problems can be solved by trigonometry since the geometry is easy to visualize. However in 3D, this is often difficult and consequently vector methods should be employed for the solution. The dot product, which defines a particular method for "multiplying" two vectors, is used to solve these types of problems.

The dot product of vectors $\vec{A}$ and $\vec{B}$, written $\vec{A}.\vec{B}$, is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle θ between their tails:



The dot product is often referred to as the **scalar** product of vectors since the result is a **scalar** and NOT a vector. The commutative law is valid for the dot product of two vectors:

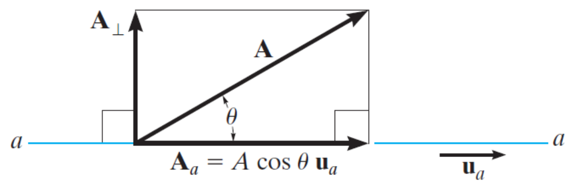
In Cartesian notation, the dot product of two Cartesian vectors $\vec{A}$ and $\vec{B}$ is defined as:

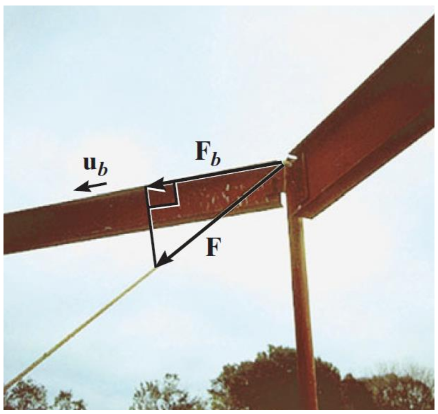
Applications of the dot product:

- The dot product can be implemented to determine the angle between 2 vectors:

- Find the projection (component) of a vector on a line:

The dot product $A_a = \vec{A}.\vec{u}_a$ determines the magnitude of the projection of vector $\vec{A}$ along the line *aa*, where line *aa* has a direction specified by $\vec{u}_a$.

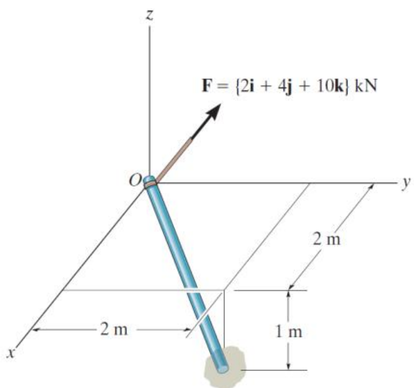




The projection of the cable force $\vec{F}$ along the beam can be determined by first finding the unit vector $\vec{u}_b$ that defines this direction. Then apply the dot product:

$$F_b = \vec{F}.\vec{u}_b$$

Example 2.7: Determine the projection of the force $\vec{F}$ along the pole.



Chapter 3

Particle Equilibrium

Chapter 3: Equilibrium of a Particle

3.1: Equilibrium of a Particle

A particle is in **static equilibrium** if it:

- remains at rest if originally at rest; or
- has a constant velocity if originally in motion.

To maintain equilibrium, it is necessary to satisfy Newton's first law of motion, which requires the resultant force acting on a particle to be equal to zero.

$$\sum F = 0$$

- A particle has mass but its size is **neglected** (*i.e.* a point mass).
- As size is neglected, the point the force is applied is unimportant (*i.e.* force cannot cause moment)
- When a finite shaped body is modelled as a particle, the forces are assumed to pass through the center of mass of the body (then the forces cannot apply a moment to the body).

3.2: The Free-Body Diagram

To apply the equilibrium equation, we must account for all known and unknown forces acting on the particle. The best way to do this is to think of the particle as isolated and "free" from its surroundings. A drawing that shows a particle with all forces acting on it is called a **free-body diagram (FBD)**.

The FBD:

- determines the forces acting on a particle
- is fundamental to statics and dynamics as it defines how to satisfy the condition for equilibrium
- defines how to set up and solve a problem.
- is simply a sketch of a particle (or body) without its **surroundings** and with **all forces** acting on it.

To complete an FBD:

1. Define a reference coordinate system (*e.g.* x,y) and indicate the direction of the gravity vector.
2. Sketch the body (*e.g. the object*)
3. Draw all forces acting on the body with **distinct** labels
   - Draw all known forces in the correct directions
   - Assume directions of unknown forces, try to assume reasonably to aid in visualizing the problem.

4. Indicate any angles between forces and the reference coordinate system
   - **Draw exaggerated angles** to help identify if sines or cosines are needed for force components.
5. For a particle, all forces act at a point (*i.e.* concurrent).
   - If you draw the body with its shape and show how the forces really act on it, assume all forces act through the center of mass when solving for equilibrium.

Force in springs:

The magnitude of force exerted on a linearly elastic spring is $F = ks$.

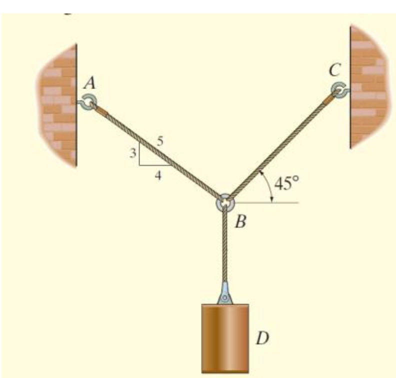
- k is stiffness of spring and $s = (l - l_0)$ is the deformed distance.
- l_0 is the unstretched length measured from its unloaded position.
- If *s* is positive, causing an elongation, then *F* must pull on the spring;
- If *s* is negative, causing a shortening, then *F* must push on it.

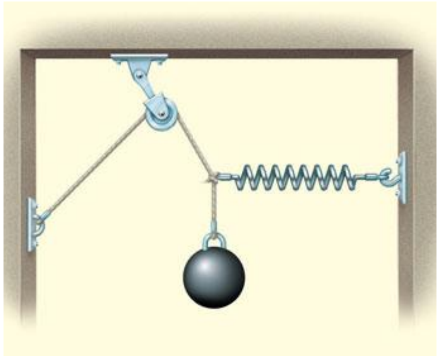
Force in cables and pulleys:

In this course it will be assumed that all cables (or cords):

- have negligible weight and they cannot stretch
- can support only a tension ("pulling" force) and this force always acts in the direction of the cable.
- the tension force developed in a continuous cable which passes over a frictionless pulley must have a constant magnitude to keep the cable in equilibrium.

FBD examples





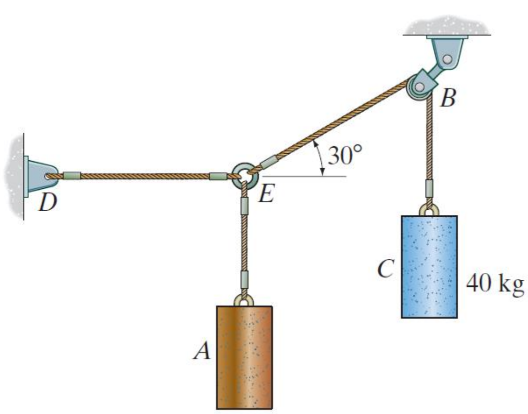
3.3: Coplanar (Two-Dimensional) Force Systems

When all the forces acting on a particle act in one plane (*e.g.* *x*-*y*) the particle is exposed to a **coplanar** force system. The condition for equilibrium can be written in its *x* and *y* components:

These two equations can be solved for at most **two unknowns**, generally represented as angles and magnitudes of forces shown on the particle's free-body diagram.

When performing the algebraic sum of forces in *x* and *y* directions, we must assume **positive directions** (the directions chosen as positive are arbitrary). If the solution of an unknown force yields a negative magnitude then the assumed force direction is wrong, *i.e.* the correct direction is opposite to that assumed.

Example 3.1: If the mass of cylinder *C* is 40 kg, determine the mass of cylinder *A* in order to hold the assembly in the position shown.

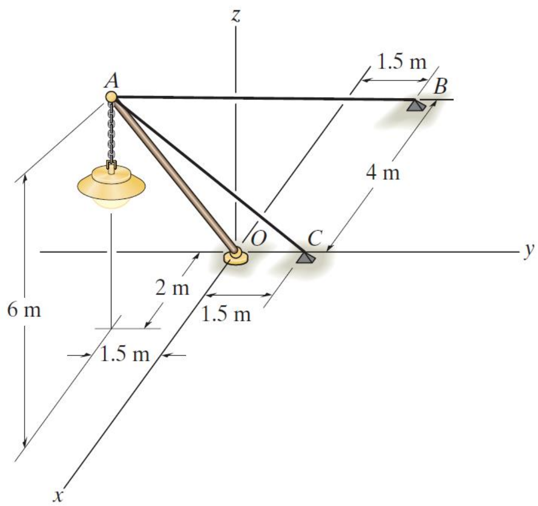


3.4: Particle Equilibrium (Three-Dimensional)

In Section 3.1 we stated that the necessary and sufficient condition for particle equilibrium is $\sum F = 0$. In the case of a three-dimensional force system we can resolve the forces into their respective *i* , *j* , *k* components:

These three equations state that the algebraic sum of the components of all the forces acting on the particle along each of the coordinate axes must be zero. Using them we can solve for at most **three unknowns** generally represented as coordinate direction angles or magnitudes of forces shown on the particle's FBD.

Example 3.2: The lamp has a mass of 15 kg and is supported by a pole AO and cables AB and AC. If the force in the pole acts along its axis, determine the forces in AO, AB, and AC for equilibrium.



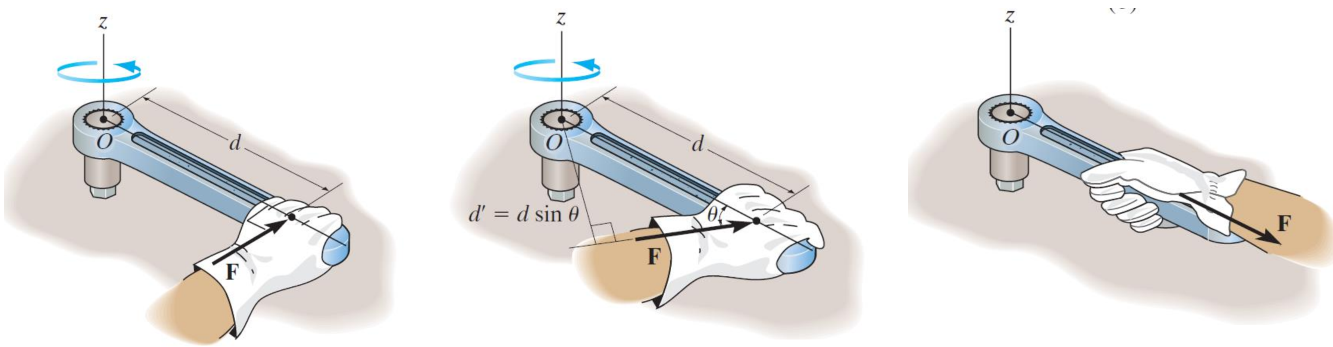
Chapter 4

Force System Resultants

Chapter 4: Force System Resultants

4.1: Moment of a Force- Scalar Formulation

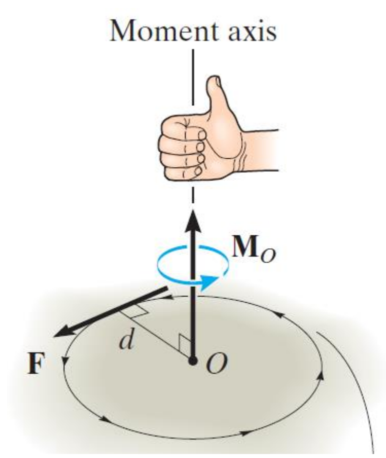
The **moment** of a force about a point or axis is a measure of the tendency of the force to cause rotation of a body about the point or axis. This tendency to rotate is sometimes called a torque, but most often it is called the moment of a force or simply the moment.

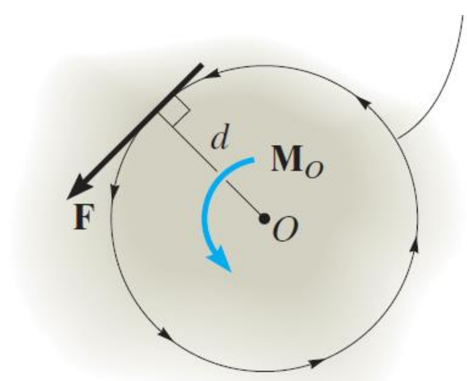


The magnitude of the moment is directly proportional to the magnitude of $\vec{F}$ and the perpendicular distance or **moment arm** d. The moment of force $\vec{F}$ about point O or an axis through O perpendicular to the plane of the force is:

The moment M_o has units of force times distance (*e.g.* N.m or lb.ft).

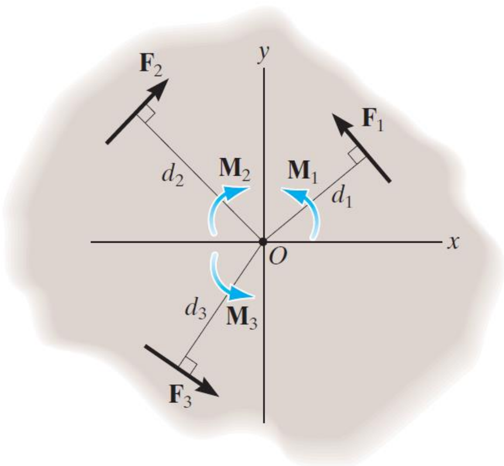
The direction of the moment is indicated by the right hand rule, where the fingers follow the rotation of the body about O caused by the force.





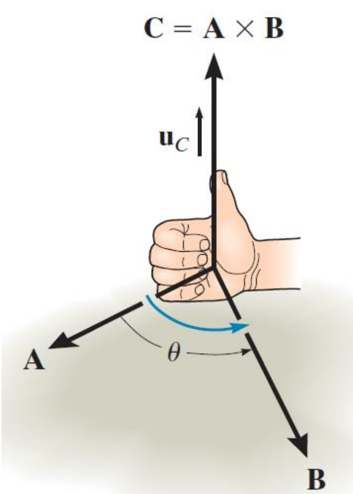
As a convention, we will generally consider **positive** moments as **counterclockwise** since they are directed along the positive *z* axis. **Clockwise** moments will be considered **negative**.

For 2D problems, the resultant moment is the algebraic sum of moments due to all forces in the system.



4.2: Moment of a Force-Vector Formulation

This section formulates the moment of a force with Cartesian vectors. First, we must introduce the **cross-product** method of vector multiplication. The cross product of two vectors $\vec{A}$ and $\vec{B}$ yields the vector $\vec{C}$:

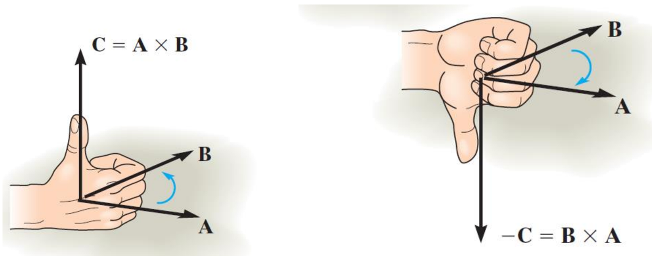


Direction of $\vec{C}$ is obtained from the right hand rule:

- push $\vec{A}$ toward $\vec{B}$ with fingers
- $\vec{C}$ is in the direction your thumb points.

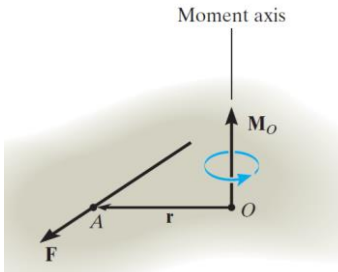
The magnitude of $\vec{C}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the sine of the angle θ between their tails.

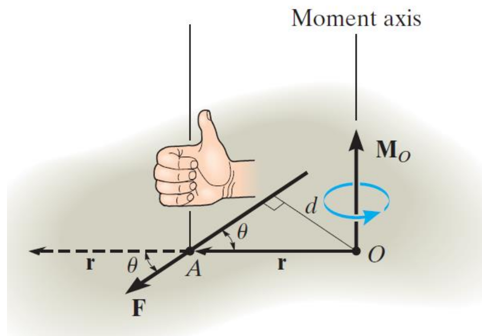
The commutative law is not valid for the cross product.



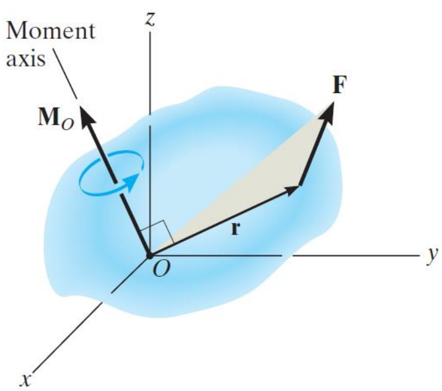
To find the cross product of any two Cartesian vectors $\vec{A}$ and $\vec{B}$, it is necessary to expand a determinant whose first row of elements consists of the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ and whose second and third rows represent the *x*, *y*, *z* components of the two vectors $\vec{A}$ and $\vec{B}$, respectively.

Using the vector cross product, the moment of a force $\vec{F}$ about point O, or actually about the moment axis passing through O and perpendicular to the plane containing O and $\vec{F}$, can be expressed as:

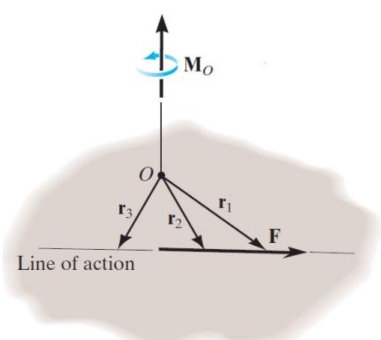




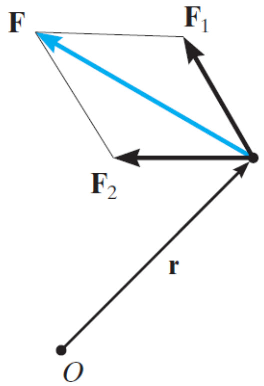
The cross product operation is useful in 3D since the perpendicular distance (or moment arm) from point O to the line of action of the force is not needed. If we establish x, y, z coordinate axes, then the position vector $\vec{r}$ and force $\vec{F}$ can be expressed as Cartesian vectors and the moment of $\vec{F}$ about O will be:

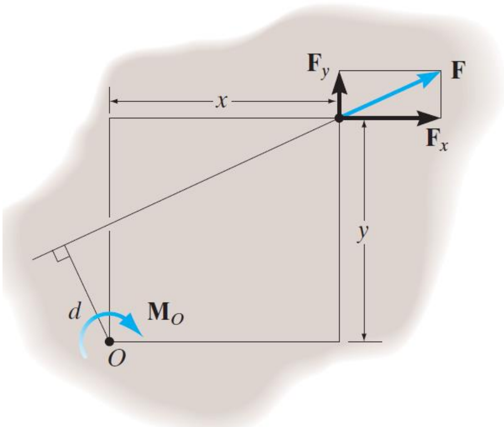


Principle of transmissibility: We can use **any** position vector $\vec{r}$ measured from point O to **any point** on the line of action of the force $\vec{F}$. Since $\vec{F}$ can be applied at any point along its line of action and still create this same moment about point O , then $\vec{F}$ can be considered a sliding vector.

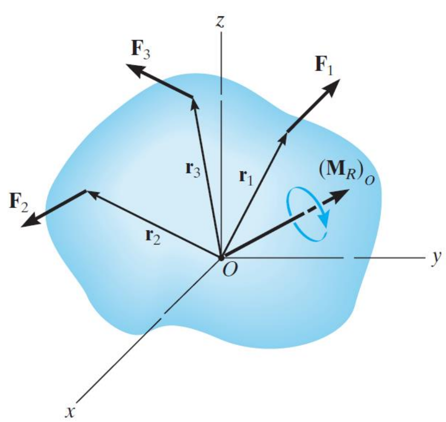


Principle of moments: The moment of a force about a point is equal to the **sum** of the moments of the **components** of the force about the point.

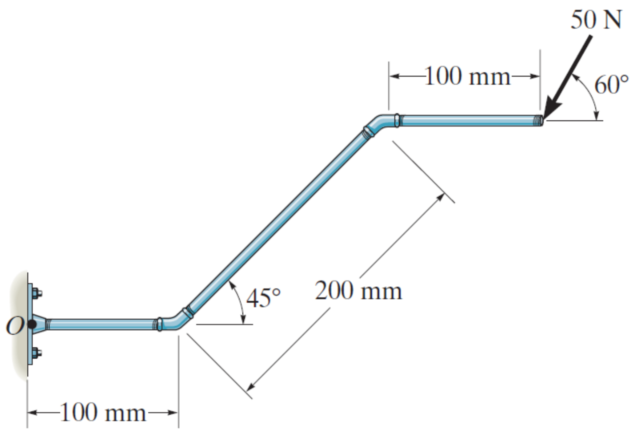




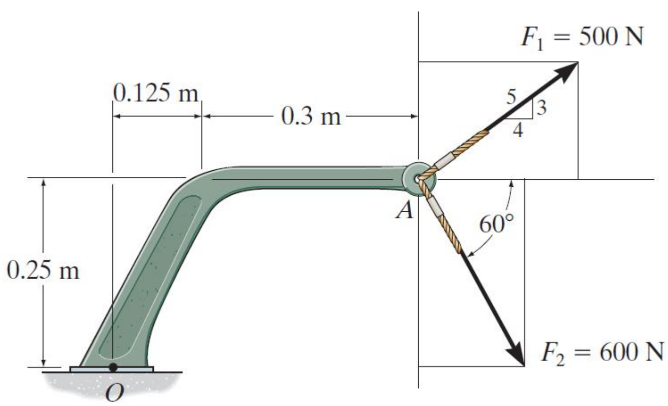
Resultant moment of a system of forces: If a body is acted upon by a system of forces, the resultant moment of the forces about point *O* can be determined by vector addition of the moment of each force.



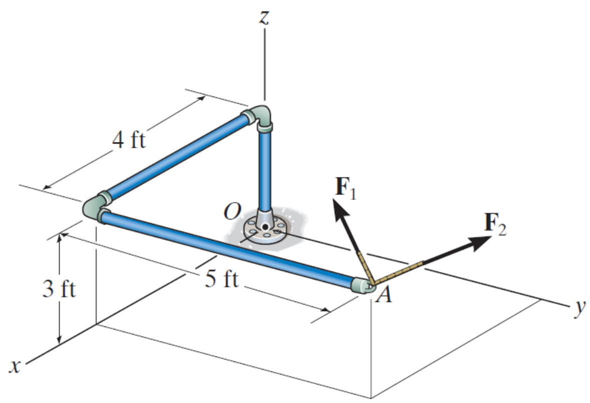
Example 4.1: Determine the moment of the force about point *O*.



Example 4.2: Determine the resultant moment produced by the forces about point O.

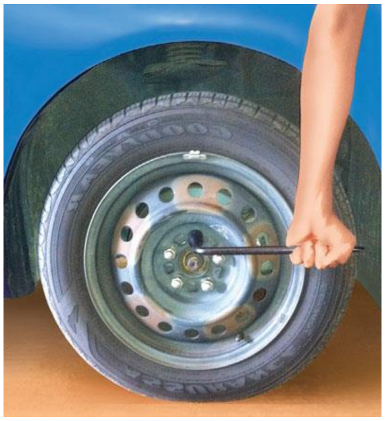


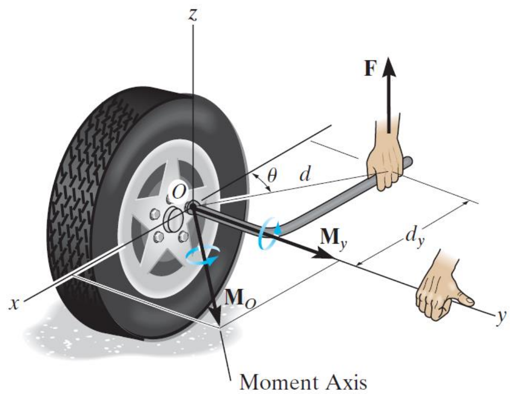
Example 4.3: If $\vec{F}_1 = 100\hat{\imath} - 120\hat{\jmath} + 75\hat{k}\ lb$ and $\vec{F}_2 = -200\hat{\imath} + 250\hat{\jmath} + 100\hat{k}\ lb$, determine the resultant moment produced by these forces about point *O*.



4.3: Moment of a Force about a Specified Axis

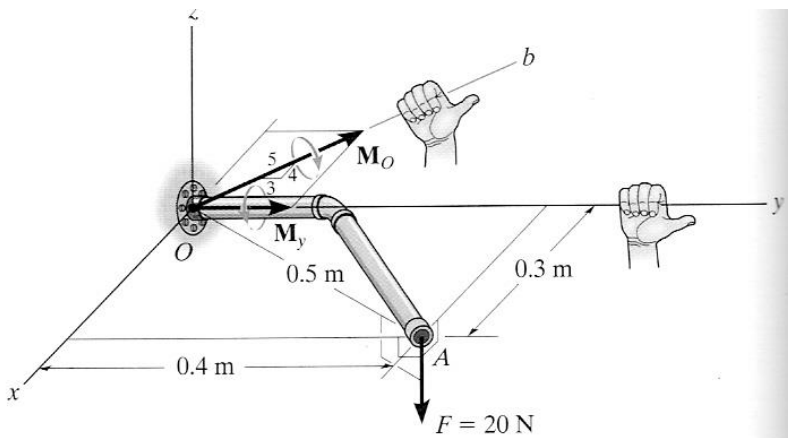
Sometimes, the moment produced by a force about a specified axis must be determined.





Scalar Formulation:

In the figure below, the force $\vec{F}$ will cause a moment about the x and y axes.



To determine the moment about an axis, multiply the component of the force **perpendicular** to the axis by the shortest (*i.e.* **perpendicular**) distance of the force from the axis.

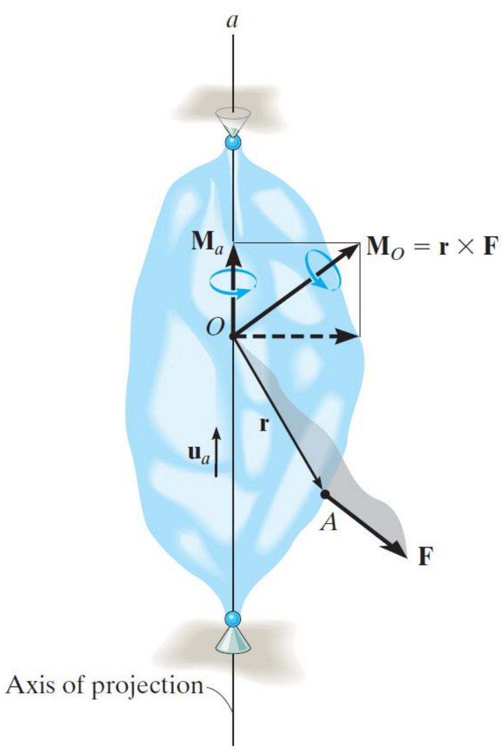
The right hand rules gives the direction (*i.e.* curl your fingers in the direction of rotation caused by force).

<u>Note 1</u>: A force will not cause a moment about an axis if the line of action of the force is: (i) parallel to the axis; or (ii) passes through the axis.

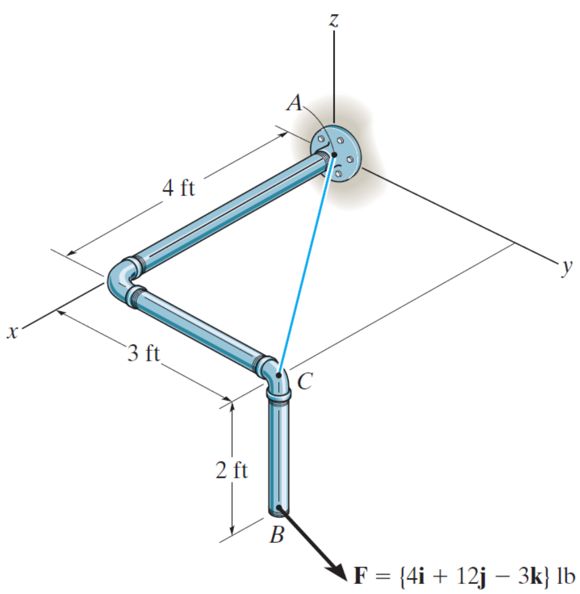
<u>Note 2</u>: Scalar formulation is useful if we can easily determine the force components & the perpendicular distances to desired axes. Generally, it is only useful if the desired axis aligns with the *x*, *y* or *z* axes.

Vector formulation:

To find the moment of a force about axis *a*, we must first use the cross product to find the moment of the force about any point on the axis. Then, the projection of the moment vector onto the axis is determined by the **dot product** of the moment vector and the **unit vector** along axis *a*. This combination is referred to as the **scalar triple product**.



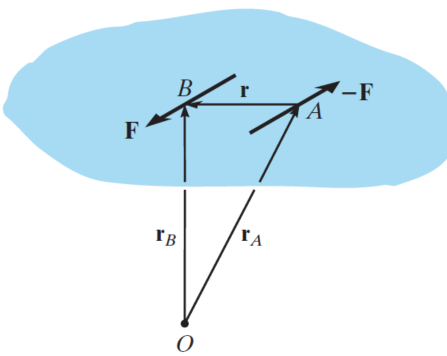
Example 4.4: Determine the moment of the force $\vec{F}$ about an axis extending between A and C.



4.4: Moment of a Couple

A couple is defined as two **parallel** forces with the **same magnitude**, but opposite directions. Since the **resultant** force is **zero**, the only effect of a couple is to produce an actual **rotation**. If no movement is possible, there is a **tendency of rotation** in a specified direction.

For example, imagine you are driving a car with both hands on the steering wheel and you make a turn. One hand will push up on the wheel while the other hand pulls down, causing the steering wheel to rotate.

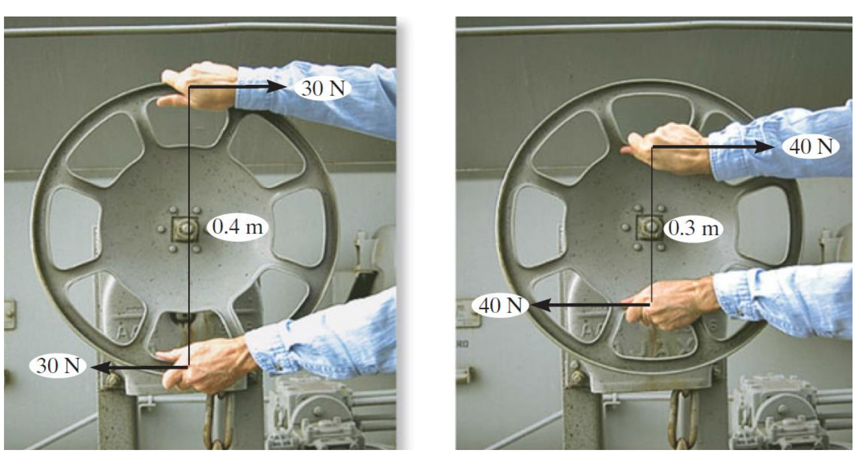


For a couple you do **not** have to take the moments about a specific point or axis. This means a couple moment is a free vector, it can act at any point, as its value is not dependent on a specific axis, only the perpendicular distance between the two forces. This is a very different from a moment due to a force, which is dependent on the axis about which the moment is determined.

Scalar formulation:

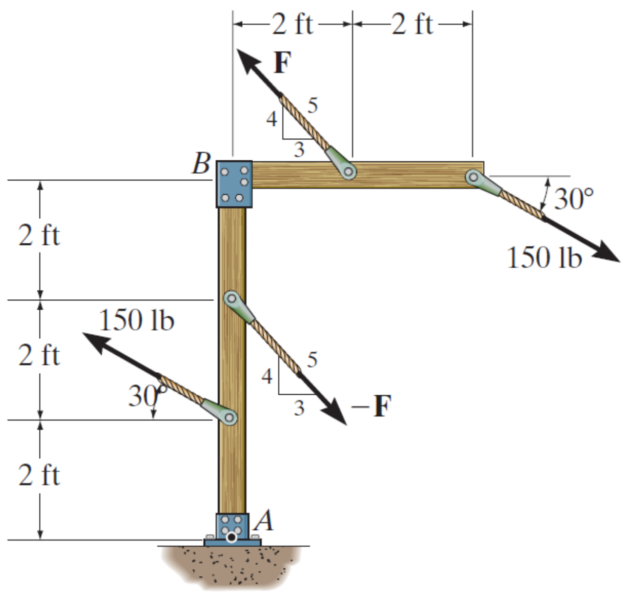
where d is the perpendicular distance between the two forces composing the couple. The direction is given by the right hand rule.

Equivalent couples:



If a body is subjected to multiple couples, each couple moment can be evaluated separately, and then added **vectorially** to give the total couple moment acting on the body. Further, since a couple moment is a **free** vector, the individual or total couple moment can be moved to any arbitrary point.

Example 4.5: If *F =200 lb*, determine the resultant couple moment.



4.5: Simplification of Force and Couple Systems

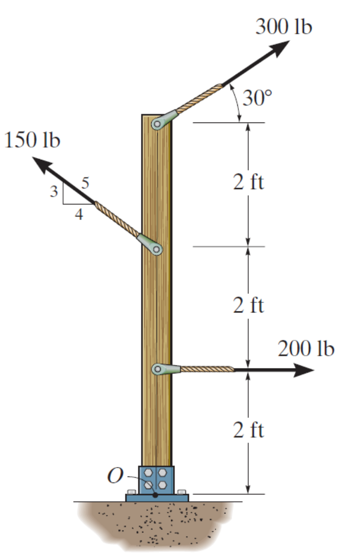
It can be convenient to simplify a system of forces and couple moments by replacing them with an equivalent system, consisting of:

- a s**ingle resultant force** acting at a specific point
- and a **resultant couple moment**.

A system is equivalent if the external effects it produces on a body are the same as those caused by the original force and couple moment system.

A force can cause translation and rotation of a body. The effect depends on where and how the force is applied. Force acting on a rigid body can be moved to any point O, provided we add a couple whose moment is equal to the moment of $\vec{F}$ about O. M_O is a free vector, so attach it anywhere (typically at O).

Example 4.6: Replace the force system acting on the post by a resultant force and couple moment at point O.

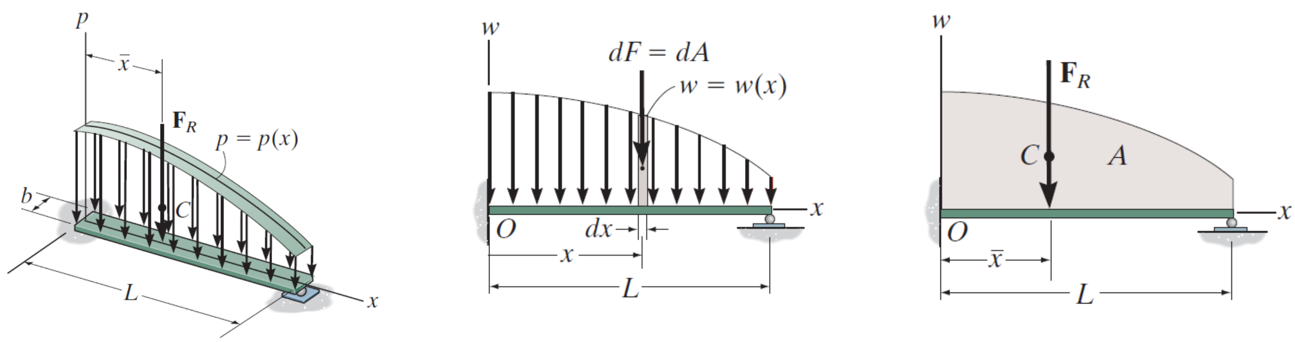


4.6: Distributed Loads

A surface may be exposed to a distributed load, *i.e.* a load acting over the whole surface area (*e.g.* wind loading, snow loading, or a hydro dam). The intensity of this type of loading at a point is pressure. A distributed load can be replaced by a concentrated (point) load.

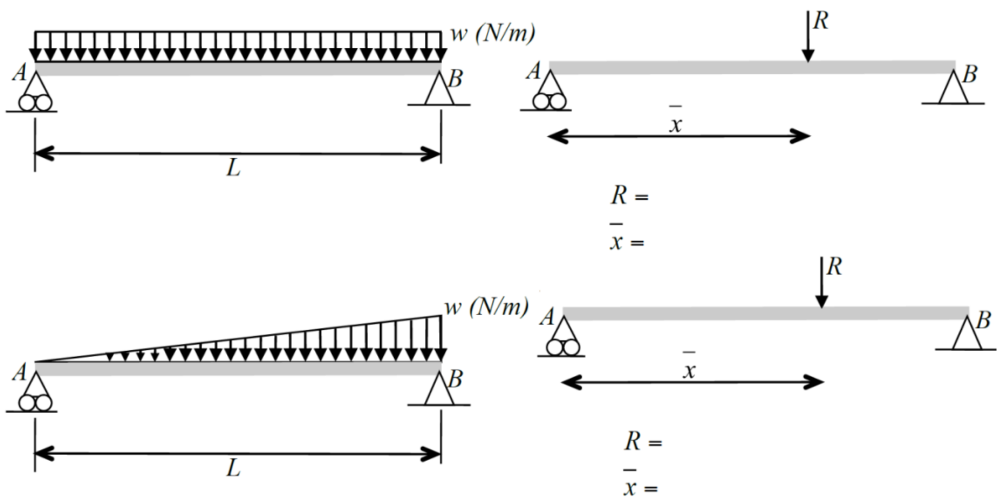
Consider the beam with a constant width is subjected to a pressure loading that varies only along the *x-axis*. The goal is to convert the distributed load into a **resultant force acting at a point**.

This loading can be described by the function $\mathrm{p} = \mathrm{p(x)}\ \frac{N}{m^2}$. Define $w(\mathrm{x}) = \mathrm{p(x)b}$, where b is the width of the beam and w(x) (distributed load) has units $\frac{N}{m}$.

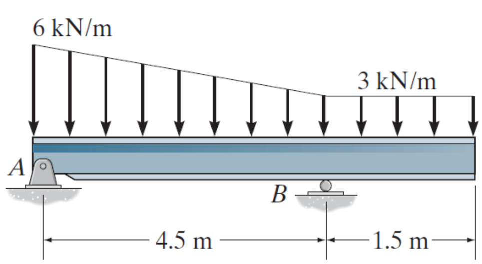


- **Magnitude of the concentrated load**: equal to the area under the loading diagram
- **Location of the concentrated force**: the centroid of the area under the distributed load

The distributed-loading diagram is often in the shape of a rectangle, triangle, or another simple geometric form. The centroid location for these common shapes is given in the table on the inside back cover.



Example 4.7: Replace the distributed loading with 2 equivalent resultant forces.



Chapter 5

Equilibrium of Rigid Bodies

Chapter 5: Equilibrium of Rigid Bodies

5.1: Equilibrium Conditions

For equilibrium (*i.e.* static or moving in a straight line with constant speed) of a rigid body there is no net force and no net moment acting on the body.

- The sum of the forces acting on the body is equal to zero.

- The sum of the moments of all forces in the system about point O, added to all the couple moments, is equal to zero.

These two equations are not only necessary for equilibrium, they are also sufficient.

5.2: Coplanar Forces or Two-Dimensional Problems

In two-dimensions it is often more convenient to work with a scalar formulation as follows:

where *O* is usually an arbitrary point on the body, often at the point of application of an unknown force. The point *O* can also be an **arbitrary** point in space, (*i.e.* not on the body). The choice is up to you.

Note: In a scalar formulation a **positive** reference direction must be chosen in the governing equations.

To successfully apply the equations of equilibrium, all the known and unknown external forces acting on the body must be completely specified. The best way to account for these forces is to draw a **free-body diagram (FBD)**. This diagram is a sketch of the outlined shape of the body, which represents it as being isolated or "free" from its surroundings, *i.e.*, a "free body". On this sketch it is necessary to show all the forces and couple moments that the surroundings exert on the body so that these effects can be accounted for when the equations of equilibrium are applied. In general, **NO equilibrium problem should be solved without FIRST drawing the FBD**.

- Apply the forces and couple moments at the correct locations
  - Weight acts at the center of mass (or gravity) of the body.

- Draw the forces and couple moments in the correct directions.
    - Assume directions of unknown forces, try to assume reasonably to aid in visualizing.
    - If the chosen direction of an unknown force or couple moment is incorrect, the solution of the magnitude will be negative, indicating the correct direction is opposite to that chosen.
- Indicate dimensions of the body, especially to the points of applied forces.
    - To calculate moments caused by forces the appropriate lever arms will be required.

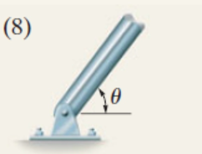
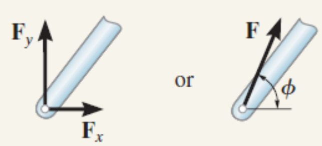
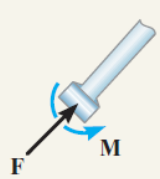
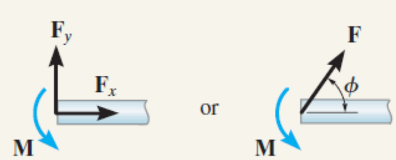
Since all FBDs require reaction forces, we must understand the reactions that exist at different supports. You will be separating a body from its surroundings, so the contact points between the body and its surroundings must have reaction forces (and/or moments), otherwise the body would behave differently.

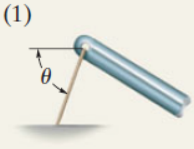
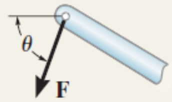
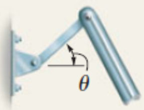
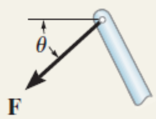
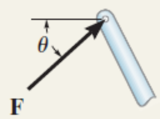
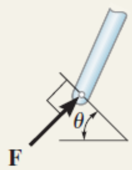
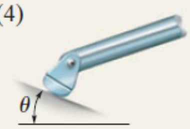
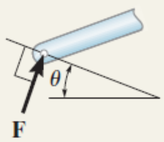
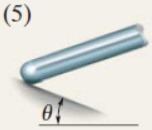
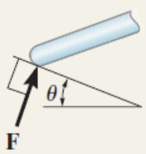
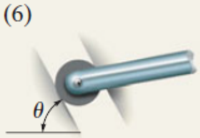
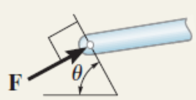
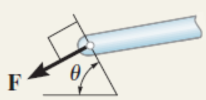
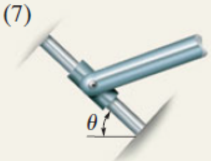
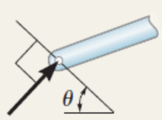
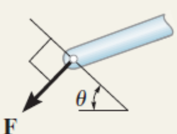
Support reactions:

There are various types of reactions that occur at supports and points of contact between bodies subjected to coplanar force systems. As a general rule:

- If a support prevents the translation of a body in a given direction, then a force is developed on the body in that direction.
- If rotation is prevented, a couple moment is exerted on the body.

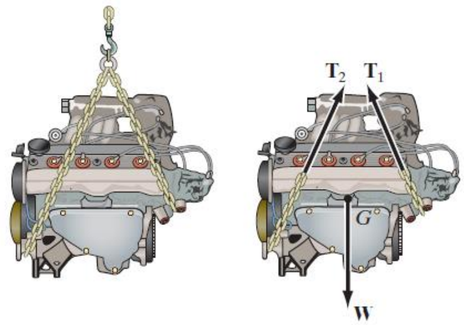
Table below lists 10 common types of supports for bodies subjected to coplanar force systems (in all cases the angle θ is assumed to be known).

Types of Connection	Reaction	Number of Unknowns
(8)  smooth pin or hinge		Two unknowns. The reactions are two components of force, or the magnitude and direction ϕ of the resultant force. Note that ϕ and θ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].
(9)  member fixed connected to collar on smooth rod		Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod.
(10)  fixed support		Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction ϕ of the resultant force.

Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.
(2)  weightless link	 or 	One unknown. The reaction is a force which acts along the axis of the link.
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4)  rocker		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(5)  smooth contacting surface		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(6)  roller or pin in confined smooth slot	 or 	One unknown. The reaction is a force which acts perpendicular to the slot.
(7)  member pin connected to collar on smooth rod	 or 	One unknown. The reaction is a force which acts perpendicular to the rod.

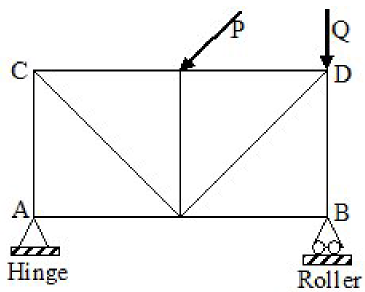
Internal forces:

The internal forces acting between adjacent particles in a body always occur in collinear pairs such that they have the same magnitude and act in opposite directions (Newton's third law). Since these forces **cancel** each other, they will not create an external effect on the body. It is for this reason that the internal forces should **NOT** be included on the free-body diagram if the **entire** body is to be considered.

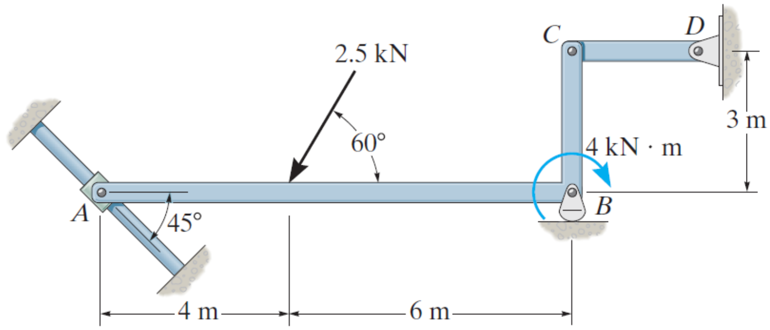


Solution strategies:

In 2D equilibrium, other sets of equations are possible as shown in the following example. However, there are only 3 unique equations of equilibrium so we can only solve for 3 unknowns.



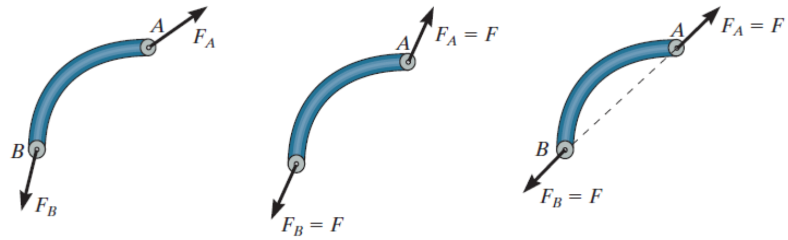
Example 5.1: Draw the free-body diagram of member *ABC* which is supported by a smooth collar at *A*, rocker at *B*, and short link *CD*.



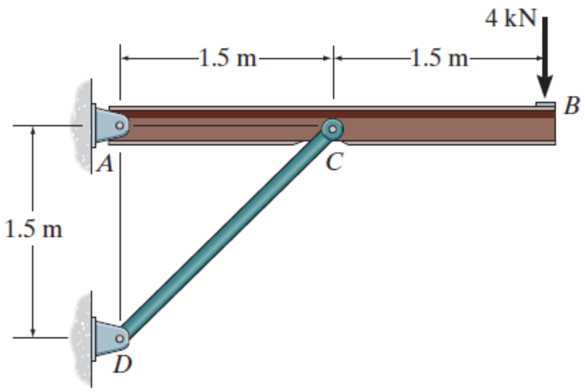
Two- force members:

As the name implies, a two-force member has forces applied at only two points on the member. For any two-force member to be in equilibrium, the two forces acting on the member must have:

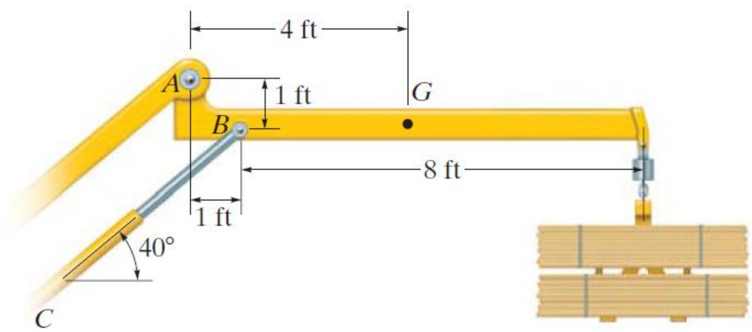
- the **same magnitude**,
- act in **opposite directions**, and
- have the **same line of action**, directed along the line joining the two points where these forces act.



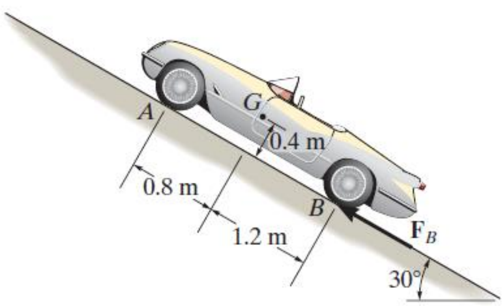
Example 5.2: Determine the horizontal and vertical components of reaction at the pin *A* and the reaction on the beam at *C*.



Example 5.3: The articulated crane boom has a weight of *125 lb* and center of gravity at *G*. If it supports a load of *600 lb*, determine the force acting at the pin *A* and the force in the hydraulic cylinder *BC* when the boom is in the position shown.



Example 5.4: The sports car has a mass of *1.5 Mg* and mass center at *G*. If the front two springs each have a stiffness of $k_A = 58\ kN/m$ and the rear two springs each have a stiffness of $k_B = 65\ kN/m$, determine their compression when the car is parked on the 30° incline. Also, what friction force $\vec{F}_B$ must be applied to each of the rear wheels to hold the car in equilibrium?



5.3. Equilibrium in 3D

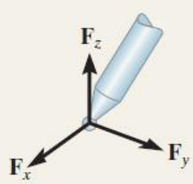
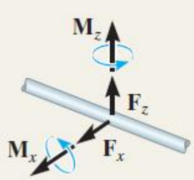
The same as in 2D problems, the conditions for equilibrium of a rigid body subjected to a 3D force system require that both the resultant force and resultant couple moment acting on the body be equal to zero.

- The sum of the external force components acting in the x, y, and z directions is zero.

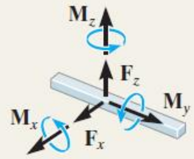
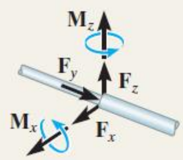
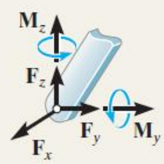
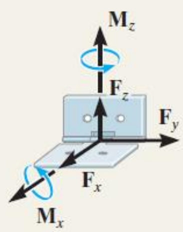
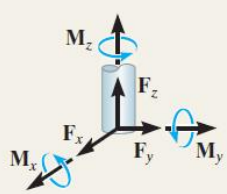
- The sum of the moment components about the *x*, y, and z axes is zero.

These six scalar equilibrium equations will solve for at most **six unknowns** shown on the FBD.

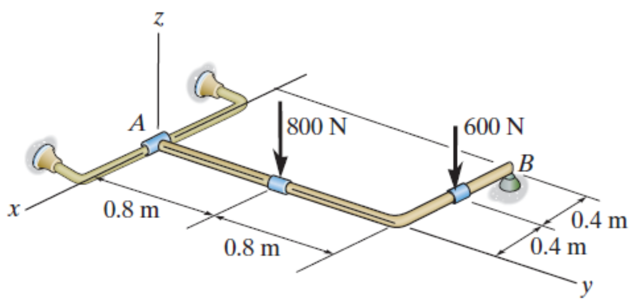
Reactions at 3D supports:

Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a force which acts away from the member in the known direction of the cable.
(2)  smooth surface support		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4)  ball and socket		Three unknowns. The reactions are three rectangular force components.
(5)  single journal bearing		Four unknowns. The reactions are two force and two couple-moment components which act perpendicular to the shaft. Note: The couple moments are *generally not applied* if the body is supported elsewhere. See the examples.

continued

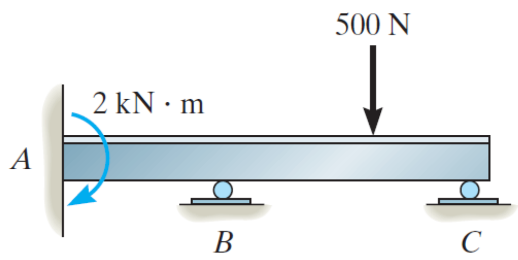
Types of Connection	Reaction	Number of Unknowns
(6)  single journal bearing with square shaft		Five unknowns. The reactions are two force and three couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(7)  single thrust bearing		Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(8)  single smooth pin		Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(9)  single hinge		Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.
(10)  fixed support		Six unknowns. The reactions are three force and three couple-moment components.

Example 5.5: Determine the support reactions at the smooth collar *A* and the normal reaction at the roller support *B*.



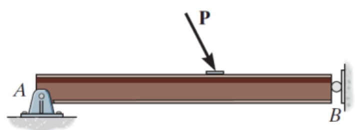
5.4. Constraints and Statical Determinacy

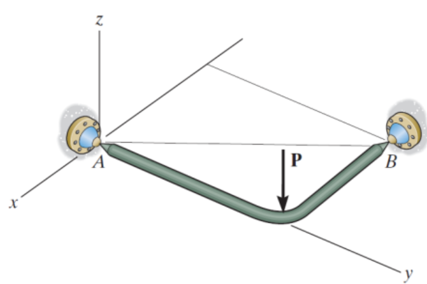
Statically indeterminate bodies: When a body has redundant supports (more supports than are necessary to hold it in equilibrium) it becomes statically indeterminate. This means there will be more unknown loadings on the body than equations of equilibrium available for their solution.

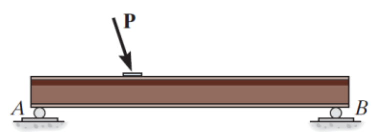


To establish an additional equation needed for solution, we must consider how points on the bar displace. An equation that specifies the conditions for displacement is referred to as a **compatibility condition**. This subject will be discussed in courses such as Solid Mechanics.

Improper constraints: Some bodies may not have enough supports, or the supports may be arranged in a way that could cause the body to move. Having the same number of unknown reactive forces as available equations of equilibrium does not always guarantee that a body will be stable when subjected to a particular loading. A body is considered improperly constrained if all the reactive forces intersect at a common point or pass through a common axis, or if all the reactive forces are parallel.







In engineering practice, these situations should always be avoided since they cause an unstable condition.

Chapter 6

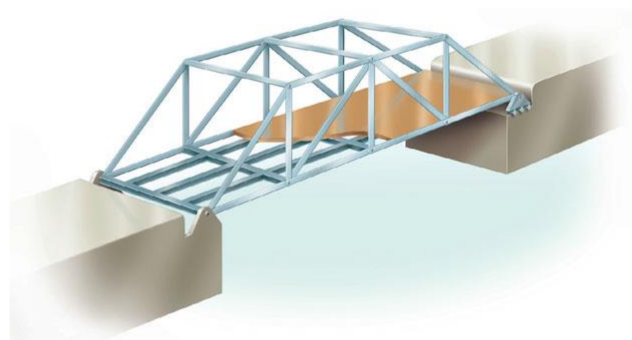
Structural Analysis

Chapter 6: Structural Analysis

Structural analysis is used to determine the forces in the individual components of a truss (*e.g.* in a bridge or building), and in the frames of machines (*e.g.* a backhoe or front-end loader).

6.1: Simple Truss

- A **truss** is a structure composed of slender members joined together at their end points.
  - The members commonly used in construction consist of wooden struts or metal bars.
  - A truss is designed to provide a light rigid structure capable of carrying large loads.
- A **planar truss** lies in a single plane and is often used to support roofs and bridges.

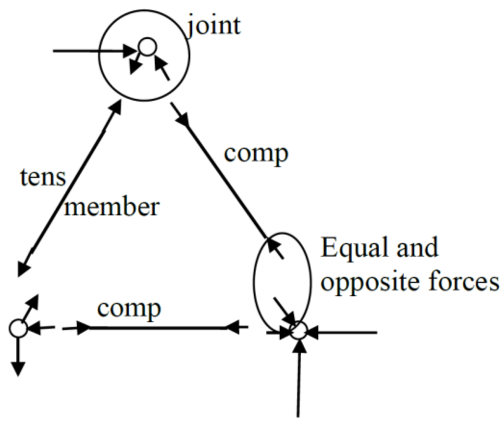


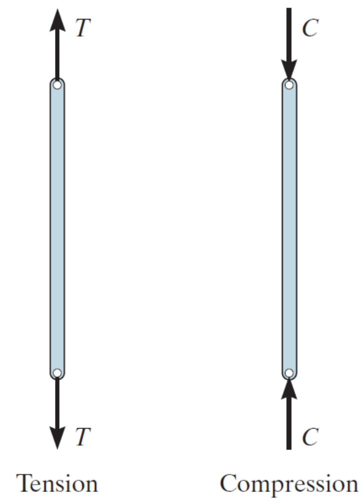
To design the members and the connections of a truss, first we need to determine the force developed in each member when the truss is subjected to a given loading. To do this we will make some assumptions:

- All loadings are applied at the joints.
- Typically, the weight of the members is neglected because the force supported by each member is usually much larger than its weight. However, if the weight is to be included in the analysis, it is generally applied as a vertical force, with half of its magnitude applied at each end of the member.
- Members are joined by smooth pins (no friction), so that moments about the joint axis are zero.
  - This is satisfied if the centerlines of the joining members are concurrent through the joint axis.
  - The truss joint is a gusset plate or a pin passing through all members (no moment resisted).

These assumptions imply all members of a truss will be two force members, *i.e.* forces are directed along the length of each member. The forces in a member can be:

- Compressive: tends to compress or shorten a member (represented by **C**)
- Tensile: tends to stretch a member (represented by **T**)



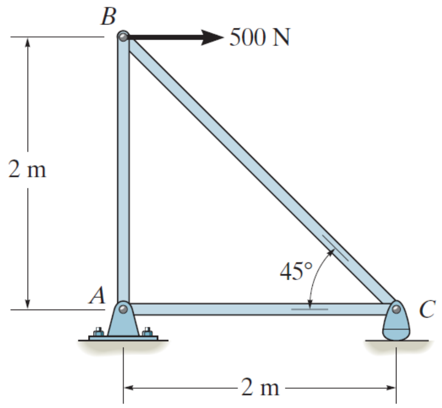


A truss must be rigid and the simplest rigid structure is a triangle, therefore, all trusses are assembled from triangles. There are two methods used in the analysis of trusses: method of joints and method of sections.

6.1.1: Method of Joints

Since the members of a plane truss are straight two-force members lying in a single plane, each joint is subjected to a force system that is coplanar and concurrent. As a result, only $\sum F_x = 0$ and $\sum F_y = 0$ need to be satisfied for equilibrium.

- Always start at a joint having at least one known force and at most two unknown forces.
- You can find the reactions at supports for the whole truss using $\sum F = 0$ and $\sum M = 0$ applied to an FBD for the whole truss.
- The correct sense of direction of an unknown member force can often be determined "by inspection".
- Always assume the unknown member forces acting on the joint's free-body diagram to be in **tension**
  - Then numerical solution of the equilibrium equations will yield positive scalars for members in tension and negative scalars for members in compression.

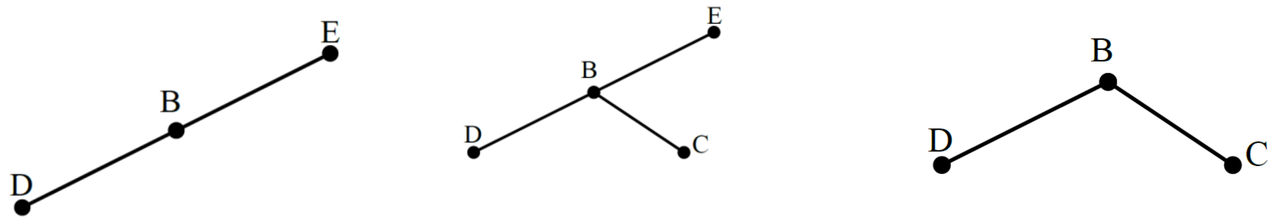


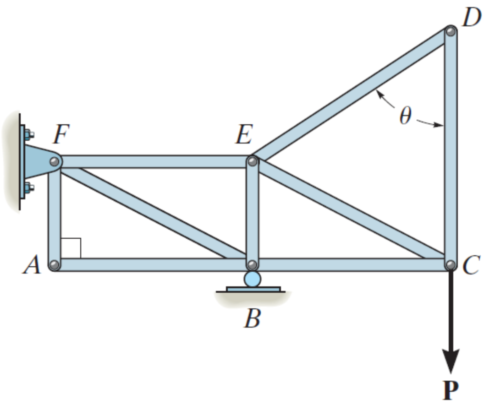
Zero-force members:

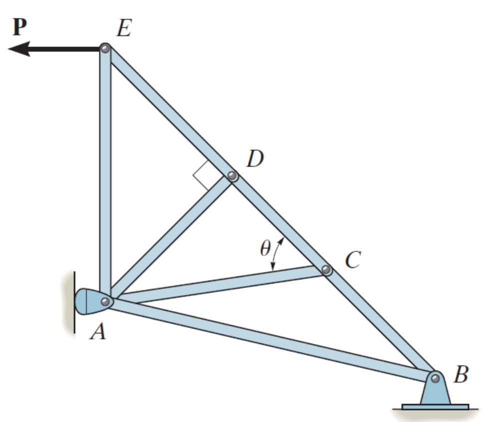
Truss analysis is simplified if we first identify the members which support **no loading**. These zero-force members increase the truss stability during construction and provide added support if loading is changed.

The zero-force members of a truss can be found by inspection of the joints. In general, if a joint is the intersection of **only two members** and **no external load** or reaction force is applied at the joint then they are zero-force members.

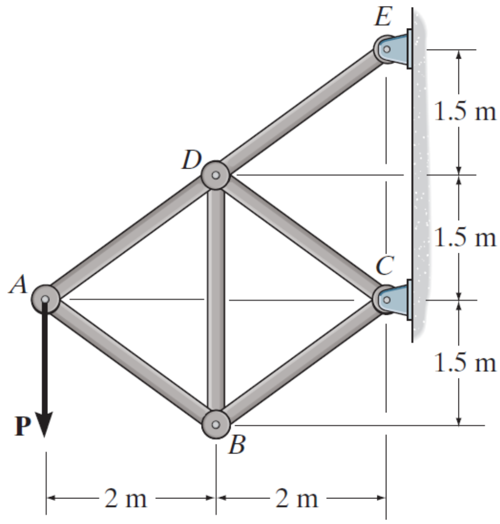
Note: Identifying zero force members can simplify analysis but be careful!! If there is an external force applied, no zero-force members can be identified.




D
θ
F
E
A
C
B
P

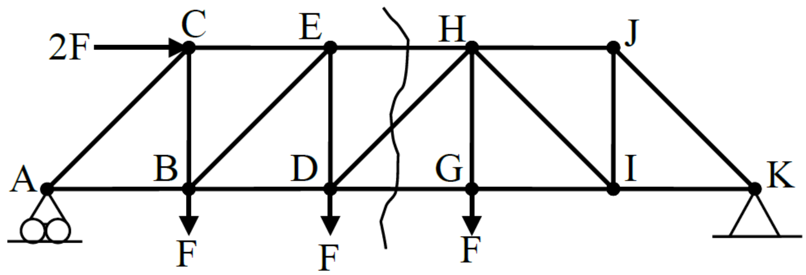

P
E
D
θ
C
A
B

Example 6.1: Determine the force in each member of the truss, and state if the members are in tension or compression. Set $P = 5\ kN$.

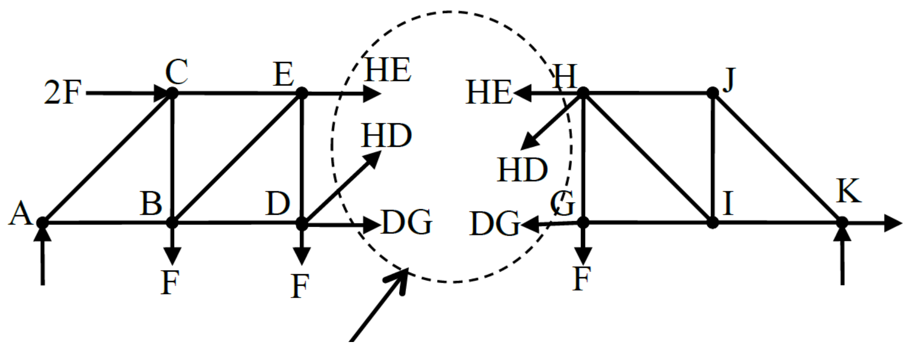


6.1.1: Method of Sections

This method is most useful if you want to know only a **few** of the member forces. As the name implies it cuts or divides a truss into sections. To maintain equilibrium, the part of the member removed must be replaced by a force (the internal force). Consider the following truss:



If we want the force in member *HE* only, the method of joints would be very tedious. Instead, "section" the truss into 2 separate parts in order to cut the desired member (makes a complete cut).

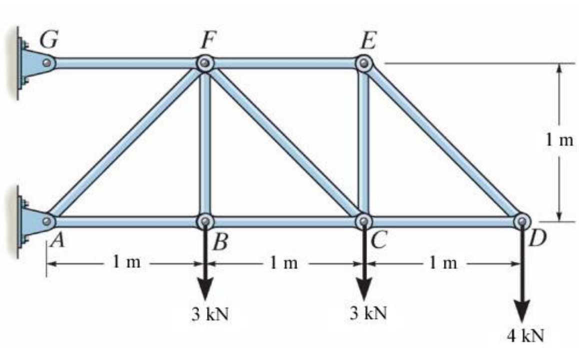


<u>Note</u>: At the cut in the two FBDs the forces are equal and opposite. This is very important, otherwise equilibrium will not be maintained.

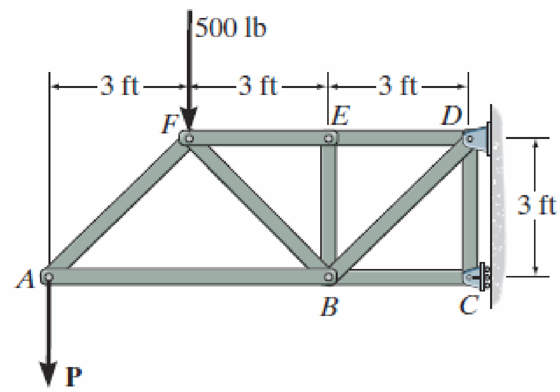
Solution procedure with the method of sections:

- Solve for the support reactions with the truss as a whole if required (only need support reactions for the section to be considered).
- Draw the FBD of the section to be analyzed (assume unknowns in tension).
- Try to pick a section that cuts through **three or less unknown** members
    - In 2D equilibrium we only have 3 equations, and therefore can only solve for 3 unknowns.
- Apply the equations of equilibrium and solve to find unknowns.
- Negative answers indicate that the correct sense is opposite to your assumption, *i.e.* the member is in compression if you assumed all unknowns in tension.

Example 6.2: Find forces in links *EF*, *FC* and *BC*. Determine if they are under tension or compression?



Example 6.3: Determine the forces in members *ED, BD* and *BC* of the truss and state if the members are in tension or compression. Set $P = 800\ lb$.



6.2: Frames and Machines

Frames and machines are two types of structures which are often composed of pin-connected multi-force members, *i.e.*, members that are subjected to more than two forces.

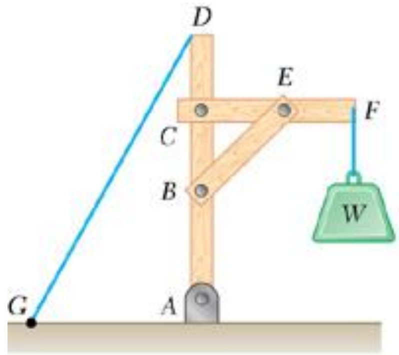
- Frames support loads
- Machines contain moving parts and are designed to transmit and alter the effect of forces.

Provided a frame or machine contains no more supports or members than are necessary to prevent its collapse, the forces acting at the joints and supports are determined by using the equations of equilibrium at each member. Once these forces are obtained, it is then possible to design the members, connections, and supports using the theory of mechanics of materials and an appropriate engineering design code.

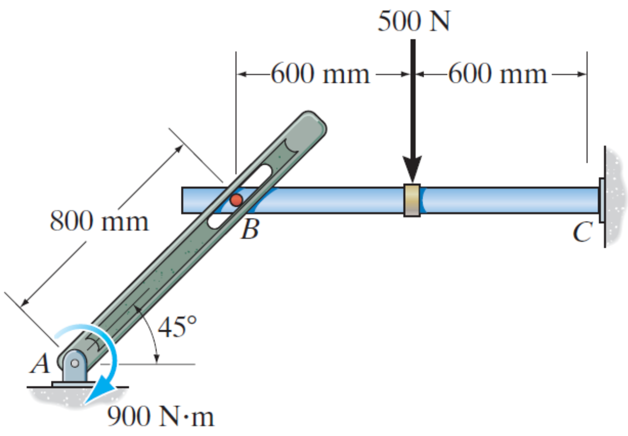
Main concept: If a structure is in equilibrium, then all parts (every joint, every member and every part of a member) are in equilibrium. This same concept was used to section trusses and find the internal forces in the truss members.

- Disassemble the structure to whatever degree necessary to find the desired forces
  - Do not separate into every member, but usually need more than one FBD to find all unknowns.
- Find support reactions by considering the FBD of the entire structure.
  - In most cases, we also need to use FBDs of internal members.
- Forces common to any two contacting members act with **equal magnitudes** but **opposite directions** on the respective members.
  - If the two members are treated as a "system" of connected members, then these forces are "internal" and are not shown on the free-body diagram of the entire system;
  - If the free-body diagram of each member is drawn, the forces are "external" and must be shown as equal in magnitude and opposite in direction on each of the two free-body diagrams.
- If a member is a **two-force member**, draw the two forces collinear! (reduces the number of unknowns)

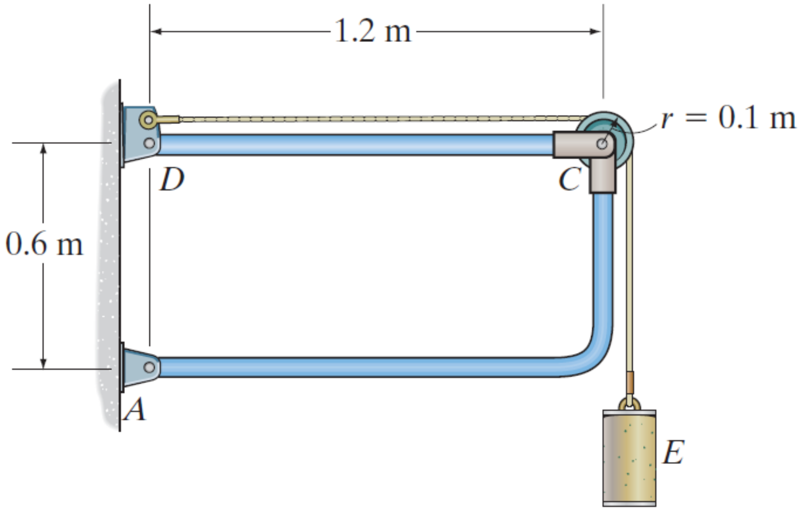
The following examples graphically illustrate how to draw the FBDs of a dismembered frame or machine. The weight of all the members is neglected.


D
E
F
C
B
W
G
A

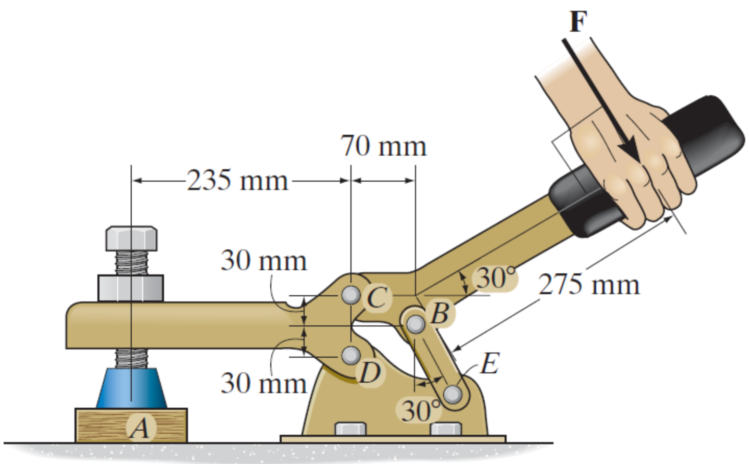
Example 6.4: If the peg at *B* is smooth, determine the components of reaction at the pin *A* and fixed support *C*.



Example 6.5: The frame is used to support the 100-kg cylinder *E*. Determine the horizontal and vertical components of reaction at *A* and *D*.



Example 6.6: If a clamping force of 300 N is required at A, determine the amount of force F that must be applied to the handle of the toggle clamp.



Chapter 7

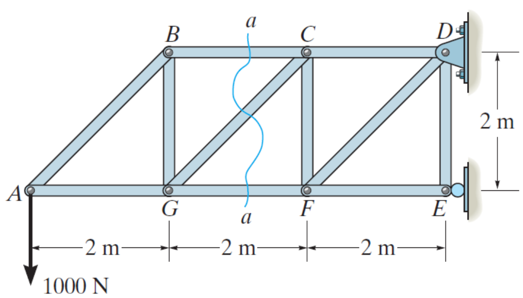
Internal Forces

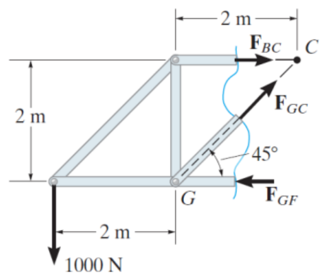
Chapter 7: Internal Forces

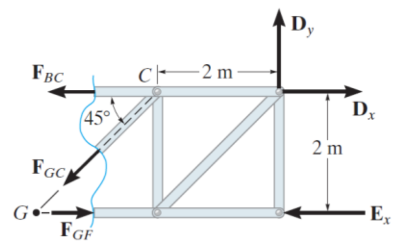
7.1: Internal Forces

To this point, we considered the external forces (Chapter 5) and the forces that hold parts of a structure together (Chapter 6). Now we will consider the forces that hold the members together (**internal forces**). We used the equilibrium concept to section trusses and find the internal forces that existed in the truss members. For straight, 2-force members, the only internal force is an **axial** force, independent of the location of the section. Remember in trusses, ALL members of a truss are straight, 2-force members.

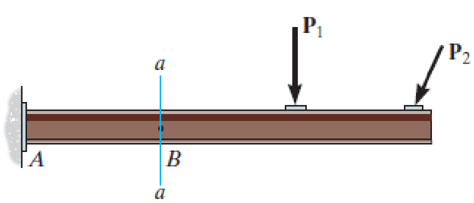
To design a structural or mechanical member, we must know the loading acting within the member to be sure the material can resist this loading. Internal loadings are determined with the method of sections.

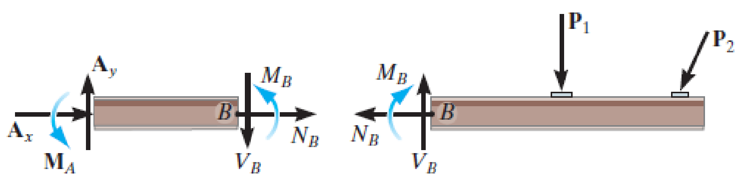






What if we apply this idea to frames/machines and section through members of these structures?





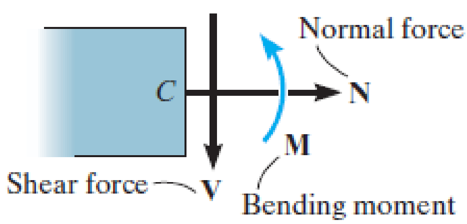
Internal forces are not limited to only a **tensile** or compressive axial force (as in straight 2-force members). In any non-straight 2-force member, or any multi-force member, they also produce **shear** and **bending**, which deforms the member. These are determined by applying the equations of equilibrium to the FBD of **either** segment.

- **Normal force:** force component $\vec{N}_B$ acting perpendicular to the cross section
- **Shear force:** force component $\vec{V}_B$ acting tangent to the cross section
- **Bending moment:** the couple moment $\vec{M}_B$

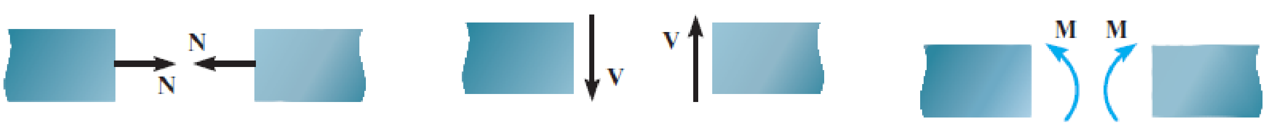
<u>**Note**</u>: The loads on the left and right sides of the section at point *B* are **equal** in **magnitude** but **opposite** in **direction**. This is so when the two sides are reconnected, the net loads are zero at the section.

Sign convention: For problems in 2D engineers generally use a sign convention to report the three internal loadings N, *V*, and *M*.

- A positive normal force will create tension
- A positive shear force will cause the beam segment on which it acts to rotate clockwise
- A positive bending moment will bend the segment on which it acts in a concave upward manner



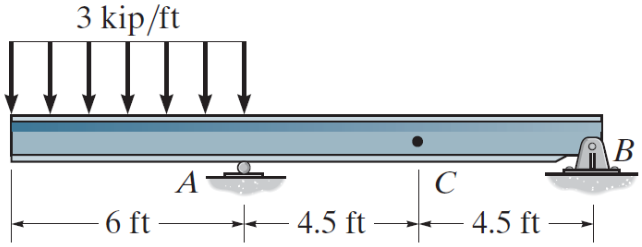
If you consider the right section, it is required to consider the positive direction as follows:



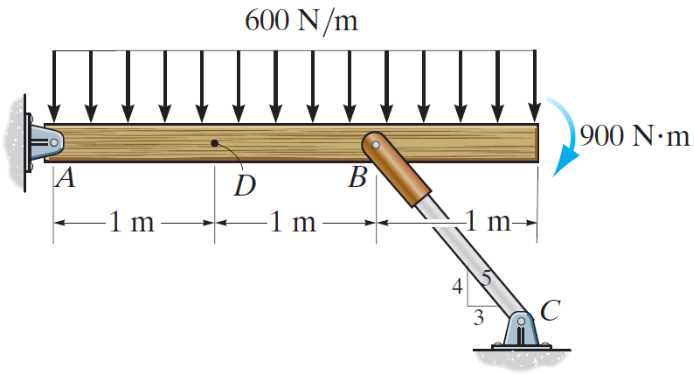
Steps for determining internal forces

- Take an imaginary cut at the point where you need to determine the internal forces. Then, decide which resulting section or piece will be easier to analyze.
- Determine any support reactions or joint forces you need by drawing a FBD of the entire structure and solving for the unknown reactions.
- Draw a FBD of the section you decided to analyze. Include the ***N***, ***V***, and ***M*** loads at the "cut" surface.
- Keep all distributed loadings, couple moments, forces, and support reactions, acting on the member in their exact locations.
- Apply the equilibrium equations to sectional FBD and solve for unknown internal loads and moment.

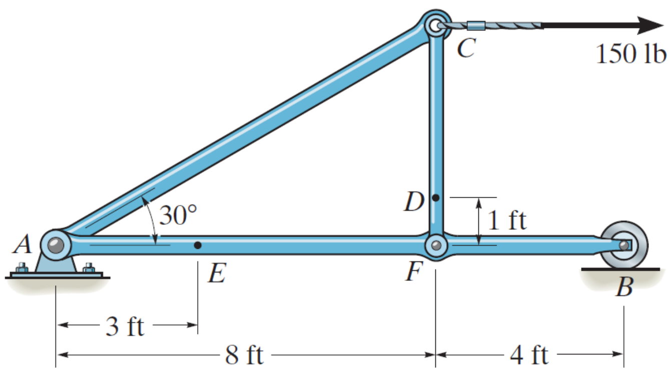
Example 7.1: Determine the normal force, shear force, and moment at point *C*.



Example 7.2: Determine the internal normal force, shear force, and moment at point *D* in the beam.



Example 7.3: Determine the internal normal force, shear force, and moment at points *D* and *E* of the frame.



Chapter 8

Friction

Chapter 8: Friction

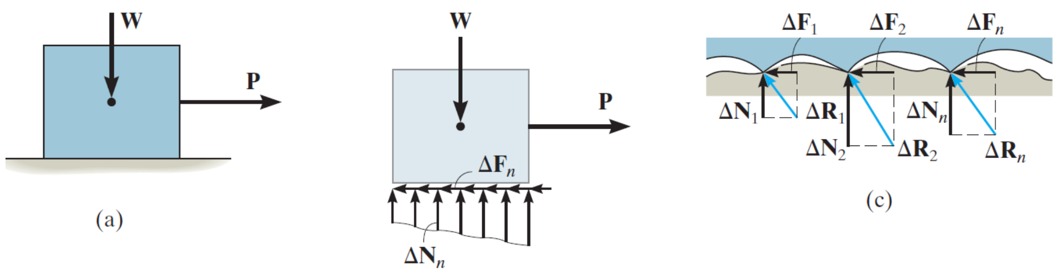
8.1: Friction

Friction is a force that resists the movement of two contacting surfaces that are sliding relative to one another.

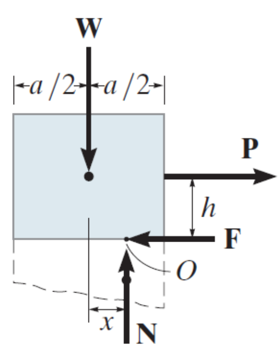
- Friction always acts tangent to the surface at the contact points
- Friction opposes the possible or existing motion between the surfaces.

In this section, we will study the effects of **dry friction** (no lubricating fluid between surfaces).

To understand the theory of dry friction, imagine pulling horizontally on a block of uniform weight W which is resting on a rough horizontal surface that is non-rigid or deformable:



The effect of the distributed normal and frictional loadings is indicated by their resultants $\vec{N}$ and $\vec{F}$ on the free-body diagram:



Resultant Normal
and Frictional Forces

<u>Note</u>: The normal force is offset from the usual location of below the center of gravity, G, to counter the tipping caused by the force $\vec{P}$. The offset "x" is determined by taking moments about the point of application of $\vec{N}$.

If $\vec{P}$ is increased slowly from zero, eventually a magnitude is reached where the box is about to move (**impending motion**). Any further increase in $\vec{P}$ will cause the box to **start sliding**. When a body is about to move, the friction force is a **maximum** and is defined as:

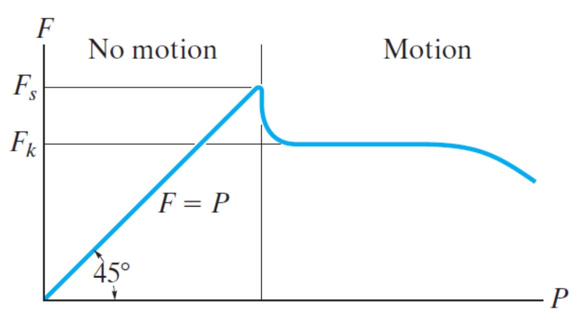
- $\vec{F}_s$ is the static friction force
- μ_s is the coefficient of static friction

$\vec{F}_s$ is the maximum possible friction force, and only exists if an object is on the **verge of sliding** (or impending motion).

If $\vec{P}$ is increased above $\vec{F}_s$, the box will begin to slide, and the friction force suddenly **drops** as the box tends to "ride" on the points of contact. The friction force in this stage will be defined as:

- $\vec{F}_k$ is the kinetic friction force
- μ_k is the coefficient of kinetic friction
- $\vec{F}_k$ is always **less** than $\vec{F}_s$, and μ_k is typically 75% of μ_s

The graph below summarizes the above effects of friction, showing the variation of the frictional force $\vec{F}$ versus the applied load $\vec{P}$.

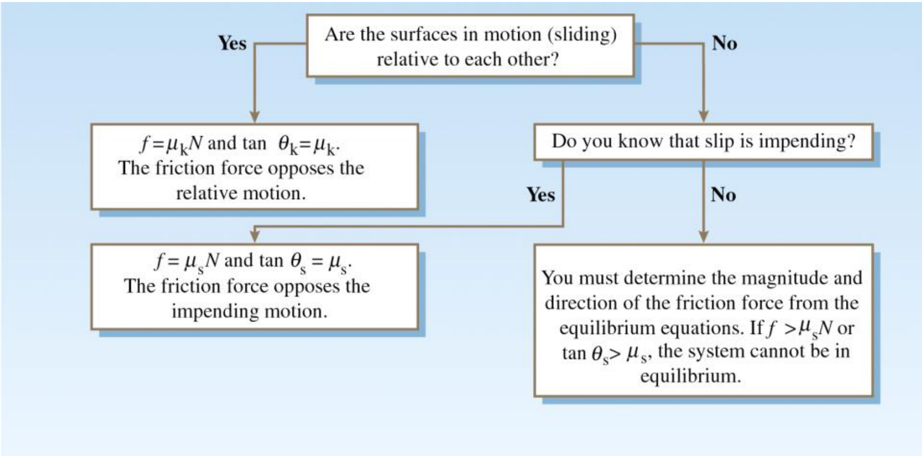


The frictional force, $\vec{F}$, is categorized in three different ways:

- **Static frictional force** when equilibrium is maintained.
- **Limiting static frictional force** ($\vec{F}_s$) when at a maximum value needed to maintain equilibrium.
- **Kinetic frictional force** ($\vec{F}_k$) when sliding occurs at the contacting surface.

Is it important to draw the correct direction of a friction force in a FBD? That depends:

- If the friction force is being solved as an unknown force, **NO**.
  - Assume an arbitrary direction for the friction force.
  - A negative sign on the result indicates the incorrect direction.
- If the friction force is defined by the equation $F = \mu N$ for either static or kinetic friction, **YES**.
  - We would know the object is on the verge of slip, or it is slipping.
  - That information would uniquely identify the direction of the friction force on an FBD.

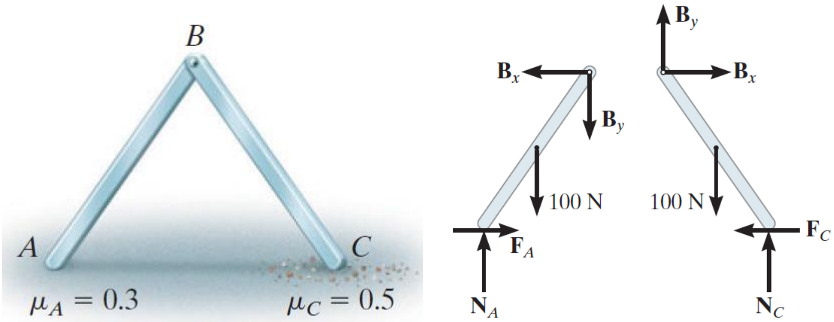


8.2: Types of Friction Problems

In general, there are **four** types of mechanics problems involving dry friction. They can easily be classified once we know the FBD, the number of unknowns and the number of equilibrium equations.

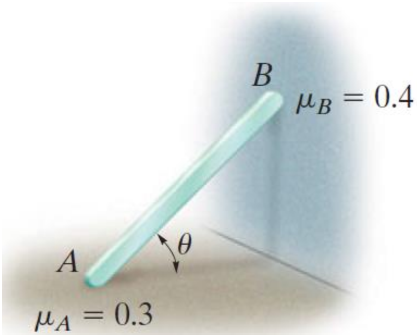
No apparent impending motion: (strictly equilibrium problems)

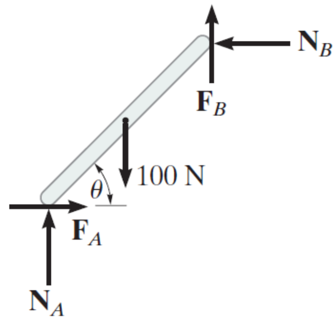
- The number of unknowns are equal to the number of available equilibrium equations.
- Determined frictional forces must be checked to ensure they satisfy the inequality $F \leq \mu_s N$
  - If untrue, slipping occurs and the body does not remain in equilibrium.
- If either check is untrue impending motion exists at that location, see next paragraph.



Impending motion at all points of contact:

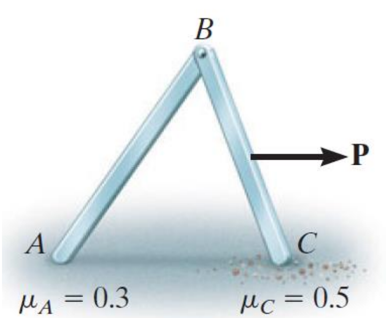
- The total number of unknowns **equals** the total number of available equilibrium equations plus the total number of available frictional equations, $F = \mu N$.
- Remember for impending motion, then $\vec{F_s} = \mu_s N$; whereas if the body is slipping, then $\vec{F_k} = \mu_k N$.

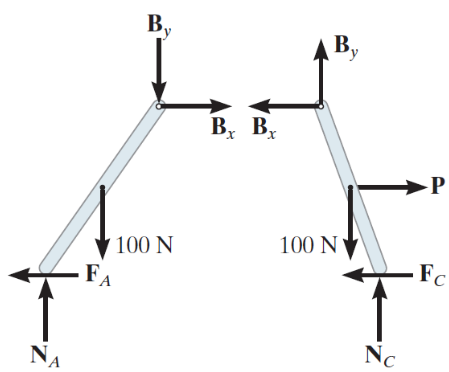




Impending motion at some points of contact:

- The number of unknowns will be **less** than the number of available equilibrium equations plus the number of available frictional equations or conditional equations for tipping.
- There are several possibilities for motion or impending motion, thus we must determine which kind of motion actually occurs.





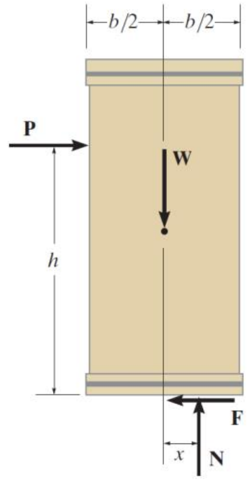
Slip or Tip:

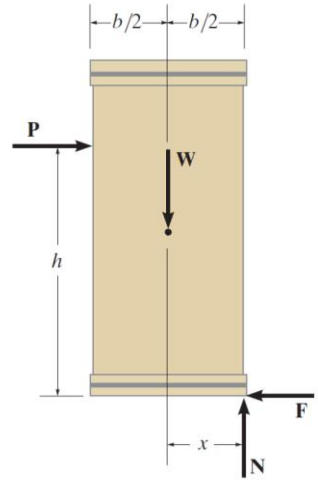
Imagine you are pushing on a crate sitting on a rough surface with a weight *W*.

- With a small magnitude of $\vec{P}$, the crate will remain in equilibrium.
- As $\vec{P}$ increases the crate will either:
  - Be on the verge of slipping on the surface ($F = \mu_s N$)
  - Or if the surface is very rough (large μ_s), then the normal force shifts to the corner ($x = b/2$). At this point the crate begins to tip over.

The crate has a greater chance of tipping if:

- the surface is very rough (large μ_s)
- $\vec{P}$ is applied at a greater height h
- its width b is relatively small





Will an applied force $\vec{P}$ cause the box to slip or tip?

- Determine the location of $\vec{N}$ (*i.e.* find x). Take moments about the point of application of $\vec{N}$.
  - If $x > b/2$ the box will tip.
  - If $x < b = 2$, the box will not tip, but will it slip? Find the magnitude of the friction $\vec{F}$.
    - ✓ If $F > \mu_s N$ the box slips.
    - ✓ if $F < \mu_s N$ the box does not slip.

What is the maximum value of $\vec{\text{P}}$ such that the box will neither slip nor tip?

- Set $x = b/2$, *i.e.* the box is about to tip. Solve for $\vec{P}$.
- But will the box slip before it tips? Solve for $\vec{F}$.
- If $F < \mu_s N$ the box will tip before slip and you are done.
- If $F = \mu_s N$ the box will slip and tip at the same instant, the location of $\vec{N}$ is correct, you are done.
- If $F > \mu_s N$ the box slips before it tips, and the location of the $\vec{N}$ is incorrect (*i.e.* $x < b/2$).
  - Set $F = \mu_s N$, and solve for x and $\vec{P}$.

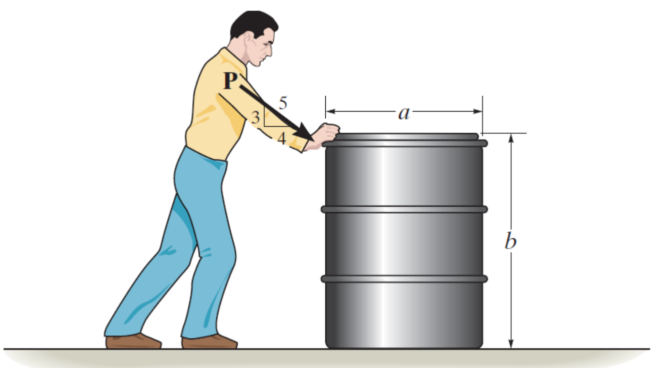
Hints for solving friction problems

- Draw the necessary FBDs, and unless stated in the problem that impending motion or slipping occurs, always show the frictional forces as unknowns (*i.e.*, do not assume $F = \mu N$).
- Determine the number of unknowns and compare to the number of available equilibrium equations
  - If there are more unknowns than equations of equilibrium, it is necessary to apply the frictional equation at some (if not all) contact points to obtain the extra equations needed.
- If the equation $F = \mu N$ is used, you must show $\vec{F}$ acting in the correct direction on the FBD.

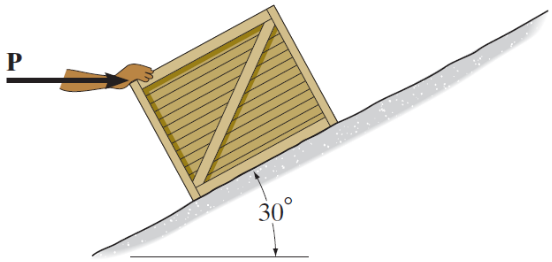
Example 8.1: The drum has a weight of 100 lb and rests on the floor for which the coefficient of static friction is $\mu_s = 0.6$.

a) If a=2 ft and b=3 ft, determine the smallest magnitude of the force $\vec{P}$ that will cause impending motion of the drum.

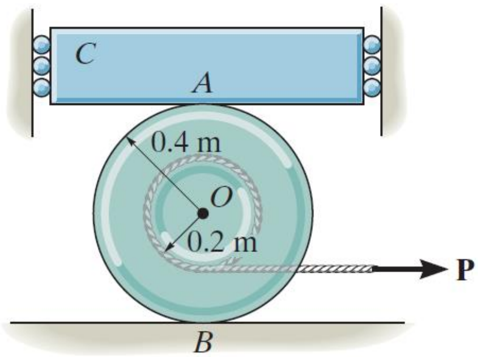
b) Assume $\mu_s = 0.5$, a=3 ft and b=4 ft and determine the smallest magnitude of the force $\vec{P}$ that will cause impending motion of the drum.



Example 8.2: A horizontal force of $\vec{P} = 100\ N$ is just sufficient to hold the crate from sliding down the plane, and a horizontal force of $\vec{P} = 300\ N$ is required to just push the crate up the plane. Determine the coefficient of static friction between the plane and the crate, and find the mass of the crate.

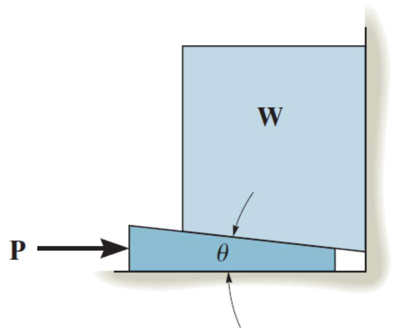


Example 8.3: Block C has a mass of 50 kg and is confined between two walls by smooth rollers. If the block rests on top of the 40-kg spool, determine the minimum cable force P needed to move the spool. The cable is wrapped around the spool's inner core. The coefficients of static friction at A and B are $\mu_A = 0.3$ and $\mu_B = 0.6$.

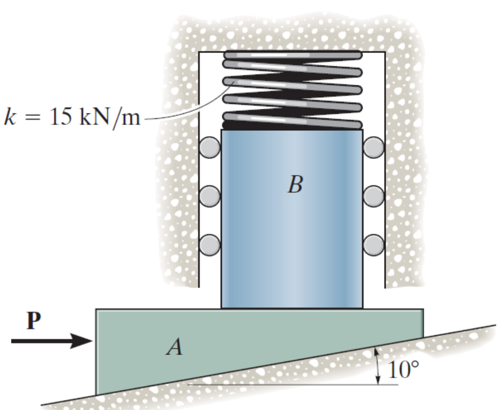


8.3: Wedges

A wedge is a simple machine that is often used to transform an applied force into much larger forces, directed at approximately right angles to the applied force. Wedges also can be used to slightly move or adjust heavy loads. Consider the wedge shown below, which is used to lift the block by applying a force to the wedge. The weight of the wedge is not usually considered in the free body diagram since it is usually small compared to the weight of the block.



Example 8.4: Determine the minimum applied force P required to move wedge A to the right. The spring is compressed a distance of 175 mm. Neglect the weight of A and B. The coefficient of static friction for all contacting surfaces is $\mu_s = 0.35$. Neglect friction at the rollers.



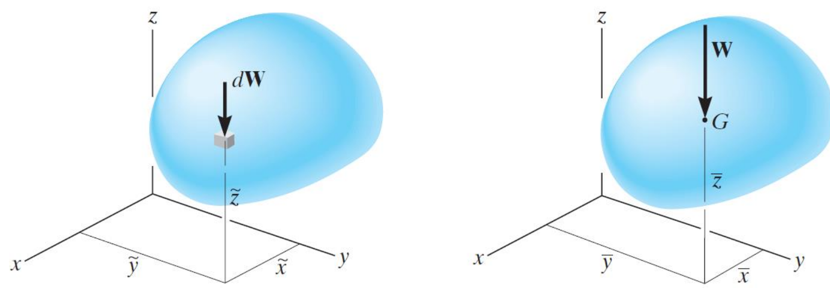
Chapter 9

Centroids, Fluid Pressure

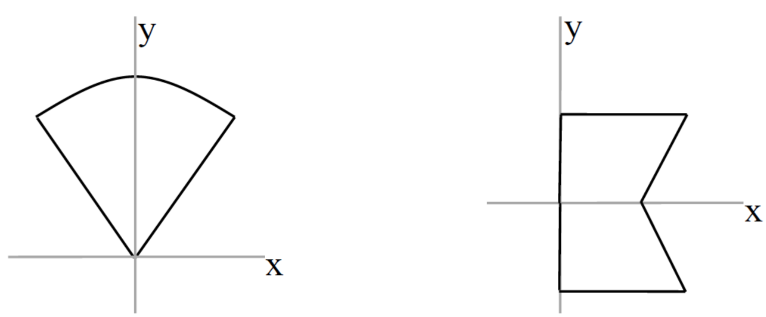
Chapter 9: Centroids, Fluid Pressure

9.1: Centroids and Center of Gravity

The weight of a body is not a point load (force) as we have assumed, but is actually **distributed** throughout the body. The weight can be represented either by a large number of forces (1 per particle) or by a **single force** (entire weight) at a single point – namely the **Center of Gravity** or Center of Mass.



- The **centroid** represents the geometric center of a body, or the Center of Area
- The Center of Gravity is located at the centroid for bodies with homogeneous and uniform material
- A body has a line (or axis) of symmetry if the points on either side of that line are mirror images.
- The centroid always lies on the line of symmetry.
- If there are 2 lines of symmetry, then the centroid is at the intersection of the lines.



A **composite body** has a series of connected "simpler" shaped bodies (rectangular, triangular, semicircular, *etc*.). These bodies can be divided into its composite parts. Provided the centroid locations for each of these parts are known, we can determine the centroid of the entire area.

- Break up the shape into sections or parts
- Establish the coordinate axes on the sketch
- Determine the coordinates of the center of gravity or centroid of each part (use the provided table).
- Compute the centroid for the entire area by the following equations

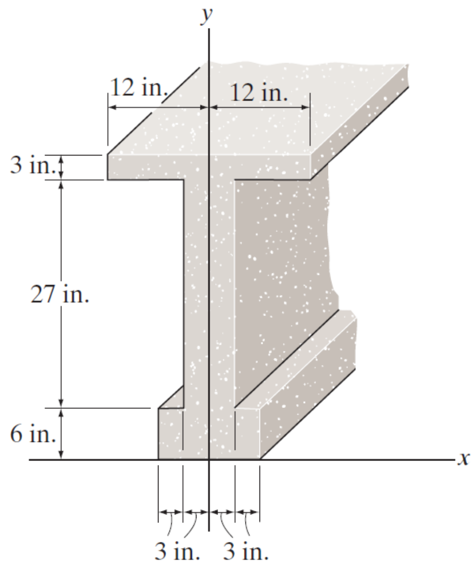
$$\bar{x} = \frac{\sum_{i=1}^{n} \tilde{x}_i A_i}{\sum_{i=1}^{n} A_i} \qquad \bar{y} = \frac{\sum_{i=1}^{n} \tilde{y}_i A_i}{\sum_{i=1}^{n} A_i} \qquad \bar{z} = \frac{\sum_{i=1}^{n} \tilde{z}_i A_i}{\sum_{i=1}^{n} A_i}$$

- $\bar{x}$, $\bar{y}$ and $\bar{z}$ represent the coordinates of the centroid of the entire body.
- $\tilde{x}_i$, $\tilde{y}_i$ and $\tilde{z}_i$ represent the coordinates of the centroid of each part of the body.
- $\sum_{i=1}^{n} A_i$ is the sum of the areas of all the parts of the body (simply the total area of the body)

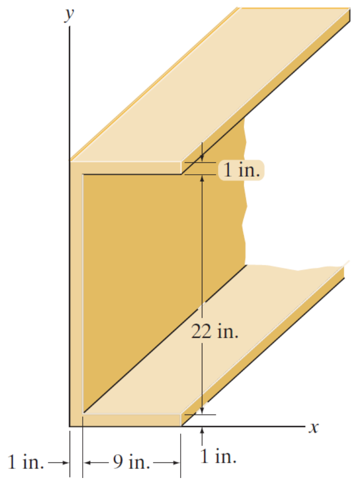
➢ If a body has a hole (a geometric region with no material), first consider the body without the hole. Then, consider the hole as an additional part with **negative** area.

➢ For clarity, the calculations can be arranged in tabular form, as shown in the following examples.

Example 9.1: Locate the centroid of the beam's cross-sectional area.

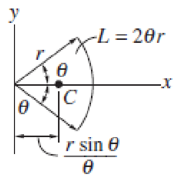


Example 9.2: Locate the centroid of the cross-sectional area of the channel.

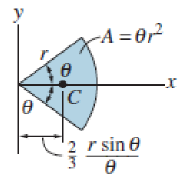


Geometric Properties of Line and Area Elements

Centroid Location	Centroid Location	Area Moment of Inertia



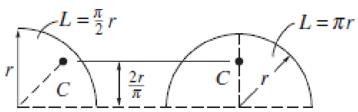
Circular arc segment



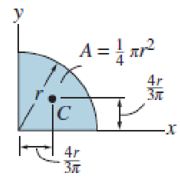
Circular sector area

$$I_x = \tfrac{1}{4} r^4 (\theta - \tfrac{1}{2} \sin 2\theta)$$

$$I_y = \tfrac{1}{4} r^4 (\theta + \tfrac{1}{2} \sin 2\theta)$$



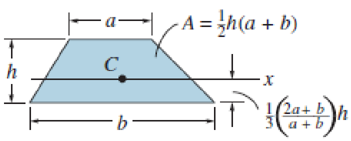
Quarter and semicircle arcs



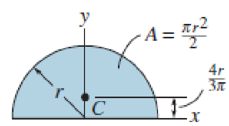
Quarter circle area

$$I_x = \tfrac{1}{16} \pi r^4$$

$$I_y = \tfrac{1}{16} \pi r^4$$



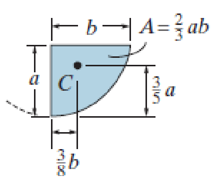
Trapezoidal area



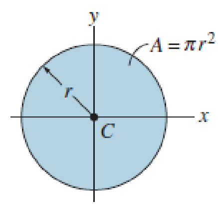
Semicircular area

$$I_x = \tfrac{1}{8} \pi r^4$$

$$I_y = \tfrac{1}{8} \pi r^4$$



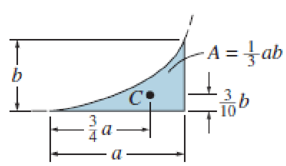
Semiparabolic area



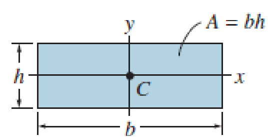
Circular area

$$I_x = \tfrac{1}{4} \pi r^4$$

$$I_y = \tfrac{1}{4} \pi r^4$$



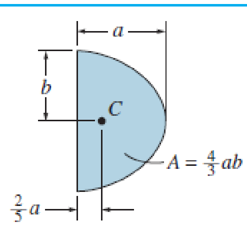
Exparabolic area



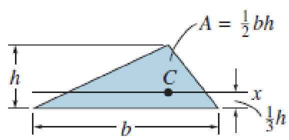
Rectangular area

$$I_x = \tfrac{1}{12} bh^3$$

$$I_y = \tfrac{1}{12} hb^3$$



Parabolic area



Triangular area

$$I_x = \tfrac{1}{36} bh^3$$

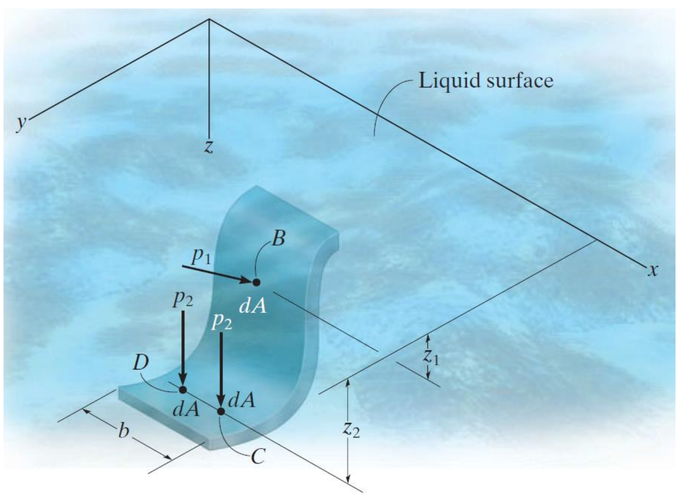
9.2: Fluid Pressure

A fluid at rest creates a pressure p at a point that is the same in all directions (Pascal's Law). The magnitude of pressure, measured as a force per unit area, depends on:

- the **specific weight** γ or **mass density** ρ of the fluid
- the **depth** z of the point from the fluid surface

where g is the acceleration due to gravity. For water, $\gamma = 9810\ \frac{\mathrm{N}}{\mathrm{m}^3}$ and $\rho = 1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$.

This equation is valid only for fluids assumed to be incompressible and for stationary fluid pressures. The common unit for pressure in SI units is the pascal (Pa), which is the same as newton per square meter $(\frac{\mathrm{N}}{\mathrm{m}^2})$. In the US customary system, fluid pressure is often expressed in pounds per square inch $(\frac{\mathrm{lb}}{\mathrm{in}^2})$ or psi.

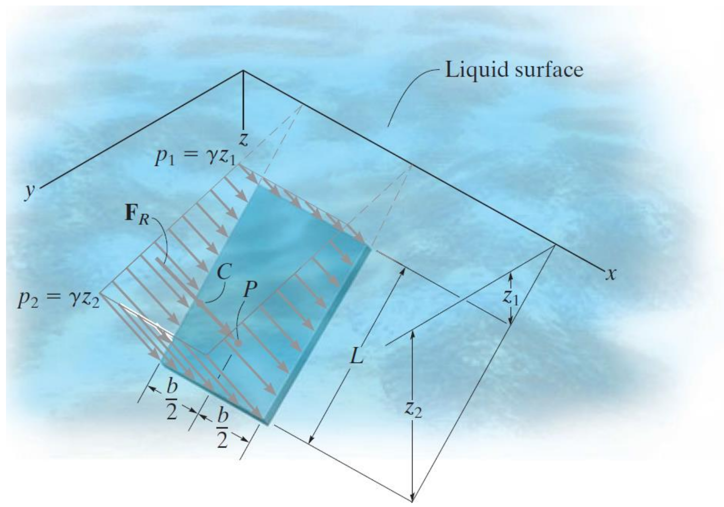


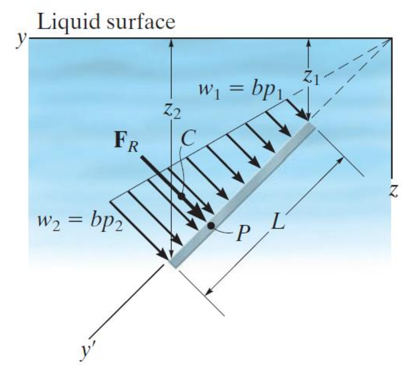
The integration constant in deriving the equation $P = \rho g z$ is neglected. The complete form of equation is:

The pressure P_0 is the pressure on the surface of the liquid where $h = 0$. If P_0 is due to atmospheric pressure and the measuring instrument records only the increment above atmospheric pressure, the measurement gives what is called **gage pressure**.

Hydrostatic Pressure on Submerged Flat Plate of Constant Width

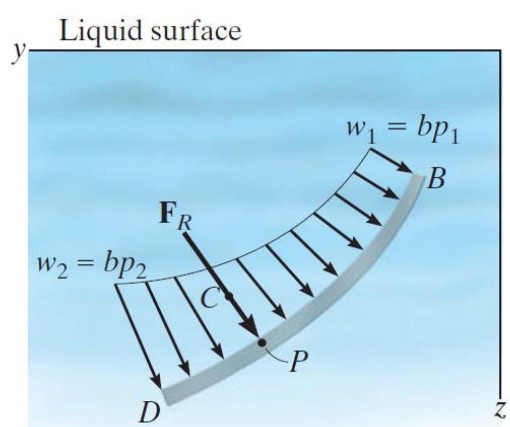
Consider a flat rectangular plate of constant width, which is submerged in a liquid having a specific weight γ. Since pressure varies linearly with depth, the distribution of pressure over the rectangular plate's surface is represented by a trapezoidal volume having an intensity of $P_1 = \gamma z_1$ at depth z_1 and $P_2 = \gamma z_2$ at depth z_2. The magnitude of the resultant pressure force is equal to the volume of this loading diagram and has a line of action that passes through the **centroid of the distributed load** and **not the centroid of the plate**. The pressure always acts normal to the surface area at each point.



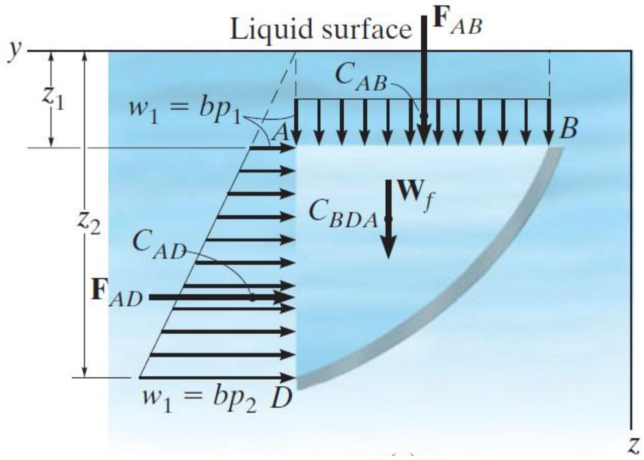


Hydrostatic Pressure on Submerged Curved Plate of Constant Width

When a submerged plate of is curved, the pressure acting normal to the plate continually changes both its magnitude and **direction**.



Integration can be implemented to determine the resultant force, but a simpler method separates calculations for horizontal and vertical components of the resultant force. Consider the following three components of the force:

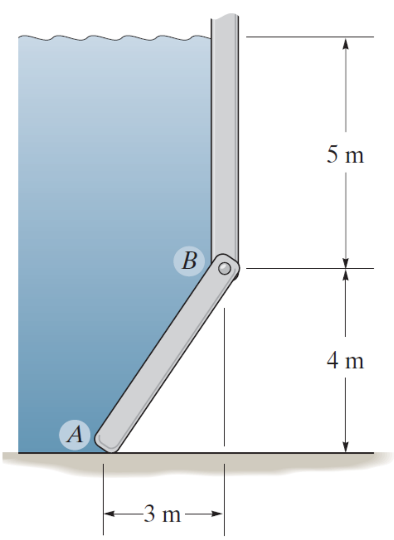


- The plate supports the weight of liquid W_f contained within the block BDA.
  - The magnitude of this force is $W_f = \gamma . b . (area_{BDA})$
  - This force acts through the centroid of BDA
- The forces F_{AB} and F_{AD} are the equivalent resultant forces of the distributed loads on AB and AD.

Summing these three forces yields:

Finally, the location of the center of pressure on the plate is determined by applying the moment of the resultant force about a convenient reference point such as *D* or *B*. This is equivalent to the sum of the moments of the above three forces about this same point.

Example 9.3: The gate *AB* is 8 m wide. Determine the horizontal and vertical components of force acting on the pin at *B* and the vertical reaction at the smooth support *A*. Density of the fluid is $1000\ \frac{\text{kg}}{\text{m}^3}$.



Example 9.4: The arched surface *AB* is shaped in the form of a quarter circle. If it is 8 *m* long, determine the horizontal and vertical components of the resultant force caused by the water acting on the surface. Density of the fluid is $1000\ \frac{\text{kg}}{\text{m}^3}$.