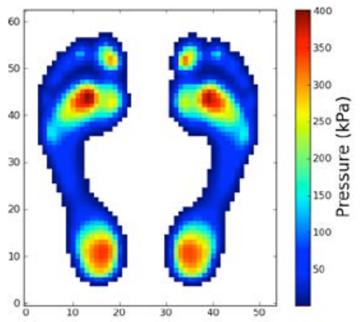
1. Distributed loads

a. Introduction

In the previous set of notes, we discussed briefly the idea of distributed forces as a motivation for studying centroids. The time has come to show that the effort was justified. You may remember that we established that concentrated (point) forces are convenient idealizations. Even when the surface of contact between two interacting bodies (such as the feet and the ground) is very small, many applications (such as the design of running shoes) demand a finer representation in terms of a *pressure*, which is measured in terms of units of force divided by units of area. We even showed (and do it again here) an experimentally obtained graph of the distribution of such pressure on the footprint of an actual subject standing upright on an instrumented measuring platform. Another important example of distributed loads is provided by the weight of an object. This weight is certainly not a concentrated force that can be represented by a single vector. The weight, rather, acts on every element of volume of the body, no matter how small. This *body-force* is measured in terms of units of force divided by units of volume. Finally, when analyzing plane frames or beams (as we shall do below) we have forces that may be distributed over the length of the members. Think, for example, of a beam supporting a brick wall (which may even be of variable height). As a result, we have a distributed line load (which may be of variable intensity) measured in terms of units of force divided by units of length. It is this last case that we want to study in greater detail.

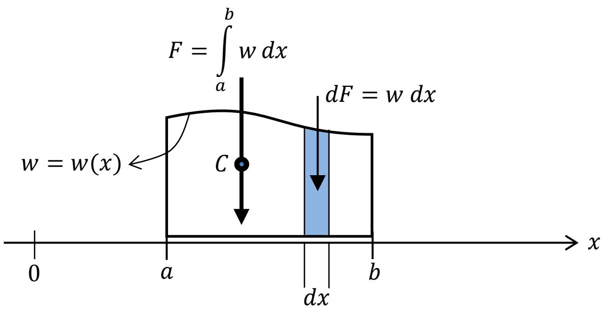


b. Distributed line-loads

Assume we have been given a load distributed over a line in terms of a function

$$w = w(x).$$

In this equation, w measures the intensity of the distributed load measured in, say, N/m or lb/ft. This intensity is a function of the running line coordinate x. In the example of the brick wall, clearly the value of w at a location x is proportional to the height of the wall at that particular point.



What is the relation between this w and the total weight of the wall? The way to answer this is to ask: How much force is being applied at a small segment of length dx? Since this is a very small interval (and we are planning to go to the limit eventually), the variation of w within this small interval is negligible and we can take the value of w at any point $\bar{x}$ of this interval, such as the mid-point of dx. We conclude that the total force dF acting on dx is given by

$$dF\,(\bar{x}) = w(\,\bar{x})\;dx$$

This value (measured now in units of force) can be regarded, if one so wishes, as the area of the thin rectangle with base dx and height $w(\bar{x})$. But we can repeat this reasoning in each one of the intervals dx of a fine subdivision of the segment $[a, b]$ on which the distributed load is acting. The total force is, therefore, the sum of all these small areas. In other words, the limit of this sum is nothing but the integral of the function $w(x)$ over the segment. Geometrically, we may say that *the area under the graph of the distributed loading represents the total force F being applied.*

Recall one of the most important concepts of this course: Two systems of forces are *statically equivalent* if, and only if, they produce the same sum of forces and the same sum of moments with respect to any directions whatever. Moreover, for a system of forces in the plane, we proved that the system is always statically equivalent to a single force (called <u>the</u> resultant) equal to the sum of all the forces and acting at a particular line of action. But we have indeed, for our distributed loading, a system of (parallel) forces, except that we have infinitely many forces of infinitesimally small magnitudes $dF(\bar{x})$. We have already determined that this resultant is given in magnitude by the area under the loading diagram. All we need to determine now is its line of action. It is not difficult to prove that the line of action must pass through the centroid C of the area under the graph of the loading diagram.

We summarize our result as follows: *A distributed load acting in a direction perpendicular to the x axis over a segment $[a, b]$ is statically equivalent to a single force (also perpendicular to this axis) of magnitude equal to the area under the graph of the loading and passing through its centroid.*

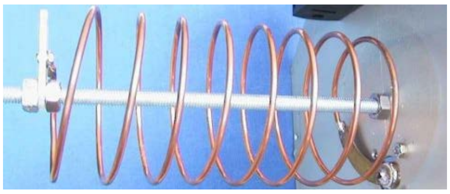
<u>Warning</u>: When we say that the distributed load is statically equivalent to the resultant (equal to the area of the diagram and passing through its centroid) <u>we don't mean that these two things are the same</u>. What we mean is that they are statically equivalent, namely, they produce the same sum of forces and the same sum of moments, and nothing

more or less than that. In particular, we cannot cavalierly replace one by the other, except when we have already drawn a free-body diagram and when all we are about to do in that diagram is to take sums of forces and sums of moments. We are not allowed to do this replacement in the structure itself. We can only do it in a correctly drawn FBD and only for the purposes of summing forces and/or moments in that particular FBD.

<u>Somewhat useful remark</u>: Some people find it convenient to relate the discussion above with what we do with the graph of a velocity as a function of time. If we are given this graph (obtained, for example, by recording the measurement of the speedometer needle of the car at regular intervals (every few seconds, say) and curve-fitting the points so obtained in a speed-versus-time Cartesian coordinate system) we have a quantity analogous to a distributed load. Here, we have a graph $v = v(t)$ measured in units of distance over units of time. If we were to ask now: what is the distance traveled between two instants of time t_0 and t_1, we would reason exactly as above. We would divide the time interval $[t_0, t_1]$ into small intervals of extent dt and calculate the small distance ds traveled by the simple formula $ds = v(t)\ dt$. The total distance traveled is given by the sum (that is, the integral) of these infinitesimal quantities, which can be regarded geometrically as the area under the velocity graph. This is only an analogy and not a very good one, but it may help to trigger the right idea.

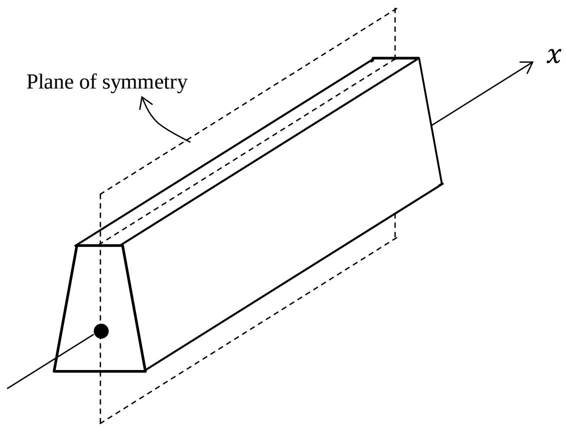
2. <u>Beams</u>

a. <u>Definition</u>



In the most general sense, a *beam* is a structure in which one dimension predominates over the other two. Thus, a curved wire is a beam. It has been said that a beam is a curve with some flesh. The curve is called the *beam axis*. A coordinate along the beam axis is said to run along the *length* of the beam. Cutting the beam with a plane perpendicular to this curve at a point we obtain a *cross section* of the beam. The dimensions of the cross sections are, according to the definition, very small when compared with the length of the beam axis.

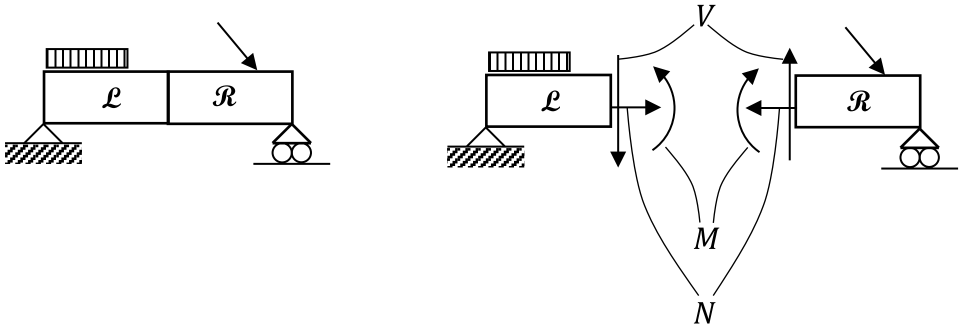
Curved spatial beams are very difficult to analyze. Consequently, we are going to limit our discussion to some of the most commonly occurring beams in structural engineering and in many other applications from the nanoscale (such as carbon nanotubes) to high-rise building and space stations. The beams we want to consider have a straight axis and, moreover, have a plane of symmetry. The figure below shows such a beam in the shape of a prism with a trapezoidal cross section. Each cross section has an axis of symmetry and the collection of these axes of symmetry forms a plane of symmetry for the whole body. The axis of the beam will be identified with the line determined by the centroids of the cross sections.



Moreover, we are going to assume that all the loads (whether concentrated or distributed) act in the plane of symmetry and the supports are arranged symmetrically with respect to this plane. Under these conditions, all the action takes place in the plane of symmetry, and we obtain what is known as a *plane beam*. For a plane beam, we need not worry about the third dimension. Our drawing of the beam can be boiled down to a line (or a slender rectangle) in the plane of symmetry, which we identify with the plane of the drawing. It is customary to place a beam horizontally along the x axis, but this is not necessary. Most of the common loads in horizontal beams are vertical forces, but, again, this is not a necessity for the analysis of plane beams. For example, a bridge beam (or girder) may be subjected to the vertical weight of a truck crossing the bridge. But, if the driver suddenly applies the brakes, a horizontal force is generated that has to be absorbed by the supports.

b. <u>Internal "forces" in plane beams</u>

When a beam is loaded, what makes the beam remain as an integral unit rather than flying apart in a zillion pieces? The answer from the physicochemical point of view is found in the chemical bonds and the inter-atomic forces responsible for the maintenance of order in a solid material. But from the engineering point of view, which approximates (very accurately, by the way) the macroscopic properties of a solid by assuming that it is a continuum, this answer is rather useless. Instead, let us assume that a beam has been broken through a cross section that splits the beam into two parts, $\mathcal{L}$ and $\mathcal{R}$. We want to repair the beam by using epoxy glue or a cyanoacrylate (such as Krazy Glue®). If the repair is successful, we conclude that *internal contact forces* provided by the glue between the two sides of the cross section are doing the job of the molecular bonds. What kind of forces is the glue subjected to? Obviously, these cannot be concentrated forces! They are distributed loads measured in units of force divided by units of area. But whatever these distributed forces are (in direction and in magnitude), we know one thing with absolute certainty: The system of these forces acting on each side of the repaired cross section can be reduced to a point (say, the centroid of the cross section) and provide us with a statically equivalent system consisting of a force and a couple. Since the distributed gluing forces on one side of the cross section are the reactions corresponding to their counterparts acting on the other side, we conclude that this total force and this couple are also mutually equilibrated. This situation is depicted in the figure below.

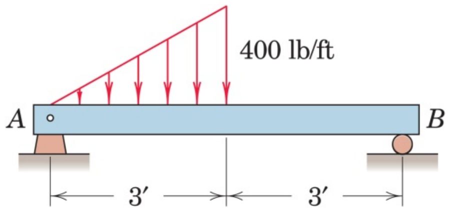


We have resolved the force into a component perpendicular to the cross section and a component tangential to it. We call the perpendicular component the *axial* or *normal force*, denoted by N. The tangential component is called the *shear force* V, and the couple is known as the *bending moment* M. There is some subtlety, however. Each one of the internal "forces" is constituted by a pair. The axial force, for example, with which we are familiar from trusses, is the pair of forces made up by the force that the part $\mathscr{R}$ (to the right of the cross section) exerts on the part $\mathscr{L}$ (to the left of the cross section), and the reaction force exerted by the left part $\mathscr{L}$ on the right part $\mathscr{R}$. If these forces point towards each other, we will call it (the pair) a positive axial force N. If the forces point away from each other, we talk about a negative axial force. This is just a sign convention. For the shear force V, we will use the convention that if the force acting on the right is upwards (and, therefore, the force acting on the left is downwards) we will call it (the shear "force", namely, the pair) positive. For the bending moment M we use the convention that it is positive if the couple on the right is clockwise. These conventions vary from country to country, and sometimes from department to department.

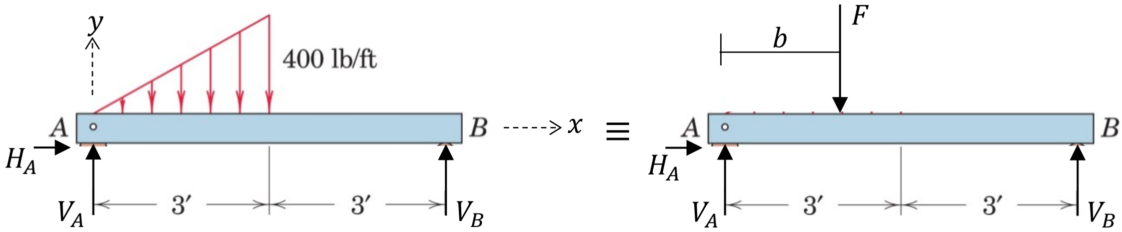
The importance of the internal forces and couples is that they are responsible for the integrity of the structure. Therefore, when designing the beam, the engineer has to choose a material and a size of the cross section that are able to withstand these internal effects. The process leading to such choices will be the subject of a further course on Mechanics of Materials (ENGG 317).

c. <u>A detailed example</u>: Determine the axial force, shear force and bending moment acting at the cross section $a - a$ located at $2'$ to the right of the support A.

<u>Solution</u>: The solution will consist of two parts, namely: (i) Find the reactions at the supports from the FBD of the whole beam; (ii) Find the internal effects at the desired cross section by means of the equilibrium of the FBD of a subsystem consisting of the part to the left (or to the right) of the cross section. The logic for part (ii) is (as in the case of a planetary system) that, if we want to calculate the internal forces, we must expose them by separating the system into two subsystems, on each of which the internal forces act externally. We have discussed these issues already in general terms before.



(i) The FBD of the beam as a whole is shown below. Once the FBD has been drawn, we are at liberty to replace the distributed force by its resultant. The total force F is given



by the area of the distributed load diagram, as we have found from the theory. Thus,

$$F = \frac{1}{2}(3ft)\left(400\frac{lb}{ft}\right) = 600\ lb.$$

According to the theory, the equivalent force passes through the centroid of the distributed load diagram. Since this diagram has the shape of a triangle, its centroid is located at a horizontal distance from the vertex equal to

$$b = \frac{2}{3}\ (3\ ft) = 2\ ft.$$

The equilibrium equations are

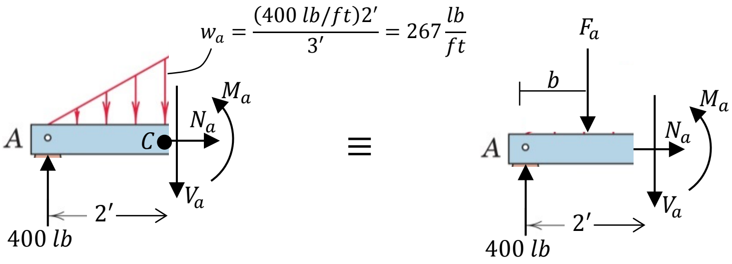
$$\Sigma\, F_x = H_A = 0$$
$$\Sigma\, F_y = V_A - 600\ lb + V_B = 0$$
$$\Sigma\, M_A = -(600\ lb)(2\ ft) + V_B(6\ ft) = 0.$$

Solving this system we obtain the reactions as

$$H_A = 0, \qquad V_A = 400\ lb\ \ (as\ shown) \qquad V_B = 200\ lb\ \ (as\ shown)$$

(ii) Now we choose either the part to the left of the indicated cross section or, equivalently, the part to the right. For the sake of pedagogy, we will do both. The results should be identical. It is very important, as we have already remarked, to draw the FBD *before doing any replacement of any forces by equivalent systems*. First we cut the beam and draw whatever part of the distributed force is captured by the chosen subsystem, and only then are we permitted to carry out any statically equivalent replacement we wish. Thus, choosing the portion of the beam to the left of the cross section indicated by the statement of the problem we obtain the free-body diagram below.

The chopped load diagram attains the maximum value w_a obtained by straightforward similarity of triangles. Only now are we allowed to replace the diagram by its resultant, as shown on the right of the drawing. The total force F_a is given by the area of this smaller triangle captured by our FBD, namely,



$$F_a = \frac{1}{2}(2\ ft)\left(267\frac{lb}{ft}\right) = 267\ lb.$$

The centroid, through which this force passes, is located at a distance

$$b = \frac{2}{3}\ (2ft) = 1.33\ ft$$

from the left support A. Enforcing the equilibrium equations, we obtain

$$\Sigma\, F_x = N_a = 0.$$
$$\Sigma\, F_y = 400\ lb - 267\ lb - V_a = 0$$
$$\Sigma\, M_C = -(400\ lb)(2\ ft) + 267\ lb\ (2' - 1.33') + M_a = 0.$$

The solution of this system is

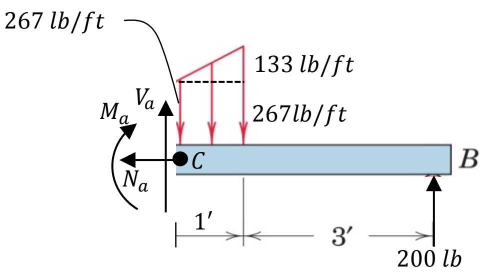
$$N_a = 0 \qquad V_a = 133\ lb\quad (as\ shown) \qquad M_a = 621\ lb.ft\quad (as\ shown).$$

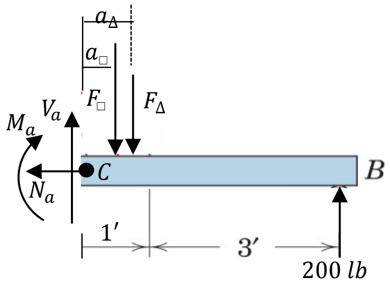
This is the required result. For the sake of illustration, however, we will now show that, had we chosen the part to the right of the cross section, we would have obtained exactly the same result. We obtain a trapezoidal loading diagram, which we can divide into a rectangle of height $267\ lb/ft$ plus a triangle of height $(400-267)lb/ft = 133\ lb/ft$. We conveniently replace the trapezoidal diagram by two forces of intensities

$$F_{\square} = (1\ ft)\left(267\ \frac{lb}{ft}\right) = 267\ lb \qquad F_{\Delta} = \frac{1}{2}(1ft)\left(133\frac{lb}{ft}\right) = 66.5\ lb$$

Their respective locations are given by

$$a_{\square} = \frac{1}{2}(1ft) = 0.5\ ft \qquad a_{\Delta} = \frac{2}{3}(1\ ft) = 0.67\ ft$$





The equilibrium equations are

$$\Sigma F_x = -N_a = 0$$
$$\Sigma F_y = V_a - 267\ lb - 66.5\ lb + 200\ lb = 0$$
$$\Sigma M_C = -M_a - (267\ lb)(0.5\ ft) - (66.5\ lb)(0.67\ ft) + (200\ lb)(4\ ft) = 0$$

The solution of this system is

$$N_a = 0 \qquad V_a = 133.5\ lb \quad (as\ shown) \qquad M_a = 622\ lb.ft \quad (as\ shown)$$

These values are identical (within the round-off error) to the ones obtained from the previous FBD. This result follows from the fact that, since the whole structure is in equilibrium, the combination of the two smaller FBDs must (by ‘cancelling’ actions and reactions) reproduce the original FBD. Notice that the results are equal in value and also in sign, as expected.