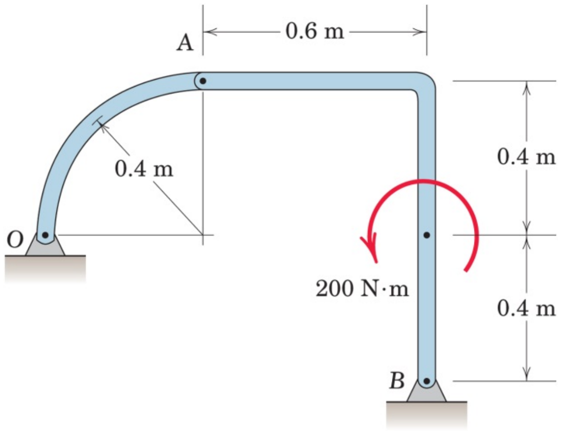
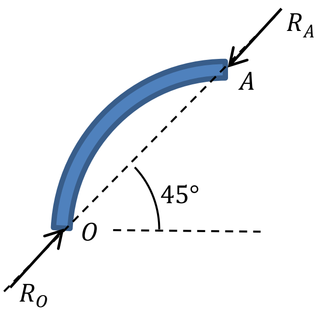
1. More examples of frames

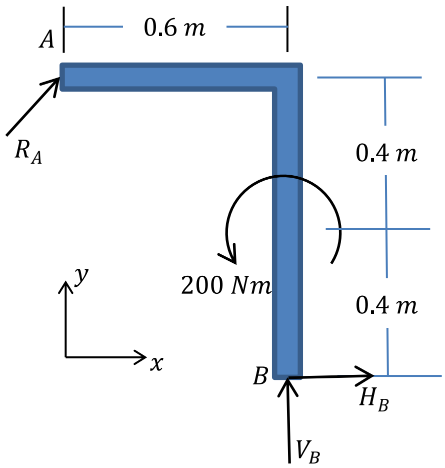
a. A very simple frame: (Problem 4.80) Determine the reactions at the supports.



Solution: Recall that our motto in these matters is "*Remember to dismember!*" In other words, if you dismember a plane frame and draw the FBD of each of its members and write the corresponding 3 equilibrium equations for each member, you have solved the problem, no matter how complicated it may seem, as long as the frame is statically determinate (which will always be the case in this course). On the other hand, there are cases for which part of the dismemberment can be done mentally. This is the case when one of the members of the frame is a two-force member Recall that a two-force member is a (rigid) object subjected to forces (no couples) acting only at two points. In that case, the line of action of the resultant of the forces at each point must coincide necessarily with the line determined by the two points. Moreover, the two resultants must be of equal magnitude and opposite orientations. When we look at our frame, therefore, we notice immediately that member OA is (for the loading shown) a two-force member. Its FBD is necessarily as shown in the figure on the left, without the need for any extra computation. Having disposed of this member, we are left with just the bracket AB, whose FBD we draw below.



Of course we can proceed with the equations of sum of forces. But a cursory inspection reveals that the resultant reaction at *B* must certainly be parallel to the reaction at A and have the same magnitude, since these two forces must amount to a couple that equilibrates the applied couple. Alternatively, we may say that this result is the direct consequence of taking equations of sum of forces in the direction AB and in the normal direction to this line. Either way, taking moments with respect to *B* yields

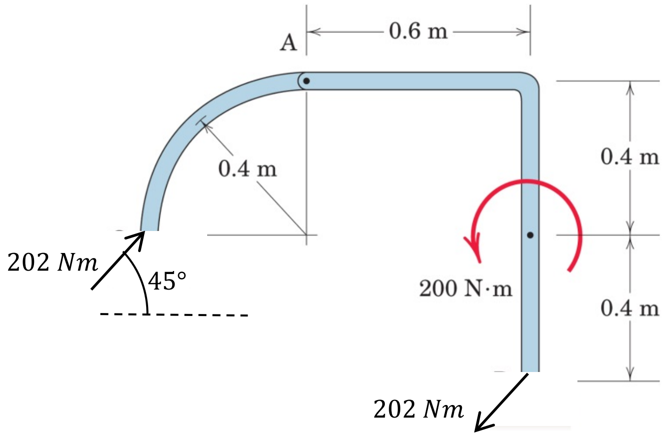


$$\Sigma M_B = -R_A \cos 45° \, (0.8m) - R_A \sin 45° \; (0.6m) + 200 \; Nm = 0$$

Thus

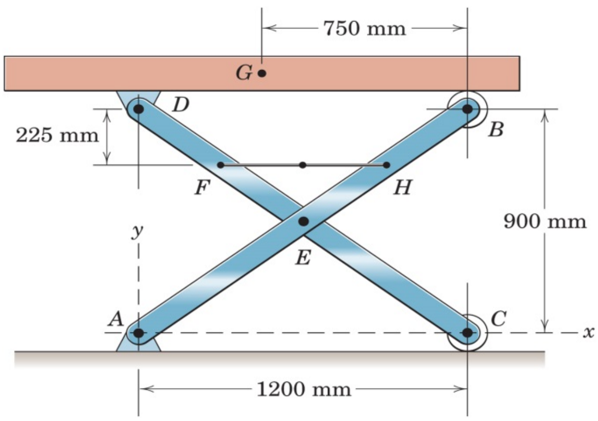
$$R_A = 202 \; N \quad (as \; shown)$$

The final reactions are shown in the FBD of the structure below.

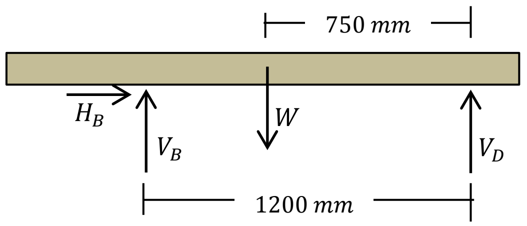


Remark: We have said in past lectures that forces abide by the principle of transmissibility and that couples are free vectors. While true, *these assertions do not mean that we can move forces and couples cavalierly*! As an example, consider the problem we just solved but move the couple from member AB to member OA. In that case, the bracket OA becomes a two-force member and the reactions will be parallel to the line AB. The magnitude of these new reactions will be different from the value we obtained above. What is meant by those principles is the following: We are not allowed to move forces and couples around in the given structure. *Only after a FBD has been drawn (capturing whatever forces and couples are applied on that subsystem) are we allowed to use the principle of transmissibility for forces and to move couples from one place to another.* Why? All that these principles say is that *as far as the equations of equilibrium of a subsystem are concerned*, forces and couples can be displaced. In particular, the internal forces of a structure are strongly dependent on the point of application of forces and couples, as intuition demands.

b. A foldable workbench: (Problem 4.112) The top of the folding workbench has a mass of 50 kg with mass centre at G. Calculate the *x*- and *y*- components of the force supported by the pin at *E*. Note that the link FH must be considered one inextensible member.



Solution: We need ultimately to reach a free-body diagram that exposes the forces at the pin *E*, such as the FBD of member CD. Let us start from the FBD of the tabletop itself, drawn below. We observe that this FBD contains only 3 unknown quantities and, therefore, we can obtain all of them from the equilibrium equations. We obtain



$$\Sigma F_x = H_D = 0$$

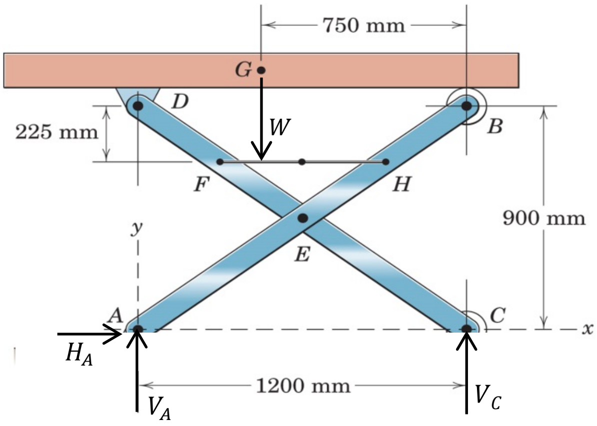
$$\Sigma F_y = V_D + V_B - W = 0$$

$$\Sigma M_B = -V_D(1200mm) + W(750mm) = 0$$

Whence

$$H_D = 0 \qquad V_D = 0.625\,W \quad (as\ shown) \qquad V_B = 0.375\,W \quad (as\ shown)$$

Next, we draw the FBD of the whole structure, as shown below.



$$\Sigma F_x = H_A = 0$$

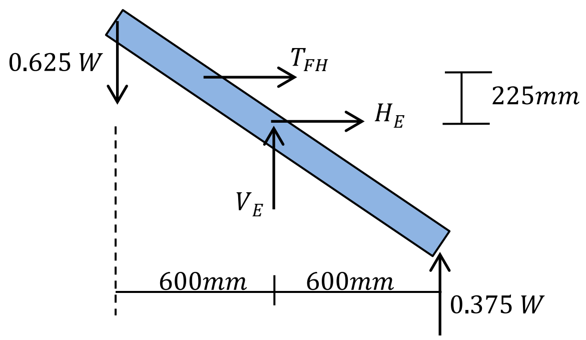
$$\Sigma F_y = V_A + V_C - W = 0$$

$$\Sigma M_C = -V_A(1200mm) + W(750mm) = 0$$

Notice that (because the hinges and the casters are vertically aligned) these equations are the same as above (except for the names of the reactions). We obtain immediately

$$H_A = 0 \qquad V_A = 0.625\,W \quad (as\ shown) \qquad V_C = 0.375\,W \quad (as\ shown)$$

We draw now the FBD for member CD, containing our desired forces.



Notice that the force at point *D* is consistent (as a reaction) with the force obtained from the first free-body diagram.

We write the corresponding equilibrium equations as

$$\Sigma F_x = T_{FH} + H_E = 0$$

$$\Sigma F_y = -0.625W + 0.375W + V_E = 0$$

$$\Sigma M_E = 0.625W(600mm) + 0.375W(600mm) - T_{FH}(225mm) = 0$$

The unknown forces are obtained as

$$T_{EH} = 2{,}67\ W\ (as\ shown) \qquad H_E = -2.67W\ \ (opp.to\ shown) \qquad V_E = 0.25W\ \ (as\ shown)$$

Since $W = 50kg\ (9.81ms^{-2}) = 490.5\ N$, we obtain the final result as

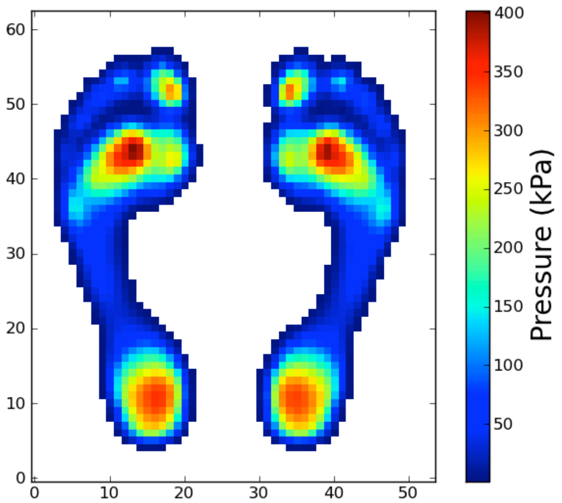
$$H_E = -1310\ N \qquad V_E = 123\ N$$

From the intuitive design point of view, we notice that the force in the link FH (usually a slender tie or wire) is the critical point. If a heavy weight were to be placed on the tabletop, the failure would occur most probably at this tension link.

2. Centroids and centres of mass

a. Introduction and motivation

We have accepted the idea of a force as a vector, without giving any thought as to how forces are actually applied to structures. In particular, we have uncritically accepted the idea that forces have a line of action and a point of application. On the other hand, with almost the same degree of credulity, we have accepted the hypothesis that matter is continuous. These two concepts, namely, the continuity of matter and the point-like nature of forces, seem somewhat incompatible. If we think of the familiar act of pushing a button in an elevator, it should be clear that, no matter how small the area of contact between the finger and the button is, the force is distributed over some non-vanishing area. It is not a point-force. In our Human Performance Laboratory, at the Faculty of Kinesiology, experiments are conducted to measure the force exerted by the foot upon the ground while a subject stands or walks on specially instrumented measuring platforms. A typical result is shown in the figure below. The main point we are trying to make is that this kind of *distributed forces* are the rule rather than the exception. They are measured in units of force per units of area, just like the pressure exerted by a gas or a liquid on the walls of a container, which provide us with another practical example of distributed forces. Another common example is the action of gravity on a body. These are forces distributed over a volume (rather than an area) and are known as *body forces*. When calculating the resultant of a system of distributed forces, and only then, we recover the concept of a *concentrated force* in the traditional sense. It is upon the passage from a distributed force (such as the contact pressure between the feet and the ground, or the gravity forces) to their resultant (a concentrated force) that the concepts of *centroid* and *centre of mass* make their appearance in Statics.



b. On weighted averages

The notions of centroid and centre of mass are intimately related to the idea of a *weighted average*. A classic example is the following. Suppose you have a marking system consisting of 5 grades, such as A, B, C, D and F. Assume, moreover, that in a class of 80 students, the results of an exam have been tabulated as shown below and that we are asked to compute the class average.

No. of students	Grade
11	A
21	B
35	C
9	D
4	F

We know exactly how to do this, but our objective is to use this as an example to make a point. The first detail to notice is that in order to calculate an average of some entity (in this case a grade) the entity in question must have a numerical measure. In other words, the letters of the alphabet cannot be averaged in any rigorous sense. We, therefore, assume that the numerical values of the grades A, B, C, D and F are, respectively, 4.0, 3.0, 2.0, 1.0 and 0.0. Let us denote these values assigned to the set of entities to be averaged by $g_1, \; g_2, \ldots$ and so on. Secondly, we must choose a *weight*. In our example the weight is given by the number of students in each category. Let us denote the weights, respectively, by $w_1, w_2, \ldots$ and so on. The weighted average is given by

$$\bar{g} = \frac{w_1 g_1 + w_2 g_2 + \cdots}{w_1 + w_2 + \cdots} = \frac{\sum w_i g_i}{\sum w_i}$$

In our example, we obtain

$$\bar{g} = \frac{(11)(4.0) + (21)(3.0) + (35)(2.0) + (9)(1.0) + (4)(0.0)}{11 + 21 + 35 + 9 + 4} = 2.325$$

The numerator can be regarded as some kind of “moment” of the weights. If this terminology is adopted, we can write

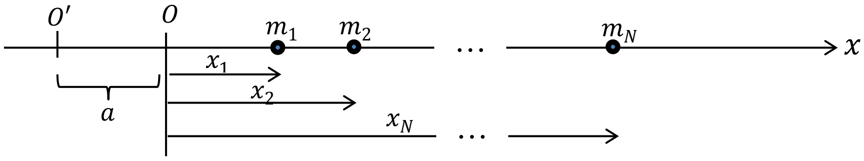
$$\bar{g} = \frac{Sum\ of\ moments\ of\ the\ weights}{Sum\ of\ weights} = \frac{Total\ moment}{Total\ weight}$$

An equivalent, and useful, way to state this formula is by saying that the average is a single object (a grade) such that when multiplied by the total weight gives us the same result as the total moment of the system. In other words, the initial set of weights $w_1, w_2, \ldots$ has been *concentrated* at the average $\bar{g}$.

c. <u>Centroids and centres of mass of discrete systems</u>

When speaking of centroids and centres of mass, the entity to be averaged is the *position*. The difference between centroids and centres of mass is the weight being used. In the first case (centroids) the weights are lengths, areas or volumes, whereas the weights used for the centre of mass are masses.

Let us start with a very simple one-dimensional example. A system of N masses $m_1, \ldots, m_N$ is arranged along the x axis occupying positions $x_1, \ldots, x_N$, respectively, with respect to an origin O.



The centre of mass of this system is the average position weighted by the masses. Following the same scheme as for the example of the average grade we obtain the position $\bar{x}$ of the centre of mass according to the formula

$$\bar{x} = \frac{m_1 x_1 + \cdots + m_N x_N}{m_1 + \cdots + m_N}$$

If we concentrate the total mass at the centre of mass obtained in this way, we obtain an "equivalent" system, at least from the point of view of taking the products that we have called "moments". A question that immediately comes to mind is the following: Is the centre of mass an *intrinsic* concept or, on the contrary, it depends of the system of coordinates used? If the latter were the case, we would not have an authentically physical entity, but just an arbitrary definition. To prove that the centre of mass is actually a physical concept, let us consider a different origin, O' say, for the coordinates, such as one located a units to the left of O. In that case, we would obtain that the coordinate (x') of any point in the new system is equal to the coordinate in the old system (x) plus a. Let us calculate the position of the centroid in the new system. We obtain

$$\bar{x}' = \frac{m_1(x_1 + a) + \cdots + m_N(x_N + a)}{m_1 + \cdots + m_N} = \frac{m_1 x_1 + \cdots + m_N x_N + (m_1 + \cdots + m_N)a}{m_1 + \cdots + m_N} = \bar{x} + a$$

What this formula proves is that the geometrical point obtained by operating on the first coordinate system is the same as that obtained by operating on the second.

If the masses are placed at arbitrary spatial positions, a similar result is obtained, except that the positions are now represented by position vectors $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N$. Thus, the position vector of the centre of mass in space is given by

$$\bar{\boldsymbol{r}} = \frac{m_1 \boldsymbol{r}_1 + \cdots + m_N \boldsymbol{r}_N}{m_1 + \cdots + m_N}$$

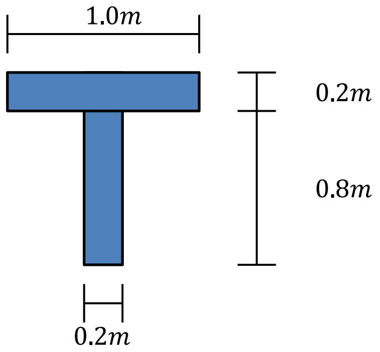
This expression can, of course, be broken into 3 component equation to calculate the coordinates $\bar{x}, \bar{y}, \bar{z}$ of the centre of mass.

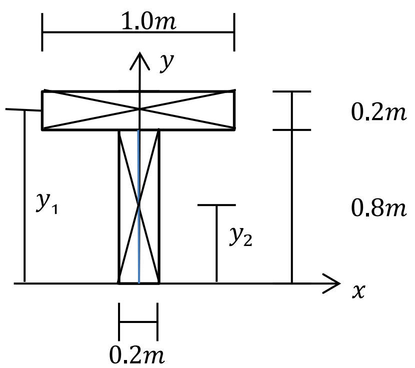
If, instead of using the masses as weights, we had taken into consideration, say, the volumes of some spheres placed at the same points as before, we would have obtained (in general) a different point called the *centroid* of the system. We will use this idea to calculate the centroid of a plane figure by using the area (rather than the mass or the volume) as weight.

d. Centroids of plane figures

If you show a plane shape to a child (such as the contour of an animal or a plant) and ask the child to point to the middle of the figure, the child's finger will most probably point at the centroid. In other words, the centroid is what we intuitively perceive as being the average position of the area of the figure. It follows that if a figure has an *axis of symmetry* the centroid must necessarily lie on it. Accordingly, if a figure is symmetric with respect to two different axes, the centroid is located at their intersection. A rectangle has its centroid at the intersection of its two symmetry axes, which coincides with the intersection of the two diagonals. The centroid of a circle is its centre.

Consider, as an example, a figure that can be regarded as the union of two separate rectangles. This is a common shape in structural engineering as the cross section of beams visibly used in concrete bridge construction (as you can observe on Crowchild Trail).[1] Let us calculate the position of the centroid of one such shape, as shown below. Clearly, this figure has a vertical axis of symmetry, which we adopt as the y axis. The centroid is located on this line. We only need to calculate the y coordinate of the centroid. Let us adopt as the x axis the line at the base of the shape (so that all y coordinates are positive, which may avoid mistakes). Let us denote the horizontal upper rectangle (called the *flange*) as component #1 and the lower (vertical) rectangle (called the *web*) as component #2. We have stated already that, as far as taking "moments" is concerned, we can concentrate the areas of each of these rectangles in the respective centroid. But, rectangles being doubly symmetric figures, their centroids are at the intersection of their two symmetry axes, which coincides with the intersection of the two diagonals. Let us redraw the figure while implementing these facts in an obvious manner. Using the areas as weights, we obtain the position of the centroid as



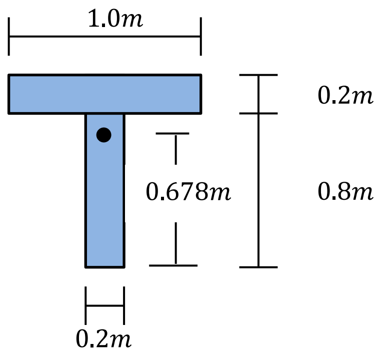


$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$$

Using the numerical values yields

[1] Structures of the kinds we study in this course are everywhere on our Campus. When you walk out of the class and take the overpass that connects to the Arts Parkade, you are treated to a truss of the kind that we have studied recently. If you walk towards the Taylor Family Digital Library and look to your right, you can see a fixed support consisting of a column bolted to a plate embedded in a concrete block. It provides all six possible components of reaction for a spatial structure. Frames abound elsewhere on Campus.

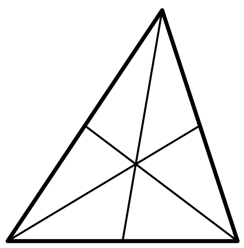
$$\bar{y} = \frac{(1.0m)(0.2m)(0.9m) + (0.2m)(0.8m)(0.4m)}{(1.0m)(0.2m) + (0.2m)(0.8m)} = 0.678m$$

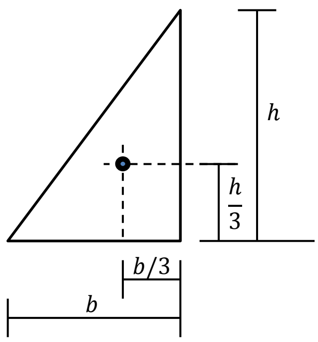


Drawing once more the figure and placing the centroid in the position found, we satisfy our intuition. Indeed, if only the flange had been there and not the web, the result would have certainly been $0.9m$. The web, therefore, lowers this value as it "attracts" the centroid towards itself, but certainly never lower than $0.4m$. Since the area of the web is somewhat smaller than that of the flange, the bias toward the larger section is dominant, so that the centroid is somewhat closer to the centroid of the flange.

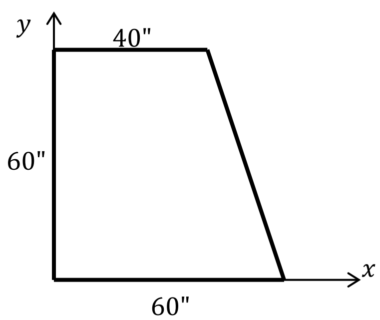
e. <u>An example with a triangle</u>

A theorem in plane geometry, proved by Heron of Alexandria 20 centuries ago, establishes that the medians of a triangle meet at a point at that this point is the centroid of the triangle.[2] Moreover, the point of intersection divides each median at 2/3 of its length from the vertex. For the case of a right-angled triangle, the two shorter sides (called catheti) can be used to determine the exact location of the centroid, as shown in the figure. Since every triangle can always be seen as the sum or difference of two right-angled triangles, we can always use the properties above to find the centroid of composite figures involving triangles.

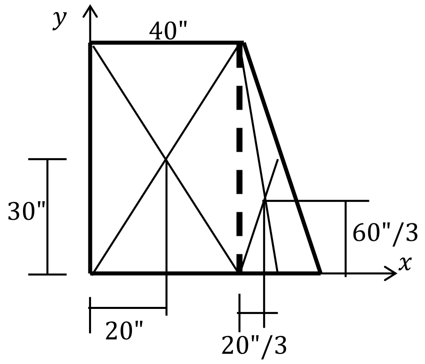




A numerical example (Problem 5.47) is shown below. We are asked to determine the coordinates of the centroid of the trapezoidal figure. We can divide this figure into two triangles or into a rectangle and a triangle. Choosing the second option, we proceed in the same fashion as in the previous problem, except that now we need to calculate both coordinates, since the figure does not enjoy any symmetry. The decomposition and the relevant distances are shown in the figure below. We obtain



[2] Recall that a median of a triangle is the segment drawn from a vertex to the midpoint of the opposite side.

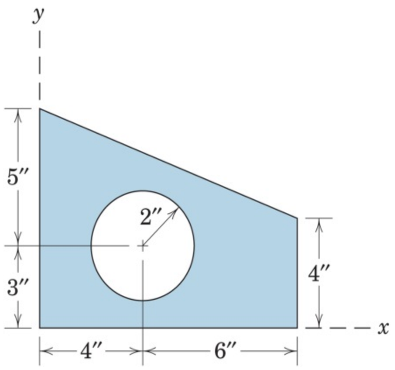


$$\bar{x} = \frac{(40")(60")(20")+1/2\ (20")(60")(40"+20"/3)}{(40")(60")+1/2(20")(60")} = 25.33"$$

$$\bar{y} = \frac{(40")(60")(30")+1/2\ (20")(60")(20")}{(40")(60") + 1/2(20")(60")} = 28"$$

To reason intuitively, if the triangle had not been there, then the centroid would have been at the point (20”, 30”). The triangle has the effect of biasing the centroid rightward and downward,

f. An example with a hole (Problem 5.51) A trapezoidal shape contains a circular hole of radius 2”. Determine the centroid of the remaining area.



Solution: Rather than trying to incorporate the difficult internal boundaries of the hole into the subdivision of the figure, we can reason that a hole is, in fact, nothing more or less than a negative area. In this spirit, we have a figure made up of three components: a rectangle, a triangle and a (negative) circle. The centroid of the circle is clearly at the centre. Following the usual procedure, we obtain

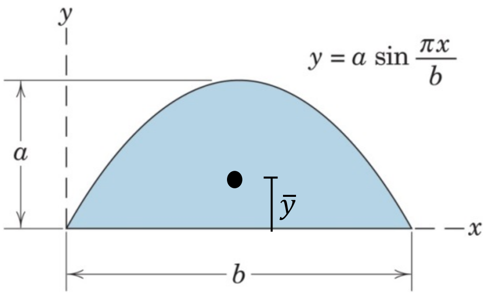
$$\bar{x} = \frac{(10")(4")(5") + \frac{1}{2}(10")(4")\left(\frac{10"}{3}\right) - \pi(2")^2(4")}{(10")(4") + \frac{1}{2}(10")(4") - \pi(2")^2} = 4.562"$$

$$\bar{y} = \frac{(10")(4")(2") + \frac{1}{2}(10")(4")\left(4" + \frac{4"}{3}\right) - \pi(2")^2(3")}{(10")(4") + \frac{1}{2}(10")(4") - \pi(2")^2} = 3.141"$$

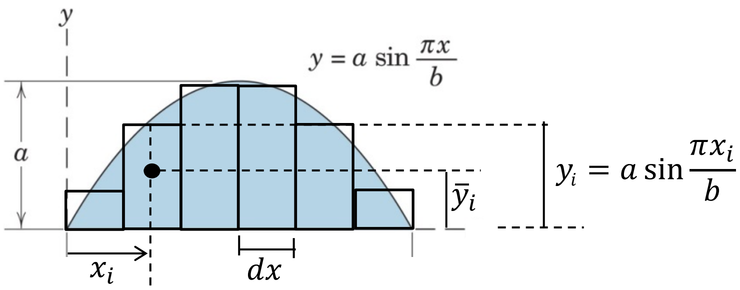
g. Centroids of arbitrary shapes

As long as the figure under consideration can be resolved into a combination of figures with known centroids, the technique described above can be used. In many cases that appear in applications, however, a direct calculation of the centroid must be carried out from first principles. What are these first principles, you may ask? Recall that Calculus was invented precisely to solve this kind of problems, namely, the expression of physical laws and of geometric properties out of a subdivision of the phenomenon into small pieces (“differentials”) and a subsequent re-composition (“integration”). Thus, Newton explained the motions of bodies by considering an infinitesimal part of the trajectory and relating its second derivative (“curvature”, say) to the instantaneous cause of the motion (“force”), thereafter obtaining the trajectory by a process of integration. This kind of ideas was already familiar to Eudoxus, Archimedes and other Hellenistic mathematicians, through a limiting process called *exhaustion*, but they lacked the algebraic and analytic tools to produce a definitive treatment. We want to follow the ideas of integral calculus in the calculation of centroids of figures that are enclosed within curves whose shapes are given in terms of functions of one variable. Let us work out a couple of examples.

<u>A half wave</u>: A shape of common occurrence in many areas of engineering is the harmonic wave. In Problem 5.6 we are asked to find the centroid of the area comprised between the x axis and a half sinusoidal wave, as shown in the figure below. Since this figure is obviously symmetric about the vertical line $x = b/2$, we only need to calculate the y coordinate of the centroid. Let us consider this area as if it had been sliced by means of an egg slicer into thin slices of thickness dx. We can then approximate the area of each slice as if it were a thin rectangle. The error of this approximation will decrease systematically as we make the slices more numerous and thinner. In other words, we are going to pass eventually to the limit as $dx \to 0$.



Subdividing the figure in the manner explained we obtain the figure below. The rectangle



number i in this subdivision has the area

$$dA_i = y_i dx = a \sin \frac{\pi x_i}{b} \; dx$$

The y coordinate of the centroid of this triangle is half its height, namely

$$\bar{y}_i = \frac{1}{2} a \sin \frac{\pi x_i}{b}$$

To obtain the position of the centroid of the whole figure we proceed as usual, that is,

$$\bar{y} \approx \frac{\bar{y}_1 dA_1 + \bar{y}_1 dA_1 + \cdots}{dA_1 + dA_2 + \cdots} = \frac{\Sigma\, \bar{y}_i dA_i}{\Sigma\, dA_i}$$

In the limit, as the subdivision gets infinitely fine, we obtain

$$\bar{y} = \lim_{dx \to 0} \frac{\Sigma\, \bar{y}_i dA_i}{\Sigma\, dA_i} = \frac{\int_0^b \bar{y} \; dA}{\int_0^b dA} = \frac{\int_0^b \frac{1}{2} a \sin \frac{\pi x}{b} \; a \sin \frac{\pi x}{b} \; dx}{\int_0^b a \sin \frac{\pi x}{b} \; dx}$$

Recall that a (Riemann) integral is, by definition, a limit of sums of the type we have just encountered. The integral sign $\int$ is nothing but the letter "S", indicating a sum. We have thus reduced the problem of calculating centroids to a ratio between two integrals. The numerator is the moment of the area with respect to one axis, while the denominator is the total area of the figure, just as before. In our particular example, the numerator involves the integral of the square of the sine function. Recalling the trigonometric identity

$$\sin^2 z = \frac{1 - \cos 2z}{2}$$

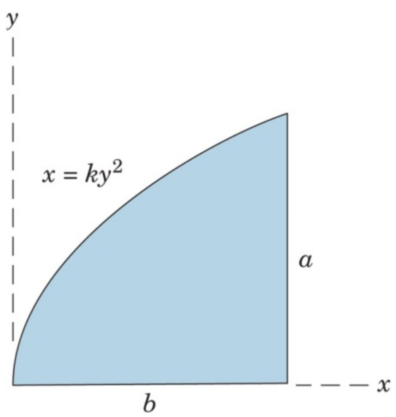
we see that the integrations in this case are rather elementary. The final answer is

$$\bar{y} = \frac{\pi}{8} a$$

Note that if we had chopped the upper part of the wave and retained a triangle with the same base as the wave and its vertex at the crest of the wave, the centroid would have been at $\bar{y} = a/3$. The presence of the two curved parts attracts the centroid slightly upwards.

h. <u>A parabolic sector</u>

Finally, we illustrate the general procedure by means of an example without any symmetry, such as that of Problem 5.14. Notice that the constant k, although not directly given, can be obtained from the condition $b = ka^2$. Following the identical procedure as in the previous example, we obtain



$$\bar{x} = \frac{\int_0^b x\sqrt{x/k}}{\int_0^b \sqrt{x/k}\ dx} = \frac{\frac{2}{5} b^{5/2}}{\frac{2}{3} b^{3/2}} = \frac{3}{5} b$$

$$\bar{y} = \frac{\int_0^b \frac{1}{2}\ \sqrt{x/k}\sqrt{x/k}}{\int_0^b \sqrt{x/k}\ dx} = \frac{\frac{1}{4k} b^2}{\frac{2}{3\sqrt{k}} b^{3/2}} = \frac{3}{8}\sqrt{\frac{b}{k}} = \frac{3}{8} a$$

It is worthwhile pointing out that we could have introduced horizontal rectangular subdivisions instead of the vertical ones. In that case, we would have obtained integrals over the variable y. The final result should be the same. Try it as an exercise. Finally, we could also have subdivided the domain into small rectangle of extent $dx\, dy$. In that case, we would have obtained a double integral, a concept that may or may not have covered in your calculus courses yet.