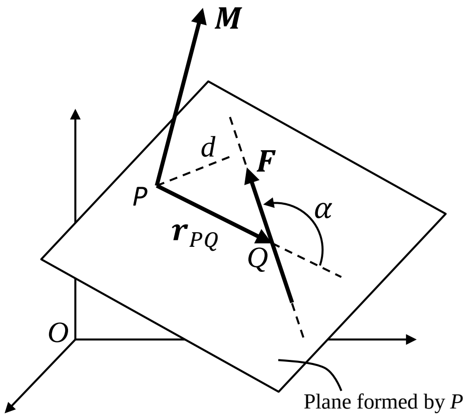
1. Recall that the moment of a force $\boldsymbol{F}$ about a point P is a vector $\boldsymbol{M}$ perpendicular to the plane determined by the line of action of $\boldsymbol{F}$ and the point P. (If P belongs to this line of action, the moment is zero). Mathematically, this vector is defined as

$$\boldsymbol{M} = \boldsymbol{r}_{PQ} \times \boldsymbol{F}$$

where $\boldsymbol{r}$ is a vector from P to *any* point Q along the line of action of $\boldsymbol{F}$.

When solving a question requiring the calculation of the moment of a force with respect to a point, there are basically two possible approaches:



Plane formed by P and $\boldsymbol{F}$

(a) Recalling that the magnitude of the cross product is given by

$$M = r\,F \sin\alpha$$

(see Figure above) and that $d = r\sin\alpha$ can be interpreted as the distance between the line of action of the force and P, in many cases we can calculate the moment of a force by inspection. In that case, the direction of the result is determined by the right-hand rule and/or by imagining the tendency to rotate of the force around the point.

(b) A more systematic approach (valid for the most general cases) consists of actually calculating the cross product using components in a Cartesian coordinate system. In other words, if $\boldsymbol{r}_{PQ} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and $\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}$, then $\boldsymbol{M} = (a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}) \times (F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k})$. Since the cross product is distributive (Varignon's lemma), we can cavalierly open brackets and remember that

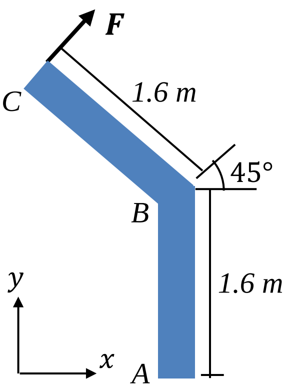
$$\boldsymbol{i}\times\boldsymbol{j} = -(\boldsymbol{j}\times\boldsymbol{i}) = \boldsymbol{k}, \quad \boldsymbol{j}\times\boldsymbol{k} = -(\boldsymbol{k}\times\boldsymbol{j}) = \boldsymbol{i}\,, \quad \boldsymbol{k}\times\boldsymbol{i} = -(\boldsymbol{i}\times\boldsymbol{k}) = \boldsymbol{j}.$$

Moreover, by skew-symmetry, $\boldsymbol{i}\times\boldsymbol{i} = \boldsymbol{j}\times\boldsymbol{j} = \boldsymbol{k}\times\ \boldsymbol{k} = \boldsymbol{0}$.

Alternatively and equivalently, we may use the determinant rule, namely,

$$\boldsymbol{M} = \boldsymbol{r}_{PQ}\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a & b & c \\ F_x & F_y & F_z \end{vmatrix} = \left(bF_z - cF_y\right)\boldsymbol{i} - (aF_z - cF_x)\boldsymbol{j} + \left(aF_y - bF_x\right)\boldsymbol{k}$$

In our example, we are given a plane bracket with a force of magnitude F = 30N acting in the plane of the bracket. The force is perpendicular to the portion BC of the bracket. We are asked to determine the moments of the force with respect to points A and B.

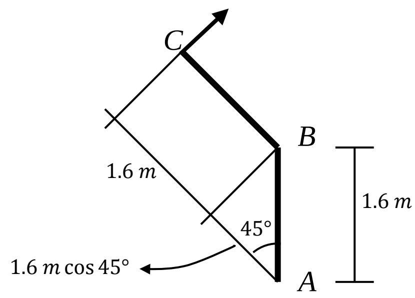


We will solve this problem in two different ways.

a. <u>First approach</u>: Since both the force and the points lie in the plane of the figure (x,y) it is easy to obtain the magnitude of the moment by just multiplying the magnitude of the force by the distances to the points in question. In both cases, the moment vector is perpendicular to the plane of the figure and points into the figure. The sense of rotation is, in both cases, clockwise (CW). If you imagine a coordinate system with x, y in the usual sense in the plane of the figure, then the z axis points towards us. The z (and only non-zero) component of the moment is, therefore, negative. We obtain immediately

$M_B = -(30N)(1.6m) = -48Nm \ \ or \ 48Nm \ CW$

To calculate the moment with respect to point A, we need the distance d between the line of action of the force and A. We obtain



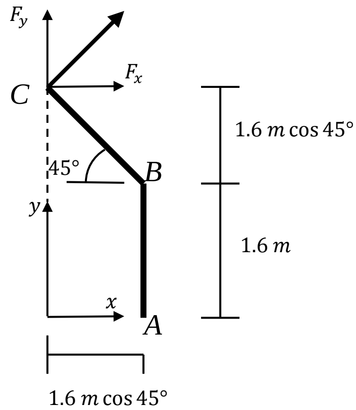
$$M_A = -(30N)d = -(30N)(1.6m + 1.6m\cos 45°) = -81.94Nm \ \ or \ 81.94Nm \ CW$$

b. <u>Second approach</u>: We will proceed in a more systematic way, as if we were preparing the data to be entered to a computer routine. This method will be valid for arbitrary situations in three dimensions. We proceed as follows:

<u>(i)</u> Choose some Cartesian coordinate system *x,y,z.*

<u>(ii)</u> Provide the coordinates of points *A*, *B*, *C* in this system as tabulated below

	x (m)	y(m)	z(m)
A	1.131	0	0
B	1.131	1.6	0
C	0	2.731	0



<u>(iii)</u> Notice that these coordinates for *A*, *B*, C are also the components of the respective position vectors $\boldsymbol{r}_A, \boldsymbol{r}_B, \boldsymbol{r}_C$. Therefore, we can write

$$\boldsymbol{r}_{AC} = \boldsymbol{r}_C - \boldsymbol{r}_A = (-1.131\ \boldsymbol{i} + 2.731\ \boldsymbol{j})\ m$$
$$\boldsymbol{r}_{BC} = \boldsymbol{r}_C - \boldsymbol{r}_B = (-1.131\ \boldsymbol{i} + 1.131\ \boldsymbol{j})\ m$$

<u>(iv)</u> Express the force in the chosen coordinate system as

$$\boldsymbol{F} = (30N)(\cos 45°\ \boldsymbol{i} + \sin 45°\ \boldsymbol{j}) = (21.21\ \boldsymbol{i} + 21.21\ \boldsymbol{j})N$$

<u>(v)</u> Calculate the cross products using the distributive property and the fact that $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} \quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i} \quad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$. We obtain

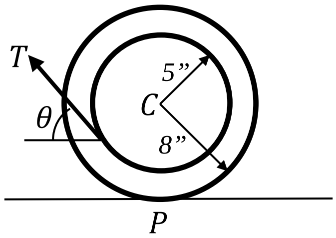
$$\boldsymbol{M}_A = \boldsymbol{r}_{AC} \times \boldsymbol{F} = (-1.131\ \boldsymbol{i} + 2.731\ \boldsymbol{j})\ m \times (21.21\ \boldsymbol{i} + 21.21\ \boldsymbol{j})N$$
$$= (-24\ \boldsymbol{i} \times \boldsymbol{j} + 57.92\ \boldsymbol{j} \times \boldsymbol{i})Nm = (-81.92\ \boldsymbol{k})\ Nm$$
$$\boldsymbol{M}_B = \boldsymbol{r}_{BC} \times \boldsymbol{F} = (-1.131\ \boldsymbol{i} + 1.131\ \boldsymbol{j})\ m \times (21.21\ \boldsymbol{i} + 21.21\ \boldsymbol{j})N$$
$$= (-24\ \boldsymbol{i} \times \boldsymbol{j} + \ 24\ \boldsymbol{j} \times \boldsymbol{i})Nm = (-48\ \boldsymbol{k})\ Nm$$

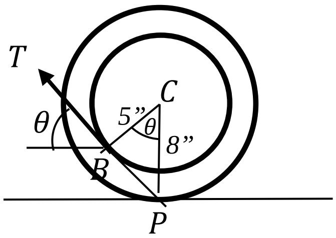
Alternatively, you may use the determinant rule for the cross product.

2. In the next question, we are first asked to calculate the moment of the force applied to a cord wound around a spool and exiting tangentially. The distance of the line of action to the centre *C* of the spool is, therefore, not affected by the particular point at which it exits. The tendency of the force to rotate about the centre is clockwise. We obtain, therefore,

$$M_C = \ -(32lb)(5'') = -160\ lb\ in \qquad (or\ 160\ lb\ in, CW)$$

The second part of the question is a bit more subtle. We want to obtain the value (or values) of the exit angle θ so that the moment with respect to the point of contact P is zero. In other words, we want the line of action of the force to contain the point P. Denoting by B the point of detachment, the triangle CBP is right-angled and the angle at C is equal to θ. Since this angle is subtended by the two radii, as indicated in the figure, we obtain

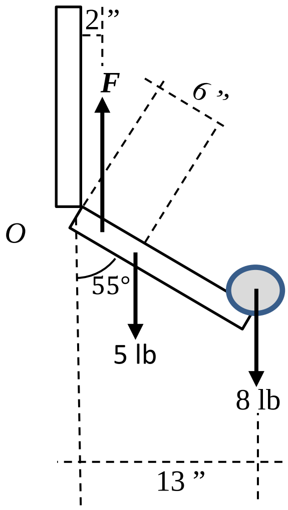




$$\cos\theta = \frac{5}{8} \qquad \therefore \qquad \theta = 51.32°$$

Note that there is another answer to the problem located symmetrically.

3. The next question is inspired in Biomechanics, a field in which our University excels. It shows an arm with the hand carrying a weight. The biceps applies a force so that the total moment with respect to elbow vanishes. We will learn pretty soon that if the elbow joint is frictionless, the vanishing of the moments of all the external forces acting on the forearm with respect to the elbow is necessary for the forearm to be in equilibrium. Please notice that the diagrams shown in all these problems are NOT free-body diagrams. (Why not?)



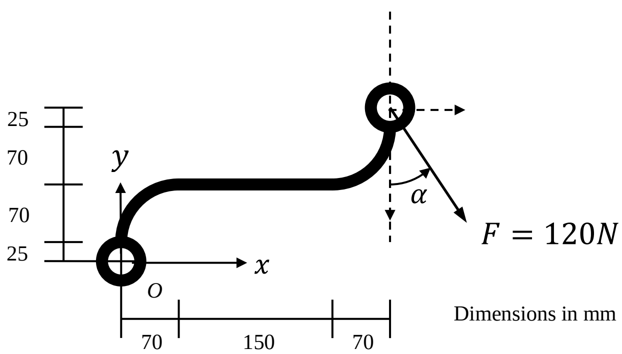
We calculate the total moment of these three forces with respect to the elbow at O as

$$M_O = F(2'') - (5lb)(6''\sin 55°) - (8lb)(13'')$$

where we measure counterclockwise (CCW) as positive. Equating this total moment to zero, as required, we obtain

$$F = \frac{128.58\ lb\ in}{2''} = 64.29\ lb$$

4. Our last question involves a wrench, as schematically shown in the figure. A force **F** of fixed magnitude $F = 120N$ is applied at one end of the wrench at an angle α with the vertical line. We are asked to find the moment of the force about O



Before anything else, let us express the force in x,y components as

$$\boldsymbol{F} = F\sin\alpha\,\boldsymbol{i} - F\cos\alpha\,\boldsymbol{j}$$

The moment of $\boldsymbol{F}$ about O is (according to Varignon's lemma) the sum of the moments of its components. By inspection, we see that both components tend to rotate clockwise about O so both contributions are negative. Moreover, the distance of the horizontal component to *O* is measured vertically, and the distance of the vertical component is measured horizontally. In other words, we have

$$M_0 = -F \sin\alpha \ (25 + 70 + 70 + 25)mm - F \cos\alpha \ (70 + 150 + 70)mm$$

The moment is, by definition, a vector. But here we are just evaluating the *z* component, since the other two components vanish. Alternatively, you could use the general cross-product approach and you would get exactly the same answer.

For any given value of the angle α, we can obtain the moment. For example, if $\alpha = 30°$, we obtain

$$M_0 = -(120N) \sin 30° \ (190\ mm) - (120N) \cos 30° \ (290\ mm) = -41.54\ Nm$$

An interesting practical question is the following: What would the optimal value of the angle α be so as to maximize the efficiency of the applied force without changing its magnitude? We can approach this problem in two ways.

(a) We can let Calculus do the job of finding maxima and minima (one of the very reasons for the invention of Calculus, as the title of Leibniz's original 1684 paper reveals). The only caveat of this procedure is that, in principle, you must make sure that what you obtain is indeed a maximum or a minimum (for instance by performing a second derivative test). Moreover, you must remember that Calculus delivers only *relative* maxima and minima. So, if your range of operation is finite (for example, if you cannot vary the angle α at will because there are some walls preventing your hand to move beyond a certain limit), then the maximum or minimum may fall at one of the ends of the interval of operation. That would be then an *absolute* maximum or minimum where the derivative may not vanish! This does not happen in our example, since there are no constraints imposed on α. Let us then calculate the derivative of the moment with respect to α and equate the result to 0. We obtain

$$\frac{dM}{d\alpha} = -F(190\ mm) cos\ \hat{\alpha} + F(290\ mm) sin\ \hat{\alpha} = 0$$

From here, we obtain

$$\hat{\alpha} = \arctan\frac{190}{290} = 33.23° \quad and \quad 213.23°$$

The corresponding values of the moment are

$$M_{min} = -41.60\ Nm \quad and \quad M_{max} = +41.60\ Nm$$

(b) The second approach is more intuitive and works well for this particular question. It just stems from the observation that by varying α, as far as the moment is concerned, what we are varying is the distance between the line of action of the force and the point *O*. For this distance to be at a maximum, clearly we need the force to be perpendicular to the line *OA*. Put differently, we need the angle α to be equal to the angle β. But, from the figure, we obtain that $\tan\beta = \frac{25+70+70+25}{70+150+70} = 0.655$, whence $\alpha = \beta = 33.23°$. This gives the line of action, but we can still apply the force in one direction or the other, which accounts for two solutions, as above.