Conversion between forms:

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Polar $M\angle\theta$	Rectangular $a+jb$
$M = \sqrt{a^2+b^2}$	$a = M\cos(\theta)$
$\theta = \tan^{-1}\left(\frac{b}{a}\right)$	$b = M\sin(\theta)$
$= \arctan\left(\frac{b}{a}\right)$	

Complex arithmetic:

Let $x = 2+j4$, $y = 4+j5$

- Addition and subtraction must be done in rectangular form

$$x+y = (2+j4)+(4+j5)$$
$$= (2+4)+j(4+5) = 6+j9$$
$$x-y = 2+j4-4-j5 = -2-j$$

- Multiplication and division can be performed in either form.

<u>Rectangular</u>:

$$x\cdot y = (2+j4)(4+j5) \qquad j^2=-1$$
$$= 8+j10+j16+j^2 20$$
$$= -12+j26$$

$$x/y = \frac{2+j4}{4+j5} = \frac{(2+j4)(4-j5)}{(4+j5)(4-j5)}$$
$$= \frac{8-j10+j16-j^2 20}{16-j20+j20-j^2 25}$$
$$= \frac{28+j6}{41} = \frac{28}{41}+j\frac{6}{41}$$

Polar: First convert x, y

$$x = \sqrt{2^2+4^2}\angle \tan^{-1}\left(\tfrac{4}{2}\right) = \sqrt{20}\angle 63.44^\circ$$

$$y = \sqrt{4^2+5^2}\angle \tan^{-1}\left(\tfrac{5}{4}\right) = \sqrt{41}\angle 51.34^\circ$$

Then: $x \cdot y = \left(\sqrt{20}\sqrt{41}\right)\angle 63.44^\circ + 51.34^\circ$ MULTIPLY MAGNITUDES, ADD ANGLES

$x/y = \left(\sqrt{20}/\sqrt{41}\right)\angle 63.44^\circ - 51.34^\circ$ DIVIDE MAGNITUDES, SUBTRACT ANGLES.

Key to the functioning of phasors is Euler's Identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Multiply both sides by M

$$\underbrace{Me^{j\theta}}_{\text{complex exponential, another way of expressing } M\angle\theta} = \underbrace{M\cos\theta + jM\sin\theta}_{\text{rectangular form}}$$

E.g., $x = 10\,e^{j30^\circ} = 10\angle 30^\circ$

By Euler's identity,

$$x = 10e^{j30^\circ} = 10\cos(30^\circ) + j10\sin(30^\circ)$$

and $\mathrm{Re}[x] = \mathrm{Re}\left[10e^{j30^\circ}\right] = 10\cos(30^\circ) = 8.66$

$\mathrm{Im}[x] = \mathrm{Im}\left[10e^{j30^\circ}\right] = 10\sin(30^\circ) = 5$

$$10e^{j30^\circ} = 8.66 + j5$$

Writing KVL/KCL equations using phasors

Motivation: Using complex exponentials is way easier than applying trig identities

The key: Express cosines as complex exponentials using Euler's identity.

$$\boxed{\cos(x) = \mathrm{Re}\left[e^{jx}\right]}$$

Back to our original KVL example:

$$v(t) = 10\cos(\omega t) + 5\cos(\omega t - 30^\circ) + 5\cos(\omega t + 90^\circ)$$

Now,

$$v_1(t) = 10\cos(\omega t) = \mathrm{Re}\left[10 e^{j\omega t}\right]$$
$$v_2(t) = 5\cos(\omega t - 30^\circ) = \mathrm{Re}\left[5 e^{j(\omega t - 30^\circ)}\right]$$
$$v_3(t) = 5\cos(\omega t + 90^\circ) = \mathrm{Re}\left[5 e^{j(\omega t + 90^\circ)}\right]$$

Giving

$$v(t) = \mathrm{Re}\left[10 e^{j\omega t}\right] + \mathrm{Re}\left[5 e^{j\omega t} e^{-j30^\circ}\right] + \mathrm{Re}\left[5 e^{j\omega t} e^{j90^\circ}\right]$$

The sum of the real parts is equal to the real part of the sum

$$v(t) = \mathrm{Re}\left[10 e^{j\omega t} + 5 e^{j\omega t} e^{-j30^\circ} + 5 e^{j\omega t} e^{j90^\circ}\right]$$
$$= \mathrm{Re}\left[\underbrace{\left(10 + 5e^{-j30^\circ} + 5 e^{j90^\circ}\right)}_{\text{addition of three complex constants (phasors!)}} e^{j\omega t}\right]$$

So

$$v(t) = \mathrm{Re}\left[\left(10 + 5e^{-j30^\circ} + 5e^{j90^\circ}\right) e^{j\omega t}\right]$$

$$\boxed{\begin{aligned} &= (10) + (4.33 - j2.5) + (j5) \\ &= 14.33 + j2.5 \\ &\doteq 14.54\angle 9.90^\circ \end{aligned}}$$

$$v(t) = \mathrm{Re}\left[(14.54\angle 9.90^\circ)\, e^{j\omega t}\right]$$

Cleaning up

$$v(t) = \mathrm{Re}\left[14.54\, e^{j(\omega t + 9.90^\circ)}\right]$$

Finally:

$$v(t) = 14.54\cos(\omega t + 9.90^\circ)$$