ENGG 319

Probability & Statistics for Engineers

Section #04

Mathematical Expectation

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Mathematical Expectation

- Let *X* be a random variable with probability distribution *f(x).* The ***mean*** or the ***expected value*** of *X* is:

$$\mu_x = E(X) = \sum_x x.f(x)$$

if X is discrete

$$\mu_x = E(X) = \int_{-\infty}^{+\infty} x.f(x)dx$$

if X is continuous

Example #1

- In a game of two players, each player is to flip a coin three times. Any player will get 6 points if the outcomes are all heads or all tails. Also, any player will be deducted 4 points if either one or two tails were the outcomes.
- What is the expected points for any player in this game?
- Indicate also if this game is a fair game or not.

Example #1 (Sol.)

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT }

N = 8

A: getting 3 tails or 3 heads → *A* = {HHH, TTT }

B : getting 1 tail or 2 tails → *B* = {HHT, HTH, HTT, THH, THT, TTH }

Use ***X*** to denote the random variable of gaining points

P(A) = $f(x_1)$ = 2/8 = 1/4

P(B) = $f(x_2)$ = 6/8 = 3/4

$$\mu_x = \sum_x x.f(x) = x_1 f(x_1) + x_2 f(x_2)$$

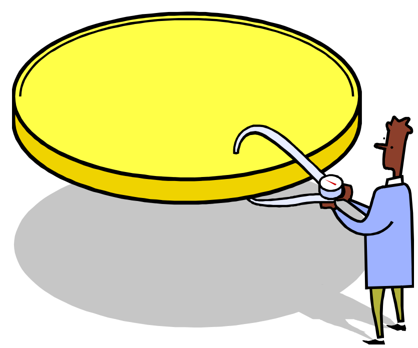
$$= 6 * \frac{1}{4} - 4 * \frac{3}{4} = -\frac{3}{2}$$

→ The game in unfair!

Example #2

(Ex. 4.4 Textbook):

- A coin is biased so that a head is three times as likely to occur as a tail. Find the expected number of tails when this coin is tossed twice.



$\omega + 3*\omega = 1$ $\Rightarrow$ $\omega = \frac{1}{4}$

$\Rightarrow$ $P(T) = \frac{1}{4}$ $P(H) = \frac{3}{4}$

$S = \{ HH, HT, TH, TT \}$

$N = 4$

Example #2 (Sol.)

Use ***X*** to denote the *No.* of tails occurring in 2 coin tosses.

x = 0, 1, 2

$$P(X=0) = P(HH) = P(H)P(H) = \frac{3}{4}*\frac{3}{4} = \frac{9}{16}$$

$$P(X=1) = P(HT)+P(TH) = P(H)P(T)+P(T)P(H) = \frac{3}{4}*\frac{1}{4}+\frac{1}{4}*\frac{3}{4} = \frac{6}{16}$$

$$P(X=2) = P(TT) = P(T)P(T) = \frac{1}{4}*\frac{1}{4} = \frac{1}{16}$$

$$\mu_x = \sum_x x.f(x) = x_1 f(x_1)+x_2 f(x_2)+x_3 f(x_3)$$

$$= 0*\frac{9}{16} + 1*\frac{6}{16} + 2*\frac{1}{16} = \frac{8}{16} = \frac{1}{2}$$

Example #3

(Ex. 4.12 Textbook):

- If a dealer's profit, in units of \$5000, on a new automobile can be looked upon as a random variable X having the density function:

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

- Find the average profit per automobile

Example #3 (Sol.)

$$f(x) = 2(1 - x), \quad 0 < x < 1$$

$$\mu_x = E(X) = \int_{-\infty}^{+\infty} x \, .f(x) dx$$

(In units of $5000)

$$\mu_x = E(X) = \int_0^1 x * 2(1 - x)\, dx = 2\int_0^1 x - x^2 \, dx$$

$$\mu_x = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Bigg|_0^1 = 2\left[\left(\frac{1}{2} - \frac{1}{3}\right) - 0\right] = 2 * \frac{1}{6} = \frac{1}{3}$$

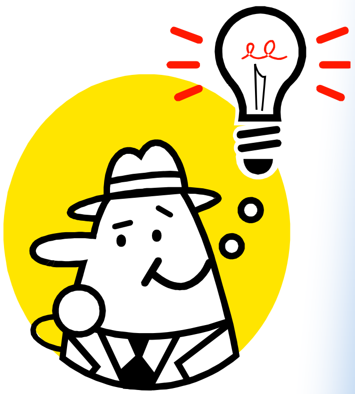
μ_x = 1/3 * $5000 = $1666.67

Mathematical Expectation

- Let *X* be a random variable with probability distribution *f(x).* The ***expected value*** of the random variable *g(X)* is:

$$\mu_{g(X)} = E[g(X)] = \sum_{x} g(x).f(x)$$ if X is discrete

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{+\infty} g(x).f(x)\,dx$$ if X is continuous



Example #4

- The probability distribution of selling luxury cars per day by a sales person is given below:

x	5	6	7	8	9
$f(x)$	0.1	0.3	0.3	0.2	0.1

- If the sales person's daily salary is \$100 + \$50 per car sold, determine the average daily salary of that sales person.

Example #4 (Sol.)

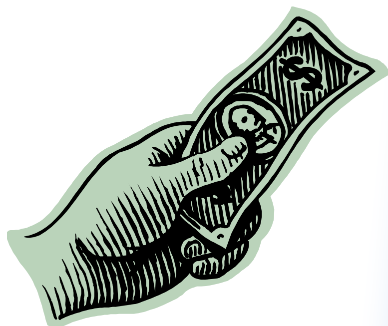
$$\mu_{g(X)} = E[g(X)] = \sum_{x} g(x).f(x)$$

$$g(X) = 100 + 50X$$

x	5	6	7	8	9
$f(x)$	0.1	0.3	0.3	0.2	0.1
$g(x)$	350	400	450	500	550

$$\mu_{g(X)} = 350*0.1 + 400*0.3 + 450*0.3 + 500*0.2 + 550*0.1$$

$$\mu_{g(X)} = \$445$$



Variance of a Random Variable

- Let *X* be a random variable with probability distribution *f(x)* and mean μ_x.
- The ***variance*** of *X* or the ***variance of probability distribution*** of X, *Var*(*X*), is:

$$\sigma_x^2 = E(X-\mu_x)^2 = \sum_x (x-\mu_x)^2 f(x)$$ if X is discrete

$$\sigma_x^2 = E(X-\mu_x)^2 = \int_{-\infty}^{+\infty} (x-\mu_x)^2 f(x)\,dx$$ if X is continuous

- The positive square root of the variance, σ_x, is the ***standard deviation*** of x.

Variance of a Random Variable

- It can be also shown that the ***variance*** of a random variable *X* is:

$$\sigma_x^2 = E\left(X^2\right) - \mu_x^2$$

Example #5:

- Using the information provided in *Example #2* (tossing a coin twice and *X* denotes the *No.* of tails occurrence in the 2 tosses), determine the variance of the occurrence of tails in that experiment.

Example #5 (Sol.)

From Example #2:

x	0	1	2
$f(x)$	9/16	6/16	1/16

$\mu_x = 1/2$

$$\sigma_x^2 = E(X-\mu_x)^2 = \sum_x (x-\mu_x)^2 f(x) = \sum_{x=0}^{2} (x-\mu_x)^2 f(x)$$

$$\sigma_x^2 = \left(0-\frac{1}{2}\right)^2 * \frac{9}{16} + \left(1-\frac{1}{2}\right)^2 * \frac{6}{16} + \left(2-\frac{1}{2}\right)^2 * \frac{1}{16} = \frac{3}{8}$$

Or: $\sigma_x^2 = E(X^2) - \mu_x^2 \Rightarrow \sigma_x^2 = \sum_{x=0}^{2} x^2 f(x) - \mu_x^2$

$$\sigma_x^2 = (0)^2 * \frac{9}{16} + (1)^2 * \frac{6}{16} + (2)^2 * \frac{1}{16} - \left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

Example #6

(Ex. 4.38 Textbook):

- The proportion of people who respond to a certain mail-order is a random variable X having the following density function:

$$f(x) = \begin{cases} \frac{2}{5}(x+2), & 0 < x < 1 \\ 0 \;, & \text{elsewhere} \end{cases}$$

- Find the variance of X.

Example #6 (Sol.)

$$\sigma_x^2 = E\left(X^2\right) - \mu_x^2$$

$$\mu_x = E(X) = \int_0^1 x \, . f(x) \, dx$$

$$E\left(X^2\right) = \int_0^1 x^2 \, . f(x) \, dx$$

$$\mu_x = E(X) = \int_0^1 x \, . \frac{2}{5}(x+2) \, dx = \frac{2}{5}\int_0^1 x^2 + 2x) \, dx = \frac{2}{5}\left[\frac{x^3}{3} + x^2\right]_0^1$$

$$\mu_x = \frac{2}{5}\left[\left(\frac{1}{3}+1\right) - 0\right] = \frac{8}{15}$$

$$E\left(X^2\right) = \int_0^1 x^2 \, . \frac{2}{5}(x+2) \, dx = \frac{2}{5}\int_0^1 x^3 + 2x^2) \, dx = \frac{2}{5}\left[\frac{x^4}{4} + \frac{2x^3}{3}\right]_0^1$$

$$E\left(X^2\right) = \frac{2}{5}\left[\left(\frac{1}{4}+\frac{2}{3}\right) - 0\right] = \frac{22}{60} = \frac{11}{30}$$

$$\sigma_x^2 = E\left(X^2\right) - \mu_x^2 = \frac{11}{30} - \left(\frac{8}{15}\right)^2 = 0.082 \cong 0.08$$

Variance of a Random Variable

- Let *X* be a random variable with probability distribution *f(x).*
- The ***variance*** of the random variable *g(X)* is:

$$\sigma^2_{g(X)} = E\left(g(X) - \mu_{g(X)}\right)^2 = \sum_x \left(g(x) - \mu_{g(X)}\right)^2 f(x)$$

if X is discrete

$$\sigma^2_{g(X)} = E\left(g(X) - \mu_{g(X)}\right)^2 = \int_{-\infty}^{+\infty} \left(g(x) - \mu_{g(X)}\right)^2 f(x)dx$$

if X is continuous

Example #7

- Using the information provided in *Example #4* (the probability distribution of selling luxury cars per day by a sales person), determine the standard deviation of the daily salary of that sales person if his daily salary is \$100 + \$50 per car sold.

Example #7 (Sol.)

From Example #4: $g(X) = 100 + 50X$

x	5	6	7	8	9
$f(x)$	0.1	0.3	0.3	0.2	0.1
$g(x)$	350	400	450	500	550

$\mu_{g(X)} = \$445$

$$\sigma^2_{g(X)} = E\left(g(x) - \mu_{g(X)}\right)^2 = \sum_{x=5}^{9} \left(g(x) - \mu_{g(X)}\right)^2 f(x)$$

$$\sigma^2_{g(X)} = (350-445)^2 * 0.1 + (400-445)^2 * 0.3 + (450-445)^2 * 0.3$$
$$+ (500-445)^2 * 0.2 + (550-445)^2 * 0.1 = \$\,3225$$

$$\sigma_{g(X)} = \sqrt{\sigma^2_{g(X)}} = \$\,56.79$$

Linear Combinations of Random Variables

- If a and b are ***constants***, then:

$$E(aX + b) = aE(X) + b$$

$$\Rightarrow \quad E(b) = b \qquad a = 0$$

$$E(aX) = aE(X) \qquad b = 0$$

Example #8

- Using the information provided in *Example #4* (the probability distribution of selling luxury cars per day by a sales person), determine the average daily salary of that sales person if his daily salary is \$100 + \$50 per car sold.

x	5	6	7	8	9
$f(x)$	0.1	0.3	0.3	0.2	0.1

Example #8 (Sol.)

$$g(X) = 100 + 50X$$

x	5	6	7	8	9
$f(x)$	0.1	0.3	0.3	0.2	0.1

$$E(g(X)) = E(100 + 50X) = E(100) + E(50X) = E(100) + 50E(X)$$

$$E(100) = 100$$

$$E(X) = \sum_{x=5}^{9} x.f(x)$$

$$E(X) = 5*0.1 + 6*0.3 + 7*0.3 + 8*0.2 + 9*0.1 = 6.9$$

$$E(g(X)) = 100 + 50*6.9 = \$445$$

(Same answer obtained in *Ex. #4* Sol.)

Linear Combinations of Random Variables

- The ***expected value*** of the sum or the difference of two or more functions of a random variables *X* is the sum or difference of the expected values of the functions.

- That is:

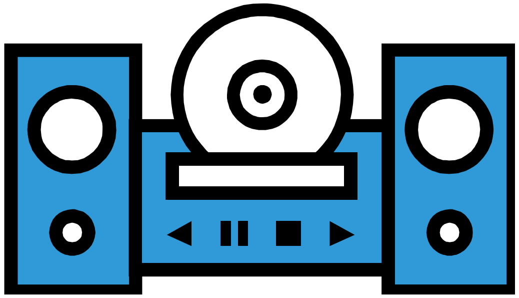
$$E\left[g(X) \pm h(X)\right] = E\left[g(X)\right] \pm E\left[h(X)\right]$$

Example #9

(Ex. 4.56 Textbook):

- The total time, measured in units of 100 hours, that a teenager runs her stereo set over a period of one year is a continuous random variable X that has the density function:

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$



- Evaluate the mean of the random variable $Y = 60X^2 + 39X$, where Y is equal to the number of kilowatt hours expended annually.

Example #9 (Sol.)

$$Y = 60X^2 + 39X$$

$$E(Y) = E(60X^2 + 39X) = 60E(X^2) + 39E(X)$$

$$E(X) = \int_{-\infty}^{+\infty} x.f(x)dx \Rightarrow E(X) = \int_0^1 x.x\,dx + \int_1^2 x.(2-x)\,dx$$

$$E(X) = \int_0^1 x^2\,dx + \int_1^2 2x - x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$E(X) = \left[\frac{1}{3} - 0\right] + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = 1$$

Example #9 (Sol.)

$$E\left(X^2\right)=\int_{-\infty}^{+\infty} x^2 . f(x)\,dx \Rightarrow E\left(X^2\right)=\int_0^1 x^2 . x\,dx+\int_1^2 x^2 .(2-x)\,dx$$

$$E\left(X^2\right)=\int_0^1 x^3\,dx+\int_1^2 2x^2-x^3\,dx=\left[\frac{x^4}{4}\right]_0^1+\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_1^2$$

$$E\left(X^2\right)=\left[\frac{1}{4}-0\right]+\left[\left(\frac{16}{3}-\frac{16}{4}\right)-\left(\frac{2}{3}-\frac{1}{4}\right)\right]=\frac{7}{6}$$

$$E(Y)=60E(X^2)+39E(X)=60*\frac{7}{6}+39*1$$

$E(Y)=109$ kilowatt hours

Linear Combinations of Random Variables

- If *a* and *b* are ***constants***, then:

$$\sigma_{aX+b}^{2} = a^{2}\sigma_{X}^{2}$$