ENGG 319 (Fall 2016) Lecture/ Tutorial # 3 (Videos 1-14, Chapter 2)

Counting techniques (Section 2.3)

Question # 1. How many ways are there that no two students will have the same birth date in a class of size 60? Consider for simplicity that no one's birthday is on Feb 29, and 1 year = 365 days.

1st 2nd 60th

365 ways 364 363 $365-59 = 306$ ways

$\text{Total} = 365 \times 364 \times \cdots \times 306$ ways

$= {}_{365}P_{60}$ or ${}^{365}P_{60}$ ways

Question # 2. In how many ways are there to select 3 candidates from 8 equally qualified recent graduates for openings in an accounting firm? **[Ans: 56]**

8 candidates

choose 3 Not choose $(8-3)=5$

order does not matter.

$${}_8C_3 = \frac{8!}{3!\,5!} = \frac{6\times7\times8}{6} = 56$$

Question # 3. (a) In how many ways 6 teachers can be assigned to 4 sections of an introductory psychology course if no teacher is assigned to more than one section? **[Ans: 360]** **(b)**. In an experiment a die is thrown and then a letter from the English alphabet is drawn at random. How many points are there in the sample space? **[Ans: 156]**

(a) order matters

6 ways 5 4 3

$N = 6\times5\times4\times3 = 360$ ways

Also, ${}_6P_4$ ways.

(b) 6×26

$= 156$ ways

Question # 4. (a) How many three-digit numbers can be formed from the digits 0,1,2,3,4,5,6 if each digit can be used only once?

6 ways 6 ways 5 ways

$N = 6\times6\times5 = 6\times30 = 180$

(b) How many of these are odd numbers? **[Ans: 75]**

$\boxed{\ \ }\boxed{\ \ }\boxed{1}$ + $\boxed{\ \ }\boxed{\ \ }\boxed{3}$ + $\boxed{\ \ }\boxed{\ \ }\boxed{5}$

5 ways 5 ways 5 ways 5 ways 5 ways 5 ways

$N = 5\times5 + 5\times5 + 5\times5$

$= 75$ ways

(c) How many are greater than 330 ? **[Ans: 105]**

Begin with 4,5 or 6 + Begin with 3

(4,5,6) 3 ways 6 ways 5 ways

$\boxed{3}\boxed{4,5,6}\boxed{\ \ }$ 3 ways 5 ways

$= 3\times6\times5 + 3\times5$

$= 90 + 15 = 105$ ways

Question # 5. If a multiple-choice test consists of 5 questions, each with 4 possible answers of which only 1 is correct.

(a) in how many different ways can a student check off one answer to each question ? **[Ans: 1024]**

Q1 ← 4 choices

Q2 ← 4 "

Q5

$N = 4 \times 4 \times \cdots \times 4$ (5 times)

$= 4^5 = 64 \times 16 = 1024$

(b) in how many different ways can a student check off one answer to each question and get all the answers wrong ? **[[Ans: 243]**

Q1 ← 3 ways to get wrong answers

Q2 ← 3 ways to get wrong answers

$N = 3^5 = 81 \times 3 = 243$

Question # 6. (a) In how many ways can 6 people be lined up to get on a bus ? **[Ans: 720]**

(b) If 3 specific persons, among 6, insist on following each other, how many ways are possible? **[A:144]**

(c) If 2 specific persons, among 6, refuse to follow each other, how many ways are possible? **[Ans: 480]**

a) 1st 2nd 3rd 4th 5th 6th $6! = 720$ ways

b) Group of 3 Consider 3 persons as a group.

Then we have ~~to arrange~~ $4!$ ways to arrange the (G+3 persons).

Within the group 3 people can be arranged in $3!$ ways.

$N = 4! \times 3! = 24 \times 6 = 144$ ways.

c) In the same manner, we have $5! \times 2! = 240$ ways where the two people stay together. Then number of arrangement where they are separated $= 720 - 240 = 480$.

Question # 7. A bag contains 6 red and 8 green balls.

(a) If one ball is drawn at random, then what is the probability of the ball being green? **[Ans: 4/7]**

(b) If two balls are drawn at random, what is the probability that one is red and the other is green? **[Ans: 48/91]**

Total $= 6 + 8 = 14$

a) $P(\text{ball green}) = \frac{8}{14} = \frac{4}{7}$

b) Total Number of ways of picking two balls from total 14 $= {}^{14}C_2 = \frac{14!}{2!\,12!} = \frac{13 \times 14}{2} = 91$

Out of those 91 ways, ~~some of the~~ there are ${}^6C_1 \times {}^8C_1 = 48$ ways where one will be green and the other red.

$P(1G, 1R) = 48/91$.

[Notice: there are ${}^8C_2 + {}^6C_2 = 28 + 15 = 43$ ways where both will be same color]

Question # 8. A bag contains 7 red, 12 white and 4 green balls. What is the probability that
(a) 3 balls drawn are all white [Ans: 0.1242]
(b) 3 balls drawn are one of each color? [Ans: 0.1897]

a) Total ways of drawing 3 balls from total $7+12+4=23$ balls $= {}^{23}C_3$

$P(\text{all white}) = {}^{12}C_3 / {}^{23}C_3 = \frac{220}{1771} = 0.1242$

b) Number of ways for one white, one red & one green $= {}^{7}C_1 \times {}^{12}C_1 \times {}^{4}C_1 = 336$

$P(\text{all different}) = 336/1771 = 0.1897$

Question # 9. There are 7 red balls and 6 black balls in a bin. In how many ways they can be arranged in a line so that the first, third, and fifth balls are of same color? [Ans: 330]

Place All Red in 1^{st}, 3^{rd}, 5^{th} place :

R		R		R								

Remaining 4R and 6B can be arranged in $\frac{10!}{4!6!} = 210$ ways

Similarly, putting all blacks in 1^{st}, 3^{rd}, 5^{th} place, we have $\frac{10!}{3!7!} = 120$ ways

Total $= 210 + 120 = 330$ ways

Bayes Rule and Theorem of Total Probability (Section 2.7)

Question # 10. In a bolt manufacturing factory, machines A, B, C manufacture 25%, 35%, and 40% of the total output respectively. Of the outputs from A, B, C, there are 5%, 4%, and 2% defective bolts. A bolt is drawn from all produced bolts and if found to be defective. What is the probability that it was manufactured by (a) machine A (b) machine B, and (c) machine C? [Ans: 0.362,0.406,0.232]

Let X_1 = bolt from machine A
X_2 = " B
X_3 = " C
D = bolt is defective.

Events	$P(X_i)$	$P(D\|X_i)$	$P(X_i) * P(D\|X_i) = P(X_i \cap D)$	$P(X_i\|D) = P(X_i \cap D)/P(D)$
X_1	0.25	0.05	$.25 \times .05 = 0.0125$	$.0125/.0345 = .362$
X_2	0.35	0.04	0.014	0.406
X_3	0.4	0.03	0.008	0.232
			Total = $\underbrace{0.0345}_{P(D)}$	

Ans: ⓐ $P(X_1|D) = 0.362$, ⓑ $P(X_2|D) = 0.406$, ⓒ $P(X_3|D) = 0.232$

Note : Theorem of total probability: $P(D) = P(X_1 \cap D) + P(X_2 \cap D) + P(X_3 \cap D)$

Baye's rule: $P(X_i|D) = \frac{P(X_i)\,P(D|X_i)}{P(D)}$

Question # 11. In a certain region of the country it is known from past experience that the probability of selecting an adult over 40 years of age with cancer is 0.05. If the probability of a doctor correctly diagnosing a person with cancer as having the disease is 0.78 and the probability of incorrectly diagnosing a person without cancer as having the disease is 0.06, (a) what is the probability that an adult over 40 years of age is diagnosed as having cancer? (b) What is the probability that a person diagnosed as having cancer actually has the disease? [Ans: (a) 0.096 (b) 0.406]

Let, A = an adult has cancer, $P(A) = 0.05$

B = doctor diagnoses as having cancer, $P(B) = ?$

Given, $P(B|A) = 0.78$, $P(B|A') = 0.06$

a) $P(B) = P(B \cap A) + P(B \cap A')$ ← theorem of total prob.

$= P(A)P(B|A) + P(A')P(B|A') = 0.05 \times 0.78 + 0.95 \times 0.06 = 0.096$

b) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)\,P(B|A)}{P(B)} = \dfrac{.05 \times 0.78}{0.096} = 0.406$

Additional problems (miscellaneous)

Question # 12 Two computers A and B are to be ~~marketed~~ sold. The probability that a salesperson will sell the computers are 0.6 for computer A and 0.4 for computer B. The two computers can be sold independently. Given that the salesperson was able to sell at least one of the computers, what is the probability that computer A was sold? [Ans: 0.7895]

Let $E1$ = computer A will be sold, $P(E1) = 0.6$

$E2$ = " B " $P(E2) = 0.4$

$P(E1 | E1 \cup E2) = ?$ $\quad P[E1 | (E1 \cup E2)] = \dfrac{P[E1 \cap (E1 \cup E2)]}{P(E1 \cup E2)}$

$= \dfrac{P(E1)}{P(E1 \cup E2)} = \dfrac{0.6}{0.6 + 0.4 - 0.6 \times 0.4}$

$= 0.7895$

Question # 13 Three research groups are working on a problem. The probability that group A, group B, and group C solves the problem are 1/2, 1/3, and 1/4 respectively. What is the probability that the problem is solved by at least one group? [Ans: 0.75]

$P(A \cap B \cap C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{12} - \frac{1}{8} + \frac{1}{24}$

$= \dfrac{12 + 8 + 6 - 4 - 2 - 3 + 1}{24} = \dfrac{\cancel{20}\ 27 - 9}{24} = \dfrac{18}{24} = 0.75$

Question # 14. A box contains 8 red, 3 blue, and 9 green balls. If 3 balls are drawn at random. What is the probability that at least one is blue? **(Ans: 0.4035)**

Total $= 8+3+9=20$

Total ways $= {}^{20}C_3$

$P(\underset{\text{2 others balls}}{\text{one blue}}) + P(\underset{\text{1 other}}{\text{2 blue}}) + P(\text{3 blue balls})$

$$= \frac{{}^3C_1 \times {}^{17}C_2}{{}^{20}C_3} + \frac{{}^3C_2 \times {}^{17}C_1}{{}^{20}C_3} + \frac{{}^3C_3 \times {}^{17}C_0}{{}^{20}C_3}$$

$$= \frac{3\times 136 + 3\times 17 + 1}{1140} = 0.4035$$

Question # 15. The probability of obtaining a high-paying job without a university degree is 20%, while the probability of obtaining a high-paying job with a degree is 60%. If 30% of jobs in Canada are high-paying, Find the percentage of workers that have a high-paying job and a university degree.**[Ans:15%]**

Let A = person holds high-paying job

B = " has university degree

$P(A)=0.3,\ P(A|B)=0.6,\ P(A|B')=0.2$

Theorem of total prob.

$P(A) = P(A\cap B) + P(A\cap B')$

$\Rightarrow P(A) = P(B)\,P(A|B) + P(B')\,P(A|B')$

$\Rightarrow 0.3 = P(B)\times 0.6 + [1-P(B)] * 0.2$

$\Rightarrow P(B) = 0.1/0.4 = 0.25$

Then, $P(A\cap B)$

$= P(B)\,P(A|B)$

$= 0.25\times 0.6 = 0.15 = 15\%$

Question # 16. An allergist claims that 50% of her patient she tests are allergic to some type of weed. What is the probability that exactly 3 of her next four patients are allergic to that weed? **[Ans: 0.25]**

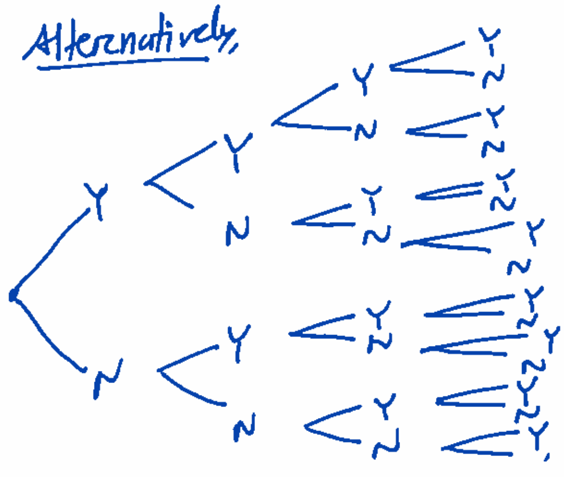
Let A_i = i-th patient is allergic, $P(A_i) = \frac{1}{2}$, $P(A_i') = \frac{1}{2}$

$P(\text{exactly 3 are allergic}) = P(1^{st} \text{ is not}) + P(2^{nd} \text{ one NOT}) + P(3^{rd} \text{ one NOT}) + P(4^{th} \text{ one NOT})$

$= P(A_1'\cap A_2\cap A_3\cap A_4) + P(A_1\cap A_2'\cap A_3\cap A_4) + P(A_1\cap A_2\cap A_3'\cap A_4) + P(A_1\cap A_2\cap A_3\cap A_4')$

$= \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} = \frac{4}{2^4} = \frac{4}{16} = \frac{1}{4} = 0.25$

Alternatively,



$P(YYYN) + P(YYNY) + P(YNYY) + P(NYYY)$

$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4}$

$= \frac{4}{2^4} = 0.25$

Each lot contains 20 items. ⇒ ~~lot~~ contains 20 $\boxed{\text{D, good, good.....}}$ lot A, contains 20 $\boxed{\text{Defective, good..}}$ lot B, ..., ...etc

Question # 17. A producer of a certain type of electronic component ships to suppliers in lots of twenty. Suppose that 60% of all such lots contain no defective components, 30% contain one defective component, and 10% contain two defective components. A lot is picked, two components from the lot are randomly selected and tested, and neither is defective. What is the probability that zero defective components exist in the lot? **[Ans: ~~0.9505~~]** .6312

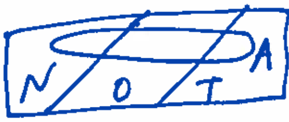
Let, O = one component in the lot is defective, $P(O) = 0.3$

T = Two " " are " , $P(T) = 0.1$

N = No " " is " $P(N) = 0.6$

A = ~~Both~~ Neither of the components tested ~~are~~ ~~do~~ is defective

Find $P(N|A) = ?$ Each lot has either no defectives, or just one or two defectives. Then N, O, T constitute a partition.



$\Rightarrow P(A) = P(A\cap N) + P(A\cap O) + P(A\cap T)$

$= P(N)\,P(A|N) + P(O)\,P(A|O) + P(T)\,P(A|T)$

$= 0.6 \times 1 + 0.3 \times \frac{{}^{19}C_2}{{}^{20}C_2} + 0.1 \times \frac{{}^{18}C_2}{{}^{20}C_2}$ $\boxed{\text{D, good 1, G2, ..., G19}}$ $\boxed{\text{D1, D2, G1, G2, ..., G18}}$

$= 0.9505$

Then, $P(N|A) = \frac{P(N\cap A)}{P(A)}$

$= \frac{P(N)\,P(A|N)}{P(A)} = \frac{0.6}{.9505} = \boxed{0.6312}$ Ans.

Question # 18. A shipment of 12 televisions contains 3 defective ones. In how many ways can a hotel purchase 5 of these ones and receive at least two defective ones? **[Ans: 288]**

$\boxed{\text{D1, D2, D3, G1, G2, ... G9}}$ G = Good, D = Defective

No of ways two defectives $= {}^3C_2 \times {}^9C_3 = \frac{3!}{2!} \times \frac{9!}{3!\,6!} = 3 \times 84 = 252$

No of ways to get 3 defectives $= {}^3C_3 \times {}^9C_2 = 1 \times 36 = 36$

Total = 252 + 36 = 288 ways.

Question # 19. The famous Monty Hall problem

Suppose a contestant is on a game show, and s/he is given the choice of three doors: Behind one door is a car; behind the others, goats. The contestant pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 2, which has a goat. He then says to you, "Do you want to pick door No. 3?" Is it advantageous to switch your choice?

[**Hint:** Define: A = the car is behind door 1, B = the car is behind door 2, and C = the car is behind door 3. Also define the following event, b = the host opens the door 2 after the contestant selects door A.

$P(b) = P(b \cap A) + P(b \cap B) + P(b \cap C) = P(A)P(b|A) + P(B)P(b|B) + P(C)P(b|C) =$

$= \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = \frac{1}{2}$

Since the host has opened door 2, the car is behind either door 1 or door 3. Based on the information that despite having two choices, the host chose to open door 2, you now revise the probability of events A and C. Find

$P(A|b) = \frac{P(A)*P((b|A)}{P(b)} = \frac{\frac{1}{3}\times\frac{1}{2}}{\frac{1}{2}} = \frac{1}{3}$ and $P(C|b) = \frac{P(C)*P(b|C)}{P(b)} = \frac{\frac{1}{3}\times 1}{\frac{1}{2}} = \frac{2}{3}$

Should the contestant change the door? The conclusion is counterintuitive. See: https://en.wikipedia.org/wiki/Monty_Hall_problem] Should switch!