ENGG 319

Probability & Statistics for Engineers

Section #02
Probability

L01

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F16

Probability

Recall:

- Probability refers to the study of ***randomness*** and ***uncertainty***.
- In any situation in which one of a number of possible ***outcomes*** may occur, the theory of probability provides methods for quantifying the ***chances*** or ***likelihoods*** associated with the various outcomes.

Sample Space (*S*):

- Is the set of ***all*** possible outcomes of a statistical experiment.
- Each outcome in a sample space is called an ***element*** or a ***member*** of the sample space (or simply a ***sample point***)

Sample Space

- If the sample space has a finite number of elements, the members can be represented as a list separated by commas and enclosed in braces.

Example #1: Flipping a coin once

$$S = \{ H, T \}$$

Where H denotes heads and T denotes tails

Sample Space

- If the sample space has a large or infinite number of elements, it is described by a ***statement*** or ***rule***.

Example #2: Set of persons in North America watching NHL

$\Rightarrow$ S = { x | x is a person in North America watching NHL }

Statement

Example #3: Tossing a die once $\Rightarrow$ $S = \{1, 2, 3, 4, 5, 6\}$

$\Rightarrow$ $S = \{ x \mid 1 \le x \le 6 \}$

Rule

Tree Diagrams

- It is helpful sometimes to list the elements of the sample space ***systematically*** using ***tree diagrams***.

Example #4: Flipping a coin once and then tossing a die once

S = { 1 H, 1 T, 2 H, 2 T, 3 H, 3 T, 4 H, 4 T, 5 H, 5 T, 6 H, 6T }

12 Elements

Tree Diagrams

Example #4:

First Outcome	Second Outcome	Sample Point	Sample Point #
1	H	1 H	#1
	T	1 T	#2
2	H	2 H	#3
	T	2 T	#4
3	H	3 H	#5
	T	3 T	#6
4	H	4 H	#7
	T	4 T	#8
5	H	5 H	#9
	T	5 T	#10
6	H	6 H	#11
	T	6 T	#12

Events

- An **event** is a subset of a sample space.
- Sometimes, an event may be a subset that includes the entire sample space ***S***.
- Also, an event may be a subset of ***S*** that contains no elements (this is known as the ***null set*** ϕ)

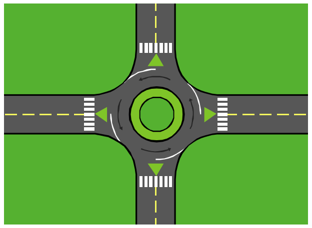
Example #5: Flipping a coin twice and we are interested in the event A of having 2 similar outcomes

S = { H H, H T, T H, T T }

A = { H H, T T }

Events

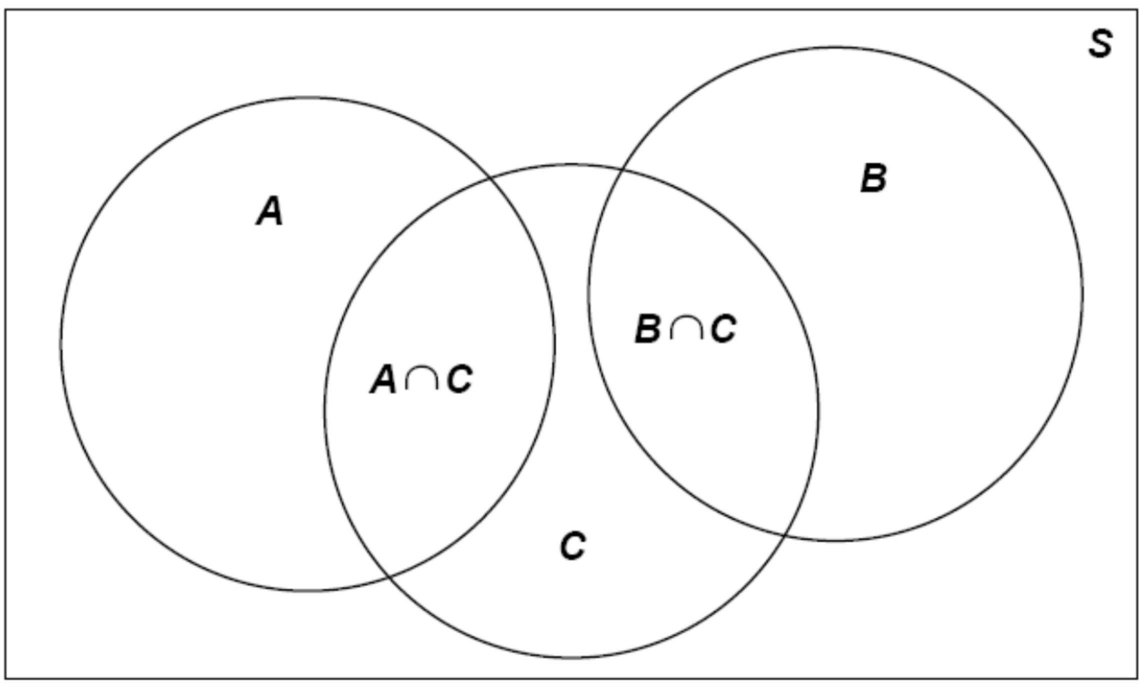
- The **complement** of an event A with respect to $\boldsymbol{S}$ is the subset of all the elements of $\boldsymbol{S}$ that is not in A. The complement of A is denoted by A'.
- The **intersection** of two events A and B, denoted by the symbol $A \cap B$, is the event containing all the elements that are common to A and B .



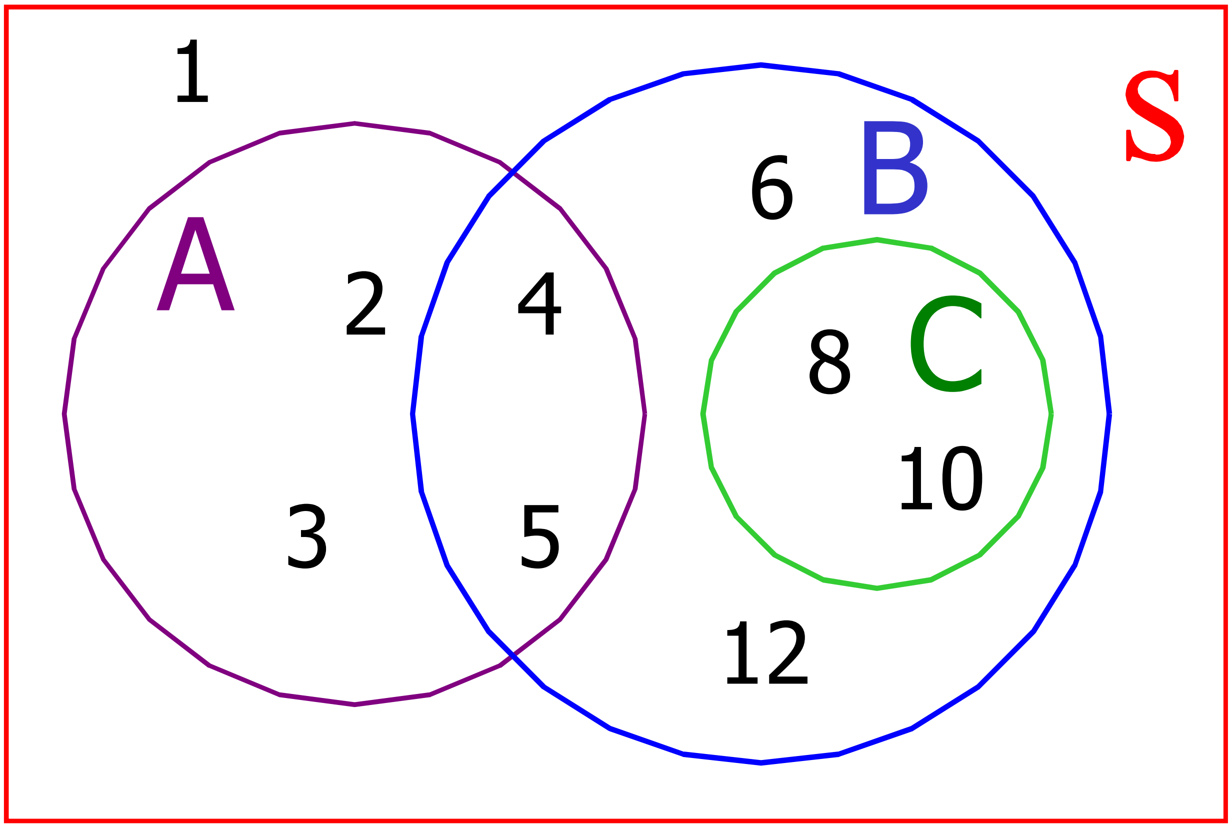
- Two events A and B are **mutually exclusive**, or **disjoint**, if $A \cap B = \phi$, that is if A and B have no elements in common.
- The **union** of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or *both*.

Venn Diagrams

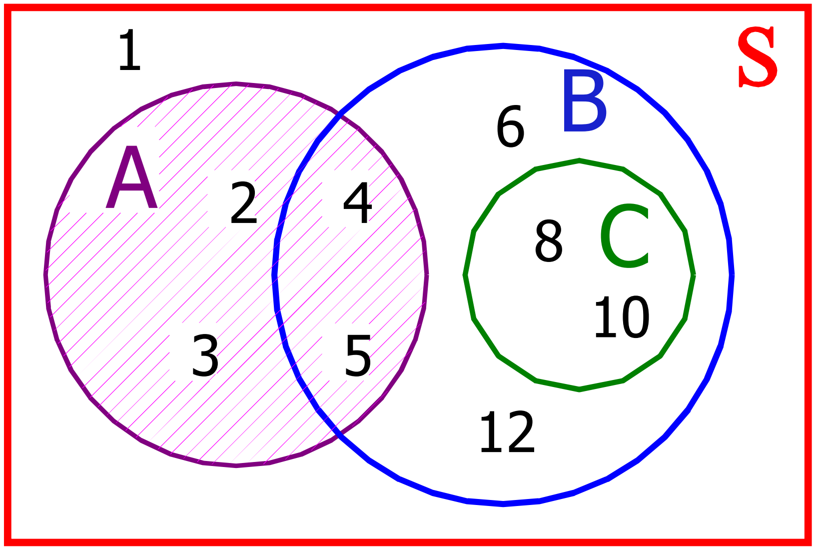
- The relationship between events and the corresponding sample space can be illustrated graphically using Venn diagrams (or set diagrams).

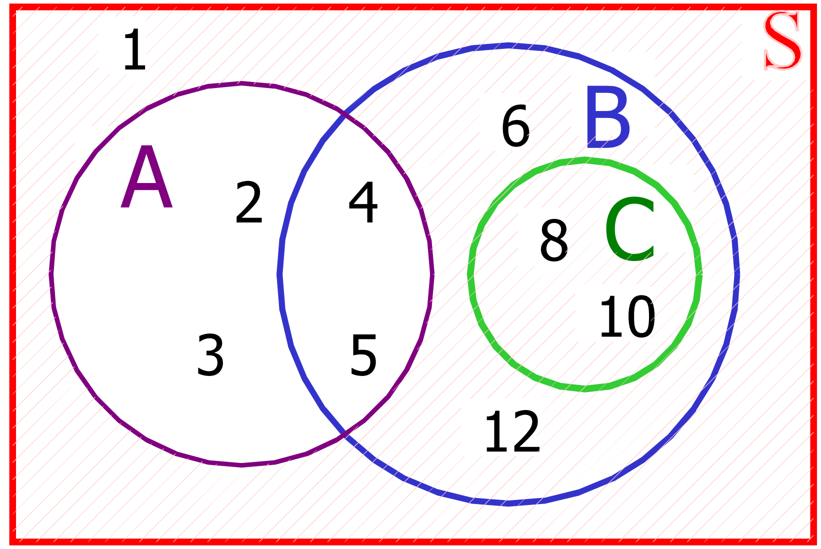


Example #6



Example #6

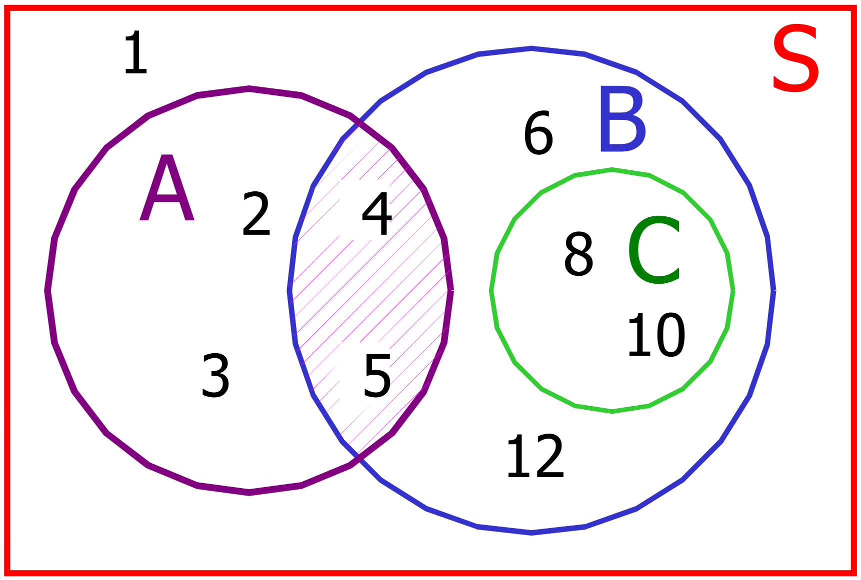




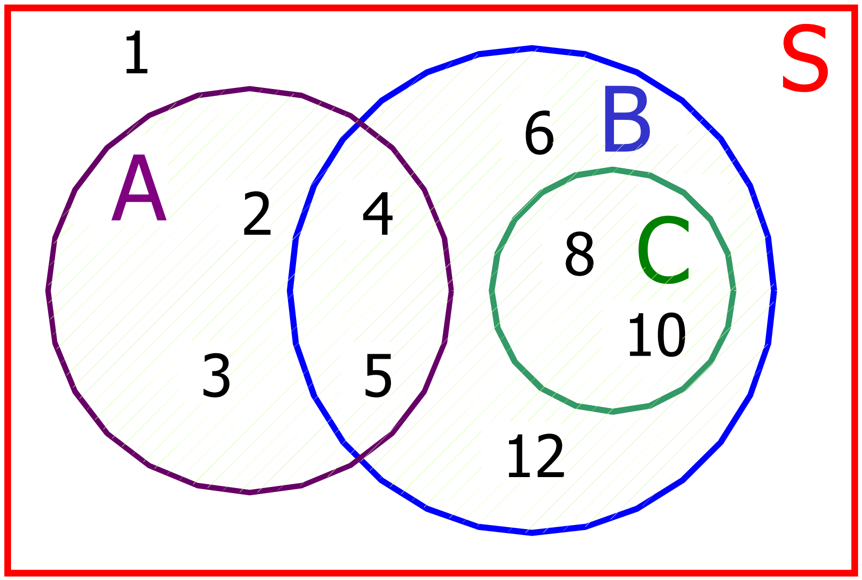
$$A = \{2, 3, 4, 5\}$$

$$A' = \{1, 6, 8, 10, 12\}$$

Example #6

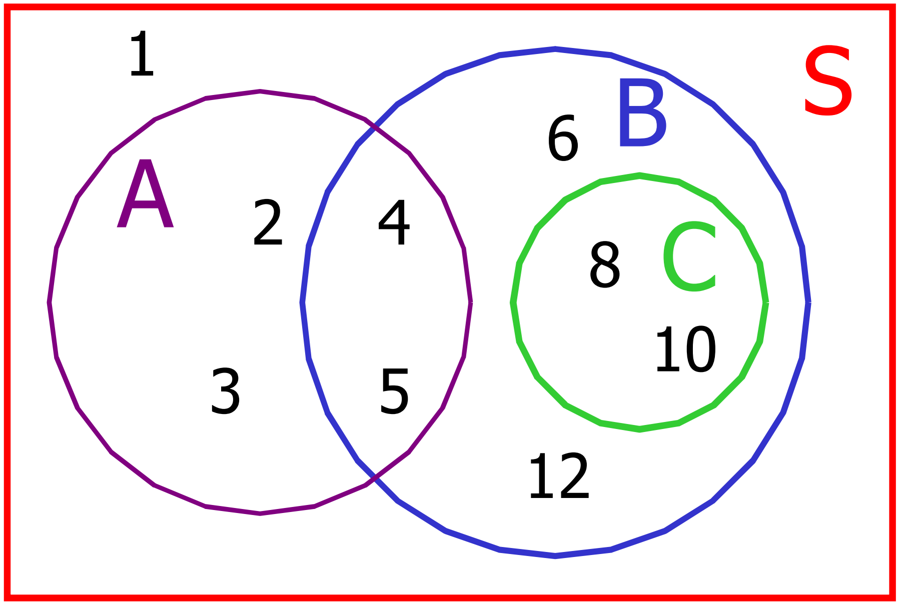


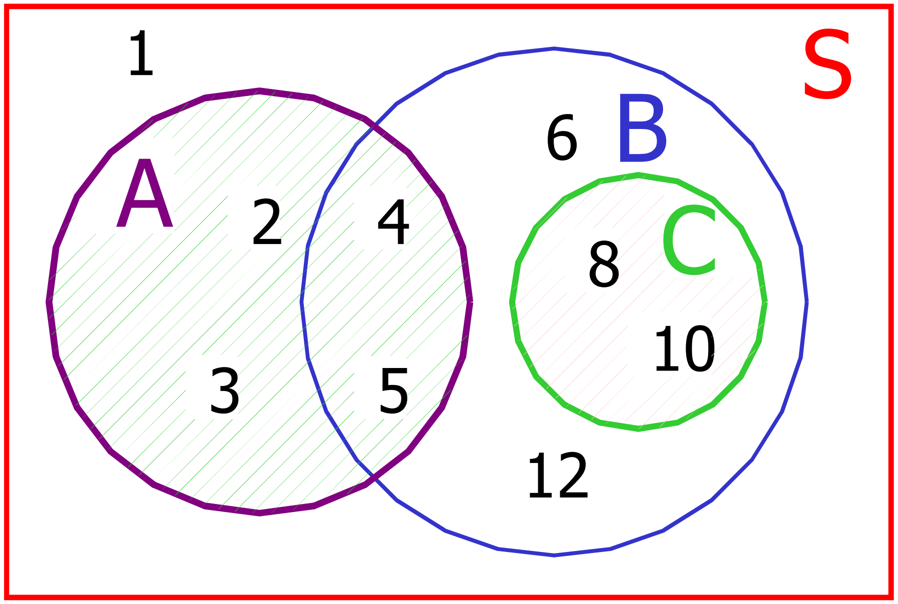
$$A \cap B = \{4, 5\}$$



$$A \cup B = \{2, 3, 4, 5, 6, 8, 10, 12\}$$

Example #6

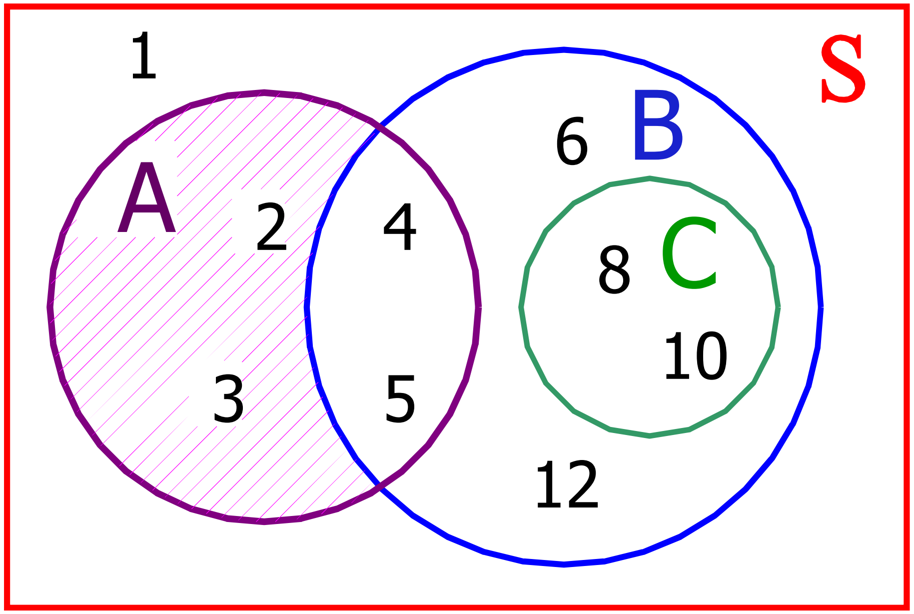


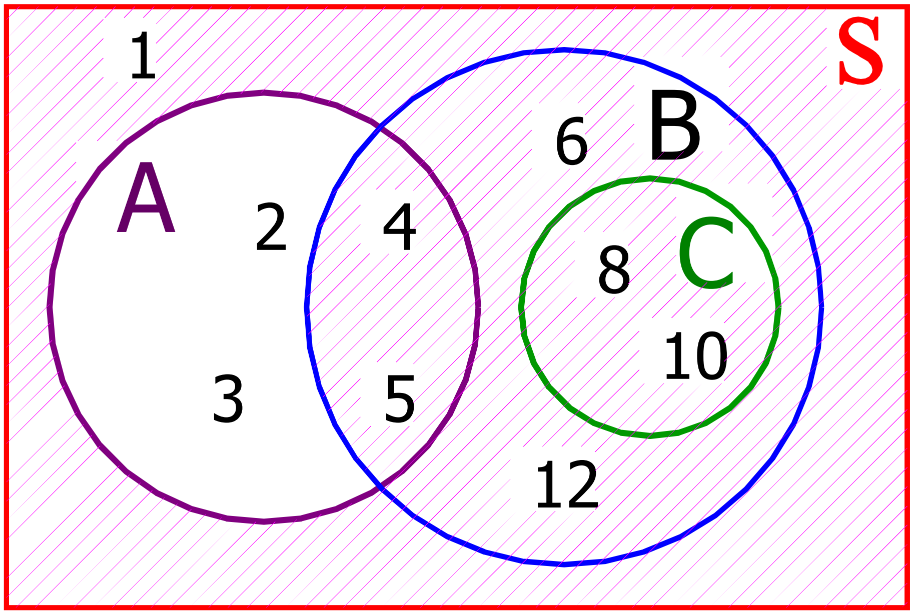


$$A \cap C = \phi$$

$$A \cup C = \{2, 3, 4, 5, 8, 10\}$$

Example #6

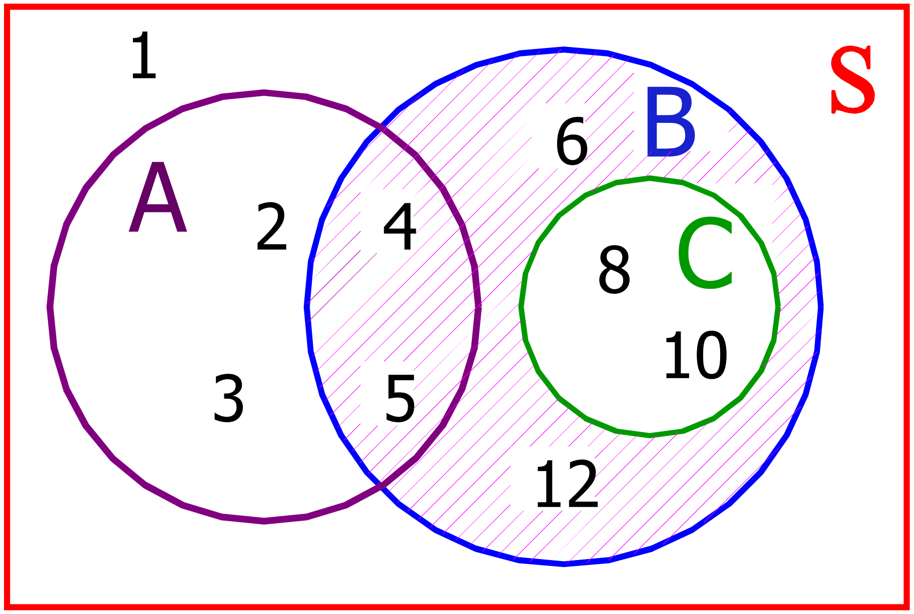


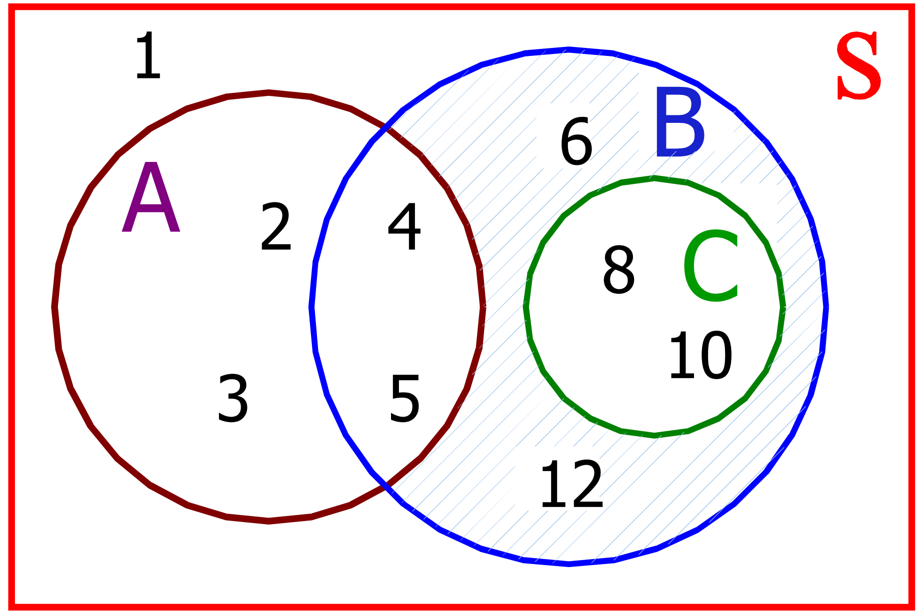


$$A \cap B' = \{2, 3\}$$

$$A' \cup B = \{1, 4, 5, 6, 8, 10, 12\}$$

Example #6

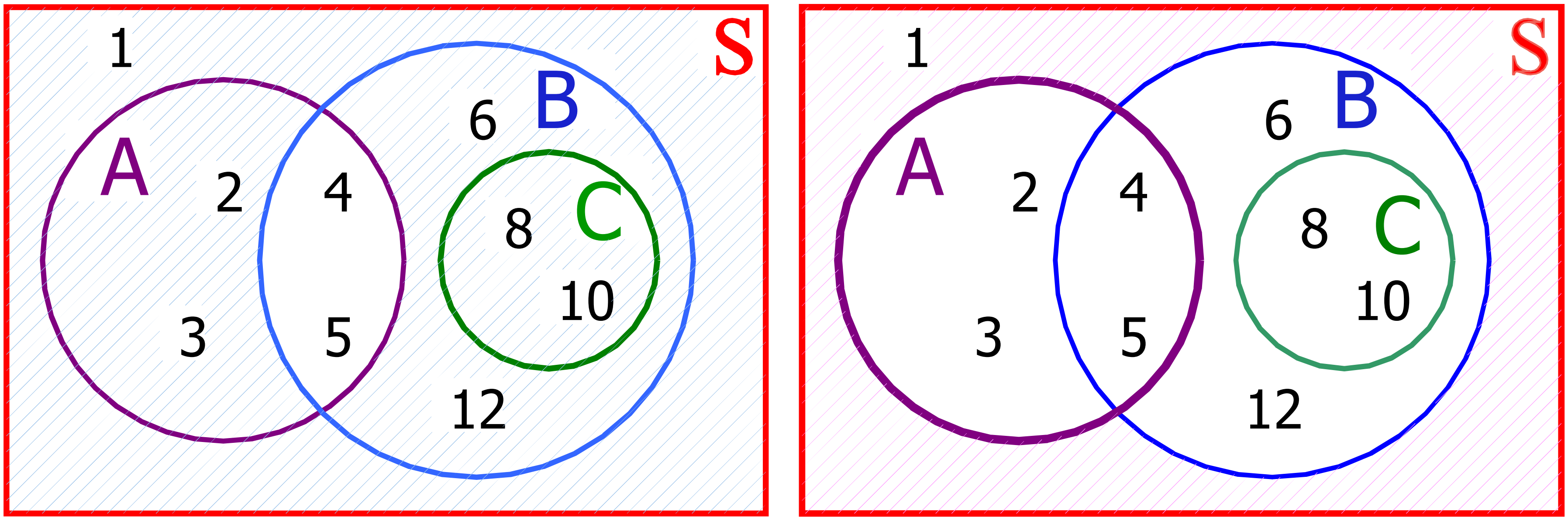




$$B \cap C' = \{4, 5, 6, 12\}$$

$$A' \cap B \cap C' = \{6, 12\}$$

Example #6



$$A' \cup B' = (A \cap B)'$$

$$= \{1, 2, 3, 6, 8, 10, 12\}$$

$$A' \cap B' = (A \cup B)'$$

$$= \{1\}$$

Some Rules

$$A \cap \phi = \phi$$

$$A \cup \phi = A$$

$$A \cup A' = S$$

$$S' = \phi$$

$$\phi' = S$$

$$(A')' = A$$

$$A' \cup B' = (A \cap B)'$$

$$A' \cap B' = (A \cup B)'$$

$$A \in S$$

A is a subset of ***S***

Example #7

(Ex. 2.16 Textbook):

- If $S = \{ x \mid 0 < x < 12\}$,
  $M = \{ x \mid 1 < x < 9\}$, and
  $N = \{ x \mid 0 < x < 5\}$,

Find:

(a) $M \cup N$

(b) $M \cap N$

(c) $M' \cap N'$

Example #7 (Sol.)

(a) $M \cup N$

$$M \cup N = \{x \mid 0 < x < 9\}$$

(b) $M \cap N$

$$M \cap N = \{x \mid 1 < x < 5\}$$

(c) $M' \cap N' = (M \cup N)'$

$$= \{x \mid 9 < x < 12\}$$

Counting Sample Points

Multiplication Rule:

- If an operation can be performed in n_1 ways and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed in $n_1 n_2$ ways.

Example #8: Throwing a pair of dice (in the same time) once

$\boxed{S_1 = \{1, 2, 3, 4, 5, 6\}}$ $n_1 = 6$

$\boxed{S_2 = \{1, 2, 3, 4, 5, 6\}}$ $n_2 = 6$

No. of sample points = $n_1 * n_2 = 6*6 = 36$

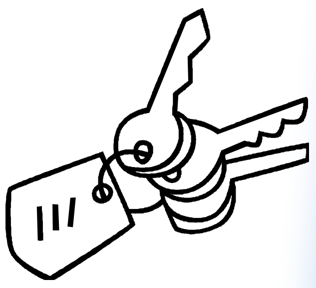
Counting Sample Points

Generalized Multiplication Rule:

- If an operation can be performed in n_1 ways and of for each of these ways a second operation can be performed in n_2 ways, and for each of the first two, a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \ldots n_k$ ways.

Example #9:

- A person wants to buy a vehicle where he can choose from the following colors: white, black, red, green, or silver. The vehicle is also available with different number of doors: 2, 4, or 5. the buyer can select the transmission type as standard or auto. If all possibilities are available, in how many different ways can this buyer select the vehicle?



Example #9 (Sol.)

$n_1 = 5$

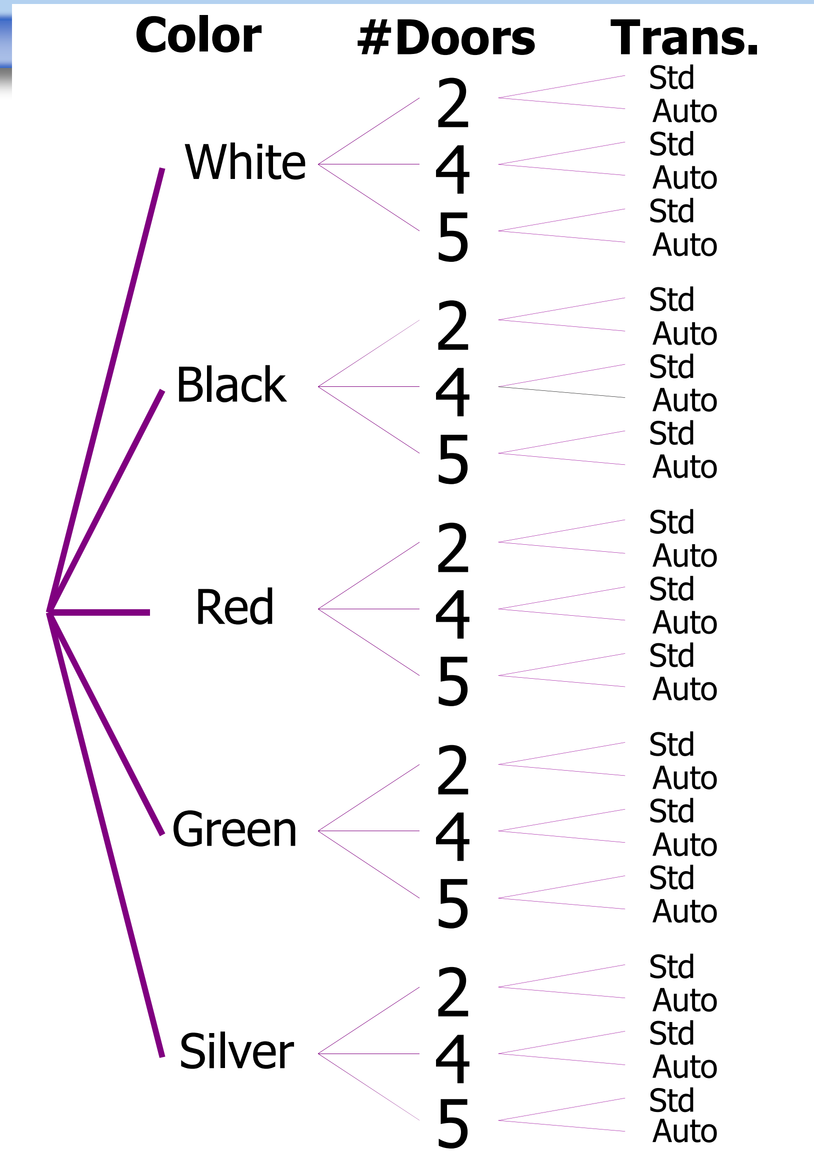
$n_2 = 3$

$n_3 = 2$

No. of possible selections

$= n_1 * n_2 * n_3$

$= 5*3*2 = 30$



Example #10

- How many **even** four-digit numbers can be formed from the digits: 0, 1, 4, 5, 8, and 9 if each digit can be used only once?

Some facts:

- We have 3 choices for the units position to make the number even (0, 4 and 8)
- We cannot have the 0 in the thousands position (since it is a four-digit number).
- Thus, we are going to have two cases for the 0 in the units position, i.e. the 0 is there or the 0 is not there!

Example #10

Case I: (**0** is in the units position)

n_1 = 1 (units position) n_2 = 5 (thousands position)

n_3 = 4 (hundreds position) n_4 = 3 (tens position)

No. of possible selections = $n_1 * n_2 * n_3 * n_4$ = 1*5*4*3 = 60

Case II: (**0** is not in the units position)

n_1 = 2 (units position) n_2 = 4 (thousands position)

n_3 = 4 (hundreds position) n_4 = 3 (tens position)

Why?

No. of possible selections = $n_1 * n_2 * n_3 * n_4$ = 2*4*4*3 = 96

Total No. of possible selections = 60 + 96 = 156

Permutations

- A **permutation** is an *arrangement* of all or part of a set of objects.
- The number of permutations of **n** objects is **n!**

$$n! = n(n-1)(n-2)\ldots(1) \qquad 0! = 1$$

Example #11:

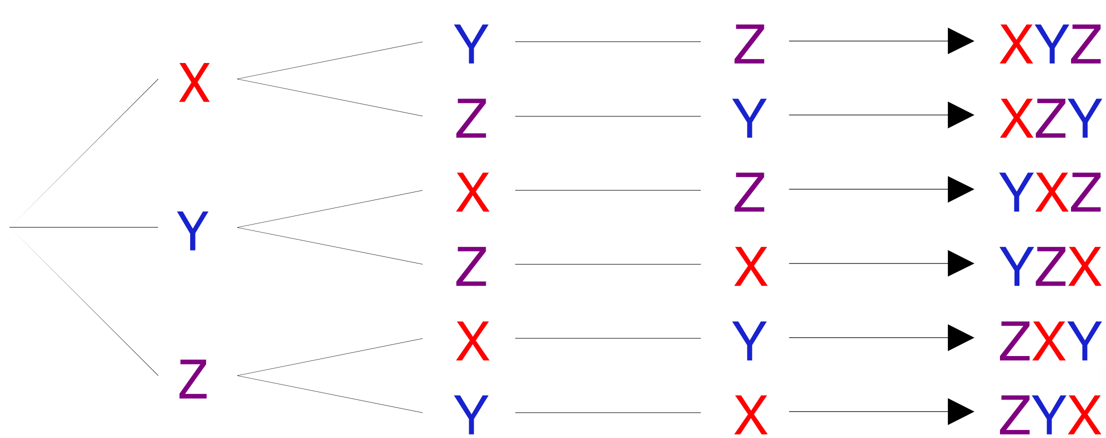
- What is the No. of possible ways (permutations) to ***arrange*** the three letters X, Y, and Z?

$n = 3$

No. of permutations

$= n! = 3!$

$= 3*2*1 = 6$



Permutations

- The number of permutations of ***n*** objects arranged in a circle is ***(n-1)!***.

- The number of permutations of ***n*** distinct objects taken ***r*** at a time is:

$$ {}_{n}P_{r} = \frac{n!}{(n-r)!} $$

Example #12:

- What is the No. of possible ways (permutations) to ***arrange*** a four-digit number using the following digits: 4, 5, 6, 7, 8, and 9?

$n = 6$ $r = 4$

No. of permutations = $ {}_{6}P_{4} = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 6*5*4*3 = 360 $

Permutations

Example #13:

- What is the No. of possible ways (permutations) to ***arrange*** a six-digit number using the following digits: 4, 5, 6, 7, 8, and 9?

$n = 6$ $r = 6$

No. of permutations = $_6P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6*5*4*3*2*1 = 720$

Example #13 (ver.2):

- What is the No. of possible ways (permutations) to ***arrange*** the following digits: 4, 5, 6, 7, 8, and 9?

$n = 6$

No. of permutations = $6! = 6*5*4*3*2*1 = 720$

Permutations

- The number of ***distinct*** permutations of ***n*** things of which n_1 are of one kind and n_2 of a second kind….n_k of a ***k***th kind is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Example #14:

- What is the No. of possible ways to ***arrange*** the following letters Y, Y, Z, and Z?

$n = 4$ $k = 2$ $n_1 = 2$ $n_2 = 2$

No. of permutations = $\dfrac{n!}{n_1!n_2!} = \dfrac{4!}{2!*2!} = \dfrac{4*3*2*1}{(2*1)*(2*1)} = 6$

Note that: possible ways are { YYZZ, YZYZ, YZZY, ZYYZ, ZYZY, ZZYY }

Partitioning

- The number of ways of ***partitioning*** a set of ***n*** objects into ***r*** cells with n_1 elements in the first cell and n_2 elements in the second, and so forth, is:

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

Where: $n = n_1 + n_2 + \ldots + n_r$

Example #15:

- What is the No. of possible ways to ***separate*** the letters X, Y, Z into 2 groups where one group has 1 letter and the other has 2 letters?

$n = 3$ $r = 2$ $n_1 = 1$ $n_2 = 2$

No. of possible separations = $\binom{3}{1, 2} = \frac{3!}{1! * 2!} = \frac{3*2*1}{(1)*(2*1)} = 3$

Note that: possible ways are { (X), (Y, Z) } , { (Y), (X, Z) } , { (Z), (X, Y) }

Combinations

- The number of ***combinations*** of ***n*** *distinct* objects taken ***r*** at a time is:

$$\binom{n}{r, n-r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example #16:

- What is the No. of possible ways to ***select*** a four-digit number from the following digits: 4, 5, 6, 7, 8 and 9?

$n = 6$ $r = 4$

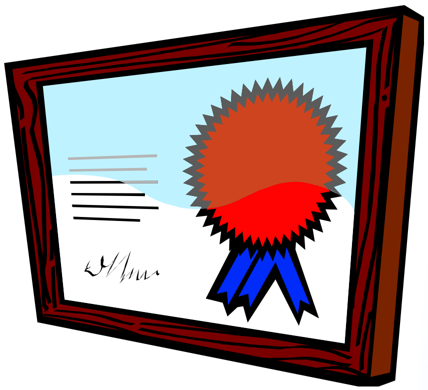
No. of possible selections = $\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!} = \frac{6*5}{2*1} = 15$

Example #17

- In an engineering class of 60 students, two awards will be given. If each student can receive at most one award, how many possible selections can be made in each of the following cases:

(a) The two awards are named: Leadership and Academic Achievement.

(b) The two awards are not named.



Example #17 (Sol.)

(a) The two awards are named, i.e. ***not similar*** $n = 60$ $r = 2$

No. of possible selections = $_{60}P_2 = \dfrac{60!}{(60-2)!} = \dfrac{60!}{58!} = 60 * 59 = 3540$

(b) The two awards are not named, i.e. ***similar***

Approach #1 (Combination): $n = 60$ $r = 2$

No. of possible selections = $\dbinom{60}{2} = \dfrac{60!}{2!(60-2)!} = \dfrac{60!}{2!58!} = \dfrac{60*59}{2*1} = 1770$

Approach #2 (Partitioning): $n = 60$ $r = 2$ $n_1 = 2$ $n_2 = 58$

No. of possible selections = $\dfrac{60!}{2!58!} = \dfrac{60*59}{2*1} = 1770$

Example #18

(Ex. 2.36 Textbook):

(a) How many three-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, and 6, if each digit can be used only once?

(b) How many of these are odd numbers?

(c) How many are greater than 330?

Example #18 (sol.)

(a)

Some facts (n = 7):

- We cannot have the 0 in the hundreds position (since it is a three-digit number).
- Thus, we have only 6 choices for the hundreds position (1, 2 , 3, 4, 5 and 6).

n_1 = 6 (hundreds position) n_2 = 6 (tens position)

n_3 = 5 (units position)

No. of possible selections = $n_1 * n_2 * n_3 = 6*6*5 = 180$

Example #18 (sol.)

(b)

Some facts (n=7):

- We cannot have the 0 in the hundreds position (since it is a three-digit number).

- We have only 3 choices for the units position to make the number odd (1, 3, and 5).

n_1 = 3 (units position)

n_2 = 5 (hundreds position)

n_3 = 5 (tens position)

No. of possible selections = $n_1 * n_2 * n_3$ = 3*5*5 = 75

Example #18 (sol.)

(c)

Some facts (n=7):

- We can have definitely 3 choices for the hundreds position to make the number greater than 330 (4, 5, and 6).
- We can have 1 choice for the hundreds position to make the number greater than 330 (3) but this is conditioned with the fact that the tens digit must be one of the following digits (4, 5, and 6).

Example #18 (sol.)

(c) **Case I:** (4 or 5 or 6 is in the hundreds position)

n_1 = 3 (hundreds position) n_2 = 6 (tens position)

n_3 = 5 (units position)

No. of possible selections = $n_1 * n_2 * n_3$ = 3*6*5 = 90

Case II: (3 is in the hundreds position)

n_1 = 1 (hundreds position) n_2 = 3 (tens position)

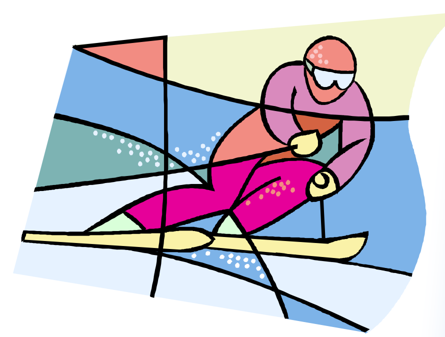
n_3 = 5 (units position)

No. of possible selections = $n_1 * n_2 * n_3$ = 1*3*5 = 15

Total No. of possible selections = 90 + 15 = 105

Example #19

- Nine people are going on a skiing trip in three cars that hold 2, 4, and 5 passengers, respectively. In how many ways is it possible to transport the nine people to the ski lodge, using all cars?



Example #19 (Sol.)

- This is a partitioning problem.

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

$n = 9$ $r = 3$ $n_1 = ?$ $n_2 = ?$ $n_3 = ?$

Possible selections of n_1, n_2 and n_3, respectively:

{ (2, 4, 3), (2, 3, 4), (2, 2, 5), (1, 4, 4), (1, 3, 5) }

No. of ways = $\binom{9}{2,4,3} + \binom{9}{2,3,4} + \binom{9}{2,2,5} + \binom{9}{1,4,4} + \binom{9}{1,3,5}$

$$= \frac{9!}{2!4!3!} + \frac{9!}{2!3!4!} + \frac{9!}{2!2!5!} + \frac{9!}{1!4!4!} + \frac{9!}{1!3!5!}$$

$$= \frac{9*8*7*6*5}{2*1*3*2*1} + \frac{9*8*7*6*5}{2*1*3*2*1} + \frac{9*8*7*6}{2*1*2*1} + \frac{9*8*7*6*5}{1*4*3*2*1} + \frac{9*8*7*6}{1*3*2*1}$$

$$= 1260 \quad + 1260 \quad + 756 \quad + 630 \quad + 504 \quad = 4410$$

Probability of an Event

- The probability of an event *A* is the sum of the weight of all sample points in *A*, thus:

$$0 \le P(A) \le 1 \qquad P(\phi) = 0 \qquad P(S) = 1$$

- if A_1, A_2, A_3...is a sequence of *mutually exclusive* events, then:

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

Example #20: Tossing a fair die once $\Rightarrow$ $S = \{1, 2, 3, 4, 5, 6\}$

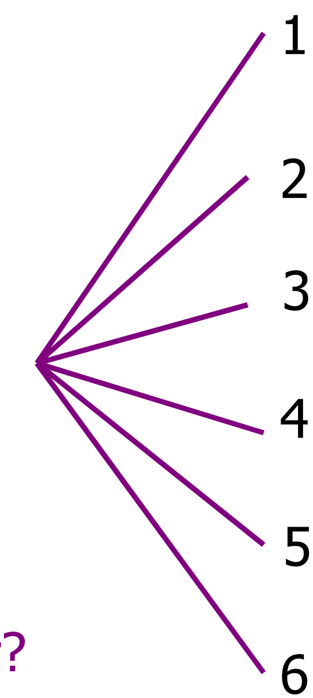
Each sample point has the same probability (weight) of ω.

$N = 6$ $\Rightarrow$ $6\omega = P(S) = 1$ $\Rightarrow$ $\omega = \dfrac{1}{6}$

Example #20

event A_1 is the occurrence of 1
event A_2 is the occurrence of 2
event A_3 is the occurrence of 3
event A_4 is the occurrence of 4
event A_5 is the occurrence of 5
event A_6 is the occurrence of 6

These events are mutually exclusive



P(1) = 1/6
P(2) = 1/6
P(3) = 1/6
P(4) = 1/6
P(5) = 1/6
P(6) = 1/6

What is the probability of getting an even number?

$$P(E) = P(A_2 \cup A_4 \cup A_6) = P(A_2) + P(A_4) + P(A_6)$$

$$= P(2) + P(4) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

Example #21

- A die is loaded in such a way that an even number is triple as likely to occur as an odd number. If A is the event that a number less than 3 occurs on a single toss of the die, find P(A).

$S = \{1, 2, 3, 4, 5, 6\}$ $O = \{1, 3, 5\}$ $E = \{2, 4, 6\}$ $A = \{1, 2\}$

$N = 6 \Rightarrow 3\omega + 3 * 3\omega = 1 \Rightarrow \omega = \frac{1}{12}$

$$\Rightarrow \mathrm{P}(\mathrm{A}) = \mathrm{P}(1) + \mathrm{P}(2) = \frac{1}{12} + \frac{3}{12} = \frac{1}{3}$$

Probability of an Event

- If an experiment can result in any one of ***N*** different *equally likely* outcomes, and if exactly ***n*** of these outcomes correspond to event *A*, then the probability of event A is:

$$P(A) = \frac{n}{N}$$

Example #22: Tossing a fair die once → $S = \{1, 2, 3, 4, 5, 6\}$

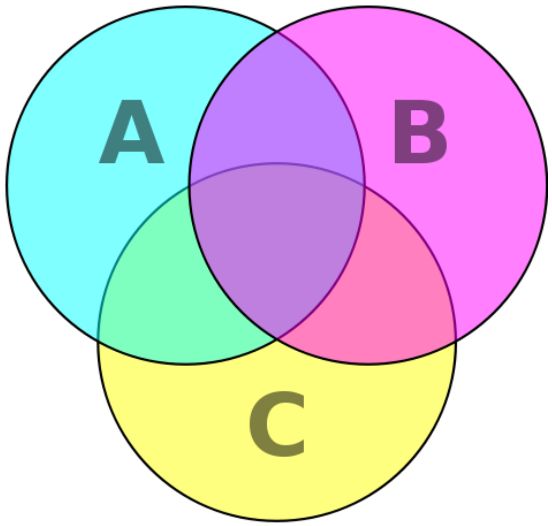
What is the probability of getting an even number?

$E = \{2, 4, 6\}$ $N = 6$ $n = 3$ → $P(E) = \frac{3}{6} = \frac{1}{2}$

Additive Rules

- If *A* and *B* are two events, then:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



http://en.wikipedia.org/

- If *A* and *B* are mutually exclusive, then:

$$P(A \cup B) = P(A) + P(B)$$

- If A_1, A_2, …., A_n are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

Additive Rules

- If A_1, A_2, …., A_n is a partition of sample space ***S***, then:

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) \quad = \quad P(S) = 1$$

- For events A, B, and C :

$$\begin{aligned} P(A \cup B \cup C) \quad = \quad & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C) \end{aligned}$$

- If A and A' are complementary events, then:

$$P(A) + P(A') = 1$$

Example #23

(Ex. 2.56 Textbook):

- An automobile manufacturer is concerned about a possible recall of its best-selling 4-door sedan. If there were a recall, there is 0.25 probability that a defect is in the brakes system, 0.18 in the transmission, 0.17 in the fuel system, and 0.40 in some other areas.

(a) What is the probability that the defect is the brakes or the fuel system if the probability of defects in both systems simultaneously is 0.15?

(b) What is the probability that there are no defects in either the brakes or the fuel system?

Example #23 (Sol.)

B = Defect in brakes system $\boxed{P(B) = 0.25}$

F = Defect in fuel system $\boxed{P(F) = 0.17}$

(a) $\boxed{P(B \cap F) = 0.15}$

$$P(B \cup F) = P(B) + P(F) - P(B \cap F)$$
$$= 0.25 + 0.17 - 0.15 = 0.27$$

(b) $P(B' \cap F')$

Recall: $A' \cap B' = (A \cup B)'$

$$= 1 - P(B \cup F)$$
$$= 1 - 0.27 = 0.73$$

Conditional Probability

- The conditional probability of *B*, given *A*, denoted by P(B|A) is defined by:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$ provided $$\mathrm{P}(\mathrm{A}) > 0$$

Example #24:

- The probability of having a heavy rain in a Canadian city during summer is 30%. The probability of a regularly scheduled flight from this city to depart on time is 84%; and the probability that it departs on time and there is no heavy rain is 63%. Find the probability that the plane departs on time given that there is no heavy rain.

Example #24 (Sol.)

H = the event of heavy rain

D = the event of departing on time

N = the event of no heavy rain

P(H) = 0.30

P(D) = 0.84

$$P(D \mid N) = ?$$

$$P(D \mid N) = \frac{P(D \cap N)}{P(N)}$$

P(N) = P(H') = 1 - P(H) = 1 - 0.30 = 0.70

$$P(D \cap N) = 0.63$$

$$\mathrm{P}(\mathrm{D} \mid \mathrm{N}) = \frac{\mathrm{P}(\mathrm{D} \cap \mathrm{N})}{\mathrm{P}(\mathrm{N})} = \frac{0.63}{0.70} = 0.90$$

Independent Events

- Two events *A* and *B* are ***independent*** if and only if:

$$P(B \mid A) = P(B) \quad \text{or} \quad P(A \mid B) = P(A)$$

provided the existence of the conditional probabilities. Otherwise, *A* and *B* are **dependent**.

Example #25:

- If two balls are drawn in succession from a bag that has 30 balls (20 white and 10 black) with replacement (i.e. the 1st ball is replaced unknown back in the bag before the 2nd ball is drawn). What is the probability of drawing the 2nd ball to be white given that the 1st one is black?

B = the 1st ball is black

W = the 2nd ball is white

$$\mathrm{P(W \mid B)} = ?$$

$$\mathrm{P(B)} = \frac{n_B}{N} = \frac{10}{30} = \frac{1}{3}$$

$$\mathrm{P(W)} = \frac{n_w}{N} = \frac{20}{30} = \frac{2}{3}$$

$$\mathrm{P(W \mid B)} = \frac{20}{30} = \frac{2}{3}$$

$$\mathrm{P(W \mid B)} = \mathrm{P(W)}$$

Multiplicative Rules

- If in an experiment, events *A* and *B* can both occur, then:

  $P(A \cap B) = P(A)\,P(B \mid A)$ provided $\mathrm{P}(\mathrm{A}) > 0$

  $\mathrm{P}(\mathrm{A} \cap \mathrm{B}) = \mathrm{P}(\mathrm{B})\,\mathrm{P}(\mathrm{A} \mid \mathrm{B})$ provided $\mathrm{P}(\mathrm{B}) > 0$

- Two events *A* and *B* are ***independent*** if and only if:

  $P(A \cap B) = P(A)\,P(B)$

- Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

Multiplicative Rules

- If in an experiment, the events A_1, A_2, …., A_k can occur, then:

$$
\begin{aligned}
&P(A_1 \cap A_2 \cap \cdots \cap A_k) \\
&= P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)\cdots P(A_k \mid A_1 \cap A_2 \cap \cdots \cap A_{k-1})
\end{aligned}
$$

- If the events A_1, A_2, …., A_k are independent, then:

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2)\cdots P(A_k)$$

Example #26

(Ex. 2.74 Textbook):

- A class in advanced physics is comprised of 10 juniors, 30 seniors, and 10 graduate students. The final grades show that 3 of the juniors, 10 of the seniors, and 5 of the graduate students received an ‘A’ for the course. If a student is chosen at random from this class and is found to have earned an ‘A’, what is the probability that he or she is a senior?

Example #26 (Sol.)

J = the event of choosing a junior

S = the event of choosing a senior

G = the event of choosing a grad student

A = the event of earning an A

$$\left.\begin{array}{l} n_J = 10 \\ n_S = 30 \\ n_G = 10 \end{array}\right\} N = 50$$

$n_A = 3_J + 10_S + 5_G = 18$

Required: $P(S \mid A) = ?$

$$P(S \mid A) = \frac{P(S \cap A)}{P(A)}$$

$$\boxed{P(A) = \frac{n_A}{N} = \frac{18}{50}}$$

$$P(S \cap A) = \frac{n_{S \cap A}}{N} = \frac{10}{50}$$

$$\Rightarrow \quad P(S \mid A) = \frac{10/50}{18/50} = \frac{10}{18} = \frac{5}{9}$$

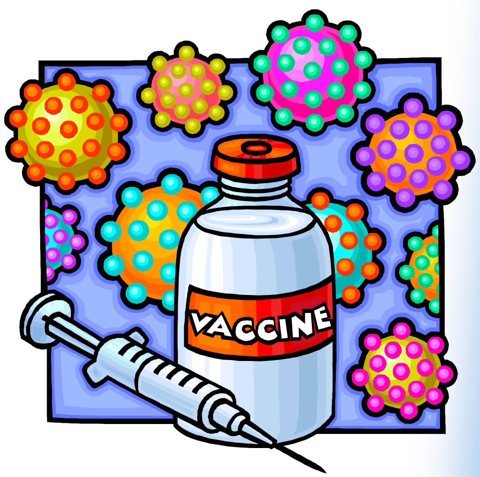
Example #27

(Ex. 2.78 Textbook):

- A manufacturer of a flu vaccine is concerned about the quality of its flu serum. Batches of serum are processed by three different departments having rejection rates of 0.10, 0.08, and 0.12, respectively. The inspections by the three departments are sequential and independent.

(a) What is the probability that a batch of serum survives the 1^{st} department inspection but is rejected by the 2^{nd} department?

(b) What is the probability that a batch of serum is rejected by the 3^{rd} department?



Example #27 (Sol.)

F = the 1st department rejects

S = the 2nd department rejects

T = the 3rd department rejects

$P(F) = 0.10$

$P(S) = 0.08$

$P(T) = 0.12$

Independent!

(a) Required: $P(F' \cap S) = ?$

$P(F' \cap S) = P(F')P(S)$

$P(F') = 1 - P(F) = 1 - 0.10 = 0.90$

$$\Rightarrow P(F' \cap S) = 0.90 * 0.08 = 0.072 \cong 0.07$$

(b) Required: $P(T \cap F' \cap S') = ?$

$P(T \cap F' \cap S') = P(T)P(F')P(S')$

$P(S') = 1 - P(S) = 1 - 0.08 = 0.92$

$P(F') = 0.90$

$$\Rightarrow P(T \cap F' \cap S') = 0.12 * 0.90 * 0.92 = 0.099 \cong 0.10$$

Example #28

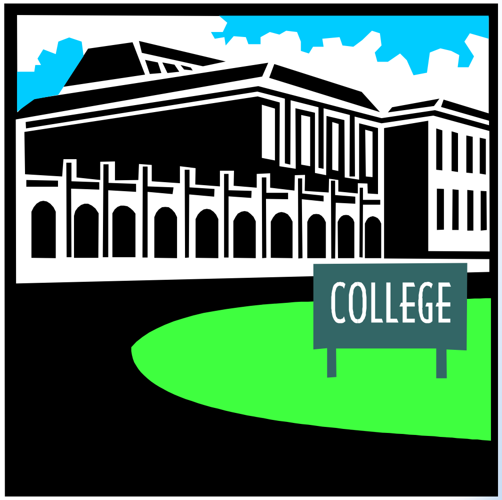
(Ex. 2.86 Textbook):

- In 1970, 11% of Americans completed four years of college; 43% of them were women. In 1990, 22% of Americans completed four years of college; 53% of them were women *(Time, Jan 19, 1996)*.

(a) Given that a person completed four years of college in 1970, what is the probability that the person was a woman?

(b) What is the probability that a woman would finish four years of college in 1990?

(c) What is the probability that in 1990 a man would not finish four years of college?



Example #28 (Sol.)

C = person completing college P(C) = 0.11 *(1970)* P(C) = 0.22 *(1990)*

W = woman completing college P(W|C) = 0.43 *(1970)* P(W|C) = 0.53 *(1990)*

M = man completing college

(a) Required: $P(W \mid C)$ *"1970"* $P(W \mid C)$ *"1970"* $= 0.43$

(b) Required: $P(W \cap C)$ *"1990"* $P(W \cap C) = P(C)P(W \mid C)$

$$P(W \cap C) = 0.22 * 0.53 = 0.1166 \cong 0.12$$

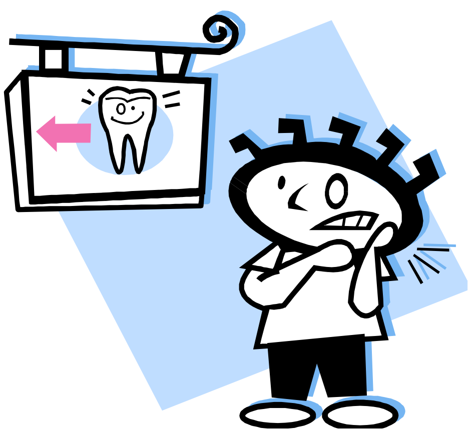
(c) Required: $P(M \cap C)'$ *"1990"* $P(M \cap C)' = 1 - P(M \cap C)$

$P(M \cap C) = P(C)P(M \mid C)$ P(M|C) = 1 - P(W|C) = 1 - 0.53 =0.47

$$P(M \cap C)' = 1 - 0.22 * 0.47 = 0.8966 \cong 0.90$$

Example #29

- The probability that a person visiting his dentist will have an-X-ray is 0.60; the probability that a person who has an X-ray will also have a cavity filled is 0.30; and the probability that a person who has had an X-ray and a cavity filled will also have a tooth extracted is 0.10. What is the probability that a person visiting his dentist will have an X-ray, a cavity filled, and a tooth extracted?



Example #29 (Sol.)

X = person with X-ray

C = person with cavity filled

E = person with tooth extracted

$P(X) = 0.60$ $P(C|X) = 0.30$ $P(E|C \cap X) = 0.10$

(a) Required: $P(X \cap C \cap E)$

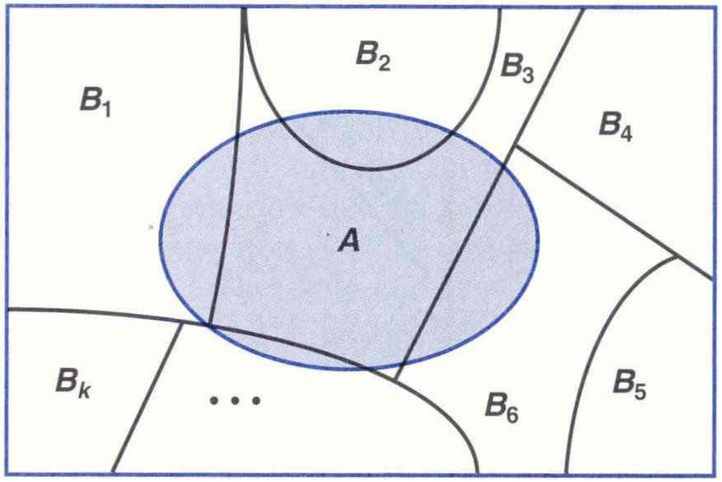
$$P(X \cap C \cap E) = P(X).P(C \mid X).P(E \mid C \cap X)$$

$$= 0.60 * 0.30 * 0.10 = 0.018 \cong 0.02$$

Rule of Elimination

- This is also sometimes called ***theorem of total probability***.
- If the events B_1, B_2, …., B_k constitute a partition of the sample space S such that $P(B_i) \neq 0$ for i = 1, 2,…,k, then for any event A of S:

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A \mid B_i)$$



Example #30

(Ex. 2.96 Textbook):

- Police plan to enforce speed limits by using radar traps at 4 different locations within the city limits. The radar traps at each of the locations L_1, L_2, L_3, and L_4 are operated 40%, 30%, 20%, and 30% of the time. If a person who is speeding has probabilities of 0.2, 0.1, 0.5, and 0.2, respectively, of passing through these locations, what is the probability that the person will receive a speeding ticket?

Example #30 (Sol.)

R_1 = event of speeding when passing through L_1 — $P(R_1) = 0.20$

R_2 = event of speeding when passing through L_2 — $P(R_2) = 0.10$

R_3 = event of speeding when passing through L_3 — $P(R_3) = 0.50$

R_4 = event of speeding when passing through L_4 — $P(R_4) = 0.20$

T = event of getting a speeding ticket

Required: $P(T) = ?$

$$P(T) = \sum_{i=1}^{4} P(R_i \cap T) = \sum_{i=1}^{4} P(R_i) P(T \mid R_i)$$

$P(T|R_1) = 0.40$ $P(T|R_2) = 0.30$ $P(T|R_3) = 0.20$ $P(T|R_4) = 0.30$

$$P(T) = 0.20 * 0.40 + 0.10 * 0.30 + 0.50 * 0.20 + 0.20 * 0.30 = 0.27$$

Bays' Rule

- If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for i = 1, 2,…,k, then for any event A of S such that $P(A) \neq 0$,

$$P(B_r \mid A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A \mid B_r)}{\sum_{i=1}^{k} P(B_i)P(A \mid B_i)}$$

for r = 1, 2,…,k.

Example #31:

- In the previous example (***Ex.#30***), if a person received a speeding ticket, what is the probability that he was found speeding when passing through location L_1?

Example #31 (Sol.)

From Example 30# Solution:

$P(R_1) = 0.20$ $P(R_2) = 0.10$ $P(R_3) = 0.50$ $P(R_4) = 0.20$

$P(T|R_1) = 0.40$ $P(T|R_2) = 0.30$ $P(T|R_3) = 0.20$ $P(T|R_4) = 0.30$

Required: $P(R_1 | T) = ?$

$$P(R_1 | T) = \frac{P(R_1)P(T | R_1)}{\sum_{i=1}^{4} P(R_i)P(T | R_i)}$$

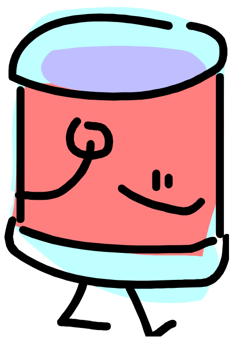
$$P(R_1 | T) = \frac{0.20 * 0.40}{0.20 * 0.40 + 0.10 * 0.30 + 0.50 * 0.20 + 0.20 * 0.30}$$

$$= \frac{0.08}{0.27} = 0.296 \cong 0.30$$

Example #32

(Ex. 2.101 Textbook):

- A paint-store chain produces and sells latex and semigloss paint. Based on long-range sales, the probability that a customer will purchase latex paint is 0.75. Of those that purchase latex paint, 60% also purchase rollers. But only 30% of semigloss paint buyers purchase rollers. A randomly selected buyer purchases a roller and a can of paint. What is the probability that the paint is latex?



Example #32 (Sol.)

L = event of buying latex paint — $P(L) = 0.75$

S = event of buying semigloss paint — $P(S) = P(L') = 1 - P(L) = 1-0.75 = 0.25$

R = event of buying a roller — $P(R|L) = 0.60$ — $P(R|S) = 0.30$

Required: $P(L \mid R) = ?$

$$P(L \mid R) = \frac{P(L)P(R \mid L)}{P(L)P(R \mid L) + P(S)P(R \mid S)}$$

$$P(L \mid R) = \frac{0.75 * 0.60}{0.75 * 0.60 + 0.25 * 0.30}$$

$$= \frac{0.45}{0.525} = 0.857 \cong 0.86$$

Textbook Sections

- 2.1
- 2.2
- 2.3
- 2.4
- 2.5
- 2.6
- 2.7