ENGG. 319 MID TERM Fall 2001

Official Solutions

UNIVERSITY OF CALGARY
FACULTY OF ENGINEERING

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THE EXAM IS OPEN TEXT AND OPEN NOTES

There are 36 questions.
Answer all 36 questions by indicating the letter of the correct answer on the scoring sheet.

Each question answered correctly is awarded 1 mark.
Each question answered incorrectly is awarded 0 marks.

Total possible for the entire exam is 36 marks.

1. Indicate which of the following statements is correct.

   a) if the random variable X has an infinite number of outcomes, then it is a continuous random variable
   b) if the joint pdf of X and Y is (2x+3y) then X and Y are independent only if the pdf is defined over a rectangular area
   c) there is no relationship between the continuous gamma random variable and the discrete poisson random variable
   d) *the sum of two normally distributed random variables is also normal, regardless of whether they are independent or dependent
   e) two mutually exclusive events are always independent

Solution: d)

2. The joint probability function of the discrete random variables S and T is:

$$f(s,t) = \begin{cases} k\dfrac{s}{t} & s = 0,1,2,4;\ \ t = 1,2 \\ 0 & \text{elsewhere} \end{cases}$$

Indicate the required value of k so that f(s,t) is a legitimate probability function.

   a) $1/10$
   b) * $2/21$
   c) $2/13$
   d) $3/13$
   e) none of the above

Solution: b)

	s =				Σ =
	0	1	2	4	
t = 1	0	k	2k	4k	7k
t = 2	•0	k/2	k	2k	3.5k
Σ =	can be ignored	1.5k	3k	6k	**10.5k**

since $\Sigma\Sigma = 1 \;\Rightarrow\; k = \frac{1}{10.5} = \frac{2}{21}$

3. For Question 2 immediately above, what is the expected value of $\frac{S}{T}$

   a) 1.75
   b) 2.00
   c) 2.25
   d) *2.50
   e) 5.00

Solution: d) Consider all $\frac{s}{t}$ values in table:

$$k\left(\frac{1}{1}\right)+2k\left(\frac{2}{1}\right)+4k\left(\frac{4}{1}\right)+\frac{k}{2}\left(\frac{1}{2}\right)+k\left(\frac{2}{2}\right)+2k\left(\frac{4}{2}\right)=26.25k=2.50=\frac{5}{2}=\text{d)}$$

4. An urn contains 14 red balls and 21 white balls. 7 balls are selected at random but replaced after each draw. Indicate which of the following is the probability (to 3 significant figures) that either 2 or 3 red balls are selected.

   a) 0.241
   b) 0.318
   c) 0.501
   d) 0.612
   e) *none of the above

Solution: e) none of the above

$$X = \#\text{ reds} \;\Rightarrow\; X \approx \text{binomial}\,(p=\frac{14}{35}=0.4, n=7)$$

$$\begin{aligned}P(2\cup 3) &= P(X=2)+P(X=3)\\ &= \binom{7}{2}(0.4)^2(0.6)^5+\binom{7}{3}(0.4)^3(0.6)^4\\ &= 21(0.16)(0.6)^5+35(\text{-------})\\ &= 0.2612+0.2903\\ &= 0.5515\end{aligned}$$

5: The probability of a certain kind of steel tie failing is 6%. A large beam includes 150 such steel ties. The beam is defined to have failed if more than 15 of these steel ties fail. Under the assumption that the probabilities of failure of these steel ties are mutually independent, indicate which of the following is the probability (to 3 places after the decimal) of the large beam failing:

a) 0.002
b) *0.020
c) 0.042
d) 0.980
e) 0.998

Solution: b) 0.020

X = binomial with p = 0.06 and n = 150
Since p is small and n is large, we can use a normal approximation with μ_X = np and σ_X = square root of npq. So $\mu_X = 150*0.06 = 9$ and $\sigma_X = 2.909$

$P(X\leq 15) = P(Z\leq(15-9/2.909)) = P(Z\leq 2.063) = 0.9803$

get failure if X>15 so,
$P(X>15) = 1 - P(X\leq 15) = 1-0.9803 = 0.0197$ or 0.020 to 3 places after decimal

6. Determine the covariance of U and V, if the standard deviation of $X = 2U - V$ is equal to 3. It is also given that U has a gamma distribution with $\alpha = 4$, $\beta = \frac{1}{2}$ and V is a normal distribution with mean $\mu = \frac{4}{5}$ and $\sigma = \sqrt{2}$

a) $-\frac{3}{2}$
b) $*-\frac{3}{4}$
c) $\frac{3}{4}$
d) $\frac{3}{2}$
e) 0

Solution: b) $-\frac{3}{4}$

$\sigma_X = 3 \Rightarrow \sigma_X^2 = 9$ or $Var(2U-V) = 9$

hence $9 = 4Var(U) + Var(V) - 4CoVar(UV)$

now $Var\ U = \alpha\beta^2 = 4(\frac{1}{4}) = 1$
$Var\ V = (\sqrt{2})^2 = 2$

so $9 = 4(1) + 2 - 4\ CoVar\ UV$

$\Rightarrow CoVar\ UV = \frac{9-4-2}{-4} = -\frac{3}{4}$

7. During your last summer holiday camping trip 2 mosquitoes penetrated your tent. This type of mosquito stings, on average every 3 hours, and the mosquitoes sting independently and according to a poisson process. Indicate which of the following is the probability (to 3 significant figures) that you can get a 4 hour continuous sleep without being stung.

   a) 0.694
   b) 0.375
   c) 0.527
   d) 0.667
   e) *none of the above

Solution: e) none of the above

Since the main characteristic of Poisson processes is the assumption of independence, and since the 2 mosquitoes act independently and not in cluster, the effect of 2 mosquitoes is the same as one with double the rate.

Hence $\lambda = 2\left(\frac{1}{3}\right) = \frac{2}{3}\,\text{hr}^{-1}$

$$\lambda t = \frac{2}{3} \times 4 = \frac{8}{3}$$

$$P(0\ \text{stings}) = e^{-\lambda t} = e^{-8/3} = 0.0694$$

8: A random variable X follows a continuous uniform distribution over the interval from 5 to 15. Which of the following is the value (to 4 significant figures) for σ_X:

   a) *2.887
   b) 3.317
   c) 8.333
   d) 11.00
   e) none of the above

Solution: a) 2.887

f(x) has a continuous uniform distribution from 5 to 15, so the area of the function is a rectangle with sides of 15 – 5 = 10 and k, a constant, and has an area of 1

$$k(15-10) = 1 \rightarrow k = \frac{1}{10} = f(x)$$

$$\mu_X = \frac{15-5}{2} = 10$$

$$VAR(X) = \int_5^{15} (x-\mu) f(x)dx = \int_5^{15} (x-10)^2 k dx = \frac{1}{10}\int_5^{15} (x-10)^2 dx = \frac{1}{10}\int_5^{15} (x^2 - 20x + 100)dx$$

integrating gives: $= \frac{1}{10}\left[\frac{x^3}{3} - \frac{20x^2}{2} + 100x\right]_5^{15}$

$$= \frac{1}{10}\left[\left(\frac{15^3}{3} - \frac{20\cdot 15^2}{2} + 100\cdot 15\right) - \left(\frac{5^3}{2} - \frac{20\cdot 5^2}{2} + 100\cdot 5\right)\right]$$

$$= \frac{1}{10}(375 - 291.67) = 8.333 \quad = \text{VAR(X)}$$

$$\sigma_X = \sqrt{\text{VAR(X)}} = \sqrt{8.333} = 2.887$$

Or, could be read from formula in summary sheets for continuous distributions, where VAR(X) = $(B-A)^2/12 = (15-5)^2/12 = 8.333$ and then $\sigma_X = \sqrt{\text{VAR(X)}} = \sqrt{8.333} = 2.887$

9: People arrive at an elevator at an average rate of 1.5 persons per minute. The number of people arriving in a given interval t is a random variable that follows a poisson distribution. Indicate which of the following is the distribution for the amount of time in minutes between 2 specific people arriving at the elevator:

a) poisson distribution with $\mu = 1.5$ persons per minute
b) gamma distribution with $\alpha = 2$ persons and $\beta = 0.667$ minutes per person
c) *exponential distribution with $\beta = 0.667$ minutes per person
d) chi-squared distribution with $\upsilon = 2$ persons
e) none of the above

Solution: c)

10. Ten light bulbs consisting of 3 red, 5 blue and 2 yellow are available to decorate a six-socket Christmas tree chord. 6 are picked at random. Indicate which of the following is the probability that there is an equal number of light bulbs of each colour.

a) * $\frac{1}{7}$
b) $\frac{4}{15}$
c) $\frac{1}{15}$
d) $\frac{9}{210}$
e) none of the above

Solution: a) $\frac{1}{7}$

2 of each colour

$$P(A)\frac{n}{N} = \frac{\binom{3}{2}\binom{5}{2}\binom{2}{2}}{\binom{10}{6}}$$

$$= \frac{3 \times 10 \times 1}{210} = \frac{3}{21} = \frac{1}{7}$$

11. Two random variables X and Y are exponentially distributed with means 1 and 3, respectively. Their covariance is −1. The joint probability density function of X and Y is:

 a) $f(x,y) = \frac{1}{3}e^{\frac{-3x-y}{3}} \quad x > 0, y > 0$
 b) $f(x,y) = \frac{1}{3}e^{\frac{-3x-y+xy}{3}} \quad x > 0, y > 0$
 c) $f(x,y) = \frac{1}{3}e^{\frac{-3x-y+xy}{3}} \quad 0 < x < \frac{y}{3} < \infty$:
 d) $f(x,y) = \frac{1}{3}e^{\frac{-3x-y+3xy}{3}} \quad x > 0, y > 0$
 e) *not enough information is available to determine the joint probability density function

 Solution: e)

 as a joint pdf cannot be re-constructed from its marginals in a unique way

12: The strength of a certain type of concrete is normally distributed with a mean of 35.5 MPa and a standard deviation of 4.1 MPa. Indicate which of the following is the probability (accurate to 3 significant figures) that a randomly selected cylinder of this type of concrete has a strength greater than 40.0 MPa:

 a) 0.0444
 b) 0.0690
 c) 0.0889
 d) *0.138
 e) 0.0153

 Solution: d) 0.138

 Let S = strength of concrete

 Given: S is normally distributed; $\mu = 35.5$ MPa; $\sigma = 4.1$ MPa

 What is $P(S > 40.0)$?

$$z = \frac{x - \mu}{\sigma} = \frac{40.0 - 35.5}{4.1} = 1.098$$

 Thus, $P(S > 40.0) = P(Z > 1.098) = 1 - P(Z \le 1.098) = 1 - 0.862 = 0.138$

13: Let X be the number obtained when tossing a standard 6-sided, fair die. Indicate which of the following is the standard deviation for X (correct to 4 significant figures):

a) 1.871
b) 2.917
c) 3.500
d) 4.183
e) *none of the above

Solution: e) none of the above

$$E[X] = \sum_i p_i x_i = \frac{1}{6}(1) + \frac{1}{6}(2) + \frac{1}{6}(3) + \frac{1}{6}(4) + \frac{1}{6}(5) + \frac{1}{6}(6) = \frac{1}{6}(1+2+3+4+5+6) = \frac{1}{6}(21) = \frac{21}{6} = 3.5 = \mu$$

$$VAR[X] = \sum_i p_i (x_i - \mu)^2 = 2.917$$

$\therefore$ σ = positive square root of 2.917 = 1.708

14: It is observed that a particular hockey player scores on average once every 3.17 shots on goal. Define the random variable R as the number of shots the player needs to take in order to score three goals. Indicate which of the following is the form of distribution for R, assuming that the probabilities of scoring on shots are mutually independent:

a) hypergeometric
b) multivariate hypergeometric
c) *negative binomial
d) geometric
e) none of the above

Solution: c)

15: Left-turning vehicles arrive at a traffic signal at an average rate of one every 2.333 seconds. If the duration of the red signal for these left-turning vehicles is 21 seconds in each cycle. indicate which of the following is the minimum required capacity for the left-turn bay for these left-turning vehicles if the bay is to overflow in less than 5% of cycles when the bay is empty at the start of the red signal.

a) 3 vehicles
b) 9 vehicles
c) 12 vehicles
d) *14 vehicles
e) none of the above

Solution: d)

Let X = number of vehicle arrivals in 21 seconds
X is poisson distributed with μ = (1 vehicle / 2.333 sec) * 21 seconds = 9 vehicles

Want x_c such that $P(X>x_c) = 0.05$ From Table A-2, x_c is slightly less than 14 and much more than 13. Rounding up to a whole number of vehicles, consistent with getting a probability of overflow of 5% as a maximum, gives 14

16. Two random variables X and Y have a joint probability density function:

$$f(x,y) = \begin{cases} kxy & 0<x<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

where k is a positive constant. In order for this to be a legitimate joint pdf, k needs to be equal to

a) k = 1
b) k = 2
c) k = 3
d) k = 4
e) *k = 8

Solution: e) k = 8

$$1=\int_0^1\int_x^1 kxy\,dy\,dx = k\int_0^1 x\left[\frac{y^2}{2}\right]_x^1 dx$$

$$= k\int_0^1 x\left(\frac{1}{2}-\frac{x^2}{2}\right)dx = \frac{k}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1$$

$$= \frac{k}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{k}{8} \qquad \rightarrow k=8$$

17. For Question 16 above, indicate which of the following is the probability that the sum of X and Y is less than 1.

a) $\frac{1}{8}$
b) * $\frac{1}{6}$
c) $\frac{1}{3}$
d) $\frac{1}{2}$
e) none of the above

Solution: b) $\frac{1}{6}$

$P(X+Y<1) = \iint_A 8xy\,dx\,dy$ where A is the shaded area

$$= \int_0^{\frac{1}{2}} \int_x^{1-x} 8xy\,dx\,dy$$

$$= 8\int_0^{\frac{1}{2}} x\left[\frac{y^2}{2}\right]_x^{1-x} dx = 4\int_0^{\frac{1}{2}} \left(x(1-x)^2 - x^3\right)dx$$

$$= 4\int_0^{\frac{1}{2}} \left(x - 2x^2\right)dx = 4\left(\frac{1}{8} - \frac{2}{3}\left(\frac{1}{8}\right)\right) = \frac{1}{6}$$

18: Indicate which of the following is NOT a binomial process:

a) tossing a coin n times
b) selecting playing cards from a standard deck and recording if they are red or black; after each selection the card is replaced and the deck is shuffled
c) checking light bulbs coming out of a manufacturing process to see if they work
d) randomly testing students and determining if they know the course material; each student has equal likelihood of being selected each time the test is performed
e) *all of the above are binomial processes

Solution: e)

19: Indicate which of the following describes a random variable with a hypergeometric distribution:

a) the number of times heads is obtained when tossing a coin n times
b) the number of vehicles arriving at a traffic signal in a specific interval of time
c) the proportion of defective light bulbs coming out of a manufacturing process
d) *the number of concrete cylinders that fail out of 6 randomly selected concrete cylinders selected from a collection of 12; in each test the cylinder is loaded to 45 MPa to see if it fails
e) none of the above

Solution: d)

20: Two random variables X and Y have a joint probability function:

$f(x,y) = (1/63)\,(2x + y^2/x)$ for $x = 1, 2, 4;\ y=2, 4$

$= 0$ otherwise

Indicate which of the following best describes the marginal distribution for X:

a) $*g(x) = (1/63)\,(4x + 20/x)$ for $x = 1, 2, 4$
b) $g(x) = (1/63)\,(14 + 7y/4)$ for $y = 2, 4$
c) $g(x) = (1/63)\,(12x + 60/x)$ for $x = 1, 2, 4$
d) $g(x) = (1/63)\,(2y + x^2/y)$ for $x = 1, 2, 4$
e) none of the above

Solution: a)

for $y = 2$ $\quad f(x,2) = \frac{1}{63}\left(2x + \frac{4}{x}\right)$ $\qquad$ for $y = 4$ $\quad f(x,4) = \frac{1}{63}\left(2x + \frac{16}{x}\right)$

$$g(x) = \frac{1}{63}\left[\left(2x + \frac{4}{x}\right) + \left(2x + \frac{16}{x}\right)\right] = \frac{1}{63}\left(4x + \frac{20}{x}\right)$$

21: For the random variables X and Y:

$E[X] = -12.884$
$E[Y] = 114.837$
$\sigma_x^2 = 84.202$
$\sigma_y^2 = 65.983$
$\sigma_{xy} = -30.291$

Given the relationship $G = 2X - 3Y$, indicate which of the following is the value of the variance for G:

a) 318.743
b) 567.163
c) 930.655
d) *1294.147
e) none of the above

Solution: d)

$$\begin{aligned} E[G] &= E[2X-3Y] \\ &= E[2X]-E[3Y] \\ &= 2E[X]-3E[Y] \end{aligned}$$

$$\begin{aligned} VAR[G] &= E[G^2]-E^2[G] \\ &= E[(2X-3Y)^2]-(2E[X]-3E[Y])^2 \\ &= E[4X^2-12XY+9Y^2]-(4E^2[X]-12E[X]E[Y]+9E^2[Y]) \end{aligned}$$

$$= E[4X^2-12XY+9Y^2]-4E^2[X]+12E[X]E[Y]-9E^2[Y]$$
$$= 4E[X^2]-12E[XY]+9E[Y^2]-4E^2[X]+12E[X]E[Y]-9E^2[Y]$$

gathering terms and using the following definitions,
$E[X^2] = \sigma_x^2 + \mu_x^2$
and $\sigma_{xy} = E[XY]-E[X]E[Y] \rightarrow E[XY] = \sigma_{xy} + E[X]E[Y]$:

gives:

$$\begin{aligned}
VAR[G] &= 4(\sigma_x^2 + \mu_x^2)-12(\sigma_{xy} + \mu_y\,\mu_y)+9(\sigma_y^2 + \mu_y^2)-4(\mu_x^2)+12(\mu_x\,\mu_y)-9(\mu_y^2) \\
&= 4\sigma_x^2 + 4\mu_x^2-12\sigma_{xy} -12\,\mu_y\,\mu_y+9\sigma_y^2 +9\,\mu_y^2-4\mu_x^2+12\mu_x\,\mu_y-9\mu_y^2 \\
&= 4\sigma_x^2 + \cancel{4\mu_x^2}-12\sigma_{xy} \cancel{-12\,\mu_y\,\mu_y}+9\sigma_y^2 \cancel{+9\,\mu_y^2}\cancel{-4\mu_x^2}\cancel{+12\mu_x\,\mu_y}\cancel{-9\mu_y^2} \\
&= 4\sigma_x^2 -12\sigma_{xy} +9\sigma_y^2 \\
&= 4(84.202)-12(-30.291)+9(65.983) \\
&= 1294.147
\end{aligned}$$

22: The covariance between 2 random variables X and Y is $\sigma_{xy} = -475$. Indicate which of the following is the most appropriate statement that can be made about the nature of the joint variation between X and Y:

a) *There is a tendency for the values of Y to be relatively smaller when the values of X are relatively larger
b) There is a tendency for the values of Y to be relatively smaller when the values of X are relatively larger and there is a high degree of joint variation in X and Y together
c) There is a tendency for the values of Y to be relatively larger when the values of X are relatively larger
d) There is a tendency for the values of Y to be relatively larger when the values of X are relatively larger and there is a high degree of joint variation in X and Y together
e) none of the above

Solution: a)

23: For the independent random variables X and Y:

$\mu_x = 3.411$
$\mu_y = 6.288$
$\sigma_x^2 = 7.838$
$\sigma_y^2 = 24.704$

Given the relationship:

$$K = 3XY^2 - 16$$

Indicate which of the following is the expected value for K (to 1 figure after the decimal):

a) 64.3
b) 80.3
c) 404.6
d) 420.6
e) *none of the above

Solution: e)

$E[K] = E[3XY^2-16]$
$= E[3XY^2]-E[16]$ X and Y are independent, so:
$= 3E[X]E[Y^2]-16$
$= 3\mu_x(\sigma_y^2+\mu_y^2)-16$
$= (3)(3.411)(24.704+6.288^2)-16$
$= 176.7$

24: A multiple choice exam has 35 questions. A particular student 'David' gets 30 of the questions correct. If 5 of the 35 questions included on David's exam paper are selected at random, and the number of correct responses for these 5 questions is defined to be the random variable X, then indicate which of the following is the variance of X (to 3 places after the decimal):

a) *0.612
b) 3.673
c) 4.286
d) 30.000
e) none of the above

Solution: a)

X = # correct responses for 5 questions = binomial

$p = \frac{30}{35} = \frac{6}{7}$ $q = 1-p = 1-\frac{6}{7} = \frac{1}{7}$

Then can read formula for variance of X from formula in summary sheets for discrete distributions, where VAR(X) = npq = 5*(6/7)*(1/7) = 0.612

Or, can calcuate VAR(X) from basic formulas as follows:

$$E[X] = \sum_i p_i x_i$$

$E[X] = P(X=0)\cdot 0 + P(X=1)\cdot 1 + P(X=2)\cdot 2 + P(X=3)\cdot 3 + P(X=4)\cdot 4 + P(X=5)\cdot 5$

$$P(X=0) = \binom{5}{0}\left(\frac{6}{7}\right)^0\left(\frac{1}{7}\right)^5 = 5.95\times 10^{-5}$$

$$P(X=1) = \binom{5}{1}\left(\frac{6}{7}\right)^1\left(\frac{1}{7}\right)^4 = 1.79\times 10^{-3}$$

$$P(X=2) = \binom{5}{2}\left(\frac{6}{7}\right)^2\left(\frac{1}{7}\right)^3 = 2.14\times 10^{-2}$$

$$P(X=3) = \binom{5}{3}\left(\frac{6}{7}\right)^3\left(\frac{1}{7}\right)^2 = 1.29\times 10^{-1}$$

$$P(X=4)=\binom{5}{4}\left(\frac{6}{7}\right)^4\left(\frac{1}{7}\right)^1=3.86\times10^{-1}$$

$$P(X=5)=\binom{5}{5}\left(\frac{6}{7}\right)^5\left(\frac{1}{7}\right)^0=4.63\times10^{-1}$$

$$\begin{aligned}E[X] &= P(X=0)\cdot 0 + P(X=1)\cdot 1 + P(X=2)\cdot 2 + P(X=3)\cdot 3 + P(X=4)\cdot 4 + P(X=5)\cdot 5\\ &= (5.95\text{x}10^{-5})(0)+(1.79\text{x}10^{-3})(1)+(2.14\text{x}10^{-2})(2)+(1.29\text{x}10^{-1})(3)+\\ &\quad (3.86\text{x}10^{-1})(4)+(4.63\text{x}10^{-1})(5)\\ &= 4.2855\end{aligned}$$

$$\begin{aligned}VAR[X] &= E[(x-\mu)^2]\\ &= \sum_i p_i(x_i-\mu)^2\\ &= (5.95\text{x}10^{-5})(0-4.2855)^2+(1.79\text{x}10^{-3})(1-4.2855)^2+(2.14\text{x}10^{-2})\\ &\quad (2-4.2855)^2+(1.29\text{x}10^{-1})(3-4.2855)^2+(3.86\text{x}10^{-1})(4-4.2855)^2+\\ &\quad (4.63\text{x}10^{-1})(5-4.2855)^2\\ &= 0.612\end{aligned}$$

25: Washing machines can have 5 kinds of major and five kinds of minor defects. Indicate the number of ways that two major and two minor defects can occur.

a) 4
b) 16
c) 20
d) 36
e) *100

Solution: e) 100

$$\#\text{ ways} = \binom{5}{2}\binom{5}{2} = 100 \text{ ways} \quad \text{(group of 4, order irrelevant)}$$

26: A person is playing a poker game. In each round of the game she is dealt a 'hand' of 5 cards randomly from a standard deck of 52 cards. Indicate which of the following is the probability (to 4 places after the decimal) that she will get a 'hand' with 3 or more red cards in 7 or more rounds when a total of 10 rounds are played.

a) *0.1719
b) 0.3251
c) 0.5000
d) 0.6749
e) none of the above

Solution: a)

P(red) = P(black) = 0.5 X is binomial

$$P(0\ \text{red}) = \binom{5}{0}(0.5)^0(0.5)^5 = 0.03125$$

$$P(1\ \text{red}) = \binom{5}{1}(0.5)^1(0.5)^4 = 0.15625$$

$$P(2\ \text{red}) = \binom{5}{2}(0.5)^2(0.5)^3 = 0.3125$$

$$P(3\ \text{red}) = \binom{5}{3}(0.5)^3(0.5)^2 = 0.3125$$

$$P(4\ \text{red}) = \binom{5}{4}(0.5)^4(0.5)^1 = 0.15625$$

$$P(5\ \text{red}) = \binom{5}{5}(0.5)^5(0.5)^4 = 0.03125$$

$\therefore P(3,4 \text{ or } 5 \text{ reds}) = 0.3125 + 0.15625 + 0.03125 = 0.5$

or you could just say that since there are 6 likely results (0,1,2,3,4,5) the probability of 3,4,5 = probability 0,1,2 = 0.5

Let R = number of rounds with 3 or more red cards

$$\begin{aligned} P(R \geq 7) &= P(R=7)+P(R=8)+P(R=9)+P(R=10) \\ &= \binom{10}{7}(0.5)^7(0.5)^3 + \binom{10}{8}(0.5)^8(0.5)^2 + \binom{10}{9}(0.5)^9(0.5)^1 + \binom{10}{10}(0.5)^{10}(0.5)^0 \\ &= 0.1172 + 0.04394 + 0.009766 + 0.0009766 = 0.1719 \end{aligned}$$

27: On a camping trip, it rains all day, so that the labels of your store of canned food are all washed off. There are 2 cans of beans, 4 cans of soup, 3 cans of tuna, 2 cans of fruit and 3 cans of cocktail wieners. All the cans look similar so you open 5 cans at random. Indicate the probability (to three significant figures) that you do not get one can of each type of food.

a) 1.30 %
b) 98.7 %
c) 86.8 %
d) *92.8 %
e) 99.4 %

Solution: d) 92.8 %

$$P(Q) = 1 - P(\text{do get all 5 different})$$

$$= 1 - \frac{\binom{2}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{3}{1}}{\binom{14}{5}} = 1 - \frac{144}{2002} = 0.928$$

since in total there are 14 cans

28: The percentage scores on an exam are normally distributed with a mean of 62.7 and a variance of 267.32 and there are 169 people in the class. Indicate which of the following is the number of students (rounded to the nearest whole number) that received a score less than 30:

a) 3
b) *4
c) 8
d) 51
e) none of the above

Solution: b)

Let X = a student's score

$\mu = 62.7\,\%$
$\sigma^2 = 267.32 \quad \Rightarrow \sigma = (267.32)^{0.5} = 16.35$
N = 169 students

$$z = \frac{x-\mu}{\sigma} = \frac{30-62.7}{16.35} = -2.0$$

Thus, $P(X \le 30) = P(Z \le -2.0) = 0.0228$

This gives the probability of 'success' (where 'success' is defined to be getting a score less than 30) for each trial in 169 trials (for 169 students) of what is a binomial process. The expected number of successes in the 169 trials is then np, and

np = (169 students)(0.0228) = 3.85 $\Rightarrow$ 4 students (rounded to nearest whole number)

29: The 2 continuous random variables X and Y have a joint pdf:

$$f(x,y) = 1 \quad -y < x < +y\ ,\ 0 < y < 1$$

$$f(x,y) = 0 \quad \text{elsewhere}$$

Indicate which of the following statements is true:

a) X and Y are independent; the covariance of X and Y is zero
b) X and Y are independent; the covariance of X and Y is not zero
c) *X and Y are not independent; the covariance of X and Y is zero
d) X and Y are not independent; the covariance of X and Y is not zero
e) not enough information is given to decide about independence and/or covariance

Solution: c) X and Y are not independent; X and Y are uncorrelated

marginals are $f(x) = 1-x$ for $0 < x < 1$
$= 1+x$ for $-1 < x < 0$

$$f(y) = 2y \quad \text{for } 0 < y < 1$$

so clearly $f(x,y) \neq f(x)\,f(y) \Rightarrow$ not independent

however $\text{Cov} = \int_0^1 \int_{-y}^{y} xy \cdot (1)\,dx\,dy - 0 = 0 \quad \Rightarrow$ uncorrelated

30: A car ferry waits until it is full of cars before it crosses the river where it operates. It has a capacity for 4 cars. Cars arrive at the ferry to cross the river at an average rate of one vehicle every 6.40 minutes. Loading and unloading of cars is very quick and the time required for these operations can be ignored. Consider a case where there is one car waiting when the ferry arrives at the edge of the river. Indicate which of the following is the distribution for the amount of additional time that this one car will have to wait until it can cross the river, that is until another 3 cars arrive and the ferry is full:

a) poisson distribution with $\mu = 0.156$ vehicles per minute
b) gamma distribution with $\alpha = 4$ vehicles and $\beta = 6.40$ minutes per vehicle
c) negative exponential distribution with $\beta = 19.2$ minutes
d) chi-squared distribution with $\upsilon = 3$ vehicles
e) *none of the above

Solution: e)

31: Indicate which of the following statements is correct:

a) $P(a < X < b) = P(a \leq X \leq b)$ for any constants a and b and any random variable X
b) the exponential distribution is a special case of the gamma distribution with $\alpha = 2$
c) *σ_x cannot be negative for any random variable X
d) σ_{xy}^2 can be positive or negative for any random variables X & Y with joint pdf=f(x,y)
e) none of the above; that is, none of the above are correct

Solution: c)

32: Three dice are enclosed in a black box; two dice are regular dice but one dice has its 6 faces labelled (2, 2, 3, 3, 4, 6). Two dice are picked at random and rolled. Given that you are told that the sum of the 2 faces is 5, indicate which of the following is the probability that the die that remained in the black box was the irregular die.

a) $^{15}/_{36}$
b) $^{1}/_{3}$
c) * $^{2}/_{7}$
d) $^{4}/_{9}$
e) none of the above

Solution: c) $^2/_7$

Let A: the die not chosen is the irregular dice
$\Leftrightarrow$ the 2 dice chosen are regular
Let B: sum of the 2 dice is 5

Question is then $P(A|B) = ?$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$$

$$\frac{\frac{4}{36}\cdot\frac{1}{3}}{\frac{4}{36}\cdot\frac{1}{3}+\frac{5}{36}\cdot\frac{2}{3}} \qquad \frac{4}{4+10} = \frac{2}{7}$$

33: Indicate which of the following statements is correct concerning $E[(X-\mu_x)^2]$ for a random variable X

a) it provides an indication of the central tendency in a particular set of realized values x_i, i from 1 to n, for the random variable X
b) it provides an indication of the central tendency of f(x)
c) it provides an indication of the degree of spread in a particular set of realized values x_i, i from 1 to n, for the random variable X
d) *it provides an indication of the degree of spread in f(x)
e) none of the above; that is, none of the above are correct

Solution: d)

34: The probability that it will rain on a given day next week is 0.24. You can't go biking if it rains or if you suffer from back pains. On a dry day (no rain) the probability that your back aches is just 10 % but this figure doubles when there is a rainy day. Indicate which of the following is the probability (to 3 significant figures) that you can't go biking on any of the next three days. Assume independence from day to day.

a) 8.51 %
b) 3.93 %
c) *3.16 %
d) 2.49 %
e) none of the above

Solution: c) 3.16 %

R = it rains B = suffer from back pain

per day: $P(\text{no biking}) = P(R \cup B) = P(R) + P(B) - P(R \cap B)$

$= P(R) + P(B|R)P(R) + P(B|R')P(R') - P(R)P(R|B)$
$= P(R) + P(B|R')P(R')$
$= 0.24 + (0.1)(0.76) = 0.316$

or alternately, $P(\text{no biking}) = 1 - P(\text{bike}) = 1 - P(R' \cap B') = 0.316$
$P(\text{no biking in 3 days}) = (0.316)^3 = 0.03156 = 3.16\ \%$

35: Suppose that for a certain population of young adults there is a 65% probability that a randomly selected member of the group smokes. Suppose there is also a test for smoking that has a 0.93 probability of correctly identifying someone who does smoke along with a 0.02 probability of incorrectly indicating 'positive for smoking' for someone who does not actually smoke. Indicate which of the following is the probability (accurate to 4 significant figures) that a person from the population who has tested 'positive for smoking' does actually smoke:

a) 0.3885
b) 0.6045
c) 0.6115
d) *0.9886
e) none of the above

Solution: d)

This is a Bayes Rule problem

Let S = person smokes $P(S) = 0.65$
Let T = tests positive for smoking

Given
$P(S) = 0.65$
$P(T|S) = 0.93$
$P(T|S') = 0.02$

Thus: $P(S') = 1 - P(S) = 1 - 0.65 = 0.35$

$$P(S \mid T) = \frac{P(S \cap T)}{P(T)}$$

$P(S \cap T) = (0.93)(0.65) = 0.6045$
$P(S' \cap T) = (0.02)(1 - 0.65) = 0.007$

$P(T) = P(S \cap T) + P(S' \cap T) = 0.6045 + 007 = 0.6115$

$$P(S \mid T) = \frac{P(S \cap T)}{P(T)} = \frac{0.6045}{0.6115} = 0.9886$$

36: Consider a poisson process where the mean number of occurrences of a particularly undesirable event is 7.2 per day. Indicate which of the following is the probability (to 3 places after the decimal) that there is at least one event between 10 AM and 1 PM given that exactly one event took place between 9 AM and 10 AM.

a) *0.593
b) 0.549
c) 0.699
d) 0.451
e) none of the above

Solution: a) 0.593

Since Poisson processes are based on independent occurrences, the conditional information does not alter the question.

So $\quad P(\text{at least } 1 \in [10 - 1\text{ pm}] \mid 1 \in [9 - 10]$

$= P(\text{at least } 1 \in [10 - 1\text{ pm}]) = 1 - P\,(0 \text{ events in 3 hrs})$

$$= 1 - \exp\left(-\frac{7.2}{24} \times 3\right) = 0.593$$