Solution to Tutorial #6

Question #1

a) $P(X<2.8)$

~~Note: From the We observe that~~

By examining $F(x)$, we identify X as a continuous random variable (CRV).

For CRV, $\boxed{P(X\le x) = P(X<x) = F(x)}$ (Page 90)

Then, $P(X<2.8) = F(2.8) = 0.2\times 2.8 = \boxed{0.56}$ Ans

b) $P(X>1.5)$, Since $F(x)$ is given, we find the probability of the complementary event,

$P(X>1.5) = 1 - P(X<1.5) = 1 - F(1.5) = 1 - 0.2\times 1.5 = 0.7$

c) $P(X<-2) = F(-2) = 0$

d) $P(X>6) = 1 - P(X<6) = 1 - F(6) = 1 - 1 = 0$

e) $P(X>6 \mid X>1.5) = \dfrac{P[(X>6)\cap(X>1.5)]}{P(X>1.5)} = \dfrac{P(X>6)}{P(X>1.5)} = \dfrac{1-F(6)}{F(1.5)} = 0$

Note: Try to interpret the results of part (d) & (e).

Question #2

Applying formula $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$ [P 112]

$$= \int_{-1}^{1} x \times 1.5x^2 dx = 1.5\int_{-1}^{1} x^4 dx$$

$$= 1.5 \times \frac{x^5}{5}\Big|_{-1}^{1} = 0.3(2) = \boxed{0.6} \text{ An}$$

Question #3

Let X = thickness in millimeters.

X follows uniform distribution on the interval $[0.95, 1.05]$.

[Note: We write it shortly as $X \sim f(x; A, B) \Rightarrow X \sim f(x; 0.95, 1.05)$]

Then,

$$f(x; 0.95, 1.05) = \begin{cases} \dfrac{1}{1.05 - .95}, & .95 \le x \le 1.05 \\ 0, & \text{elsewhere} \end{cases} = \begin{cases} 10, & .95 \le x \le 1.05 \\ 0, & \text{elsewhere} \end{cases}$$

a) $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = \int_{A}^{x} f(x)\,dx \quad A \le x \le B$

$= \int_{0.95}^{x} 10\,dx = 10(x - 0.95)$

$$F(x) = \begin{cases} 10(x-0.95), & 0.95 \le x \le 1.05 \\ 0, & \text{elsewhere} \end{cases}$$

b) $P(X > 1.02) = 1 - P(X \le 1.02) = 1 - F(1.02) = 1 - 10(1.02 - .95)$
$= 1 - 0.7 = \boxed{0.3}$, Then, $\boxed{30\%}$ of flanges exceed 1.02 mm.

[Note: you can also use the pdf to compute probability]

c) Let, the thickness is t, and assume $0.95 \le t \le 1.05$
Then, $P(X > t) = 90\%$

$\Rightarrow 1 - F(t) = 0.9 \Rightarrow 0.1 = F(t) = 10(t - 0.95)$ ~~assuming~~

$\Rightarrow 0.01 = t - 0.95$

$\Rightarrow \boxed{t = 0.96 \text{ mm}}$ → ~~d~~

(d) $E(X) = \frac{A+B}{2} = \frac{0.95 + 1.05}{2} = \frac{2}{2} = 1$ $\boxed{P172}$

$\sigma^2 = \frac{(B-A)^2}{12} = \frac{(1.05-0.95)^2}{12} = 8.333 \times 10^{-4}$.

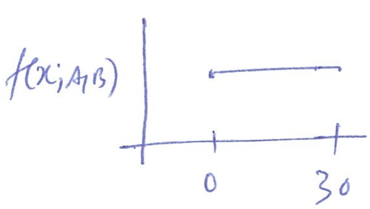
Question # 4.

X = Instantaneous time for the arrival of the bus.

Given, X is uniformly distributed between 2:00pm & 2:30pm
(i.e. the arrival time)

Choose: $A = 0$ min & $B = 30$ min after 2 pm.

$X \sim f(x; 0, 30)$



$$f(x) = \begin{cases} \frac{1}{30-0} = \frac{1}{30}, & 0 \le x \le 30 \\ 0, & \text{otherwise} \end{cases}$$

$F(x) = \int_{0}^{x} \frac{1}{30}\,dx = \frac{1}{30} x \Big|_{0}^{x} = \frac{x}{30}, \quad 0 \le x \le 30$

~~$F(x) = \{ x^2/60, 0 \le x \le 30$; 0, otherwise~~ $F(x) = \begin{cases} x/30, & 0 \le x \le 30 \\ 0, & \text{otherwise} \end{cases}$

$$P(15 < x < 20) = P(X < 20) - P(X < 15)$$
$$= F(20) - F(15) = \frac{20}{30} - \frac{15}{30} = \frac{5}{30} = \frac{1}{6}$$

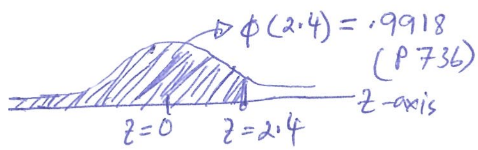
Question #5

X = line width

$X \sim n(x; \mu, \sigma)$ $\mu = 0.5\ \mu m$, $\sigma = 0.05\ \mu m$

a) $P(X > 0.62)$ Convert $X \to Z$

$$= P\left(\frac{X-\mu}{\sigma} > \frac{0.62-\mu}{\sigma}\right) = P\left(Z > \frac{0.62-0.5}{0.05}\right) = P(Z > 2.4)$$



$$= 1 - P(Z < 2.4) = \cancel{1 - 0.0793}$$
$$= 1 - 0.9918 \ [P736]$$
$$= 8.2 \times 10^{-3}$$

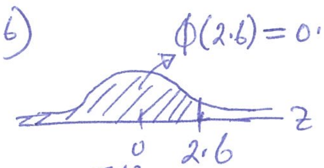
b) $P(0.47 < X < 0.63)$

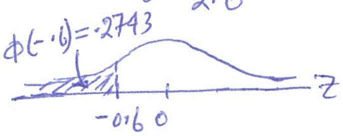
$$= P\left(\frac{0.47-0.5}{0.05} < Z < \frac{0.63-0.5}{0.05}\right) = P(-0.6 < Z < 2.6)$$

$$= P(Z < 2.6) - P(Z < -0.6)$$

$$= 0.9953 - 0.2743$$

$$= 0.721$$





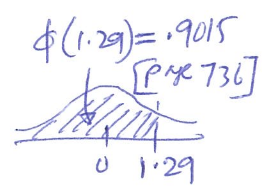
c) Let $X = a$ then, $P(X < a) = 0.9015$

$$\Rightarrow P\left(Z < \frac{a-0.5}{0.05}\right) = 0.9015$$

$$\Rightarrow \phi\left(\frac{a-0.5}{0.05}\right) = 0.9015 = \phi(1.29)$$

$$\Rightarrow \frac{a-0.5}{0.05} = 1.29$$

$$\Rightarrow a - 0.5 = 0.0645 \Rightarrow \boxed{a = 0.5645}$$



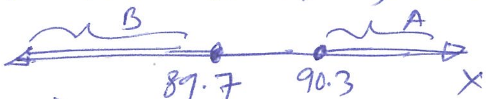
Question # 6

X = length of plastic case

$X \sim n(x; \mu, \sigma)$ $\mu = 90.2$ mm, $\sigma = 0.1$ mm

$P(X > 90.3 \text{ or } X < 89.7) = P(X > 90.3) + P(X < 89.7)$

[Note: $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{\to 0 \text{ in the present case}} = P(A) + P(B)$]



$= [1 - P(X < 90.3)] + P(X < 89.7)$

$\cancel{= 1 - \Phi(90.3) + \Phi(89.7)}$

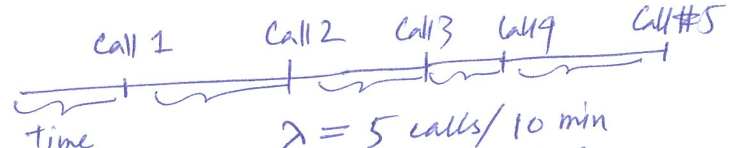
$= 1 - P\left(Z < \frac{90.3 - 90.2}{0.1}\right) + P\left(Z < \frac{89.7 - 90.2}{0.1}\right)$

$= 1 - P(Z < 1) + P(Z < -5)$

$= 1 - \Phi(1) + \Phi(-5) = 1 - 0.8413 + 0 = 0.1587$

Question # 7

Poisson process



$\lambda = 5$ calls/10 min
$= 0.2$ call/min

a) X = time lapses until the 1st call

$X \sim \text{exponential}(x; \beta)$ $\beta = \frac{1}{\lambda} = 2$ min/call

$F(x) = P(X \le x) = 1 - e^{-\lambda x}$ (P 196)

$= 1 - e^{-0.2x}$, for $x > 0$

$P(X < 5 \text{ min})$

$= F(5) = 1 - e^{-0.2 \times 5} = 1 - e^{-1} = 1 - \frac{1}{2.71828} = 0.632$

~~b) T = time elapses until there are 2 calls.~~

b) Let X = Number of calls in the first minute

$X \sim p(x; \lambda t)$ where $\lambda t = E(X) = \text{Avg value of } X = \frac{5 \times 1}{10} = 0.5$

$X \sim p(x; 0.5)$

$P(X = 2) = \frac{e^{-0.5}(0.5)^2}{2!} = \boxed{0.076}$

Question #8

(a)

$N = 1000$ $n = 25$

$K = 100$ (success)	$N-K = 900$ (failur)

$\longrightarrow$

$x = 0$	$n-x = 25$

X = number of defective in the sample of 25 selected connectors.

$X \sim h(x; N, n, K)$ $x=0,\ N=1000,\ n=25,\ K=100$

$= h(0; 1000, 25, 100)$

$$P(X=0) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} = \frac{\binom{100}{0}\binom{900}{25}}{\binom{1000}{25}} = 0.0694$$

$\boxed{P154}$

(b) Approximation using binomial

$X \sim b(x; n, p)$ $n = 25,\ p \simeq \frac{K}{N} = \frac{100}{1000} = 0.1$ (probability of defective)

$$P(X=0) = b(0; 25, 0.1) = \binom{n}{x} p^x q^{n-x}$$
$$= \binom{25}{0}(0.1)^0(0.9)^{25} = .9^{25} = 0.072$$

~~Using~~ Usually, this approximation is good when $\frac{n}{N} \leq 0.05$. In the present case $\frac{n}{N} = 0.1$, still the approximation is close. $\boxed{P155}$

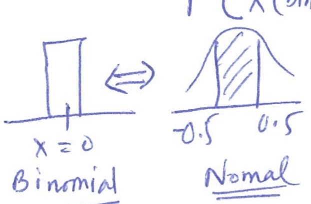
(c) The mean and variance of the binomial random variable X are $\mu = np = 25 \times 0.1 = 2.5$ and

$\sigma^2 = npq = 25 \times .1 \times .9 = 2.25,\ \sigma = 1.5$

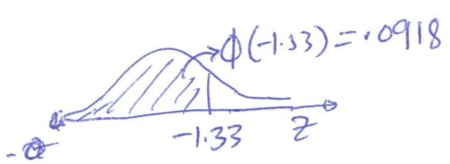
However, a binomial RV can also be approximated by a normal random variable

$\cancel{P(X \leq x) \simeq P\left(Z < \frac{x+0.5-np}{\sqrt{npq}}\right)}$

$$P(X(\text{binomial}) = 0) = P(0-0.5 < X(\text{normal}) < 0+0.5)$$
$$= P(-0.5 < X(\text{normal}) < 0.5)$$



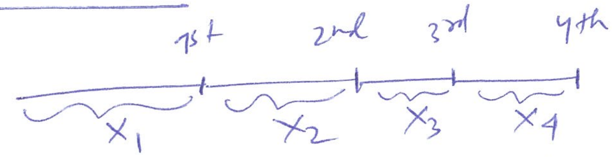
$$= \left(\frac{-0.5-\mu}{\sigma} < Z < \frac{0.5-\mu}{\sigma}\right)$$
$$= \left(\frac{-0.5-2.5}{1.5} < Z < \frac{+0.5-2.5}{1.5}\right)$$



$$= (-2 < Z < -1.33) = \phi(-1.33) - \phi(-2)$$
$$= P(Z < -1.33) - P(Z < -2) = 0.0918 - .0228$$
$$= \boxed{0.069}$$

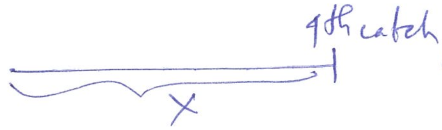
Question # 9

[Note:



$\lambda = \frac{1}{30\,min} = \frac{1}{0.5\,hour}$

X_1, X_2, X_3, X_4 are random variables denoting time to catch the next fish. They all follow exponential random variables. $E(X_1) = E(X_2) = E(X_3) = E(X_4) = \beta = 1/\lambda$]



Let, X = ~~time~~ waiting time until the 4th catch. This is a gama random variable with $\alpha = 4$. and $\beta = \frac{1}{\lambda} = 30$ min $= 0.5$ hours.

$X \sim gamma(x; \alpha = 4, \beta = 0.5)$

$P(2 < X < 4) = P(X<4) - P(X<2)$

$= F(4/\beta, \alpha) - F(2/\beta, \alpha)$, F = incomplete gamma function

$= F(4/\beta, \alpha)$ ~~$- F(4/\beta, \alpha)$~~

$= F(8,4) - F(4,4)$

$= 0.958 - 0.567$ (P 767)

$= 0.391$

Question # 10

X = life time of the disk

$X \sim Weibull(x; \alpha, \beta)$; $\beta = 0.5$

$E(X) = \mu = \alpha^{-1/\beta}\,\Gamma(1+\frac{1}{\beta}) = \alpha^{-1/0.5}\,\Gamma(3) = \alpha^{-2}(3-1)! = \frac{2}{\alpha^2}$

[Note: $\Gamma(n) = (n-1)!$ when n is integer]

$\mu = 600 = \frac{2}{\alpha^2}$

$\Rightarrow \alpha^2 = 0.0033 \Rightarrow \alpha = 0.0577$

$X \sim Weibull(x; \alpha = 0.0577, \beta = 0.5)$, $F(x) = 1 - e^{-\alpha x^\beta}$ [page 209]

a) $P(X > 500\,hor) = 1 - P(X < 500) = 1 - F(500)$

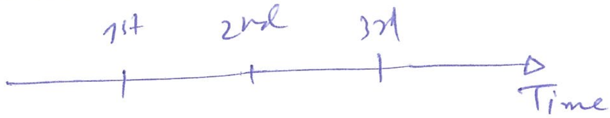
$= 1 - 1 + e^{-.0577 \times (500)^{.5}}$

$= e^{-1.2902} = \boxed{0.275}$

b) $P(X < 400) = F(400)$

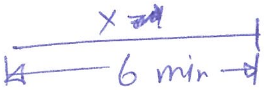
$= 1 - e^{-\alpha x^\beta} = 1 - e^{-0.0577 \times 20} = \boxed{0.685}$.

Question #11



$\lambda = \frac{25}{60\ min}$

Let, X = Number of log ins in an interval of t=6 min

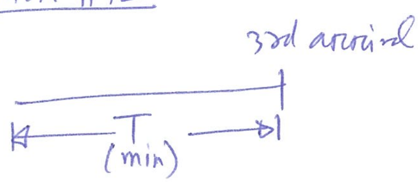


$E(X) = \lambda t = \frac{25 \times 6\ min}{60\ min} = 2.5$ logon/6 min

$X \sim poisson(x; \lambda t = 2.5) = p(x; \lambda t = 2.5) = \frac{e^{-2.5}(2.5)^x}{x!}$

$P(X=0) = p(0; 2.5) = \frac{e^{-2.5}\, 2.5^0}{0!} = e^{-2.5} = \boxed{0.082}$

Question #12



$T \sim Gamma(x; \alpha = 3, \beta)$

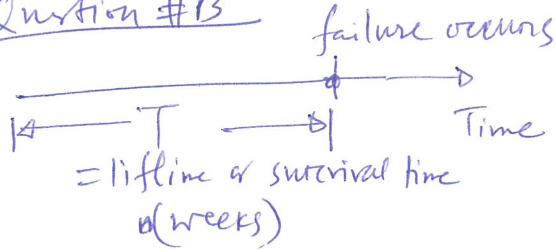
$\beta = \frac{1}{\lambda}$, $\lambda = \frac{0.2}{1\ min}$, $\beta = 5$ min/per arrival

$P(T \leq 20) = F(20/\beta, \alpha) = F(20/5, \alpha = 3) = F(4, \alpha = 3) = \boxed{0.762}$ Ans.

F = incomplete gamma function $\boxed{P767}$

Question #13

(a)



$T \sim Gamma(x; \alpha, \beta)$

$\mu = \alpha\beta = 10$, $\sigma^2 = \alpha\beta^2 = 50$

Then, $\beta = 50/10 = 5$, $\alpha = 10/\beta = 2$

(i) $P(T \leq 50 \text{ weeks}) = F(50/\beta, \alpha) = F(10, 2) \simeq 1$

ii) $P(T \leq 10) = F(10/\beta, \alpha) = F(2, 2) = 0.594$

F = incomplete gamma function.

(b) $\mu = 10$, $\sigma = \sqrt{50}$

(i) $P(T \leq 50) = P\left(Z < \frac{50-10}{\sqrt{50}}\right) = P(Z < 5.65) \simeq 1$

ii) $P(T < 10) = P\left(Z < \frac{10-10}{\sqrt{50}}\right) = P(Z < 0) = 0.5$

(c) Results in part a(i) & b(i) are same, for large values of T.

Question #14

(a)

Failure rate $= 0.01$ device per hour $\Rightarrow \lambda$

Average life time of these devices $= \dfrac{1 \text{ hor}}{0.01} = 100$ hours.

(b) 1st failure

T

$T \sim \text{exponential}(x; \beta)$

$\beta = E(X) = $ Avg time for 1 failure $= 100$ hour.

$$P(T \geq 200) = 1 - P(T < 200) = 1 - F(200) = 1 - [1 - e^{-x/\beta}]$$

$$= e^{-x/\beta} = e^{-200/100}$$

$$= e^{-2} = 0.1353$$

(c) Reliability at 200h

$= R(200h) = P(T > 200h) = 0.1353$ (from part 'b') $= 13.5\%$

Reliability at 400h

$= R(400h) = P(T > 400h) = e^{-T/\beta} = e^{-400/100} = e^{-4} = 0.018$

$= 1.8\%$

Question #15

fails

T Time

$P(T > 8)$

$T \sim \text{exponential}(x; \beta)$

$\beta = E(T) = 5$ years

$= 1 - F(8)$

$= e^{-T/\beta} = e^{-8/5} \simeq 0.2 =$ the prob. that a function will operate at the end of 8 years.

$X =$ Number of components

$X \sim b(x; n=5, p=.2)$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1) \cancel{- P(X=2)}$$

$$= 1 - \left[\binom{5}{0}(.2)^0(.8)^5 + \binom{5}{1}(.2)^1(.8)^4 \cancel{+ \binom{5}{2}(.2)^2(.8)^3}\right]$$

$$= 1 - [0.32768 + 0.4096] \cancel{+ .2048}$$

$$= 1 - 0.73728 = \boxed{0.263}$$

[Note: Using binomial sums:

$$P(X \geq 2) = \sum_{x=2}^{5} b(x; 5, 0.2) = \sum_{x=0}^{r=5} b(x; 5, 0.2) - \sum_{x=0}^{r=1} b(x; 5, 0.2)$$

$$= \cancel{1 - 0.9421} \; 1 - 0.7373 \text{ [page 726]}$$

$$= \boxed{0.263}$$