ENGG 319

Probability & Statistics for Engineers

Section #10
One & Two-Sample Tests of Hypotheses

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L01

F16

Hypothesis & Significance Tests

- A **hypothesis** is a statement about a population, usually of the form that a certain parameter takes a particular numerical value **or** falls in a certain range of values.
- The main goal in many studies is to check whether the data support certain hypotheses.
- A **significance test** is a method of using data to summarize the evidence about a hypothesis.
- A **significance test** about a hypothesis has **4 steps**.

Step 1: Assumptions

- A (**significance**) test assumes that the data production used **randomization**
- **Other assumptions may include:**
  - Assumptions about the sample size
  - Assumptions about the shape of the population distribution

Step 2: Hypotheses

- **Each significance test has 2 hypotheses:**
  - The **null hypothesis** is a statement that the parameter takes a particular value.
  - The **alternative hypothesis** states that the parameter falls in some alternative range of values.

Null and Alternative Hypotheses

- The value in the **null** hypothesis usually represents **no effect.**
  - The symbol H_o denotes **null hypothesis.**
- The value in the **alternative** hypothesis usually represents **an effect of some type.**
  - The symbol H_1 (or H_a) denotes **alternative hypothesis.**

Null and Alternative Hypotheses

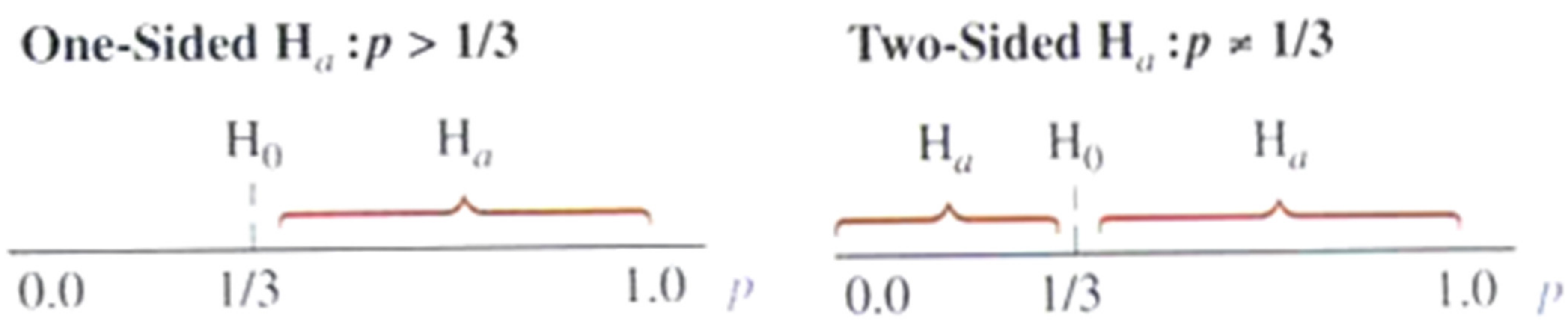
- A **null** hypothesis has a **single** parameter value, such as:
  1. H_0: $p = 1/3$
  2. H_0: defendant is innocent (analogy)

- An **alternative** hypothesis has a range of values that are alternatives to the one in H_0 such as:
  1. H_1: $p \neq 1/3$ or H_1: $p > 1/3$ or H_1: $p < 1/3$
  2. H_1: defendant is guilty (analogy)

One- and Two- Tailed tests

- The **null** hypothesis has the form:
  - $H_0: \theta = \theta_0$
- The **alternative** hypothesis has the form:
  - $H_1: \theta > \theta_0$ or $H_1: \theta < \theta_0$ (**one-tailed test**)

  or
  - $H_1: \theta \neq \theta_0$ (**two-tailed test**)



Step 3: Test Statistic

- The parameter to which the hypotheses refers has a point estimate: **the sample statistic**

- A **test statistic** describes how far that estimate (the sample statistic) falls from the parameter value given in the null hypothesis.

Type I and Type II Errors

- When H_0 is true, a **Type I Error** occurs when H_0 is rejected.
- When H_0 is false, a **Type II Error** occurs when H_0 is not rejected.

Significance Test Results

TABLE 8.6: The Four Possible Results of a Decision in a Significance Test

Type I and Type II errors are the two possible incorrect decisions. We make a correct decision if we do not reject H_0 when it is true or if we reject it when it is false.

Reality About H_0	**Decision** Do not reject H_0	Reject H_0
H_0 true	Correct decision	Type I error ← Type I error occurs if reject H_0 when it is actually true.
H_0 false	Type II error	Correct decision

An Analogy: Decision Errors in a Legal Trial

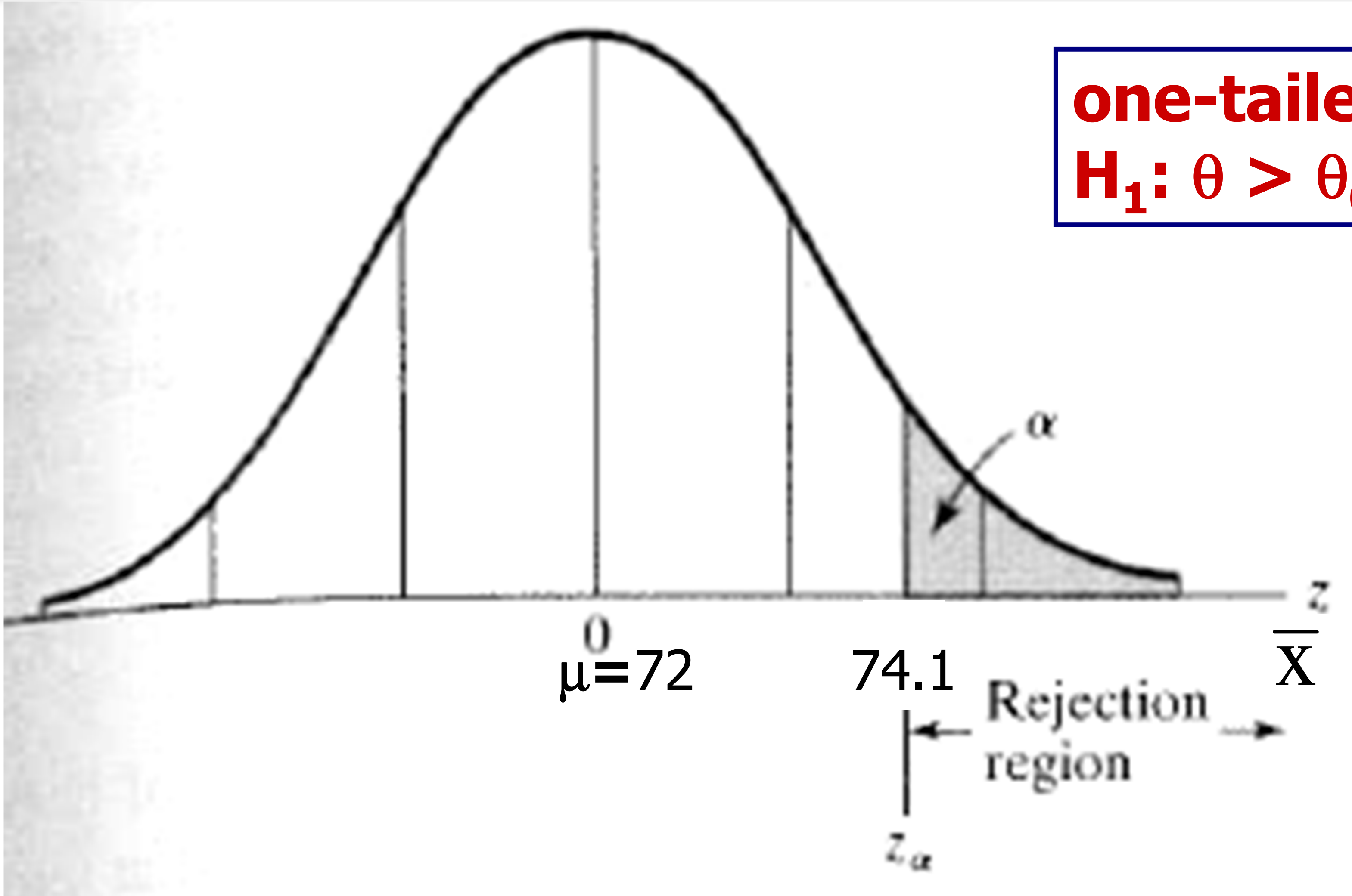
TABLE 8.7: Possible Results of a Legal Trial

	Legal Decision	
Defendant	Acquit	Convict
Innocent (H_0)	Correct decision	Type I error
Guilty (H_a)	Type II error	Correct decision

P(Type I Error)

- Suppose $\mathbf{H_0}$ is true.
- The probability of rejecting $\mathbf{H_0}$, i.e. committing a **Type I error**, equals the significance level α for the test.
- We can control the probability of a **Type I** error by our choice of the significance level.
- The more serious the consequences of a **Type I error**, the smaller α should be

P(Type I Error)



one-tailed:
H_1: θ > θ_0

P(Type II Error)

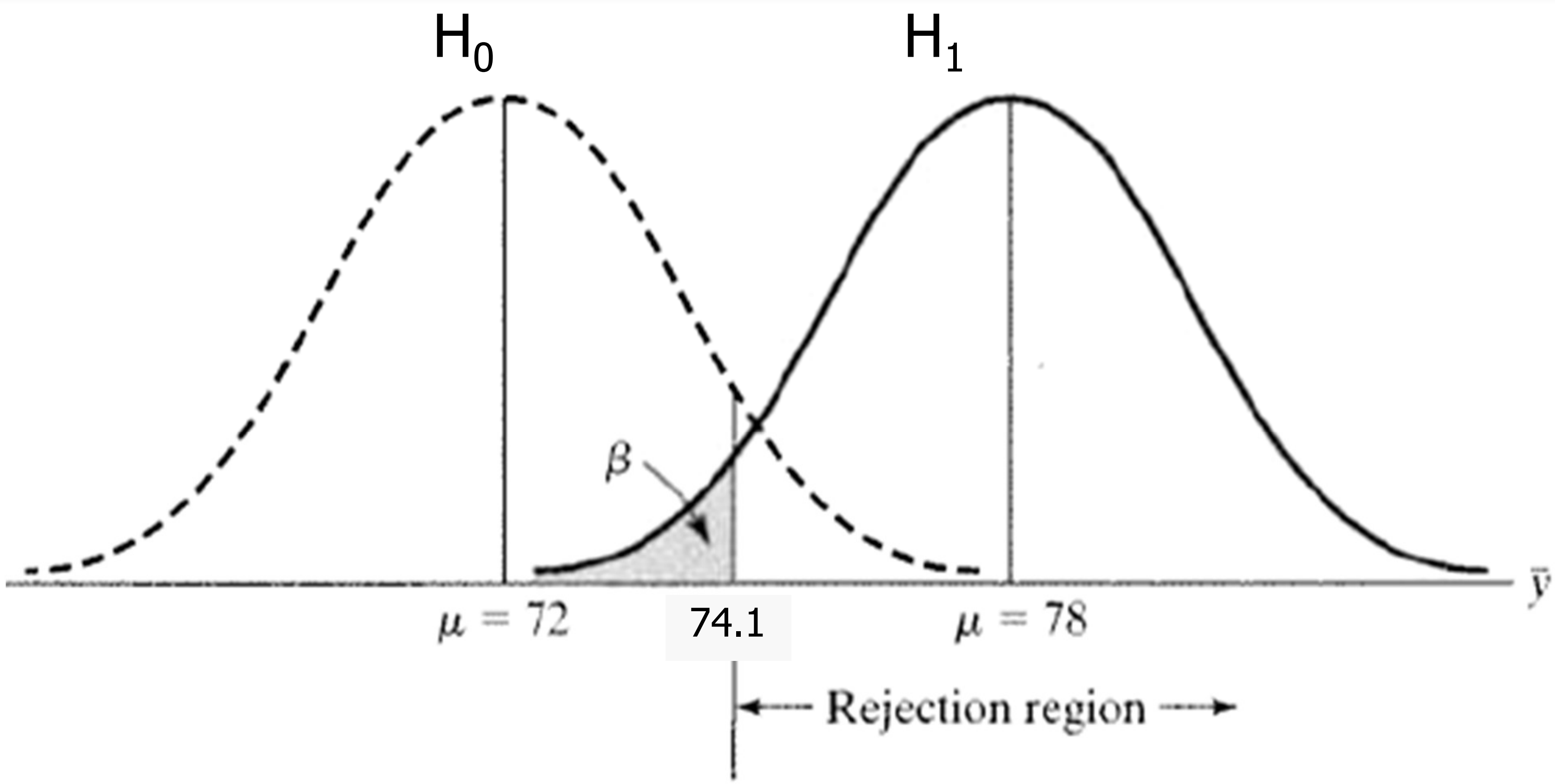
- The probability of committing a **Type II** error, denoted by β, is impossible to compute unless we have a specific alternative hypothesis.

- As P(**Type I** Error) goes Down,

  P(**Type II** Error) goes Up

  - The two probabilities are inversely related.

P(Type II Error)

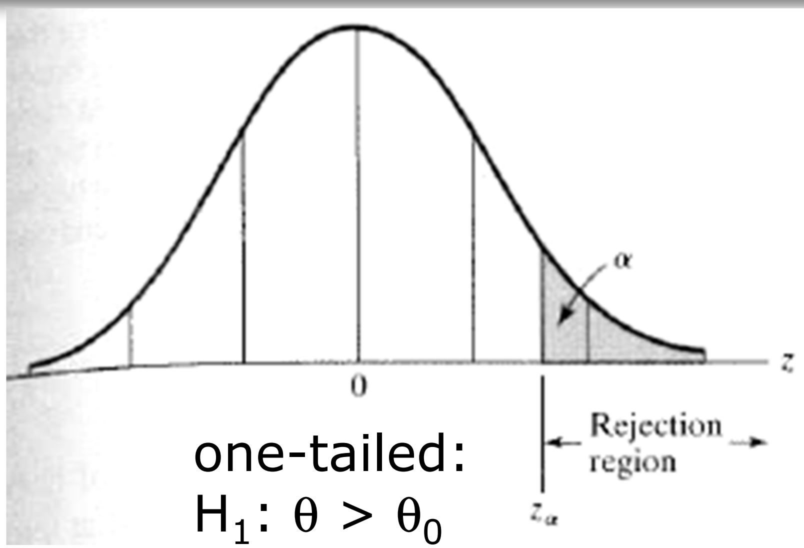


Probability β of making a **Type II** error if $\mu = 78$

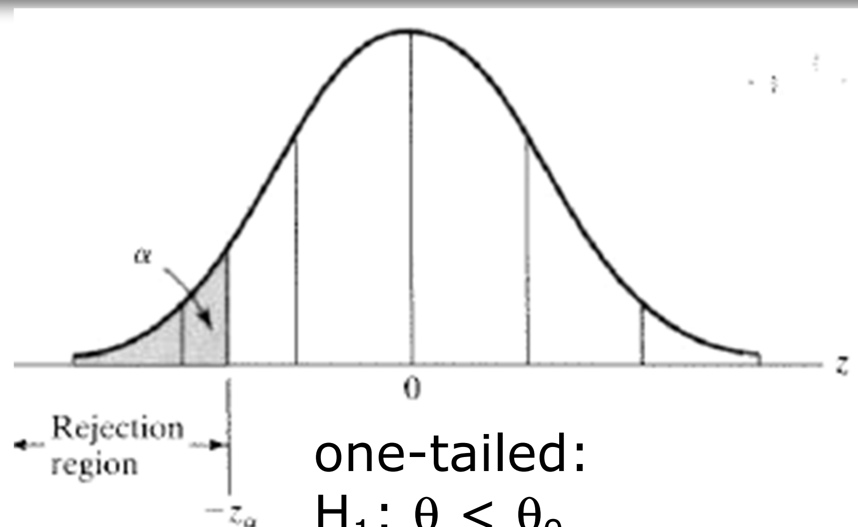
Step 4: Decision: the Significance Level

- We need to make a decision about whether the data provide sufficient evidence to reject H_0.
- This cutoff point is called the **significance level**
- It tells us how strong the evidence must be.

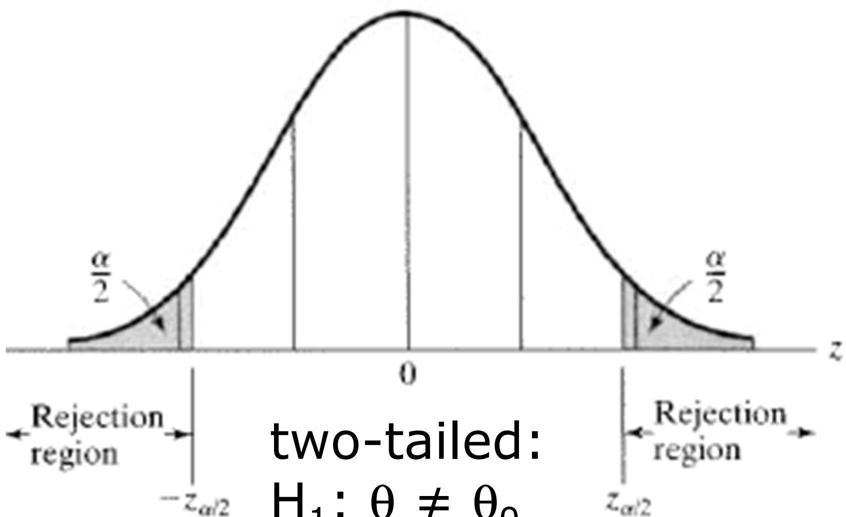
Step 4: Significance level



one-tailed:
$H_1: \theta > \theta_0$

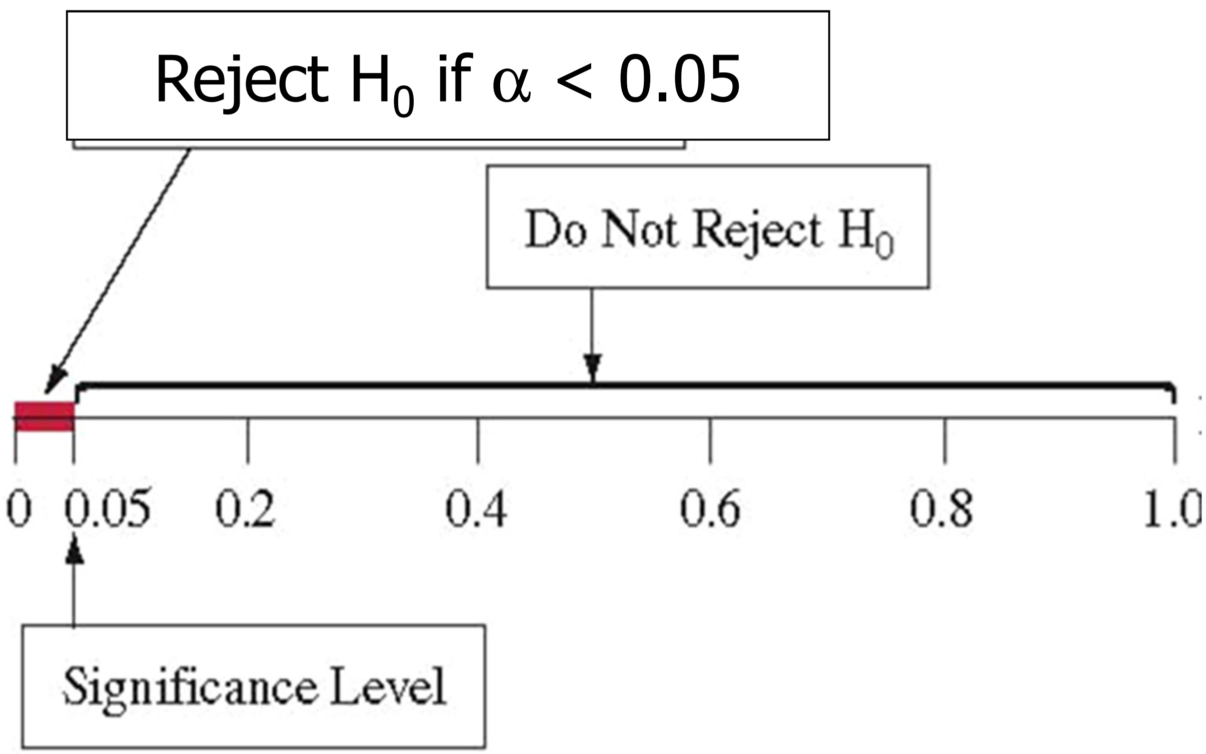


one-tailed:
$H_1: \theta < \theta_0$



two-tailed:
$H_1: \theta \neq \theta_0$

The Significance Level Tells Us How Strong the Evidence Must Be



- In practice, the most common significance level is 0.05
- When we reject H_0 we say the results are **statistically significant**

Possible Decisions in a Test with Significance Level = 0.05

α	**Decision about H_0**
≤ 0.05	Reject H_0
> 0.05	Fail to reject H_0

"Do Not Reject H_0" Is NOT the Same as "Accept H_0"

- **Analogy**: Legal trial
  - **Null Hypothesis:** Defendant is Innocent
  - **Alternative Hypothesis:** Defendant is Guilty
  - If the jury acquits the defendant, this does not mean that it accepts the defendant's claim of innocence.
  - Innocence is plausible, because guilt has not been established **beyond a reasonable doubt**

Summary: The Four Steps of A Significance Test

- Assumptions
- Hypotheses
- Test Statistic
- Decision: use of the Significance Level

Example #1 (1/2) (Type I Error)

The Dept. of Highway Improvements wants to design a surface for repairing a highway that will be structurally efficient. One important consideration is the volume of heavy freight traffic on the highway.

State weight station reports that the average number of heavy-duty trailers traveling on that highway is 72 per hour. However, the section of the highway to be repaired is located in an urban area and the Dept. engineers believe that the volume of heavy freight traffic for this particular section is greater than the average reported for the entire highway.

Example #1 (2/2) (Type I Error)

To validate this theory, the Dept. monitors the highway for 50 1-hour periods randomly selected throughout the month.

Suppose the sample mean and standard deviation of the heavy freight traffic for the 50 sampled hours are:

$$\bar{y} = 74.1 \quad \& \quad S = 13.3$$

- What is the probability of committing Type **I** error?

Example #1 (Sol.)

- **Step 1: Assumptions**
  - Parameter of interest: μ
  - If we used $\bar{y}$ to estimate μ and we know that n>30, then $\bar{y}$ follows an approximately normal sampling distribution.

- **Step 2: Hypotheses**
  - $H_0: \mu = 72$
  - $H_1: \mu > 72$

Example 1 (Sol.)

- **Step 3: Test Statistic**

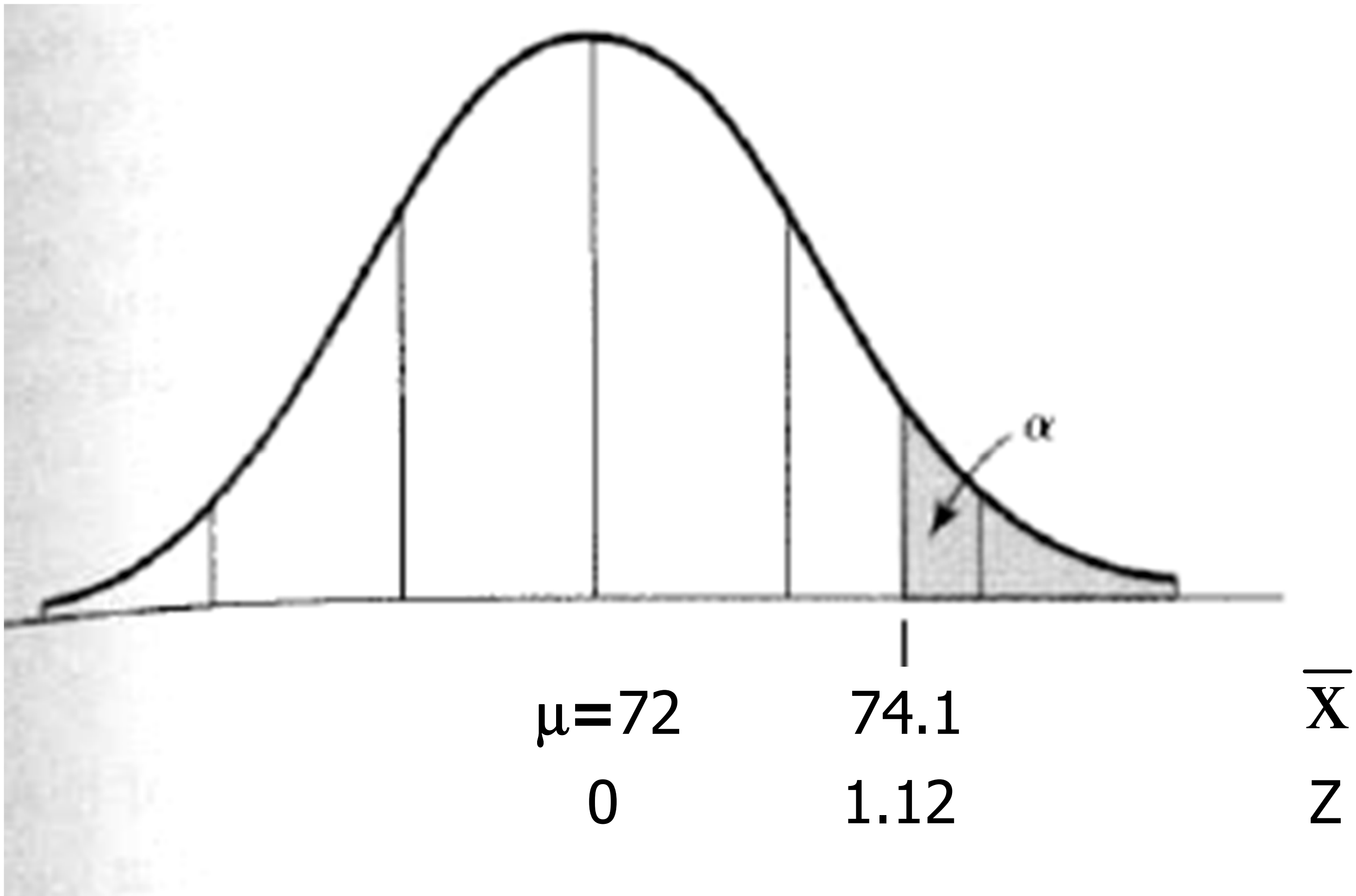
$$Z = \frac{\bar{y} - \mu}{\sigma_{\bar{y}}} = \frac{\bar{y} - 72}{S/\sqrt{n}}$$

$$= \frac{74.1 - 72}{13.3/\sqrt{50}} = 1.12$$

- **Step 4: Compute α**

$$\alpha = P(Z > 1.12) = 1 - P(Z < 1.12) = 1 - 0.8686 = 0.1314$$

Example #1 (Sol.)



Example #2 (Reject H_0 or not)

From the data given in Example # 1:

Do the data support the Department's theory?

Use a significance level of 10% (Step 4)

Example #2 (Sol.)

- **Step 1: Assumptions**
  - Parameter of interest: μ
  - If we used $\bar{y}$ to estimate μ and we know that n>30, then $\bar{y}$ follows an approximately normal sampling distribution.

- **Step 2: Hypotheses**
  - H_0: $\mu = 72$
  - H_1: $\mu > 72$

Example #2 (Sol.)

- **Step 3: Test Statistic**

$$Z = \frac{\bar{y} - \mu}{\sigma_{\bar{y}}} = \frac{\bar{y} - 72}{s/\sqrt{n}}$$

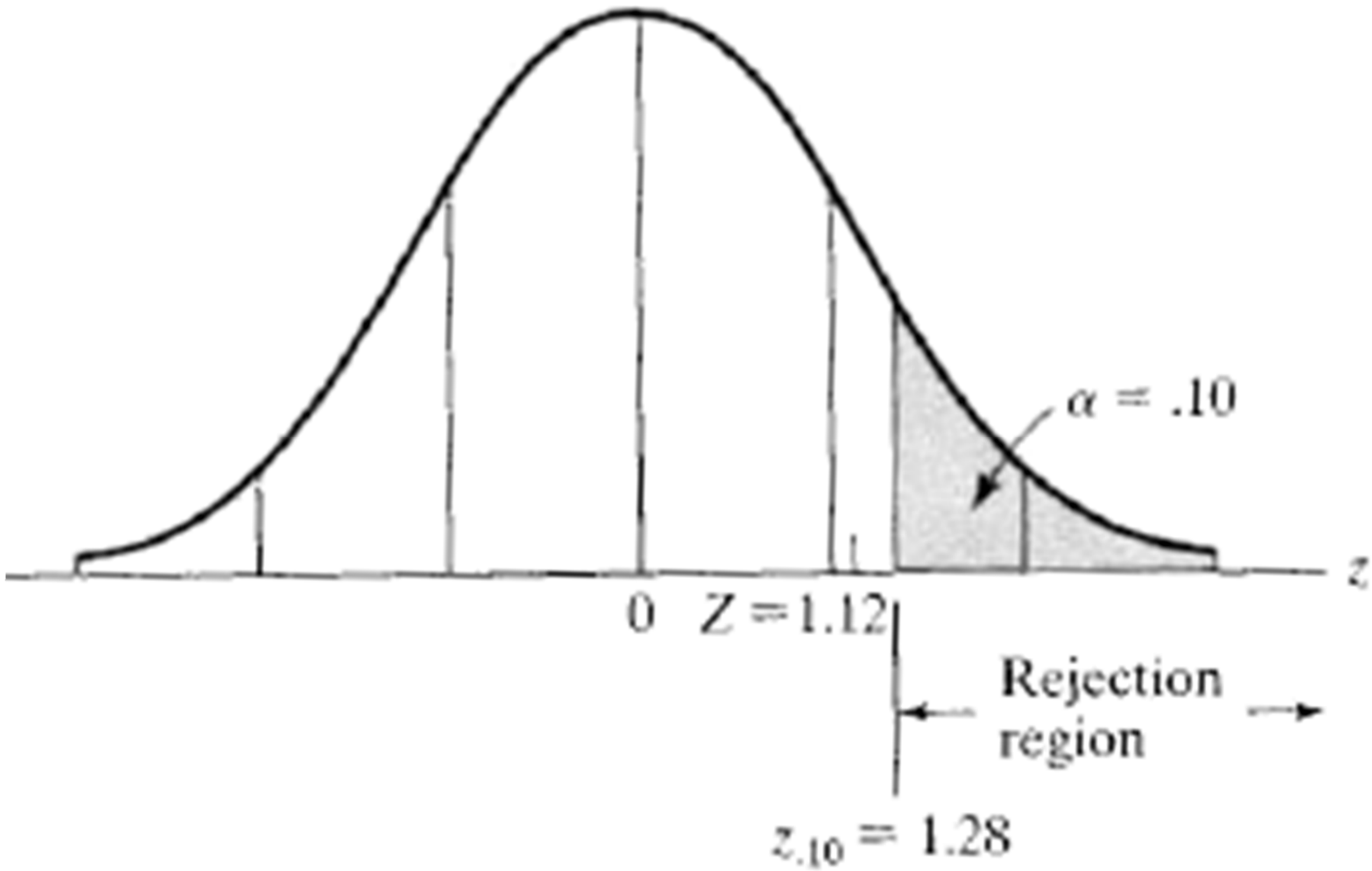
$$= \frac{74.1 - 72}{13.3/\sqrt{50}} = 1.12$$

- **Step 4: Decision**
  - Significance level α **= 0.10**

    1 - α = 0.90 (from Table A.3, z = 1.28)

    => Rejection region: z (critical value) > 1.28

Example #2 (Sol.)



Critical value

Example #2 (Sol.)

Although the average number of heavy freight trucks per hour in the sample exceeds the highway's average by more than 2, the **Z** value of 1.12 does NOT fall in the **rejection region**.

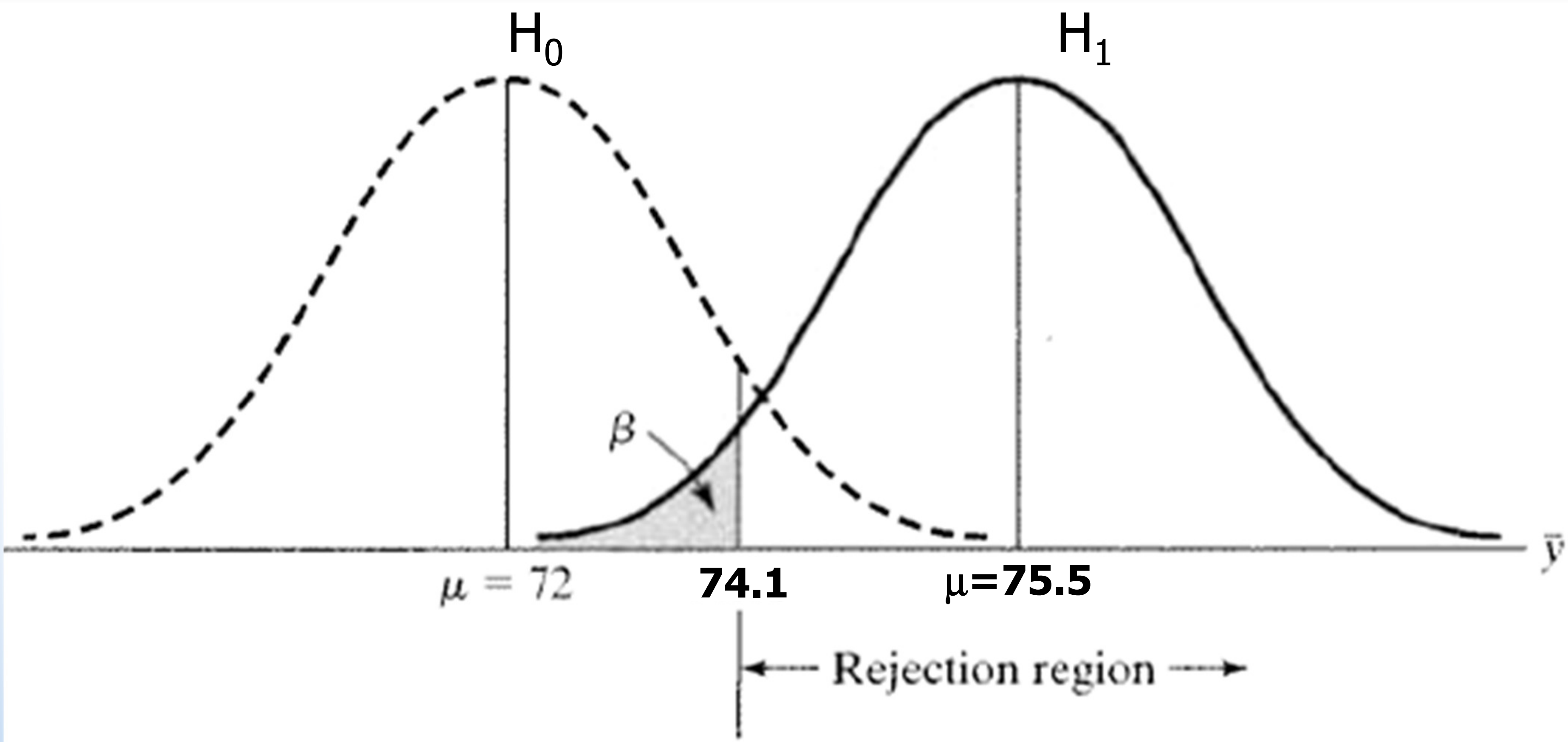
This sample does not provide enough evidence at $\alpha = 0.10$ to support the Department's theory.

Example #3 (Type II Error)

Using the data in Example #1:

If the mean number μ of heavy freight trucks travelling that particular section of the highway is in fact 75.5 per hour, what is the probability that the previous test procedure would fail to detect it? That is, what is the probability β that we would fail to reject (H_0: μ = 72) in this one-tailed test if μ is actually equal to 75.5?

Example #3 (Sol.)



Dot curve: sampling distribution of $\bar{y}$ if (H_0: μ=72) is true

Solid curve: sampling distribution of $\bar{y}$ if μ =75.5

Example #3 (Sol.)

- We want to find β if H_0 is in fact false and $\mu = 75.5$

=> we want to find the probability that $\bar{y}$ does not fall in the rejection region if $\mu = 75.5$

=> shaded area under the solid curve for values $\bar{y} < 74.1$

=> we need to find:

$$P\left[Z < \frac{74.1 - 75.5}{13.3/\sqrt{50}} = -0.74\right] = 0.2296 = \beta$$

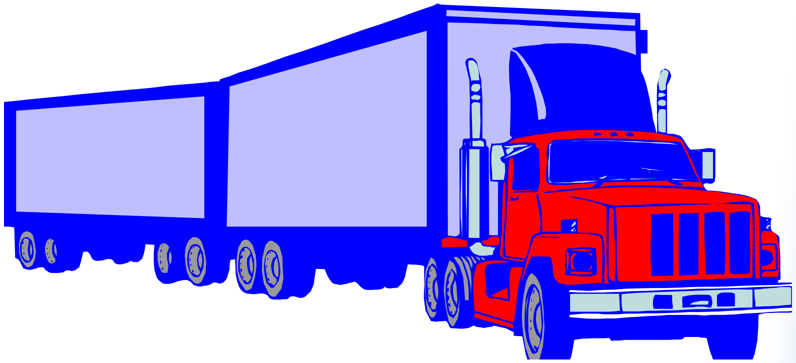
Therefore, the probability of failing to reject H_0: $\mu = 72$ if μ is 75.5, is $\beta = 0.2296$ (22.96%)

Example #4 (Effect of n)

Using the data in Example #1 & Example #3:

If the sample size was 100:

- What is the probability of committing Type **I** error?
- What is the probability of committing Type **II** error?



Example #4 (Sol.)

Computing α

$$Z = \frac{\bar{y} - \mu}{\sigma_{\bar{y}}} = \frac{\bar{y} - 72}{s/\sqrt{n}}$$

$$= \frac{74.1 - 72}{13.3/\sqrt{100}} = 1.58$$

$$\alpha = P(Z > 1.58) = 1 - P(Z < 1.58) = 1 - 0.9429 = 0.0571$$

Computing β

$$Z = \frac{\bar{y} - \mu}{\sigma_{\bar{y}}} = \frac{\bar{y} - 75.5}{s/\sqrt{n}}$$

$$= \frac{74.1 - 75.5}{13.3/\sqrt{100}} = -1.05$$

$$\beta = P(Z < -1.05) = 0.1469$$

Comment!

Example #5

Using Data from Example #1:

We want to find β if H_0 is in fact false and μ =73.

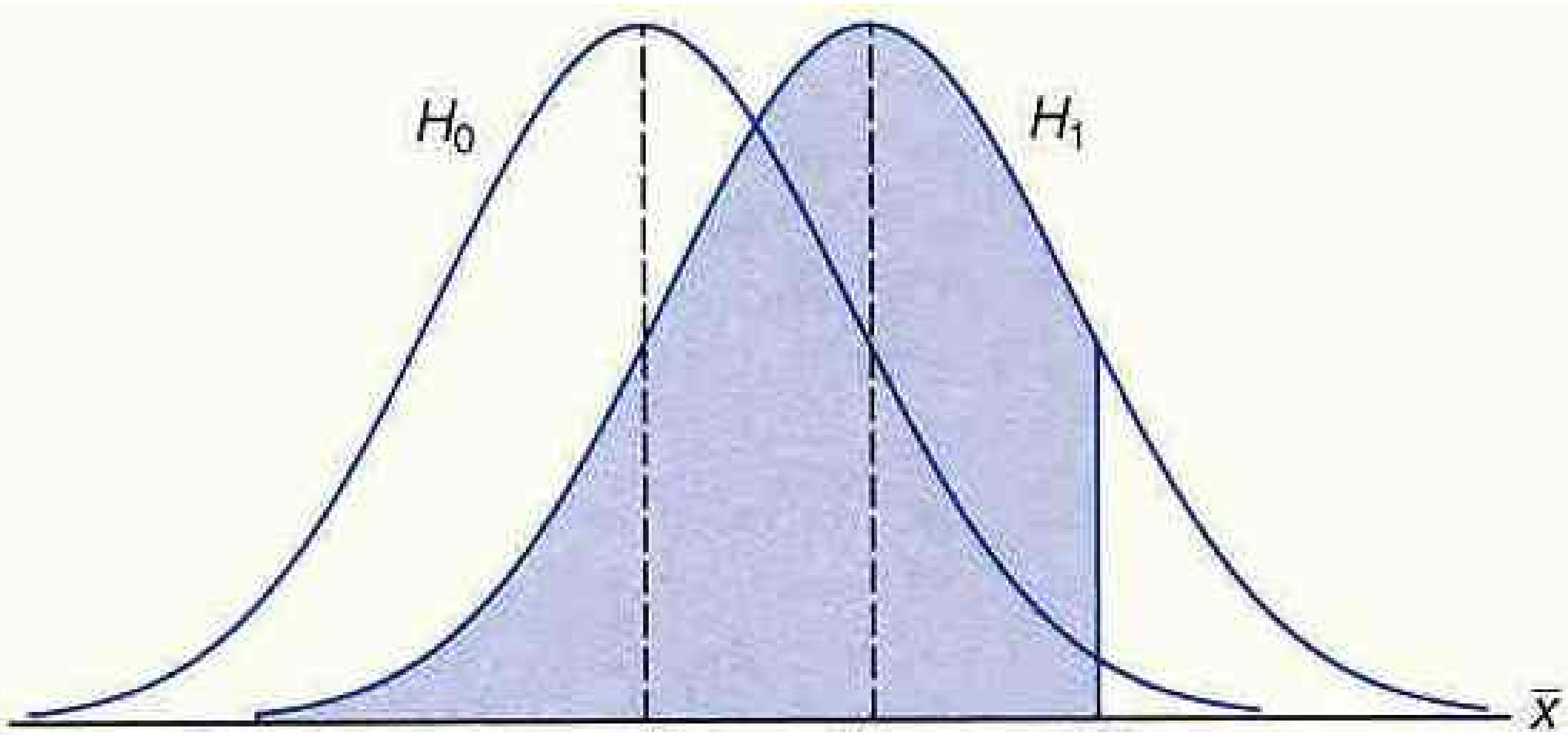
Example #5 (Sol.)

$$P\left[Z < \frac{74.1 - 73}{13.3/\sqrt{50}} = 0.58\right]$$

$$= 0.7190$$
$$= \beta$$

Therefore, the probability of failing to reject H_0: $\mu = 72$ if μ is 73, is $\beta = 0.7190$ (71.90%)

Example #5 (Sol.)



μ=72 μ= 73 74.1

Formulating Hypothesis Testing Problems

Hypotheses about a random variable **X** are often formulated in terms of its distributional properties.

Example, if property is a:

Null hypothesis: $\mathbf{H_0: a = a_0}$

Alternative hypothesis: $\mathbf{H_1: a < a_0 \;||\; a > a_0 \;||\; a \neq a_0}$

Objective of hypothesis testing is to decide whether or not to reject this hypothesis.

The decision is based on estimator $\hat{a}$ of a.

Reject H_0: If observed estimate $\hat{a}$ lies in rejection region R_{a0} (i.e. $\hat{a} \in R_{a0}$)

Do not reject H_0: Otherwise (i.e. $\hat{a} \notin R_{a0}$)

Null & Alternative Hypotheses

A metal lathe is checked periodically by quality control inspectors whether it is producing machine bearings with a mean diameter of 0.5 inch. If the mean diameter of the bearings is larger or smaller than 0.5 inch, then the process is out of control and needs to be adjusted.

- Formulate the null and alternative hypotheses that could be used to test whether the bearing production process is out of control.

$$\mathbf{H_0}: \mu = 0.5$$
$$\mathbf{H_1}: \mu \neq 0.5$$

Constructing Alternative Hypotheses

Claim	H_1
$\theta \geq \theta_0$ (At least)	$\theta < \theta_0$
$\theta \leq \theta_0$ (At most) (Does not exceed)	$\theta > \theta_0$
$\theta > \theta_0$	$\theta < \theta_0$
$\theta < \theta_0$	$\theta > \theta_0$
Show strong evidence for	**H_1**
$\theta > \theta_0$ (Greater than)	$\theta > \theta_0$
$\theta < \theta_0$ (Less than)	$\theta < \theta_0$

Hypothesis Testing on the Population Mean (case 1 and case 2)

1. σ **known or** σ **unknown** (but with a large sample: $n \geq 30$)
2. **Hypotheses:**

One-tailed Test	Two-tailed Test
H_0: $\mu = \mu_0$	H_0: $\mu = \mu_0$
H_1: $\mu > \mu_0$ (or $\mu < \mu_0$)	H_1: $\mu \neq \mu_0$

3. **Test statistic**

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} \text{ if } \sigma \textbf{ known} \text{ or } Z = \frac{\overline{X} - \mu_0}{\sigma_{\overline{X}}} = \frac{\overline{X} - \mu_0}{s / \sqrt{n}} \text{ if } \sigma \textbf{ unknown} \text{ but } n \geq 30$$

3. **Rejection Region:** using significance level α

One-tailed Test	Two-tailed Test
$Z > z_\alpha$ (or $Z < z_\alpha$)	$\lvert Z \rvert > z_{\alpha/2}$

Example #6

The burning rate of a rocket propellant is being studied. Specifications require that the mean burning rate must be 40 cm/s. Furthermore, suppose that we know that the standard deviation of the burning rate is approximately 2 cm/s. The experimenter decides to specify a Type I Error probability α = 0.05, and he will base the test on a random sample of size n=25. The sample mean burning rate obtained is 41.25 cm/s.

Do the data support the specifications?

Example #6 (Sol.)

Given (known): $\mu_0 = 40$ $\sigma = 2$ $\bar{x} = 41.25$ $n = 25$ $\alpha = 0.05$

Hypothesis: $H_0 : \mu = \mu_0 = 40$ $H_1 : \mu \neq 40$ **(two tailed)**

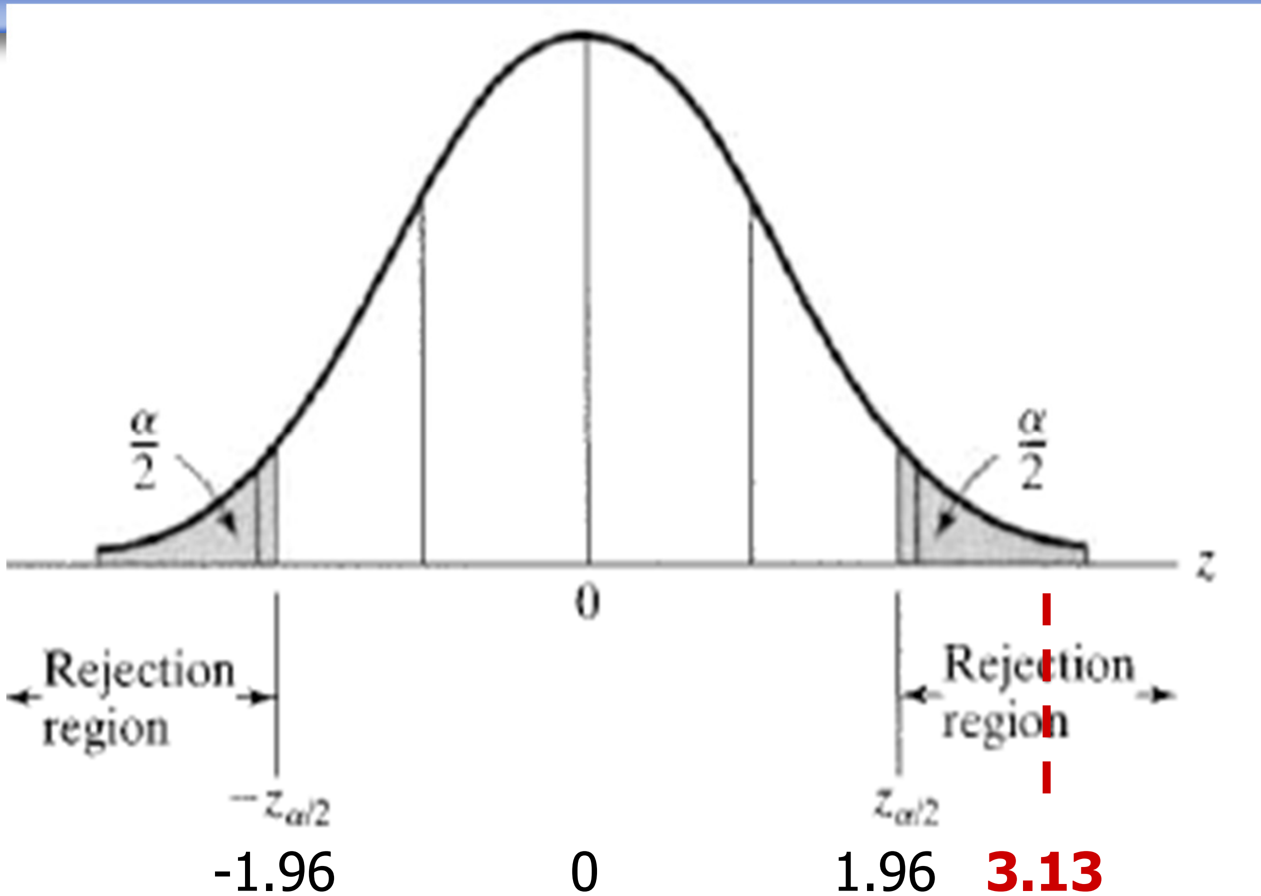
What to Use (Test statistic): $Z = \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{41.25 - 40}{2 / \sqrt{25}} = 3.13$$

Rejection Region: $|Z| > z_{\alpha/2}$

$\alpha = 0.05 \rightarrow \alpha/2 = 0.025 \rightarrow$ $z_{\alpha/2} = 1.96$

Example #6 (Sol.)



Therefore, H_0 is rejected and we can conclude that the mean burning rate is not equal to 40 cm/s.

Example #7

Hypothesis Testing & Relationship with Confidence Interval (CI)

Using all data given in ***Example #6***, determine The 100(1-α)% confidence interval of the burning rate.

Then conclude: do the data support the specifications?

Example #7 (Sol.)

Given (known): $\mu_0 = 40$ $\sigma = 2$ $\bar{x} = 41.25$ $n = 25$ $\alpha = 0.05$

Required: 95% CI of the mean

What to Use:

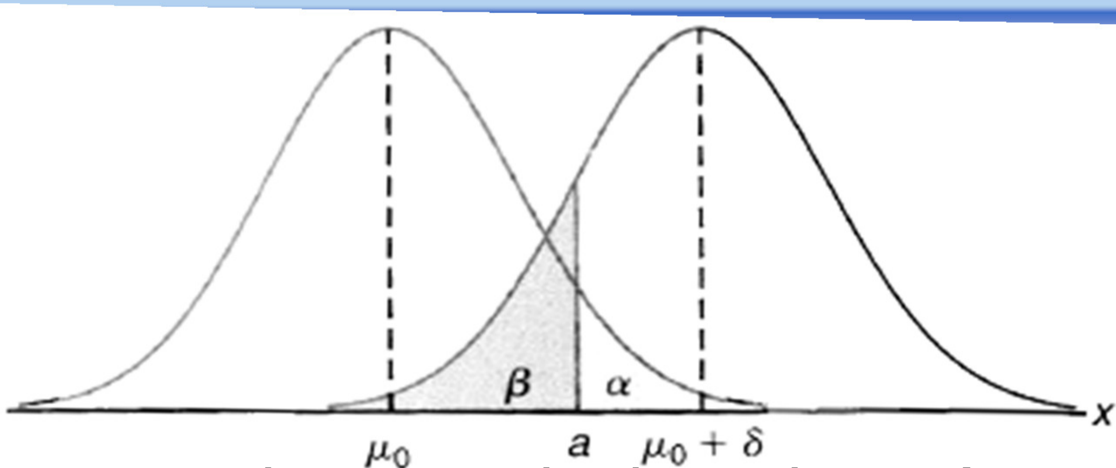
$$CI = \bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$\alpha = 0.05$ ➡ $\alpha/2 = 0.025$ ➡ $z_{\alpha/2} = 1.96$

$$CI = 41.25 \pm 1.96 \frac{2}{\sqrt{25}} = 41.25 \pm 0.784 = [40.466 \ , \ 42.034]$$

We are 95% confident that the true mean burning rate will fall within the interval [40.47, 42.03], and since μ_0=40 cm/s is not included in this interval, the data do not support the specifications.

Choice of Sample Size for Testing Mean



Suppose that we wish to test the hypothesis (H_0: $\mu=\mu_0$; H_1: $\mu>\mu_0$) with a significance level α when σ^2 is known. For a specific alternative, say $\mu=\mu_0+\delta$, the power of our test is: **1-β** = P[$\bar{x}$>a when $\mu=\mu_0+\delta$], which is equivalent to:

$$\beta = P\left[Z < z_\alpha - \frac{\delta}{\sigma/\sqrt{n}}\right] = P\left[Z < -z_\beta\right] \Rightarrow$$

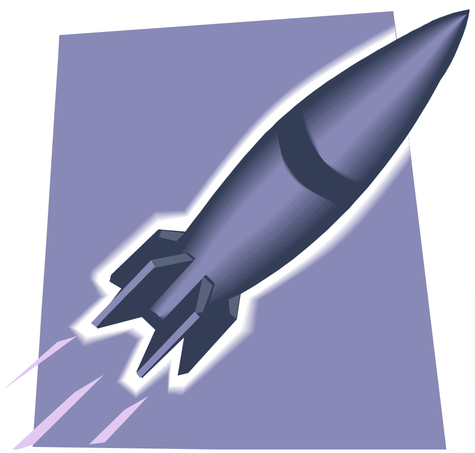
$$n = \frac{(z_\alpha + z_\beta)^2\sigma^2}{\delta^2} \quad \text{(one-tailed)}$$

$$n \approx \frac{(z_{\alpha/2} + z_\beta)^2\sigma^2}{\delta^2} \quad \text{(two-tailed)}$$

Example #8

Using data given in Example #6

Find the sample size required if the power of the test is to be 0.90 when the true mean burning rate is actually 41 cm/s.



Example #8 (Sol.)

Given: $\mu_0 = 40$ $\sigma = 2$ $\alpha = 0.05$ $\mu_0 + \delta = 41$ $(1-\beta) = 0.90$

Required: n

What to Use: $n \approx \dfrac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{\delta^2}$ (two-tailed)

$\alpha = 0.05 \rightarrow \alpha/2 = 0.025 \rightarrow z_{\alpha/2} = 1.96$

$1-\beta = 0.90 \rightarrow z_\beta = 1.28$

$\delta = 41 - 40 = 1$

$$n \approx \frac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{\delta^2} \approx \frac{(1.96 + 1.28)^2 * 2^2}{1^2} \approx 41.99 \rightarrow n = 42$$

Example #9

The alkalinity level of water specimens collected from the Han River in Seoul (Korea) has a mean of 50 mg per liter (*Env. Science & Eng.*, Sept. 2000). Consider a random sample of 100 water specimens collected from a tributary of the Han River. The mean and standard deviation of the alkalinity levels of the sample are 67.8 mg/l and 14.4mg/l respectively.

Is there sufficient evidence (at α=0.01) to indicate that the population mean alkalinity level of water in the tributary exceeds 50 mg/l?

Example #9 (Sol.)

Given (known): $\mu_0 = 50$ $\bar{x} = 67.8$ $S = 14.4$ $n = 100$ $\alpha = 0.01$

Hypothesis: $H_0 : \mu = \mu_0 = 50$ $H_1 : \mu > 50$ **(one-tailed)**

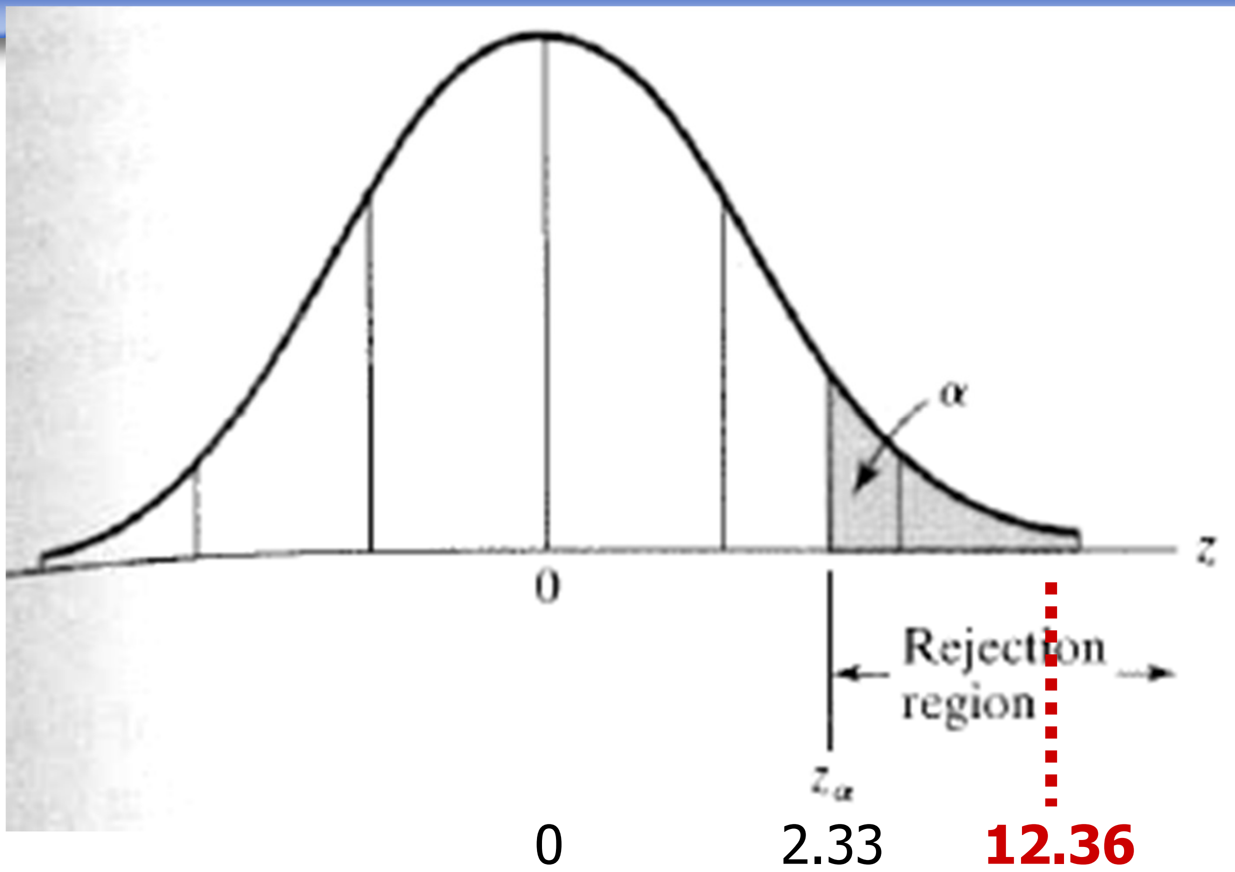
What to Use (Test statistic): $Z = \dfrac{\bar{X} - \mu_0}{s / \sqrt{n}}$

$$Z = \frac{\bar{X} - \mu_0}{s / \sqrt{n}} = \frac{67.8 - 50}{14.4 / \sqrt{100}} = 12.36$$

Rejection Region: $Z > z_\alpha$

$\alpha = 0.01$ ➡ $z_\alpha \approx 2.33$ **(one-tailed)**

Example #9 (Sol.)



Therefore, H_0 is rejected and we can conclude that there is enough evidence that the mean alkalinity level of water in the tributary exceeds 50 mg/l

Example #10

Using data given in Example #9

Find the sample size required for a 5% probability of committing Type II error when the true mean alkalinity level of water in the tributary is actually equal to 52 mg/l.

Example #10 (Sol.)

Given: $\mu_0 = 50$ $S = 14.4$ $\alpha = 0.01$ $\mu_0 + \delta = 52$ $\beta = 0.05$

Required: n

What to Use: $n = \dfrac{(z_\alpha + z_\beta)^2 s^2}{\delta^2}$ (one-tailed)

$\alpha = 0.01$ ➡ $z_\alpha \approx 2.33$ **(one-tailed)**

$\beta = 0.05$ ➡ $1-\beta = 0.95$ ➡ $z_\beta = 1.645$

$\delta = 52 - 50 = 2$

$$n = \frac{(z_\alpha + z_\beta)^2 s^2}{\delta^2} = \frac{(2.33 + 1.645)^2 * 14.4^2}{2^2} = 819.10$$ ➡ $n = 820$

Hypothesis Testing on the Population Mean (case 3)

1. σ **unknown**, population approximately normal, and small sample size ($n<30$)
2. **Hypotheses:**

One-tailed Test	Two-tailed Test
H_0: $\mu=\mu_0$	H_0: $\mu=\mu_0$
H_1: $\mu>\mu_0$ (or $\mu<\mu_0$)	H_1: $\mu\neq\mu_0$

3. **Test statistic:**

$$T=\frac{\bar{X}-\mu_0}{s/\sqrt{n}}$$

4. **Rejection Region:** using significance level α

One-tailed Test	Two-tailed Test
$T>t_\alpha$ (or $T<-t_\alpha$)	$\lvert T\rvert>t_{\alpha/2}$

t-distribution based on df=n-1

Example #11

The following numbers are particle (contamination) counts for a sample of **10** semiconductor silicon wafers:

50 48 44 56 61 52 53 55 67 51

The mean $\bar{x}$ = 53.7 counts and the standard deviation S = 6.567 counts have been computed. Over a long run, the process average for wafer particle counts has been 50 counts per wafer, and on the basis of the sample, we want to test whether a change has occurred.

For a significance level of $\alpha = 0.05$, can we assume now that the process mean is not equal to 50 counts?

Example #11 (Sol.)

Given: $\mu_0 = 50$ $\quad \bar{x} = 53.7$ $\quad s = 6.567$ $\quad n = 10$ $\quad \alpha = 0.05$

Hypothesis: $H_0 : \mu = \mu_0 = 50$ $\quad H_1 : \mu \neq 50$ **(two-tailed)**

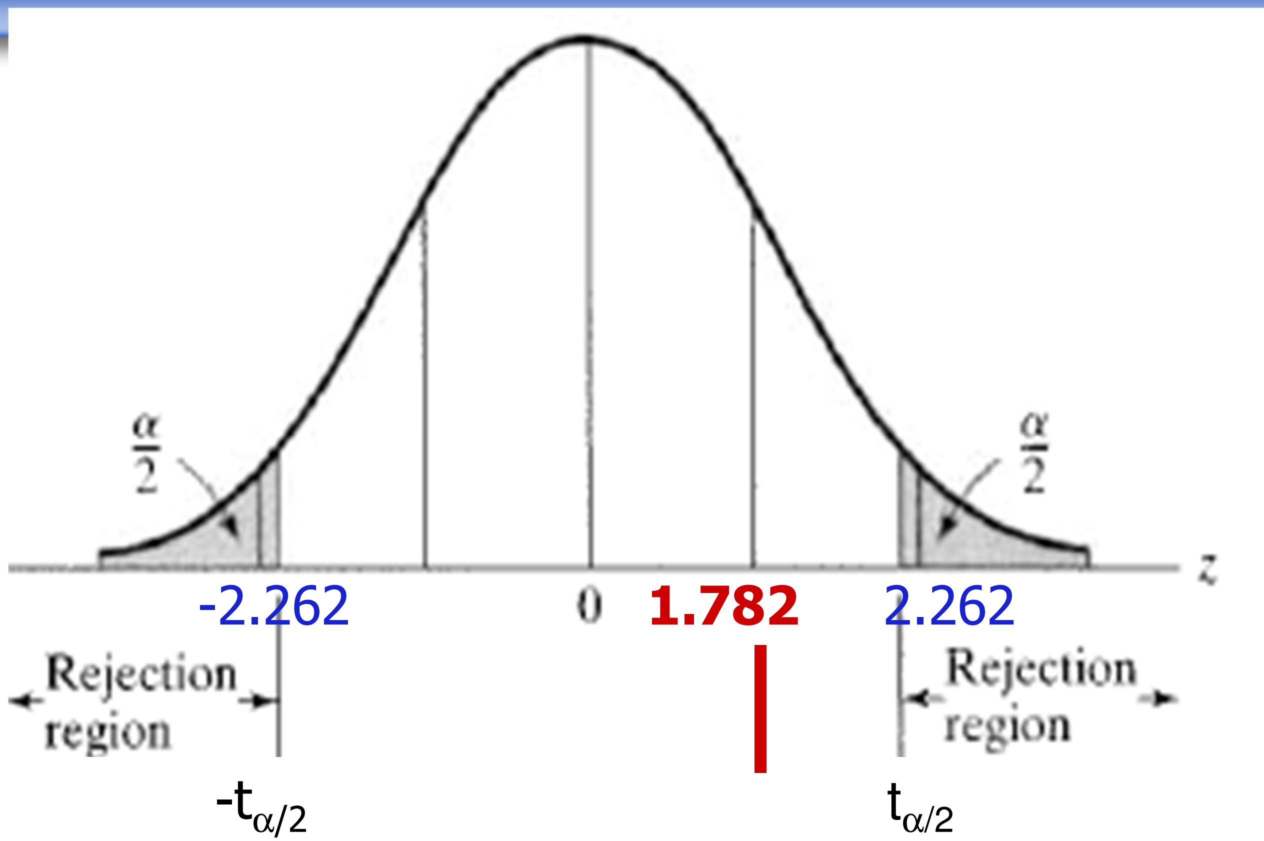
What to Use (Test statistic): $T = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$

$$T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{53.7 - 50}{6.567/\sqrt{10}} = 1.782$$

Rejection Region: $|T| > t_{\alpha/2}$

$\alpha = 0.05 \rightarrow \alpha/2 = 0.025 \rightarrow t_{\alpha/2} = 2.262$ (df = 9)

Example #11 (Sol.)



We are unable to reject the null hypothesis and, therefore, we **CANNOT** assume now that the process mean is not equal to 50 counts

Example #12

The maximum allowable level of benzene in a workplace is 1 ppm. Suppose a steel manufacturing plant, which exposes its workers to benzene daily, is under investigation. 20 air samples, collected over a period of 1 month and examined for benzene content, yielded a sample mean of 2.143 with a sample variance of 3.013.

Is the steel manufacturing plant in violation of the standards?

Test the hypothesis that the mean level of benzene at the plant is greater than 1 ppm, using $\alpha = 0.10$.

Example #12 (Sol.)

Given: $\mu_0 = 1$ $n = 20$ $\bar{x} = 2.143$ $s^2 = 3.013$ $\alpha = 0.10$

Hypothesis: $H_0 : \mu = \mu_0 = 1$ $H_1 : \mu > 1$ **(one-tailed)**

What to Use (Test statistic):

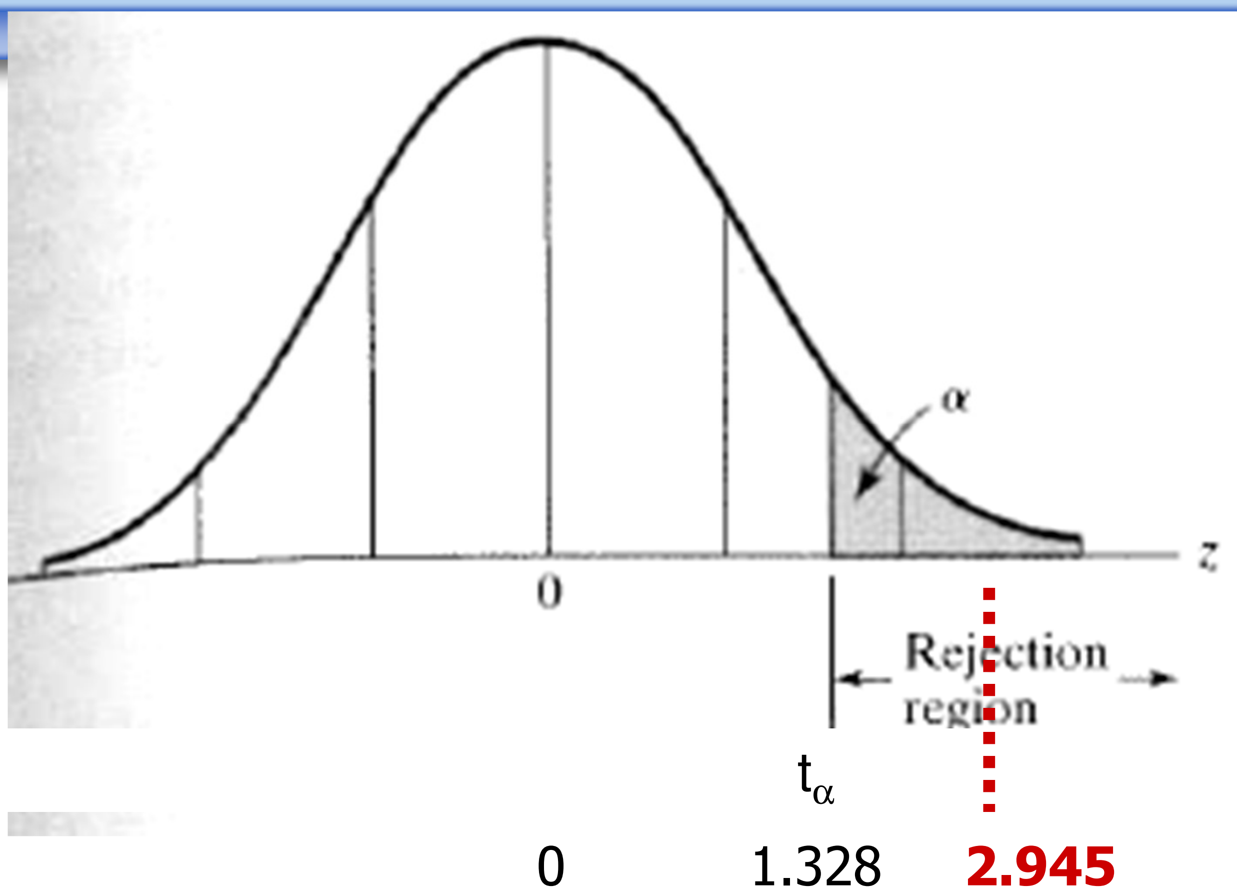
$$T = \frac{\bar{X} - \mu_0}{s / \sqrt{n}}$$

$$T = \frac{\bar{X} - \mu_0}{s / \sqrt{n}} = \frac{2.143 - 1}{\sqrt{3.013} / \sqrt{20}} = 2.945$$

Rejection Region: $T > t_\alpha$

$\alpha = 0.10$ ⇨ $t_\alpha = 1.328$ **(df = 19)**

Example #12 (Sol.)



Therefore, H_0 is rejected and we can conclude that there is enough evidence to say that the steel manufacturing plant is in violation of the standards.

Hypothesis Testing on the Difference between Two Population Means (case 1)

1. σ_1 and σ_2 **known**; two samples of size n_1 and n_2 selected randomly and independently from the target populations.

2. **Hypotheses:**

One-tailed Test	Two-tailed Test
H_0: $\mu_1 - \mu_2 = D_0$	H_0: $\mu_1 - \mu_2 = D_0$
H_1: $\mu_1 - \mu_2 > D_0$ (**or** $\mu_1 - \mu_2 < D_0$)	H_1: $\mu_1 - \mu_2 \neq D_0$

3. **Test statistic:**

$$Z = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - D_0}{\sigma_{\overline{X}_1 - \overline{X}_2}} = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

4. **Rejection Region:** using significance level α

One-tailed Test	Two-tailed Test
$Z > z_\alpha$ (or $Z < -z_\alpha$)	$\lvert Z \rvert > z_{\alpha/2}$

Hypothesis Testing on the Difference between Two Population Means (case 2)

1. σ_1 and σ_2 **unknown**; two **large** samples: size $n_1 \geq 30$ & $n_2 \geq 30$ selected randomly & independently from the target populations

2. **Hypotheses:**

   One-tailed Test

   H_0: $\mu_1 - \mu_2 = D_0$

   H_1: $\mu_1 - \mu_2 > D_0$ (**or** $\mu_1 - \mu_2 < D_0$)

   Two-tailed Test

   H_0: $\mu_1 - \mu_2 = D_0$

   H_1: $\mu_1 - \mu_2 \neq D_0$

3. **Test Statistic:**

$$Z = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - D_0}{\sigma_{\overline{X}_1 - \overline{X}_2}} = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - D_0}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

4. **Rejection Region:** using significance level α

   One-tailed Test

   $Z > z_\alpha$ (or $Z < -z_\alpha$)

   Two-tailed Test

   $|Z| > z_{\alpha/2}$

Hypothesis Testing on the Difference between Two Population Means (case 3)

1. σ_1 and σ_2 **unknown** but **equal**; two random and independent samples of size n_1 & n_2 selected from the approximately normal target populations.
2. **Hypotheses:**

   One-tailed Test

   $H_0: \mu_1 - \mu_2 = D_0$

   $H_1: \mu_1 - \mu_2 > D_0$ (**or** $\mu_1 - \mu_2 < D_0$)

   Two-tailed Test

   $H_0: \mu_1 - \mu_2 = D_0$

   $H_1: \mu_1 - \mu_2 \neq D_0$
3. **Test Statistic:**

$$T = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - D_0}{s_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$$

with

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

4. **Rejection Region:** using significance level α

   One-tailed Test

   $T > t_\alpha$ (or $T < -t_\alpha$)

   Two-tailed Test

   $|T| > t_{\alpha/2}$

   $df = n_1 + n_2 - 2$

Hypothesis Testing on the Difference between Two Population Means (case 4)

1. σ_1 and σ_2 **unknown** and **unequal**; two random and independent samples of size n_1 & n_2 selected from the approximately normal target populations.

2. **Hypotheses:**

One-tailed Test	Two-tailed Test
$H_0: \mu_1 - \mu_2 = D_0$	$H_0: \mu_1 - \mu_2 = D_0$
$H_1: \mu_1 - \mu_2 > D_0$ (**or** $\mu_1 - \mu_2 < D_0$)	$H_1: \mu_1 - \mu_2 \neq D_0$

3. **Test Statistic:**

$$T = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$df = [s_1^2/n_1 + s_2^2/n_2]^2 \Bigg/ \left[\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}\right]$$

If not Integer, round it **Down**

4. **Rejection Region:** using significance level α

One-tailed Test	Two-tailed Test
$T > t_\alpha$ (or $T < -t_\alpha$)	$\lvert T \rvert > t_{\alpha/2}$

Hypothesis Testing on the Difference between Two Population Means (case 5)

1. Population of differences is approximately normal; a sample of **paired** differences of size n is randomly selected.

2. **Hypotheses:**

One-tailed Test	Two-tailed Test
$H_0: \mu_1-\mu_2=D_0$	$H_0: \mu_1-\mu_2=D_0$
$H_1: \mu_1-\mu_2>D_0$ (**or** $\mu_1-\mu_2<D_0$)	$H_1: \mu_1-\mu_2 \neq D_0$

3. **Test Statistic:**

$$T=\frac{\bar{d}-D_0}{\sigma_d/\sqrt{n}} \approx \frac{\bar{d}-D_0}{s_d/\sqrt{n}}$$

4. **Rejection Region:** using significance level α

One-tailed Test	Two-tailed Test
$T>t_\alpha$ (or $T<-t_\alpha$)	$\lvert T\rvert>t_{\alpha/2}$

df = n - 1

Note:
In many applications, we want to hypothesize that there is no difference between the population means, i.e. $D_0=0$.

Example #13

To reduce costs, a bakery has implemented a new leavening process for preparing commercial bread loaves. Loaves of bread were randomly sampled and analyzed for calorie content both before and after implementation of the new process. A summary of the results of the 2 samples is shown in the table. Do these samples provide sufficient evidence to conclude that the mean number of calories per loaf has decreased since the new leavening process was implemented? Use $\alpha=0.05$

New Process	Old Process
$n_1=50$	$n_2=30$
$\bar{x}_1=1255$ calories	$\bar{x}_2=1330$ calories
$s_1=215$ calories	$s_2=238$ calories

Example #13 (Sol.)

Given:

$\bar{x}_1 = 1255$ $\quad s_1 = 215$ $\quad n_1 = 50$ $\quad \alpha = 0.05$

$\bar{x}_2 = 1330$ $\quad s_2 = 238$ $\quad n_2 = 30$

Hypothesis: $H_0 : \mu_1 - \mu_2 = D_0 = 0$ $\quad H_1 : \mu_1 - \mu_2 < 0$

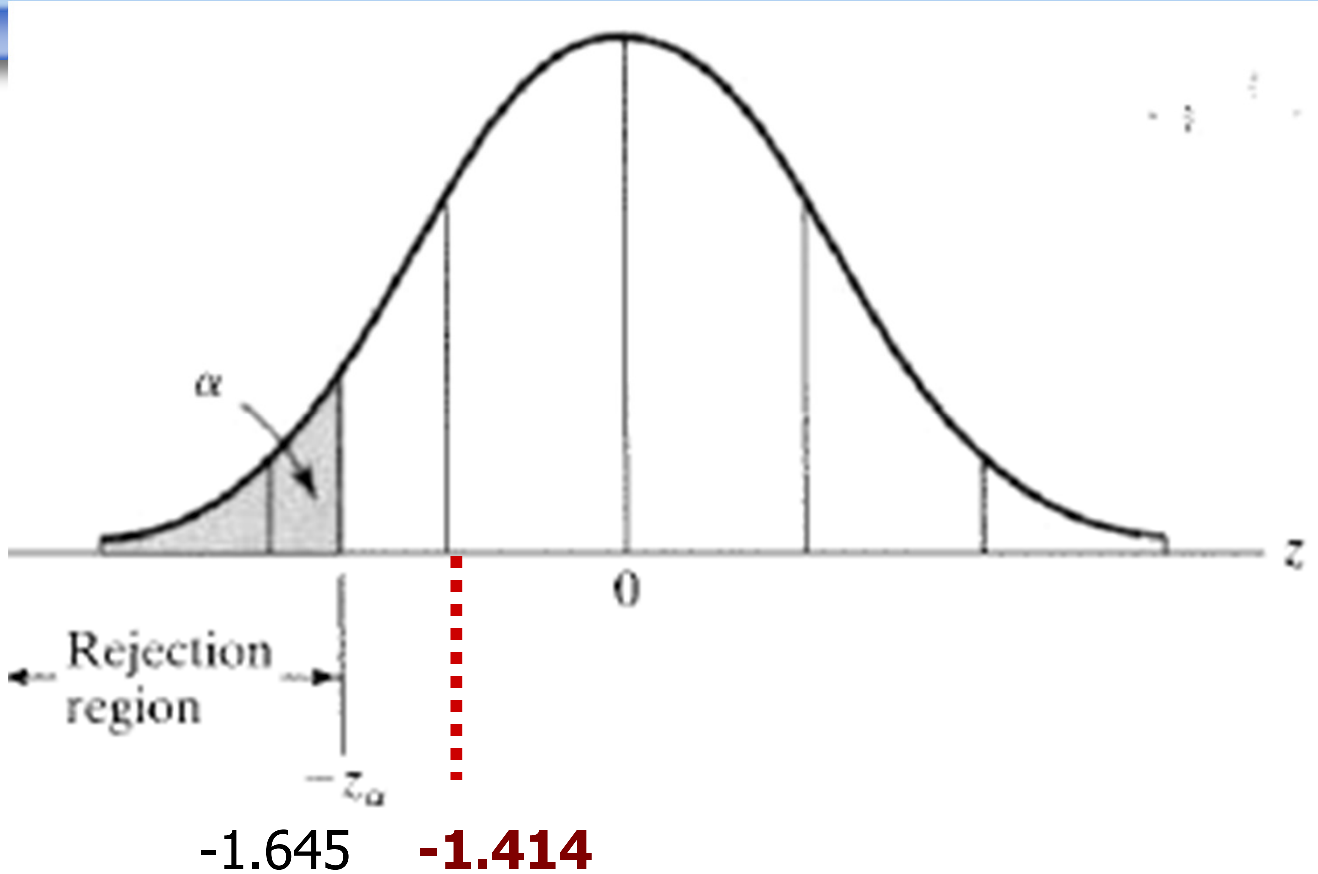
What to Use (Test statistic):

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{(1255 - 1330) - 0}{\sqrt{\frac{215^2}{50} + \frac{238^2}{30}}} = -1.414$$

$\alpha = 0.05 \Rightarrow - z_\alpha = -1.645$

Example #13 (Sol.)



Therefore, H_0 is not rejected; there is not enough evidence, at α=0.05, to conclude that the new process yields a loaf with fewer mean calories

Example #14

D0	D60	Diff
57	62	5
27	49	22
32	30	-2
31	34	3
34	38	4
38	36	-2
71	77	6
33	51	18
34	45	11
53	42	-11
36	43	7
42	57	15
26	36	10
52	58	6
36	35	-1
55	60	5
36	33	-3
42	49	7
36	33	-3
54	59	5
34	35	1
29	37	8
33	45	12
33	29	-4

A study was carried out to test a certain drug. Patients were given various dosages of a drug and then completed a specific task with a certain score. The given data shows the scores of 24 patients where each patient was tested using two dosage levels (D0 level and D60 level). The score difference will be positive for patients who performed better in the D60 level (*Population 1*) and negative for those who scored better in the D0 level (*Population 2*). If increasing the dosage level has a positive effect, then the mean difference score in the population will be positive. The null hypothesis is that the mean difference score in the population is 0.0. Use α=0.05.

Example #14 (Sol.)

Given: $n_d = 24$ $\alpha = 0.05$

Hypothesis: $H_0 : \mu_1 - \mu_2 = D_0 = 0$ $H_1 : \mu_1 - \mu_2 > 0$

Computed: $\bar{d} = 4.96$ $s_d = 7.54$

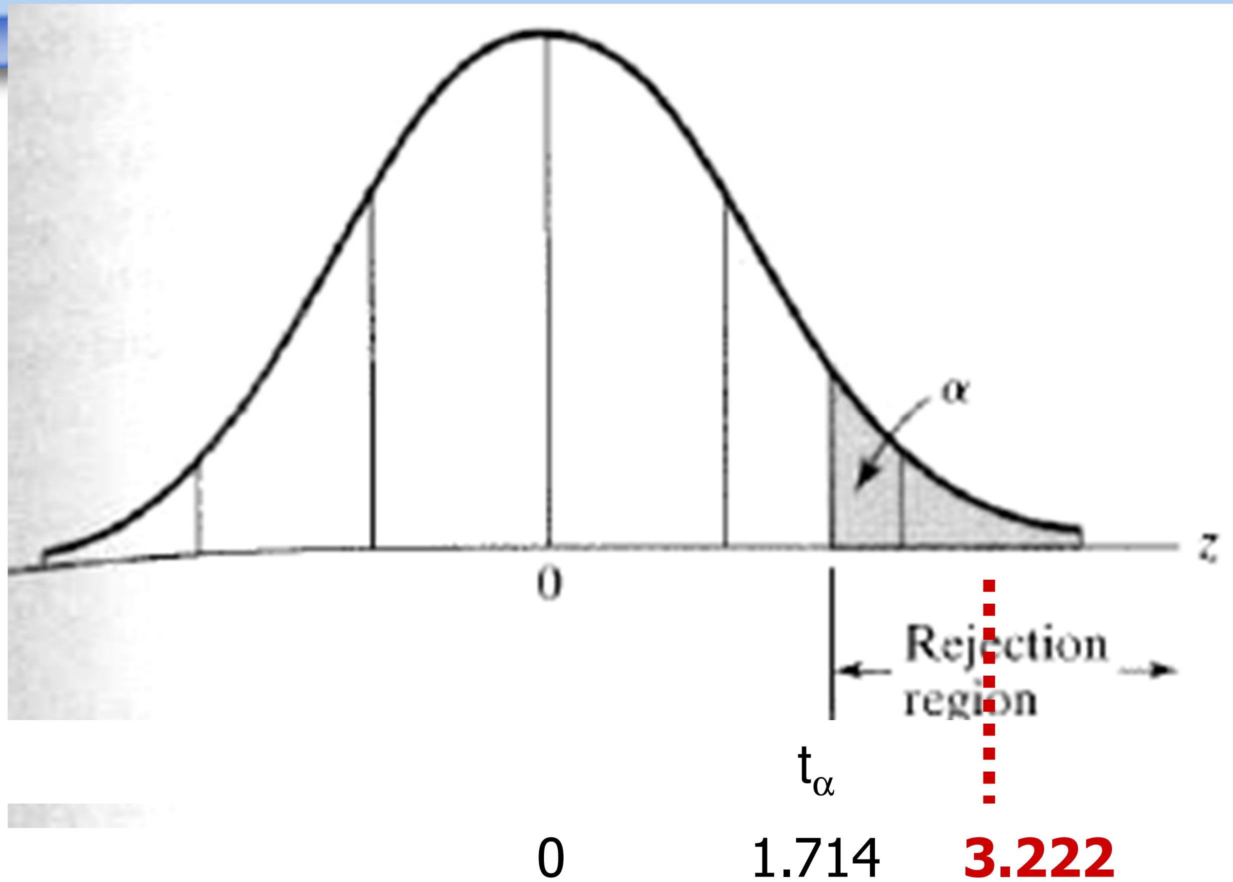
What to Use:

$$T = \frac{\bar{d} - D_0}{s_d / \sqrt{n}}$$

$$T = \frac{\bar{d} - D_0}{s_d / \sqrt{n}} = \frac{4.96 - 0}{7.54 / \sqrt{24}} = 3.222$$

$\alpha = 0.05 \Rightarrow$ $t_\alpha = 1.714$ (df=23)

Example #14 (Sol.)



Therefore, H_0 is rejected

Example #15

An industrial plant wants to determine which of 2 types of fuel (gas or electric) will produce more useful energy at the lower cost. Independent random samples of 11 plants using electrical utilities and 16 plants using gas utilities were taken, and the plant investment/quad ratio was calculated for each. (the smaller the ratio, the less the industrial plant pays for its delivered energy). The obtained data are listed in the table.

Electric	Gas
$\bar{x}_1 = 52.43$	$\bar{x}_2 = 37.74$
$s_1 = 62.43$	$s_2 = 49.05$

Is there enough evidence at α=0.05 to indicate a difference in the average investment/quad between all plants using gas and those using electric utilities?

Example #15 (Sol. I)

Given:

$n_1 = 11$ $\quad n_2 = 16$

$\bar{x}_1 = 52.43$ $\quad \bar{x}_2 = 37.74$

$s_1 = 62.43$ $\quad s_2 = 49.05$

$\alpha = 0.05$

σ_1 : unknown $\quad \sigma_2$: unknown

Assumption: $\sigma_1 = \sigma_2$

Hypothesis: $H_0 : \mu_1 - \mu_2 = D_0 = 0$ $\quad H_1 : \mu_1 - \mu_2 \neq 0$

What to Use (Test statistic):

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - D_0}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

with:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

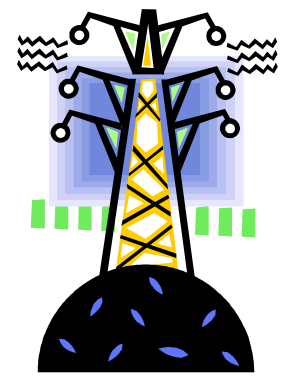
df = n_1+n_2-2

Example #15 (Sol. I)

$$s_p^2 = \frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2} = \frac{(11-1)*(62.43)^2+(16-1)*(49.05)^2}{11+16-2}$$

$$s_p^2 = 3002.543 \Rightarrow s_p = 54.795$$

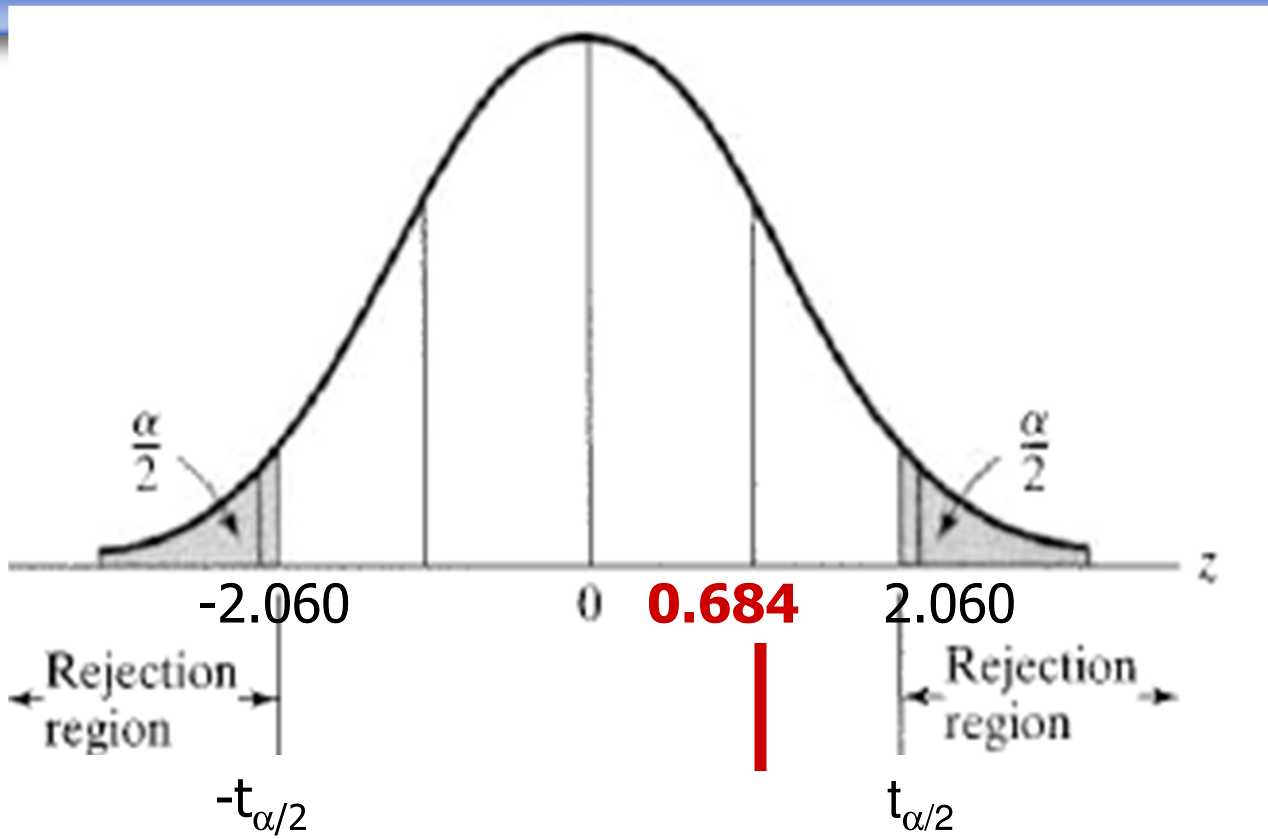
$$T = \frac{\left(\overline{X}_1-\overline{X}_2\right)-D_0}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{(52.43-37.74)-0}{54.795\sqrt{\frac{1}{11}+\frac{1}{16}}} = 0.684$$



Rejection Region: $|T| > t_{\alpha/2}$

$\alpha = 0.05 \Rightarrow \alpha/2 = 0.025 \Rightarrow t_{\alpha/2} = 2.060$ (df = 11+16-2=25)

Example #15 (Sol. I)



Therefore, H_0 is not rejected; there is not enough evidence, at α=0.05, to conclude that the mean investment/quad levels for those plants with electric and gas utilities are different.

Example #15 (Sol. II)

Given:

$n_1 = 11$, $n_2 = 16$

$\bar{x}_1 = 52.43$, $\bar{x}_2 = 37.74$

$s_1 = 62.43$, $s_2 = 49.05$

$\alpha = 0.05$

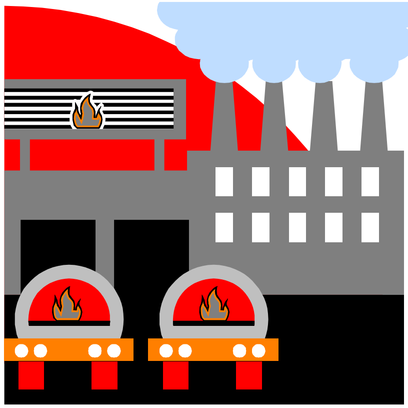
σ_1 : unknown, σ_2 : unknown

Assumption: $\sigma_1 \neq \sigma_2$

Hypothesis:

$H_0 : \mu_1 - \mu_2 = D_0 = 0$

$H_1 : \mu_1 - \mu_2 \neq 0$



What to Use (Test statistic):

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

with:

$$df = [s_1^2/n_1 + s_2^2/n_2]^2 \Bigg/ \left[\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}\right]$$

Example #15 (Sol. II)

$$T = \frac{(\overline{X}_1 - \overline{X}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{(52.43 - 37.74) - 0}{\sqrt{\frac{(62.43)^2}{11} + \frac{(49.05)^2}{16}}} = 0.654$$

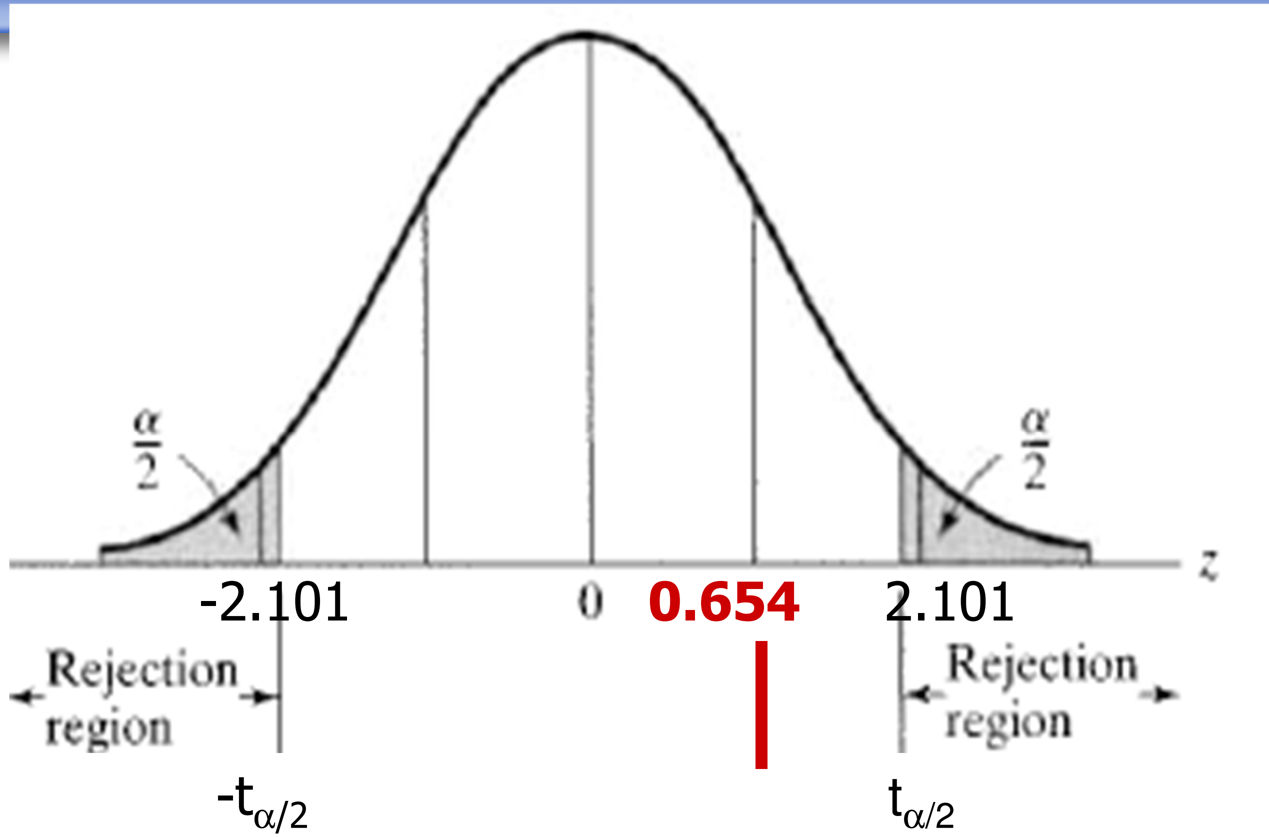
$$df = [s_1^2/n_1 + s_2^2/n_2]^2 \Bigg/ \left[\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}\right] =$$

$$\left[\frac{(62.43)^2}{11} + \frac{(49.05)^2}{16}\right]^2 \Bigg/ \left[\frac{((62.43)^2/11)^2}{11-1} + \frac{((49.05)^2/16)^2}{16-1}\right] = 18.11 \Rightarrow 18$$

Rejection Region: $|T| > t_{\alpha/2}$

$\alpha = 0.05 \Rightarrow \alpha/2 = 0.025 \Rightarrow t_{\alpha/2} = 2.101$ (df = 18)

Example #15 (Sol. II)



H_0 is still not rejected; there is not enough evidence, at α=0.05, to conclude that the mean investment/quad levels for those plants with electric and gas utilities are different.

Example #16

For a sample of 15 adult Europeans picked at random, the mean weight was 154 lb, whereas for a sample of 18 adults in the United States the mean weight was 162 lb.

From past surveys it is known that the variance of weight in Europe is 100 lb^2 and in the United States it is 169 lb^2.

Is it true that there is a significant difference between weights in the two places?

Use $\alpha = 0.05$ (assume that the weights are normally distributed)

Example #16 (Sol.)

Given:

$\bar{x}_1 = 154$ $\quad n_1 = 15$ $\quad \sigma_1^2 = 100$

$\bar{x}_2 = 162$ $\quad n_2 = 18$ $\quad \sigma_2^2 = 169$

$\alpha = 0.05$

Hypothesis: $H_0 : \mu_1 - \mu_2 = D_0 = 0$ $\qquad H_1 : \mu_1 - \mu_2 \neq 0$

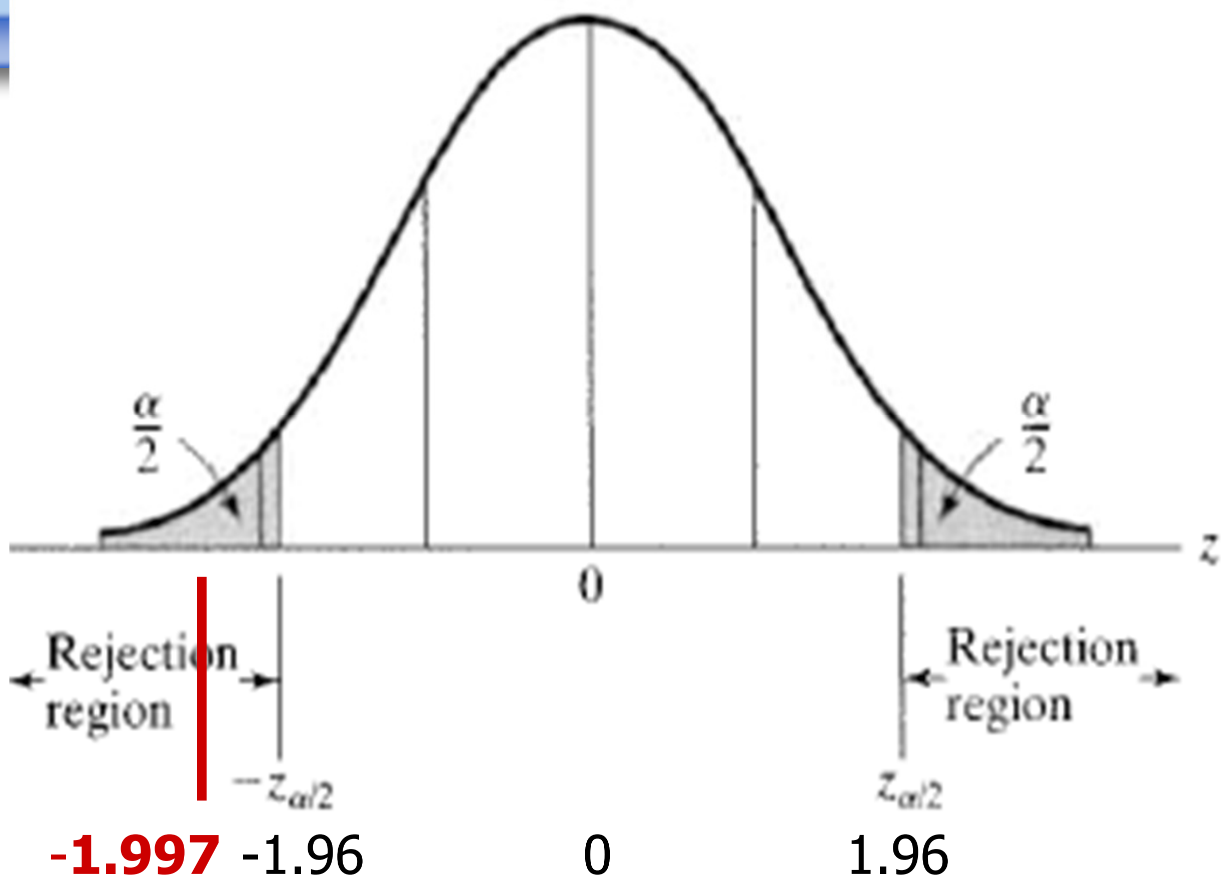
What to Use (Test statistic):

$$Z = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$Z = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{(154 - 162) - 0}{\sqrt{\frac{100}{15} + \frac{169}{18}}} = -1.997$$

$\alpha = 0.05 \Rightarrow \alpha/2 = 0.025 \Rightarrow$ $z_{\alpha/2} = 1.96$

Example #16 (Sol.)



Therefore, H_0 is rejected; there is an evidence, at α=0.05, to conclude that there is a difference between weights in the two places

Example #17

A nitrogen fertilizer was used on 10 plots with a mean yield per plot of 82.5 bushels with a standard deviation of 10 bushels. On the other hand, 15 plots treated with phosphate fertilizer gave a mean yield of 90.5 bushels per plot with a standard deviation of 20 bushels.

At the 5% level of significance, are the two fertilizers significantly different?

(Assume that the ratio of population variances is equal to 1.)

Example #17 (Sol.)

Given:

$$n_1 = 10 \qquad n_2 = 15$$

$$\overline{x}_1 = 82.5 \qquad \overline{x}_2 = 90.5$$

$$s_1 = 10 \qquad s_2 = 20$$

$$\alpha = 0.05$$

$$\sigma_1 = \sigma_2$$

Hypothesis:

$$H_0 : \mu_1 - \mu_2 = D_0 = 0$$

$$H_1 : \mu_1 - \mu_2 \neq 0$$

What to Use (Test statistic):

$$T = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - D_0}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

with:

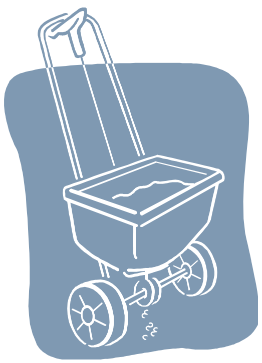
$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

df = $n_1 + n_2 - 2$

Example #17 (Sol.)

$$s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} = \frac{(10-1)*(10)^2 + (15-1)*(20)^2}{10+15-2}$$

$$s_p^2 = 282.608 \Rightarrow s_p = 16.811$$

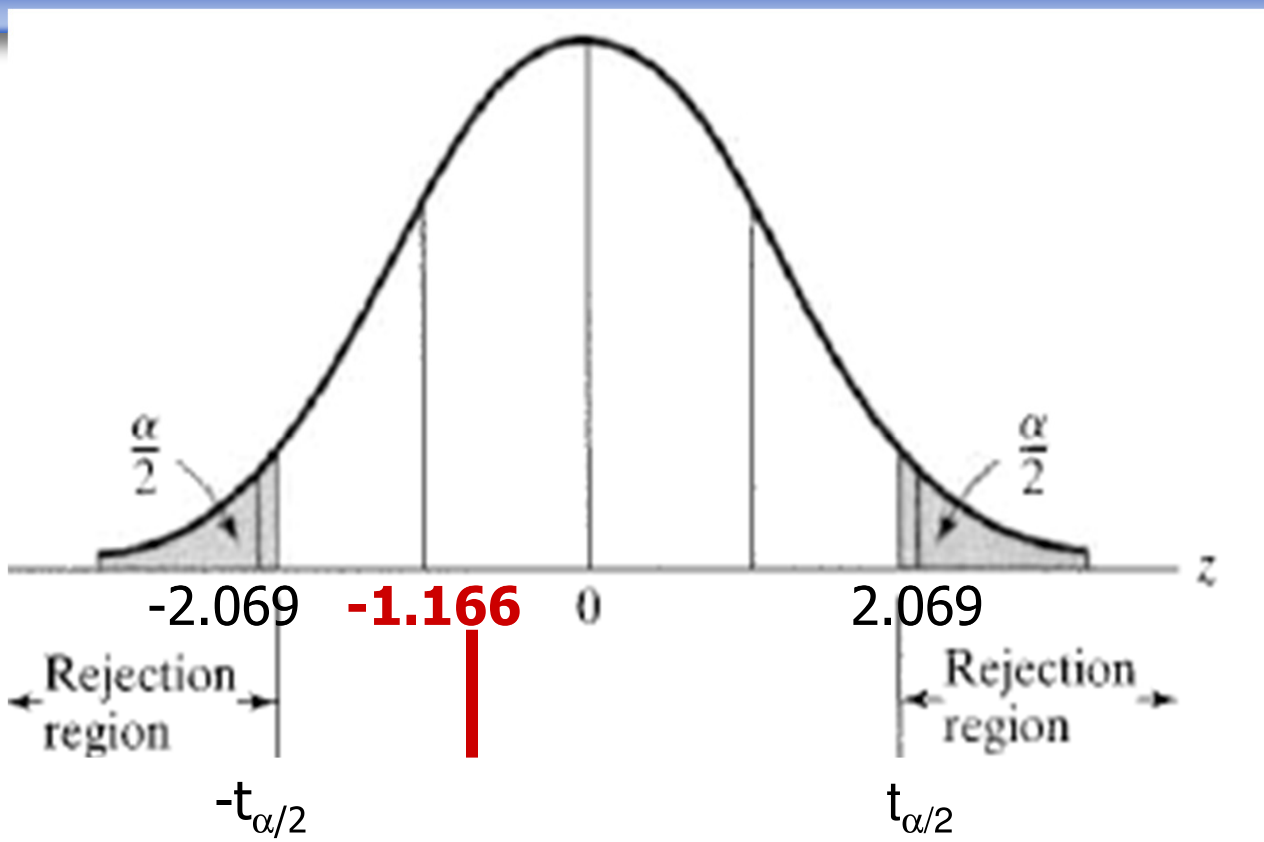


$$T = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - D_0}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{(82.5-90.5)-0}{16.811\sqrt{\frac{1}{10}+\frac{1}{15}}} = -1.166$$

Rejection Region: $|T| > t_{\alpha/2}$

$\alpha = 0.05 \Rightarrow \alpha/2 = 0.025 \Rightarrow t_{\alpha/2} = 2.069$ (df = 10+15-2=23)

Example #17 (Sol.)



H_0 is not rejected; there is not enough evidence, at α=0.05, to conclude that the two fertilizers are significantly different.

Example #18

It is desired to test the claim that a steady diet of wolfbane will cause a lycanthrope (werewolf) to lose 10 lb over 5 months.

A random sample of 49 lycanthropes was taken yielding an average weight loss over 5 months of 12.5 lb, with a variance of 49 lb^2.

Is there enough evidence at 2% significance level to back up that claim?

Example #18 (Sol.)

Given: $\mu_0 = 10$ $\quad \bar{x} = 12.5$ $\quad s^2 = 49$ $\quad n = 49$ $\quad \alpha = 0.02$

Hypothesis: $H_0 : \mu = \mu_0 = 10$ $\quad H_1 : \mu \neq 10$

What to Use (Test statistic):

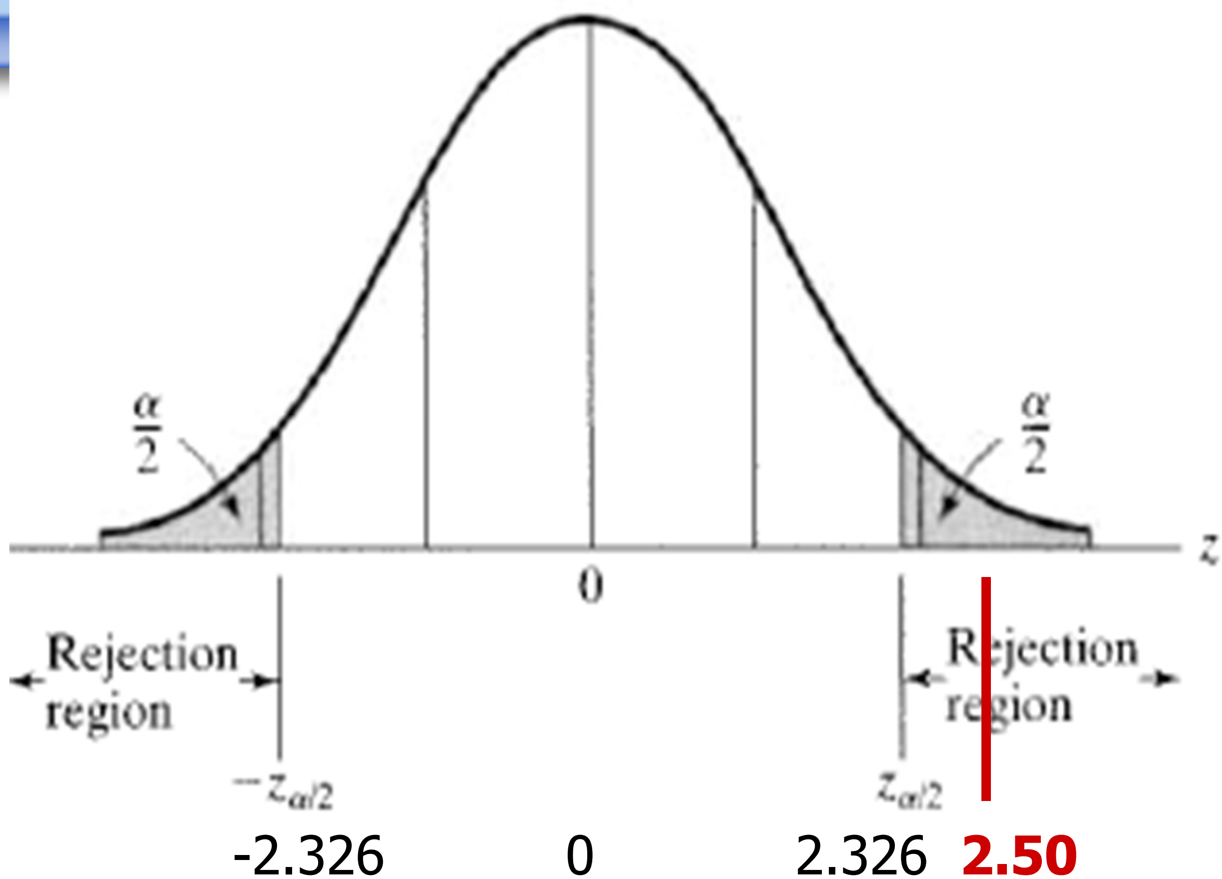
$$Z = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

$$Z = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{12.5 - 10}{\sqrt{49}/\sqrt{49}} = 2.50$$

Rejection Region: $Z > z_{\alpha/2}$

$\alpha = 0.02 \rightarrow \alpha/2 = 0.01 \rightarrow z_{\alpha/2} \approx 2.326$

Example #18 (Sol.)



Therefore, H_0 is rejected; there is an evidence, at α=0.02, to conclude that a steady diet of wolfbane will not cause a lycanthrope (werewolf) to lose "**exactly**" 10 lbs over 5 months

Hypothesis Testing on the Population Variance

1. Sample of size n selected at random from an approximately normal population.

2. **Hypotheses:**

One-tailed Test	Two-tailed Test
H_0: $\sigma^2 = \sigma_0^2$	H_0: $\sigma^2 = \sigma_0^2$
H_1: $\sigma^2 > \sigma_0^2$ (**or** $\sigma^2 < \sigma_0^2$)	H_1: $\sigma^2 \neq \sigma_0^2$

3. **Test Statistic:**

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

4. **Rejection Region:** using significance level α

One-tailed Test	Two-tailed Test
$\chi^2 > \chi_\alpha^2$ or $\chi^2 < \chi^2_{1-\alpha}$	$\chi^2 < \chi^2_{1-\alpha/2}$ or $\chi^2 > \chi^2_{\alpha/2}$

where χ_α^2 and $\chi_{1-\alpha}^2$ are values of χ^2 that locate an area of α to the right and to the left, respectively, of a chi-squared distribution based on df = n-1.

Hypothesis Testing on the Ratio of Two Population Variances

1. Samples of size n_1 and n_2 selected at random and independently from 2 approximately normal populations.

2. **Hypotheses:**

One-tailed Test

$$H_0: \sigma_1^2/\sigma_2^2 = 1$$
$$H_1: \sigma_1^2/\sigma_2^2 > 1 \text{ (or } < 1)$$

Two-tailed Test

$$H_0: \sigma_1^2/\sigma_2^2 = 1$$
$$H_1: \sigma_1^2/\sigma_2^2 \neq 1$$

3. **Test Statistic:**

$$F = \frac{\text{larger sample variance}}{\text{smaller sample variance}} \Rightarrow F = \frac{s_1^2}{s_2^2} \text{ if } s_1^2 > s_2^2 \text{ or } F = \frac{s_2^2}{s_1^2} \text{ if } s_1^2 < s_2^2$$

4. **Rejection Region:** using significance level α

One-tailed Test

$F > f_\alpha (df_1, df_2)$

Two-tailed Test

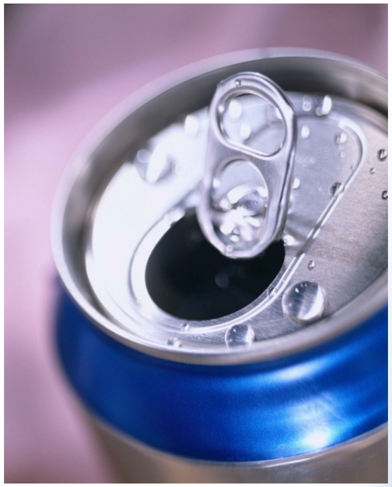
$F > f_{\alpha/2} (df_1, df_2)$

where f_α and $f_{\alpha/2}$ are values locating an area of α and $\alpha/2$, respectively, in the upper tail of the F-distribution with numerator df_1= df for the sample variance in the numerator and denominator df_2 = df for the sample variance in the denominator

Example #19

Regulatory agencies specify that the standard deviation of the amount of fill should be less than 0.1 ounce. The quality control supervisor sampled 10 cans and measured the amount of fill in each. He found that the sample mean was 7.989 ounce and that the sample standard deviation was 0.043 ounce.

Does this information provide sufficient evidence to indicate that the standard deviation of the fill measurements is less than 0.1 ounce? (use α=0.05 and assume normal population)

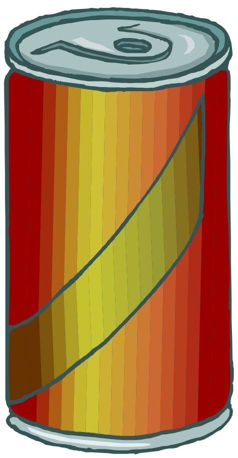


Example # 19 (Sol.)

Given: $\sigma_0 = 0.10$ $\bar{x} = 7.989$ $s = 0.043$ $n = 10$ $\alpha = 0.05$

Hypothesis: $H_0 : \sigma = \sigma_0 = 0.10$ $H_1 : \sigma < 0.10$

What to Use (Test statistic): $\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$

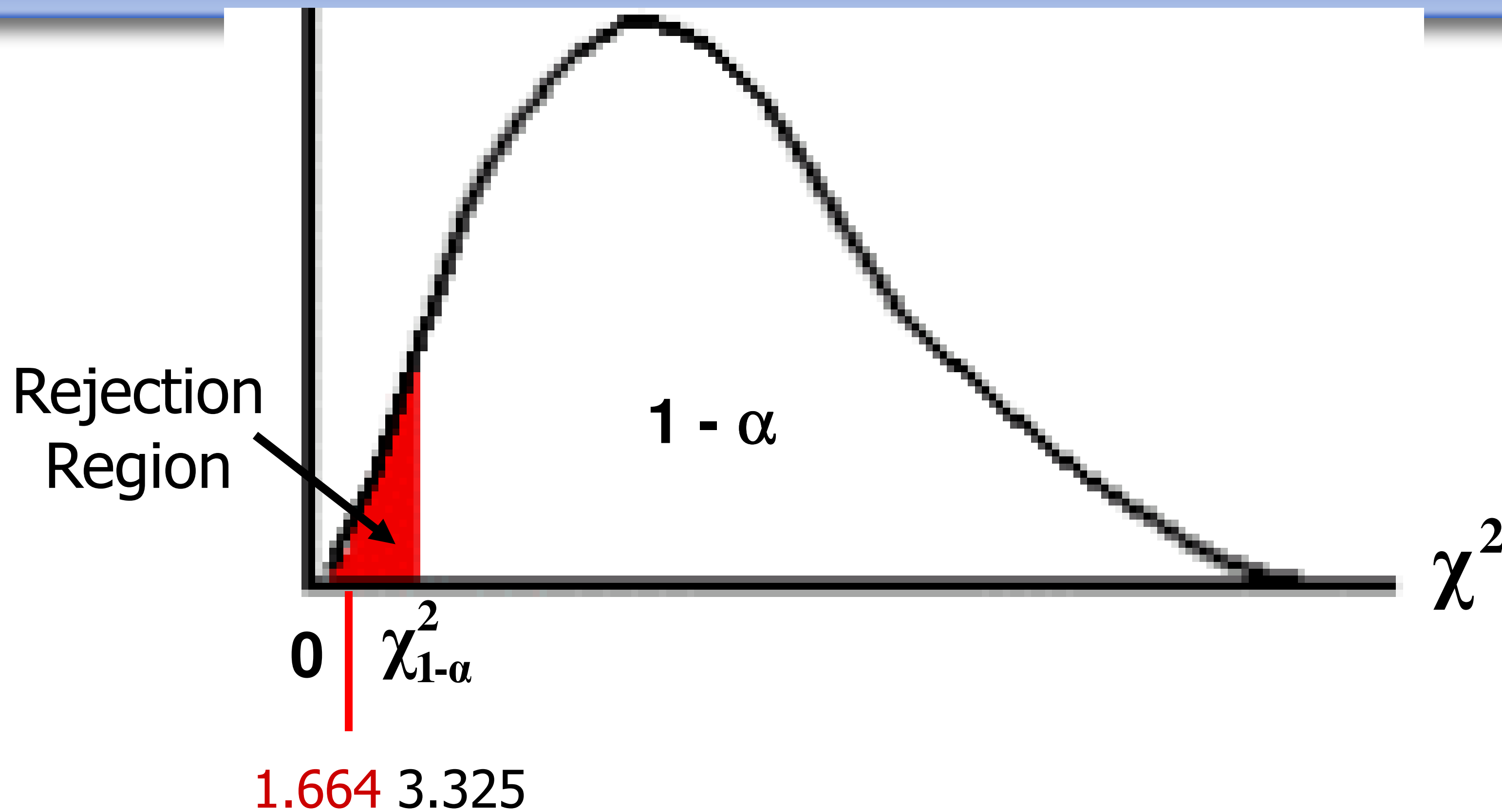


$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(10-1) * 0.043^2}{0.10^2} = 1.664$$

Rejection Region: $\chi^2 < \chi^2_{1-\alpha}$

$\alpha = 0.05 \Rightarrow (1-\alpha) = 0.95 \Rightarrow$ $\chi^2_{1-\alpha} = 3.325$ (df = 10-1 = 9)

Example # 19 (Sol.)



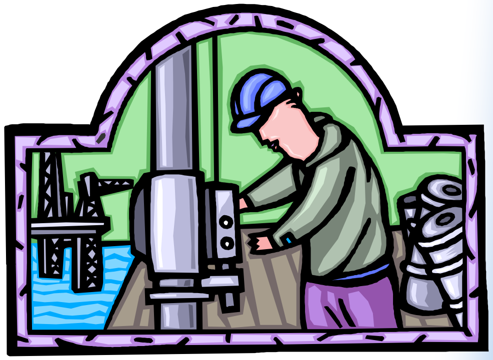
Therefore, H_0 is rejected; there is a sufficient evidence at α=0.05 to indicate that the standard deviation σ of the fill measurements is less than 0.10 ounce

Example #20

"Deep hole" drilling is a family of drilling process used when the ratio of hole depth to hole diameter exceeds 10. Successful deep hole drilling depends on the satisfactory discharge of the drill chip. An experiment was conducted to investigate the performance of deep hole drilling when chip congestion exists (*Journal of Eng. For Industry, May 1993*). The length of 51* drill chips resulted in the following summary statistics: $\bar{x} = 81.2$ mm and S = 50.2 mm.

Test to determine whether the true standard deviation of drill chip lengths differs from 75 mm. (use a 1% significance level and assume normal population)

*Original number is 50



Example #20 (Sol.)

Given: $\sigma_0 = 75$ $\bar{x} = 81.2$ $S = 50.2$ $n = 51$ $\alpha = 0.01$

Hypothesis: $H_0 : \sigma = \sigma_0 = 75$ $H_1 : \sigma \neq 75$

What to Use (Test statistic): $\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(51-1)*50.2^2}{75^2} = 22.400$$

Rejection Region: $\chi^2 < \chi^2_{1-\alpha/2}$ or $\chi^2 > \chi^2_{\alpha/2}$

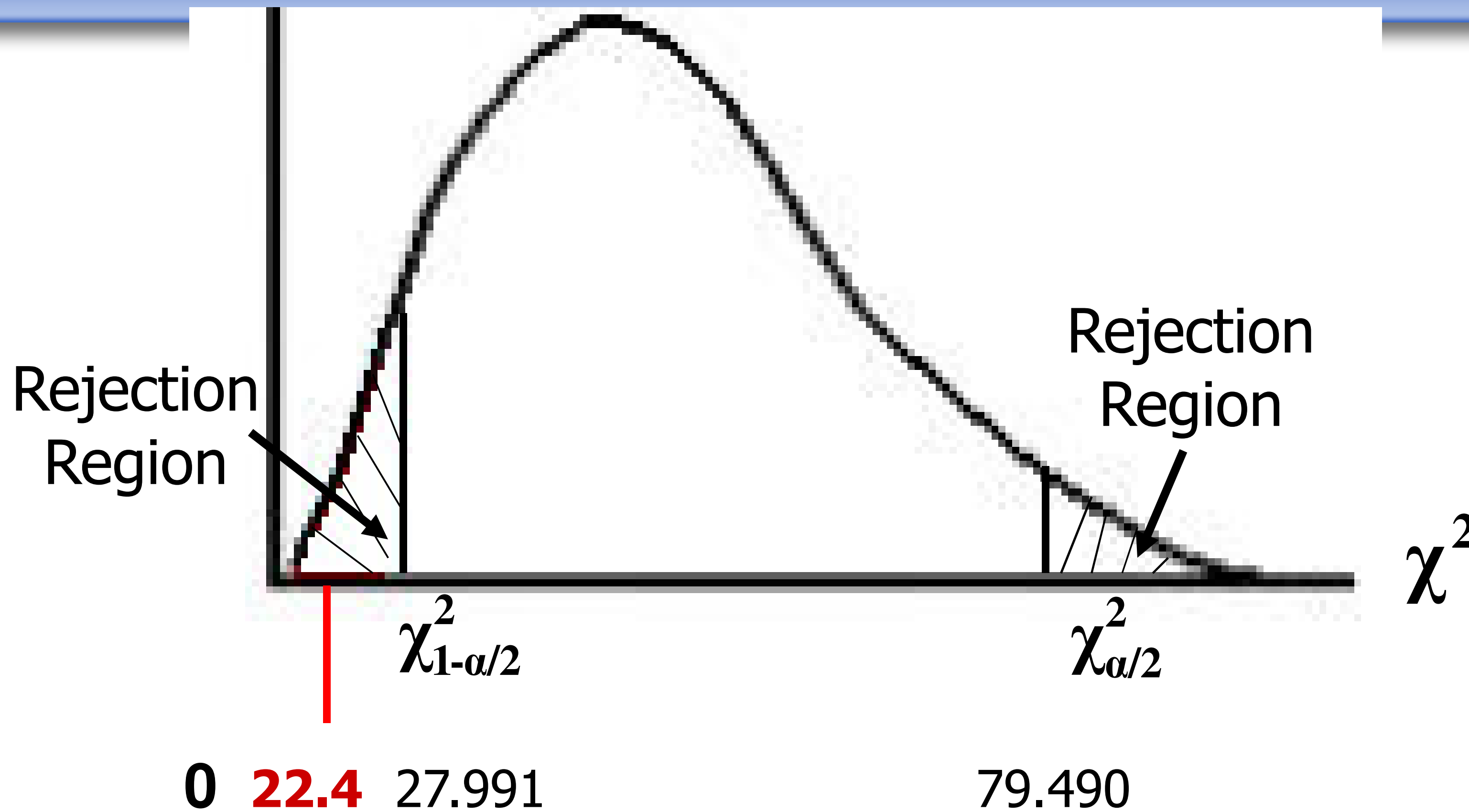
$\alpha = 0.01 \Rightarrow \alpha/2 = 0.005 \Rightarrow$

$\chi^2_{0.005} = 79.490$

$\chi^2_{0.995} = 27.991$

(df = 51-1 = 50)

Example #20 (Sol.)



Therefore, H_0 is rejected; there is a sufficient evidence at α=0.01 to determine that the true standard deviation of drill chip lengths differs from 75 mm

Example #21 (1/2)

In Ireland, the majority of commercial forests are located in remote areas on predominantly peat soils. These roads exhibit rapid deterioration when traversed by logging vehicles or other heavy machinery. A study of forest access roads in Ireland was published in the *Int. Journal of Forest Eng.* (July 1999). One measure of the strength of pavement is transient surface deflection (the higher the surface deflection, the weaker the pavement). The type of pavement (mineral or peat subgrade) was determined for a sample of 72 forest access roads, then each was analyzed for surface deflection. The measures are summarized in the given Table:

Example #21 (2/2)

	Pavement Subgrade	
	Mineral	Peat
Number of roads	31*	41*
Mean surface deflection (mm)	1.53	3.80
Standard Deviation	3.39	14.3

Compare the surface deflection variances of the two pavement types with a two-tailed test of hypothesis using $\alpha=0.05$.

*Original numbers are 32 and 40

Example #21 (Sol.)

Given:

$n_1 = 31$
$n_2 = 41$

$\bar{x}_1 = 1.53$
$\bar{x}_2 = 3.80$

$s_1 = 3.39$
$s_2 = 14.3$

$\alpha = 0.05$

Hypothesis:

$H_0 : \sigma_1^2 = \sigma_2^2$

$H_1 : \sigma_1^2 \neq \sigma_2^2$

(Test statistic):

$$F = \frac{s_1^2}{s_2^2} \quad \text{if } s_1^2 > s_2^2 \quad \text{or} \quad F = \frac{s_2^2}{s_1^2} \quad \text{if } s_1^2 < s_2^2$$

$$F = \frac{s_2^2}{s_1^2} = \frac{14.3^2}{3.39^2} = 17.79$$

Rejection Region: $F > f_{\alpha/2}$

$\alpha = 0.05 \Rightarrow \alpha/2 = 0.025 \Rightarrow$ $f_{0.025} = 2.01$

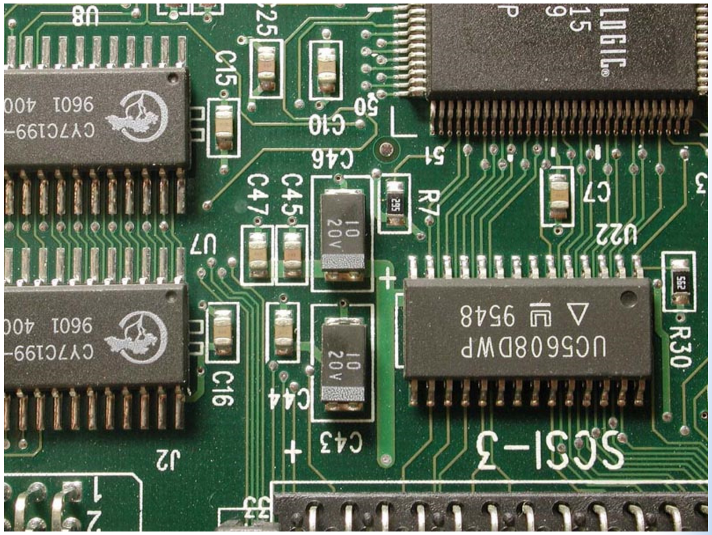
Remember:
($df_1 = 40$, $df_2 = 30$)

H_0 is rejected; there is a sufficient evidence at α=0.05 to determine that the variances of the two pavement types are not equal

Example #22

Chemical etching is used to remove copper from printed circuit boards. Two process yields were used with different concentrations where $n_1 = n_2 = 8$ and $s_1^2 = 4.02$, $s_2^2 = 3.89$.

Is there enough evidence to indicate that $\sigma_1^2 \neq \sigma_2^2$ at 5% significance level?



Example #22 (Sol.)

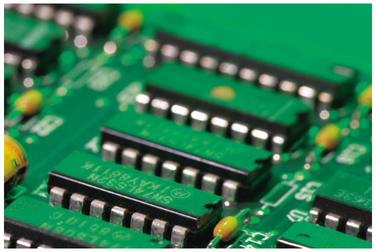
Given: $n_1 = 8$, $n_2 = 8$ $\quad s_1^2 = 4.02$, $s_2^2 = 3.89$ $\quad \alpha = 0.05$

Hypothesis: $H_0 : \sigma_1^2 = \sigma_2^2$ $\quad H_1 : \sigma_1^2 \neq \sigma_2^2$

(Test statistic):

$$F = \frac{s_1^2}{s_2^2} \text{ if } s_1^2 > s_2^2 \quad \text{or} \quad F = \frac{s_2^2}{s_1^2} \text{ if } s_1^2 < s_2^2$$

$$F = \frac{s_1^2}{s_2^2} = \frac{4.02}{3.89} = 1.03$$



Rejection Region: $F > f_{\alpha/2}$

$\alpha = 0.05 \Rightarrow \alpha/2 = 0.025 \Rightarrow f_{0.025} = 4.99$ $\quad (df_1 = 7, df_2 = 7)$

H_0 is not rejected; there is no sufficient evidence at α=0.05 to say that the variance of the yield is affected by the concentration

Goodness-of-Fit Test

- Test to see how good a fit we have between the frequency of occurrence of observations, **o_i** ,in an observed sample of **k** cell frequencies and the expected frequencies, **e_i** , obtained from the hypothesized distribution.

- **Test statistic:**

$$\chi^2 = \sum_{i=1}^{k} \frac{(o_i - e_i)^2}{e_i}$$

- **Rejection Region:** using significance level α:

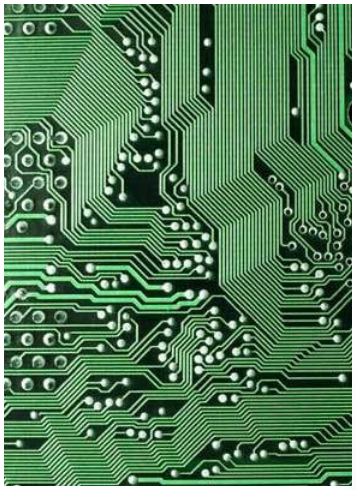
Reject H_0 if $\chi^2 > \chi_\alpha^2$ with k-1 df

Note:
If the expected frequencies are less than **5**, collapse the classes where necessary.

Example #23

The number of defects in printed circuit boards is hypothesized to follow a Poisson distribution. A random sample of n = 60 printed boards have been collected and the number of defects were observed (see table below).

Do we have enough information to back up that claim at 0.05 significance level?



Number of defects	Observed Frequency	Expected Frequency
0	32	28.32
1	15	21.24
2	9	7.98
3	4	1.98

=13

=9.96

Example #23 (Sol.)

i	o_i	e_i	$o_i - e_i$
1	32	28.32	3.68
2	15	21.24	-6.24
3	13	9.96	3.04

Test Statistic

$$\chi^2 = \sum_{i=1}^{3} \frac{(o_i - e_i)^2}{e_i} = \frac{3.68^2}{28.32} + \frac{(-6.24)^2}{21.24} + \frac{3.04^2}{9.96}$$

$$\Rightarrow \quad \chi^2 = 3.239$$

k=3

Rejection Region :

$$\chi^2 > \chi^2_\alpha$$

$$\alpha = 0.05 \Rightarrow \chi^2_{\alpha=0.05\ (df=2)} = 5.991$$

Hence, H_0 is not rejected

Textbook Sections

- 10.1
- 10.2
- 10.4
- 10.5
- 10.6
- 10.10
- 10.11