MT 2015 Solutions

1 (i.) If 1st, 3rd, 5th are red : # ways $= \frac{10!}{4!6!}$

$= 210$

If 1st, 3rd, 5th are blue : # ways $= \frac{10!}{7!3!}$

$= 120$

Total # ways $= 210 + 120 = 330.$

(ii) # ways $= \binom{28}{0}\binom{33}{12} + \binom{28}{1}\binom{33}{11} + \binom{28}{2}\binom{33}{10}$

Total # ways $= \binom{61}{12}$

Prob $= \frac{40761918600}{1742058970275} = 0.023$

(iii) A = high paying job , B = university degree

$P(A \mid B) = 0.6, \quad P(A \mid B') = 0.2$

$P(A) = 0.2$ 0.3

$\underbrace{P(A)}_{=0.3} = \underbrace{P(A \mid B)}_{=0.6} P(B) + \underbrace{P(A \mid B')}_{=0.2}(1 - P(B))$

$\Rightarrow P(B) = 0.25$

$P(AB) = P(A \mid B)\, P(B) = 0.6 \times 0.25 = 0.15$

(iv) $P(\text{correct}) = \frac{1}{5}$

$P(x \geq 2) = 1 - \binom{4}{0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^4 - \binom{4}{1}\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^3$

$= 0.18$

2(a) $P(x > 2 \text{ years}) = \exp\left[-\left(\frac{2}{2.5}\right)^{1.8}\right] = \underbrace{0.512}_{\text{Reliability at 2 years}}$

comput., sensors

$$P(x \geq 3 \text{ working}) = \binom{4}{3} 0.512^3 0.488^1$$

$$+ \binom{4}{4} 0.512^4 = 0.331$$

(b) Reliability of the device is worse than the sensor's

$R = 0.512$ as before for the sensor.

$$P(x \geq 3) = 1 - \binom{N}{0} 0.512^0 0.488^N$$

$$- \binom{N}{1} 0.512^1 0.488^{N-1}$$

$$- \binom{N}{2} 0.512^2 0.488^{N-2} = 0.75$$

$$0.75 = 1 - 0.488^N - 0.512 \times 0.488^{N-1} N$$

$$- 0.262 \frac{N(N-1)}{2} 0.488^{N-2}$$

Solve it numerically,

N	Prob
4	0.33
5	0.52
6	0.68
7	0.79

Thus, 7 sensors have to be installed.

(c) Reliability at device's average lifetime $= \exp\left[-\left(\dfrac{2.23}{2.5}\right)^{1.8}\right]$

$= 0.443$

Prob of failure $= 1 - 0.443 = 0.557$

We use normal to approximate binomial.

$\mu = np = 125{,}000 \times 0.557 = 69{,}625$

$\sigma^2 = np(1-p) = 30{,}844$

$$P(x < 35{,}000) = P\left(z < \frac{35{,}000 + 0.5 - 69{,}625}{\sqrt{30{,}844}}\right)$$

$$= P(z < -197.2)$$

$$= \phi(-197.2)$$

$$= 0.$$

Note: Q2 is written using notations β and δ. In Walpole textbook, Weibull distributions is written using β and α notation.

$P(\cancel{x > 35{,}000})$ ~~The relationship between~~ ~~the notations:~~

Note: Using notations of Walpole, $\beta = 1.8$ and $\alpha = 0.19218$ (will be given)

[Note: $\alpha = 0.19218$ yrs is equivalent to $\delta = 2.5$ yrs, just different notation].

2(a) $P(X > 2 \text{ yrs}) = 1 - F(2) = e^{-\alpha x^{\beta}} = e^{-.19218 \times 2^{1.8}} = e^{-0.6692} = 0.512$

2(b) same as before

2(c) $R(2.23) = P(x > 2.23) = 1 - F(2.23) = e^{-\alpha x^{\beta}} = e^{-.19218 \times 2.23^{1.8}}$

$= e^{-0.814} = 0.443.$

3. $P(\text{widget}) = 0.64$

$P(\text{gizmo}) = 0.36$

(a) $$P(\text{gizmo} \geq 3) = 1 - \binom{5}{0} 0.36^0 0.64^5 - \binom{5}{1} 0.36^1 0.64^4 - \binom{5}{2} 0.36^2 0.64^3$$

$$= 0.25$$

(b) $$P(2^{nd}\ \text{gizmo}, x = 5) = \binom{5-1}{2-1} 0.36^3 0.64^2$$

$$= 0.076$$

4. $\lambda^{-1} = 0.5\,m \rightarrow \lambda = 2/\text{meter}$

(a) $$P(x > \lambda^{-1}) = \int_{\lambda^{-1}}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \left[-e^{-\lambda x}\right]_{x=\lambda^{-1}}^{x=\infty}$$

$$= e^{-1}$$

$$= 0.368..$$

(b) $$P(x > 3, r = 4) = \int_{3}^{\infty} \frac{\lambda^{4} x^{4-1} e^{-\lambda x}}{4!}\,dx$$

$$= 1 - \underbrace{P(\lambda z = 2 \times 3, r = 4)}_{\cancel{0.71}\ 0.849 \text{ Gamma distribution table}}$$

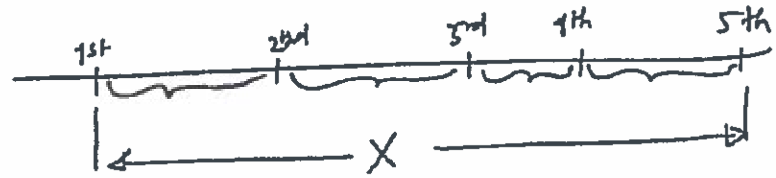
$$= \cancel{0.29}\ 0.151$$

(c) $$P(x \geq 2) = 1 - \frac{\lambda'^{0} e^{-\lambda'}}{0!} - \frac{\lambda' e^{-\lambda'}}{1!}, \quad \lambda' = 2 \times 3$$

$$= 0.983.$$

4(b) Using Notations of Walpole:

Let X = the distance between the 1st and 5th fracture



Each gap between two adjacent fractures is a random variable that follows exponential distribution. Since X is the sum of 4 such gaps, X has a gamma distribution with parameters $\alpha = 4, \lambda = 2$.

$$P(X>3) = 1 - P(X \leq 3) = 1 - \int_0^3 \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx$$

$$= 1 - F(6, \alpha) = 1 - F(6, 4)$$

$$= 1 - 0.849 \text{ (Table A.23)}$$

$$= \underline{\underline{0.151}} \quad \text{Ans}$$

Note:

Alternatively, this probability can also be calculated as $P(Y \leq 3)$ where Y has a Poisson distribution with parameter $2 \times 3 = 6$, which is the average number of fractures in a 3-m length.

$$P(Y \leq 3) = P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3)$$

$$= e^{-6}\left[\frac{6^0}{0!} + \frac{6^1}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!}\right]$$

$$= 61 \times e^{-6} = \underline{\underline{0.151}} \quad \text{Ans}$$