Solution to Quiz 2

Question 1.

Let X be a random variable with the following probability distribution function (pdf)

$$f(x)=\begin{cases} cx^2, |x|\le 1 \\ \\ 0, otherwise \end{cases}$$

What is the value of the constant c?

Solution:

$$\int_{-\infty}^{\infty} f(x)dx = 1 = \int_{-1}^{1} cx^2 dx = \frac{2}{3}c \Leftrightarrow c = \frac{3}{2}$$

Question 2.

Let X be a continuous random variable with the following probability distribution function (pdf)

$$f(x)=\begin{cases} 2(1-x),\ 0<x\le 1 \\ 0, otherwise \end{cases}$$

What is the probability that X will be less than 0.5 given that X is greater than or equal to 0.25?

$$P(X<0.5 \mid X \geq 0.25) = \frac{P(0.25 \leq X < 0.5)}{P(X \geq 0.25)} = \frac{\int_{0.25}^{0.5}(2-2x)dx}{\int_{0.25}^{1}(2-2x)dx}$$

$$= \frac{(2x-x^2)\Big|_{0.25}^{0.5}}{(2x-x^2)\Big|_{0.25}^{1}} = \frac{(1-0.5^2)-(0.5-0.25^2)}{1-(0.5-0.25^2)} = \frac{0.3125}{0.5625} = \boxed{0.556}$$

Alternatively, $P(X<0.5 \mid X \geq 0.25) = \frac{P(0.25 \leq X < 0.5)}{P(X \geq 0.25)} = \frac{F(0.5)-F(0.25)}{1-F(0.25)} = 0.556.$

Where, $F(0.5) = \int_{-\infty}^{0.5} f(x)dx = \int_{0}^{0.5}(2-2x)dx = 0.75$, $F(0.25) = \int_{0}^{0.25}(2-2x)dx = 0.4375$

Question 3. A shipment from SAMSUNG contains 20 similar televisions (TVs). The shipment contains 3 TVs that are defective. A customer makes a random purchase of 2 TVs from the shipment. Let X be a random variable whose values are the possible numbers of defective televisions purchased by the customer. What is the variance of X?

$$f(0) = P(X=1) = \frac{\binom{3}{0}\binom{17}{2}}{\binom{20}{2}} = \frac{136}{190}$$

$$f(1) = P(X = 1) = \frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}} = \frac{51}{190}$$

$$f(2) = P(X = 2) = \frac{\binom{3}{2}\binom{17}{0}}{\binom{20}{2}} = \frac{3}{190}$$

$$\mu = E(X) = \sum_x xf(x) = (0)(\frac{136}{190}) + (1)(\frac{51}{190}) + (2)(\frac{3}{190}) = \frac{57}{190} = 0.3$$

$$\sigma^2 = \sum_x x^2 f(x) - \mu^2 = E(X^2) - \mu^2$$

$$\sum_x x^2 f(x) = (0)^2(\frac{68}{95}) + (1)^2(\frac{51}{190}) + (2)^2(\frac{3}{190}) = \frac{63}{190}$$

$$\sigma^2 = \frac{63}{190} - \left(\frac{57}{190}\right)^2 = \frac{11970}{36100} - \frac{3249}{36100} = \frac{8721}{36100}$$

$$\sigma = \sqrt{\frac{8721}{36100}} = 0.491$$

Question 4

Suppose X is a random variable with the following cumulative probability density function (cdf)

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.25, & 0 \le x < 1 \\ 0.5, & 1 \le x < 2 \\ 1, x \ge 2 \end{cases}$$

Find the following probability.

$$P(0 < X \le 2) = ?$$

Solution:

$$P(0 < X \le 2) = P(X \le 2) - P(X \le 0) = 1 - 0.25 = 0.75$$

Question 5. For a box of 5 green, 3 black, and 2 red balls, you randomly pick 2 balls without replacement. Green, black, and red balls are worth 1, 2, and 3 points respectively. What is the expected value of your point? Ans: 3.4

Solution:

X : Your point at this game

$x = \{2, 4, 6, 3, 5\}$ = possible draws = {**GG, BB, RR, GR, GB, RB**}

$$P(X=2)=\frac{\binom{5}{2}}{\binom{10}{2}}=0.222$$

$$P(X=4)=\frac{\binom{3}{2}}{\binom{10}{2}}+\frac{\binom{5}{1}\binom{2}{1}}{\binom{10}{2}}=0.29$$

$$P(X=6)=\frac{\binom{2}{2}}{\binom{10}{2}}=0.022$$

$$P(X=3)=\frac{\binom{5}{1}\binom{3}{1}}{\binom{10}{2}}=0.333$$

$$P(X=5)=\frac{\binom{3}{1}\binom{2}{1}}{\binom{10}{2}}=0.133$$

$$E(X)=\mu=\sum xf(x)=[2*0.222+4*0.29+6*0.022+3*0.333+5*0.133]=3.4$$

Question 6. Let X denote the time in milliseconds for the completion of a chemical reaction. The cumulative distribution function of X is

$$F(x)=\begin{cases}0, x<0\\ 1-e^{-0.05x}, x\geq 0\end{cases}$$

The probability that the reaction completes within 40 milliseconds is:

Solution: $P(X \leq 40) = F(40) = 1 - e^{-2} = 0.865$