ENGG 319 (Fall 2016)

Lecture/ Tutorial # 2 (Videos 1-8, Chapter 2, Sections 2.1, 2.2, 2.4, 2.5, 2.6)

Sample space and event

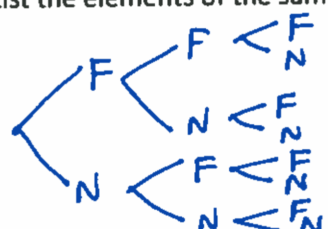
Question # 1. A coin is tossed until a tail or three heads appear. List the elements of the sample space.

H = Head T = Tail

$S = \{T, HT, HHT, HHH\}$

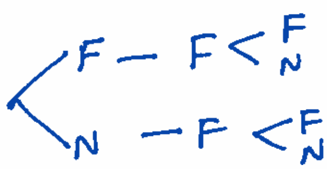
Question # 2. An engineering firm is hired to determine if certain waterways in Virginia are safe for fishing. Samples are taken from three rivers.

(a) List the elements of the sample space S, using the letters F for safe to fish and N for not safe to fish



$S = \{FFF, FFN, FNF, NFF, NFN, NNF, NNN\}$

(b) List the element of S corresponding to the event E that the second river was safe for fishing



$E = \{FFF, FFN, NFF, NFN\}$

Question # 3. Consider the sample space S = {Cu, Na, N, K, U, O, Zn} and the following events

A = {Cu, Na, Zn}

B = {Na, N, K}

C = {O}

List the elements of the sets corresponding to the event (A' U B') ∩ (A' ∩C)

$A' = \{\cancel{Cu}, \cancel{Na}, N, K, U, O, \cancel{Zn}\} = \{N, K, U, O\}$

$B' = \{Cu, U, O, Zn\}$

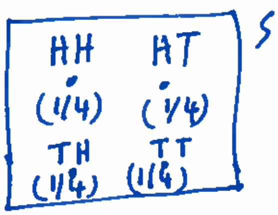
$A' \cup B' = \{N, K, U, O, Cu, Zn\}$

$A' \cap C = \{N, K, U, O\} \cap \{O\} = \{O\}$

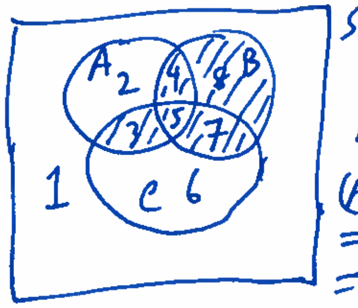
$(A' \cup B) \cap (A' \cap C) = \{O\}$

Question # 4. (a) Draw the sample space of the experiment of tossing two coins in a Venn diagram with the probabilities of each sample point. (b) Let A, B, C be events relative to the sample space S. Using a Venn diagram, shade the areas reprinting the event (A∩C) U B. Assume that A, B, C share few common elements.

Label areas as 1, 2, 3, 4, 5, 6, 7, 8.



Note:
Sample space → Rectangle
" points → dots
Event → Circle



$A \cap C = 3 + 5$

$B = 4 + 5 + 7 + 8$

$(A \cap C) \cup B$
$= 3 + 4 + 5 + 7 + 8$
= Areas labelled as 3, 4, 5, 7, 8.

Probability of an event: application of additive rules

Question # 5. The probability that a Canadian industry will locate in Shanghai, China, is 0.7, the probability that it will locate in Beijing, China is 0.4, and the probability that it will locate in either Shanghai or Beijing or both is 0.8. What is the probability that the industry will locate

(a) in both cities

(b) in neither city (**Ans:** 0.2)

Let, S = Industry will locate in ~~Sa~~ Shanghai
B = " in Beijing

Given, $P(S) = 0.7$, $P(B) = 0.4$, $P(S \cup B) = 0.8$

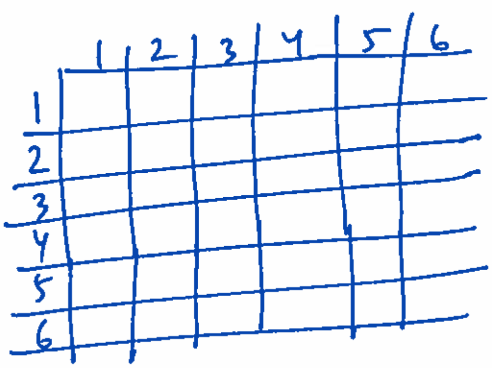
(a) $P(S \cap B) = P(S) + P(B) - P(S \cup B) = 0.7 + 0.4 - 0.8 = 0.3$

(b) $P(S' \cap B') = 1 - P(S \cup B) = 1 - 0.8 = 0.2$

Question # 6. A pair of fair dice is tossed. Find the probability that

(a) the sum of two scores is 6;

(b) Niether die records a 6.



a) Let A = sum of two scores is 6
$= \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$P(A) = \frac{5}{36}$

(b) Let B = At least one die record a 6

$B = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,1), (6,2), (6,3), (6,4), (6,5)\}$

$P(B) = 11/36$

$P(B') = 1 - 11/36 = \frac{25}{36}$.

<u>Question # 7</u>. It is common in many industrial areas to use a filling machine to fill boxes full of product. These machines are not perfect, and indeed they may A, fill to specification, B, underfill, and C, overfill. Let P(B) =0.001 while P(A) =0.99

(a) Give P(C)

(b) What is the probability that the machine does not underfill ?

(c) What is the probability that the machine either overfills or underfills ? **(Ans:** 0.01)

Now suppose 50,000 boxes of detergent are produced per week and suppose also that those underfilled are "sent back," with customers requesting reimbursement of the purchase price. Suppose also that the cost of production is known to be $ 4 per box while the purchase price is $ 4.50 per box.

(d) What is the weekly profit under the condition of no defective boxes?

(e) What is the loss in profit expected due to underfilling ? **(Ans:** $ 225)

(a) A = fill to specification

B = underfill

C = overfill

A, B, C are mutually exclusive and collectively exhaustive.

Then, $P(A) + P(B) + P(C) = 1$

$\Rightarrow P(C) = 1 - P(A) - P(B) = 1 - 0.99 - 0.001 = 0.009$

(b) B and B', these two events are mutually exclusive & collectively exhaustive.

$P(B) + P(B') = 1 \quad \Rightarrow P(B') = 1 - P(B) = 0.999$

(c) either B or C

$P(B \cup C) = P(B) + P(C) - \underbrace{P(B \cap C)}_{\to 0}$ [because B & C are mutually exclusive, $B \cap C = \emptyset$, $P(\emptyset) = 0$

$= P(B) + P(C) = 0.01$

(d) $(\$4.50 - \$4.00) \times 50{,}000 = \$\ 25{,}000$

(e) $P(B) = 0.001$

Then, $50{,}000 \times 0.001 = 50$ boxes are expected to be underfilled.

Loss in profit $= 50 \times \$0.5 + 50 \times \$4.00$

$= \$\ 225$

Probability of an event: Application of conditional probability, independence and product rule

Question # 8. In an experiment to study the relationship of hypertension and smoking habits, the following data are collected for 180 individuals (H = Hypertesnion, NH =Nonhypertension):

	Nonsmokers	Moderate smokers	Heavy smokers	Total
H	21	36	30	87
NH	48	26	19	93
			49	

If one of these individuals is selected at random,
(a) find the probability that the person is experiencing hypertension, given that the person is a heavy smoker. **(Ans: 30/49)**
(b) a nonsmoker, given that the person is experiencing no hypertension. **(Ans: 16/31)**

(a) Let A = person experiencing hypertension
B = " is heavy smoker
C = " is non-smoker

a) P(experiencing hypertension GIVEN heavy smoker) (Using reduced sample space method)
$= P(A|B) = 30/49$ (Using reduced sample space method)
b) $P(C|A') = 48/93 = 16/31$ (Using reduced sample space method)

Question # 9. An extrusion die is used to produce aluminium rods. Specifications are given for the length and the diameter of the rods. For each rod, the length is classified as too short, too long, or OK, and the diameter is classified as too thin, too thick, or OK. In a population of 1000 rods, the number of rods in each class is as follows:

	Diameter			
Length	**Too thin**	**OK**	**Too thick**	Total
Too short	10	3	5	18 = too short = N(S)
OK	38	900	4	
Too long	2	25	13	40
Total	50	928	22	

A rod is sampled at random from this population.

(a)What is the probability that it is too short? **(Ans: 0.018)**
(b) What is the probability that it is either too short or too thick? **(Ans: 0.035)**
(c) Compute the conditional probability P (diameter OK | length too long). Is this the same as the unconditional probability P(diameter OK) ? **(Ans: 0.625, 0.928)**
(d) Find P(too long) and P (too long | too thin). Are the two events 'too long' and 'too thin' independent?

diameter OK = D

Let too short = S, " thick = T, too thin = N, " long = L

a) $P(\text{short}) = \frac{18}{1000} = 0.018$
b) $P(\text{"short" or "thick"}) = P(S \cup T) = P(S) + P(T) - P(S \cap T) = \frac{18}{1000} + \frac{22}{1000} - \frac{5}{1000} = 0.035$

c) $P(D|L) = 25/40 = 0.625$, and $P(D) = 928/1000 = .928$, Not same.
d) $P(L) = \frac{40}{1000} = 0.04$, $P(L|N) = 2/50 = 0.04$
Independent because $P(L) = P(L|N)$.

Question # 10. An electrical system consists of two components as illustrated in the figure below. The system works if either of the components B or C works. The reliability (probability of working of each component is also shown in the figure. What is the probability that the system works? Assume that the component operates (or fails) independently. **(Ans: 0.94)**

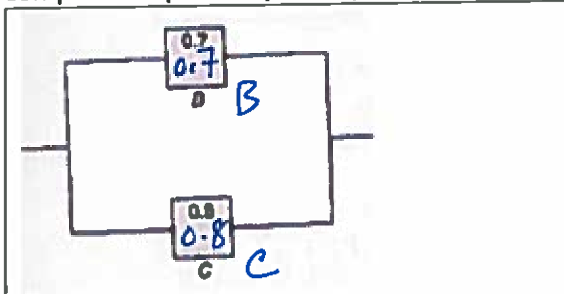


Figure: Circuit for problem 10

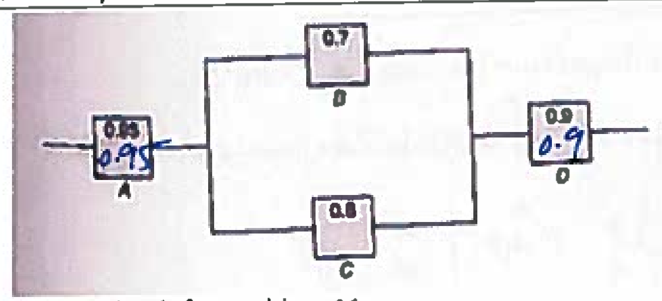


Figure: Circuit for problem 11

$P(\text{system works}) = P(B \text{ or } C) = P(B \cup C) = P(B) + P(C) - P(B \cap C)$

B, & C are independent, then by product rule, $P(B \cap C) = P(B)P(C) = 0.56$

So, $P(B \cup C) = 0.7 + 0.8 - 0.56 = \underline{0.94}$ ← parallel circuit, high probability to function. $P(B \cup C) > P(B)$ or $P(C)$

Question # 11. Consider an extension of the circuit in problem 10 that consists of four components as illustrated in the figure above. The system works if component A and D work and either of the components B or C works. The reliability (probability of working of each component is also shown in the figure. What is the probability that the system works? Assume that the component operates (or fails) independently.**(Ans: 0.8037)**

Let $X = B \cup C$, Then system works if A, X, and D work together.

$P(A \cap X \cap D) = P(A)P(X)P(D)$, because independent.

But, $P(X) = P(\text{either } B \text{ or } C) = P(B \cup C) = P(B) + P(C) - P(B \cap C) = 0.7 + 0.8 - 0.7 \times 0.8 = 0.94$

Then, $P(A \cap X \cap D) = \cancel{0.7}\ 0.95 \times 0.94 \times 0.9 = 0.8037$

Question # 12. Consider a river, and the following three events: A: the river is polluted, B: a sample of water tested detects pollution, and C: fishing is permitted. Given, P(A) =0.3, P(B|A)=0.75, P(C|A∩B')=0.8, P(B'∩C)=0.564. Find the probability that the river is polluted, given that fishing is permitted and the sample of water tested did not detect pollution. **(Ans: 0.1064)**

A | C ∩ B'

$P(A \mid C \cap B')$

$= \dfrac{P(A \cap C \cap B')}{P(C \cap B')} = \dfrac{P(A \cap B' \cap C)}{0.564}$

$P(A \cap B' \cap C) = P(A)\, P(B' \mid A)\, P(C \mid A \cap B') = 0.3 \times 0.25 \times 0.8 = 0.06$

Then, $P(A \mid C \cap B') = 0.06 / .564 = 0.1064$

[~~Note: P(A∩B∩C) = P(A)P(B|A)P(C|A∩B) = P(B)P(B|A)P(C) = P(C)P(B|C)P(C) = P(B)P(C|B)P(A|B∩C) = P(C)P(B|C)P(C~~]

Additional problems (miscellaneous)

Question # 13 Consider the experiment of tossing two balanced coins. Find the probability of observing
(a) exactly one head, and
(b) at least one head (**Ans: ¾**)

$S = \{HH, HT, TH, TT\}$ Let, A = exactly one head $= \{HT, TH\}$

(a) $P(\text{exactly one head}) = P(A) = \frac{2}{4} = \frac{1}{2}$

Let, B = at least one head $= \{HH, HT, TH\}$

$P(B) = 3/4$

Question # 14 Consider the experiment of tossing a fair die, and let A is the event that an even number is observed, and B is the event that a number less than or equal to 4 is observed. Is A and B independent event? [Hint: Calculate and compare P(A) and P(A|B)] (**Ans: Independent**)

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{\text{even number}\} = \{2, 4, 6\}$

$B = \{x \leq 4\} = \{1, 2, 3, 4\}$

$P(A) = 3/6 = \frac{1}{2}$, $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{2,4\})}{P(B)} = \frac{2/6}{4/6} = \frac{1}{2}$

Since, $P(A) = P(A|B)$, they are independent.

Question # 15. The probability that a doctor correctly diagnoses a particular illness is 0.7. Given that the doctor makes an incorrect diagnosis, the probability that the patient files a lawsuit is 0.9. What is the probability that the doctor makes an incorrect diagnosis and the patient sues? (**Ans: 0.27**)

Let A = doctor correctly diagnoses
B = patient sues

Given, $P(A) = 0.7$, $P(B|A') = 0.9$

$P(A' \cap B) = P(A') * P(B|A')$ (product rule)

$= 0.3 \times 0.9$

$= 0.27$.

Question # 16 A town has two fire engines operating independently. The probability that a specific engine is available when needed is 0.96.

(a) What is the probability that neither is available when needed

(b) What is the probability that a fire engine is available when needed (Ans: 0.9984)

Let, A = engine A (first engine) is available when needed

B = second engine is " "

(b) $P(\text{either } A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ✓ independent

$= 0.96 + 0.96 - 0.96 \times 0.96$

$= 0.9984$

(a) $P(\text{Neither } A \text{ nor } B) = P(A' \cap B') = 1 - P(A \cup B)$

$= 1 - 0.9984 = 0.0016$

Also, $P(A' \cap B') = P(A')P(B') = (1 - 0.96)(1 - 0.96) = 0.04 \times 0.04 = 0.0016$.

Question # 17 Assume A is the event that a person is smoker and B is the event that a person develops cancer. Given P(A∩B) = 0.05, P(A∩B') =0.2, P(A'∩B) = 0.03, P(A'∩B') =0.72, Calculate the probability that (a) a smoker develops cancer, and (b) a nonsmoker develops cancer, and (c) Does the conditional probabilities obtained in parts (a) and (b), suggest that there is a link between smoking and cancer? (d) What is the probability that a nonsmoker does not develop cancer? (**Ans**: part (a) =0.2, part (d) =0.96))

Let, A = a person is smoker

B = a " develops cancer

(a) $P(\text{develops cancer} \mid \text{person is smoker}) = P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

Law of total probability: $P(A) = P(A \cap B) + P(A \cap B')$

$= 0.05 + 0.2 = 0.25$

Then, $P(B|A) = \dfrac{0.05}{0.25} = 0.2$

(c) $P(B|A) = 5 * P(B|A')$

Yes, there is a link.

(b) $P(B|A') = \dfrac{P(B \cap A')}{P(A')} = \dfrac{0.03}{1 - 0.25} = 0.04$

(d) $P(B'|A') = \dfrac{P(B' \cap A')}{P(A')} = \dfrac{.72}{.75} = 0.96$

Question # 18. Consider a river, and the following three events: A: the river is polluted, B: a sample of water tested detects pollution, and C: fishing is permitted. Given, P(A) =0.3, P(B|A)=0.75, P(B|A')=0.2, P(C|A∩B)=0.2, P(C|A'∩B)=0.15, P(C|A ∩B')=0.8, and P(C|A'∩B')=0.9. Find

(a) P(A∩B∩C), and (b)The probability that the sample of water did not detect the pollution and fishing is permitted. (**Ans (b):** 0.564)

(a) $P(A \cap B \cap C)$

$= P(A)\, P(B|A)\, P(C|A \cap B) = 0.3 \times 0.75 \times 0.2 = 0.045$

(b) $P(B' \cap C) = ?$ Let $X = B' \cap C$

$P(X) = P(A \cap X) + P(A' \cap X)$ ← theorem of total probability

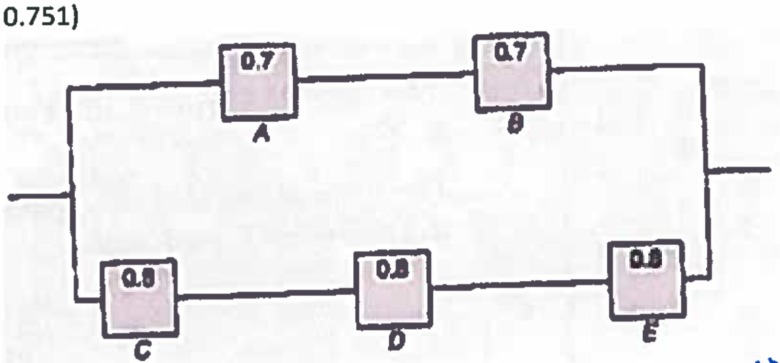
$= P(A \cap B' \cap C) + P(A' \cap B' \cap C)$

$= P(A)\,P(B'|A)\,P(C|A \cap B') + P(A')\,P(B'|A')\,P(C|A' \cap B')$

$= 0.3 \times 0.25 \times 0.8 + 0.7 \times 0.8 \times 0.9$

$= 0.06 + 0.504 = 0.564.$

Question # 19 The following circuit works if at least one of the following two conditions are met (1) top route works, i.e., both A and B works (2) bottom route works, i.e., all of C, D and E works. The components A, B, C, D, E can function or fail independently. What is the probability that the entire system works? **(Ans: 0.751)**



Let X = Top route works i.e., A and B work.

Y = Bottom " i.e., C and D and E work.

$P(X) = P(A \cap B) = P(A)P(B) = 0.7 \times .7 = 0.49$

$P(Y) = P(C \cap D \cap E) = P(C)P(D)P(E) = 0.8 \times 0.8 \times 0.8 = \cancel{0.572}$ 0.512

$P(\text{system works}) = P(\text{either } X \text{ or } Y)$

$= P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$ independent

$= P(X) + P(Y) - P(X)P(Y)$

$= 0.49 + 0.512 - 0.49 \times 0.512$

$= 0.751$ Ans