ENGG 319

Probability & Statistics for Engineers

Section #03

Random Variables
&
Probability Distributions

L01

Dr. Sameh Nassar

F16

Random Variables

- A ***random variable*** is a ***function*** that associates a real number with each element in the sample space.
- We usually use a capital letter, say X, to denote a random variable and its corresponding small letter, i.e. x in this case, for one of its values.

Example #1: Tossing a coin three times

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT }

Use ***X*** to denote the random variable occurrence of number of heads

Use ***x*** to denote the possible values for the occurrence of number of heads, i.e. 0, 1, 2, 3

Example #1

First Outcome	Second Outcome	Third Outcome		x
H	H	H	HHH	3
		T	HHT	2
	T	H	HTH	2
		T	HTT	1
T	H	H	THH	2
		T	THT	1
	T	H	TTH	1
		T	TTT	0

Random Variables

- If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a ***discrete sample space***.

- If a sample space contains an infinite number of possibilities equal to the points on a line segment, it is called a ***continuous sample space***.

Discrete Probability Distributions

- The set of ordered pairs $(x, f(x))$ is a ***probability function***, ***probability mass function***, or ***probability distribution*** of the discrete random variable X if, for each possible outcome x:

  1. $f(x) \geq 0$
  2. $\sum_{x} f(x) = 1$
  3. $P(X = x) = f(x)$

- Thus, a discrete random variable assumes each of its values with a certain probability.

Example #2

- If a coin is flipped three times, determine the following:

(1) The probability distribution of the random variable representing the number of heads.

(2) The probability of the occurrence of 3 heads

Use ***X*** to denote the random variable occurrence of number of heads

Use ***x*** to denote the possible values for the occurrence of number of heads, i.e. 0, 1, 2, 3

This is a discrete sample space with finite number of possibilities

Example #2 (Sol.)

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT }

N = 8

(1)

x	$f(x)$
0	1/8
1	3/8
2	3/8
3	1/8

Note that: $\sum_{x} f(x) = 1$

(2) $P(X = 3) = f(3) = \frac{1}{8}$

Example #3

- If a coin is flipped until a head occurs, show the sample space.

S = {H, TH, TTH, TTTH, TTTTH, TTTTTH, ……. }

This is a discrete sample space with an unending sequence with as many elements as there are whole numbers

(although we may have to continue indefinitely)

Example #4

- The heights of students in a class of 80 are given as follows:

$h < 150$ cm	6
$150 \leq h < 160$ cm	10
$160 \leq h < 170$ cm	26
$170 \leq h < 180$ cm	24
$180 \leq h < 190$ cm	9
$190 \text{ cm} \leq h$	5

We only can find the number of students *within* a given *range* of heights

This is a continuous sample space

Discrete Cumulative Distribution Function

- The cumulative distribution function *F(x)* of a discrete random variable X with probability distribution *f(x)* is:

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t) \qquad \text{for } -\infty < x < \infty$$

Example #5:

- If a coin is tossed three times, determine the following:

(1) The cumulative probability distribution function of the random variable representing the number of heads.

(2) Using the results from (1), verify that the probability distribution of the occurrence of 3 heads is 1/8.

Example #5 (Sol.)

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT }

N = 8

$$F(x) = P(X \le x) = \sum_{t \le x} f(t)$$

(1)

x	***f(x)***	***F(x)***
< 0	0	0
0	1/8	1/8
1	3/8	4/8 = 1/2
2	3/8	7/8
3	1/8	8/8 = 1

Note that:

F(0) = f(0)

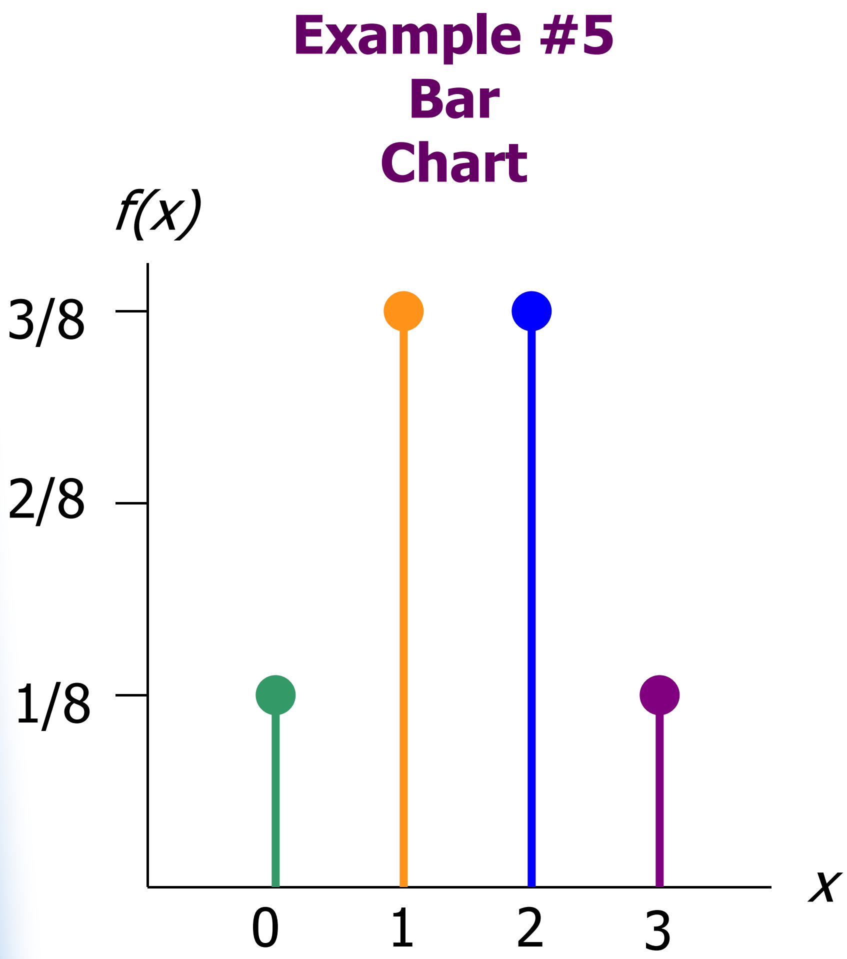
F(1) = f(0) + f(1)

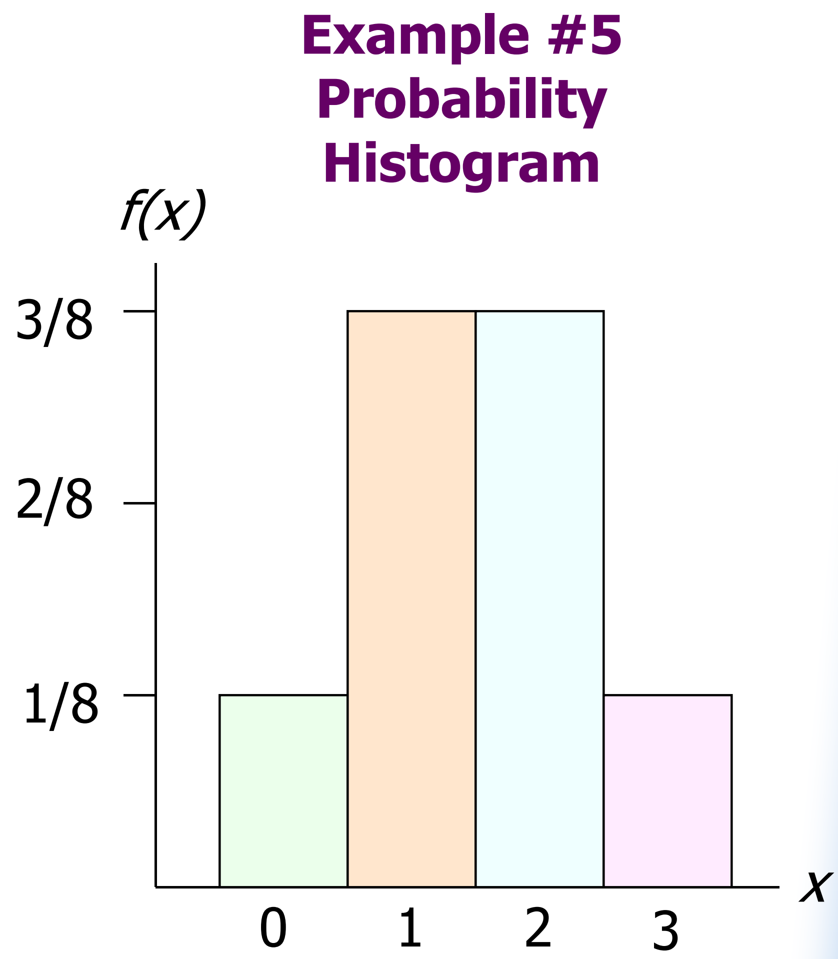
F(2) = f(0) + f(1) + f(2)

F(3) = f(0) + f(1) + f(2) + f(3)

(2) *f(3) = F(3) – F(2)* = 1 - 7/8 = 1/8

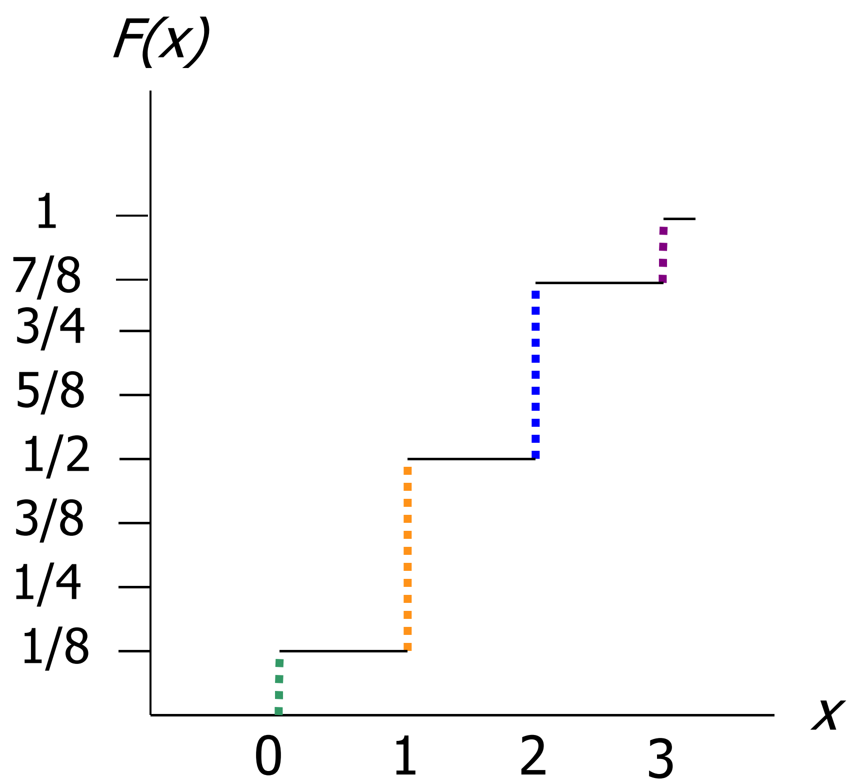
Probability Histogram





Discrete Cumulative Distribution Function

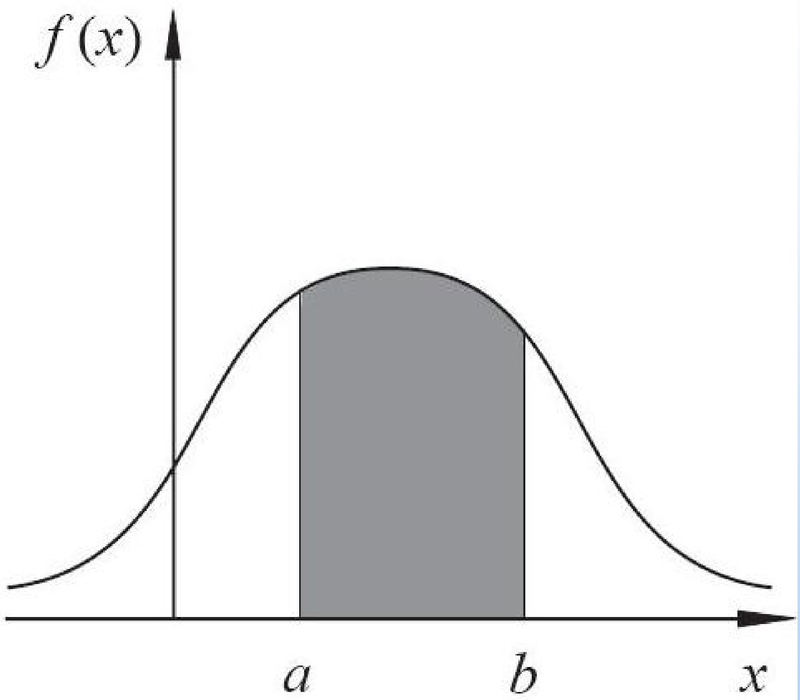
Example #5
Cumulative Distribution Function



Continuous Probability Distributions

- The function f(x) is a ***probability density function*** or ***density function*** for the continuous random variable X, defined over the set of real numbers R, if:

1. $f(x) \geq 0$ for all $x \in R$
2. $\int_{-\infty}^{+\infty} f(x)\,dx = 1$
3. $P(a < X < b) = \int_{a}^{b} f(x)\,dx$

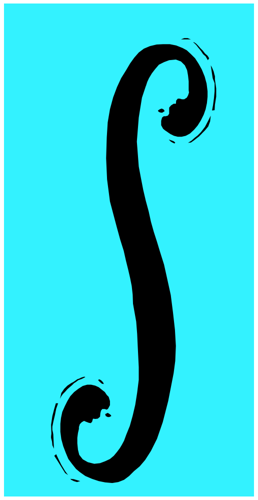


Continuous Cumulative Distribution Function

- Cumulative distribution function *F(x)* of a continuous random variable X with density function *f(x)* is:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

for $-\infty < x < \infty$



Example #6

(Ex. 3.6 Textbook):

- The shelf life, in days, for bottles of a certain prescribed medicine is a random variable having the density function:

$$f(x)=\begin{cases} \dfrac{20{,}000}{(x+100)^3}, & x>0 \\ 0, & \text{elsewhere} \end{cases}$$

- Find the probability that a bottle of this medicine will have a shelf life of:

(a) at least 200 days.

(b) anywhere from 80 to 120 days.

Example 6 (Sol.)

(a) Required: $P(x \geq 200) = ?$

$$P(x \geq 200) = \int_{200}^{\infty} f(x)\,dx$$

$$P(x \geq 200) = \int_{200}^{\infty} \frac{20,000}{(x+100)^3} dx = \left(\frac{20,000}{(x+100)^2} * \frac{1}{-2} \right) \Bigg|_{200}^{\infty}$$

$$P(x \geq 200) = \frac{-10,000}{(x+100)^2} \Bigg|_{200}^{\infty} = 0 + \frac{10,000}{(200+100)^2} = \frac{1}{9}$$

(b) Required: $P(80 \leq x \leq 120) = ?$

$$P(80 \leq x \leq 120) = \int_{80}^{120} f(x)\,dx$$

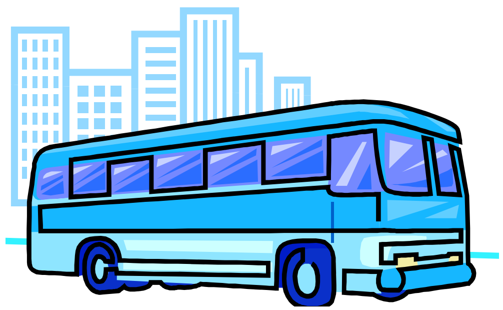
$$P(80 \leq x \leq 120) = \int_{80}^{120} \frac{20,000}{(x+100)^3} dx = \left(\frac{20,000}{(x+100)^2} * \frac{1}{-2} \right) \Bigg|_{80}^{120} = \frac{-10,000}{(x+100)^2} \Bigg|_{80}^{120}$$

$$= \frac{-10,000}{(120+100)^2} + \frac{10,000}{(80+100)^2} = -0.2066 + 0.3086 = 0.102 \cong 0.10$$

Example #7

- For a bus station, the waiting time between successive arrivals of a specific bus in hours is a random variable X having the following density function:

$$f(x)=\begin{cases}\dfrac{5-27x^2}{2}, & 0\le x<\dfrac{1}{3},\\[2ex] \dfrac{3}{2}(1-x), & \dfrac{1}{3}\le x<1\\[2ex] 0, & \text{elsewhere}\end{cases}$$



(a) Verify condition (2) of the continuous probability density function.

(b) Determine the probability of waiting 15 to 30 minutes.

(c) Obtain the corresponding cumulative distribution function.

(d) Use (c) to obtain the probability of waiting 15 to 30 minutes, and then compare the results to what you obtained in (b).

Example #7 (Sol.)

(a) Required: $\int_{-\infty}^{+\infty} f(x)dx = 1$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{0}^{1/3} \frac{5-27x^2}{2}dx + \int_{1/3}^{1} \frac{3}{2}(1-x)dx$$

$$= \left[\frac{1}{2}\left(5x - 27\frac{x^3}{3}\right)\right]_0^{1/3} + \left[\frac{3}{2}\left(x - \frac{x^2}{2}\right)\right]_{1/3}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{5}{3} - 9*\left(\frac{1}{3}\right)^3\right) - 0\right] + \frac{3}{2}\left[\left(1-\frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}*\left(\frac{1}{3}\right)^2\right)\right] = \frac{2}{3} + \frac{1}{3} = 1$$

Example #7 (Sol.)

(b) Required: $P\left(\frac{1}{4} \le x \le \frac{1}{2}\right) = ?$

$$P\left(\frac{1}{4} \le x \le \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)dx = \int_{1/4}^{1/3} \frac{5 - 27x^2}{2} dx + \int_{1/3}^{1/2} \frac{3}{2}(1 - x)dx$$

$$= \left[\frac{1}{2}\left(5x - 27\frac{x^3}{3}\right)\right]_{1/4}^{1/3} + \left[\frac{3}{2}\left(x - \frac{x^2}{2}\right)\right]_{1/3}^{1/2}$$

$$= \frac{1}{2}\left[\left(\frac{5}{3} - 9 * \left(\frac{1}{3}\right)^3\right) - \left(\frac{5}{4} - 9 * \left(\frac{1}{4}\right)^3\right)\right] + \frac{3}{2}\left[\left(\frac{1}{2} - \frac{1}{2} * \left(\frac{1}{2}\right)^2\right) - \left(\frac{1}{3} - \frac{1}{2} * \left(\frac{1}{3}\right)^2\right)\right]$$

$$= \frac{1}{2}\left[\frac{4}{3} - \frac{71}{64}\right] + \frac{3}{2}\left[\frac{3}{8} - \frac{5}{18}\right] = \frac{43}{384} + \frac{7}{48} = \frac{99}{384} = \frac{33}{128} \cong 0.26$$

Example #7 (Sol.)

(c) Required: $F(x) = ?$ $\quad F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

$0 \leq x < \frac{1}{3}$ $\Rightarrow$

$$F(x) = \int_0^x \frac{5-27t^2}{2}dt = \left[\frac{5t-27\,t^3/3}{2}\right]_0^x = \frac{x}{2}\left(5-9x^2\right)$$

$\frac{1}{3} \leq x < 1$ $\Rightarrow$

$$F(x) = \int_0^{1/3} \frac{5-27t^2}{2}dt + \int_{1/3}^{x} \frac{3}{2}(1-t)dt$$

$$= \left[\frac{1}{2}\left(5t - 27t^3/3\right)\right]_0^{1/3} + \left[\frac{3}{2}\left(t - \frac{t^2}{2}\right)\right]_{1/3}^{x}$$

$$= \frac{2}{3} + \frac{3x}{2}\left(1-\frac{x}{2}\right) - \left(\frac{3}{2}\right)\left(\frac{5}{12}\right) = \frac{1}{4} + \frac{3x}{2}\left(1-\frac{x}{2}\right) = \frac{1+6x-3x^2}{4}$$

Example #7 (Sol.)

(c)

$$F(x)=\begin{cases} 0 & , \quad x<0 \\ \frac{x}{2}\left(5-9x^2\right) & , \quad 0\le x<\frac{1}{3} \\ \frac{1+6x-3x^2}{4} & , \quad \frac{1}{3}\le x<1 \\ 1 & , \quad 1\le x \end{cases}$$

Example #7 (Sol.)

(d) Required: $P\left(\frac{1}{4} \le x \le \frac{1}{2}\right) = ?$ **Using F(X) obtained in (c)**

$$P\left(\frac{1}{4} \le x \le \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{1}{4}\right) = \left[\frac{1+6x-3x^2}{4}\right]_{x=\frac{1}{2}} - \left[\frac{x}{2}\left(5-9x^2\right)\right]_{x=\frac{1}{4}}$$

$$= \frac{1}{4} * \left[1 + 6 * \frac{1}{2} - 3 * \left(\frac{1}{2}\right)^2\right] - \frac{1}{2} * \left[\frac{1}{4}\left(5 - 9 * \left(\frac{1}{4}\right)^2\right)\right]$$

$$= 0.8125 - 0.5547 = 0.2578 \cong 0.26$$

same as obtained in (b)

Textbook Readings

- 3.1
- 3.2
- 3.3