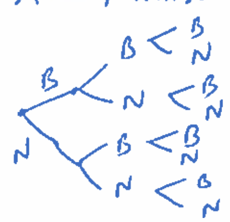
Concept of Random Variables

Question 1. Classify the following random variables as discrete or continuous

X: the number of automobile accidents per year in Virginia D
Y: the length of time to play 18 holes of golf C
M: the amount of milk produced yearly by a particular cow C
N: Number of bacteria in per cubic centimeter in drinking water D
T: Time spent by a school bus to reach school from a certain location C

Question 2. An overseas shipment of 6 foreign automobiles contains 3 that have slight paint blemishes. If an agency receives 3 of these automobiles at random, list the elements of the sample space S, using the letters B and N for blemished and nonblemished, respectively. Then, to each sample point, assign a value of x of the random variable X representing the number of automobiles with paint blemishes purchased by the agency.

N = Nonblemished
B = Blemished
X = Number of Bs



Sample Space	x
BBB	3
BBN	2
BNB	2
~~BNN NBB~~	~~0~~
BNN	1
NBB	2
NBN	1
NNB	1
NNN	0

Developing Probability Distribution Function

Question 3. Suppose that a day's production of 850 manufactured parts contains 50 parts that do not conform to customer requirements. Two parts are randomly selected in succession, without replacement, from the batch. Let the random variable X equal the number of nonconforming parts in the sample. (a) Find the distribution function of X in a tabular form and show it graphically. ~~(b) Find the probability distribution function and plot it.~~ (b) What is the cumulative distribution function of X? Graph the cumulative distribution function of X.

Events / ~~Sample~~   C = Conforming   N = Nonconforming   X = # of nonconforming

Sample point	X
CC	0
CN	1
NC	1
NN	2

$$P(X=0) = \frac{800}{850} \times \frac{799}{849} = 0.886$$

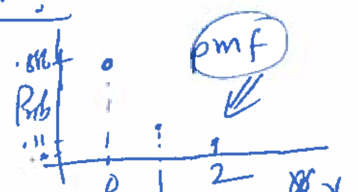
$$P(X=1) = \frac{800}{850} \times \frac{50}{849} + \frac{50}{850} \times \frac{800}{849} = .111$$

$$P(X=2) = \frac{50}{850} \cdot \frac{49}{849} = .003$$

(a) Probability distribution fn (pdf) in tabular form:

x	$f(x)$
0	.886
1	.111
2	.003

pmf



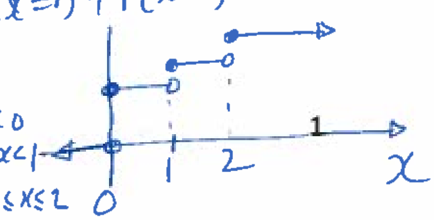
Cum. distri. fn (cdf)

$$F(x) = P(X \le x)$$

$$\begin{cases} F(0) = P(X \le 0) = .886 \\ F(1) = P(X=0) + P(X=1) = .997 \\ F(2) = 1 \end{cases}$$

(b) cdf:

$$F(x) = \begin{cases} 0, & x < 0 \\ .886, & 0 \le x < 1 \\ .997, & 1 \le x \le 2 \\ 1, & 2 \le x \end{cases}$$



Question 4. From a box containing 4 black balls and 2 green balls, 3 balls are drawn in succession, each ball being replaced in the box before the next draw is made. Find the probability distribution for the number of green balls.

$\boxed{\begin{matrix}4B\\2G\end{matrix}}$ $P(B) = 4/6 = 2/3$, $P(G) = \frac{2}{6} = 1/3$ X = number of green balls

Sample points	x	$f(x)$
BBB	0	$P(BBB) = P(B) * P(B) * P(B) = 2/3 \cdot 2/3 \cdot 2/3 = 8/27$
GBB	1	$P(G) * P(B) * P(B) = 1/3 * 2/3 * 2/3 = 4/27$
BGB	1	4/27
BBG	1	4/27
BGG	2	2/27
GBG	2	2/27
GGB	2	2/27
GGG	3	1/27

x	$f(x)$
0	8/27
1	12/27
2	6/27
3	1/27

Question 5. (a) Suppose the measurement error X of a certain physical quantity is decided by the density function. Determine k that renders f(x) a valid probability density function.

$$f(x) = \begin{cases} k(3 - x^2), & -1 \le x \le 1 \\ 0, & elesewhere \end{cases}$$

X is continuous. $\int_{-\infty}^{\infty} f(x) = 1 \Rightarrow \int_{-1}^{1} f(x)\,dx = 1 \Rightarrow \int_{-1}^{1} k(3-x^2)\,dx = 1$
$\Rightarrow k[3x - x^3/3]_{-1}^{1} = 1 \Rightarrow \frac{16k}{3} = 1$
$\Rightarrow k = 3/16$

(b) Determine the value of c so that the following function can serve as a probability distribution function of the discrete random variable X

$$f(x) = c\binom{2}{x}\binom{3}{3-x}, \qquad for\ x = 0,1,2$$

$\sum_{x=0}^{2} f(x) = 1 \Rightarrow c\left[\binom{2}{0}\binom{3}{3} + \binom{2}{1}\binom{3}{2} + \binom{2}{2}\binom{3}{1}\right] = 1 \Rightarrow c[1+6+3] = 1$
$\Rightarrow c = 1/10$.

Calculation of probability from distribution functions

Question 6. The total number of hours, measured in units of 100 hours, that a family runs a vacuum cleaner over a period of one year is a random variable X that has the density function

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & elesewhere \end{cases}$$

Find the probability that over a period of one year, a family runs their vacuum cleaner

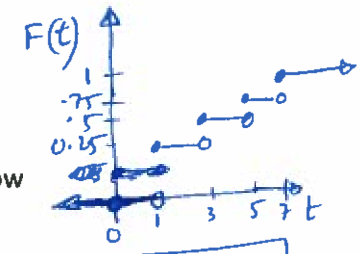
(a) Less than 120 hours $120/100 = 1.2$
(b) Between 50 and 100 hours
(c) At least 200 hours

a) $P(x < 1.2) = \int_{-\infty}^{1.2} f(x)\,dx = \int_{0}^{1} x\,dx + \int_{1}^{1.2} (2-x)\,dx = \frac{x^2}{2}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^{1.2} = 0.68$

b) $P(0.5 \le x \le 1) = \int_{0.5}^{1} x\,dx = \frac{x^2}{2}\Big|_{0.5}^{1} = 0.375$

$\boxed{P(a<x<b) = \int_a^b f(x)dx}$

c) $P(x > 2) = \int_2^{\infty} f(x)\,dx = 0$



Question 7. The cumulative distribution function of a random variable T is given below

F(t) is a step function, and therefore T is a discrete random variable that can have the values T = 1, 3, 5 and 7. The probability $P(T \neq 1 \text{ or } 3 \text{ or } 5 \text{ or } 7) = 0$

$$F(t) = \begin{cases} 0, & t < 1 \\ \frac{1}{4}, & 1 \le t < 3 \\ \frac{1}{2}, & 3 \le t < 5 \\ 3/4, & 5 \le t < 7 \\ 1, & t \ge 7 \end{cases}$$

$\boxed{F(x) = P(X \le x)}$

Find

(a) P (T=5) $= P(T \le 5) - P(T \le 4) - \underbrace{P(4 < T < 5)}_{0} = F(5) - F(4) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

(b) P(T>5) $= 1 - P(T \le 5) = 1 - F(5) = 1 - \frac{3}{4} = 1/4$

(c) P(1.4<T<6) $= P(T \le 6) - P(T \le 1.4) - P(T = 6) \cancel{- P(T = 1.4)}$
$= F(6) - F(1.4) - [F(6) - F(5)] = \frac{3}{4} - \frac{1}{4} - [\frac{3}{4} - \frac{3}{4}] = \frac{2}{4} = \frac{1}{2}$

(d) $P(T \le 5 | T \ge 2)$

$$= \frac{P(T \le 5 \cap T \ge 2)}{P(T \ge 2)} = \frac{P(2 \le T \le 5)}{P(T \ge 2)} = \frac{P(T \le 5) - P(T \le 2) + P(T = 2)}{1 - P(T \le 2)}$$
$$= \frac{F(5) - F(2) + [F(2) - F(1)]}{1 - F(2)} = \frac{\frac{3}{4} - \frac{1}{4}}{1 - 1/4} = \frac{2}{3}$$

Question 8. A shipment of 7 television contains 2 that are defective. A hotel makes a random purchase of 3 televisions. If x is the number of defective televisions purchased by the hotel, (a) find the probability distribution of X. (b) Express the results graphically as a histogram. (c) Find F(x) (d) Find P (X=1) (e) Find $P(0<X\le 2)$

as a formula

5 good	2 defective
3	0
2	1
1	2

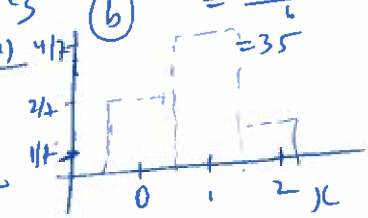
a) Let x = Number of defective
Then $x = 0, 1, 2$
We choose x defective from 2 defective and $(3-x)$ good from 5 good
$n = {}^2C_x \cdot {}^5C_{3-x}$, Total, $N = {}^7C_3$

(a) Then, $P(X = x) = f(x) = \frac{n}{N} = \frac{\binom{2}{x}\binom{5}{3-x}}{{}^7C_3}$, ${}^7C_3 = \frac{7!}{3!4!} = \frac{5 \times 6 \times 7}{6} = 35$

Then pdf:

x	0	1	2
$f(x)$	$\frac{\binom{2}{0}\binom{5}{3}}{{}^7C_3} = \frac{5!}{3!2!} \times \frac{1}{35} = 10/35 = 2/7$	$\binom{2}{1}\binom{5}{2}/35 = 2 \times \frac{5!}{2!3!} \times \frac{1}{35} = 4/7$	$\binom{2}{2}\binom{5}{1}/35 = 5/35 = 1/7$

(b)



c) $F(0) = P(X \le 0) = 2/7$
$F(1) = P(X \le 1) = 2/7 + 4/7 = 6/7$
$F(2) = 2/7 + 4/7 + 1/7 = 1$

$$F(x) = \begin{cases} 0, & x < 0 \\ 2/7, & 0 \le x \le 1 \\ 6/7, & 1 \le x \le 2 \\ 1, & x \ge 2 \end{cases}$$

d) $P(X = 1) = P(X \le 1) - P(X \le 0) = F(1) - F(0) = 6/7 - 2/7 = 4/7$

e) $P(0 < X \le 2) = P(X \le 2) - P(X \le 0) = F(2) - F(0) = 1 - 2/7 = 5/7$.

Question 9. (a) Find the cumulative distribution function (cdf) for the following probability density function (pdf) of the continuous random variable X.

$$f(x) = \begin{cases} (\frac{3}{16})(3 - x^2), & -1 \le x \le 1 \\ 0, & elesewhere \end{cases}$$

(b) Calculate P(X<1/2) using the cdf and the pdf.

a) $\boxed{F(x) = \int_{-\infty}^{\infty} f(t)\,dt = P(X \le x)}$

For $-1 \le x \le 1$, $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-1}^{x} \frac{3}{16}(3-t^2)\,dt = \frac{3}{16}\left[3t - \frac{t^3}{3}\right]_{-1}^{x}$

$= \frac{3}{16}\left[3x - \frac{x^3}{3} - (-3 + \frac{1}{3})\right]$

$= \frac{1}{2} + \frac{9}{16}x - \frac{x^3}{16}, \quad -1 \le x \le 1$

Then, $F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} + \frac{9}{16}x - \frac{x^3}{16} & -1 \le x < 1 \\ 1, & 1 \le x \end{cases}$

b) Using cdf $P(X < \frac{1}{2}) = P(X \le \frac{1}{2}) = F(\frac{1}{2}) = \frac{99}{128}$, Using pdf, $P(X < \frac{1}{2}) = \int_{-1}^{1/2} \frac{3}{16}(3 - x^2)\,dx$

$= \frac{3}{16}\left[3x - x^3/3\right]_{-1}^{1/2} = \frac{99}{128}$

Question 10. The total number of hours, measured in units of 100 hours, that a family runs a vacuum cleaner over a period of one year is a random variable X that has the density function

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & elesewhere \end{cases}$$

(a) Find the cdf and use the cdf to calculate the probability that over a period of one year, a family runs their vacuum cleaner (i) less than 120 hours (ii) Between 50 and 100 hours (Compare answers with Q6).

(b)Find the average of X

(c) Find the variance and standard deviation of X

$F(x) = \begin{cases} 0, & x \le 0 \\ x^2/2, & 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$

a) $F(x) = 0$ for $x \le 0$

$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x t\,dt = \frac{x^2}{2}$ for $0 < x < 1$

$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^1 t\,dt + \int_1^x (2-t)\,dt = \frac{1}{2} + \left[2t - \frac{t^2}{2}\right]_1^x$

$= \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2} = 2x - \frac{x^2}{2} - 1$

for $1 \le x < 2$

$F(x) = 1, \quad x \ge 2$

(i) $P(X < 1.2) = P(X \le 1.2) = F(1.2) = 2 \times 1.2 - \frac{1.2^2}{2} - 1 = 0.68$

(ii) $P(.5 < X < 1) = F(1) - F(.5) = (2 \times 1 - \frac{1}{2} - 1) - \frac{1}{8} = \frac{1}{2} - \frac{1}{8} = .375$

b) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 (2x - x^2)\,dx = \frac{x^3}{3}\Big|_0^1 + \left(x^2 - \frac{x^3}{3}\right)\Big|_1^2$

$= 1 = \mu$

Avg. # of hours/yr $= 1 \times 100 = 100$ hrs

(c) $E(X^2) = \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2(2-x)\,dx = 7/6$

$\sigma^2 = E(X^2) - \mu = 7/6 - 1 = 1/6$