ENGG 319, F16
Midterm Solution

University of Calgary
Schulich School of Engineering

Thursday, November 3, 2016
5:00 pm -7:00 pm

ENGG 319: Probability and Statistics for Engineers
Midterm Examination (Fall 2016)

Instructions:

- Write your name and ID at the top of every page
- This exam is open book, closed note, one cheat sheet is allowed
- You must show all of your work to receive full marks

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Section A (Multiple Choice Questions, Q1-Q13)

[Instructions: Each question in Section A carries 2 marks. Circle the letter of the correct answer and show your work to justify your answer. You must show your work to receive full marks. Partial marks will be awarded for partially correct calculations.]

Q # 1

The probability density function of a random variable X is given by:

$$f(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

Find the variance of X.

(a) 0.05
(b) 0.025
(c) 0.03
(d) 0.02
(e) None of the above

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$
$$= \int_0^1 6x^2(1-x)\, dx$$
$$= \int_0^1 (6x^2 - 6x^3)\, dx$$
$$= \left(2x^3 - \tfrac{3}{2}x^4\right)\Big|_0^1 = \tfrac{1}{2}$$

$$E(X^2) = \int_0^1 (6x^3 - 6x^4)\, dx = \left(\tfrac{3}{2}x^4 - \tfrac{6}{5}x^5\right)\Big|_0^1 = \tfrac{3}{10}$$

$$\sigma^2 = E(X^2) - \mu^2 = \tfrac{3}{10} - \tfrac{1}{4} = \tfrac{1}{20} = \boxed{0.05}$$

Alternatively, $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_0^1 \left(x - \tfrac{1}{2}\right)^2 (6x - 6x^2)\, dx$
$$= \tfrac{1}{20} = \boxed{0.05}$$

Q # 2

For the random variable and probability density function given in Question # 1, find the probability, P(0.059 < X < 1.059). Indicate the nearest number.

(a) 0.98
(b) 0.95
(c) 0.92
(d) 0.89
(e) 0.88

$$P(0.059 < X < 1.059) = \int_{0.059}^{1.059} (6x - 6x^2)\, dx$$
$$= (3x^2 - 2x^3)\Big|_{0.059}^{1.059}$$
$$= 0.989 - 0.01 = 0.979 \simeq \boxed{0.98}$$

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Q # 3

You are given a set of 4 keys, only one opens a certain door. The keys are unknown to you. You try the keys successively without replacement until the door is opened. What is the probability that the right key appears the third time? Indicate the nearest number.

(a) 1/4
(b) 3/8
(c) 3/4
(d) 3/32
(e) None of the above

Keys: Key 1, K2, K3, K4

Doors: | Fail | Fail | Success |

$P(3^{rd}\ time) = \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \boxed{\frac{1}{4}}$ Ans

Q # 4

It is known that 70% of the products of a manufacturing process are of grade A and the remaining are of grade B. 5 products are chosen randomly for inspection. Let X = number of products that are of grade A. What is the variance of X?

(a) 3.5
(b) 3/8
(c) 3/4
(d) 3/32
(e) None of the above

$X \sim b(x; n, p)$ $\quad p(\text{grade } A) = 0.7$

$\sigma^2 = npq = np(1-p) = 5 \times 0.7 \times 0.3 = \boxed{1.05}$

Q # 5

Entry to a summer school is determined by a test. Scores in this test are normally distributed with mean, $\mu = 500$ and standard deviation, $\sigma = 100$. Determine what percentage of students will probably score higher than 700. Indicate the nearest number.

(a) 1.6%
(b) 2.9%
(c) 1.9%
(d) 2.6%
(e) 2.3%

$\mu = 500, \sigma = 100$

$P(X > 700) = P\left(Z > \frac{700-\mu}{\sigma}\right) = P(Z > 2)$

$= 1 - P(Z < 2)$

$= 1 - \phi(2)$

$= 1 - 0.9772$

$= 0.0228$

$= 2.28\% \simeq \boxed{2.3\%}$

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Q # 6

The continuous random variable X follows an exponential distribution. The probability density function of X is given by

$$f(x) = \begin{cases} e^{-x}, x > 0 \\ 0, otherwise \end{cases}$$

Solution #1:

For exponential distribution,

$\mu = \beta$ and $\sigma^2 = \beta^2$

$\Rightarrow \sigma = \beta$

Then, $\boxed{\mu = \sigma}$

Which of the following is true?

(a) The mean of X is equal to its standard deviation
(b) The mean of X is twice of variance
(c) The mean of X is half of its variance
(d) The mean of X is greater than its standard deviation
(e) None of the above

Sol^n #2

Given, $f(x) = e^{-x} = \frac{1}{1} e^{-x/1}$

So, $\beta = 1$

Then, $\mu = \beta = 1$

$\sigma^2 = \beta^2 = 1 \Rightarrow \sigma = 1$

So, $\boxed{\mu = \sigma}$

Q # 7

For a biased coin the probability of observing a head is not equal to the probability of observing a tail. Let X = number of heads observed when a biased coin is flipped three times. Given that the probability of X=1 is 4/9. What is the probability of observing a head if the coin is thrown once?

(a) 2/3
(b) 1/3
(c) 2/27
(d) 1/27
(e) None of the above

Let, p = prob of observing head

Then, $P(X=1) = b(x=1, n=3, p)$

$= \binom{3}{1} p(1-p)^2 = 3p(1-p)^2$

Given, $3p(1-p)^2 = \frac{4}{9}$

$\Rightarrow p(1-p)^2 = \frac{4}{27}$

Upon substitution of the given values,

For, (a), $p = 2/3$, $L.S. = \frac{2}{3} \times \left(\frac{1}{3}\right)^2 = \frac{2}{27} \neq R.S.$

For, (b), $p = 1/3$, $L.S. = \frac{1}{3} \times \left(\frac{2}{3}\right)^2 = \frac{4}{27} = R.S.$

So, $p = \frac{1}{3}$

Then, $P(X=1, n=1) = p = \boxed{\frac{1}{3}}$

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Q # 8

10% of all the components produced in a manufacturing process are defective. An engineer decides to test a sequence of these components one by one until she finds a defective component. Find the probability that she finds a defective component at some time after the second attempt.

(a) 0.01
(b) 0.9025
(c) 0.81
(d) 0.09
(e) None of the above

Let 1st ($k=1$) success occurs on x^{th} trial.

Then, $x \sim b^*(x, k=1, p) = pq^{x-1}$ (geometric)

P(1st success after 2nd trial)

$= P$(1st success does not occur at $x=1$ & $x=2$)

$= 1 - P(x=1) - P(x=2)$

$= 1 - pq^0 - pq = 1 - p - pq = 1 - 0.1 - 0.1 \times 0.9$

$= 1 - 0.19$

$= \boxed{0.81}$

Q # 9

On a garage sale, a box containing 12 classic music CDs and 10 country music CDs was offered with a very good deal. If a buyer chose five CDs randomly from the box without replacement, what is the probability that less than three of the chosen CDs were country music? Choose the nearest number.

(a) 0.5685
(b) 0.5940
(c) 0.5639
(d) 0.5984
(e) 0.5635

$N = 22$

$\boxed{K=10 \quad N-K=12}$ $\longrightarrow$ $n=5$ $\boxed{x = 0, 1, 2 \quad n-x = 5, 4, 3}$

P(less than 3 country music)

$$= \frac{\binom{10}{0}\binom{12}{5} + \binom{10}{1}\binom{12}{4} + \binom{10}{2}\binom{12}{3}}{\binom{22}{5}}$$

$= \frac{15642}{26334} = \boxed{0.5940}$

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Q # 10

A CD player company claims that only 10% of its manufactured devices will need any repairs within the free warranty period of 12 months. If this claim is true, find the probability that 5 out of 20 of its manufactured devices will require any repairs within the first year of usage. Indicate the nearest number.

(a) 3.4 %
(b) 6.8%
(c) 5.5 %
(d) 6.4%
(e) 3.2%

Sol#1: X = Number of devices requring repair

$X \sim b(x; n=20, p=0.1)$

$$P(X=5) = \binom{20}{5}(0.1)^5(0.9)^{15}$$

$$= 15504 \times 10^{-5} \times 0.2058 = 0.0319 = 3.19\% \simeq \boxed{3.2\%}$$

Sol#2: Alternatively,

$$P(X=5) = \sum_{x=0}^{r=5} b(x; 20, 0.1) - \sum_{x=0}^{r=4} (x; 20, 0.1)$$

$$= 0.9887 - 0.9568 \quad \text{(P731)}$$

$$= 0.0319 = 3.19\% \simeq 3.2\%$$

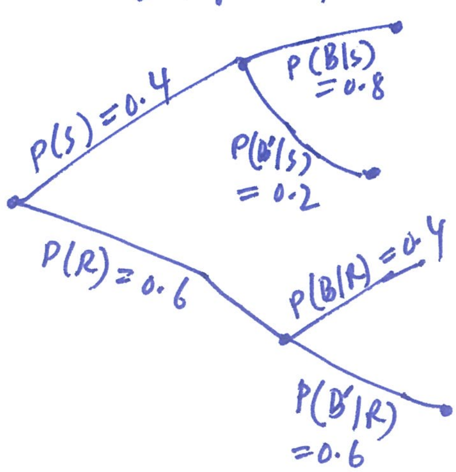
Q # 11

Based on the weather forecast, it is going to be sunny or rainy tomorrow with 40% and 60% chance, respectively. If it is sunny, you go biking with 80% probability, otherwise, there is a 40% chance you go biking. What is the probability that it is going to be rainy, given that you don't actually go biking? Indicate the nearest number.

(a) 0.16
(b) 0.15
(c) 0.82
(d) 0.116
(e) 0.083

Sample space = { sunny (40%), rainy (60%) }

B = biking
S = sunny
R = rainy



$$P(R|B') = \frac{P(R \cap B')}{P(B')}$$

$$= \frac{P(B'|R)\,P(R)}{P(B'|R)P(R) + P(B'|S)P(S)}$$

$$= \frac{0.6 \times 0.6}{0.6 \times 0.6 + 0.2 \times 0.4}$$

$$= \frac{0.36}{0.44} = \boxed{0.82}$$

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Q # 12

In a battleship video game, a single torpedo has the following mutually exclusive events and corresponding probabilities when fired at a ship: hitting and sinking it (0.50), hitting and damaging it (0.20), missing it (0.30). In addition, the ship will sink if it is damaged twice. What will be the probability of sinking the ship when four torpedoes are fired at it at the same time? Indicate the nearest number.

(a) 0.02
(b) 0.150
(c) 0.008
(d) 0.97
(e) 0.083

(S,D,M) (S,D,M) (S,D,M) (S,D,M)

S = Sinking
D = Damage
M = Miss

$3 \times 3 \times 3 \times 3 = 81$ sample points.

Out of 81 sample points, ship will NOT SINK in the following cases {MMMM, DMMM, MMDM, MMMD}

$P(\text{Not sink}) = P(4\text{ miss}) + P(3 \text{ miss \& one damage})$

$= (0.3)^4 + \binom{4}{1}(0.3)^3(0.2) = 0.0081 + 0.0216$

$= 0.0297$

$P(\text{sink}) = 1 - 0.0297 = \boxed{0.9703}$

Q # 13

If the random variable X has the following probability distribution function:

$$f(x) = \begin{cases} mx, & -1 < x < 0 \\ nx, & 0 \le x < 1 \\ 0, & otherwise \end{cases}$$

The mean value of X is 1/3. Indicate the values of the constants 'm' and 'n'.

(a) m = -1/2, n = -3/2
(b) m = -1/2, n = 3/2
(c) m = 1/2, n = 3/2
(d) m = 1/2, n = -3/2
(e) None of the above

$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-1}^{0} mx\,dx + \int_{0}^{1} nx = 1$

$\Rightarrow m\frac{x^2}{2}\Big|_{-1}^{0} + n\frac{x^2}{2}\Big|_{0}^{1} = 1$

$\Rightarrow m[0 - \frac{1}{2}] + n[\frac{1}{2} - 0] = 1$

$\Rightarrow n = 2 + m$

$\mu = \int_{-1}^{0} mx^2 dx + \int_{0}^{1} nx^2 dx$

$= m\frac{x^3}{3}\Big|_{-1}^{0} + n\frac{x^3}{3}\Big|_{0}^{1} = \frac{m}{3} + \frac{n}{3} = \frac{1}{3}$

$\Rightarrow m + n = 1$

$\Rightarrow m + 2 + m = 1$

$\Rightarrow \boxed{m = -\frac{1}{2},\ n = \frac{3}{2}}$

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SECTION B (Q14-Q15)

Q # 14 (6 marks)

Assume fishing is a Poisson process with the average of 0.6 fish per hour. Suppose that you go fishing for 2 hours and if no fish got caught in the first two hours, you continue fishing until the first catch.

(2 marks) (a) What is the probability that you go home after 2 hours of fishing?
(2 marks) (b) What is the probability that you catch at least 2 fish in the first 2 hours?
(2 marks) (c) What is the average fishing time?

Let X = Number of fish(es) caught ~~in~~ within the 1st two hours ($t = 2h$).

$\lambda = 0.6$ fish/hr

$E(X) = \mu = \lambda t = 1.2$ fish

(a) $P(\text{go home after } 2\text{hr}) = P(X \geq 1)$
$= 1 - P(X=0) = 1 - \frac{e^{-\lambda t}(\lambda t)^0}{0!}$
$= 1 - e^{-1.2} = \boxed{0.699}$

(b) $P(X \geq 2) = 1 - P(X=0) - P(X=1)$
$= 1 - e^{-\lambda t}\left[\frac{(\lambda t)^0}{0!} + \frac{(\lambda t)^1}{1!}\right] = 1 - e^{-1.2} \times 2.2$
$= 1 - 0.6626$
$= \boxed{0.337}$

(c) T = total fishing time

$T_{avg} = 2h$

69.9 % chance, from part (A)

Case I: go home after 2 hrs.

$T_{avg} = 2 + \beta = 2 + 1/.6 = 3.6667$

2h | $\beta = \frac{1}{\lambda} = 1/.6$ hours | 1st catch

$(100 - 69.9) = 30.1$ % chance

Case II: fish for more than two hours.

In case II, the extra time is an exponential random variable with mean $= \beta = 1/\lambda = 1/0.6$ hours.

$T_{avg} = (0.699)(2h) + (0.301)(\cancel{1/.6}\ 2 + 1/.6 \text{ hr}) = \boxed{2.5 \text{ hr}}$

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Q # 15 (8 marks)

Suppose that you are working in a company manufacturing a certain type of component. The average time to failure for these components is 5 years (assuming that all components' lifetimes follow exponential distribution). One of your clients ordered a system including 5 of these components working independently. This system is functioning well if at least 3 of the components work well.

(2 marks) (a) Calculate the reliability of a component at its average time to failure.

(2+2 marks) (b) What is the probability that the system fails after 8 years?

(2 marks) (c) What is the probability that two out of 5 such systems will fail after 8 years?

(a) Let X = component lifetime in yrs

$X \sim$ exponential with $E(X) = \beta = 5$ yrs

Reliability at $X = X_{avg} = \beta$ $= R(\text{at } X=\beta) = P(X>\beta)$

$= 1 - F(\beta)$

$= e^{-X/\beta} = e^{-\beta/\beta}$

$= e^{-1} = \boxed{0.3679}$

(b) ~~Let~~

P(component fails after 8 yrs)

$= P(X \geq 8) = 1 - P(X<8) = 1 - F(8) = e^{-8/\beta} = e^{-8/5}$

$= 0.201$ [2 marks]

Let Y = Number of components that fails after 8 yrs out of 5 components

Then, $Y \sim b(y; n=5, p=\cdot 201)$

$P(\text{system fails}) = P(Y \leq 2) = P(Y=0) + P(Y=1) + P(Y=2)$

$= \binom{5}{0}(\cdot 201)(\cdot 799)^5 + \binom{5}{1}(\cdot 201)(\cdot 799)^4 + \binom{5}{2}(\cdot 201)^2 \times (\cdot 799)^3$

$= 0.3256 + \cdot 4096 + \cdot 2061$

$= \boxed{0.9413}$

[Alternatively, using binomial sums, $P(Y\leq 2) = \sum_{y=0}^{r=2} (y, n=5, p\simeq 0.2)$

$= 0.9421$ (P 726)]

(c) Let Z = Number of systems that survives out of five.

$Z \sim b(z, n=5, p=\cdot 9413)$ $\binom{5}{2}$ $P(Z=2) = \binom{5}{2}(\cdot 9413)^2(\cdot 0587)^3$

$= \boxed{0.0018}$ Ans.